

Research Article

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Homogenization of a net of periodic critically scaled boundary obstacles related to reverse osmosis “nano-composite” membranes

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Abstract: One of the main goals of this paper is to extend some of the mathematical techniques of some previous papers by the authors showing that some very useful phenomenological properties which can be observed at the nano-scale can be simulated and justified mathematically by means of some homogenization processes when a certain critical scale is used in the corresponding framework. Here the motivating problem in consideration is formulated in the context of the reverse osmosis. We consider, on a part of the boundary of a domain $\Omega \subset \mathbb{R}^n$, a set of very small periodically distributed semipermeable membranes having an ideal infinite permeability coefficient (which leads to Signorini-type boundary conditions) on a part Γ_1 of the boundary. We also assume that a possible chemical reaction may take place on the membranes. We obtain the rigorous convergence of the problems to a homogenized problem in which there is a change in the constitutive nonlinearities. Changes of this type are the reason for the big success of the nanocomposite materials. Our proof is carried out for membranes not necessarily of radially symmetric shape. The definition of the associated critical scale depends on the dimension of the space (and it is quite peculiar for the special case of $n = 2$). Roughly speaking, our result proves that the consideration of the critical case of the scale leads to a homogenized formulation which is equivalent to having a global semipermeable membrane, at the whole part of the boundary Γ_1 , with a “finite permeability coefficient of this virtual membrane”, which is the best we can get, even if the original problem involves a set of membranes of any arbitrary finite permeability coefficients.

Keywords: Homogenization, critical scale, reverse osmosis, Signorini boundary conditions, elliptic partial differential equations, strange term

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1 Introduction and statement of results

We present new results concerning the asymptotic behavior, as $\varepsilon \rightarrow 0$, of the solution u_ε of a family of boundary value problems formulated in a cavity (or plant) represented by a bounded domain $\Omega \subset \mathbb{R}^n$ in which a linear diffusion equation is satisfied. The boundary $\partial\Omega$ is split into two regions. On one of the regions,

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homogeneous Dirichlet conditions are specified. On the other one, some small subsets G_ε are ε -periodically distributed and some unilateral boundary conditions are specified on them. We also assume that a possible “reaction” may take place on a net G_ε of small pieces of the boundary given by the periodic repetition of a rescaled particle G_0 .

There are several relevant problems in a wide spectrum of applications leading to such type of formulations, ranging from water and wastewater treatment to food and textile engineering as well as pharmaceutical and biotechnology applications (for a recent review see [29]). One of them concerns the reverse osmosis when we apply it, for instance, to desalination processes (see, e.g., [24] and the references therein). Without intending to use here a “realistic model”, we shall present an over-simplified formulation that, nonetheless, preserves most of the mathematical difficulties concerning the passing to the limit as $\varepsilon \rightarrow 0$. Some examples of more complex formulations covering different aspects of the problems considered here can be found, for instance, in [17] and the many references therein.

We start by recalling that, roughly speaking, semipermeable membranes allow the passing of a certain type of molecules (the so-called “solvents”) but block another type of molecules (the “solutes”). The solvents flow from the region of smaller concentration of solutes to the region of higher concentration (the difference of the concentrations produces the phenomenon known as *osmotic pressure*). Nevertheless, by creating a very high pressure it is possible to produce an inverse flow, such as the one being used in desalination plants: it is the so-called “reverse osmosis”. Since in many cases the semipermeable membrane contains some chemical products (e.g., polyamides; see [18]), our formulation will contain also a nonlinear kinetic reaction term in the flux given by a continuous nondecreasing function $\sigma(s)$. Let us call w_ε the solvent concentration corresponding to the membrane periodicity scale ε . Let us modulate the intensity of the reaction in terms of a factor ε^{-k} , with k to be analyzed later. So, for a critical value of the solvent concentration ψ (associated to the osmotic pressure) the flux (including the reaction kinetic term) is an incoming flux with respect to the solvents plant Ω if the concentration of the solvent molecules $w(x)$ on the semipermeable membrane $G_\varepsilon \subset \partial\Omega$ is smaller than or equal to this critical value, but it remains isolated (with no boundary flow, excluding the reaction term, on the membrane, i.e. when the concentration is $w(x) < \psi$). So, if ν is the exterior unit normal vector to the membrane surface, we have

$$\begin{aligned}\partial_\nu w_\varepsilon + \varepsilon^{-k} \sigma(\psi - w_\varepsilon) &= 0 && \text{on } \{x \in G_\varepsilon \subset \partial\Omega : w_\varepsilon(x) > \psi\}, \\ \partial_\nu w_\varepsilon + \varepsilon^{-k} \sigma(\psi - w_\varepsilon) &= -\varepsilon^{-k} \mu(\psi - w_\varepsilon) && \text{on } \{x \in G_\varepsilon : w_\varepsilon(x) \leq \psi\},\end{aligned}$$

for some parameter $\mu > 0$ called the “finite permeability coefficient of the membrane” (usually, in practice, μ takes big values). We assume a simplified linear diffusion equation on the solvent concentration

$$-\Delta w = F \text{ in } \Omega,$$

and some boundary conditions on the rest of the boundary $\partial\Omega$. For instance, we can distinguish some sub-regions where Dirichlet or Neumann types of boundary conditions hold, and so, if we introduce the partition $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and assume that, in fact, $G_\varepsilon \subset \Gamma_1$, then we can imagine that

$$\partial_\nu w_\varepsilon(x) = h(x) \quad \text{on } x \in \Gamma_1 \setminus \overline{G_\varepsilon}$$

and

$$w_\varepsilon(x) = g(x) \quad \text{on } x \in \Gamma_2.$$

Figure 1 presents a simplified case of the above-mentioned framework.

We are especially interested in the study of new behaviors arising in the reverse osmosis membranes having a periodicity ε of the order of nanometers (see, e.g., [4] and its references). Mathematically, we shall give a sense to those extremely small scales by demanding that the diameter of these subsets included in G_ε is of order a_ε , where $a_\varepsilon \ll \varepsilon$.

Sometimes it is interesting to consider semipermeable membranes with an “infinite permeability coefficient” (formally $\mu = +\infty$, but only for the case $w_\varepsilon(x) = \psi$), and thus ψ becomes an obstacle which is

that, by using some different methods of proof, the critical case was solved for $n \geq 3$ in [35] (see also [21]). The consideration of the case $n = 2$ and for arbitrary shaped domains G_ε was carried out in [33]. More recently, many results concerning the asymptotic behavior of solutions of problems similar to (1.1) were published in the literature [9–12, 19, 20, 23, 35]. Nevertheless, in all the above-mentioned works the particles (or perforations, according to the physical model used as motivation of the mathematical formulation) of subsets G_ε were assumed to be balls (having a critical radius). We also mention here the paper [14] that describes the asymptotic behavior of some related problem for the case of arbitrary shaped sets G_ε and for $n \geq 3$. One of the main goals of this paper is to extend some of the techniques of [14] to problem (1.1), where the periodically distributed reactions arise merely on some part of the global boundary $\partial\Omega$ for the critical scaled case. This is the case for which some phenomenological properties which arise at the nano-scale can be simulated and justified by means of homogenization processes.

Case $n \geq 3$. In order to present the main results of this paper, and their application to the reverse osmosis framework, we need to introduce some auxiliary notations. We start by considering the case $n \geq 3$. We assume that Ω is a bounded domain in $\mathbb{R}^n \cap \{x_1 > 0\}$, $n \geq 3$, with a piecewise-smooth boundary $\partial\Omega$ that consists of two parts Γ_1 and Γ_2 , with the property that

$$\Gamma_1 = \partial\Omega \cap \{x \in \mathbb{R}^n : x_1 = 0\} \neq \emptyset.$$

We consider a model G_0 such that $\overline{G_0} \subset \{x \in \mathbb{R}^n : x_1 = 0, |x| < \frac{1}{4}\}$ with $\overline{G_0}$ being diffeomorphic to a ball. We define $\delta B = \{x : \delta^{-1}x \in B\}$, $\delta > 0$. Let

$$\widetilde{G_\varepsilon} = \bigcup_{j \in \mathbb{Z}'} (a_\varepsilon G_0 + \varepsilon j) = \bigcup_{j \in \mathbb{Z}'} G_\varepsilon^j,$$

where

$$\mathbb{Z}' = \{0\} \times \mathbb{Z}^{n-1}$$

and

$$a_\varepsilon = C_0 \varepsilon^k, \quad k = \frac{n-1}{n-2} \text{ and } C_0 > 0. \quad (1.2)$$

A justification of the above choice of exponent k can be found, for instance, in [28] (see also [32]). We define the net of sets G_ε as the union of sets $G_\varepsilon^j \subset \widetilde{G_\varepsilon}$ such that $G_\varepsilon^j \subset \Gamma_1$ and $\rho(\partial\Gamma_1, \overline{G_\varepsilon^j}) \geq 2\varepsilon$, i.e.

$$G_\varepsilon = \bigcup_{j \in Y_\varepsilon} G_\varepsilon^j,$$

where

$$Y_\varepsilon = \{j \in \mathbb{Z}' : \rho(\partial\Gamma_1, \overline{G_\varepsilon^j}) \geq 2\varepsilon\}.$$

Notice that $|Y_\varepsilon| \cong d\varepsilon^{1-n}$, $d = \text{const} > 0$. Later it will be useful to observe that if we denote by $T_r(x_0)$ the ball in \mathbb{R}^n of radius r centered at a point x_0 , and if we define the boundary points

$$P_\varepsilon^j = \varepsilon j = (0, P_{\varepsilon,2}^j, \dots, P_{\varepsilon,n}^j) \quad \text{for } j \in \mathbb{Z}',$$

and the set $T_{\varepsilon/4}^j = T_{\varepsilon/4}(P_\varepsilon^j)$, then we have $\overline{G_\varepsilon^j} \subset T_{\varepsilon/4}^j$. In this geometrical setting (see Figure 2), the so-called “strong formulation” of the problem for which we want to study the asymptotic behavior of its solutions is the following:

$$\begin{cases} -\Delta u_\varepsilon = f(x), & x \in \Omega, \\ u_\varepsilon \geq 0, \quad \partial_\nu u_\varepsilon + \varepsilon^{-k} \sigma(u_\varepsilon) \geq 0, \quad u_\varepsilon (\partial_\nu u_\varepsilon + \varepsilon^{-k} \sigma(u_\varepsilon)) = 0, & x \in G_\varepsilon, \\ \partial_\nu u_\varepsilon = 0, & x \in \Gamma_1 \setminus \overline{G_\varepsilon}, \\ u_\varepsilon = 0, & x \in \Gamma_2, \end{cases} \quad (1.3)$$

where $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is a locally Hölder continuous nondecreasing function and, at most, super-linear at infinity, i.e. such that

$$k_1 |s - t| \leq |\sigma(t) - \sigma(s)| \leq K_1 |t - s|^{\rho_1} + K_2 |s - t|^{\rho_2} \quad \text{for some } \rho_1, \rho_2 \in (0, 2] \quad (1.4)$$

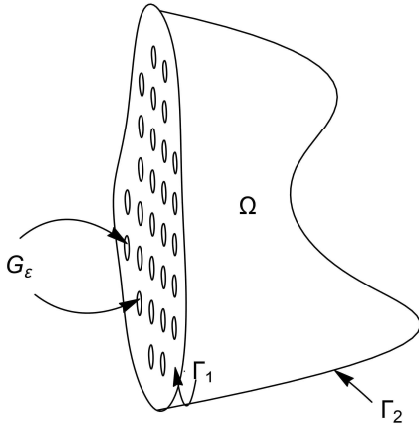


Figure 2: Domain Ω when $n \geq 3$.

for all $t, s \geq 0$, where $k_1, K_1, K_2 > 0$, $\sigma(0) = 0$. In problem (1.3), $v = (-1, 0, \dots, 0)$ is the unit outward normal vector Ω at $\{x_1 = 0\}$ and $\partial_\nu u = -\frac{\partial u}{\partial x_1}$ is the normal derivative of u at this part of the boundary.

Example 1.1. Notice that examples of such functions cover

$$\sigma(s) = \sqrt{s}.$$

Furthermore, the behavior at infinity may be superlinear as, for example, in

$$\sigma(s) = \begin{cases} \sqrt{s}, & 0 \leq s \leq s_0, \\ \sqrt{s_0} + (s - s_0)^2, & s > s_0. \end{cases}$$

The weak formulation of (1.3) is the following definition (see, e.g., [16]).

Definition 1.2. We say that u_ε is a weak solution of (1.1) if

$$u_\varepsilon \in K_\varepsilon = \{g \in H^1(\Omega, \Gamma_2) : g \geq 0 \text{ a.e. on } G_\varepsilon\}$$

and

$$\int_{\Omega} \nabla u_\varepsilon \nabla (\varphi - u_\varepsilon) dx + \varepsilon^{-k} \int_{G_\varepsilon} \sigma(u_\varepsilon) (\varphi - u_\varepsilon) dx' \geq \int_{\Omega} f(\varphi - u_\varepsilon) dx \quad (1.5)$$

for all $\varphi \in K_\varepsilon$.

By $H^1(\Omega, \Gamma_2)$ we denote the closure in $H^1(\Omega)$ of the set of infinitely differentiable functions in $\overline{\Omega}$, vanishing on the boundary Γ_2 .

It is well known (see, e.g., some references in [10]) that problem (1.5) has a unique weak solution $u_\varepsilon \in K_\varepsilon$. From (1.5) we immediately deduce that

$$\|\nabla u_\varepsilon\|_{L_2(\Omega)} \leq K, \quad (1.6)$$

where, here and below, the constant K is independent of ε . Hence, there exists a subsequence (denoted as the original sequence by \tilde{u}_ε) such that, as $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} \tilde{u}_\varepsilon &\rightharpoonup u_0 \quad \text{weakly in } H^1(\Omega, \Gamma_2), \\ \tilde{u}_\varepsilon &\rightarrow u_0 \quad \text{strongly in } L_2(\Omega). \end{aligned} \quad (1.7)$$

By using the monotonicity of the function $\sigma(u)$ one can show (see some references in [10]) that u_ε satisfies the following “very weak formulation”:

$$\int_{\Omega} \nabla \varphi \nabla (\varphi - u_\varepsilon) dx + \varepsilon^{-k} \int_{G_\varepsilon} \sigma(\varphi) (\varphi - u_\varepsilon) dx' \geq \int_{\Omega} f(\varphi - u_\varepsilon) dx, \quad (1.8)$$

where φ is an arbitrary function from K_ε .

The main goal of this paper is to consider this critical relation between parameters. This scale is characterized by the fact that the resulting homogenized problem will contain a so-called “strange term” expressing the fact that the character of some nonlinearity arising in the homogenized problem differs from the original nonlinearity appearing in the boundary condition of (1.3). Still focusing first on the case $n \geq 3$, one can show that the critical scale of the size of the holes is given by (see, e.g., the arguments used in [10, 28])

$$\alpha = \frac{n-1}{n-2}.$$

The appropriate scaling of the reaction term so that both the diffusive and nonlinear characters are preserved at the limit is, as usual, driven by $\varepsilon^k \sim |G_\varepsilon|$. Therefore,

$$k = \frac{n-1}{n-2}.$$

In the present paper, we construct a homogenized problem with a nonlinear Robin-type boundary condition that contains a new nonlinear term, and prove the corresponding theorem stating that the solution of the original problem converges, as $\varepsilon \rightarrow 0$, to the solution of the homogenized problem.

We point out that the main difficulties to get a homogenized problem associated to (1.3) come from the following different aspects:

- (i) The low differentiability assumed on the function σ (since it is non-Lipschitz continuous at $u = 0$ and it has quadratic growth at infinity).
- (ii) The unilateral formulation of the boundary conditions on G_ε .
- (iii) The general shape assumed on the sets G_ε .
- (iv) The critical scale of the sets G_ε^j .

Some of those difficulties were already in the previous short presentation paper by the authors [13], but only for $n = 2$, without (ii) and by assuming that σ is Lipschitz continuous. Our main goal is to extend our techniques to the above-mentioned more general framework.

To build the homogenized problem we still need to introduce some “capacity-type” auxiliary problems. Given $u \in \mathbb{R}$, for $y \in (\mathbb{R}^n)^+ = \mathbb{R}^n \cap \{y_1 > 0\}$ we introduce the new auxiliary function $\widehat{w}(y; u)$, depending also on G_0 and σ , as the (unique) solution of the problem

$$\begin{cases} -\Delta \widehat{w} = 0, & y \in (\mathbb{R}^n)^+, \\ \partial_\nu \widehat{w} - C_0 \sigma(u - \widehat{w}) = 0, & y \in G_0, \\ \partial_\nu \widehat{w} = 0, & y \notin G_0, y_1 = 0, \\ \widehat{w} \rightarrow 0 & \text{as } |y| \rightarrow \infty. \end{cases} \quad (1.9)$$

Remember that $C_0 > 0$ was given in the structural assumption (1.2). The existence and uniqueness of $\widehat{w}(y; u)$ is given in Lemma 2.2 below. Let us introduce also the auxiliary function $\widehat{\kappa}(y)$, $y \in (\mathbb{R}^n)^+$, as the unique solution of the problem

$$\begin{cases} \Delta \widehat{\kappa} = 0, & y \in (\mathbb{R}^n)^+, \\ \widehat{\kappa} = 1, & y \in G_0, \\ \partial_\nu \widehat{\kappa} = 0, & y \notin G_0, y_1 = 0, \\ \widehat{\kappa} \rightarrow 0, & \text{as } |y| \rightarrow \infty. \end{cases}$$

We then define, for $u \in \mathbb{R}$, the possibly nonlinear function

$$H_{G_0}(u) := \int_{G_0} \partial_\nu \widehat{w}(u, y') dy' = C_0 \int_{G_0} \sigma(u - \widehat{w}(u, y')) dy' \quad (1.10)$$

and the scalar

$$\lambda_{G_0} := \int_{G_0} \partial_\nu \widehat{\kappa}(y') dy', \quad (1.11)$$

where in both definitions $y' = (0, y_2, \dots, y_n)$. Notice that $\hat{\kappa}(y)$ can be extended by symmetry to $\mathbb{R}^n \setminus \overline{G_0}$ as a harmonic function. Moreover, by the maximum principle, $\hat{\kappa}(y)$ reaches its maximum in G_0 , and so by the strong maximum principle $\partial_\nu \hat{\kappa} = -\partial_{x_1} \hat{\kappa} > 0$. Then we know that

$$\lambda_{G_0} > 0.$$

Some properties of the function H_{G_0} will be presented later. In particular, in Lemma 2.8 below we will show that H_{G_0} is always a Lipschitz continuous function (even if σ is merely Hölder continuous) such that

$$0 \leq H'_{G_0} \leq \lambda_{G_0}. \quad (1.12)$$

The following theorem gives a description of the limiting function u_0 obtained in (1.7).

Theorem 1.3. *Let $n \geq 3$, let $\alpha = k = \frac{n-1}{n-2}$ and let u_ε be a weak solution of problem (1.3). Then the function u_0 defined in (1.7) is a weak solution of the following problem:*

$$\begin{cases} -\Delta u_0 = f, & x \in \Omega, \\ \partial_\nu u_0 + C_0^{n-2} H_{G_0}(u_{0,+}) - \lambda_{G_0} C_0^{n-2} u_{0,-} = 0, & x \in \Gamma_1, \\ u_0 = 0, & x \in \Gamma_2, \end{cases}$$

where H_{G_0} is defined by (1.10) and λ_{G_0} by (1.11). Here, as usual, $u_{0,+} := \max\{u_0, 0\}$ and $u_{0,-} = \max\{-u_0, 0\}$, so that $u_0 = u_{0,+} - u_{0,-}$.

Case $n = 2$. As mentioned before (see also [13]), the case $n = 2$ requires to introduce some slight changes. The domain is given now in the following way: We consider Ω to be a bounded domain in $\mathbb{R}^2 \cap \{x_2 > 0\}$, the boundary of which consists of two parts $\partial\Omega = \Gamma_1 \cup \Gamma_2$ and $\Gamma_1 = \partial\Omega \cap \{x_2 = 0\} = [-l, l]$, $l > 0$, $\Gamma_2 = \partial\Omega \cap \{x_2 > 0\}$. We set

$$Y_1 = \left\{ (y_1, 0) : -\frac{1}{2} < y_1 < \frac{1}{2} \right\}, \quad \hat{l}_0 = \{(y_1, 0) : -l_0 < y_1 < l_0\} \subset Y_1, \quad l_0 \in \left(0, \frac{1}{2}\right).$$

For a small parameter $\varepsilon > 0$ and $0 < a_\varepsilon < \varepsilon$, we introduce the sets

$$\widetilde{G}_\varepsilon = \bigcup_{j \in \mathbb{Z}'} (a_\varepsilon \hat{l}_0 + \varepsilon j) = \bigcup_{j \in \mathbb{Z}'} l_\varepsilon^j,$$

where \mathbb{Z}' is a set of vectors $j = (j_1, 0)$ and j_1 is a whole number. Set

$$Y_\varepsilon = \{j \in \mathbb{Z}' : \overline{l_\varepsilon^j} \subset \{x = (x_1, 0) : x_1 \in [-l + 2\varepsilon, l - 2\varepsilon]\}\}.$$

Consider $Y_\varepsilon^j = \varepsilon Y_0 + \varepsilon j$ and

$$l_\varepsilon = \bigcup_{j \in Y_\varepsilon} l_\varepsilon^j.$$

It is easy to see that $\overline{l_\varepsilon^j} \subset Y_\varepsilon^j$. Set $\gamma_\varepsilon = \Gamma_1 \setminus \overline{l_\varepsilon}$. Note that for all $j \in \mathbb{Z}'$ we have $|l_\varepsilon^j| = 2a_\varepsilon l_0$ and $|l_\varepsilon| \cong da_\varepsilon \varepsilon^{-1}$; see Figure 3.

The formulation of the problem starts by searching

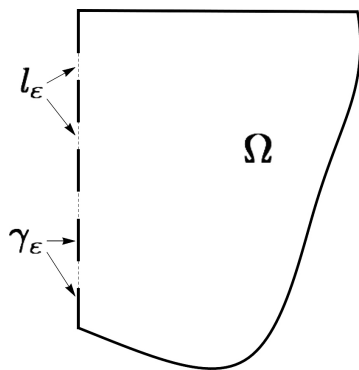
$$u_\varepsilon \in K_\varepsilon = \{g \in H^1(\Omega, \Gamma_2) : g \geq 0 \text{ a.e. on } l_\varepsilon\}, \quad (1.13)$$

being a solution to the following variational inequality:

$$\int_{\Omega} \nabla u_\varepsilon \nabla (\varphi - u_\varepsilon) dx + e^{\frac{\sigma^2}{\varepsilon}} \int_{G_\varepsilon} \sigma(u_\varepsilon) (\varphi - u_\varepsilon) dx_1 \geq \int_{\Omega} f(\varphi - u_\varepsilon) dx, \quad (1.14)$$

where φ is an arbitrary function from K_ε . This time, for simplicity, we assume merely that the function $\sigma : \mathbb{R} \rightarrow \mathbb{R}$ is continuously differentiable, $\sigma(0) = 0$ and there exist positive constants k_1, k_2 such that the following condition is satisfied:

$$k_1 \leq \partial_u \sigma(u) \leq k_2 \quad \text{for all } u \in \mathbb{R}. \quad (1.15)$$

Figure 3: Domain Ω in two dimensions.

(more general terms $\sigma(u)$ were considered in [13], but without the Signorini-type constraints). Therefore, we have

$$k_1 u^2 \leq u \sigma(u) \leq k_2 u^2 \quad \text{for all } u \in \mathbb{R}.$$

Note that (1.13), (1.14) is a weak formulation of the following strong formulation of the problem:

$$\begin{cases} -\Delta u_\varepsilon = f, & x \in \Omega, \\ u_\varepsilon \geq 0, \quad \partial_{x_2} u_\varepsilon - \beta(\varepsilon) \sigma(u_\varepsilon) \geq 0, \quad u_\varepsilon (\partial_{x_2} u_\varepsilon - \beta(\varepsilon) \sigma(u_\varepsilon)) = 0, & x \in l_\varepsilon, \\ \partial_{x_2} u_\varepsilon = 0, & x \in \gamma_\varepsilon, \\ u_\varepsilon = 0, & x \in \Gamma_1. \end{cases} \quad (1.16)$$

Our homogenized result in this case is the following theorem.

Theorem 1.4. Let $a_\varepsilon = C_0 \varepsilon e^{-a^2/\varepsilon}$, $\beta(\varepsilon) = e^{a^2/\varepsilon}$, $\alpha \neq 0$, $C_0 > 0$ and let u_ε be a solution to problem (1.16). Then there exists a subsequence such that $u_\varepsilon \rightarrow u_0$, strongly in $L^2(\Omega)$ and weakly in $H^1(\Omega, \Gamma_1)$ as $\varepsilon \rightarrow 0$, and the function $u_0 \in H^1(\Omega, \Gamma_1)$ is a weak solution to the following boundary value problem:

$$\begin{cases} -\Delta u_0 = f, & x \in \Omega, \\ \partial_{x_2} u_0 = \frac{\pi}{\alpha^2} (H_{l_0}(u_{0,+}) - u_{0,-}), & x \in \Gamma_1, \\ u_0 = 0, & x \in \Gamma_2, \end{cases}$$

where $H_{l_0}(u)$ verifies the functional equation

$$\pi H_{l_0}(u) = 2l_0 \alpha^2 C_0 \sigma(u - H_{l_0}(u)). \quad (1.17)$$

The special case of dimension $n = 2$ is illustrative in order to get a complete identification of the function H_{G_0} . Curiously enough, the characterization condition for H_{G_0} and H_{l_0} (see, e.g., the functional equations (1.10) and (1.17), respectively) is quite related (but not exactly the same) to the one obtained in [10], although the problem under consideration in that paper was not the same as problem (1.3). It was shown in [10] that if $H : \mathbb{R} \rightarrow \mathbb{R}$ is the solution of a problem

$$H(u) = C \sigma(u - H(u)),$$

then H is given by

$$H(u) = (I + (C\sigma)^{-1})^{-1}(u) \quad (1.18)$$

for $u \in \mathbb{R}$. When the Signorini condition is included, σ can be generalized to the maximal monotone graph

$$\bar{\sigma}(u) = \begin{cases} \sigma(u), & u > 0, \\ (-\infty, 0], & u = 0, \\ \emptyset, & u < 0. \end{cases}$$

It was proven in [10] that the corresponding zero-order term is given by

$$\tilde{H}(u) = \begin{cases} H(u), & u > 0, \\ u, & u \leq 0. \end{cases}$$

This behavior is interesting because \tilde{H} matches (1.18) formally. The maximal monotone graph

$$\gamma(u) = \begin{cases} 0, & u > 0, \\ (-\infty, 0], & u = 0, \\ \emptyset, & u < 0, \end{cases}$$

has, formally, the inverse

$$\gamma^{-1}(u) = \begin{cases} \emptyset, & u > 0, \\ [0, +\infty), & u = 0, \\ 0, & u < 0. \end{cases}$$

In particular,

$$H_\gamma(u) := (I + (C\gamma)^{-1})^{-1}(u) = \begin{cases} \emptyset, & u > 0, \\ [0, +\infty), & u = 0, \\ u, & u < 0. \end{cases}$$

In this way, it is clear that

$$\tilde{H}(u) = \begin{cases} H(u), & u > 0, \\ H_\gamma(u), & u < 0. \end{cases}$$

For the case $n = 3$ we will make further comments in this direction in Section 6.1.

Coming back to the framework of the semipermeable membranes problems, what we can conclude is that the homogenization of a set of periodic semipermeable membranes with an “infinite permeability coefficient”, in the critical case, leads to a homogenized formulation which is equivalent to having a global semipermeable membrane, at Γ_1 , with a “finite permeability coefficient of this virtual membrane” μ_∞ , which is *the best we can get*, even if the original problem involves a finite permeability μ . Indeed, this comes from the properties of the function H_{G_0} (which can also be computed for the case of a microscopic membrane with finite permeability), for instance, on the subpart of the boundary $\{x \in \Gamma_1 : w_0(x) \leq \psi\}$ (with $w_0(x) := \psi - u_0(x)$, i.e. where $u_0(x) \leq 0$). Moreover, by using that $H_{G_0}(0) = 0$ and the decomposition $u_0 = u_{0,+} - u_{0,-}$, we know now that $\partial_\nu w_0 = \lambda_{G_0} C_0^{n-2} w_0$, and thus, for the case of a microscopic finite permeability membrane, it is not difficult to show that we get a homogenized permeability membrane coefficient given by $\lambda_{G_0} C_0^{n-2}$, which is larger than μ (see more details in Section 6.1 below).

The plan of the rest of the paper is the following: Part I (containing Sections 2–6) is devoted to the study of the case $n \geq 3$ and contains also some comments and possible extensions (for instance, we give some link between the present homogenization results and the homogenization of some problems involving non-local fractional operators). Part II (containing Section 7) is devoted to the proof of the convergence result for $n = 2$.

Part I:

The case $n \geq 3$

2 Estimates on the auxiliary functions

2.1 On the auxiliary function $\hat{\kappa}$

By using the method of sub- and supersolutions as in [14], the following result holds.

Lemma 2.1. *There exists a unique solution $\widehat{\kappa} \in \mathbb{X}$ of problem (1.9), where*

$$\mathbb{X} = \left\{ v \in L^2_{\text{loc}}(\overline{(\mathbb{R}^n)^+}) : \nabla v \in L^2((\mathbb{R}^n)^+)^n, |v| \leq \frac{K}{|y|^{n-2}} \right\}$$

such that

$$0 \leq \widehat{\kappa}(y) \leq 1 \quad \text{for a.e. } y \in \overline{(\mathbb{R}^n)^+}$$

and

$$\widehat{\kappa}(y) \leq \frac{K}{|y|^{n-2}} \quad \text{for a.e. } y \in \overline{(\mathbb{R}^n)^+}.$$

We define also the family of auxiliary functions

$$\widehat{\kappa}_\varepsilon^j = \kappa \left(\frac{x - P_\varepsilon^j}{a_\varepsilon} \right).$$

2.2 On the auxiliary function \widehat{w}

Arguing as above, in terms similar to the paper [14], we get the following lemma.

Lemma 2.2. *Let σ be a maximal monotone graph. There exists a unique solution $\widehat{w}(u, \cdot) \in \mathbb{X}$ of problem (1.9). Furthermore, the following assertions hold:*

- *If $u \geq 0$, then $0 \leq \widehat{w}(u, y) \leq u\widehat{\kappa}(y) \leq u$.*
- *If $u \leq 0$, then $0 \geq \widehat{w}(u, y) \geq u\widehat{\kappa}(y) \geq u$.*

Hence,

$$|\widehat{w}(u, y)| \leq |u|\widehat{\kappa}(y) \quad \text{for a.e. } y \in \overline{(\mathbb{R}^n)^+}.$$

Concerning the dependence with respect to the parameter u , we start by considering the case in which σ is a maximal monotone graph associated to a continuous function.

Lemma 2.3. *Let σ be a nondecreasing continuous function such that $\sigma(0) = 0$. Then*

$$|\widehat{w}(u_1, y) - \widehat{w}(u_2, y)| \leq |u_1 - u_2|$$

for all $u_1, u_2 \in \mathbb{R}$.

Proof. Let $\widehat{w}(u_1, y), \widehat{w}(u_2, y)$ be two solutions of problem (1.9) with parameters $u_1, u_2 \in \mathbb{R}$. Consider the function $v = w(u_1, y) - w(u_2, y)$. This function is a solution of the following exterior problem:

$$\begin{cases} \Delta v = 0, & y \in (\mathbb{R}^n)^+, \\ \partial_{\nu_y} v = C_0(\sigma(u_1 - \widehat{w}(u_1, y)) - \sigma(u_2 - \widehat{w}(u_2, y))), & y \in G_0, \\ v \rightarrow 0 & \text{as } |y| \rightarrow \infty. \end{cases} \quad (2.1)$$

First consider the case $u_1 > u_2$. If we choose v^- as a test function in the integral identity for problem (2.1), we arrive at

$$\int_{(\mathbb{R}^n)^+} |\nabla v^-|^2 dx + C_0 \int_{G_0} (\sigma(u_1 - \widehat{w}(u_1, y)) - \sigma(u_2 - \widehat{w}(u_2, y))) v^- ds = 0.$$

The second integral in the obtained expression can be nonzero only if $v < 0$, i.e. $\widehat{w}(u_1, y) - \widehat{w}(u_2, y) < 0$. By combining this inequality with the condition $u_1 > u_2$, we get $u_1 - \widehat{w}(u_1, y) > u_2 - \widehat{w}(u_2, y)$. This inequality and the monotonicity of the function σ imply that the second integral is non-negative. Hence, two integrals must be equal to zero, so $v^- = 0$ \mathcal{L}^{n-1} -a.e. in G_0 and $v^- = c$ in $(\mathbb{R}^n)^+$. But we have $v \rightarrow 0$ as $|y| \rightarrow \infty$, and hence $c = 0$, i.e. $\widehat{w}(u_1, y) - \widehat{w}(u_2, y) \geq 0$.

One can construct a function $\varphi(r) \in C_0^\infty(\mathbb{R})$ such that $\varphi = 0$ if $|r| > 1$, and $\varphi = 1$ if $|r| < 0.5$. We take $(u_1 - u_2 - v)^- \varphi(\rho(x, G_0)/R)$ as a test function in the integral identity for problem (2.1) and obtain

$$\begin{aligned} & - \int_{(\mathbb{R}^n)^+} \nabla v \nabla (u_1 - u_2 - v)^- \varphi(\rho(x, G_0)/R) dx - \int_{(\mathbb{R}^n)^+} \frac{\varphi'(\rho(x, G_0)/R)}{R} (u_1 - u_2 - v)^- \nabla v \nabla \rho dx \\ & \quad + C_0 \int_{G_0} (\sigma(u_1 - \widehat{w}(u_1, y)) - \sigma(u_2 - \widehat{w}(u_2, y)))(u_1 - u_2 - v)^- ds \\ & = I_{1,R} + I_{2,R} + I_3 = 0. \end{aligned}$$

Since σ is monotone $I_3 \leq 0$.

For the first integral we have

$$I_{1,R} \leq - \int_{\mathcal{D}_{1,R}} |\nabla(u_1 - u_2 - v)^-|^2 dx \leq 0,$$

where $\mathcal{D}_{1,R} = ((\mathbb{R}^n)^+) \cap \{x \in \mathbb{R}^n : \rho(x, G_0) < R\}$. We have that

$$I_1 = - \int_{(\mathbb{R}^n)^+} |\nabla(u_1 - u_2 - v)^-|^2 dx = \lim_{R \rightarrow \infty} I_{1,R} \leq 0.$$

For the second integral we derive the estimation

$$|I_{2,R}| \leq K_1 \int_{\mathcal{D}_{2,R}} \frac{|v|}{R} |\nabla v| dx \leq \frac{K_1}{R} \|v\|_{L_2(\mathcal{D}_{2,R})} \|\nabla v\|_{L_2((\mathbb{R}^n)^+)} \leq \frac{K_2}{R^{\frac{n-2}{2}}} \|\nabla v\|_{L_2((\mathbb{R}^n)^+)} \rightarrow 0 \quad \text{as } R \rightarrow \infty,$$

where $\mathcal{D}_{2,R} = ((\mathbb{R}^n)^+) \cap \{x \in \mathbb{R}^n : \frac{R}{2} < \rho(x, G_0) < R\}$. Thereby, as $R \rightarrow \infty$, we have $I_1 + I_3 = 0$, $I_1 \leq 0$, $I_3 \leq 0$, and so

$$I_1 = 0, \quad I_3 = 0.$$

Taking into account that $v \rightarrow 0$ as $|y| \rightarrow \infty$, we derive from the last corollary that $(u_1 - u_2 - v)^- \equiv 0$, i.e. $0 \leq v < u_1 - u_2$ in $(\mathbb{R}^n)^+$. Moreover, we have that $v < u_1 - u_2$ \mathcal{L}^{n-1} -a.e. in G_0 .

The case $u_1 < u_2$ is analogous to the one above, so we have $u_1 - u_2 \leq v \leq 0$ in $(\mathbb{R}^n)^+$ and \mathcal{L}^{n-1} -a.e. in G_0 . This concludes the proof. \square

The use of the comparison principle leads to an additional conclusion.

Lemma 2.4. *Let $u_1 > u_2$. Then*

$$0 \leq \widehat{w}(u_1, y) - \widehat{w}(u_2, y) \leq (u_1 - u_2) \widehat{\kappa}(y). \quad (2.2)$$

Proof. The functions $\varphi_1(y) = \widehat{w}(u_1, y) - \widehat{w}(u_2, y)$ and $\varphi_2(y) = (u_1 - u_2) \widehat{\kappa}(y)$ can be extended (by symmetry) as harmonic functions to $\mathbb{R}^n \setminus \overline{G_0}$ such that $\lim_{|y| \rightarrow \infty} (\varphi_2 - \varphi_1) = 0$ and $\varphi_2 - \varphi_1 \geq 0$ on G_0 . The comparison principle proves the result. \square

A more regular dependence with respect to u can also be proved under additional regularity of the function σ .

Lemma 2.5 (Differentiable dependence of solutions). *Suppose that $\sigma \in \mathcal{C}^1$ and $\sigma' \geq k_1 > 0$. Then the map $u \in \mathbb{R} \mapsto \widehat{w}(u, \cdot) \in L_{\text{loc}}^2(K)$ is differentiable for every smooth bounded set K such that $G_0 \subset K \subset (\mathbb{R}^n)^+$. Furthermore, if we define*

$$\widehat{W}(u, y) = \frac{\partial \widehat{w}(u, y)}{\partial u},$$

then

$$\int_{(\mathbb{R}^n)^+} \nabla \widehat{W}(u, y) \nabla \varphi dy = C_0 \int_{G_0} \sigma'(u - \widehat{w}(u, y)) (1 - \widehat{W}(u, y)) \varphi(y) dS_y \quad (2.3)$$

for $\varphi \in C_c^\infty(\overline{(\mathbb{R}^n)^+})$, $\nabla \widehat{W} \in L^2(\Omega)^n$ and $0 \leq \widehat{W}(u, y) \leq \widehat{\kappa}(y)$. In particular,

$$0 \leq \frac{\partial \widehat{w}}{\partial u} \leq \widehat{\kappa}(y).$$

Proof. Considering the difference of two solutions, we obtain

$$\begin{aligned} \int_{(\mathbb{R}^n)^+} \nabla \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h} \nabla \varphi \, dy &= C_0 \int_{G_0} \frac{\sigma(u+h - \widehat{w}(u+h, y)) - \sigma(u - \widehat{w}(u, y))}{h} \varphi \, dS_y \\ &= C_0 \int_{G_0} \sigma'(\xi_h(y)) \left(1 - \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h}\right) \varphi(y) \, dS_y \end{aligned}$$

for some ξ_h in between $u+h - \widehat{w}(u+h, y)$ and $u - \widehat{w}(u, y)$. From this we obtain

$$\begin{aligned} \int_{(\mathbb{R}^n)^+} \nabla \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h} \nabla \varphi \, dy &+ C_0 \int_{G_0} \sigma'(\xi_h(y)) \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h} \varphi(y) \, dS_y \\ &= C_0 \int_{G_0} \sigma'(\xi_h(y)) \varphi(y) \, dS_y. \end{aligned}$$

Taking $\varphi = \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h}$, and using the fact that $\widehat{w}(u, y)$ can be bounded and σ' is continuous, we obtain

$$\left\| \nabla \left(\frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h} \right) \right\|_{L^2(\Omega)^n}^2 + k_1 C_0 \left\| \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h} \right\|_{L^2(G_0)}^2 \leq C$$

for h small. Thus $\frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h}$ admits a weak limit as $h \rightarrow 0$ in $H^1(K)$; let it be denoted by $\widehat{W}(u, y)$. Thus, up to a subsequence, it admits a pointwise limit and strong limit in $L^2(K)$. It is clear that

$$\xi_h(y) \rightarrow u - \widehat{w}(u, y) \quad \text{pointwise as } h \rightarrow 0.$$

By passing to the limit for $\varphi \in \mathcal{C}_c^\infty(\overline{(\mathbb{R}^n)^+})$ fixed, we characterize (2.3). From (2.2) we deduce that, for $h > 0$,

$$0 \leq \frac{\widehat{w}(u+h, y) - \widehat{w}(u, y)}{h} \leq \kappa(y).$$

As $h \rightarrow 0$, we deduce the result $0 \leq \widehat{W}(u, y) \leq \kappa(y)$. □

Remark 2.6. Notice that \widehat{W} is the unique solution of

$$\begin{cases} -\Delta \widehat{W} = 0, & y \in (\mathbb{R}^n)^+, \\ \partial_\nu \widehat{W} + C_0 \sigma'(u - \widehat{w}) \widehat{W} = C_0 \sigma'(u - \widehat{w}), & y \in G_0, \\ \partial_\nu \widehat{W} = 0, & y \notin G_0, y_1 = 0, \\ \widehat{W} \rightarrow 0, & |y| \rightarrow +\infty. \end{cases}$$

Remark 2.7. Assume $\sigma(u) = \mu u$. Then $\sigma'(u - \widehat{w}) \equiv \mu$, and \widehat{W} does not depend on μ . Therefore, $\widehat{w}(x, u) = u \widehat{W}(x)$. Furthermore,

$$H(u) = \lambda_\mu u.$$

2.3 On the regularity of the function H

Lemma 2.8. Let σ be a maximal monotone graph. The function $H_{G_0}(u)$ defined by (1.10) is nondecreasing and Lipschitz continuous of constant λ_{G_0} given by (1.11), i.e. if $u_1 > u_2$, then

$$0 \leq H(u_1) - H(u_2) \leq \lambda_{G_0}(u_1 - u_2), \quad (2.4)$$

i.e. in the notation of weak derivatives,

$$0 \leq H'(u) \leq \lambda_{G_0} \quad \text{for a.e. } u \in \mathbb{R}.$$

Remark 2.9. Notice that $\widehat{\kappa}$ (and thus λ_{G_0}) does not depend on σ , but only on G_0 .

Proof. First, let σ be smooth and $\sigma' \geq k_1 > 0$. Again, let $\widehat{W} = \frac{\partial \widehat{w}}{\partial u}$. Taking derivatives in (1.10), we have that

$$H'(u) = C_0 \int_{G_0} \sigma'(u - \widehat{w})(1 - \widehat{W}) dy'.$$

Since $\widehat{W} \leq 1$, we have that $H' \geq 0$. Using $\widehat{\kappa}$ as a test function in (2.3), we obtain that

$$\begin{aligned} H'(u) &= C_0 \int_{G_0} \sigma'(u - \widehat{w})(1 - \widehat{W}) dy' \\ &= \int_{G_0} (\sigma'(u - \widehat{w})(1 - \widehat{W})) \widehat{\kappa} dy' \\ &= \int_{(\mathbb{R}^n)^+} \nabla \widehat{W} \nabla \widehat{\kappa} dy \\ &= \int_{G_0} \widehat{W} (\partial_\nu \widehat{\kappa}) dy' \\ &\leq \int_{G_0} (\partial_\nu \widehat{\kappa}) dy' \\ &= \lambda, \end{aligned}$$

using the facts that $\partial_\nu \widehat{\kappa} \geq 0$ and $0 \leq \widehat{W} \leq 1$.

If σ is a general maximal monotone graph, estimate (2.4) is maintained by approximation by a smooth sequence of the function σ_k . \square

3 Convergence of the boundary integrals where $u_\varepsilon \geq 0$

3.1 On the auxiliary function w_ε^j

We introduce a function $w_\varepsilon^j(u, x)$ as a solution of the boundary value problem

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+, \\ \partial_\nu w_\varepsilon^j = \varepsilon^{-k} \sigma(u - w_\varepsilon^j), & x \in G_\varepsilon^j, \\ \partial_\nu w_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^0 \setminus \overline{G_\varepsilon^j}, \\ w_\varepsilon^j = 0, & x \in (\partial T_{\varepsilon/4}^j)^+, \end{cases} \quad (3.1)$$

where $u \in \mathbb{R}$ is a parameter. We will compare this auxiliary function with the functions

$$\widehat{w}_\varepsilon^j(u, x) = \widehat{w}\left(u, \frac{x - P_\varepsilon^j}{a_\varepsilon}\right).$$

The function $w_\varepsilon^j \in H_0^1(T_{\varepsilon/4}^j)$ is a weak solution of problem (3.1) if it satisfies the integral identity

$$\int_{(T_{\varepsilon/4}^j)^+} \nabla w_\varepsilon^j \nabla \varphi dx - \varepsilon^{-k} \int_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j) \varphi dx' = 0$$

for an arbitrary function $\varphi \in H_0^1(T_{\varepsilon/4}^j)$.

From the uniqueness of problem (3.1) and the method of sub- and supersolutions, we have the following lemma.

Lemma 3.1. *The function w_ε^j satisfies the following estimations:*

- (i) *If $u \geq 0$, then $0 \leq w_\varepsilon^j(u, x) \leq \widehat{w}_\varepsilon^j(u, x) \leq u$.*
- (ii) *If $u \leq 0$, then $u \leq \widehat{w}_\varepsilon^j(u, x) \leq w_\varepsilon^j(u, x) \leq 0$.*

Remark 3.2. Since σ is monotone, from the previous result we obtain

$$|\sigma(u - w_\varepsilon^j)| \leq |\sigma(u)|. \quad (3.2)$$

We define the function

$$W_\varepsilon = \begin{cases} w_\varepsilon^j(u, x), & x \in (T_{\varepsilon/4}^j)^+, j \in Y_\varepsilon \\ 0, & x \in (\mathbb{R}^n)^+ \setminus \bigcup_{j \in Y_\varepsilon} \overline{(T_{\varepsilon/4}^j)^+}. \end{cases} \quad (3.3)$$

Notice that $W_\varepsilon \in H^1(\Omega, \Gamma_2)$ for all $u \in \mathbb{R}$. The following lemma proposes an estimate of the introduced function and its gradient.

Lemma 3.3. *The following estimations for the function W_ε , which was defined in (3.3), are valid:*

$$\begin{cases} \|\nabla W_\varepsilon\|_{L_2(\Omega)}^2 \leq K|u| |\sigma(u)|, \\ \|W_\varepsilon\|_{L_2(\Omega)}^2 \leq K\varepsilon^2 |u| |\sigma(u)|. \end{cases} \quad (3.4)$$

Proof. From the weak formulation of problem (3.1) for $j \in Y_\varepsilon$ we know

$$\int_{(T_{\varepsilon/4}^j)^+} \nabla w_\varepsilon^j \nabla \varphi \, dx - \varepsilon^{-k} \int_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j) \varphi \, dx' = 0.$$

We take $\varphi = w_\varepsilon^j$ as a test function in this expression and obtain

$$\int_{(T_{\varepsilon/4}^j)^+} |\nabla w_\varepsilon^j|^2 \, dx - \varepsilon^{-k} \int_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j) w_\varepsilon^j \, dx' = 0.$$

Then we can transform the obtained relation to the following expression:

$$\int_{(T_{\varepsilon/4}^j)^+} |\nabla w_\varepsilon^j|^2 \, dx + \varepsilon^{-k} \int_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j)(u - w_\varepsilon^j) \, dx' = \varepsilon^{-k} \int_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j) u \, dx'.$$

By using the monotonicity of $\sigma(u)$, we derive the following inequality:

$$\int_{(T_{\varepsilon/4}^j)^+} |\nabla w_\varepsilon^j|^2 \, dx + \varepsilon^{-k} \int_{G_\varepsilon^j} \sigma(u - w_\varepsilon^j)(u - w_\varepsilon^j) \, dx' \leq \varepsilon^{-k} \int_{G_\varepsilon^j} |\sigma(u - w_\varepsilon^j)| |u| \, dx'.$$

Due to the monotonicity of σ and (3.2), we have that

$$\|\nabla w_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 \leq \varepsilon^{-k} \int_{G_\varepsilon^j} |\sigma(u - w_\varepsilon^j)| |u| \, dx' \leq k_2 |u| |\sigma(u)| \varepsilon^{-k} |G_\varepsilon^j|.$$

Hence, the following estimate is valid:

$$\|\nabla w_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 \leq K|u| |\sigma(u)| \varepsilon^{n-1}.$$

Adding over all cells, we get

$$\|\nabla W_\varepsilon\|_{L_2(\Omega)}^2 \leq K|u| |\sigma(u)|.$$

Friedrich's inequality implies

$$\|w_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 \leq K\varepsilon^2 \|\nabla w_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2.$$

Summing over all cells and using the obtained estimations, we derive

$$\|W_\varepsilon\|_{L_2(\Omega)}^2 \leq K\varepsilon^2 \|\nabla W_\varepsilon\|_{L_2(\Omega)}^2 \leq K\varepsilon^2 |u| |\sigma(u)|,$$

which concludes the proof. \square

Hence, as $\varepsilon \rightarrow 0$ we have

$$\begin{aligned} W_\varepsilon &\rightharpoonup 0 \quad \text{weakly in } H^1(\Omega), \\ W_\varepsilon &\rightarrow 0 \quad \text{strongly in } L_2(\Omega). \end{aligned}$$

3.2 The comparison between w_ε^j and $\widehat{w}_\varepsilon^j$

As an immediate consequence of Lemma 3.1 we have the following lemma.

Lemma 3.4. *For all $u \in \mathbb{R}$ and a.e. $x \in (T_{\varepsilon/4}^j)^+$, we have*

$$|w_\varepsilon^j(u, x)| \leq |\widehat{w}_\varepsilon^j(u, x)|.$$

The following lemma gives an estimate of the proximity of the functions w_ε^j and \widehat{w} .

Lemma 3.5. *For the introduced functions $w_\varepsilon^j(u, x)$ and $\widehat{w}(y, u)$ following estimations hold:*

$$\begin{aligned} \|\nabla(v_\varepsilon^j(u, x))\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 &\leq K|u|^2\varepsilon^n, \\ \|v_\varepsilon^j(u, x)\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 &\leq K|u|^2\varepsilon^{n+2}, \end{aligned}$$

where $v_\varepsilon^j(u, x) = w_\varepsilon^j(u, x) - \widehat{w}_\varepsilon^j(u, x)$.

Proof. The function v_ε^j is a solution to the following boundary value problem:

$$\begin{cases} \Delta v_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{G_\varepsilon^j}, \\ \partial_\nu v_\varepsilon^j = \varepsilon^{-k}(\sigma(u - \widehat{w}_\varepsilon^j) - \sigma(u - w_\varepsilon^j)), & x \in G_\varepsilon^j, \\ \partial_\nu v_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^0 \setminus \overline{G_\varepsilon^j}, \\ v_\varepsilon^j = -\widehat{w}_\varepsilon^j, & x \in (\partial T_{\varepsilon/4}^j)^+. \end{cases}$$

Applying the comparison principle, we have $|v_\varepsilon^j(u, x)| \leq |\widehat{w}_\varepsilon^j(u, x)|$ a.e. in $(T_{\varepsilon/4}^j)^+$. We take v_ε^j as a test function in the corresponding weak solution integral expression for the above problem:

$$\|\nabla v_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 + \varepsilon^{-k} \int_{G_\varepsilon^j} (\sigma(u - w_\varepsilon^j) - \sigma(u - \widehat{w}_\varepsilon^j)) v_\varepsilon^j dx' = - \int_{(\partial T_{\varepsilon/4}^j)^+} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j ds. \quad (3.5)$$

We transform the right-hand side expression of the inequality in the following way:

$$- \int_{(\partial T_{\varepsilon/4}^j)^+} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j ds = - \int_{(T_{\varepsilon/4}^j)^+ \setminus (T_{\varepsilon/8}^j)^+} \nabla v_\varepsilon^j \nabla \widehat{w}_\varepsilon^j dx + \int_{(\partial T_{\varepsilon/8}^j)^+} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j ds.$$

Let us estimate the obtained terms. We can extend v_ε^j by the symmetry $v_\varepsilon^j(x, u) = v_\varepsilon^j(-x, u)$ for $x \in (T_{\varepsilon/4}^j)^-$, which is harmonic in $T_{\varepsilon/4}^j \setminus \overline{G_0}$. By using some estimates on the derivatives of harmonic functions and the maximum principle, for $\tilde{x} \in \partial T_{\varepsilon/8}^j$, we get

$$|\partial_{x_i} v_\varepsilon^j(\tilde{x})| \leq \frac{1}{|T_{\varepsilon/16}(\tilde{x})|} \left| \int_{T_{\varepsilon/16}(\tilde{x})} \frac{\partial v_\varepsilon^j}{\partial x_i} dx \right| = \frac{K}{\varepsilon^n} \left| \int_{\partial T_{\varepsilon/16}(\tilde{x})} v_\varepsilon^j \nu_i dx \right| \leq K|u|$$

since, on $\partial T_{\varepsilon/16}(\tilde{x})$, from the maximum principle we have

$$|v_\varepsilon^j| \leq |\widehat{w}_\varepsilon^j| \leq \frac{K|u|}{|(x - P_\varepsilon^j)/a_\varepsilon|^{n-2}} = K|u|\varepsilon^{2-n}a_\varepsilon^{n-2} = K|u|\varepsilon^{2-n}\varepsilon^{n-1} = K|u|\varepsilon.$$

This last estimate implies that $|\nabla v_\varepsilon^j(\tilde{x})| \leq K$ for $\tilde{x} \in \partial T_{\varepsilon/8}^j$. Therefore, we get an estimate of the second term:

$$\left| \int_{(\partial T_{\varepsilon/8}^j)^+} \partial_\nu v_\varepsilon^j \widehat{w}_\varepsilon^j ds \right| \leq K|u| \max_{\partial T_{\varepsilon/8}^j} |\widehat{w}_\varepsilon^j| |\partial T_{\varepsilon/8}^j| \leq K|u|^2 \varepsilon^n.$$

Then we estimate the first term:

$$\left| \int_{(T_{\varepsilon/4}^j)^+ \setminus (T_{\varepsilon/8}^j)^+} \nabla v_\varepsilon^j \nabla \widehat{w} dx \right| \leq K|u|^2 \varepsilon a_\varepsilon^{n-2} = K|u|^2 \varepsilon^n.$$

Combining the obtained estimates and using the properties of the function σ , we get

$$\|\nabla v_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 \leq K|u|^2 \varepsilon^n.$$

Friedrich's inequality implies that

$$\|v_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 \leq K|u|^2 \varepsilon^{n+2}.$$

This concludes the proof. \square

Lemma 3.6. *We have that¹*

$$\frac{1}{|G_\varepsilon^j|} \int_{G_\varepsilon^j} |v_\varepsilon^j(u, x')|^2 dx' \leq \varepsilon.$$

Proof. From (3.5), using (1.4), we deduce that

$$k_1 a_\varepsilon^{-1} \int_{G_\varepsilon^j} |v_\varepsilon^j|^2 dx' \leq K|u|^2 \varepsilon^n.$$

Thus,

$$\begin{aligned} \frac{1}{|G_\varepsilon^j|} \int_{G_\varepsilon^j} |v_\varepsilon^j(u, x')|^2 dx' &\leq K a_\varepsilon^{1-n} \int_{G_\varepsilon^j} |v_\varepsilon^j(u, x')|^2 dx' \\ &\leq K a_\varepsilon^{2-n} a_\varepsilon^{-1} \int_{G_\varepsilon^j} |v_\varepsilon^j(u, x')|^2 dx' \\ &\leq K|u|^2 a_\varepsilon^{2-n} \varepsilon^n \\ &= K|u|^2 \varepsilon. \end{aligned}$$

This completes the proof. \square

3.3 Convergence to the “strange term”

The following result plays a crucial role in the proof of Theorem 1.3.

Lemma 3.7. *Let H be the function defined by (1.10), and let φ be an arbitrary function in $C^\infty(\Omega)$. Then for any test function $h \in H^1(\Omega, \Gamma_2)$ we have*

$$\left| \sum_{j \in Y_\varepsilon} \int_{(\partial T_{\varepsilon/4}^j)^+} \partial_\nu \widehat{w}_\varepsilon^j(\varphi(\tilde{P}_\varepsilon^j), s) h(s) ds + C_0^{n-2} \int_{\Gamma_1} H(\varphi(x')) h(x') dx' \right| \rightarrow 0 \quad (3.6)$$

as $\varepsilon \rightarrow 0$, where $\tilde{P}_\varepsilon^j \in G_\varepsilon^j$ and $\nu = (-1, 0, \dots, 0)$ is the unit outward normal to $(\partial T_{\varepsilon/4}^j)^+$.

¹ The function v_ε^j in this section depends on u .

Proof. Consider the cylinder

$$Q_\varepsilon^j = \left\{ x \in \mathbb{R}^n : 0 < x_1 < \varepsilon, -\frac{\varepsilon}{2} < x_i - (\tilde{P}_\varepsilon^j)_i < \frac{\varepsilon}{2}, i = 2, \dots, n \right\}.$$

We define the auxiliary function θ_ε^j as the unique solution to the following boundary value problem:

$$\left\{ \begin{array}{ll} \Delta \theta_\varepsilon^j = 0, & Y_\varepsilon^j = Q_\varepsilon^j \setminus \overline{(T_{\varepsilon/4}^j)^+}, \\ \partial_\nu \theta_\varepsilon^j = -\partial_\nu \tilde{w}_\varepsilon^j(\varphi(\tilde{P}_\varepsilon^j), x), & x \in (\partial T_{\varepsilon/4}^j)^+, \\ \frac{\partial \theta_\varepsilon^j}{\partial x_1} = \mu_\varepsilon^j, & x \in Y_\varepsilon^j = \partial Q_\varepsilon^j \cap \{x : x_1 = \varepsilon\}, \\ \frac{\partial \theta_\varepsilon^j}{\partial x_1} = 0, & \text{on the rest of the boundary } \partial Q_\varepsilon^j, \\ \langle \theta_\varepsilon^j \rangle_{Y_\varepsilon^j} = 0. \end{array} \right. \quad (3.7)$$

The constant μ_ε^j is defined from the solvability condition for problem (3.7):

$$\mu_\varepsilon^j = -C_0^{n-2} H(\varphi(\tilde{P}_\varepsilon^j)).$$

We take θ_ε^j as a test function in the integral identity associated to problem (3.7) and obtain

$$\int_{Y_\varepsilon^j} |\nabla \theta_\varepsilon^j|^2 dx = -\mu_\varepsilon^j \int_{Y_\varepsilon^j} \theta_\varepsilon^j dx' + \int_{S_{\varepsilon/4}^{j,+}} \partial_\nu \tilde{w}_\varepsilon^j \theta_\varepsilon^j ds.$$

Using the embedding theorems, we obtain the estimate

$$\int_{Y_\varepsilon^j} |\theta_\varepsilon^j| dx' \leq K \varepsilon^{(n-1)/2} \|\theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)} \leq K \varepsilon^{n/2} \|\nabla \theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)}. \quad (3.8)$$

Taking into account that

$$\max_{(\partial T_{\varepsilon/4}^j)^+} |\partial_\nu \tilde{w}_\varepsilon^j(\varphi(\tilde{P}_\varepsilon^j), x)| \leq K \frac{a_\varepsilon^{-1}}{|\frac{x-P_\varepsilon^j}{a_\varepsilon}|^{n-1}} = K a_\varepsilon^{n-2} \varepsilon^{1-n} \leq K$$

and using some estimates proved in [31], we derive

$$\begin{aligned} \int_{(\partial T_{\varepsilon/4}^j)^+} |(\partial_\nu \tilde{w}_\varepsilon^j(\varphi(\tilde{P}_\varepsilon^j), s)) \theta_\varepsilon^j| ds &\leq K \int_{(\partial T_{\varepsilon/4}^j)^+} |\theta_\varepsilon^j| ds \\ &\leq K \varepsilon^{(n-1)/2} \|\theta_\varepsilon^j\|_{L_2((\partial T_{\varepsilon/4}^j)^+)} \\ &\leq K \varepsilon^{(n-1)/2} \{ \varepsilon^{-1/2} \|\theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)} + \sqrt{\varepsilon} \|\nabla \theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)} \} \\ &\leq K \varepsilon^{n/2} \|\nabla \theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)}. \end{aligned} \quad (3.9)$$

From the above estimates (3.8) and (3.9) we get

$$\|\nabla \theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)}^2 \leq K \varepsilon^n. \quad (3.10)$$

From estimate (3.10) it follows that

$$\sum_{j \in Y_\varepsilon} \|\theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)}^2 \leq K \varepsilon.$$

Adding all the above integral identities for problem (3.7), we derive that for $h \in H^1(\Omega)$ the following inequality holds:

$$\begin{aligned} &\left| \sum_{j \in Y_\varepsilon} \int_{(\partial T_{\varepsilon/4}^j)^+} \partial_\nu \tilde{w}_\varepsilon^j(\varphi(\tilde{P}_\varepsilon^j), s) h ds + C_0^{n-2} \int_{\Gamma_1} H(\varphi(x')) h dx' \right| \\ &\leq \left| \sum_{j \in Y_\varepsilon} \int_{\tilde{Y}_\varepsilon^j} \nabla \theta_\varepsilon^j \nabla h dx \right| + \left| \sum_{j \in Y_\varepsilon} \mu_\varepsilon^j \int_{Y_\varepsilon^j} h dx' + C_0^{n-2} \int_{\Gamma_1} H(\varphi(x')) h dx' \right|. \end{aligned} \quad (3.11)$$

Let us estimate the terms on the right-hand side of inequality (3.11). By using estimate (3.10), we get the following inequality for the first term:

$$\left| \sum_{j \in Y_\varepsilon} \int_{Y_\varepsilon^j} \nabla \theta_\varepsilon^j \nabla h \, dx \right| \leq K \sqrt{\varepsilon} \|h\|_{H^1(\Omega)}. \quad (3.12)$$

Set

$$\tilde{\gamma}_\varepsilon^j = (Q_\varepsilon^j)^0, \quad \Gamma_1^\varepsilon = \bigcup_{j \in Y_\varepsilon} \tilde{\gamma}_\varepsilon^j.$$

Then we have

$$\begin{aligned} & \left| \sum_{j \in Y_\varepsilon} \mu_\varepsilon^j \int_{Y_\varepsilon^j} h \, dx' + C_0^{n-2} \int_{\Gamma_1} H(\varphi(x')) h \, dx' \right| \\ & \leq \left| C_0^{n-2} \sum_{j \in Y_\varepsilon} \left(\int_{Y_\varepsilon^j} H(\varphi(\tilde{P}_\varepsilon^j)) h \, dx' - \int_{Y_\varepsilon^j} H(\varphi(x')) h \, dx' \right) \right| \\ & \leq C_0^{n-2} \left| \sum_{j \in Y_\varepsilon} \int_{\tilde{\gamma}_\varepsilon^j} (H(\varphi(\tilde{P}_\varepsilon^j)) - H(\varphi(x'))) h \, dx' \right| + C_0^{n-2} \left| \sum_{j \in Y_\varepsilon} \left(\int_{Y_\varepsilon^j} H(\varphi(\tilde{P}_\varepsilon^j)) h \, dx' - \int_{\tilde{\gamma}_\varepsilon^j} H(\varphi(\tilde{P}_\varepsilon^j)) h \, dx' \right) \right|. \end{aligned}$$

Let us estimate the terms on the right-hand side of the obtained inequality. For the first term we have

$$\begin{aligned} & \left| \sum_{j \in Y_\varepsilon} \int_{\tilde{\gamma}_\varepsilon^j} (H(\varphi(\tilde{P}_\varepsilon^j)) - H(\varphi(x'))) h \, dx' \right| \leq K \|h\|_{L_2(\Gamma_1^\varepsilon)} \max_{\substack{j \in Y_\varepsilon \\ x' \in \tilde{\gamma}_\varepsilon^j}} |H(\varphi(\tilde{P}_\varepsilon^j)) - H(\varphi(x'))| \\ & \leq K \|h\|_{L_2(\Gamma_1^\varepsilon)} \max_{\substack{j \in Y_\varepsilon \\ x' \in \tilde{\gamma}_\varepsilon^j}} |\varphi(\tilde{P}_\varepsilon^j) - \varphi(x')| \\ & \leq K \varepsilon \|h\|_{H^1(\Omega)}. \end{aligned}$$

By using the continuity in L^2 -norm on the hyperplanes of the functions from $H^1(\Omega)$, we estimate the second term:

$$\left| \sum_{j \in Y_\varepsilon} \left(\int_{Y_\varepsilon^j} H(\varphi(\tilde{P}_\varepsilon^j)) h \, dx' - \int_{\tilde{\gamma}_\varepsilon^j} H(\varphi(\tilde{P}_\varepsilon^j)) h \, dx' \right) \right| \leq K \sqrt{\varepsilon} \|h\|_{H^1(\Omega)}.$$

Hence we have

$$\left| \sum_{j \in Y_\varepsilon} \mu_\varepsilon^j \int_{Y_\varepsilon^j} h \, dx' + C_0^{n-2} \int_{\Gamma_1} H(\varphi(x')) h \, dx' \right| \leq K \sqrt{\varepsilon} \|h\|_{H^1(\Omega)}. \quad (3.13)$$

Combining estimates (3.12) and (3.13), we conclude the proof. \square

4 Convergence of the boundary integrals where $u_\varepsilon \leq 0$

4.1 The auxiliary function κ_ε^j

We introduce the function κ_ε^j as the unique solution of the following problem:

$$\begin{cases} \Delta \kappa_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{G_\varepsilon^j}, \\ \kappa_\varepsilon^j = 1, & x \in G_\varepsilon^j, \\ \partial_\nu \kappa_\varepsilon^j = 0, & x \in (\partial T_{\varepsilon/4}^j)^0 \setminus \overline{G_\varepsilon^j}, \\ \kappa_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j, \end{cases}$$

and then we define

$$\kappa_\varepsilon = \begin{cases} \kappa_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+, j \in Y_\varepsilon, \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j \in Y_\varepsilon} \overline{(T_{\varepsilon/4}^j)^+}. \end{cases}$$

It is easy to see that $\kappa_\varepsilon \in H^1(\Omega)$ and

$$\kappa_\varepsilon \rightharpoonup 0 \quad \text{weakly in } H^1(\Omega) \text{ as } \varepsilon \rightarrow 0. \quad (4.1)$$

4.2 Estimate of the difference between κ_ε^j and $\widehat{\kappa}_\varepsilon^j$

Lemma 4.1. *Let κ_ε^j and $\widehat{\kappa}$ be as above. Then*

$$\sum_{j \in Y_\varepsilon} \|\kappa_\varepsilon^j - \widehat{\kappa}_\varepsilon^j\|_{H^1((T_{\varepsilon/4}^j)^+)}^2 \leq K\varepsilon.$$

Proof. The function $v_\varepsilon^j = \kappa_\varepsilon^j - \widehat{\kappa}$ satisfies the following problem:

$$\begin{cases} \Delta v_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+, \\ v_\varepsilon^j = 0, & x \in G_\varepsilon^j, \\ \partial_\nu v_\varepsilon^j = 0, & x \in (T_\varepsilon^j)^0 \setminus \overline{G_\varepsilon^j}, \\ v_\varepsilon^j = -\widehat{\kappa}_\varepsilon^j, & x \in (\partial T_{\varepsilon/4}^j)^+. \end{cases}$$

We take v_ε^j as a test function in an integral identity for the above problem:

$$\|\nabla v_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)} = - \int_{\partial(T_{\varepsilon/4}^j)^+} \partial_\nu v_\varepsilon^j \widehat{\kappa} \, ds.$$

We transform the right-hand side expression of the identity in the following way:

$$- \int_{\partial(T_{\varepsilon/4}^j)^+} \partial_\nu v_\varepsilon^j \widehat{\kappa} \, ds = - \int_{(T_{\varepsilon/4}^j)^+ \setminus \overline{T_{\varepsilon/8}^j}} \nabla v_\varepsilon^j \nabla \widehat{\kappa} \, dx + \int_{\partial T_{\varepsilon/8}^j} \partial_\nu v_\varepsilon^j \widehat{\kappa} \, ds.$$

For an arbitrary point $x_0 \in \partial T_{\varepsilon/8}^j$ we have

$$|\partial_{x_i} v_\varepsilon^j(x_0)| \leq \frac{1}{|T_{\varepsilon/16}(x_0)|} \left| \int_{T_{\varepsilon/16}(x_0)} \frac{\partial v_\varepsilon^j}{\partial x_i} \, dx \right| = \frac{K}{\varepsilon^n} \left| \int_{\partial T_{\varepsilon/16}(x_0)} v_\varepsilon^j \nu_i \, dx \right| \leq K.$$

The last estimate implies that $|\nabla v_\varepsilon^j(\tilde{x})| \leq K$ for $\tilde{x} \in \partial T_{\varepsilon/8}^j$. Therefore, we can estimate the second term in the following way:

$$\left| \int_{\partial T_{\varepsilon/8}^j} \partial_\nu v_\varepsilon^j \widehat{\kappa} \, ds \right| \leq K \max_{\partial T_{\varepsilon/8}^j} |\widehat{\kappa}| |\partial T_{\varepsilon/8}^j| \leq K\varepsilon^n.$$

Then we estimate the first term:

$$\left| \int_{(T_{\varepsilon/4}^j)^+ \setminus \overline{T_{\varepsilon/8}^j}} \nabla v_\varepsilon^j \nabla \widehat{\kappa} \, dx \right| \leq K\varepsilon a_\varepsilon^{n-2} = K\varepsilon^n.$$

By combining acquired estimations, we derive

$$\|\nabla v_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 \leq K\varepsilon^n$$

Friedrich's inequality implies

$$\|v_\varepsilon^j\|_{L_2((T_{\varepsilon/4}^j)^+)}^2 \leq K\varepsilon^{n+2}.$$

This concludes the proof. \square

4.3 Convergence to the “strange term”

Lemma 4.2. Let λ_{G_0} be given by (1.11). Then for all functions $h \in H^1(\Omega, \Gamma_2)$ we have

$$\left| \sum_{j \in Y_\varepsilon} \int_{(\partial T_{\varepsilon/4}^j)^+ \cap \Omega} \partial_{v_x} \tilde{\kappa}_\varepsilon^j(s) h(s) ds + C_0^{n-2} \lambda_{G_0} \int_{\Gamma_1} h dx' \right| \rightarrow 0$$

as $\varepsilon \rightarrow 0$, where v is the unit outward normal to $\partial T_{\varepsilon/4}^j \cap \Omega$.

Proof. By analogy with the proof of Lemma 3.7, we define the function θ_ε^j as a solution to the following boundary value problem:

$$\begin{cases} \Delta \theta_\varepsilon^j = 0, & Y_\varepsilon^j, \\ \partial_v \theta_\varepsilon^j = -\partial_v \tilde{\kappa}_\varepsilon^j, & x \in (\partial T_{\varepsilon/4}^j)^+, \\ \frac{\partial \theta_\varepsilon^j}{\partial x_1} = \mu, & x \in Y_\varepsilon^j, \\ \frac{\partial \theta_\varepsilon^j}{\partial x_1} = 0, & \partial Q_\varepsilon^j \setminus ((\partial T_{\varepsilon/4}^j)^+ \cup Y_\varepsilon^j), \\ \langle \theta_\varepsilon^j \rangle_{Y_\varepsilon^j} = 0. \end{cases} \quad (4.2)$$

The constant μ is defined from the solvability condition for problem (4.2):

$$\mu = -C_0^{n-2} \lambda.$$

By using the same technique as in the proof of the Lemma 3.6, we have

$$\|\nabla \theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)}^2 \leq K \varepsilon^n, \quad \sum_{j \in Y_\varepsilon} \|\theta_\varepsilon^j\|_{L_2(Y_\varepsilon^j)}^2 \leq K \varepsilon.$$

Summing up all integral identities for problem (4.2), we derive that for the arbitrary function from $H^1(\Omega)$ the following inequality is true:

$$\begin{aligned} & \left| \sum_{j \in Y_\varepsilon} \int_{(\partial T_{\varepsilon/4}^j)^+} (\partial_v \tilde{\kappa}_\varepsilon^j) h ds + C_0^{n-2} \lambda \int_{\Gamma_1} h dx' \right| \\ & \leq \left| \sum_{j \in Y_\varepsilon} \int_{Y_\varepsilon^j} \nabla \theta_\varepsilon^j \nabla h dx \right| + \left| \sum_{j \in Y_\varepsilon} \mu \int_{Y_\varepsilon^j} h dx' + C_0^{n-2} \lambda \int_{\Gamma_1} h dx' \right|. \end{aligned} \quad (4.3)$$

Let us estimate the terms on the right-hand side of inequality (4.3). By using estimate (3.10), we get following estimation of the first term:

$$\left| \sum_{j \in Y_\varepsilon} \int_{Y_\varepsilon^j} \nabla \theta_\varepsilon^j \nabla h dx \right| \leq K \sqrt{\varepsilon} \|h\|_{H^1(\Omega)}. \quad (4.4)$$

Then we have

$$\left| \sum_{j \in Y_\varepsilon} \mu \int_{Y_\varepsilon^j} h dx' + C_0^{n-2} \lambda \int_{\Gamma_1} h dx' \right| \leq \left| C_0^{n-2} \lambda \sum_{j \in Y_\varepsilon} \left(\int_{Y_\varepsilon^j} h dx' - \int_{\tilde{Y}_\varepsilon^j} h dx' \right) \right|.$$

By using the continuity in L_2 -norm on the hyperplanes of the functions from $H^1(\Omega)$, we estimate the second term:

$$\left| \sum_{j \in Y_\varepsilon} \left(\int_{Y_\varepsilon^j} h dx' - \int_{\tilde{Y}_\varepsilon^j} h dx' \right) \right| \leq K \sqrt{\varepsilon} \|h\|_{H^1(\Omega)}.$$

Hence we have

$$\left| \sum_{j \in Y_\varepsilon} \mu_\varepsilon \int_{Y_\varepsilon^j} h dx' + C_0^{n-2} \lambda \int_{\Gamma_1} h dx' \right| \leq K \sqrt{\varepsilon} \|h\|_{H^1(\Omega)}. \quad (4.5)$$

Combining estimations (4.4) and (4.5), we conclude the proof. \square

5 Proof of Theorem 1.3

For different reasons it is convenient to introduce some new notation: instead of using the decomposition $u_0 = u_{0,+} - u_{0,-}$ mentioned in Section 1 (see the statement of Theorem 1.3), we shall use the alternative decomposition $u_0 = u_0^+ + u_0^-$ (i.e. $u_0^+ = u_{0,+}$, but $u_0^- = -u_{0,-}$).

Proof. Let $\varphi(x)$ be an arbitrary function from $C_0^\infty(\Omega)$. We choose a point $\tilde{P}_\varepsilon^j \in \overline{G_\varepsilon^j}$ such that

$$\min_{x \in \overline{G_\varepsilon^j}} \varphi^+(x) = \varphi^+(\tilde{P}_\varepsilon^j),$$

where $\varphi^+ = \max\{0, \varphi(x)\}$ and $\varphi^-(x) = \varphi(x) - \varphi^+(x)$. Define the function

$$\mathcal{W}_\varepsilon(\varphi^+, x) = \begin{cases} w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x), & x \in (T_{\varepsilon/4}^j)^+, j \in Y_\varepsilon \\ 0, & x \in \mathbb{R}^n \setminus \bigcup_{j \in Y_\varepsilon} \overline{(T_{\varepsilon/4}^j)^+}. \end{cases}$$

From estimates (3.4) we conclude that

$$\mathcal{W}_\varepsilon(\varphi^+, x) \rightarrow 0 \quad (5.1)$$

in $H^1(\Omega, \Gamma_2)$ as $\varepsilon \rightarrow 0$. We set

$$v = \varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^-$$

as a test function in the integral inequality (1.8), where φ is an arbitrary function from $C_0^\infty(\Omega)$. Notice that $v \in K_\varepsilon$. Indeed, according to Lemma 3.1 and using that $\kappa_\varepsilon \equiv 1$ in G_ε , we have for all $x \in G_\varepsilon^j$ that

$$v = \varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^- \geq \varphi^+(\tilde{P}_\varepsilon^j) - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) \geq 0.$$

Hence we get

$$\begin{aligned} & \int_{\Omega} \{ \nabla(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^-) \nabla(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^- - u_\varepsilon) \} dx \\ & \quad + \varepsilon^{-k} \sum_{j \in Y_\varepsilon} \int_{G_\varepsilon^j} \sigma(\varphi^+ - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x))(\varphi^+ - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) - u_\varepsilon) dx' \\ & \geq \int_{\Omega} f(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) + (1 - \kappa_\varepsilon)\varphi^- - u_\varepsilon) dx. \end{aligned}$$

Considering the first integral of the right-hand side of the inequality above, we have

$$\begin{aligned} & \int_{\Omega} \nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^-) \nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) dx \\ & = \int_{\Omega} \nabla \varphi \nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+(\tilde{P}_\varepsilon^j), x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) dx - \int_{\Omega} \nabla \mathcal{W}_\varepsilon(\varphi^+, x) \nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) dx \\ & \quad - \int_{\Omega} \nabla(\kappa_\varepsilon \varphi^-) \nabla(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) dx \\ & = \sum_{i=1}^3 J_\varepsilon^i. \end{aligned} \quad (5.2)$$

By using (4.1) and (5.1), we have

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon^1 = \int_{\Omega} \nabla \varphi \nabla(\varphi - u_0) dx. \quad (5.3)$$

Then we proceed by transforming J_ε^2 in the following way:

$$\begin{aligned}
 J_\varepsilon^2 &= - \sum_{j \in Y_\varepsilon} \int_{(T_{\varepsilon/4}^j)^+} \nabla w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) \cdot \nabla(\varphi - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) dx \\
 &= - \sum_{j \in Y_\varepsilon} \int_{(T_{\varepsilon/4}^j)^+} \{ \nabla(w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) - \tilde{w}_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x)) \nabla(\varphi - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) \} dx \\
 &\quad - \sum_{j \in Y_\varepsilon} \int_{(T_{\varepsilon/4}^j)^+} \nabla \tilde{w}_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) \nabla(\varphi - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) dx \\
 &= I_\varepsilon^1 + I_\varepsilon^2.
 \end{aligned} \tag{5.4}$$

Lemma 3.5 implies that

$$I_\varepsilon^1 \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{5.5}$$

By using Green's formula, we have the following decomposition of the second integral:

$$\begin{aligned}
 I_\varepsilon^2 &= - \sum_{j \in Y_\varepsilon} \int_{(\partial T_{\varepsilon/4}^j)^+} \{ \partial_\nu \tilde{w}_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) (\varphi^+ - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) - u_\varepsilon) \} ds \\
 &\quad - \varepsilon^{-k} \sum_{j \in Y_\varepsilon} \int_{G_\varepsilon^j} \{ \sigma(\varphi^+(\tilde{P}_\varepsilon^j) - \tilde{w}_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x)) (\varphi^+ - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x) - u_\varepsilon) \} dx' \\
 &= \mathcal{I}_\varepsilon^1 + \mathcal{I}_\varepsilon^2.
 \end{aligned} \tag{5.6}$$

From Lemma 3.7 we have

$$\lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^1 = C_0^{n-2} \int_{\Omega} H(\varphi^+(x)) (\varphi^+ - u_0) dx. \tag{5.7}$$

Combining (5.4)–(5.7) implies that

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon^2 = C_0^{n-2} \int_{\Omega} H(\varphi^+(x)) (\varphi^+ - u_0) dx + \lim_{\varepsilon \rightarrow 0} \mathcal{I}_\varepsilon^2. \tag{5.8}$$

Now we consider the third term of identity (5.2). By using the fact that

$$\begin{aligned}
 \nabla(\kappa_\varepsilon^j \varphi^-) \cdot \nabla \rho_\varepsilon &= (\nabla \kappa_\varepsilon^j \cdot \nabla \rho_\varepsilon) \varphi^- + \kappa_\varepsilon^j \nabla \varphi^- \cdot \nabla \rho_\varepsilon \\
 &= \nabla \kappa_\varepsilon^j \cdot \nabla(\varphi^- \rho_\varepsilon) - (\nabla \kappa_\varepsilon^j \cdot \nabla \varphi^-) \rho_\varepsilon + \kappa_\varepsilon^j \nabla \varphi^- \cdot \nabla \rho_\varepsilon \\
 &= \nabla \tilde{\kappa}_\varepsilon^j \cdot \nabla(\varphi^- \rho_\varepsilon) - \nabla(\tilde{\kappa}_\varepsilon^j - \kappa_\varepsilon^j) \cdot \nabla(\varphi^- \rho_\varepsilon) - (\nabla \kappa_\varepsilon^j \cdot \nabla \varphi^-) \rho_\varepsilon + \kappa_\varepsilon^j \nabla \varphi^- \cdot \nabla \rho_\varepsilon.
 \end{aligned}$$

we deduce that

$$\begin{aligned}
 J_\varepsilon^3 &= - \sum_{j \in Y_\varepsilon} \int_{(T_{\varepsilon/4}^j)^+} \nabla \tilde{\kappa}_\varepsilon^j \nabla(\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon)) dx \\
 &\quad - \sum_{j \in Y_\varepsilon} \int_{(T_{\varepsilon/4}^j)^+} \nabla(\kappa_\varepsilon^j - \tilde{\kappa}_\varepsilon^j) \nabla(\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon)) dx \\
 &\quad + \int_{\Omega} (\nabla \kappa_\varepsilon \cdot \nabla \varphi^-) (\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon)) dx \\
 &\quad - \int_{\Omega} \kappa_\varepsilon \nabla \varphi^- \cdot \nabla(\varphi^-(\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon)) dx \\
 &= \mathcal{Q}_\varepsilon^1 + \mathcal{Q}_\varepsilon^2 + \mathcal{Q}_\varepsilon^3 + \mathcal{Q}_\varepsilon^4.
 \end{aligned}$$

Lemma 4.1 implies that $\mathcal{Q}_\varepsilon^2 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then we use (1.6), (1.7), (4.1) and (5.1) to derive that $\mathcal{Q}_\varepsilon^3 \rightarrow 0$ and $\mathcal{Q}_\varepsilon^4 \rightarrow 0$ as $\varepsilon \rightarrow 0$. Then we transform $\mathcal{Q}_\varepsilon^1$ using the Green's formula:

$$\mathcal{Q}_\varepsilon^1 = - \sum_{j \in Y_\varepsilon} \int_{(\partial T_{\varepsilon/4}^j)^+} (\partial_\nu \hat{\kappa}_\varepsilon^j) \varphi^- (\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) ds - \sum_{j \in Y_\varepsilon} \int_{G_\varepsilon^j} (\partial_\nu \hat{\kappa}_\varepsilon^j) \varphi^- (\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) dx'.$$

By using the fact that $\partial_\nu \hat{\kappa} \geq 0$ and $\varphi^- \mathcal{W}_\varepsilon(\varphi^+, x) \leq 0$, $\varphi^- u_\varepsilon \leq 0$ a.e. in G_ε , we have that

$$- \sum_{j \in Y_\varepsilon} \int_{G_\varepsilon^j} (\partial_\nu \hat{\kappa}_\varepsilon^j) \varphi^- (\varphi - \mathcal{W}_\varepsilon(\varphi^+, x) - \kappa_\varepsilon \varphi^- - u_\varepsilon) dx' \leq 0.$$

Hence we conclude

$$\mathcal{Q}_\varepsilon^1 \leq - \sum_{j \in Y_\varepsilon} \int_{\partial T_{\varepsilon/4}^j \cap \Omega} (\partial_\nu \hat{\kappa}_\varepsilon^j) \varphi^- (\varphi - u_\varepsilon) ds.$$

Lemma 4.2 implies

$$\lim_{\varepsilon \rightarrow 0} J_\varepsilon^3 = \lim_{\varepsilon \rightarrow 0} \mathcal{Q}_\varepsilon^1 \leq \lambda C_0^{n-2} \int_{\Omega} \varphi^- (\varphi - u_\varepsilon) dx. \quad (5.9)$$

From (5.2), (5.3), (5.8) and (5.9) we derive that

$$\begin{aligned} & \int_{\Omega} \nabla \varphi \nabla (\varphi - u_0) dx + C_0^{n-2} \int_{\Gamma_1} H(\varphi^+) (\varphi - u_0) dx' + \lambda C_0^{n-2} \int_{\Gamma_1} \varphi^- (\varphi - u_0) dx' \\ & + \lim_{\varepsilon \rightarrow 0} \left\{ \varepsilon^{-k} \sum_{j \in Y_\varepsilon} \int_{G_\varepsilon^j} \sigma(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x)) (\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) - u_\varepsilon) dx' \right. \\ & \left. - \varepsilon^{-k} \sum_{j \in Y_\varepsilon} \int_{G_\varepsilon^j} \sigma(\varphi^+ (\hat{P}_\varepsilon^j) - \hat{w}_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x)) (\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x) - u_\varepsilon) dx' \right\} \\ & \geq \int_{\Omega} f(\varphi - u_0) dx. \end{aligned} \quad (5.10)$$

We first notice that $\varphi^+ - w_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x) \geq \varphi^+ (\hat{P}_\varepsilon^j) - \hat{w}_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x)$, and so

$$\{\sigma(\varphi^+ - w_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x)) - \sigma(\varphi^+ (\hat{P}_\varepsilon^j) - \hat{w}_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x))\} u_\varepsilon \geq 0.$$

Thus we only need to study the term

$$\varepsilon^{-k} \int_{G_\varepsilon^j} \{\sigma(\varphi^+ - w_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x)) - \sigma(\varphi^+ (\hat{P}_\varepsilon^j) - \hat{w}_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x))\} (\varphi^+ - w_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x)) dx'.$$

On the other hand,

$$\begin{aligned} |\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x)| & \leq |\varphi^+| \leq K, \\ w_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x) & \leq \hat{w}_\varepsilon^j(\varphi^+ (\hat{P}_\varepsilon^j), x) \leq |\varphi^+| \leq K. \end{aligned}$$

Since σ is Hölder continuous and $\varphi \in \mathcal{C}^1(\overline{\Omega})$, we have that, for a.e. $x \in G_\varepsilon^j$,

$$\begin{aligned} |\sigma(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x)) - \sigma(\varphi^+ (\hat{P}_\varepsilon^j) - \mathcal{W}_\varepsilon(\varphi^+, x))| & \leq K \sum_{i=1}^2 |\varphi^+(x) - \varphi^+ (\hat{P}_\varepsilon^j)|^{\rho_i} \\ & \leq K \sum_{i=1}^2 |x - \hat{P}_\varepsilon^j|^{\rho_i} \\ & = K \sum_{i=1}^2 a_\varepsilon^{\rho_i}. \end{aligned}$$

By using the same reasoning, estimate (3.4) implies that

$$\begin{aligned} & \left| \varepsilon^{-k} \sum_{j \in Y_\varepsilon} \int_{G_\varepsilon^j} (\sigma(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x)) - \sigma(\varphi^+(\tilde{P}_\varepsilon^j) - \mathcal{W}_\varepsilon(\varphi^+, x)))(\varphi^+ - w_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x)) dx' \right| \\ & \leq K_2 \varepsilon^{-k} |G_\varepsilon| \sum_{i=1}^2 a_\varepsilon^{\rho_i} \leq K \sum_{i=1}^2 a_\varepsilon^{\rho_i}. \end{aligned} \quad (5.11)$$

Then by Lemma 3.5 we have that

$$\begin{aligned} & \varepsilon^{-k} \left| \sum_{j \in Y_\varepsilon} \int_{G_\varepsilon^j} (\sigma(\varphi^+(\tilde{P}_\varepsilon^j) - \mathcal{W}_\varepsilon(\varphi^+, x)) - \sigma(\varphi^+(\tilde{P}_\varepsilon^j) - \tilde{w}_\varepsilon^j))(\varphi^+ - \mathcal{W}_\varepsilon(\varphi^+, x)) dx' \right| \\ & \leq K \varepsilon^{-k} \sum_{j \in Y_\varepsilon} \sum_{i=1}^2 \int_{G_\varepsilon^j} |v(\varphi^+(\tilde{P}_\varepsilon^j), x)|^{\rho_i} dx' \\ & \leq K \varepsilon^{-k} \sum_{j \in Y_\varepsilon} |G_\varepsilon^j| \sum_{i=1}^2 \frac{1}{|G_\varepsilon^j|} \int_{G_\varepsilon^j} |v(\varphi^+(\tilde{P}_\varepsilon^j), x)|^{\rho_i} dx', \end{aligned}$$

which, by applying the $L^2(G_0) \rightarrow L^{\rho_i}(G_0)$ embedding for $0 < \rho_i \leq 2$, can be estimated as

$$\dots \leq K \varepsilon^{-k} \sum_{j \in Y_\varepsilon} |G_\varepsilon^j| \sum_{i=1}^2 \left(\frac{1}{|G_\varepsilon^j|} \int_{G_\varepsilon^j} |v_\varepsilon^j(\varphi^+(\tilde{P}_\varepsilon^j), x)|^2 dx' \right)^{\frac{\rho_i}{2}}.$$

By Lemma 3.6 and using $0 < \rho \leq 2$, we obtain

$$\dots \leq K \varepsilon^{-k} \sum_{j \in Y_\varepsilon} |G_\varepsilon^j| \sum_{i=1}^2 \varepsilon^{\frac{\rho_i}{2}} \leq K \varepsilon^{-k} |Y_\varepsilon| |G_\varepsilon^j| \sum_{i=1}^2 \varepsilon^{\frac{\rho_i}{2}} \leq K \varepsilon^{-k+1-n+k(n-1)} \sum_{i=1}^2 \varepsilon^{\frac{\rho_i}{2}} = K \sum_{i=1}^2 \varepsilon^{\frac{\rho_i}{2}} \rightarrow 0. \quad (5.12)$$

Combining these estimates with (5.10), we derive, since $\rho > 0$, that

$$\int_{\Omega} \nabla \varphi \nabla (\varphi - u_0) dx + C_0^{n-2} \int_{\Gamma_1} H(\varphi^+) (\varphi - u_0) dx' + \lambda C_0^{n-2} \int_{\Gamma_1} \varphi^- (\varphi - u_0) \geq \int_{\Omega} f(\varphi - u_0) dx \quad (5.13)$$

holds for any $\varphi \in H^1(\Omega, \Gamma_2)$.

Finally, given $\psi \in H^1(\Omega, \Gamma_2)$, we consider the test function $\varphi = u_0 \pm \delta \psi$, $\delta > 0$ in (5.13) and we pass to the limit as $\delta \rightarrow 0$. By doing so, we get that u_0 satisfies the integral condition

$$\int_{\Omega} \nabla u_0 \nabla \psi dx + C_0^{n-2} \int_{\Gamma_1} H(u_0^+) \psi dx' + \lambda C_0^{n-2} \int_{\Gamma_1} u_0^- \psi dx' = \int_{\Omega} f \psi dx$$

for any $\psi \in H^1(\Omega, \Gamma_2)$. This concludes the proof. \square

6 Possible extensions and comments

6.1 Extension to the case of σ as a maximal monotone graph

In [10], the authors showed that a similar problem, although restricted to the case of spherical particles distributed through the whole domain, could be treated in the general framework of maximal monotone graphs σ , which allow for a *common roof* between the Dirichlet, Neumann and Signorini boundary conditions and many more. We have restricted ourselves here to the case of Hölder continuous σ (see (1.4)), but this condition is only used at the very end, in estimates (5.11) and (5.12) to compute the last term of (5.10). The superlinearity condition is only used to obtain Lemma 3.6. These seem to be only technical difficulties, and can probably be avoided. Let us introduce what results can be expected if these problems could be circumvented.

Maximal monotone graph of \mathbb{R}^2 . A monotone graph of \mathbb{R}^2 is a map (or operator) $\sigma : D(\sigma) \subset \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R}) \setminus \{\emptyset\}$ such that

$$(\xi_1 - \xi_2)(x_1 - x_2) \geq 0 \quad \text{for all } x_i \in D(\sigma) \text{ and all } \xi_i \in \sigma(x_i).$$

The set $D(\sigma)$ is called domain of the multivalued operator σ . Some authors define maximal monotone graphs as maps $\sigma : \mathbb{R} \rightarrow \mathcal{P}(\mathbb{R})$ and define $D(\sigma) = \{x \in \Omega : \sigma(x) \neq \emptyset\}$.

A monotone graph σ is extended by another monotone graph $\bar{\sigma}$ if $D(\sigma) \subset D(\bar{\sigma})$ and $\sigma(x) \subset \bar{\sigma}(x)$ for all $x \in D(\sigma)$. A monotone graph is called maximal if it admits no proper extension; for further references, see [2].

Definition of solution. The solution u_ε is also well-defined, although the set K_ε must now be written as

$$K_\varepsilon = \{v \in H^1(\Omega, \Gamma_2) : \text{for all } x' \in G_\varepsilon \text{ one has } v(x') \in D(\sigma)\}.$$

We will have that the integral condition (1.8) turns into

$$\int_{\Omega} \nabla \varphi \nabla (\varphi - u_\varepsilon) dx + \varepsilon^{-k} \int_{G_\varepsilon} \xi(x) (\varphi - u_\varepsilon) dx' \geq \int_{\Omega} f(\varphi - u_\varepsilon) dx$$

for all $\varphi \in K_\varepsilon$ and $\xi \in L^2(G_\varepsilon)$ such that $\xi(x') \in \sigma(\varphi(x'))$ for a.e. $x' \in G_\varepsilon$. The existence and uniqueness of this solution follows as in [10] and the references therein.

The auxiliary functions. The equation of \widehat{w} is well-defined when σ is a maximal monotone graph. As we have proved in this paper, the estimate $0 \leq H' \leq \lambda_{G_0}$ is independent of σ , and so H is Lipschitz continuous for any maximal monotone graph σ .

Signorini boundary conditions. This is the case under study in this paper. Nonetheless, let us study in the general setting. For this kind of boundary condition, we need to consider the following maximal monotone graph:

$$\bar{\sigma}(s) = \begin{cases} \sigma(s), & s > 0, \\ (-\infty, 0], & s = 0, \\ \emptyset, & s < 0, \end{cases} \quad (6.1)$$

and $D(\sigma) = [0, +\infty)$. Let us compute \widetilde{H}_{G_0} in this setting:

- For $u < 0$, we can see what happens explicitly. We have that $u \leq \widehat{w}(u, \cdot) \leq 0$. Thus $u - \widehat{w} \leq 0$. Since $D(\sigma) = [0, +\infty)$, we must have that $\widehat{w}(u, y) = u$ on G_0 . But then $\widehat{w}(u, y) = u\widehat{\kappa}(y)$. Hence $\widetilde{H}_{G_0}(u) = \lambda_{G_0}u$ when $u < 0$.
- When $u > 0$, we have that $0 \leq u - \widehat{w}(u, \cdot)$. Thus only the values of σ affect $H_{G_0}(u)$.

We conclude that

$$\widetilde{H}_{G_0}(u) = \begin{cases} H_{G_0}(u), & u > 0, \\ \lambda_{G_0}u, & u \leq 0. \end{cases}$$

The computations with maximal monotone graphs yield precisely Theorem 1.3. Notice that the bound on \widetilde{H}' given by (1.12) is sharp.

Dirichlet boundary conditions. In this case, we would have $D(\sigma) = \{0\}$ and

$$\bar{\sigma}(0) = (-\infty, +\infty).$$

By the same reasoning, we have that

$$\widetilde{H}_{G_0}(u) = \lambda_{G_0}u$$

for all $u \in \mathbb{R}$. In this case of Dirichlet boundary conditions, the critical case generates a linear term in the homogenized equation. This type of phenomena was already noticed by Cioranescu and Murat [6].

Cases of finite and infinite permeable coefficient. The Signorini boundary condition imposed as a maximal monotone graph (6.1) is the extreme case of *infinite permeability*, aiming to represent the behavior of very large *finite* permeability given by a reaction term of the form

$$\tilde{\sigma}_\mu(u) = \begin{cases} \sigma(u), & u > 0, \\ \mu u, & u \leq 0, \end{cases}$$

where μ is very large. As in Remark 2.7, it is easy to show that the corresponding kinetic will be of the form

$$\tilde{H}_\mu(u) = \begin{cases} H(u), & u > 0, \\ \lambda_\mu u, & u \leq 0. \end{cases}$$

Moreover, since we have proven that the Signorini boundary condition is an extremal case (i.e. $\tilde{H}'(u) \leq \lambda_{G_0}$), we have that $\lambda_\mu \leq \lambda_{G_0}$.

6.2 On the super-linearity condition

The condition

$$|\sigma(s) - \sigma(t)| \geq k_1 |s - t|$$

is only used in the proof of Lemma 3.6. However, it is our belief that this condition can be removed and that the result can still be obtained. We provide here a proof for $n = 3$ and a ball G_0 .

We define the auxiliary function w_ε as the unique solution of

$$\begin{cases} -\Delta w_\varepsilon^j = 0, & T_\varepsilon^0, \\ w_\varepsilon = 0, & G_\varepsilon^0, \\ w_\varepsilon = 1, & \partial T_\varepsilon^0, \end{cases}$$

where T_ε^0 is given by

$$T_\varepsilon^0 = \left\{ x \in \mathbb{R}^3 : \frac{x_1^2}{1 - (a_\varepsilon \varepsilon^{-1})^2} + x_2^2 + x_3^2 < \varepsilon^2 \right\}.$$

Using prolate ellipsoidal coordinates, we can give an explicit expression of w_ε . These coordinates are given by

$$\begin{aligned} x_1 &= a_\varepsilon \sinh \psi \cos \theta_1, \\ x_2 &= a_\varepsilon \cosh \psi \sin \theta_1 \cos \theta_2, \\ x_3 &= a_\varepsilon \cosh \psi \sin \theta_1 \sin \theta_2, \end{aligned}$$

where $0 \leq \psi < \infty$, $0 \leq \theta_1 \leq \pi$ and $0 \leq \theta_2 \leq 2\pi$. By defining $\sigma = \sinh \psi$, it can be proven through symmetry that $w_\varepsilon(x) = V_\varepsilon(\sigma)$. Furthermore, V_ε is the unique solution of the one-dimensional problem

$$\begin{cases} \frac{d}{d\sigma} \left((1 + \sigma^2) \frac{dV}{d\sigma} \right) = 0, & \sigma \in (0, \sqrt{(a_\varepsilon^{-1} \varepsilon)^2 - 1}), \\ V(0) = 0, \\ V(\sqrt{(a_\varepsilon^{-1} \varepsilon)^2 - 1}) = 1. \end{cases}$$

Integrating this simple one-dimensional boundary value problem, we obtain

$$V(\sigma) = \frac{\arctan \sigma}{\arctan(\sinh \sqrt{(a_\varepsilon^{-1} \varepsilon)^2 - 1})}$$

since we can recover from the change in variable

$$\sigma = \sinh \psi = \sqrt{\frac{|x|^2 - a_\varepsilon^2 + \sqrt{(a_\varepsilon^2 - |x|^2)^2 + 4x_1^2 a_\varepsilon^2}}{2a_\varepsilon^2}}.$$

Due to mirror symmetry it is clear that $\partial_{x_1} w_\varepsilon|_{\{x_1=0\}} = 0$. Thus we have

$$\int_{(T_\varepsilon^0)^+} \nabla w_\varepsilon \nabla(u^2) dx = \int_{(\partial T_\varepsilon^0)^+} \partial_{x_1} w_\varepsilon u^2 dS - \int_{G_\varepsilon^0} \partial_{x_1} w_\varepsilon u^2 dS.$$

Using the explicit expression of w_ε , we can compute that

$$\begin{aligned} \partial_\nu w_\varepsilon|_{(\partial T_\varepsilon^0)^+} &\sim a_\varepsilon \varepsilon^{-2}, \\ \partial_\nu w_\varepsilon|_{G_\varepsilon^0} &\sim -\frac{1}{\sqrt{a_\varepsilon^2 - |x|^2}}. \end{aligned}$$

Now let

$$\begin{aligned} T_\varepsilon^j &= P_\varepsilon^j + T_\varepsilon^0, \\ T_\varepsilon &= \bigcup_{j \in Y_\varepsilon} T_\varepsilon^j, \\ W_\varepsilon(x) &= w_\varepsilon(x - P_\varepsilon^j) \quad \text{for } x \in T_\varepsilon^j. \end{aligned}$$

Summing over Y_ε , we deduce that

$$\int_{(T_\varepsilon)^+} \nabla W_\varepsilon \nabla(u^2) ds = \sum_{j \in Y_\varepsilon} \int_{(\partial T_\varepsilon^j)^+} \partial_\nu w_\varepsilon^j u^2 ds - \int_{G_\varepsilon} \partial_{x_1} W_\varepsilon u^2 ds. \quad (6.2)$$

It is easy to prove that

$$\int_{(\partial T_\varepsilon^j)^+} \partial_\nu w_\varepsilon^j h^2 ds \leq K \sum_{j \in Y_\varepsilon} \|h\|_{H^1((T_\varepsilon^j)^+)}^2 \quad (6.3)$$

for any $h \in H^1(\Omega)$. We now apply that

$$\|u\|_{L^2(G_\varepsilon)}^2 \leq K(\varepsilon^{-1} \|u\|_{L^2(T_\varepsilon^+)} + \varepsilon \|\nabla u\|_{L^2(T_\varepsilon^+)}^2). \quad (6.4)$$

With (6.2)–(6.4) we can prove Lemma 3.6 for $k_1 = 0$.

6.3 Connections to fractional operators

Let us consider a domain $\Omega = \Omega' \times (0, +\infty)$, where $\Omega' \subset \mathbb{R}^{n-1}$ is a smooth bounded domain. Then $\Gamma_1 = \Omega'$ and $\Gamma_2 = \partial\Omega \times (0, +\infty)$. The related problem

$$\begin{cases} -\Delta u_\varepsilon = 0, & (x, y) \in \Omega' \times (0, +\infty), \\ \partial_\nu u_\varepsilon + \varepsilon^{-k} \sigma(u_\varepsilon) = \varepsilon^{-k} g_\varepsilon, & x \in G_\varepsilon, \\ \partial_\nu u_\varepsilon = 0, & x \in \Omega' \setminus \overline{G_\varepsilon}, \\ u_\varepsilon = 0, & (x, y) \in \partial\Omega' \times (0, +\infty), \\ u_\varepsilon \rightarrow 0, & |y| \rightarrow +\infty, \end{cases} \quad (6.5)$$

is very relevant because it can be linked to the study of the fractional Laplacian $(-\Delta)^{1/2}$. In fact, the boundary conditions on Ω' can be written compactly as

$$\partial_\nu u_\varepsilon + \varepsilon^{-k} \chi_{G_\varepsilon} \sigma(u_\varepsilon) = \varepsilon^{-k} \chi_{G_\varepsilon} g_\varepsilon, \quad x \in \Omega', \quad (6.6)$$

where χ is the indicator function. This boundary condition can be written as an equation of Ω' not involving the interior part of the domain, $\Omega' \times (0, +\infty)$, by understanding the normal derivative of problem (6.5) as the fractional Laplace operator $(-\Delta)^{1/2}$ in Ω' (see [3, 15] and the references therein). Then (6.6) can be written as

$$(-\Delta)^{\frac{1}{2}} u_\varepsilon + \varepsilon^{-k} \chi_{G_\varepsilon} \sigma(u_\varepsilon) = \varepsilon^{-k} \chi_{G_\varepsilon} g_\varepsilon, \quad x \in \Omega'. \quad (6.7)$$

Thus the study of the limit of (6.5) will provide a homogenization result for (6.7). By applying similar techniques to this paper and previous results in the literature [12], the homogenized problem

$$(-\Delta)^{\frac{1}{2}} u_0 + CH(x, u_0) = Ch, \quad x \in \Omega',$$

is expected, where H and h will depend on σ and g_ε . This could provide some new results of critical size homogenization for the fractional Laplacian (in the spirit of the important work [3], where some random aspects on the net, and for a general fractional power of the Laplacian, are also considered).

Part II:

The case $n = 2$

7 Proof of Theorem 1.4

It is well known that problem (1.13), (1.14) has a unique weak solution $u_\varepsilon \in H^1(\Omega, \Gamma_2)$. By using (1.14) and condition (1.15), which was set on the function σ , we get the following estimates:

$$\begin{aligned} \|\nabla u_\varepsilon\|_{L_2(\Omega)} &\leq K, \\ e^{\frac{a^2}{\varepsilon}} \|u_\varepsilon\|_{L_2(G_\varepsilon)}^2 &\leq K_1. \end{aligned}$$

Here and below, the constants K and K_1 are independent of ε .

Hence there exists a subsequence (denote as the original sequence u_ε) such that, as $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} u_\varepsilon &\rightharpoonup u_0 \quad \text{weakly in } H_0^1(\Omega), \\ u_\varepsilon &\rightarrow u_0 \quad \text{strongly in } L_2(\Omega). \end{aligned}$$

We introduce auxiliary functions w_ε^j and q_ε^j as weak solutions to the following problems:

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in T_{\varepsilon/4}^j \setminus \overline{T_{a_\varepsilon}^j}, \\ w_\varepsilon^j = 1, & x \in \partial T_{a_\varepsilon}^j, \\ w_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j, \end{cases}$$

and

$$\begin{cases} \Delta q_\varepsilon^j = 0, & x \in T_\varepsilon^j \setminus \overline{l_\varepsilon^j}, \\ q_\varepsilon^j = 1, & x \in l_\varepsilon^j, \\ q_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j. \end{cases} \quad (7.1)$$

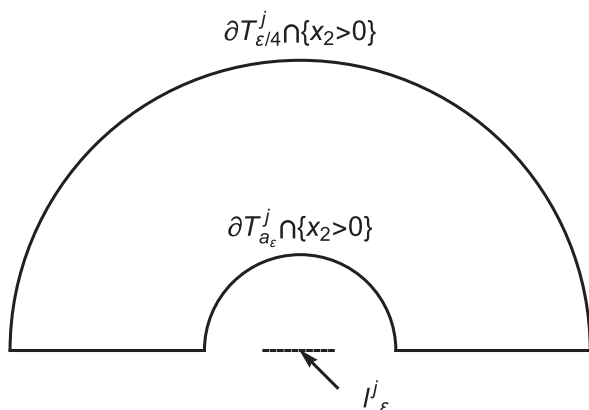


Figure 4: Domain $T_{\varepsilon/4}^{+,j} \setminus \overline{T_{a_\varepsilon}^j}$ and l_ε^j .

Note that w_ε^j and q_ε^j are also solutions of the following boundary value problems in the domains $(T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}$ and $(T_{\varepsilon/4}^j)^+$ (see Figure 4), respectively, where $(T_r^j)^+ = T_r^j \cap \{x_2 > 0\}$,

$$\begin{cases} \Delta w_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}, \\ w_\varepsilon^j = 0, & x \in \partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}, \\ w_\varepsilon^j = 1, & x \in \partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}, \\ \partial_{x_2} w_\varepsilon^j = 0, & x \in \{x_2 = 0\} \cap (T_{\varepsilon/4}^j \setminus \overline{T_{a_\varepsilon}^j}), \end{cases}$$

$$\begin{cases} \Delta q_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j)^+, \\ q_\varepsilon^j = 0, & x \in (\partial T_{\varepsilon/4}^j)^+, \\ q_\varepsilon^j = 1, & x \in l_\varepsilon^j, \\ \partial_{x_2} q_\varepsilon^j = 0, & x \in (T_{\varepsilon/4}^j \cap \{x_2 = 0\}) \setminus \overline{l_\varepsilon^j}, \end{cases}$$

where $j \in Y_\varepsilon$ and $l_\varepsilon^j = a_\varepsilon \hat{l}_0 + \varepsilon j$. Define

$$W_\varepsilon(x) = \begin{cases} w_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a_\varepsilon}^j}, j \in Y_\varepsilon, \\ 1, & x \in (T_{a_\varepsilon}^j)^+, \\ 0, & x \in \mathbb{R}_+^2 \setminus \bigcup_{j \in Y_\varepsilon} \overline{T_{\varepsilon/4}^j}^+, \end{cases} \quad (7.2)$$

where $\mathbb{R}_+^2 = \{x_2 > 0\}$, and

$$Q_\varepsilon(x) = \begin{cases} q_\varepsilon^j(x), & x \in (T_{\varepsilon/4}^j)^+, j \in Y_\varepsilon, \\ 0, & x \in \mathbb{R}_+^2 \setminus \bigcup_{j \in Y_\varepsilon} \overline{T_{\varepsilon/4}^j}^+. \end{cases} \quad (7.3)$$

We have $W_\varepsilon, Q_\varepsilon \in H_0^1(\Omega)$ and

$$W_\varepsilon \rightharpoonup 0 \quad \text{weakly in } H_0^1(\Omega) \text{ as } \varepsilon \rightarrow 0.$$

Lemma 7.1. Let W_ε be a function defined by formula (7.2), and let Q_ε be a function defined by formula (7.3). Then

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \leq K\sqrt{\varepsilon}.$$

Proof. Note that for an arbitrary function $\psi \in H^1(T_{\varepsilon/4}^j)$ such that $\psi = 0$ on l_ε^j , we have

$$\int_{(T_{\varepsilon/4}^j)^+} \nabla q_\varepsilon^j \nabla \psi \, dx_1 \, dx_2 = 0.$$

We consider $\psi = w_\varepsilon^j - q_\varepsilon^j$ as a test function in the obtained equality and get

$$\int_{(T_{\varepsilon/4}^j)^+} \nabla q_\varepsilon^j \nabla (w_\varepsilon^j - q_\varepsilon^j) dx_1 dx_2 = 0. \quad (7.4)$$

In addition, we have

$$\int_{(T_{\varepsilon/4}^j)^+} \nabla w_\varepsilon^j \nabla (w_\varepsilon^j - q_\varepsilon^j) dx_1 dx_2 = \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} \partial_\nu w_\varepsilon^j (w_\varepsilon^j - q_\varepsilon^j) ds. \quad (7.5)$$

By subtracting (7.4) from (7.5), we derive

$$\int_{(T_{\varepsilon/4}^j)^+} |\nabla (w_\varepsilon^j - q_\varepsilon^j)|^2 dx_1 dx_2 = \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} \partial_\nu w_\varepsilon^j (w_\varepsilon^j - q_\varepsilon^j) ds. \quad (7.6)$$

Note that

$$w_\varepsilon^j(x) = \frac{\ln(4r/\varepsilon)}{\ln(4a_\varepsilon/\varepsilon)} \quad \text{and} \quad \partial_\nu w_\varepsilon^j = -\frac{1}{a_\varepsilon \ln(4a_\varepsilon/\varepsilon)}.$$

Hence, (7.6) implies that

$$\begin{aligned} \|\nabla (w_\varepsilon^j - q_\varepsilon^j)\|_{L^2((T_{\varepsilon/4}^j)^+)}^2 &\leq \frac{1}{a_\varepsilon |\ln(4a_\varepsilon/\varepsilon)|} \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} |w_\varepsilon^j - q_\varepsilon^j| ds \\ &= \frac{1}{|\ln(4a_\varepsilon/\varepsilon)|} \int_{\partial T_1^j \cap \{y_2 > 0\}} |w_\varepsilon^j - q_\varepsilon^j| ds_y \equiv J_\varepsilon. \end{aligned}$$

Given that $w_\varepsilon^j - q_\varepsilon^j = 0$ if $y \in \widehat{l}_0$ and using the embedding theorem, we get

$$J_\varepsilon \leq \frac{K}{|\ln(4a_\varepsilon/\varepsilon)|} \left(\int_{(T_1^j)^+} |\nabla_y (w_\varepsilon^j - q_\varepsilon^j)|^2 dy \right)^{1/2} \leq K\varepsilon \|\nabla (w_\varepsilon^j - q_\varepsilon^j)\|_{L^2((T_{\varepsilon/4}^j)^+)}.$$

From here we derive the estimate

$$\|\nabla (w_\varepsilon^j - q_\varepsilon^j)\|_{L^2((T_{\varepsilon/4}^j)^+)} \leq K\varepsilon.$$

From this estimation it follows that

$$\|W_\varepsilon - Q_\varepsilon\|_{H^1(\Omega)} \leq K\sqrt{\varepsilon}.$$

This concludes the proof. \square

We introduce the function $m(y) \in H^1((T_1^0)^+)$ as the weak solution to the following boundary value problem:

$$\begin{cases} \Delta_y m = 0, & y \in T_1^0 \cap \{y_2 > 0\} = (T_1^0)^+, \\ \partial_{y_2} m = 1, & y \in \widehat{l}_0, \\ \partial_\nu m = \frac{2l_0}{\pi}, & y \in \partial T_1^0 \cap \{y_2 > 0\} = (\partial T_1^0)^+, \\ \partial_{y_2} m = 0, & y \in \partial(T_1^0)^+ \setminus \widehat{l}_0 \cup (\partial T_1^0)^+. \end{cases}$$

Consider

$$m_\varepsilon^j(x) = \varepsilon m\left(\frac{x - P_\varepsilon^j}{a_\varepsilon}\right), \quad x \in (T_{a_\varepsilon}^j)^+.$$

The function $m_\varepsilon^j(x)$ verifies the problem

$$\begin{cases} \Delta_x m_\varepsilon^j = 0, & x \in (T_{a_\varepsilon}^j)^+, \\ \partial_\nu m_\varepsilon^j = \frac{\varepsilon a_\varepsilon^{-1} 2l_0}{\pi}, & x \in (\partial T_{a_\varepsilon}^j)^+, \\ \partial_{x_2} m_\varepsilon^j = \varepsilon a_\varepsilon^{-1}, & x \in \{x_2 = 0 : |x - P_\varepsilon^j| \leq a_\varepsilon l_0\} = l_\varepsilon^j, \\ \partial_{x_2} m_\varepsilon^j = 0 & \text{on the rest of the boundary.} \end{cases}$$

Lemma 7.2. Let $n = 2$ and let $h_\varepsilon \in H^1(\Omega, \Gamma_2)$ be a bounded sequence. Then the following estimate holds:

$$\left| \frac{2l_0\varepsilon}{\pi a_\varepsilon} \sum_{j \in Y_\varepsilon} \int_{(\partial T_{a_\varepsilon}^j)^+} h_\varepsilon ds - \frac{\varepsilon}{a_\varepsilon} \int_{l_\varepsilon} h_\varepsilon dx_1 \right| \leq K\sqrt{\varepsilon}.$$

Proof. We have that

$$\left| \frac{2l_0\varepsilon a_\varepsilon^{-1}}{\pi} \int_{(\partial T_{a_\varepsilon}^j)^+} h_\varepsilon ds - \varepsilon a_\varepsilon^{-1} \int_{l_\varepsilon} h_\varepsilon dx_1 \right| = \left| \int_{(T_{a_\varepsilon}^j)^+} \nabla_x m_\varepsilon^j \nabla h_\varepsilon dx \right| \leq \|\nabla_x m_\varepsilon^j\|_{L^2((T_{a_\varepsilon}^j)^+)} \|\nabla h_\varepsilon\|_{L^2((T_{a_\varepsilon}^j)^+)}. \quad (7.7)$$

Due to the fact that

$$\|\nabla_x m_\varepsilon^j\|_{L^2((T_{a_\varepsilon}^j)^+)}^2 = \varepsilon^2 \|\nabla_y m(y)\|_{L^2((T_1^0)^+)}^2 \leq K\varepsilon^2,$$

we have

$$\sum_{j \in Y_\varepsilon} \|\nabla_x m_\varepsilon^j\|_{L^2((T_{a_\varepsilon}^j)^+)}^2 \leq K\varepsilon. \quad (7.8)$$

From (7.7), (7.8) we derive

$$\left| e^{\alpha^2/\varepsilon} \frac{\pi}{2l_0} \int_{l_\varepsilon} h_\varepsilon dx_1 - e^{\alpha^2/\varepsilon} \sum_{j \in Y_\varepsilon} \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} h_\varepsilon ds \right| \leq \delta^{-1} \sum_{j \in Y_\varepsilon} \|\nabla_x m_\varepsilon^j\|_{L^2((T_{a_\varepsilon}^j)^+)}^2 + \delta \|\nabla h_\varepsilon\|_{L^2(\Omega)}^2 \leq K\sqrt{\varepsilon}$$

if $\delta = \sqrt{\varepsilon}$. □

Proof of Theorem 1.4. First of all, equation (1.17) has a unique solution $H(u)$ that is a Lipschitz continuous function in \mathbb{R} and satisfies

$$(H(u) - H(v))(u - v) \geq \tilde{k}_1 |u - v|, \quad |H(u)| \leq |u|, \quad (7.9)$$

for all $u, v \in \mathbb{R}$ and a certain constant $\tilde{k}_1 > 0$.

We take $v = \psi - Q_\varepsilon(H(\psi^+) + \psi^-)$ as a test function in (1.14), where $\psi \in C^\infty(\bar{\Omega})$, $\psi(x) = 0$ in the neighborhood of Γ_2 and $H(u)$ satisfies the functional equation (1.17). Note that from (7.1), (7.3) and (7.9) we have $v \geq 0$ on l_ε , so $v \in K_\varepsilon$. Hence we get

$$\begin{aligned} & \int_{\Omega} \nabla(\psi - Q_\varepsilon(H(\psi^+) + \psi^-)) \nabla(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon) dx + e^{\frac{\alpha^2}{\varepsilon}} \int_{l_\varepsilon} \sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_\varepsilon) dx_1 \\ & \geq \int_{\Omega} f(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon) dx. \end{aligned} \quad (7.10)$$

We rewrite inequality (7.10) in the following way:

$$\begin{aligned} & \int_{\Omega} \nabla(\psi - W_\varepsilon(H(\psi^+) + \psi^-)) \nabla(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon) dx \\ & - \int_{\Omega} \nabla((Q_\varepsilon - W_\varepsilon)(H(\psi^+) + \psi^-)) \nabla(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon) dx \\ & + e^{\frac{\alpha^2}{\varepsilon}} \int_{l_\varepsilon} \sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_\varepsilon) dx_1 \\ & \geq \int_{\Omega} f(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon) dx. \end{aligned} \quad (7.11)$$

From the fact that $Q_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$ weakly in $H^1(\Omega, \Gamma_2)$, we have

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} f(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon) dx = \int_{\Omega} f(\psi - u_0) dx, \quad (7.12)$$

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla \psi \nabla(\psi - Q_\varepsilon(H(\psi^+) + \psi^-) - u_\varepsilon) dx = \int_{\Omega} \nabla \psi \nabla(\psi - u_0) dx. \quad (7.13)$$

Lemma 7.1 implies that

$$\lim_{\varepsilon \rightarrow 0} \int_{\Omega} \nabla((Q_{\varepsilon} - W_{\varepsilon})(H(\psi^+) + \psi^-)) \nabla(\psi - Q_{\varepsilon}(H(\psi^+) + \psi^-) - u_{\varepsilon}) dx = 0. \quad (7.14)$$

Consider the remaining integrals in (7.11). Set

$$\begin{aligned} I_{\varepsilon} &\equiv - \int_{\Omega} \nabla(W_{\varepsilon}(H(\psi^+) + \psi^-)) \nabla(\psi - Q_{\varepsilon}(H(\psi^+) + \psi^-) - u_{\varepsilon}) dx \\ &= - \int_{\Omega} \nabla W_{\varepsilon} \nabla\{(H(\psi^+) + \psi^-)(\psi - Q_{\varepsilon}(H(\psi^+) + \psi^-) - u_{\varepsilon})\} dx + \alpha_{\varepsilon}, \end{aligned}$$

where $\alpha_{\varepsilon} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

It is easy to see that

$$\begin{aligned} I_{\varepsilon} &= - \int_{\Omega} \nabla W_{\varepsilon} \nabla\{(H(\psi^+) + \psi^-)(\psi - Q_{\varepsilon}(H(\psi^+) + \psi^-) - u_{\varepsilon})\} dx \\ &= - \sum_{j \in Y_{\varepsilon}} \int_{(T_{\varepsilon/4}^j)^+ \setminus \overline{T_{a\varepsilon}^j}} \nabla w_{\varepsilon}^j \nabla\{(H(\psi^+) + \psi^-)(\psi - q_{\varepsilon}^j(H(\psi^+) + \psi^-) - u_{\varepsilon})\} dx + \widetilde{\alpha}_{\varepsilon} \\ &= - \sum_{j \in Y_{\varepsilon}} \int_{\partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}} \partial_{\nu} w_{\varepsilon}^j (H(\psi^+) + \psi^-)(\psi - u_{\varepsilon}) ds \\ &\quad - \sum_{j \in Y_{\varepsilon}} \int_{\partial T_{a\varepsilon}^j \cap \{x_2 > 0\}} \partial_{\nu} w_{\varepsilon}^j (H(\psi^+) + \psi^-)(\psi^+ - H(\psi^+) - u_{\varepsilon}) ds + \widetilde{\alpha}_{\varepsilon}, \end{aligned} \quad (7.15)$$

where $\widetilde{\alpha}_{\varepsilon} \rightarrow 0$, $\varepsilon \rightarrow 0$.

Since

$$\partial_{\nu} w_{\varepsilon}^j|_{\partial T_{\varepsilon/4}^j} = \frac{4}{\varepsilon \ln(4a_{\varepsilon}/\varepsilon)} = \frac{4}{-\alpha^2 + \varepsilon \ln(4C_0)},$$

using the results of [27], we derive

$$\begin{aligned} & - \lim_{\varepsilon \rightarrow 0} \sum_{j \in Y_{\varepsilon}} \int_{\partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}} \partial_{\nu} w_{\varepsilon}^j (H(\psi^+) + \psi^-)(\psi - u_{\varepsilon}) ds \\ &= \lim_{\varepsilon \rightarrow 0} \frac{4}{\alpha^2 - \varepsilon \ln(4C_0)} \sum_{j \in Y_{\varepsilon}} \int_{\partial T_{\varepsilon/4}^j \cap \{x_2 > 0\}} (H(\psi^+) + \psi^-)(\psi - u_{\varepsilon}) ds \\ &= \frac{\pi}{\alpha^2} \int_{\Gamma_2} (H(\psi^+) + \psi^-)(\psi - u_0) ds. \end{aligned} \quad (7.16)$$

Let us find the limit of the expression

$$\begin{aligned} & - \sum_{j \in Y_{\varepsilon}} \int_{\partial T_{a\varepsilon}^j \cap \{x_2 > 0\}} \partial_{\nu} w_{\varepsilon}^j (H(\psi^+) + \psi^-)(\psi^+ - H(\psi^+) - u_{\varepsilon}) ds + e^{\frac{\alpha^2}{\varepsilon}} \int_{l_{\varepsilon}} \sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_{\varepsilon}) dx_1 \\ &= \sum_{j \in Y_{\varepsilon}} \frac{(\alpha^2 C_0)^{-1} e^{\alpha^2/\varepsilon}}{1 - \varepsilon \alpha^{-2} \ln(4C_0)} \int_{\partial T_{a\varepsilon}^j \cap \{x_2 > 0\}} (H(\psi^+) + \psi^-)(\psi^+ - H(\psi^+) - u_{\varepsilon}) ds \\ &\quad + e^{\alpha^2/\varepsilon} \int_{l_{\varepsilon}} \sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_{\varepsilon}) dx_1 \end{aligned}$$

$$\begin{aligned}
&= e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} \sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_\varepsilon) dx_1 \\
&\quad - \frac{e^{\alpha^2/\varepsilon}}{\alpha^2 C_0} \sum_{j \in Y_\varepsilon} \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} (H(\psi^+) + \psi^-)(\psi^+ - H(\psi^+) - u_\varepsilon) ds + \hat{\alpha}_\varepsilon \\
&\equiv D_\varepsilon + \hat{\alpha}_\varepsilon,
\end{aligned} \tag{7.17}$$

where $\hat{\alpha}_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

To conclude the proof we will estimate the limit of D_ε . We have

$$\begin{aligned}
D_\varepsilon &= \left\{ \frac{\pi}{2l_0 \alpha^2 C_0} e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} (H(\psi^+) + \psi^-)(\psi^+ - H(\psi) - u_\varepsilon) dx_1 \right. \\
&\quad \left. - \frac{e^{\alpha^2/\varepsilon}}{\alpha^2 C_0} \sum_{j \in Y_\varepsilon} \int_{\partial T_{a_\varepsilon}^j \cap \{x_2 > 0\}} (H(\psi^+) + \psi^-)(\psi^+ - H(\psi^+) - u_\varepsilon) ds \right\} \\
&\quad + e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} \left\{ \sigma(\psi^+ - H(\psi^+)) - \frac{\pi}{2l_0 \alpha^2 C_0} H(\psi^+) \right\} (\psi^+ - H(\psi^+) - u_\varepsilon) dx_1 \\
&\quad - \frac{\pi}{2l_0 \alpha^2 C_0} e^{\alpha^2/\varepsilon} \int_{l_\varepsilon} \psi^- (\psi^+ - H(\psi) - u_\varepsilon) dx_1 \\
&= \mathcal{J}_\varepsilon^1 + \mathcal{J}_\varepsilon^2 + \mathcal{J}_\varepsilon^3.
\end{aligned} \tag{7.18}$$

Lemma 7.2 implies that

$$|\mathcal{J}_\varepsilon^1| \leq K\sqrt{\varepsilon}. \tag{7.19}$$

Then $\mathcal{J}_\varepsilon^2$ vanishes due to equation (1.17). Using that $u_\varepsilon \geq 0$, $\psi^- \leq 0$ on l_ε and the fact that $\psi^-(\psi^+ - H(\psi^+)) \equiv 0$, we have

$$\mathcal{J}_\varepsilon^3 \leq 0. \tag{7.20}$$

Hence we have that $\lim_{\varepsilon \rightarrow 0} D_\varepsilon \leq 0$ and

$$\lim_{\varepsilon \rightarrow 0} \left(e^{\frac{\alpha^2}{\varepsilon}} \int_{l_\varepsilon} \sigma(\psi^+ - H(\psi^+))(\psi^+ - H(\psi^+) - u_\varepsilon) dx_1 + I_\varepsilon \right) \leq \frac{\pi}{\alpha^2} \int_{\Gamma_2} (H(\psi^+) + \psi^-)(\psi - u_0) ds. \tag{7.21}$$

Therefore, from (7.10)–(7.21) we conclude that $u_0 \in H^1(\Omega, \Gamma_2)$ satisfies the inequality

$$\int_{\Omega} \nabla \psi \nabla (\psi - u_0) dx + \frac{\pi}{\alpha^2} \int_{\Gamma_1} (H(\psi^+) + \psi^-)(\psi - u_0) dx_1 \geq \int_{\Omega} f(\psi - u_0) dx$$

for any $\psi \in H^1(\Omega, \Gamma_2)$, where $H(u)$ satisfies the functional equation (1.17). This concludes the proof. \square

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