Differential Games of Exhaustible Resource Extraction



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To my parents

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Abstract

This thesis is concerned with game-theoretic models of oligopoly resource markets. They revolve around an open market, on which a number of firms sell a common resource. The market price-demand relationship means that the price (demand) that results from the firm's production (pricing) decisions is a function of the decisions of all firms selling to that market. This means that firms must generally anticipate the actions of competing firms when determining their own strategies, which means that these models often need to be analysed using game theory.

We focus on games in which the resource is exhaustible, with the exception of Chapter 5, in which the majority of the analysis is carried out in an inexhaustible resources setting. Exhaustibility introduces an additional complication into these games; that of allocating the extraction and sale of a limited resource pool over time. We consider several separate areas of extension, which we outline below.

In Chapter 2, we consider a dynamic Stackelberg game. Stackelberg competition is an asymmetric form of competition in which one player (the leader) has the ability to pre-commit to and announce a strategy in advance. The ability to pre-commit to a strategy is almost always highly valuable, and in this case allows the leader to drive down the follower's production by pre-committing to drive up their own. We follow the framework used in [62] to analyse Cournot competition to derive our results.

In Chapter 3, we compare the two settings in which resource extraction models are usually formulated: Open-Loop, in which the players determine their strategies as functions of time and the *initial* resource levels of the players only; and Feedback-Loop, in which the players determine their strategies at each point in time as a function of the *current* resource levels at that time. Our focus is on the investigation of the relationship between the difference in the production or value of a firm under these two models, and the distribution of resources across the firms.

In Chapter 4, we consider a common property resource game. These involve multiple firms which can extract from a common resource pool. We study a widely-used Open-

Loop model, as formulated in [79]. We examine the result that analysis of the problem by standard methods results in two candidate equilibria, and argue that one of these equilibria can be ruled out by construction of a superior response.

In Chapter 5, we analyse joint constraints on production, namely constraints which are met when the *total* production is above or below a certain level. It is a well-established result that these constraints can result in multiple equilibria. We provide several brief extensions to existing uniqueness results. We also demonstrate methods by which these results can be utilised to analyse games with piecewise-linear windfall taxes or congestion charges. Finally, we discuss the problems of extending these results to games with resource exhaustibility.

Statement of Originality

This is to certify that, to the best of my knowledge, the content of this thesis is original and is my own work, unless otherwise specified. This thesis has not been submitted for examination for any other degree or award, although parts of this thesis are being prepared for publication, in collaboration with Prof. Sam Howison.

I certify that the intellectual content of this thesis is the product of my own work and that all of the assistance received in preparing this thesis has been acknowledged.

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Chapter 1

Introduction

The exhaustibility of the natural resources which drive manufacturing and energy generation is becoming an increasingly important consideration in markets worldwide.[1] Recent years have been punctuated by growing scarcity in resources such as oil,[2] coal, and rare-earth metals, while booming economic development in China, India, Brazil and Russia have led to rising demand for those scarce resources. Given this, it is unsurprising that there has also been a surge of activity by countries such as China and Russia in acquiring strategic resource deposits around the world,[3] and resource wars are frequently forecast as a growing threat in the coming decades.[4] It is therefore to be expected that modelling exhaustibility in resource markets is a topic which has been widely analysed in the literature, from both top-down and bottom-up approaches.

From a top-down perspective, there is a large body of work which seeks to analyse these markets using stochastic processes to model and predict future prices. Oil prices are fitted to a stochastic process using past price data in [124]. Econometric models, which use a variety of indicators, such as GDP and domestic FDI levels, to predict prices and regional production levels are considered in [75] and [81] respectively. A variety of rules of thumb are also used in the valuation of specific mining sites by industry practitioners, which incorporate economic indicators as well as site-specific indicators arising from geological surveys; several of these are considered in [3] and [2].

[1] See [84], [11].

[2] There is an especially rich literature on oil scarcity, driven by the 'Peak-Oil' thesis— see [38] and [10].

[3] See [39].

[4] See [83] and [9].

These models can be highly useful in predicting day-to-day fluctuations in markets,[5] but are less useful in gaining an understanding of the mechanisms and behaviours which drive them. Moreover, these behaviours are often highly complex. Resource markets are frequently oligopolistic, requiring firms to consider the impact of their own production as well as that of their competitors on the market. Furthermore, the resources in question are also exhaustible, at least insofar as the stock of an individual firm is concerned. This exhaustibility creates the additional problem of optimally spreading the extraction and sale of limited resource pools over time.

Resource markets, like any other, are susceptible to large changes and shocks. An understanding of the behaviours driving market participants is therefore key to predicting the impact of these changes on the market. Externally, governments and regulators can intervene in markets via a number of policy tools: they can impose taxes, constrain production, set limits on the market power of firms, or direct investment towards resource substitutes. Internally, the competitive structure of the market changes with the entry and exit of competitors; while investment in extraction technology, or the exploration and discovery of new resource pools can result in sudden changes to the relative concentration of resources across firms. Understanding the strategic behaviour underpinning energy markets can not only aid in predicting the effects of these changes, but in doing so may inform the design and implementation of policy changes in the market by regulators.

Analysing these markets from a bottom-up perspective is a useful way of addressing this problem. This entails modelling strategic behaviour in resource markets, and the formation of prices within them, as a game played by resource holders, resource buyers, and governments. This area of research has a rich history, beginning with the original Hotelling model of [71],[6] and it is this area which our thesis focuses on.

We note that this is a very broad area of research, and the areas of extension which we aim to cover are specialised. We therefore devote this introduction to providing an overview of the field and introducing its key concepts. We begin by providing a general discussion of game-theoretic models in Section 1.1. In Section 1.2 we introduce and discuss the derivation of the demand curves linking supply and demand in markets. In

[5] However, this has been disputed; [72] is one of many works which criticise the predictive capacity of these models in response to large-scale shocks.

[6] A detailed background of developments on this work from 1931-1981 is provided in [40].

Section 1.3 we discuss the economic models of market competition which we base our resource market games on. In Section 1.4 we discuss the impact of resource exhaustibility in more detail. In Section 1.5 we discuss differential games, in particular focusing on an important technical issue for differential games of the type we cover—the period of commitment. We conclude this introduction by providing a brief roadmap of the areas of extension covered by this thesis in Section 1.6.

1.1 Game Theoretic Components

The models which we study in this thesis are a class of economic game. Broadly speaking, these are models of the behaviour of a number of economic agents. These agents perform strategic actions and receive a pay-off based on the strategies undertaken by all agents. Because this pay-off depends on the strategies of all agents, each one must anticipate the likely strategies of the other agents when selecting their own, and so on. This iterative reasoning is one of the key features of a game-theoretic model.

Games have been used to model a diverse set of problems, from missile targeting[7] to the merits of the Doctrine of Papal Infallibility.[8] In this section, we introduce the key building blocks of a game-theoretic model in a non-cooperative setting, and present several illustrative examples. We then conclude by briefly discussing cooperative games.[9]

1.1.1 Components of a Non-cooperative Game

A non-cooperative game is one in which it is assumed that each player is only concerned with their own welfare, and in which the players must determine their strategies independently. This prevents players from agreeing to undertake combinations of strategies which are only mutually advantageous if all players adhere to them, even if they would benefit from defecting, because there are no means by which the players can be bound to cooperative agreements. Games are defined by the following components:

[7] See [139].

[8] See [45].

[9] All of the areas of extension of our thesis relate to non-cooperative games. Cooperative games are discussed for completeness and, while we draw on the results of this section briefly in Chapter 5, it can be readily skipped.

1.1.1.1 Players

We define the players of the games covered in this thesis as those agents which determine strategies which have an impact on the pay-offs allocated in the game; these are generally resource firms in our thesis. We note that whilst this definition of player is common, the word is sometimes also used as a more general term of reference. In [23], for example, the term player is also used to refer to passive agents which have no ability to determine strategies which impact the course of the game, but who are still impacted by the decisions made by other players of the game. Other works such as [104] and [64] go even further, and incorporate asymmetric information across players via the inclusion of an additional player, Nature, which randomly allocates attributes to players which are not known by the full set of other players.

1.1.1.2 State Space

The state space is a set of variables which denotes the 'position' of the game and its players. In chess for example, the state space would be the position of each piece on the board, while in poker it would be the cards in each player's hand and their previous bids. We note that this may also incorporate player attributes, such as fighting weight in boxing.

In this thesis, the only state variables we use are the resource levels of each player. These resource levels inform production insofar as we later show that the production level of a player increases as the 'scarcity cost' they face for extracting the resource decreases. This in turn increases as the resource pool becomes smaller and the resource becomes more scarce. Furthermore, we show in Section 1.3 that the profit each firm receives is generally dependent on the actions of all firms in the market, meaning that the resource levels of *all* players are usually relevant to each firm in determining their optimal strategies. Exhaustibility and its impact on the actions of a firm are discussed in greater detail in Section 1.4.1.

1.1.1.3 Information Set

The information set describes the knowledge available to each player in the game. We can broadly classify the information set by two properties. The first of these is the set of variables of the state-space which each player is aware of. In many games, players have

information which is not known to other players; in poker, each player at least aims to keep knowledge of the cards in their hand to themselves. There may also be variables of interest to the players which are known by no one. Games of either type are called games of *incomplete* information. Games of the former type, in which the players have different knowledge about the values in the state-space, are called games of *asymmetric* information.[10]

In either type of game, it is generally assumed that the other players form expectations (or 'priors') about the possible values of these unknown state variables, and update them based on the progression of the game and the actions of the other players. Returning to the example of poker, a community card pool containing three aces makes it considerably less sound to believe another player may hold two aces than if the pool held none. The games we discuss are limited to games of *complete*, or *perfect* information, in which information known to one player is known to all other players as well.

The second property relates to how much knowledge the players are assumed to have on the progression of state-variables and player strategies through time. This can be considered as analogous to the filtrations of each variable in the information set.

Many games involve players incorporating information about the past behaviour of other players in the determination of their strategies, either allowing them to punish or reward other players for their past behaviour, or establish reputations about their own behaviour with the intention of deterring their opponents from certain strategies.

However, most of these games are highly complex and are generally restricted to a discrete setting as a result. Most of the literature on dynamic games of resource extraction assumes that players can at most only act on current values of the resource levels, therefore precluding history-dependent strategies.[11] Consequently, the models we consider in this thesis also exclude history-dependent strategies.

Even with this caveat, the frequency with which the players' knowledge of the state-space is updated is a crucial consideration. This is because it is also the frequency with which they can change the course of their actions, insofar as a change in strategies is only sensible if the information on which the players are determining those strategies also changes. Because of this, it can also be viewed as the *period of commitment* of the game.

[10] See [117] for a detailed introduction to both types of game.

[11] There are notable exceptions, such as the cooperative model of [29].

This property is one which is key to differential games, and one central to the analysis of Chapter 3. This consideration is best explained in the context of resource exhaustibility and so we leave a detailed discussion of it to Section 1.5 after discussing exhaustibility in Section 1.4. However, we briefly introduce the two most important models here:

- **Open-Loop** games allow strategies to be a function only of time and the *initial* level of the reserve. This effectively requires the players each to determine their production trajectory at the beginning of the game, and then commit to it for the game's duration. Furthermore, it requires each player to anticipate the actions of the competing players over the entire course of the game.

- **Feedback-Loop** games allow strategies to be functions of the *current* level of the reserve at any time t. This removes the requirement that the players commit to their production trajectories in advance, and allows them to determine their strategies freely at each time.

The difference between these two models is subtle, and we provide an additional illustration in Example 1.1.2.2 via a two period discrete-time dynamic resource extraction example before discussing it in a differential games setting in Section 1.5.

1.1.1.4 Strategy Set

The strategy set is the set of actions that can be taken by each player. We note that this is often dependent on the values of variables in the information set. In chess for example, the queen can only be moved forward if there is an empty square in its path to move it to and the queen is still on the board. There are many different characteristics of admissible strategies within a game. In *static* or one-shot games such as the Prisoners' Dilemma in Example 1.1.2.1, the strategy is simply an action undertaken. *Dynamic* or multi-stage games, such as the cake-eating game of Example 1.4.1.1, on the other hand, involve taking multiple actions over time depending on how the game evolves. Games can either require *pure* strategies, which are restricted to deterministic functions of the state-space, or allow for *mixed* strategies, in which the players assign a probability to the execution of each element of their set of admissible strategies and then choose randomly.

In this thesis, we primarily deal with strategies that take the form of production levels, and we always require pure strategies as opposed to mixed ones. We add that the

production rates available to the players are constrained. Some of these constraints are independent of the other players—we require that production rates are non-negative, for example. However, we also find in Example 1.3.1.1 and Chapter 5 that the production of a player can be constrained by the production of the other players.

Both the results and solution method of the game change greatly based on the order in which strategies are played. A key consideration is whether the players move simultaneously or sequentially and, if sequentially, whether or not they are aware of the actions undertaken by players on previous turns. If the players move simultaneously, then we assume that players look for strategies which are mutual best responses. This is also referred to as a Nash Equilibrium, and we illustrate it below in Example 1.1.2.1. If the players move sequentially, then we analyse each stage of the game in reverse-sequential order, eventually arriving at an equilibrium sequence of strategies from the beginning of the game. We illustrate the solution method for a sequential game in the Stackelberg competition of Example 1.3.2.1.

1.1.1.5 Pay-Off Function

The pay-off function takes the state values of the game and the strategies of the players as inputs, and allocates a pay-off to each player as an output. We assume that the players determine their strategies in such a way as to maximise their own pay-off, and assume the same of the other players in turn. The pay-off function can reference a number of outcomes, including individual or social utility, but most of the work in this thesis assumes that the pay-off in question is the present value of profits from the sale of the resource, over the duration of the game.

1.1.2 Examples

1.1.2.1 Nash-Equilibrium: Prisoner's Dilemma

The Prisoner's Dilemma is a widely cited game which is often used as an example to illustrate why cooperation between economic agents can be difficult even when it is in the best interests of all concerned. The game was first framed in [115], but formalised in [47] and presented as follows:

> Two members of a criminal gang are arrested and imprisoned. Each prisoner is in solitary confinement with no means of speaking to or exchanging

> messages with the other. The police admit they don't have enough evidence to convict the pair on the principal charge. They plan to sentence both to a year in prison on a lesser charge. Simultaneously, the police offer each prisoner a Faustian bargain. If he testifies against his partner, he will go free while the partner will get three years in prison on the main charge. Oh, yes, there is a catch ... If both prisoners testify against each other, both will be sentenced to two years in jail.

The players in this game are the two criminals. They have a symmetric strategy set, meaning that they each have access to the same strategies; in this case to cooperate or defect. This formulation of the prisoner's dilemma is a one-shot game. In games of this type, the players each only make one move. Therefore, the game can only ever be in one position, the starting position, and there is no need for state variables. For this reason, both the state-space and the information set are null sets. The objective function in this case is the jail time allocated to each criminal, and is a function of the binary cooperate/defect decisions made by the two criminals. We represent these pay-offs below in table 1.1.

Table 1.1: Pay-off in a Prisoners' Dilemma

	Player B cooperates	Player B defects
Prisoner A cooperates	Player A: 1 year Player B: 1 year	Player A: 3 years Player B: goes free
Prisoner A defects	Player A: goes free Player B: 3 years	Player A: 2 years Player B: 2 years

To solve this game, each player considers each strategy that the other player can make and looks for the best-response to that strategy. In this case the best-response to either strategy is to defect.[12] Consequently, the pair of strategies in which both players defect is the only one in which the players are each responding optimally to the other. This is the unique Nash-Equilibrium.

[12]When this occurs, we also say that the strategy *strictly dominates*.

1.1.2.2 The Information Set and Dynamic Games: Resource Extraction

We discuss resource market models in Section 1.3, and the considerations of resource exhaustibility in Section 1.4.1. Here, we briefly consider a highly simplified game of resource extraction to illustrate the difference between Open-Loop and Feedback-Loop models in a discrete setting; we discuss this difference in detail in a continuous-time setting in Section 1.5. We note that the model we use is a simplified form of Cournot competition. We discuss this model and its solution method in greater detail in Section 1.3.1.

We consider a game in which there are two firms, indexed by $i = 1, 2$, in the market for a resource. We assume that player 1 has a finite stockpile of the resource of amount $0 \leq x \leq 2/3$, but player 2 has an infinite amount. The model consist of two periods, which we reference by $t = 1, 2$. In each period, the two players simultaneously decide their own production. We define q_i^t as the production of player i in period t. Player 2 has no constraints on their production in each period, while player 1 is constrained by the exhaustibility of their resource so that $q_1^1 + q_1^2 \leq x$. We also define x_2 as the quantity of player 1's resource remaining at the beginning of period 2, such that $x_2 = x - q_1^1$.

In each period t, the players receive a payoff π_i^t where

$$\pi_i^t(q_1^t, q_2^t) = q_i^t(1 - q_1^t - q_2^t), \qquad (t, i) \in \{1, 2\} \times \{1, 2\}.$$

We assume that the payoff the players receive in period 2 is discounted by a factor $0 < r < 1$ such that the total value each player receives is given by

$$v_i = \pi_i^1(q_1^1, q_2^1) + r\pi_i^2(q_1^2, q_2^2). \tag{1.1.1}$$

We first consider the game without exhaustibility to justify our focus on $x \leq 2/3$. In the absence of exhaustibility, the optimisation of profit in each period can be performed independently. In either period, if player i assumes that their competitor will produce at a rate of q_{-i}, then their optimal response is given by

$$\begin{aligned} q_i &= \arg\max_{q_i \geq 0} \{q_i(1 - q_i - q_{-i})\}, \\ &= \frac{1}{2}(1 - q_{-i}). \qquad i = 1, 2. \end{aligned}$$

The Nash Equilibrium is given by the intersection of these two best-response functions, so that both players would produce at $(q_1, q_2) = (1/3, 1/3)$. In other words, even given

an infinite supply of x, player 1 would not choose to consume more than $2/3$ of it in the course of the game. Similarly, this result also shows that the resource constraint is binding for $x \leq 2/3$, so that

$$q_1^1 + q_1^2 = x. \tag{1.1.2}$$

We now consider the solution of the game under the Open-Loop model and Feedback-Loop model in turn, beginning with the Open-Loop game.

Open-Loop Solution We recall that under an Open-Loop model, the two players must determine their production over both periods and are then essentially bound to it, in that neither player receives any new information on which to adjust their strategies. Player 1's knowledge of their own resource level in period 2 is implicitly updated by virtue of the exhaustibility constraint, but player 2 must determine their own strategy based on an expectation of what player 1's resource level and production in period 2 will be. Player 2's strategy must therefore be *anticipative* of player 1's.

Incorporating the resource exhaustibility constraint in (1.1.2) results in only three controls to be determined, (q_1^1, q_2^1, q_2^2). Player 2 must determine their production in each period, while player 1 only freely determines their production in period 1—their production in period 2 is given by the resource exhaustibility constraint. The optimal controls are therefore given by

$$\begin{aligned}
q_1^1 &= \arg \max_{0 \leq q_1^1 \leq x} \left\{ q_1^1(1 - q_1^1 - q_2^1) + r(x - q_1^1)(1 - x + q_1^1 - q_2^2) \right\}, \\
q_2^1 &= \arg \max_{q_2^1 \geq 0} \left\{ q_2^1(1 - q_1^1 - q_2^1) + r q_2^2(1 - x + q_1^1 - q_2^2) \right\}, \\
q_2^2 &= \arg \max_{q_2^2 \geq 0} \left\{ q_2^1(1 - q_1^1 - q_2^1) + r q_2^2(1 - x + q_1^1 - q_2^2) \right\}.
\end{aligned}$$

Each of these optimisations is straightforward, and so the optimal set of controls is given by

$$\begin{aligned}
q_1^1 &= \min \left(\frac{1}{3(1+r)} \left(1 - r(1 - 3x)\right), x \right), \\
q_2^1 &= \max \left(\frac{1}{6(1+r)} \left(2 + 4r - 3rx\right), \frac{1}{2}(1 - x) \right), \\
q_1^2 &= \max \left(\frac{1}{3(1+r)} \left(3x - 1 + r\right), 0 \right), \\
q_2^2 &= \min \left(\frac{1}{6(1+r)} \left(4 + 2r - 3x\right), \frac{1}{2} \right).
\end{aligned}$$

The value functions are found by (1.1.1), and plotted in Figure 1.1 for $r = 0.9$.

Feedback-Loop Solution In a Feedback-Loop game, player 2 is aware of the amount of resource that player 1 has remaining in period 2. They therefore determine their production in period 2 as a function of x_2. The exhaustibility constraint (1.1.2), coupled with our earlier analysis of the inexhaustible game, implies $q_1^2 = \min(x_2, 1/3)$, so that player 2's optimal period 2 production is given by

$$\begin{aligned} q_2^2 &= \arg\max_{q_2^2 \geq 0} \left\{ q_2^2 (1 - \min\left(x_2, \frac{1}{3}\right) - q_2^2) \right\}, \\ &= \max\left(\frac{1}{2}(1 - x + q_1^1), \frac{1}{3}\right). \end{aligned}$$

Incorporating this into the value functions for each player and maximising in q_i^1 gives

$$\begin{aligned} q_1^1 = \arg\max_{0 \leq q_1^1 \leq x} \{ & q_1^1 (1 - q_1^1 - q_2^1) \\ & + r \min\left(x - q_1^1, \frac{1}{3}\right) \left(1 - \min\left(x - q_1^1, \frac{1}{3}\right) - \max\left(\frac{1}{2}(1 - x + q_1^1), \frac{1}{3}\right)\right) \Big\}, \\ q_2^1 = \arg\max_{q_2^1 \geq 0} \{ & q_1^1 (1 - q_1^1 - q_2^1) \\ & + r \max\left(\frac{1}{2}(1 - x + q_1^1), \frac{1}{3}\right) \left(1 - \min\left(x - q_1^1, \frac{1}{3}\right) - \max\left(\frac{1}{2}(1 - x + q_1^1), \frac{1}{3}\right)\right) \Big\}. \end{aligned}$$

For $x \leq (1 - r)/3$, the resultant period 1 and period 2 production rates (and therefore value functions) for both players coincide with the Open-Loop solution; player 1 consumes their entire reserve in period 1, with player 2 then producing as a monopolist in period 2. For $x \geq (1 - r)/3$, the period 1 production rates are given by

$$\begin{aligned} q_1^1 &= \max\left(\frac{1}{3 + 2r}\left(1 - r(1 - 2x)\right), x - \frac{1}{3}\right), \\ q_2^1 &= \min\left(\frac{1}{2\left(3 + 2r\right)}\left(2 + 3r - 2rx\right), \frac{2}{3} - \frac{x}{2}\right). \end{aligned}$$

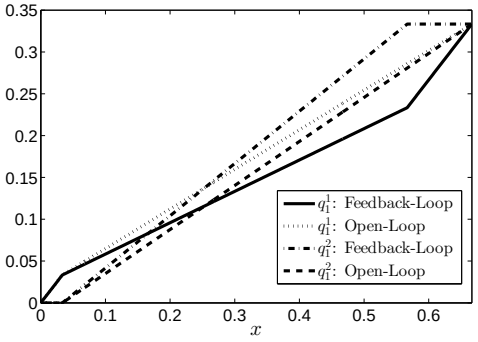
Period 2 production rates are then given by

$$\begin{aligned} q_1^2 &= \min\left(\frac{1}{3 + 2r}\left(3x - 1 + r\right), \frac{1}{3}\right), \\ q_2^2 &= \max\left(\frac{1}{2\left(3 + 2r\right)}\left(4 + r - 3x\right), \frac{1}{3}\right). \end{aligned}$$

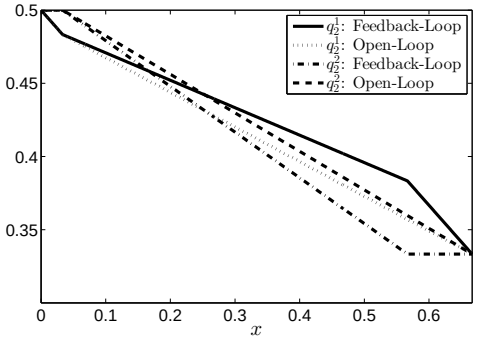
The value functions are given by (1.1.1) as before, and are plotted in Figure 1.1 for $r = 0.9$.

Comparing the Open-Loop and Feedback-Loop solutions allow us to make the following observations

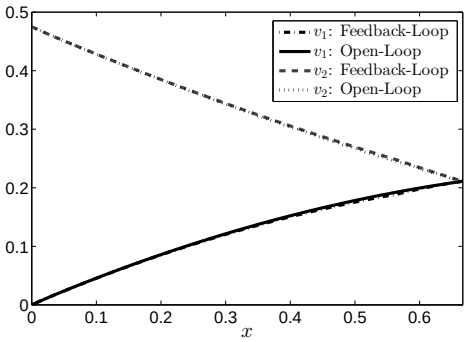
- The Open-Loop and Feedback-Loop games coincide when $x < (1 - r)/3$, and the resource is completely depleted in period 1.
- Both q_1^1 and q_1^2 increase with x under both games, as the resource becomes less scarce. We discuss this effect in detail in Section 1.4.
- Both q_2^1 and q_2^2 decrease as player 1's resource level increases. This is because player 2 anticipates higher production from player 1 and responds by reducing their own to increase the price. We discuss the effect of competition in market models in Section 1.3.
- In the Open-Loop game we have $q_1^1 \geq q_1^2$, while in the Feedback-Loop game we have $q_1^1 < q_1^2$ for suitably large values of x. The first of these results is a function of exhaustibility and the discounting effect, while the second requires more explanation. In this particular game, by setting production low in period 1, player 1 is able to pre-commit to higher production in period 2 via the game's implicit time-limit on the sale of the reserve; any reserves remaining in period 2 are either sold or lost. Doing so compels player 2 to lower their period 2 production in response. In this way player 1 can implicitly direct player 2's period 2 production via their depletion of the resource. In general, the ability to commit to a strategy in advance is highly advantageous in games. We note that in this particular case, player 1's ability to pre-commit is implicitly limited. Player 1 cannot pre-commit to a higher level of production than they would set with an infinite supply of the resource. This is because increasing production beyond this point would make them worse off than simply not producing those reserves, via the resultant decrease in price. We consider a market model with explicit market-leadership in Section 1.3.2.
- The value is lower for player 1 and higher for player 2 under the Open-Loop game, but they are extremely close. This finding is similar to that we observe and discuss in Chapter 3, though in this case may be partly mitigated by the implicit market-leadership effect we previously discussed.



(a) Player 1 production



(b) Player 2 production



(c) Total value

Figure 1.1: Open-Loop vs Feedback-Loop in discrete time

1.1.3 Cooperative Games

We have thus far only considered non-cooperative games. Whilst the games we consider in our thesis are all non-cooperative, there is also rich literature of cooperative games, which we discuss here for completeness.[13] Cooperative games allow the players to confer and determine a set of strategies to their mutual advantage. Apart from this distinction, they are defined by the same components as non-cooperative games.

Moreover, there is a great deal of work in this field specifically relating to resource exhaustibility. This work broadly lies on one of two strands: conservation and cartelisation. The first of these relates to cooperative agreements for the sustainable extraction from renewable but exhaustible commons resources,[14] while the second considers cartels which aim to maximise the profit to their members, and the allocation of production across the cartel.

The literature on conservation programmes largely relates to two problems: socially-optimal or sustainable extraction schedules for the cooperative; and punishment mechanisms when the players defect from these schedules. Optimal management of common property fisheries is considered in [5] and [132], while [29] and [31] focus on optimal punishment mechanisms for players that defect from cooperative extraction schedules.

The other strand of common property resource literature is focused on dominant cartels in exhaustible resource markets in which players often have access to separate resource pools.[15] Here, the aim of cooperation is primarily focused on the mutual gain of profit from the sale of the resource on the market, as opposed to questions of sustainability. OPEC in particular is considered in a number of models. Cooperative game theory is used to measure the gains to cartelisation to individual member countries in [52], [67] and [17], while [55], [74], and [6] measure 'cheating' by member countries in the form of over-production. The finding that gains from cartelisation seem larger for countries that are believed to have higher resource levels[16] while smaller countries are more likely to defect, is offered as one explanation for the fact that countries with lower resource levels, such as Algeria, regularly do well in OPEC agreements, despite OPEC's

[13] A detailed analysis of cooperative games, with a focus on cooperative differential games, is given in [147].

[14] See Section 1.4.2.2 for a more detailed introduction to common-property pool games.

[15] See Section 1.4.2.1 for a more detailed introduction to separate pool resource extraction games.

[16] It is reported in [123] that many of these countries do not allow external audits of their resource stockpiles, and so imperfect estimates are relied upon.

decision-making structure being dominated by countries with larger stockpiles, such as Saudi Arabia.

We begin this section by discussing solution methods for co-operative games. The types of solution acceptable in a cooperative game differ greatly depending on whether or not the players can make transfers between themselves, and so we consider games with transfers and games without transfers separately in Sections 1.1.3.1 and 1.1.3.1 respectively. We discuss how this changes when there are multiple players by discussing the formation of cooperative cartels in multiple player games (i.e. how the set of players which cooperate is determined) in Section 1.1.3.2. Finally, although we restrict our discussion in the majority of this section to games in which the cooperative agreements are binding on all players, we discuss models which allow players to defect on these arrangements in Section 1.1.3.3.

1.1.3.1 Two Player Cooperative Bargaining

With Transfers By transfers, we refer to an in-game mechanism by which players can compensate other players for taking certain strategies which are not immediately optimal to them, but which provide proportionally more benefit to the other player. For this to be possible, there must firstly be some medium which the players can trade (which we henceforth refer to as cash), and secondly the pay-offs to each player must be convertible to that medium. In practice, this means that most of the games in the literature of this type feature pay-offs strictly in the form of cash which can then be transferred across players. Utility based pay-offs are still permissible, but there must be some financial measure of utility by which a loss of utility by one player can be compensated for by the other player with cash.

For games such as this, the solution of the cooperative game becomes simple. The players work together to maximise total utility across all players, as measured in cash, and then devise of an allocation of that cash across the players. This consists of two problems. The first is maximising the total pay-off across all players using all available controls as a one-player optimisation problem. The second of these is devising an allocation of that total across the players.

This second problem is less trivial, and is not one for which there is generally a unique solution. What can be said is that we can rule out any outcome in which any of the cooperating players receive less than they would under the Nash-Equilibrium of the

game played against a coalition of the remaining players, as they could achieve that pay-off by not cooperating. Beyond that there is no optimal way of resolving the problem of allocating the additional total pay-off. Various methods have been proposed, the most widely used being the Nash, Egalitarian, and Kalai-Smorodinsky bargaining methods. These models can be used in games with or without transfers, but their formulation is more general if we assume that transfers are not allowed. We therefore define them formally in Section 1.1.3.1 below.

Without Transfers Where there are no transfers, outcomes which are detrimental to an individual player cannot be offset by a pay-off from the other cooperating players. This means that the game's solution becomes slightly complicated, insofar as the pay-off to each player needs to be considered independently.

The determination of the solution set hinges on the concept of *Pareto Efficiency*. Pareto Efficiency, first formulated in [111], describes a set of pay-offs in which no pay-off for an individual can be improved by an alteration in the actions of any player, without that alteration making one or more other players worse off.

In the Prisoner's Dilemma in 1.1.2.1, it is Pareto Efficient for both players to cooperate as doing so improves the welfare of each of them. It would not, however, be Pareto Efficient for either player to then defect on that arrangement. Doing so would increase their pay-off, but come at the cost of reducing the utility of the other player. In this way, Pareto Efficiency provides a definition of minimal fairness in a cooperative game.

From this, we can construct a minimal definition of a valid cooperative equilibrium. We say that we require an equilibrium of a cooperative game without transfers to be Pareto Efficient and for each player in the cartel to receive at least the value they would receive under a Nash-Equilibrium, i.e. the value they would receive if they did not cooperate. This point is also called the *threat point*.

We illustrate this in Figure 1.2a for two players. The Pareto curve is given by $A-B$. The point E is the threat point. The set of feasible cooperative outcomes is therefore given by $C-D$. Any point outside this curve on the Pareto Frontier would result in one player being made worse off by cooperating, and they would therefore not cooperate.

There are a number of different bargaining solutions postulated in the literature; we outline and graphically illustrate three of the most popular. The Symmetric or *Egalitarian* solution is the simplest of the three. It was first formulated in [77] for a two

player game. Here, the players work to maximise the minimum pay-off to either player. Graphically, the solution is illustrated in Figure 1.2b. A 45° line is drawn from the threat point to the Pareto Frontier; the point at which they intersect is the Egalitarian Solution, S.

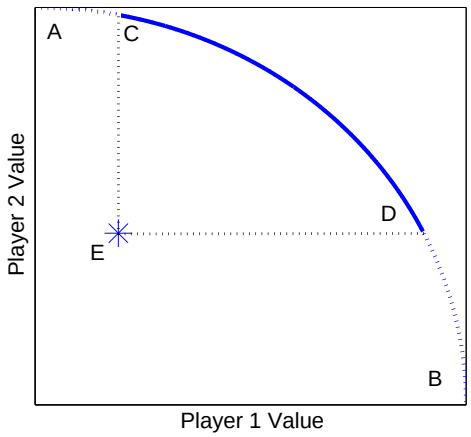
The *Nash* Bargaining Solution is formulated for two players in [103]. In it, player 1 and player 2 choose the point (v, w) on the Pareto curve that maximises $(v-v^*)(w-w^*)$, where v is the value for player 1, w is the value for player 2, and (v^*, w^*) are the player 1 and player 2 values at the threat point, E. Graphically, this point can be found as the apex of the isosceles triangle with one side tangent to the Pareto Frontier, and the other side passing through the threat point.

Finally, the *Kalai-Smorodinsky* Bargaining Solution was first formulated in [78]. Here, it is assumed that the players aim to maximise the ratio of maximal gain between them. We illustrate this method in Figure 1.2d. Graphically, the players first find the point on the frontier of admissible outcomes $C - D$ that maximises their own value. In other words, they identify the best possible outcome for them which the other player could plausibly agree to. If this point is w_1 for player 1 and w_2 for player 2, the cooperative solution is given by the point $K = (k_1, k_2)$ on the Pareto Frontier, such that $k_1/k_2 = w_1/w_2$. The point K can therefore be found by drawing a line from the point E to the point $W = (w_1, w_2)$ and identifying its point of intersection with the Pareto Frontier.

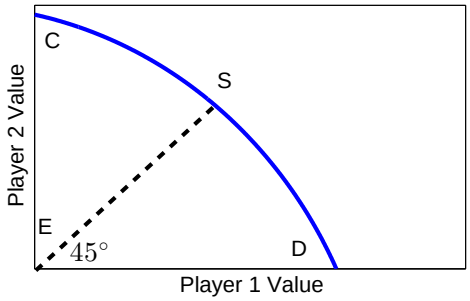
1.1.3.2 Coalitions

In a game with more than two players, it is important to note that the optimum cartel may not always contain all of the players in the game. A cooperative game between N players is specified by a characteristic function v relating each possible coalition to a total value for the coalition. Given that there are 2^N possible coalitions, we have $v : \{1, ..., 2^N\} \rightarrow \mathbf{R}$. The players then determine which coalitions form, according to the total value the coalition receives and the way that value is divided amongst the players.

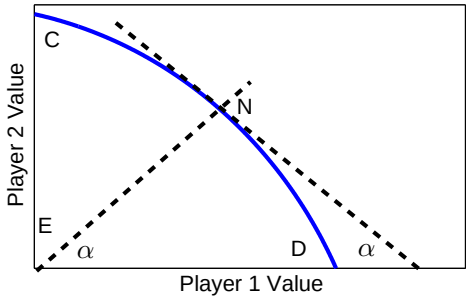
This comprises two problems, that of determining which coalition(s) will form, and that of determining how the value of that coalition should be distributed to each participating player. However, the vast majority of cooperative models, and indeed almost all models involving cooperation in resource markets assume two properties which simplify the first problem. These properties are *super-additivity* and *monotonicity*.



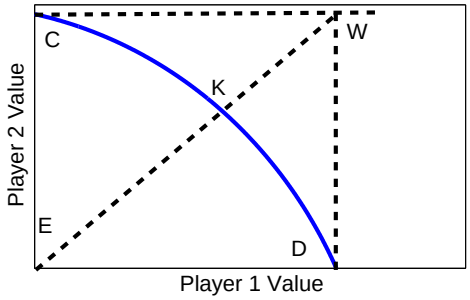
(a) The Pareto Frontier



(b) Symmetric (Egalitarian) Bargaining Solution



(c) Nash Bargaining Solution



(d) Kalai-Smorodinsky Bargaining Solution

Figure 1.2: Graphical Illustration of Two Player Bargaining Solutions

Super-additivity means that the value from a coalition of two disjoint sets of players is greater than or equal to the sum of the values of each separate coalition, so that it is always optimal for separate coalitions to merge. *Monotonicity* can be shown to follow from super-additivity and simply means that coalitions always gain or hold value with the inclusion of an additional player.

From this it is trivial to see that when these two properties hold, all N players cooperate, forming *the grand coalition*, which we call G. The problem of the game then reduces to that of apportioning the total value across the players. We can determine a minimum feasible set of solution payment vectors (also called an *imputation*) by requiring that they are *efficient*, so that all of the value is distributed across the players, and *individually-rational*, such that each player receives at least the value that they would receive if they did not cooperate with a coalition of the remaining players. This is analogous to the threat point value in the two player bargaining game. We can also require that the game attributes equal payments to *symmetric* players. Two players, i and j, are symmetric if, for any coalition S that does not contain either player, $v(S \cup \{i)\}) = v(S \cup \{j\})$. In other words, every possible coalition that does not contain the two players is equally improved by adding either player and not the other.

Beyond this, it seems reasonable that a fair allocation should consider the value a player brings to the coalition. However, this is complicated by the fact that the value they add to the coalition is highly sensitive to the order in which the coalition is formed. A popular imputation method, first proposed in [126], is given by allocating each player their *Shapley Value*. This is given by

$$\phi_i(v) = \sum_{S \subseteq G \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} \left(v(S \cup \{i\}) - v(S)\right).$$

This imputation is intuitively appealing, insofar as it allocates each player the average value they bring to the coalition, across all ways in which that coalition can be formed.

1.1.3.3 Cheating and Cooperation

Up to now we have assumed that the players are bound to the cooperative strategy. In practice, this is often not the case. In the Prisoner's Dilemma of 1.1.2.1 the players may come to an amicable agreement to cooperate, but that agreement is completely unenforceable once the players are taken into their separate interrogation chambers.

An important area of cooperative game research has been built up around the problem of reducing the likelihood of cheating in cartels. In general these models allow players some latitude to cheat by adopting a different strategy to that which they have agreed to. These models then either incorporate a probability that the cheater's duplicity will be discovered,[17] or guarantee its discovery after a certain amount of time, or number of periods, has elapsed.[18] We note that this is an important area in both the conservation and the cartel literature.

The result of these models is a calculus of defection which is dependent on the risk of being caught, the time-lag to being caught, and the punishment meted out to the player if they are detected. This can then be used to assess the likelihood and cost of cheating in cartel arrangements [19] and design optimal mechanisms for reducing its likelihood. Because the easiest variable to change out of those listed above is usually the punishment meted out to the player, this often becomes the crux of the modelling problem.

1.2 Price Functions

We begin by briefly considering the function linking price and the quantity that can be sold on the market at that price. The development of these functions (demand or price functions) in the literature relies heavily on the principle, first argued in publications such as [73], that, in a market economy, it is the marginal utility of a given quantity of an object that determines its price as opposed to the intrinsic importance or value of that class of object.

This school of thought partly arose as a response to the *paradox of value* posed by early economic thinkers such as Adam Smith in [130]. The paradox lies in the fact that items such as jade and diamonds are considerably more valuable in the market place than items such as bread and water, despite the fact that only bread and water are necessary for sustenance. The marginalist argument addresses this paradox by explaining that the relative scarcity of the former two goods means that the marginal utility they provide at that level of scarcity is very high, whereas the marginal utility of the staple goods is rendered low by their abundance.

[17] As is the case in [137].
[18] This is the case in the model of fisheries in [29].
[19] Such as in the analysis of OPEC in [55], [74], and [6].

Given that price functions rely heavily on the theory of marginal utility, we begin by introducing measures of utility. We then consider the relationship between price and utility functions, before introducing and discussing several popular examples of utility functions and their respective price curves.

1.2.1 Price Functions and Utility Curves

1.2.1.1 Utility Functions

Utility is the perceived value attached to a good, and a utility function is a real valued function that provides a mapping between real wealth as measured in cash, and the perceived value attached to a good by an economic agent. In other words, we adopt dollars as the units of utility in this thesis. It is important to note that utility is only a relative measure; it denotes relative preferences between different options but does not describe the magnitude of those preferences. In the context of wealth, for a good to have utility of x for an agent means that they would be indifferent between receiving the good and receiving x dollars in cash.

We only discuss utility functions in the context of a single resource market. If we use x to denote the resource level, and $u(x)$ to denote the utility associated with a resource of level x then we require u to be a continuous function with domain $\mathbf{R}^+$. Furthermore we require the utility function to be *transitive*. That is, for three resource levels x_1, x_2, and x_3, we require that if $u(x_1) > u(x_2)$ and $u(x_2) > u(x_3)$, then $u(x_1) > u(x_3)$.

The marginal utility of the resource is given by $u'(x) = \mathrm{d}u(x)/\mathrm{d}x$. It is clear that the marginal utility of a resource is dependent on its current level. If $u'(x) > 0$, then the marginal utility is positive, and the agent would be better off with additional resources. Furthermore, if marginal utility is positive for all x, then the agent will always prefer to have more to less. This property is referred to as *non-satiation*. However, as anyone who has had too much chocolate will know, it is sometimes possible for marginal utility to become negative. Although non-satiation is therefore a common assumption in utility models, it is not universal.

It is also generally the case that as an agent receives more of a good, each additional unit of that good provides less value than the unit preceding it. Giving £50 to a person with no money will almost always have a substantially greater impact on their welfare than if they had a million dollars in the bank. This is called *decreasing marginal utility*, and means $u''(x) \leq 0$. We require this property to hold in all of the models we consider.

1.2.1.2 Price Curve Derivation

Here, we briefly demonstrate how a demand curve can be derived from an agent's utility function. First, consider a buyer's utility function given by $u(q)$, where q is the level of the resource taken by the buyer. Now assume that the resource is being made available on the market at a price p by one or more supplying firms. We wish to determine the quantity that the agent will purchase at this price. To do this, first note that the net utility the agent receives after purchasing the resource is given by the cash value of the utility they receive from the resource itself minus the amount of cash they are required to spend on purchasing it. If we call this net utility $\tilde{u}(q)$, then it is given by

$$\tilde{u}(q) = u(q) - qp,$$

Maximising this in q gives

$$u'(q) = p,$$

In other words, the price is given by the marginal utility of the resource as expected. We note that taking the inverse of the above also yields the quantity demanded by the market at a given price. Accordingly, demand curves may relate the market price to a certain quantity sold on the market, or a price set by a firm to a quantity demanded on the market. Which of these forms of demand curve a market model uses depends on whether we assume the firms set their price or production rate as their strategic variable.

1.2.2 Popular Utility Functions and their Associated Price Curve Models

1.2.2.1 Utility

An exponential utility function takes the form

$$u(q) = 1 - \mathrm{e}^{-\alpha q},$$

where $\alpha > 0$ is a constant parametrising the demand curve. It is clear that the utility function implies non-satiation and marginal utility is decreasing for all levels of q. The relation 1.2.1.2 gives the price as a function of q:

$$p(q) = \alpha \mathrm{e}^{-\alpha q}.$$

Note that $p'(q) = -\alpha^2 \mathrm{e}^{-\alpha q} < 0$, so that the price is decreasing in the supply as expected. The cash received by a monopoly seller is given by

$$\pi(q) = \alpha q \mathrm{e}^{-\alpha q}.$$

1.2.2.2 Log Utility

A log utility function is given by

$$u(q) = \log(q),$$

so that the resultant price is given by

$$p(q) = \frac{1}{q}. \tag{1.2.1}$$

It is clear that the marginal utility, and therefore the price, is positive for all $q > 0$ so that the utility function implies non-satiation. Furthermore, $p'(q) = -1/q^2$, so that the price is decreasing in quantity. However, we note that the price approaches infinity as the supply approaches 0. While there are some highly demand-inelastic goods which approximate this kind of behaviour (such as life-saving medicines for example), this is clearly not possible in reality. Moreover, it results in an unusual revenue curve for a monopoly seller. We have

$$\pi(q) = q \times \frac{1}{q} = 1,$$

which implies that the revenue they receive is independent of their production rate. While this is highly implausible for a monopoly example, we note that for a market with N producers, indexed by i, the revenue an individual producer receives is given by

$$\pi_i(q_1, ..., q_N) = q_i \frac{1}{\sum_{i=1}^{N} q_i},$$

so that the firms receive revenue at the rate of their market-share of production. We also show in Example 1.5.2.3 that this isoelastic price curve is singular in yielding a simple closed-form solution for a highly complex two-player exhaustible resource extraction problem.

1.2.2.3 Power Utility

A power utility function takes the form

$$u(q) = \frac{1}{\gamma}\left(q^{\gamma} - 1\right),$$

where $0 < \gamma < 1$ is a a constant. The price is then given by

$$p(q) = q^{\gamma - 1},$$

which is positive for all $q > 0$. We have $-1 < \gamma - 1 < 0$, so that $p'(q) = (\gamma - 1)q^{\gamma - 2} < 0$, and price is decreasing in quantity. The rate of revenue for a monopoly producer is given by

$$\pi = q^{\gamma}.$$

1.2.2.4 Isoelastic Utility

The isoelastic utility function is similar in form to the power utility, and is given by

$$u(q) = \frac{q^{1-\rho}}{1-\rho},$$

where ρ is a parametrising constant and we assume $\rho > 0$, $\rho \neq 1$. We therefore have that the price is given by

$$p(q) = q^{-\rho},$$

so that the price is again positive for all $q > 0$. Furthermore, we have $p'(q) = -\rho q^{-(\rho+1)} < 0$ so that price is decreasing in quantity as required.

Quadratic Utility The quadratic utility function is given by

$$u(q) = \alpha q - \frac{\beta}{2}q^2,$$

where $\alpha > 0$, $\beta > 0$. The price curve is given by

$$p(q) = \alpha - \beta q. \tag{1.2.2}$$

The resultant price curve is therefore a linear price curve. The linearity of this price function makes it extremely advantageous in solving the games we consider in this thesis.

Furthermore, it is used widely in the literature, and it has been verified by empirical testing in work such as [114], [7], and [112] that price data from oil, hydrocarbon, and electricity markets display a suitable fit to linear pricing curves. Therefore, we use it almost exclusively.

However, we note that it differs from the previous functions in one important way. Namely, it is clear that for $q > \alpha/\beta$, we have $p < 0$. The price becomes negative and the marginal utility received by an agent for an extra unit of the resource is negative. In other words, they would prefer not to have more beyond this point.

This point is called the saturation point, and the utility function ceases to be non-satiating beyond this point. We note that it is theoretically possible, though implausible, for markets to demonstrate this kind of behaviour. This is because markets are sometimes unable to readily accommodate manufacturing and energy production inputs beyond a certain point in the short-run, and storage and transportation costs may make this excess stock more trouble than it's worth holding long enough to be able to use.

Electricity markets provide one example of a market in which negative prices can occur. When demand is low, the long time it takes to shut down the plant results in excess supply, which may require negative prices to sell on. However, we note that in practice it is rare, if not unheard of, for resource markets such as those we consider to ever reach this point on the demand curve, even if it exists as a theoretical possibility.

1.2.3 Exogenous demand curve based constraints

We note that in this set-up, we have only considered various models for the direct interaction between buyers and sellers in the market. There can be other actors and exogenous constraints on demand. Governments can set policies which limit maximum production, or impose taxes in such a way as to distort equilibrium production or prices. Firms can experience both individual and shared capacity constraints which can impact the entire set of firms. We discuss how some of these complications—specifically caps and tariffs on price and total production in the market—impact game solutions in an inexhaustible resource setting for the Cournot market defined in Section 1.3.1 in Chapter 5.

1.3 Models of Market Competition

The games we analyse in this thesis all feature the sale of a resource on a market by a number of oligopoly firms. We must therefore have a model of this market, and how firms compete in it. We discuss several alternative models in this section. This is an area of research with a long history, and some of these models pre-date much of modern game theory. The Cournot model, which we discuss first, was the earliest to be formulated in [34] in 1838, pre-dating Nash's work in [102] by over a century.

While these models differ in many respects, there are several common features. All of these games feature a commodity, where the resources extracted and sold by different firms are at least partly homogenised, such that they are sold on a common market. Because buyers within the market can buy from any producer, the producers' pricing and production decisions are interdependent. The specific relationship between supply and demand within the market varies across the models, but it holds true that consumers prefer a lower price to a higher one, and that supply and demand are inversely related.

The key game-theoretic aspect in each of these models is the means by which firms act to maximise their profits, given the presence of other producers and the relationship between supply and demand. The models below vary in the strategic variable(s) the firms set in performing this maximisation, the order in which the producers make decisions, and the way in which the market responds to them. In this section, we discuss several popular models of market competition, and provide several illustrative examples in a two-player setting.

1.3.1 Cournot

In this form of market competition the firms' strategic variables are their production rates. The firms independently set their own production, and receive revenue at the rate of price times production. The price in this case is determined by an inverse demand curve of the type discussed in Section 1.2. In general, we assume that this curve takes the linear form given in (1.2.2). This greatly simplifies analysis, but is also consistent with empirical data from several commodity markets, such as oil,[20] coal,[21] and

[20] See [114].
[21] See [7].

hydrocarbons.[22] For a market with multiple producers, this is given by

$$P(Q) = \alpha - \beta Q, \tag{1.3.1}$$

where P is price, Q is total production across all producers, and α and β are constants parametrising the demand curve. In Section 1.3.1.1, we demonstrate a scaling argument that can be used to show that the games of this type which we discuss can be analysed under the assumption $\alpha = 1$, $\beta = 1$ without loss of generality. We therefore do so throughout this thesis.

The dependence of price on total supply is the key component of this game. Firms need to anticipate the production rates of the other firms in order to predict the price which will result from their own production. To illustrate the solution method, we solve a Cournot game with two players for an inexhaustible resource in Section 1.3.1.1 below.

1.3.1.1 Two player illustration

We assume two players, indexed by $i = 1, 2$. The Cournot competition operates as follows at each point in time, given by $t > 0$:

- Each player, indexed by $i = 1, 2$, simultaneously chooses a dynamic production rate, given by $q_{i,t}$. They pay extraction costs at the rate c_i per unit produced, where we assume without loss of generality that $0 \leq c_1 \leq c_2$.
- The price is determined according to the linear demand curve (1.3.1).
- The players receive profit, at the rate $\pi_t(q_{1,t}, q_{2,t}) = \max(q_{i,t}(\alpha - \beta(q_{1,t} + q_{2,t}) - c_i), 0)$.

We assume that the players aim to select their production rates so as to maximise the present value of profits, discounted at a common interest rate, r, such that the value player i receives over the course of the game is given by

$$v_i = \int_0^\infty \mathrm{e}^{-rt} q_{i,t}(\alpha - \beta(q_{1,t} + q_{2,t}) - c_i)\mathrm{d}t.$$

[22] See [112].

If we scale such that

$$\begin{aligned} v_i &= \alpha \tilde{v}_i, \\ q_{i,t} &= \frac{\alpha}{\beta} \tilde{q}_{i,t}, \\ c_i &= \alpha \tilde{c}_i, \end{aligned}$$

then the problem for each player reduces to maximising

$$\tilde{v}_i = \int_0^\infty \mathrm{e}^{-rt} \tilde{q}_{i,t}(1 - \tilde{q}_{1,t} - \tilde{q}_{2,t} - \tilde{c}_i)\mathrm{d}t. \tag{1.3.2}$$

Hence it is sufficient to solve for $\alpha = 1$, $\beta = 1$; from this point we therefore do so.[23] Under the assumption that the resource is inexhaustible, the actions of the players at each time do not constrain their future production choices. Therefore, the players maximise value by maximising their rate of profit at each point in time. The maximisation problem at each instant therefore becomes a static optimisation of $\pi_{i,t}$. Taking q_2 (q_1) as given, player 1's (player 2's) optimal response is given by

$$q_1 = \max\left(\frac{1}{2}(1 - q_2 - c_1), 0\right), \tag{1.3.3}$$

$$q_2 = \max\left(\frac{1}{2}(1 - q_1 - c_2), 0\right). \tag{1.3.4}$$

We identify equilibria by looking for mutual best-responses. We demonstrate this graphically as the intersection of the best-response functions (1.3.3)-(1.3.4) in Figure 1.3a for $c_1 = 0$, $c_2 = 0$. Solving (1.3.3) and (1.3.4) simultaneously gives

$$q_1 = \begin{cases} \frac{1}{3}(1 - 2c_1 + c_2) & \text{for } 2c_2 - 1 \leq c_1 \leq \frac{1}{2}(1 + c_2), \\ \frac{1}{2} & \text{for } c_1 > 2c_2 - 1, \\ 0 & \text{for } c_1 < \frac{1}{2}(1 + c_2), \end{cases}$$

with q_2 given by reversing c_1 and c_2 in the above. We note that for certain combinations of c_1 and c_2 it is possible that the optimal strategy for the player with higher costs (player 2, by assumption) is not to produce at all. This occurs because the lower cost faced by player 1 makes it profitable for them to increase production to a rate high enough to reduce the price below c_2. At this point, the maximum price player 2 can

[23] In addition, we note that if both players extract at the same cost, given by c, then we can set $\alpha - c = 1$ by the same scaling argument. However, this does not apply when $c_1 \neq c_2$.

receive for producing is lower than their extraction costs, and any production by player 2 will create negative revenue. Consequently, their optimal strategy is not to play.

Because it is the production of player 1 that prevents player 2 from producing on the market at a profit, we say player 2 is *blockaded* by player 1. Blockading in Cournot competitions is discussed in more detail in [62] for a dynamic duopoly game and [88] for a multiple player inexhaustible resource game.

Cournot competitions are used in much of Chapters 3, 4, and 5, while Chapter 2 is primarily devoted to extending the work performed in a Cournot setting in [62] to a Stackelberg one.

1.3.2 Stackelberg

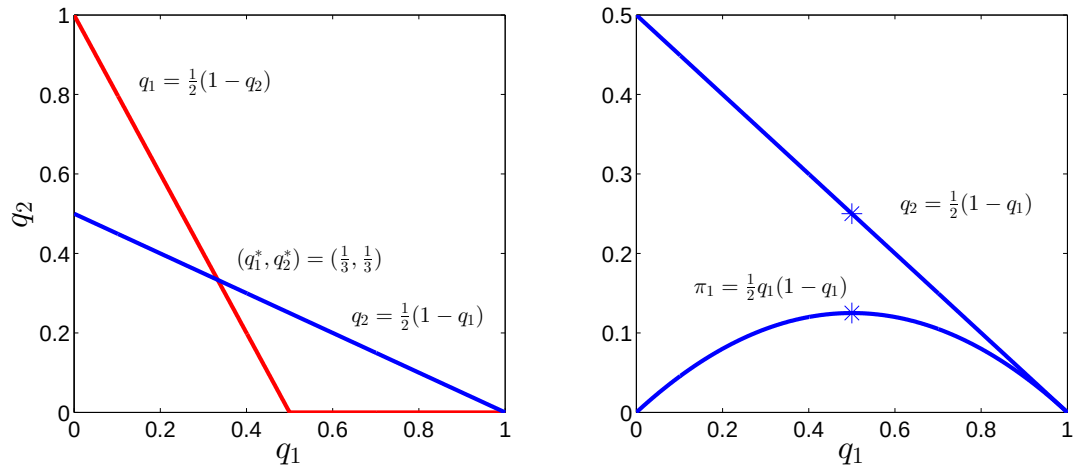
A central assumption in the Cournot model of competition is that the players make their production decisions simultaneously and without knowledge of the decisions made by the competing players, but assuming that they all act optimally. This assumption is criticised in [145] on the grounds that such a model treats all producers as equally credible in their ability to influence the price.

The challenge to this assumption lies in the fact that certain oligopoly markets are highly asymmetric in the distribution of market power. Oil markets are cited as one such example in [52], in which a dominant cartel (OPEC) competes against a fringe of competitor firms, none of whom can be said to have the same degree of credibility in their ability to influence the market price.

Stackelberg competition was introduced in [145] to allow for asymmetry in the way in which the players influence the market price. The original model assumes that one dominant player in the market takes the role of leader, and the remaining players take the role of followers.[24] The model works along similar lines to that of the Cournot model save for one aspect: the leader sets their production strategy first and makes it publicly known to the followers. The followers then set their production rates simultaneously to each other, but taking the leader's strategy as given. The solution of the game takes place across three steps:

- The leader determines the followers' best-responses to each strategy in the leader's set of admissible strategies;

[24] This is extended to models with multiple leaders in work such as [100].



(a) Cournot Equilibrium

(b) Stackelberg Equilibrium

Figure 1.3: The players look for mutual best-responses under Cournot competition. Under Stackelberg, the player calculates the follower's best-response for each possible q_1, and picks the (q_1, q_2) pair that maximises their value.

- The leader selects the strategy and best-response pair that maximises their value function and announces their selected strategy to the followers;
- The followers reacts optimally to this strategy, selecting their best-responses.

We illustrate the solution method by solving a Stackelberg game with one leader and one follower in Section 1.3.2.1 below, and provide a graphical illustration of the difference between Cournot and Stackelberg competition for a two player game in Figure 1.3. Chapter 2 is entirely devoted to Stackelberg duopoly competition, as is the second half of Chapter 3.

1.3.2.1 Two-Player Illustration

We note that the only factor which differentiates this game from the two player Cournot example is the order in which the players determine their strategies. The objective function they aim to maximise is therefore still given by (1.3.2) by the same scaling argument, and it is maximised by performing a static optimisation for the profit rate at an arbitrary time as before. For this reason, we omit the subscript t in describing the solution below.

The strategies are computed as follows. First, taking an arbitrary q_1 as given, player 2 computes their best-response, q_2^*, by maximising their value function, given by

$$rv_2 = q_2\,(1 - q_1 - q_2 - c_2)\,,$$

over all admissible $0 \le q_2 \le 1$. This gives player 2's optimal strategy as a function of q_1 in the form

$$q_2^*(q_1) = \max\left\{\frac{1}{2}\,(1 - c_2 - q_1)\,, 0\right\}.$$

The first possibility on the right here arises from an interior maximum of v_2 as q_2 varies, and holds for $0 \le q_1 \le 1 - c_2$, an interval which we denote by L; the second possibility, in which player 2 is blockaded, arises from an end-point maximum and holds for $1 - c_2 \le q_1 \le 1$, an interval which we denote by R.

Once the dependency of the response is known via $q_2^*(q_1)$, the optimal strategy for player 1, q_1^*, is given by maximising their value function, given by

$$rv_1 = q_1\,(1 - q_1 - q_2^*(q_1) - c_1)\,.$$

over all $0 \le q_1 \le 1$.

To ease the notation, we define

$$F_L(q_1) = q_1\,(1 - 2c_1 + c_2 - q_1)\,/2, \qquad F_R(q_1) = q_1\,(1 - c_1 - q_1)\,,$$

which are the results of substituting for the two possible cases for $q_2^*(q_1)$ into rv_1. Thus we need to maximise

$$F(q_1) = \begin{cases} F_L(q_1) & \text{for } 0 \le q_1 \le 1 - c_2, \text{ i.e. } q_1 \in L, \\ F_R(q_1) & \text{for } 1 - c_2 \le q_1 \le 1, \text{ i.e. } q_1 \in R. \end{cases}$$

Note that $F_L(0) = F_R(0) = 0$, and that $F_L(1-c_2) = F_R(1-c_2) = (1-c_2)(c_2-c_1)$. The global maximum value of $F_L(q_1)$ is $(1 - 2c_1 + c_2)^2/8$, realised at $q_1 = (1 - 2c_1 + c_2)/2$, which lies in $[0, 1]$ for all $0 \le c_1, c_2 \le 1$, although is is only in L when $0 \le c_2 \le (1+2c_1)/3$. The global maximum value of $F_R(q_1)$ is $(1 - c_1)^2/4$, realised at $q_1 = (1 - c_1)/2$; this point is also always in $[0, 1]$, but only lies in R if $(1 + c_1)/2 \le c_2 \le 1$.

There are various cases, depending on the values of the costs c_1 and c_2. We classify them according to which (if any) player is blockaded.

Proposition 1.3.1. *Player 1 is blockaded for all costs* $0 \leq c_1, c_2 \leq 1$ *satisfying* $(1 + c_2)/2 \leq c_1 \leq 1$ *(see Region 4 in Figure 1.4a). Then player 2 adopts the monopolist strategy* $q_2^* = (1 - c_2)/2$ *and has value function* $v_2 = (1 - c_2)^2/4r$.

Proof. If player 1 is blockaded, then we must have $F(q_1) < 0$ for all $0 < q_1 \leq 1$, so that player 1 chooses $q_1^* = 0$. Now $F_L(q_1) < 0$ for all $q_1 \in L$ if and only if $F_L(0) \leq 0$, so that $1 - 2c_1 + c_2 \leq 0$. It is then automatic that $F_R(q_1) < 0$ for all $q_1 \in R$ because $F_R(1 - c_2) = F_L(1 - c_2) < 0$. Rearranging the inequality on the costs gives the stated result. The strategy and value function are obtained by substitution into (1.3.2.1) and (1.3.2.1). □

Note that player 2 always plays when player 1 is blockaded, so we never have a situation when both players are blockaded.

Proposition 1.3.2. *Player 2 is blockaded for all costs* $0 \leq c_1, c_2 \leq 1$ *satisfying* $(1 + 2c_1)/3 \leq c_2 \leq 1$. *Furthermore, there are two cases:*

1. *For* $(1 + 2c_1)/3 \leq c_2 \leq (1 + c_1)/2$, *player 1 adopts the strategy* $q_1^* = 1 - c_2$, *and has value function* $v_1 = (1 - c_2)(c_2 - c_1)/r$.

2. *For* $(1 + c_1)/2 \leq c_2 \leq 1$, *player 1 adopts the monopolist strategy* $q_1^* = (1 - c_1)/2$, *and has value function* $v_1 = (1 - c_1)^2/4r$.

Proof. If player 2 does not play then, by (1.3.2.1), player 1's optimal strategy must be chosen from the interval R. If the global maxima of both the quadratics F_L and F_R lie in L, then q_1^* lies in L too, which is not possible. Hence there are three cases to consider:

(i) The global maximum of F_L lies in L and that of F_R in R. For the first, we require $0 \leq c_2 \leq (1 + 2c_1)/3$ and for the second we require $(1 + c_1)/2 \leq c_2 \leq 1$. These inequalities are inconsistent and so this case cannot occur.

(ii) The global maximum of F_L lies in R and that of F_R in L, which occurs for $(1 + 2c_1)/3 \leq c_2 \leq (1 + c_1)/2$, namely Region 2 in Figure 1.4a. In this case, the maximum value of F_L *in* L is achieved at $q_1 = 1 - c_2$ (an endpoint maximum), and the maximum value of F_R in R is achieved at the same point. The optimal strategy is $q_1^* = 1 - c_2$ and player 1's value function follows immediately.

(iii) Both global maxima lie in R. This requires both $(1 + c_1)/2 \leq c_2 \leq 1$ and $(1 + 2c_1)/3 \leq c_2 \leq 1$, and as the first of these implies the second, it occurs only when the first holds, namely in Region 1 of Figure 1.4a. In this instance, the maximum of $F_L(q_1)$ on L is at $q_1 = 1 - c_2$ and is necessarily less than the maximum of F_R. Hence the optimal strategy, obtained at the global maximum of F_R, is the monopoly strategy $q_1^* = (1 - c_1)/2$, and the value function follows immediately. □

The remaining cases are covered in the following:

Proposition 1.3.3. *For all values of c_1, c_2 not covered in the preceding propositions (that is, Region 3 in Figure 1.4a), both players produce. The optimal strategies are*

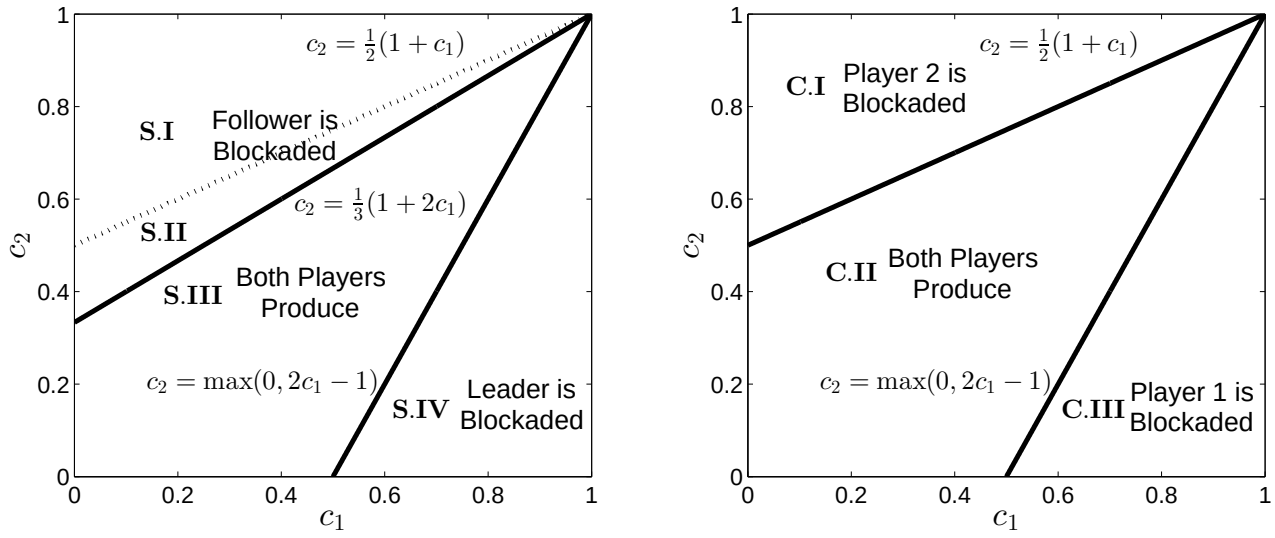
$$q_1^* = \frac{1}{2}(1 - 2c_1 + c_2),$$
$$q_2^* = \frac{1}{4}(1 - 3c_2 + 2c_1),$$

and the associated value functions are

$$v_1 = \frac{1}{8r}(1 - 2c_1 + c_2)^2,$$
$$v_2 = \frac{1}{16r}(1 - 3c_2 + 2c_1)^2,$$

We recall that in the analogous Cournot competition of Section 1.3.1.1, where there is no blockading, each player i produces at the rate $q_{i,t} = \frac{1}{3}(1-2c_{-i}+c_i)$, where subscript $-i$ denotes the competitor to player i. From this, it is clear that the cost pairs at which the leader is blockaded are the same cost pairs as those at which players in a Cournot competition are blockaded, but different from those at which the follower is blockaded. Figure 1.4 shows for each player i, the cost of the competing player c_{-i} at which they are blockaded when their own cost is c_i under the two-player Stackelberg and Cournot competitions respectively.

A key observation is that, while a cost advantage is a necessary condition for either player to be able to blockade their competitor and therefore produce in the market as a monopolist, the leader requires a much lower cost advantage than the follower in order to blockade the other player from the market. Because the point at which the leader is blockaded is the same as that for either player in a Cournot competition, this also



(a) Blockading in a Stackelberg Competition

(b) Blockading in a Cournot Competition

Figure 1.4: Blockading regions under Cournot and Stackelberg competition

indicates that we can expect blockading to occur more readily when the player with a cost advantage is also in a position of market dominance, as the Stackelberg leader is in this case.

We have shown that within region **S.II** of Figure 1.4a the leader blockades the follower by producing above the rate they would as a natural monopolist. Because the monopoly rate of profit is decreasing in q_1 beyond this point, the leader is producing above it for the sole purpose of blockading the follower with an artificially high rate of production.

We remark here that the possibility of an artificially high production rate for the purpose of blockading is a feature that does not enter into the analogous Cournot game of 1.3.1.1. We argue that this is because it requires the leader to commit to a strategy that is not a best-response to that of the follower. In the Cournot game, if player 1 attempted a strategy of artificially high production for the purpose of blockading, player 2's best-response would still be not to produce. However, the subsequent best response of player 1 to player 2's halt in production would be to lower production back to its natural monopolist level, making the blockade inadmissible as a mutual best-response. In the Stackelberg game however, by being able to commit to producing at above the natural monopolist rate, even after the follower is successfully blockaded, the leader makes that

blockade possible.

1.3.3 Bertrand

Both of the above models have assumed that the firms' strategic variables are their production rates. This results in tractable models, but does not necessarily mesh with the reality of markets. In [18] it was argued that, in some markets,[25] firms are more likely to compete on the basis of price than on quantity. Following from this, models of Bertrand competition were first formalised in [43]. These are a class of model in which the producers compete by setting price as the strategic variable. Under the original formulation:

- the resource is assumed to be homogeneous, such that consumers do not differentiate between the output of producers, as in the Cournot game;
- there are N firms which set their prices simultaneously, where we refer to the price set by firm i as p_i;
- all buyers in the market buy from the firm with the lowest price, which we call P;
- capacity constraints on the production of an individual firm are assumed not to apply, so that the company which sets the lowest price then produces to meet the entire demand of the market;
- this demand is given by a demand curve $D(P)$, which is a non-increasing function of the price, of the type discussed in 1.2;
- it is generally assumed that where prices are equal, demand is split equally between competing companies;
- extraction costs are at a constant per unit rate for each firm, which we again denote by c_i.

As we have done previously, we illustrate the solution of this model with a two-player example, and discuss the model in more detail in the context of that illustration.

[25] It is argued in [13] this is especially likely for markets for consumables such as fish.

1.3.3.1 Two Player Illustration

Without loss of generality, we assume that $c_1 \leq c_2$ as before. By the same reasoning as the Cournot game, if the resource is inexhaustible, then the optimal strategy over the whole game can be found by solving for the profit rate as a static game. The profit rate function for player 1 is given by

$$\pi_1(p_1, p_2) = \begin{cases} \frac{1}{2}p_1 D(p_1) - \frac{1}{2}c_1 D(p_1) & \text{for } p_1 = p_2, \\ p_1 D(p_1) - c_1 D(p_1) & \text{for } p_1 < p_2, \\ 0 & \text{for } p_1 > p_2, \end{cases}$$

with that of player 2 being given by symmetry. The company receives all of the market demand, and therefore profit, when $p_1 < p_2$; no profit when $p_1 > p_2$; and market demand is split evenly between the two companies where $p_1 = p_2$, and market revenue is therefore split also. If $c_1 < c_2$, then it is trivial to show that the Nash Equilibrium has player 1 setting their price just below c_2 and capturing the demand for the entire market. Consider that if player 1 sets their price above c_2, player 2 will underbid them and capture all market demand. We therefore have blockading of the player with higher costs follow automatically as a consequence of any difference in extraction costs under this model, a departure from both the Cournot and Stackelberg models.

If $c_1 = c_2$ then the same reasoning shows that both firms set their prices at c_1, the rate of marginal cost. Under Bertrand competition, the market is therefore pushed to the same pricing outcome as that of perfect competition, despite having only two firms. In a Cournot setting, this would only occur as the number of firms approached infinity.[26]

Whilst it can be argued that firms are more likely to compete on price than quantity in reality, this model of competition is still based on a number of questionable assumptions. The assumption that all market demand goes to the firm with the lowest price-bid, and the exclusion of capacity constraints, are both clearly problematic. Moreoever, the discontinuities in the above profit function serve to significantly complicate any attempt to extend this model by incorporating resource exhaustibility.

There are many extensions to the original model that have been designed to address these criticisms. Approaches for incorporating capacity constraints have been suggested

[26] See [121].

in [89] and [105],[27] and below we discuss a model of differentiated Bertrand competition, which relaxes the assumption that all market demand goes to the firm with the lower price. However, the classical Bertrand model cannot be meaningfully extended to resource exhaustibility. Consequently, it is not considered in the main body of our research.

1.3.4 Differentiated Bertrand Competition

One widely analysed extension of Bertrand competition is that of product differentiation. Moreover, it is one which can be used to model problems of resource exhaustibility. First proposed in [125] and [90], these models allow for a degree of differentiation between the product offered by each producer. This differentiation can either be by actions of the firm—differentiating their product on the basis of quality or reliability of service for example—or via exogenous differentiating factors such as transportation and search costs faced by buyers in the market. Given that many large-scale resource operations involve considerable investments in specialised transport and storage infrastructure, this is not unreasonable to assume, even when the resource itself is homogeneous. This is especially the case for energy resources, in which there are a wide range of resources of varying quality which can be used as substitutes with different production and switching costs.

The effect of this differentiation is to imbue consumer preferences with a degree of 'stickiness', such that consumers do not uniformly switch to the cheaper product. A common way of incorporating this stickiness is via a demand curve for player i of the form

$$q_i(p_1, ..., p_N) = \alpha - \beta p_i + \gamma \sum_{j=1, j\neq i}^{N} p_j,$$

where α, β, and γ are non-negative constants parametrising the demand curve. It is clear that the quantity demanded decreases as the producer's price increases, and increases as the price charged by competitors increases. In effect, as long as γ is positive, such a demand curve results in the production of competing firms having a similar impact to that of an imperfect substitute good. We add that by making γ negative this model can also be used to analyse markets with complementary goods.[28]

[27] It is also shown that the result that price is set at marginal costs is not robust to the inclusion of capacity constraints or product differentiation.

[28] See [95].

For an inexhaustible resource, the analysis of this game follows the same steps as a Cournot game, with the exception that the maximisation is performed in p_i and not q_i. We therefore omit a two-player illustration. This model is analysed in depth in [88] in a static setting, and in [87] in a dynamic setting under exhaustibility.

1.3.4.1 Bertrand-Stackelberg

It is also possible to have a Stackelberg form of Bertrand competition. However, this type of game is clearly trivial in the classical setting. Research on this game is therefore focussed around the differentiated model. Because the literature of differentiated Bertrand competition is already a relatively small subset of the literature on Bertrand competition, the literature on this model is quite narrow. It has, however, been analysed in [140], [76], and [54].

1.4 Resource Exhaustibility

So far we have only discussed market games in the context of an inexhaustible resource. In practice, most resources are limited. Even where a resource is relatively plentiful, the stockpile of an individual firm is often still finite. In determining optimal production in the market, there are therefore two important problems. We have discussed the need to anticipate the production rates of other firms, but resource exhaustibility adds the additional problem of optimally spreading the extraction of a limited pool of resources over time.

These two problems can create conflicting objectives. Under resource exhaustibility, the discounting effect of interest creates an incentive to frontload production. In maximising the profit rate however, the fact that the demand curve is a decreasing function of quantity, and that competing producers are also likely to frontload production, is an incentive to spread extraction more evenly over time.

There are two important considerations with regards to the impact of resource exhaustibility on a game. The first of these is the nature of the resource exhaustibility faced by the firm(s). The second is the way in which the firms' property rights over the resource pool(s) are structured. The former concerns the extent to which the resource is 'scarce' and the implied scarcity costs of extraction, while the latter concerns the way in which the scarcity costs associated with the depletion of the resource are allocated

to the firms. In this section we focus on discussing resource exhaustibility qualitatively. We discuss how it is incorporated into the differential games we analyse in this thesis in greater detail in Section 1.5.

1.4.1 Exhaustibility

When the total pool of resources available to a firm is limited, this adds an additional scarcity cost (or shadow cost) to production. By extracting a quantity of the resource from the pool and selling it onto the market, the producer implicitly pays the 'cost' of not having that reserve available for future production. The scarcity cost is therefore akin to an opportunity cost, insofar as it represents the cost of not being able to sell that reserve at a later date.

Similar to other extraction costs, higher scarcity costs generally result in lower production rates. It is therefore important to gauge the scarcity costs faced by a firm in order to model their production. The level of the shadow cost itself relates to the degree of 'scarcity' of the resource and the impact of its depletion on the firm, which in turn depends on a number of factors. Chief among these is the size of the resource pool available to the firm. The discounting effect and the negative linear relationship between supply and price mean that a firm with a small pool of reserves will face a higher scarcity cost and produce less, all things being equal, than a similar firm with a much larger pool. We illustrate how this impacts the extraction profile of a resource with the cake-eating example described in Example 1.4.1.1 and the monopoly extraction game of Example 1.4.1.2.

It is important to note that there is not a binary divide between a resource being inexhaustible and being of a known, finite pool. Even where a resource pool can be depleted by extraction, there are a number of extensions which can counteract this depletion. Resource pools can replenish naturally. Models of fisheries such as [13] and [132] discuss optimal production where fisheries can be exhausted by overfishing, but replenish naturally over time through breeding. Resource pools can also be increased by the activities of the extracting firm. In [97], firms can undertake exploration to discover additional resource pools; whereas improvements in extraction technology via directed

R & D or via learning-by-doing[29] can decrease the relative scarcity of a given pool by making extraction or refinement more efficient.

One extension which we frequently use in our thesis is the availability of a backstop technology. This is an alternative resource supply, which is infinite (or of sufficient size to be considered infinite) but more expensive to produce from than the original resource pool. For example, backstops for fossil fuels include alternative energy sources such as solar or wind power. We consider the impact of a backstop technology on a monopolist firm in Example 1.4.1.2.

Backstop technologies have a dual purpose within this thesis. Firstly, they allow us to consider games between producers from finite resource pools against producers from backstops. This is particularly useful in an energy resources context, where this becomes a game between fossil fuels firms and green energy firms. This has been analysed in [62] in a Cournot setting, but we extend this analysis to a Stackelberg one in Chapter 2. The second purpose is a numerical one and is discussed in more detail in Section 1.5.

Finally, we note that there are also factors which can serve to deplete a resource beyond the amount extracted by a producer. It is sometimes possible for negative jumps to occur in the resource level as well as positive ones. Jump losses due to natural disasters are considered in [50]. It is also possible for extraction from a resource pool to yield greater levels of information about the size of that resource pool, and these corrections can be either positive or negative. One example of this is where a resource pool comprises multiple layers, and breaking into a new layer yields more accurate information about the amount of resources present in that layer and the ones below. These games are the subject of [82]. The impact of uncertainty on production in games without this form of learning-by-doing is considered in [20].

1.4.1.1 Example: Cake-Eating Problem

We assume that Georgie has been given a large cake. We construct an arbitrary measure of cake-size and say that on this measure, Georgie's cake is of size 10. We further assume that the cake will spoil after 7 days. The problem Georgie faces is that of optimally spreading her consumption of the cake over the coming days. We assume that, over one

[29] Learning-by-doing refers to cases where the cost of production decreases over time via the act of producing, as the firm develops more efficient production processes from their experience. The early production of electric cars is considered as such a case in [27].

day, the utility Georgie receives from eating an amount of cake given by c is expressed by the log utility function

$$u(c) = \log(c),$$

so that cake is always utility creating, but marginal utility is decreasing. The fact that $u(0) = -\infty$ illustrates that it is not optimal to consume all cake at time 0. We also assume that Georgie prefers cake now to the same amount of cake later, so that we can express the utility she receives over the entire life of the cake by

$$V(c_1, ..., c_7) = \sum_{t=1}^{7} \beta^t u(c_t),$$

where c_t is the amount of cake consumed on day t, and β is a discounting factor which is applied to each period. Georgie's preference for immediate over future consumption means that $0 < \beta < 1$. For this example, we use $\beta = 0.8$.

Solution We first solve the problem for an arbitrary cake size x, discount rate β, and arbitrary time to spoilage, T. The problem can be expressed formally as

$$V = \max_{(c_1,..,c_T)} \left\{ \sum_{t=1}^{T} \beta^t \ln(c_t) \right\}, \tag{1.4.1}$$

with the constraint

$$\sum_{t=1}^{T} c_t \leq x. \tag{1.4.2}$$

If we incorporate (1.4.2) into (1.4.1) with the Lagrange multiplier λ, then we have

$$V = \max_{(c_1,..,c_T)} \left\{ \sum_{t=1}^{T} \beta^t \ln(c_t) + \lambda \left(x + \sum_{t=1}^{T} c_t \right) \right\}. \tag{1.4.3}$$

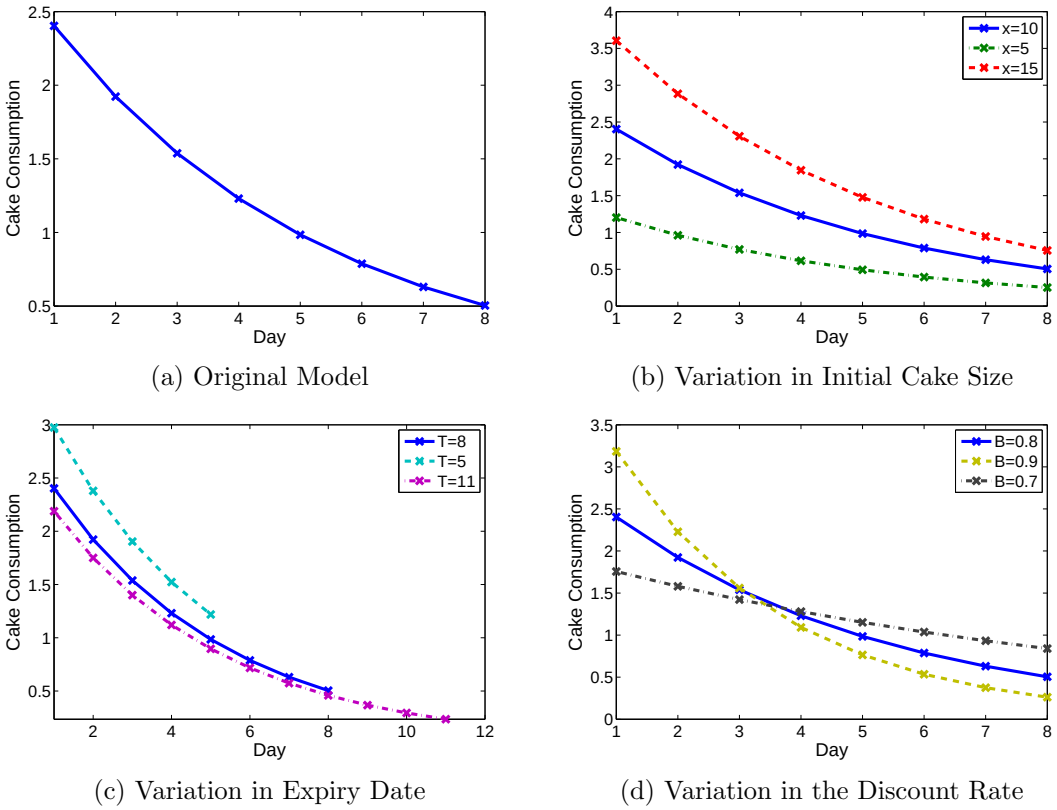
Maximising (1.4.3) in c_t gives

$$\beta^t u'(c_t) = \lambda \qquad t = 1, ..., T,$$

so that the optimal set of controls is given by the Euler equation for consumption

$$u'(c_{t-1}) = \beta u'(c_t). \tag{1.4.4}$$

We can use (1.4.4) to determine the optimal consumption pattern recursively. If we use x_t to denote the amount of cake remaining at the beginning of day t, then we have



(a) Original Model

(b) Variation in Initial Cake Size

(c) Variation in Expiry Date

(d) Variation in the Discount Rate

Figure 1.5: Consumption Profiles for a Cake-Eating Problem

$c_T = x_T$, i.e. Georgie should eat whatever cake is remaining on the final day before it spoils. Applying (1.4.4) to this recursively gives

$$c_t = \frac{\beta^t(1-\beta)}{1-\beta^{T+1}}x_0,$$

so that $c_0 = (1-\beta)x_0/(1-\beta^{T+1})$.

In Figure 1.5, we plot the optimal consumption path for the parameter values $x = 10$, $T = 7$. The optimal schedule is clearly decreasing, so that the player consumes less and less as her cake becomes increasingly scarce. In Figure 1.5b-1.5d, we also show that we can induce similar effects by altering the scarcity cost of consumption at the beginning of the game, by either increasing or decreasing x and T. Increasing (decreasing) x makes the cake less (more) scarce and leads to higher (lower) consumption. Increasing or decreasing T has the opposite effect; by increasing (decreasing) the horizon of the

game, the relative scarcity of the fixed amount of cake increases (decreases) because the same amount must last over more (less) time.

We make the additional observation that the relationship (1.4.4) is analogous to the *Hotelling rule* of [71], which says that the most socially and economically profitable extraction path of a non-renewable resource is one along which the price of the resource increases at the rate of interest. Given the relationship between price and marginal utility explained in Section 1.2, it is to be expected that the optimal consumption schedule in this Example is one in which the marginal utility of consumption increases at the rate of interest.

1.4.1.2 Example: Monopolist Resource Extraction

In this section we consider a monopolist firm, extracting a resource of initial size x_0 for sale on a Cournot market with a linear price curve. We assume that profits are discounted at the constant rate of r, and that the firm aims to maximise the present value of their profits. We use the notation of Example 1.3.1.1, but drop the player index subscript notation. The depletion of the resource means that the dynamics of the resource at time t, x_t, are given by

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = -q_t. \tag{1.4.5}$$

We solve the game with and without a backstop technology. The backstop we consider is infinite in supply, but costly to produce from, such that the firm pays costs at the rate of c per unit production from the backstop. Consequently, the firm only switches to backstop production once their supply of the free resource is exhausted. We use the following lemma, from [97], in the solution of both games.

Lemma 1.4.1. *The solution to the ODE*

$$\left(\alpha - \frac{\mathrm{d}v}{\mathrm{d}x}\right)^2 = \kappa v \quad x > 0,$$
$$v(0) = v_{0,}$$

where α, $\kappa > 0$ *and* $v_0 \geq 0$, *is*

$$v(x) = \frac{\alpha^2}{\kappa}\left(1 + \mathbf{W}(\theta(x))\right)^2,$$

where

$$\theta(x) = \beta \mathrm{e}^{\beta} \mathrm{e}^{-\frac{\kappa x}{2\alpha}},$$
$$\beta = -1 + \frac{\sqrt{\kappa v_0}}{\alpha},$$

and where $\mathbf{W}(x)$ *is the Lambert-W function defined by the equation* $x = \mathbf{W}(x)\mathrm{e}^{W(x)}$*, with the restriction* $x > \mathrm{e}^{-1}$*.*

We begin by considering the game in the absence of a backstop.

Without a Backstop. The optimisation problem for the firm is given by

$$v = \sup_{q \geq 0} \left\{ \int_0^{\infty} \mathrm{e}^{-rt} q_t (1 - q_t) \mathrm{d}t \right\},$$

where the depletion of x is given by (1.4.5), and with the constraint $x_t \geq 0$. This is a straightforward optimisation problem, and can be solved using the approaches of either Appendix A or Appendix B; we use the latter in this example. The optimal strategy satisfies the HJB equation

$$rv = \max_{q \geq 0} \left\{ q(1 - q) - q \frac{\mathrm{d}v}{\mathrm{d}x} \right\},$$

with the initial condition $v(0) = 0$. Performing this maximisation gives

$$q = \frac{1}{2} \left(1 - \frac{\mathrm{d}v}{\mathrm{d}x} \right). \tag{1.4.6}$$

Substituting this into 1.4.1.2 gives the ODE

$$rv = \frac{1}{4} \left(1 - \frac{\mathrm{d}v}{\mathrm{d}x} \right)^2, \qquad v(0) = 0.$$

This problem can be solved explicitly using Lemma 1.4.1. The solution is given by

$$v(x) = \frac{1}{4r} \left(1 + \mathbf{W}(-\mathrm{e}^{-(2rx+1)}) \right)^2.$$

The solution can also be given by the implicit expression

$$2\sqrt{rv} + \log\left(1 - 2\sqrt{rv}\right) = -2rx. \tag{1.4.7}$$

With a Backstop. The optimisation problem for the firm is now given by

$$v = \sup_{q \geq 0} \left\{ \int_0^\infty \mathrm{e}^{-rt} q_t (1 - q_t - c\mathbf{1}_{x_t=0}) \mathrm{d}t \right\},$$

where the depletion of x is given by (1.4.5), and with the constraint $x_t \geq 0$ as before. The switch to the backstop on exhaustion of the resource is incorporated via the indicator function $c\mathbf{1}_{x_t=0}$. Following the same steps as for the game without the backstop shows that the HJB equation for the game is once again given by 1.4.1.2. The backstop only changes the initial condition of the game. This can be determined by the static optimisation of $rv = q(1 - q - c)$, so that $q(0) = (1 - c)/2$, and $v(0) = (1 - c)^2/4r$. The optimal value for the firm is given by the ODE 1.4.1.2 with the initial condition $v(0) = (1 - c)^2/4r$. Again, this ODE can be solved explicitly using Lemma 1.4.1. The solution is given by

$$v(x) = \frac{1}{4r} \left(1 + \mathbf{W}(-c\mathrm{e}^{-(2rx+c)}) \right)^2.$$

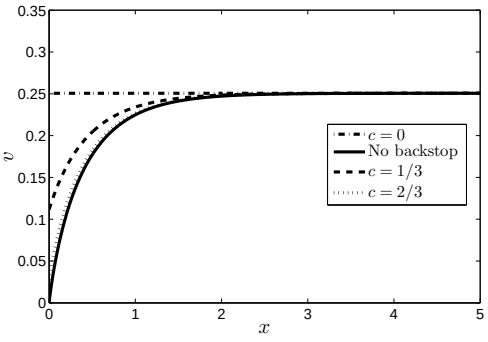
The solution is also given implicitly by

$$\left(2\sqrt{rv} - (1 - c)\right) + \log\left(\frac{1}{c} \left(1 - 2\sqrt{rv}\right) \right) = -2rx.$$

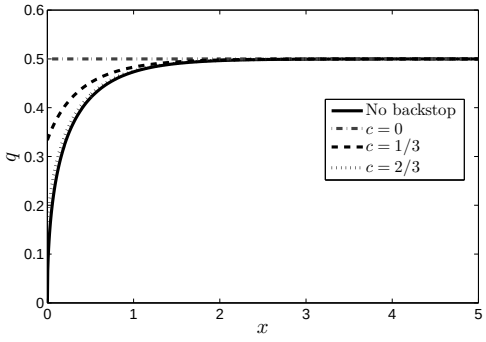
We plot the value and production functions, and the shadow costs, for a range of backstop costs and for $r = 1$ in Figure 1.6, noting that the case $c = 1$ is equivalent to the game without a backstop, insofar as the backstop cannot profitably be used to produce. We make the following observations:

- The case $c = 0$ is a special case; the shadow cost is zero, and the production and value functions are constant in x. If the backstop is free, then the firm receives the same profit from producing from it as it would from the free resource. Accordingly, the firm's total profit is not impacted by the total depletion of the resource at all, and the shadow cost is zero. As we saw for the inexhaustible game of Example 1.3.1.1, this means that the value function and production rate are given by a static optimisation, and are both independent of the resource level.

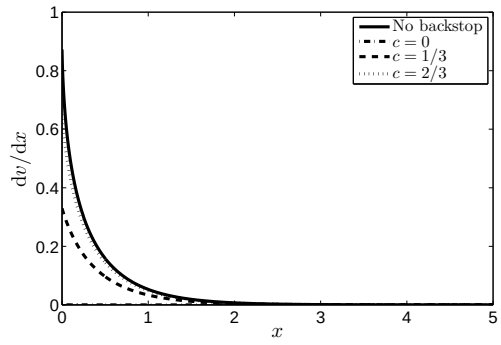
- Aside from this special case, the relationship between production and the resource level closely mirrors that of Example 1.4.1.1, so that the optimal extraction path follows the Hotelling rule as we would expect.



(a) Optimal value



(b) Optimal production



(c) The shadow cost, $\mathrm{d}v/\mathrm{d}x$

Figure 1.6: The impact of backstop cost on monopolist resource extraction

- We also see that the shadow cost of the resource at all resource levels falls with the backstop cost, while the production and value functions rise. This is intuitive. The lower the cost of the backstop, the less the firm is impacted by the exhaustion of their free resource. This means we can expect the scarcity cost to decrease, which in turn results in an expansion of production by the firm, in the same way as the reduction of an explicit extraction cost would.

1.4.2 Property Rights

In addition to the innate scarcity value of a resource, it is also important to consider how property rights to resource pools are allocated, insofar as this informs the competing demands on a resource by other agents, and how scarcity costs are actually spread across the players. We return to Georgie from Example 1.4.1.1, and now assume that she has a slice of cake kept in a fridge she shares with James. If Georgie can reasonably expect that James will not eat her food, or that he has an aversion to cake, then by eating the cake she can be certain that she is denying herself the opportunity to consume it later, and it therefore has a high scarcity value. If on the other hand, James has a fondness for sweet desserts and wanton disregard for fridge etiquette then it is unlikely the cake will be there for later consumption, even if Georgie herself decides to save it for later. Consequently, the cake now has a significantly lower shadow cost, despite being no less scarce in absolute terms. Given this, she would be well advised to eat it sooner rather than later, or she will neither have her cake nor eat it.

An important divide in the literature on exhaustible resources is whether the players extract from a common property resource pool, or whether they each have separate pools from which to produce. All of the models we consider in our research are of one of these two categories, and so we discuss them in more detail below in Sections 1.4.2.1 and 1.4.2.2.

However, we do note that there are other intermediate models of property rights. Models of the hunting or fishing of interconnected wildlife populations, such as that of [28], frequently have separate pools for different groups of players and species, but have renewal rates for the stock of each pool which are a function of the populations across all pools, due to migration and predator-prey relationships. Consequently, the pools are separate in the short term, but closely linked in the long term. Models in which players draw from a common pool, but have some form of allocated extraction rights are studied

in [116] and [91]. These models sometimes also allow for a degree of 'cheating', namely for players to have the option to extract more than they are entitled, but risk being penalised if they are discovered doing so; with the aim usually being the design of an optimal penalty system.

Whether or not models allow for the entry of new players can also dramatically change the solution of a game. This is especially so for common property models such as [99], [98], and [35]. The possibility for market entry often results in existing firms over-extracting to either deter other players from entry, or maximise their profits before the entry of additional players dilutes the price and speeds up the exhaustion of the resource.

The impact of this behaviour can be dramatic. In [99] and [98], a common property renewable game is studied. Where future entry by other players is barred, the equilibrium results in sustainable extraction of the resource, so that it is never depleted. It is shown that the risk of entry of other players leads to the existing firms pre-emptively extracting all of the common resource pool, deterring the entry of other players. In other words, the threat of entry alone is enough to result in the exhaustion of the resource, even though no additional players actually enter in equilibrium.

In addition to the allocation of property rights across players, expropriation risks by governments can also be a consideration. In [136], [59], [36], and [138], models are considered in which governments have the option to expropriate the resource from private firms via renationalisation. This risk is an important one, especially for resource extraction projects in countries such as Russia, Nigeria, and Colombia, which are resource rich but have a long history of politically motivated expropriations.

Expropriation risk is generally either modelled as a stochastic shock, or as a decision rule applied by the government, determined via a game between the government and extracting firms. In either case, the effect is usually the lowering of the extracting firm's perceived likelihood of holding the resource pool to depletion. This in turn decreases the shadow cost of production and spurs on extraction. A similar result is shown in [22], where the firm's extraction rights expire after a set time. When this time limit reduces their exposure to the scarcity costs of depleting the resource their extraction rises dramatically.

Even where expropriations are not total or even explicit, other policies can have a similar result. Government taxes are one such example. In particular, it has been argued

that 'green' taxes on the extraction and sale of fossil-fuels fall into this category. There has been much work in recent years devoted to investigating the existence of 'Green Paradoxes'. This term was first coined in [128],[30] and refers to the possibility that firms will expect future increases in emissions taxes and alternative energy investment to undermine the profitability of fossil fuel extraction in the long term. If they see these increases as likely, they may choose to increase extraction in the short-medium term and "make hay while the sun shines". In other words, future increases in these taxes may be accounted for by firms as an implicit expropriation risk.

There are two types of Green Paradox identified in the literature. A weak Green Paradox is where future green taxes result in an increase in production in the short term. A strong Green Paradox occurs when the total environmental harm caused by emissions over the duration of the game increases as well.[31]

The existence of a strong Green Paradox in fossil fuels production under expected future emissions taxes is demonstrated under certain conditions in [129], [94], [127], and [134], while [143] shows a similar result from investments into reducing the cost of alternative energy sources.[32] In [27] it is argued that fossil fuels producers have an additional incentive to extract pre-emptively when competing against substitutes that benefit from learning-by-doing, as minimising their production in the short-term also decreases their cost reductions in the long term. In [70], Green Paradoxes are used as a justification for their argument that carbon-capture and sequestration technology[33] should be prioritised by governments over alternative energy investments.

We note, however, that the existence in practice of these Green Paradoxes is still a subject of much debate. In [69] and [106], it is shown that the existence of these paradoxes is dependent on the expected future rate of increase of green taxes and energy investments, and that current levels of increase can be safely continued. In [144], it is shown that they are also dependent on the remaining resource level being above a certain point. In [143] and [106], it is further argued that this is effect is diminished by including resource exploration in the strategy set of the firm, as doing so means

[30]The possibility of this effect was earlier analysed in papers such as [142].

[31]This is compounded by the fact that many of the environmental harms of CO_2 emissions increase when the emissions are concentrated as opposed to spread evenly over time; see [131], or [129].

[32]Earlier papers such as [66] discuss the effects of backstop technologies on extraction profiles, but do so divorced from the context of emissions policies.

[33]This is technology designed to reduce the environmental harm from fossil fuels based energy production by storing the resultant emissions. See [85] for a detailed introduction.

that firms can decrease exploration efforts as well as increase short-term production as a means of mitigating the anticipated effects of green taxes.

1.4.2.1 Separate Pools

By separate pools, we mean models in which each player has access to their own resource pool, and the extraction of competing players therefore does not impact the player's resource level directly. Following the example of Georgie and James, this would be akin to each having their own cake safely locked away in a separate fridge.

The resource levels of the other players are still of relevance to each player even when they have separate pools. This is because they inform the scarcity costs faced by each competing player, which affects their choice of strategy. This in turn impacts the optimal strategy of the original player via the market competition mechanism. In the duopoly competition examples illustrated in 1.3.1.1, 1.3.3.1 and 1.3.2.1, we showed that the optimal strategy of any player is dependent on the full set of extraction costs paid by all players. In the same way, optimal strategies here are informed by the full set of shadow costs.

This dependency makes the state-space multidimensional; with the dimension being equal to the number of resource pools, and thereby the number of players, in the game. This significantly complicates the solution of the game as the number of players increases. As a result, most of the analysis we perform under a separate pools setting is restricted to two-dimensional (duopoly) games.

Because of this complexity, separate pool games are generally less widely discussed than common property ones. Accordingly, a great deal of our thesis is concerned with games of this type with the aim of partly addressing this gap. Chapters 2 and 3 both discuss separate pool games.

1.4.2.2 Common Property Resources

In general, models of this class assume a resource pool from which a number of firms each have free rein to extract, and there are no explicit limits on the extraction of any one producer. This model is generally used to model common property problems, but it has also been argued to be representative of the long-term state of resource extraction within a country or region. This is because the large pool of undiscovered or unclaimed resources is often effectively common, even though producers may hold small individual

deposits from which they extract in the short term.[34] There are numerous examples of common property resources. Fisheries are discussed in [132], [29], and [13]. Water aquifers are considered in [116]. Forestry management is considered in [4].

In these games, the exhaustion of the resource is not only a function of the relevant player's production, but that of all competing players as well. This induces an effect coined in [61] as the *Tragedy of the Commons*. The fact that other players can extract from the stock effectively adds a pre-emptive incentive to front-load production and get as much stock as possible before the other players take it. In other words, there is less of an incentive to spread extraction evenly if stock reserved for later extraction ends up being taken by the competing players.

On a technical note, we add that a useful feature of these games is that the computational complexity of the game tends not to increase with the number of players. This is because there remains only one resource pool, and so the state-space remains one-dimensional. Some work, such as [27], therefore assumes a common property pool as a means of simplifying computation even where a separate pools model may adhere more closely to the reality of the market being modelled.

1.5 Differential Games and the Period of Commitment

The multiple-player games we have thus far considered have all been in discrete time, with the exception of our market models 1.3.1.1, 1.3.2.1, and 1.3.3.1. Even there, the inexhaustibility of the resource in question allowed for the games to be solved as a static optimisation problem. Here we discuss differential games, or games in continuous time, in detail. We begin by providing a formal definition, and then discuss the relationship between the period of commitment and the solution method of the game.

1.5.1 Differential Games

In a differential game, the state space is given by $x \in \mathbf{R}^n$ where x is a dynamical system given by the ODE

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(t, x_t, u_{1,t}, ..., u_{N,t}),$$

where t is time, f is a real-valued continuously differentiable function describing the dynamics of x_t, $(u_{1,t}, ..., u_{N,t})$ are the players' controls and with initial data given by

[34] See [101] for example.

$x_0 \in \mathbf{R}^n$. There are N players are indexed by i, and each player aims to maximise their objective function:

$$J_i(u_1, ...u_N) = \psi_i(x_T) + \int_0^T \pi_i(t, x_t, u_{1,t}, ..., u_{N,t})\mathrm{d}t, \tag{1.5.1}$$

where ψ_i is the terminal pay-off, π_i is a running pay-off over the course of the game, such that π_i and ψ_i are continuously differentiable, and where T is the terminal time of the game. This may itself depend on x and $(u_1, ..., u_N)$, or it may be infinite if the game has no terminal time. We assume that player i has a set of admissible controls, $U_i \subset R^{m_i}$, and that x has a set of admissible values, $X \subset \mathbf{R}^n$, such that each player maximises J_i over $u_i \in U_i$, $x \in X$. We have not yet given a complete description of the game, as we have not discussed the information set, and in particular, the aspect of the information set defining the period of commitment of the game. This is an important property of differential game models, and impacts on the solution method of the game. We devote the following section to discussing this consideration in more depth.

1.5.2 The Period of Commitment

We introduced the information structure in Section 1.1.1.3 as the set of information on which the players of the game can base their strategies. Some of this information is common to all types of model which we introduce. In particular, we always assume that all players have perfect knowledge of:

- the initial state of the dynamical system representing the state space, x_0;
- the time, t at any point in the game;
- the payoff functions of all players;
- the form of the function f which determines the evolution of the dynamical system x; and
- the set of admissible controls available to each player.

In addition to this, the games we study are limited in the sense that the variables in x are almost always restricted to the resource levels of the players (or resource allocation). However, a highly significant source of variation between these games is the frequency with which the players' knowledge of x is updated.

This property can also be considered as analogous to the period of commitment present in the game. For it to be optimal for a player to change their strategy, there must be some change in the information set on which they are determining their strategies. Therefore, the frequency with which this information set is updated is also the frequency with which the players of the game can alter their strategies.

Little work has been done on the full range of possible commitment periods in games of resource extraction, though there are several significant exceptions. The impact of the period of commitment on a dynamic common property game is considered in [118]. A game in which players observe the production levels of their competitors continuously but with a lag is considered in [31]. Aside from notable examples such as these, most work assumes one of two cases: Open-Loop games, and Feedback-Loop games, which we illustrated in 1.1.2.2 in a two period setting. We recall from Section 1.1.1.3 the following distinction:

Open-Loop games allow strategies to be a function only of time and the *initial* value of x. This effectively requires the players each to determine their production trajectory at the beginning of the game, and then commit to it for the game's duration. Furthermore, it requires each player to anticipate the production trajectories of the competing players over the entire course of the game.

A **Feedback-Loop** information structure allows the players to make their production decisions as functions of the *current* level of x_t at any time t. This removes the requirement that the players commit to their production trajectories in advance, and allows them to determine their strategies freely at each time.

These two model types are a key component of our work in Chapters 2, 3 and 4. In this section we therefore give a formal definition of both Nash and Stackelberg equilibria in Open-Loop and Feedback-Loop settings, and illustrate their solution methods by example, before concluding with a comparative discussion. We note that we define Nash equilibria for N players, but define Stackelberg for one leader and one follower. This is done firstly because our work on common property Nash-Cournot games in Chapter 4 uses an N player equilibria, while our work on Stackelberg games is restricted to duopoly competition; and secondly to simplify the presentation of the solution method of the Stackelberg game.

1.5.2.1 Open-Loop Nash Equilibrium

Recalling that an Open-Loop control for player i is a function $u_i : t \to \mathbf{R}^{m_i}$, we define the Open-Loop Nash Equilibrium.

Definition A set of Open-Loop controls $(u_1^*, ..., u_N^*)$ is an Open-Loop Nash Equilibrium (OLNE) for the game given by (1.5.1) if, for each i, the control u_i^* maximises

$$J_i(u_i, \mathbf{u}_{-i}^*) = \psi(x_T) + \int_0^T \pi_i(t, x_t, u_i(t), \mathbf{u}_{-i}^*(t))\mathrm{d}t,$$

with initial data x_0 and where the evolution of x_t given by

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = f(t, x_t, u_i(t), \mathbf{u}_{-i}^*(t)),$$

over all controls such that $u_i \in U_i$, $x \in X$, where $\mathbf{u}_{-i} = \left\{u_j^*\right\}_{j=1, j\neq i}^{N}$.

We use the notation $u_{i,t} = u_i(t)$ when describing the Open-Loop game in order to highlight that the controls are functions of time alone. Finding a Nash Equilibrium for an N player game means simultaneously solving N optimal control problems. Each player considers an arbitrary set of admissible control paths for the other players, and finds their best-response as a control problem using the theory of Appendix A, with the OLNE given by the simultaneous solution of these control problems. We illustrate this below.

Example: Resource Extraction with Isoelastic Price Function. In this Example and in Example 1.5.2.3 we consider a resource extraction game in an Open-Loop and Feedback-Loop setting respectively. We assume a Cournot duopoly market for an exhaustible resource, with no extraction costs and an isoelastic pricing function in the form given in (1.2.1). In other words, the players receive revenue at the rate of their market-share of total production. We noted in Section 1.2 that this price function is highly singular, and that the form of revenue under this pricing function leads to questionable extraction behaviour, especially in a monopoly setting. However, in a multiple player setting both the Feedback and Open-Loop games conveniently yield the same solution. This model therefore serves as a useful example for the purpose of illustrating the difference between the Feedback and Open-Loop solution methods.

Defining the resource level of player i at time t as $x_{i,t}$ and the instantaneous extraction rate for player i as $q_i(t)$, player i's optimization problem is given by

$$v_i = \max_{q_i \geq 0} \left\{ \int_0^\infty \mathrm{e}^{-rt} \frac{q_i(t)}{q_i(t) + q_{-i}(t)} \mathrm{d}t \right\},$$

for an arbitrary $q_{-i}(t)$ and under the constraint $x_{i,t} \geq 0$ where

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -q_i(t). \tag{1.5.2}$$

Applying the theory of Appendix A, the Hamiltonian for player 1's control problem is given by

$$H_1(t, \lambda(t), x_{1,t}, q_1) = \mathrm{e}^{-rt} \frac{q_1}{q_1 + q_2} + \lambda q_1, \tag{1.5.3}$$

where λ is an adjoint function used to incorporate (1.5.2) into (1.5.3). This results in the first order conditions

$$\frac{q_2}{(q_1 + q_2)^2} + \lambda \mathrm{e}^{rt} = 0, \qquad \frac{\mathrm{d}\lambda}{\mathrm{d}t} = 0,$$

so that λ is a constant. If player 1 exhausts their reserve at a finite time, T_1, then the transversality condition is given by $q_1(T_1) = 0$; otherwise it is given by $\lim_{t\to\infty} q_1(t) = 0$. By similar reasoning, the Hamiltonian for player 2's control problem is given by

$$H_2(t, \mu(t), x_{2,t}, q_2) = \mathrm{e}^{-rt} \frac{q_2}{q_1 + q_2} + \mu q_2, \tag{1.5.4}$$

where μ is an adjoint function used to incorporate (1.5.2) into (1.5.4). This results in the first order conditions

$$\frac{q_1}{(q_1 + q_2)^2} + \mu \mathrm{e}^{rt} = 0, \qquad \frac{\mathrm{d}\mu}{\mathrm{d}t} = 0,$$

so that μ is a constant. If player 2 exhausts their reserve at a finite time, T_2, then the transversality condition is given by $q_2(T_2) = 0$, otherwise it is given by $\lim_{t\to\infty} q_2(t) = 0$. The OLNE is given by the simultaneous solution of the two control problems. Rearranging gives

$$q_1 = \frac{-\mu}{(\mu + \lambda)^2} \mathrm{e}^{-rt}, \qquad q_2 = \frac{-\lambda}{(\mu + \lambda)^2} \mathrm{e}^{-rt}. \tag{1.5.5}$$

We note that $\mathrm{d}x/\mathrm{d}y = q_1/q_2 = \mu/\lambda$. This is a constant; the solution trajectories therefore approach the boundary on a line with gradient λ/μ. In order to determine the

constants μ and λ, we first determine whether or not the players exhaust their reserves in finite time. From the transversality conditions and (1.5.5), it is clear that exhaustion in finite time for either player results in production at zero for the duration of the game. We therefore have that neither player exhausts their reserve in finite time.

We next show that the boundary point the solution trajectories approach is the origin. If we assume that player i uses $\hat{\mathbf{x}}_i \leq x_{i,0}$ of their resource pool over the course of the game, we can determine the values of λ and μ by integrating (1.5.2) over $t \geq 0$ for $i = 1, 2$. Doing so gives

$$\hat{\mathbf{x}}_1 = \frac{-\mu}{r(\lambda+\mu)^2}, \qquad \hat{\mathbf{x}}_2 = \frac{-\lambda}{r(\lambda+\mu)^2},$$

so that

$$\begin{aligned} q_1(t) &= r\hat{\mathbf{x}}_1 \mathrm{e}^{-rt}, & q_{2,t} &= r\hat{\mathbf{x}}_2 \mathrm{e}^{-rt}, \\ x_{1,t} &= \hat{\mathbf{x}}_1 \mathrm{e}^{-rt}, & x_{2,t} &= \hat{\mathbf{x}}_2 \mathrm{e}^{-rt}, \\ v_1 &= \frac{\hat{\mathbf{x}}_1}{r(\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2)}, & v_2 &= \frac{\hat{\mathbf{x}}_2}{r(\hat{\mathbf{x}}_1+\hat{\mathbf{x}}_2)}. \end{aligned}$$

Clearly each v_i is increasing in $\hat{\mathbf{x}}_i$, so that the players will choose strategies which result in their extracting their total reserve in infinite time, and we have $\hat{\mathbf{x}}_1 = x_{1,0}$, $\hat{\mathbf{x}}_2 = x_{2,0}$ in equilibrium. This means that the solution trajectories approach the origin along the line $x_{1,t}/x_{2,t} = \mu/\lambda$. The players produce at the level of their own resource at time t, scaled by r. The players therefore each decrease their absolute rates of production as their stockpiles decline so that neither player runs out in finite time, but they maintain the same ratio of production. Because of this, each player receives a perpetual stream of revenue at the constant rate of their share of the total resources across the two players.

This value can be interpreted as a perpetuity of 1, the rate of revenue received by a monopoly player, split across the two players according to their share of the total resources in the market. From this, we also observe that there are no gains to cartelisation for either player in this game, because the combined revenue across the two players at all times is the maximal rate of revenue possible under this price-curve.

1.5.2.2 Open-Loop Stackelberg Equilibrium

Under both the Open-Loop and Feedback-Loop equilibrium, we assume without loss of generality that the Stackelberg leader is player 1, and that player 2 is the follower. To define the Stackelberg Equilibrium we first define the follower's best-response function.

Definition If player 1 selects strategy $\mathbf{u}_1$, then we define the set of best-responses of player 2 as $\mathcal{R}(\mathbf{u}_1)$, where $u_2^* \in \mathcal{R}(\mathbf{u}_1)$ if it maximises

$$J_i(u_2, \mathbf{u}_1) = \psi_2(x_T) + \int_0^T \pi_2(t, x_t, u_2(t), \mathbf{u}_1(t))\mathrm{d}t,$$

with initial data x_0 and the evolution of x_t given by

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = f(t, x_t, u_2(t), \mathbf{u}_1(t)),$$

over all Open-Loop controls such that $u_2 \in U_2$, $x \in X$.

To determine their optimal strategy, player 1 needs to compute the best-response of each strategy in U_1, and select that which optimises their pay-off.

Definition A set of controls (u_1^*, u_2^*) is an Open-Loop Stackelberg Equilibrium if the following hold:

1. $u_2^* \in \mathcal{R}(u_1^*)$,
2. Given any $\tilde{u}_1 \in U_1$ and $\tilde{u}_2 \in \mathcal{R}(\tilde{u}_1)$, we have

$$J_1(\tilde{u}_1, \tilde{u}_2) \leq J_1(u_1^*, u_2^*),$$

for J defined in (1.5.1).

Example: Economic Growth Game. We consider a model of capitalism in which the government is a Stackelberg leader and the capitalists in the market act as a single follower. The players have asymmetric controls, such that the government's control, given by $u_1(t)$, is the capital tax rate at time t, and the capitalists' control, given by $u_2(t)$, is their instantaneous rate of consumption at time t. The players' payoffs depend on a state variable of one dimension, the total wealth of the capitalists at time t. The value of this state variable at any time is given by x_t where

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = ax_t - u_1 x_t - u_2,$$

where $a > 0$ is a constant growth rate, and with $x_0 = x(0)$. We assume that the payoffs for the government and capitalists respectively are given by

$$J_1 = bx_T + \int_0^T k_1 \ln(u_1(t)x_t)\mathrm{d}t, \tag{1.5.6}$$
$$J_2 = x_T + \int_0^T k_2 \ln(u_2(t))\mathrm{d}t.$$

Finding the game's solution requires determining the capitalists' best-response for a given $u_1(t)$. Following the theory of OLNEs provided indicates that the capitalists' best-response is given by

$$u_2(t, u_1, \lambda_2) = \arg\max_{u_2 \geq 0} \{k_2 \ln(u_2) - \lambda_2 u_2\} = \frac{k_2}{\lambda_2},$$
$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \lambda_2(a - u_1),$$
$$\lambda_2(T) = 1.$$

The government, as the Stackelberg leader, incorporates the above best-response into (1.5.6), and their maximisation problem becomes

$$\text{maximise} \qquad bx_T + \int_0^T k_1 \ln(u_1(t)x_t)\mathrm{d}t,$$

with

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = ax_t - u_1 x_t - u_2,$$
$$u_2 = \frac{k_2}{\lambda_2},$$
$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \lambda_2(a - u_1).$$

Following Appendix A, the controls are given by

$$u_1(t, \lambda_1, \lambda_2, \mu_1) = \arg\max_{u_1 \geq 0} \{k_1 \ln(u_1 x_t) - \lambda_1 u_1 x_t + \mu_1 \lambda_2 u_1\} = \frac{k_1}{\lambda_1 x_t - \mu_1 \lambda_2},$$

where $(\lambda_1, \lambda_2, \mu_1, x_t)$ are given by the ODE system:

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = (a - u_1)x_t - \frac{k_2}{\lambda_2} = \left(a - \frac{k_1}{\lambda_1 x_t - \mu_2 \lambda_2}\right) x_t - \frac{k_2}{\lambda_2},$$
$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = -\frac{k_1}{x_t} - \lambda_1(a - u_1) = -\frac{k_1}{x_t} + \lambda_1 \left(\frac{k_1}{\lambda_1 x_t - \mu_2 \lambda_2} - a\right),$$
$$\frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = -\lambda_2(a - u_1) = \lambda_2 \left(\frac{k_1}{\lambda_1 x_t - \mu_2 \lambda_2} - a\right),$$
$$\frac{\mathrm{d}\mu_1}{\mathrm{d}t} = -\lambda_1 \frac{k_2}{\lambda_2} + \mu_2(a - u_1) = -\lambda_1 \frac{k_2}{\lambda_2} + \lambda_2 \left(a - \frac{k_1}{\lambda_1 x_t - \lambda_2 \mu_2}\right),$$

with boundary conditions

$$x_0 = x(0), \quad \mu_2(T) = 1, \quad \lambda_1(T) = b, \quad \mu_2(0) = 0.$$

1.5.2.3 Feedback-Loop Nash Equilibrium

The Feedback-Loop games analysed in this thesis are infinite-horizon time-autonomous games, for both the Nash and Stackelberg models. Consequently, the controls of the game are stationary Feedback-Loop strategies such that the strategy space for player i is given by

$$S_i = \{u_i(x_t) : u_i(x_t) \text{ is continuous in } x_t \text{ and } u_i(x_t) \in U_i\}.$$

The profit function for player i is given by $\pi_i(x_t, u_1(t), ..., u_N(t))$, and depends on the state variables and controls. The control problem for player i is given by

$$\sup_{u_i \in S_i} J_i = \int_0^\infty \mathrm{e}^{-rt} \pi_i(x_t, u_{1,t}, ..., u_{N,t}) \mathrm{d}t,$$

such that

$$\frac{\mathrm{d}x_{i,t}}{\mathrm{d}t} = f_i(x_t, u_{1,t}, ..., u_{N,t}), \qquad x_{i,0} = x_i(0), \qquad i = 1, ..., n.$$

Definition A Feedback-Loop Nash Equilibrium is given by a set of Feedback-Loop strategies $(u_1^*, ..., u_N^*) \in S_1 \times ... \times S_N$ such that

$$J_i(u_1^*, ..., u_i^*, ..., u_N^*) \geq J_i(u_1^*, ..., u_i, ..., u_N^*),$$

for all $u_i \in U_i$, for each $i = 1, ..., N$, and for every possible $x \in X$.

We use the theory of Appendix B to show that this equilibrium is found as a system of partial differential equations (PDEs) in x. We first note that the equilibrium strategies must satisfy the HJB equations

$$rv_i(x) = \sup_{u_i \in S_i} \left\{ \pi_i(x, u_1, ..., u_N) + \sum_{j=1}^{n} \frac{\partial v_i}{\partial x_j} f_j(x, u_1, ..., u_N) \right\}, \quad i = 1, ..., N,$$

where $v_i(x)$ is the value for player i of the game 1.5.2.3 starting at x. Optimising in the controls gives the players' best-response functions.

Definition The players' best-response functions are given by

$$B(x, \mathbf{u}_{-i}, \nabla v_i) = \arg\max_{u_i \in S_i} \left\{ \pi_i(x, u_i, \mathbf{u}_{-i}) + \sum_{j=1}^{n} \frac{\partial v_i}{\partial x_j} f_j(x, u_i, \mathbf{u}_{-i}) \right\}, \quad i = 1, ..., N,$$

where $\nabla v_i = (\partial v_i / \partial x_1, ..., \partial v_i / \partial x_n)$ is the gradient of v_i and $\mathbf{u}_{-i} = \{u_j\}_{j=1, j \neq i}^{N}$.

Performing this maximisation for each player gives the system of $\sum_{i=1}^{N} m_i$ equations

$$\frac{\partial \pi_i(x, u_1, ..., u_N)}{\partial u_i^k} + \sum_{j=1}^{n} \frac{\partial v_i(x)}{\partial x_j} \frac{\partial f_j(x, u_1, ..., u_N)}{\partial u_i^k} = 0,$$

for $k = 1, ...m_i$ and $i = 1, ..., N$. If we assume that π_i and f_i are such that this system has a unique solution, then we write the stationary Feedback strategies as

$$u_i^* = \phi_i(x; \nabla v_1, ..., \nabla v_N), \qquad i = 1, ..., N.$$

Substituting into 1.5.2.3 gives

$$\begin{aligned} rv_i &= \pi_i(x, \phi_1, ..., \phi_N) + \sum_{j=1}^{n} \frac{\partial v_i(x)}{\partial x_j} f_j(x, \phi_1, ..., \phi_N), \\ &= \pi_i(x, \nabla v_1, ..., \nabla v_N) + \sum_{j=1}^{n} \frac{\partial v_i(x)}{\partial x_j} f_j(x, \nabla v_1, ..., \nabla v_N), \qquad i = 1, ..., N. \end{aligned}$$

Solving this system gives the value functions, which can then be used to find the equilibrium production strategies.

Example: Feedback-Loop reformulation of Example 1.5.2.1 We revisit the game of Example 1.5.2.1 under a Feedback-Loop setting. We now define the instantaneous extraction rate for player i as $q_i(x_{1,t}, x_{2,t})$, and the problem for player i now becomes that of finding an optimal feedback q_i for the control problem

$$v_i = \max_{q_i \geq 0} \left\{ \int_0^\infty \mathrm{e}^{-rt} \frac{q_{i,t}}{q_{1,t} + q_{2,t}} \mathrm{d}t \right\},$$

under the constraints $x_{1,t} \geq 0$, $x_{2,t} \geq 0$, where

$$\frac{\mathrm{d}x_{i,t}}{\mathrm{d}t} = -q_{i,t}(x_{1,t}, x_{2,t}), \qquad i = 1, 2.$$

Here we use the t subscript to refer to functions of time t along a game trajectory $(x_{1,t}, x_{2,t})$. Throughout this thesis, we also refer to x_1, x_2 as independent variables labelling the resource-space, and use $v_i(x_1, x_2)$ and $q_i(x_1, x_2)$ to refer to the value function and production strategy of player i as functions that satisfy the Bellman PDEs of the Feedback-Loop game. The two players solve their control problems simultaneously. Following Appendix B, their Bellman equations are given by

$$rv_i = \max_{q_i} \left\{ \frac{q_i}{q_1 + q_2} - q_i v_{i,x_i} - q_{-i} v_{i,x_{-i}} \right\} \qquad i = 1, 2,$$

where $v_{i,x_j} = \partial v_i / \partial x_j$. Maximising in q_i gives

$$\frac{1}{q_1 + q_2} - \frac{q_i}{(q_1 + q_2)^2} - v_{i,x_i} = 0, \tag{1.5.7}$$

Summing this over i gives

$$q_1 + q_2 = \frac{1}{v_{1,x_1} + v_{2,x_2}},$$

Substituting this into (1.5.7) gives

$$q_1 = \frac{v_{2,x_2}}{(v_{1,x_1} + v_{2,x_2})^2}, \qquad q_2 = \frac{v_{1,x_1}}{(v_{1,x_1} + v_{2,x_2})^2}.$$

Substituting this into the Bellman equations gives the PDEs

$$rv_1 = \frac{1}{(v_{1,x_1} + v_{2,x_2})^2} \left[\frac{v_{2,x_2}\,(v_{1,x_1} + v_{2,x_2} - 1)}{v_{1,x_1} + v_{2,x_2}} - v_{1,x_1} v_{2,x_2} - v_{1,x_1} v_{1,x_2} \right],$$
$$rv_2 = \frac{1}{(v_{1,x_1} + v_{2,x_2})^2} \left[\frac{v_{1,x_1}\,(v_{1,x_1} + v_{2,x_2} - 1)}{v_{1,x_1} + v_{2,x_2}} - v_{2,x_1} v_{2,x_2} - v_{1,x_1} v_{2,x_2} \right],$$

with the boundary conditions $v_1(0, x_2) = 0$, $v_2(x_1, 0) = 0$. Solving these gives

$$rv_1 = \frac{x_1}{x_1 + x_2}, \qquad rv_2 = \frac{x_2}{x_1 + x_2},$$
$$q_1 = rx_1, \qquad q_2 = rx_2.$$

This solution results in the same value functions and production trajectories as the Open-Loop game. This follows because of a property proven in [44]. It is shown that the Open-Loop and Feedback-Loop equilibria of resource extraction games such as this coincide when the Feedback-Loop production rates of each player are independent of the resource levels of the competing players. In other words, the production rates of a player must be functions of their own resource level and time alone.

This would not have been the case if the strategies exhibited dependence on the resource levels of the competing players. These cannot be determined as a function of time alone by the player, as they are dependent on the strategies which the other players select, and these are unknown. Because of this, strategies of this type cannot usually be written in an Open-Loop form. The isoelastic price function is therefore somewhat anomalous. In Chapter 3, we perform similar analysis for the game with the more realistic linear pricing curve, and demonstrate that the equilibrium strategies do indeed differ.

1.5.2.4 Feedback-Loop Stackelberg Equilibrium

As we did for the Open-Loop game, we begin by defining the follower's best-response function in a Feedback-Loop setting. As we did for the Feedback-Loop Nash game in 1.5.2.3, we rely on the theory of Appendix B.

Definition The follower's best-response function, $R : \mathbf{R}^n \times \mathbf{R}^{m_1} \times \mathbf{R}^n \to \mathbf{R}^{m_2}$ is given by

$$R(x, \mathbf{u}_1, \nabla v_2) = \arg\max_{u_2 \in U_2} \left\{ \pi_2(x, \mathbf{u}_1, u_2) + \sum_{j=1}^{n} \frac{\partial v_2(x)}{\partial x_j} f_j(x, \mathbf{u}_1, u_2) \right\}.$$

From this, we can define the stationary Feedback-Loop Stackelberg Equilibrium. The definition we provide, and the derivation of the final HJB equations follows that of [12][35] and [120].

Definition A pair of strategies $(u_1^*, u_2^*) \in (U_1, U_2)$ is a Feedback-Loop Stackelberg Equilibrium if, for every possible initial condition x_0:

1. $u_2^* = R(x, u_1^*, \nabla v_2)$; and
2. $u_1^* = \arg\sup_{u_1 \in U_1} \{v_1(x, u_1, R(x, u_1, \nabla v_2))\}$.

The equilibrium strategy of the leader must therefore satisfy the HJB equation

$$rv_1 = \max_{u_1 \in U_1} \left\{ \pi_1(x, u_1(x), R(x, u_1, \nabla v_2)) + \sum_{j=1}^{n} \frac{\partial v_1(x)}{\partial x_j} f_j(x, u_1, R(x, u_1, \nabla v_2)) \right\}.$$

This maximisation gives

$$\frac{\partial \pi_1}{\partial u_1^k} + \sum_{l=1}^{m_2} \frac{\partial \pi_1}{\partial u_2^l} \frac{\partial R_l}{\partial u_1^k} + \sum_{j=1}^{n} \frac{\partial v_1}{\partial x_j} \left(\frac{\partial f_j}{\partial u_1^k} + \sum_{l=1}^{m_2} \frac{\partial f_j}{\partial u_2^l} \frac{\partial R_l}{\partial u_1^k} \right) = 0, \qquad k = 1, ..., m_1.$$

This system of equations implicitly defines the leader's strategy. If it has a unique solution, then the leader's strategy can be written as a function of the state variables, $u_1^* = \psi(x, \nabla v_1, \nabla v_2)$, with the follower's response given by $u_2^* = R(x; \psi, \nabla v_2) = \omega(x, \nabla v_1, \nabla v_2)$. Substituting them into the HJB equations gives

$$\begin{aligned} rv_i &= \pi_i(x, \psi, \omega) + \sum_{j=1}^{n} \frac{\partial v_i}{\partial x_j} f_j(x, \psi, \omega), \\ &= \pi_i(x, \nabla v_1, \nabla v_2) + \sum_{j=1}^{n} \frac{\partial v_i(x)}{\partial x_j} f_j(x, \nabla v_1, \nabla v_2), \qquad i = 1, 2. \end{aligned}$$

[35] Pages 413-415.

The equilibrium value functions are given by the solution of this PDE system, from which the equilibrium strategies can be calculated. We illustrate this below in Example 1.5.2.4.

Example: Pesticide Usage. We present the following example from [120], in which the joint exploitation of a pesticide by two firms is considered. Pesticides become less effective with increasing use, so that the rate of usage of one firm impacts the effectiveness of usage by the other. We define $u_{i,t}$ as the rate of usage at time t by player i, and the level of effectiveness of the pesticide as x_t, with dynamics given by

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = \begin{cases} -u_{1,t} - u_{2,t} & \text{for } x_t > 0, \\ 0 & \text{for } x_t = 0. \end{cases}$$

The value function of each player is given by

$$v_i = \int_0^\infty \mathrm{e}^{-rt} \left(u_{i,t} x_t\right)^\alpha \mathrm{d}t,$$

where $0 < \alpha < 1/2$ is a parametrising constant. Taking u_1 as given, the best-response function of player 2 is

$$\begin{aligned} u_2 &= \arg\max_{u_2 \geq 0} \left\{ (u_2 x)^\alpha - {v_2}' \left(u_1 + u_2\right) \right\}, \\ &= \left(\frac{\alpha x^\alpha}{{v_2}'} \right)^{\frac{1}{1-\alpha}}, \end{aligned}$$

where ${v_2}' = \mathrm{d}v_2/\mathrm{d}x$. Incorporating the follower's response function, the leader's Bellman equation is therefore given by

$$r v_1 = \max \left\{ (u_1 x)^\alpha - {v_1}' \left(u_1 + \left(\frac{\alpha x^\alpha}{{v_2}'} \right)^{\frac{1}{1-\alpha}} \right) \right\},$$

so that

$$u_1 = \left(\frac{\alpha x^\alpha}{{v_1}'} \right)^{\frac{1}{1-\alpha}}.$$

Substituting the controls into the Bellman equations for each player, and using the symmetry of the solution across the two players, gives

$$r v_i \left({v_i}'\right)^{\frac{\alpha}{1-\alpha}} = (1 - 2\alpha) \left(\alpha x\right)^{\frac{\alpha}{1-\alpha}}, \qquad i = 1, 2,$$

with boundary conditions $v_1(0) = 0$, $v_2(0) = 0$. This can be solved explicitly by

$$u_i = \frac{rx}{2 - 4\alpha}.$$

It is noted in [120] that, for this particular game, the Feedback-Stackelberg solution coincides with that of the Feedback-Nash solution. This is because the follower's response function in this case is independent of the leader's chosen control; it is a function of x alone. Because the leader cannot induce the follower to alter their strategy by changing their own, there is no incentive for them to deviate from the Nash Equilibrium. This is not generally true, and does not occur for the Stackelberg game considered in Chapter 2. However, here this property results in a readily calculable solution, which is useful for the purpose of this illustrative example.

1.5.2.5 Discussion

Having given a formal definition of each solution method, we can now discuss the differences between them in greater detail, and in the context of resource extraction games. First, it is important to consider how these two models approach commitment as they represent the two extremes of possible commitment models.

On the one hand, Open-Loop games require players to determine the extraction profile of their entire stock even before the first unit is extracted. Moreover, determining such a strategy also requires that each player assume a production path for each other player over the entire game. On the other hand, Feedback-Loop strategies assume that not only do players have continuously and perfectly updated knowledge of the resource stocks of *all* players in the market, but complete freedom to update their strategies instantaneously in response.

In reality, it is likely that information flow and commitment in the market place operates somewhere between these extremes. Firms almost always have a degree freedom to alter production in response to new information as it emerges, and do invest considerable resources in ensuring a reliable and steady flow of that information. They conduct rigorous and frequent surveys of (say) their own mining sites, and closely pay attention to the announcements and activities of other companies—sometimes going so far as to procure information about their competitors by illicit means.

However, it is also true that there are ineliminable frictions in the market–both in the flow of information and in the constraints placed on changes in strategy by the firm.

Companies only gain periodic updates in the information they receive about market resource allocations. Furthermore, sales contracts place short to medium term commitments on the firm's output, and changing production often requires changes to the allocation of labour and infrastructure, even though these are generally inflexible in the short term.

The firms within the market also respond differently to each other's actions. We first reiterate that because x_t is a function of time, the control strategies under either model are ultimately functions of time as well. However, the dependence on the current level of x in the Feedback-Loop game induces a difference in the way in which players are able to adapt to variations of the strategies of their opponents, and this means that the underlying strategies themselves are also different. We explain this below in more detail for a duopoly setting:

- When calculating a Nash equilibrium strategy, each player assumes that the other player adopts their own equilibrium strategy, and then looks for strategies that are best responses. A part of doing this necessarily involves considering variations from a potential equilibrium strategy and confirming that these variations decrease the player's welfare.

- Under either an Open-Loop or a Feedback-Loop game, these variations in (say) player 1's strategy as a function of t result in a shift in the trajectory of the game. However, in an Open-Loop game, the path of player 2's control remains fixed as a function of t, because the strategy of player 2 has no dependence on x_1.

- In a Feedback-Loop game, although we still assume that player 2 does not alter their strategy, there is now an implicit response, which we dub the Feedback-Loop effect. Player 1's shift in strategy drives x_1 away from what it otherwise would have been. The fact that player 2's strategy is a function of x_1 means that player 2 now responds strategically to this change in x_1, even though we retain the assumption that the functional form of their control does not change in response to the player 1's shift in strategy per se.

The impact of this Feedback-Loop effect changes depending on whether or not the resources are common-property or separate pools. In a common property setting, production is generally higher under a Feedback-Loop game than under an Open-Loop one,

because the resource-level dependency in the Feedback-Loop game creates an additional pre-emptive incentive to extract. This result is discussed in detail in [79], but we briefly consider it in our work on common property games in Chapter 4.

In a separate pools setting, the impact of the Feedback-Loop effect on a player's production levels and value function is more complex; both the size and direction of it depend on the size of the player's resource level in comparison to those of their competitors. This is the main subject of our work in Chapter 3, in which we consider the relationship between relative resource size and the difference between the Open-Loop and Feedback-Loop equilibria for duopoly resource firms under both Cournot-Nash and Stackelberg solution methods.

On a more technical level, we have shown in Examples 1.5.2.1 and 1.5.2.3 that the solution method is dependent on the choice of information set. This consideration is relevant for all resource extraction games, but especially so for separate pools games. The state-space for many common property games are one dimensional. As we later see in Chapter 4, this means that both the Feedback-Loop and Open-Loop solution methods are tractable for the simple models we consider.[36]

In a separate pool setting, however, the state space is not one dimensional but increases with the number of players. In an Open-Loop game this makes solving the optimal control problems more tedious, but as we saw in Example 1.5.2.1, this does not preclude closed-form solutions in and of itself. For a Feedback-Loop system however, we see throughout this thesis that the results for Feedback-Loop games with separate pools are given by nonlinear PDE systems.[37] The increase in the dimension of these systems generally complicates the process of solving them numerically.

It is partly for this reason that there has been relatively little analysis of these games. A Cournot-Nash Feedback-Loop game is analysed in [122], but its results were brought into question by [110]. It also relied upon the inclusion of a resource dependent extraction cost in generating its solutions. More recently, work in [62] has yielded numerical solutions to a Cournot game with separate pools through the inclusion of a backstop technology. An asymptotic expansion in the cost of this backstop is performed, yielding an analytic approximation to the game's solution which is used as the starting point for a numerical scheme. This approach is modified for a differentiated Bertrand game in [87].

[36] See [93] and [79] for more general examples.
[37] See [62] and [122] for supporting examples.

We extend on this work in two ways in this thesis. Firstly, we extend this analysis to a Feedback-Loop Stackelberg model in Chapter 2. Secondly, we make a small alteration to the numerical scheme to allow for the computation of the Feedback-Loop game without a backstop, and compare the solution to that of the Open-Loop game in Chapter 3.

We add that the treatment of time in each case means that models with time-dependent factors are also treated differently across the two games. In Open-Loop games, the strategies are already considered as functions of time and so the inclusion of relatively simple time-dependent factors into a game[38] is relatively trivial. For a Feedback-Loop game however, including time-dependent factors requires treating time as an additional variable in the state-space, in addition to the resource level(s). This increases the dimension of the problem and increases the difficulty of its solution in a similar way to the addition of another player. As a result, there has been relatively little analysis of time-dependent complications such as Green Paradoxes and time-dependent property rights in a Feedback-Loop setting.

While Open-Loop problems are often more tractable than Feedback-Loop ones, this ease of computation does not come without a cost. Open-Loop methods can carry a number of problems, one of which is time-inconsistency of solutions. A strategy is weakly time-consistent[39] if, at all points along the solution trajectory from the initial resource allocation, neither player would change their strategy if given the opportunity. In [12] it is shown that for some Open-Loop games, solutions display weak time-inconsistency. In other words, there are solution trajectories in which the players would later opt for different strategies were they able to determine their strategies based on the resource allocation at that point in time. This means that the strategies are only plausible if the commitment implicit in the Open-Loop game is externally enforced.

Feedback-Loop strategies, on the other hand, induce strong-time consistency by design. They are called strongly time-consistent[40] because consistency holds not only along the solution trajectory, but also outside of it. In other words, the solution functions still hold in the event of exogenous shifts in the supply of any player's resource that moves the game from its current solution trajectory, be it by unexpected discoveries or by expropriations.

[38] Examples of these include future tax increases, or expiry dates on extraction rights to a resource pool.

[39] See [12].

[40] This term was first coined in [12].

1.6 Roadmap for this Thesis

Each Chapter of this thesis focuses on a largely self contained question, but there are a number of cross-overs between the models analysed in each chapter. We outline each chapter below, and summarise their usage of the models we have described in this Introduction in Table 1.2.

Chapter 2 focuses on Feedback-Loop Stackelberg games in a separate pools setting. We demonstrate that the Feedback-Loop game displays a highly counterintuitive non-monotonicity in its equilibrium value functions and production rates. In particular, this implies that there are resource levels for which the follower's marginal value from their stock is negative.

We consider the game with and without a backstop technology. The game without a backstop is considered primarily because the limiting case as the resource level of the leader approaches infinity replicates the non-monotonicities we describe in a one dimensional setting.

The primary focus of our work in a two dimensional setting involves a backstop. Here we first consider the boundary conditions of the game in which at least one player is dependent on the backstop and demonstrate that the nature of blockading in this kind of game differs a great deal from the Cournot case in [62] in interesting ways. We then use an asymptotic expansion in the backstop cost to develop an analytic approximation to the value functions at a low backstop cost. This not only demonstrates analytically the same non-monotonicities we have referred to but also serves as the starting point for a numerical scheme converging to the exact value functions.

Chapter 3 focuses on the differences between Open-Loop and Feedback-Loop games in a separate pools setting as described in Section 1.4.2.1. The chapter comprises two halves, relating to Cournot competition and Stackelberg competition respectively. In this Chapter, we also develop an adjusted form of the numerical schemes used in [62] and [87] which allows us to solve for the value functions of the Feedback-Loop game without a backstop under Cournot and Stackelberg competition. We derive the Open-Loop values analytically, and compare them to the results of the numerical scheme for the Feedback-Loop game.

Chapter 4 analyses a widely used common property Cournot game in an Open-Loop setting. We build on the result in [79] that there are in fact two solutions to the terminal

condition of the optimal control problem associated with the game, and that each one relates to a different equilibrium solution. We challenge the validity of one of these candidate equilibria, by demonstrating that it results in value functions for the players which are non-monotonic in the resource level, and by showing that a better response can be constructed.

Our criticism of this candidate equilibrium lies in the fact that it is not robust to certain strategies in which players anticipate different terminal times for the game. In particular, we show that by extracting more than is dictated by this candidate equilibrium path, a player can attain a higher level of value than they would by following it. Doing so results in the resource being depleted, and the game ending, earlier than would be anticipated by other players adhering to the candidate equilibrium path. The findings of this Chapter therefore have broader implications for the use of traditional Open-Loop solution methods in the analysis of common property resource extraction games.

Finally, Chapter 5 considers Cournot games with multiple players in a static setting, under joint constraints of the kind introduced in Section 1.2.3. Here we build on work in [109], [60], [68], and [24], in which it is shown that the imposition of price caps and joint constraints on production can preclude the existence of unique Nash equilibria. We extend on this result by formulating uniqueness results for a game with joint constraints on production, as well as the game in which both price caps and joint production caps exist. We also show that combining these results makes it possible to analyse the uniqueness of solutions of a game with price-dependent windfall taxes or congestion charges. In particular, we find that the uniqueness-disrupting effect of these joint constraints can still hold even when their violation is permitted but penalised.

Table 1.2: Model components by Chapter

	Chapter 2	Chapter 3	Chapter 4	Chapter 5
Cournot or Stackelberg	Stackelberg	Both	Cournot	Cournot
Exhaustibility	✔	✔	✔	✖
Backstop	✔	✖	✖	✖
Open-Loop or Feedback-Loop	Feedback-Loop	Both	Open-Loop	N/A
Common-Property or Separate Pools	Separate Pools	Separate Pools	Common Property	N/A

Chapter 2

Feedback-Loop Stackelberg Competition

2.1 Introduction

We introduced Cournot models in Section 1.3.1. We recall that these models are popular and highly tractable, but are open to a number of criticisms. In particular, they treat all producers as equally credible in their ability to influence the price. This is questionable, given that many resource markets involve asymetric competition between a powerful cartel and a competitive fringe of smaller producers.

Modelling 'cartel versus fringe' games, in which one firm occupies a position of market leadership, requires incorporating the asymmetry of market power between the cartel and the firms in the competitive fringe. One widely used method adopts Stackelberg competition of the type introduced in Section 1.3.2.[1] Here, the dominant cartel is the Stackelberg leader and has the ability to announce and commit to their strategy before the remaining firms determine their own strategies. Much of the literature analyses such games in an Open-Loop setting, in which the players commit to their production trajectories for the duration of the game at the point that the game begins.

As we discussed in Section 1.5.2.5, Open-Loop games may allow closed-form strategies to be calculated, but suffer from the problem of time-inconsistency. One way of avoiding this problem, used in [56], is to derive the solution using dynamic programming, which achieves time-consistency by construction of the solution. Doing so, however, changes the information structure of the problem and assumes a Feedback-Loop game is played

[1] See [56], [141], and [58].

instead, in which the players' strategies become functions of the current resource levels of all of the players.

These strategies are also called Markov-perfect strategies. They are referred to as strongly time-consistent in [12] because consistency holds not only along the solution trajectory, but also outside of it. In other words, the solution functions still hold in the event of exogenous shifts in the supply of any player's resource that moves the game from its current solution trajectory, whether by unexpected resource discoveries or by expropriations.

In this Chapter, we study a two-player Feedback-Loop Stackelberg game with one leader and one follower, following the formulation in Section 1.5.2.4 and [12]. The model considered in [56] follows the same formulation, but assumes a competitive fringe of infinitely many followers with negligible influence on the resource price that results in value functions that are linear in their resource levels. Here, we make no assumptions about the form of the value functions of either the leader or follower, and we calculate the solution numerically. We also incorporate and examine the impact of a backstop technology, an alternative resource that is infinite in supply but costly to produce. We derive the HJB equations of the game, and make use of the methods of [62] which developed numerical solutions for Feedback-Loop Cournot competitions, in order to calculate a numerical solution from the HJB equations.

In doing so, we show that the Feedback-Loop Stackelberg strategies exhibit highly counterintuitive non-monotonicities in the follower's value function. In particular, we show that there are parts of the solution space for which the follower's value function even declines as its resource level increases.

Such non-monotonicities are rare, but not unheard of in the literature. In [15], it is shown that the total social welfare arising from an Open-Loop cartel vs fringe game can be decreasing in the resource level of the fringe by inducing overproduction from the cartel. Temporary cartels, in which the firms within the cartel revert to non-cooperative competition after a specified time period, are analysed in [14]. It is shown that a greater level of resources for the cartel can result in a lower value for the firms, as a result of the larger resource pool making competition after the dissolution of the cartel more aggressive.

In Section 2.2 we consider the game in the absence of a backstop. Doing so allows us to demonstrate the non-monotonicities in the follower resource level in a one-dimensional

setting, verifying that they are not an artefact of a particular numerical scheme for the two-dimensional game. In Section 2.3 we introduce a backstop technology in the duopoly game, including the derivation of the HJB equations and the solution of the game on the boundaries. In Section 2.4 we use an asymptotic expansion in the cost of a backstop supply to derive an analytic approximation to the value functions, and then use this approximation as the starting point for a full numerical solution. We conclude in Section 2.5.

2.2 Duopoly Game without a Backstop

The game operates under a Stackelberg competition as defined in Section 1.3.2, with a Feedback-Loop information structure as defined in Section 1.1.1.3; so that the game is solved by Feedback-Loop Stackelberg equilibria as defined in Section 1.5.2.4. We assume that player 1, the leader, and player 2, the follower, begin the game with initial capacities x_1 and x_2 respectively. Each player chooses a Markov Perfect dynamic production rate $q_{i,t} = q_i(x_{1,t}, x_{2,t})$, where $x_{i,t}$ is the resource level of player i at time t, and depletes with the dynamics

$$\frac{\mathrm{d}x_{i,t}}{\mathrm{d}t} = -q_i(x_{1,t}, x_{2,t}), \quad x_{i,t} > 0. \tag{2.2.1}$$

As this section does not allow for the existence of a backstop technology, the total exhaustion of a player's reserve means that player can no longer produce. We assume a linear demand curve as defined in (1.3.1), and that producing from the exhaustible resource is costless. Given these assumptions, player i seeks to maximise their discounted lifetime profit:

$$v_i = \int_0^\infty \mathrm{e}^{-rt} q_{i,t}(1 - q_{1,t} - q_{2,t})\mathrm{d}t. \tag{2.2.2}$$

We begin by considering the games in which one player has run out, and either x_1 or x_2 is equal to zero, which we refer to as the *axis games*. We then consider the limiting cases where either player 1 or player 2 has an infinite supply of the resource, while the other player has a finite quantity. Finally, having considered all of the *boundary games*, we consider the game for $0 < x_1, x_2 < \infty$, which we refer to as the *interior game*. In each case, we begin by deriving the follower's response function and then use this to derive the leader's strategy.

2.2.1 The Axis Games

Because players cease production at the point at which they run out of the resource, each of the axis games is equivalent to the monopoly game for the player which has stock remaining on the axis. The monopoly game is solved explicitly in Section 1.4.1.2. On the $x_1 = 0$ axis, we have $v_1 = 0$, $q_1 = 0$. The value function for player 2 as a function of x_2 is given by (1.4.7) and their production is given by (1.4.6). Reversing (x_1, x_2), and (v_1, v_2) in the above gives the solution on the $x_2 = 0$ axis.

2.2.2 $x_1 = \infty$

In this case, player 1 has unlimited resources while player 2 faces exhaustion at $x_2 = 0$. The value functions are functions of x_2 alone and are given by (2.2.2), with the depletion of x_2 given by (2.2.1).

Proposition 2.2.1. *The value functions are given by the solution of the system of ordinary differential equations*

$$\begin{aligned} \frac{\mathrm{d}v_1}{\mathrm{d}x_2} &= -\frac{1}{2} - 4\sqrt{rv_2} + 3\sqrt{\frac{1}{4} + 2rv_2 - rv_1}, \\ \frac{\mathrm{d}v_2}{\mathrm{d}x_2} &= \frac{1}{2} - \sqrt{\frac{1}{4} + 2rv_2 - rv_1}, \end{aligned} \tag{2.2.3}$$

with initial conditions

$$v_1(0) = \frac{1}{4r}, \qquad v_2(0) = 0.$$

The solution of this system exists and is unique. The solution cycles around a stable focus in the (v_1, v_2) plane at $(v_1, v_2) = (1/8r, 1/16r)$ as $x_2 \to \infty$. This implies that there are values of x_2 for which $\mathrm{d}v_2/\mathrm{d}x_2 < 0$ and $\mathrm{d}v_1/\mathrm{d}x_2 > 0$.

Proof. Taking the player 1 strategy $q_{1,t}$ as given, the value function for the follower, $v_2(x_2, q_{1,t})$, is defined by the optimization problem

$$v_2 = \sup_{q_{2,t} \geq 0} \left\{ \int_0^\infty \mathrm{e}^{-rt} q_{2,t}(1 - q_{1,t} - q_{2,t}) \mathrm{d}t \right\}.$$

By a standard dynamic programming argument such as that of [133] and Appendix B, this value function must satisfy

$$rv_2 = \sup_{q_2 \geq 0} \left\{ q_2(1 - q_1 - q_2) - q_2 \frac{\mathrm{d}v_2}{\mathrm{d}x_2} \right\}.$$

Maximising the above in q_2 then gives the follower's optimal response function as

$$q_2^* = \frac{1}{2}\left(1 - q_1 - \frac{\mathrm{d}v_2}{\mathrm{d}x_2}\right).$$

We can now find the value function for the leader, assuming that the follower reacts optimally to their announced strategy, by optimizing

$$v_1 = \sup_{q_{1,t}\geq 0}\left\{\int_0^\infty \mathrm{e}^{-rt}\frac{q_{1,t}}{2}\left(1 - q_{1,t} + \frac{\mathrm{d}v_2}{\mathrm{d}x_2}\right)\mathrm{d}t\right\},$$

which results in

$$rv_1 = \sup_{q_1\geq 0}\left\{\frac{q_1}{2}\left(1 - q_1 + \frac{\mathrm{d}v_2}{\mathrm{d}x_2}\right) - \frac{1}{2}\left(1 - q_1 - \frac{\mathrm{d}v_2}{\mathrm{d}x_2}\right)\frac{\mathrm{d}v_1}{\mathrm{d}x_2}\right\},$$

Maximising the above in q_1 gives

$$q_1^* = \frac{1}{2}\left(1 + \frac{\mathrm{d}v_1}{\mathrm{d}x_2} + \frac{\mathrm{d}v_2}{\mathrm{d}x_2}\right).$$

Finally, substituting the above into the follower's response function to calculate their strategy in equilibrium gives

$$q_2^* = \frac{1}{4}\left(1 - 3\frac{\mathrm{d}v_2}{\mathrm{d}x_2} - \frac{\mathrm{d}v_1}{\mathrm{d}x_2}\right).$$

We therefore have ODEs for the value functions at equilibrium given by

$$rv_1 = q_1(1 - q_1 - q_2) - q_2\frac{\mathrm{d}v_1}{\mathrm{d}x_2}, \tag{2.2.4}$$

$$rv_2 = q_2(1 - q_1 - q_2) - q_2\frac{\mathrm{d}v_2}{\mathrm{d}x_2}, \tag{2.2.5}$$

for $x_2 > 0$, where

$$q_1 = \frac{1}{2}\left(1 + \frac{\mathrm{d}v_1}{\mathrm{d}x_2} + \frac{\mathrm{d}v_2}{\mathrm{d}x_2}\right), \qquad q_2 = \frac{1}{4}\left(1 - 3\frac{\mathrm{d}v_2}{\mathrm{d}x_2} - \frac{\mathrm{d}v_1}{\mathrm{d}x_2}\right).$$

These can be put in the form (2.2.3) by solving the quadratics above, and always taking the root that implies $v_1 \geq 0$, $v_2 \geq 0$. The initial conditions on v_1 and v_2 can be found by letting $x_1 \to \infty$ in the axis game on $x_2 = 0$, so that we have $(v_1(0), v_2(0)) = (1/4r, 0)$.

The equations in this form allow us to confirm existence and uniqueness of the solution using general results such as those in [32]. To identify the singular point note

that if $\mathrm{d}v_1/\mathrm{d}x_2 = 0$ then $v_1 = 2v_2$. This can be substituted into the expression for $\mathrm{d}v_1/\mathrm{d}x_2$ to show that both $\mathrm{d}v_1/\mathrm{d}x_2$ and $\mathrm{d}v_2/\mathrm{d}x_2$ vanish at $(v_{1,e}, v_{2,e}) = (1/8r, 1/16r)$ as required. We note that this is the value obtained in the limiting case where $x_2 \to \infty$ and the game approaches the inexhaustible game played repeatedly. To determine the nature of this singular point, we linearise about the solution, using:

$$v_1 = v_{1,e} + \delta v_1, \quad v_2 = v_{2,e} + \delta v_2,$$

finding the Jacobian

$$\begin{pmatrix} -3 & 2 \\ 1 & -2 \end{pmatrix},$$

with eigenvalues given by

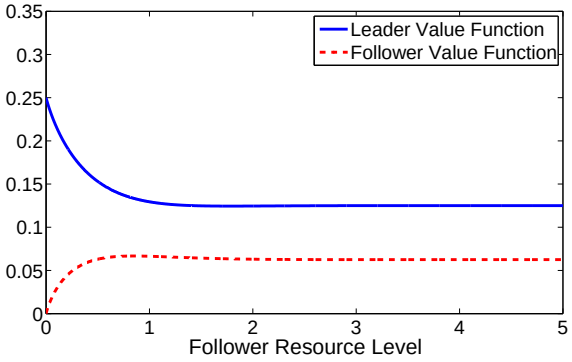
$$\lambda_{1,2} = \frac{1}{2}\left(-5 \pm \mathrm{i}\sqrt{7}\right).$$

As both λ_1 and λ_2 are complex with negative real parts, the singular point is a stable focus. The existence of oscillation in v_1 and v_2 with respect to x_2 follows trivially from the fact that the phase plane trajectory of v_1, v_2 with respect to x_2 shown in Figure 2.1d is in a spiral orbit around $(1/8r, 1/16r)$ as $x_2 \to \infty$. □

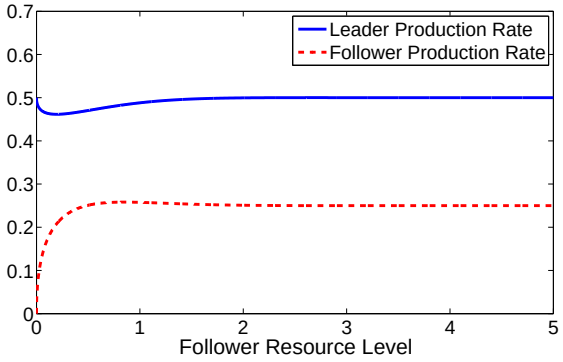
While the ODE system (2.2.3) cannot be solved explicitly, a numerical solution is straightforward and can be achieved in Matlab with Runge-Kutta methods. In Figure 2.1 we present the numerical results for the value functions, production strategies and first derivatives in x_2.

The primary result we wish to draw attention to, and the most counter-intuitive, is the non-monotonicity of v_2 in x_2. Intuitively, we would always expect the follower to gain non-negative value from additional resources and $\mathrm{d}v_2/\mathrm{d}x_2$ to be positive but this is clearly not the case. Inspection of the production rates also highlights similar non-monotonicities in the production strategy of the leader. We have confirmed for all x_2 that the price is greater than player 2's shadow cost, $\mathrm{d}v_2/\mathrm{d}x_2$, and that this non-monotonicity is not the result of undetected blockading of player 2 in the interior.

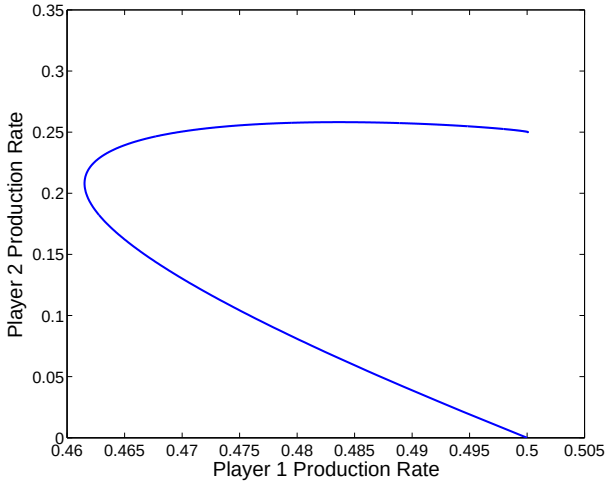
This is a highly counterintuitive result, and violates the free-disposal condition described in [8]. We argue that the oscillatory behaviour is related to the leader switching between actively driving down the follower's production rate with their own production, and accommodating them in the market to some extent for the purpose of bringing forward the time at which they exhaust their stock of the resource and the leader becomes a monopolist.



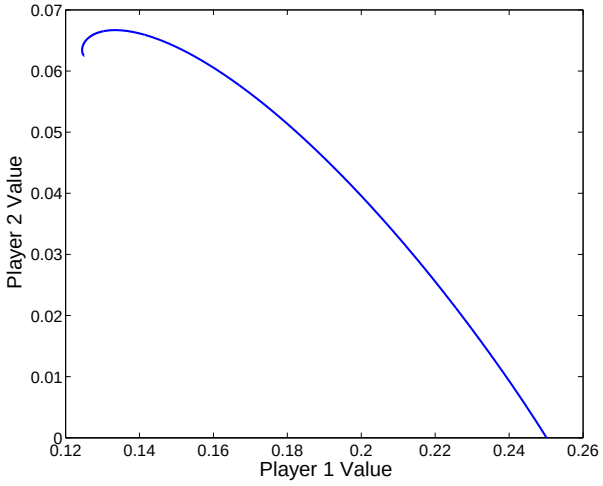
(a) Value Functions



(b) Production Rates



(c) q_1 vs q_2



(d) v_1 vs v_2

Figure 2.1: Numerical results for $x_1 = \infty$

This seems especially likely in the case of the initial change in the sign of $\mathrm{d}v_2/\mathrm{d}x_2$ from positive to negative. At low levels of x_2, it is profitable for player 1 to produce less for a higher given level of x_2, because doing so induces player 2 to produce more, and thereby reach total depletion faster. This benefits the leader, as the production rate of the follower declines quite rapidly for small x_2, meaning that as it nears total exhaustion of its stock, player 2's depletion creates proportionately more space in the market for the leader to produce profitably. For larger values of x_2, the effects of depletion on player 2's production, and thereby player 1's price, are significantly lowered, and it becomes profitable for player 1 then to revert to the strategy of driving up their own production rate to drive down that of player 2.

We also make note of the significance of demonstrating this result for $x_1 = \infty$. Introducing exhaustibility for both players makes the game a nonzero-sum continuous-time game with a multidimensional state space. Analysis therefore generally suffers from a lack of general existence and uniqueness results for games of this type.[2] When $x_1 = \infty$, the resource is effectively only exhaustible for the follower, and the game reduces to one with a one-dimensional state space in which general existence and uniqueness results do exist.[3] This allows us to confirm that the non-monotonicity we observe is an intrinsic feature of the game, and not an idiosyncracy of a particular numerical solution method.

2.2.3 $x_2 = \infty$

Using the same approach as for the case $x_1 = \infty$, the ODEs for the game at $x_2 = \infty$ are given by

$$rv_1 = q_1(1 - q_1 - q_2) - q_1\frac{\mathrm{d}v_1}{\mathrm{d}x_1}, \qquad rv_2 = q_2(1 - q_1 - q_2) - q_1\frac{\mathrm{d}v_2}{\mathrm{d}x_1}, \tag{2.2.6}$$

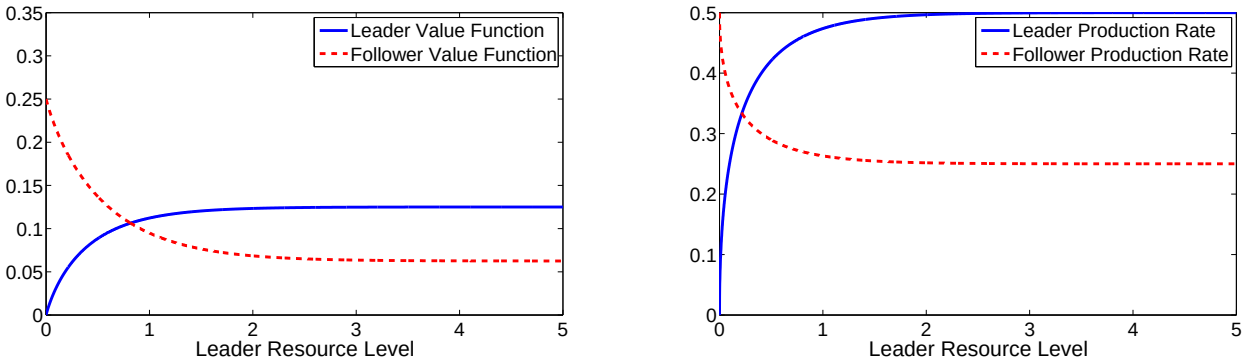
where v_1 and v_2 are functions of x_1 alone, and where

$$q_1 = \frac{1}{2}\left(1 - 2\frac{\mathrm{d}v_1}{\mathrm{d}x_1}\right), \qquad q_2 = \frac{1}{4}\left(1 + 2\frac{\mathrm{d}v_1}{\mathrm{d}x_1}\right),$$

with initial conditions

$$v_1(0) = 0, \qquad v_2(0) = \frac{1}{4r}.$$

[2] See [21] for a detailed explanation of these problems.
[3] See [32].



(a) (b)

Figure 2.2: (2.2a) Value functions and (2.2b) production strategies for $x_2 = \infty$.

This can be solved explicitly by

$$v_1 = \frac{1}{8r}\left[1 + \mathbf{W}(-\mathrm{e}^{-(2rx_1+1)})\right], \qquad v_2 = \frac{1}{48r}\left(3 - 12\frac{\mathrm{d}v_1}{\mathrm{d}x_1} - 4\frac{\mathrm{d}^2 v_1}{\mathrm{d}x_1^2}\right).$$

Player 1's value function can also be represented implicitly by

$$-4rx_1 = 2\sqrt{2rv_1} - 1 + \log\left(1 - 2\sqrt{2rv_1}\right).$$

We plot the value functions and production rates for both players in Figure 2.2. We note that the non-monotonicities observed for $x_1 = \infty$ do not occur for $x_2 = \infty$. The fact that the non-monotonicities in the solution are present for $x_1 = \infty$ when strategies depend only on x_2, but not here when strategies only depend on x_1, supports the argument that the non-monotonicities in the solution are driven by the ability of both players to dynamically condition their strategy on the resource level of the follower.

We argue that the ability to condition on the follower's resource level is especially important in Stackelberg competition, in that the subsequent optimisation introduces the cross-derivative term $\mathrm{d}v_1/\mathrm{d}x_2$ into the equilibrium strategies; something which does not occur in the Cournot game. When $x_2 = \infty$ and strategies are conditioned only on x_1, the only derivative term which features in the production strategies is the shadow cost $\mathrm{d}v_1/\mathrm{d}x_1$. This is also why we are able to derive an explicit solution for this case but not for the case $x_1 = \infty$. The analysis of the Cournot game in [62] shows that shadow costs are a feature of the equilibrium Cournot strategies as well and in no way unique to Stackelberg games.

2.2.4 The Interior Game

Having considered the limiting cases, we now consider the interior game. Following the same steps as for the limiting case at $x_1 = \infty$; the value function for the follower, $v_2(\mathbf{x}, q_1)$, taking the player 1 strategy q_1 as given, is defined by the optimization problem

$$v_2 = \sup_{q_2 \geq 0} \left\{ \int_0^\infty \mathrm{e}^{-rt} q_{2,t}(1 - q_{1,t} - q_{2,t}) \mathrm{d}t \right\}.$$

Because we assume that the players adopt Markov perfect strategies, we adopt the notation $q_{2,t} = q_2(x_{1,t}, x_{2,t})$. By a standard dynamic programming argument such as that of Appendix B, or [133], this value function must satisfy

$$rv_2 = \sup_{q_2 \geq 0} \left\{ q_2(1 - q_1 - q_2) - q_1 \frac{\partial v_2}{\partial x_1} - q_2 \frac{\partial v_2}{\partial x_2} \right\}.$$

Maximising the above in q_2 then gives the follower's optimal response function in the interior as

$$q_2^* = \frac{1}{2} \left(1 - q_1 - \frac{\partial v_2}{\partial x_2} \right).$$

We can now define the value function for the leader, assuming that the follower reacts optimally to their announced strategy, by the optimisation problem

$$v_1 = \sup_{q_{1,t} \geq 0} \left\{ \int_0^\infty \frac{1}{2} q_{1,t} \left(1 - q_{1,t} + \frac{\partial v_2}{\partial x_2} \right) \mathrm{d}t \right\},$$

which results in

$$rv_1 = \sup_{q_1 \geq 0} \left\{ \frac{1}{2} q_1 \left(1 - q_1 + \frac{\partial v_2}{\partial x_2} \right) - q_1 \frac{\partial v_1}{\partial x_1} - \frac{1}{2} \frac{\partial v_1}{\partial x_2} \left(1 - q_1 - \frac{\partial v_2}{\partial x_2} \right) \right\}.$$

Maximising in q_1 gives

$$q_1^* = \frac{1}{2} \left(1 - 2\frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_2} \right).$$

Substituting the above into the follower's response function to calculate their strategy in equilibrium gives

$$q_2^* = \frac{1}{4} \left(1 - 3\frac{\partial v_2}{\partial x_2} - \frac{\partial v_1}{\partial x_2} + 2\frac{\partial v_1}{\partial x_1} \right).$$

Finally, substituting the expressions for the equilibrium strategies into the equations for the value functions,

$$rv_1 = q_1(1-q_1-q_2) - q_1\frac{\partial v_1}{\partial x_1} - q_2\frac{\partial v_1}{\partial x_2}, \qquad rv_2 = q_2(1-q_1-q_2) - q_1\frac{\partial v_2}{\partial x_1} - q_2\frac{\partial v_2}{\partial x_2},$$

gives the following PDE's in the interior:

$$\begin{aligned} rv_1 =& \frac{1}{8}\left(1-2\frac{\partial v_1}{\partial x_1}+\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_2}\right)\left(1+2\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}-\frac{\partial v_1}{\partial x_2}\right) \\ &-\frac{1}{2}\frac{\partial v_1}{\partial x_1}\left(1-2\frac{\partial v_1}{\partial x_1}+\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_2}\right)-\frac{1}{4}\frac{\partial v_1}{\partial x_2}\left(1-3\frac{\partial v_2}{\partial x_2}-\frac{\partial v_1}{\partial x_2}+2\frac{\partial v_1}{\partial x_1}\right), \end{aligned}$$

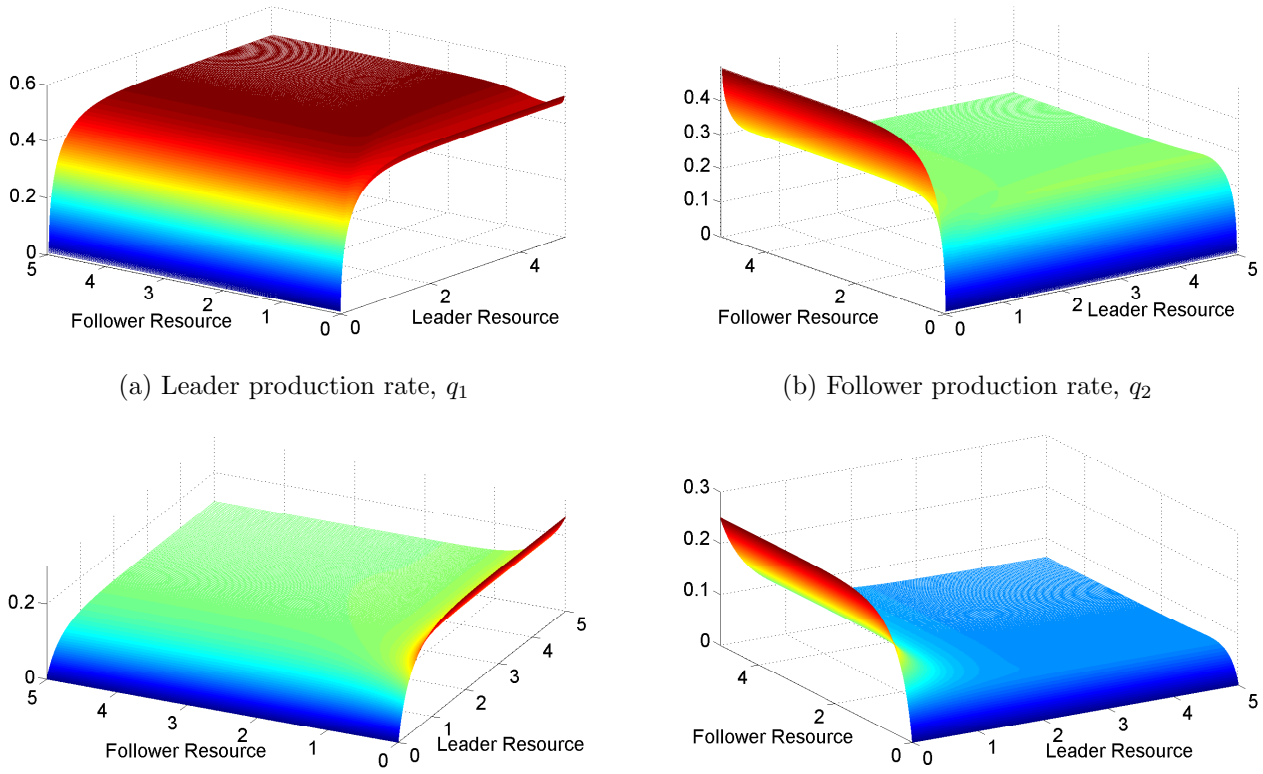
$$\begin{aligned} rv_2 =& \frac{1}{16}\left(1-3\frac{\partial v_2}{\partial x_2}-\frac{\partial v_1}{\partial x_2}+2\frac{\partial v_1}{\partial x_1}\right)\left(1+2\frac{\partial v_1}{\partial x_1}+\frac{\partial v_2}{\partial x_2}-\frac{\partial v_1}{\partial x_2}\right) \\ &-\frac{1}{2}\frac{\partial v_2}{\partial x_1}\left(1-2\frac{\partial v_1}{\partial x_1}+\frac{\partial v_1}{\partial x_2}+\frac{\partial v_2}{\partial x_2}\right)-\frac{1}{4}\frac{\partial v_2}{\partial x_2}\left(1-3\frac{\partial v_2}{\partial x_2}-\frac{\partial v_1}{\partial x_2}+2\frac{\partial v_1}{\partial x_1}\right). \end{aligned}$$

Allowing $\partial v_1/\partial x_1 \to 0$ and $\partial v_2/\partial x_1 \to 0$ as $x_1 \to \infty$ (and $\partial v_1/\partial x_2 \to 0$, $\partial v_2/\partial x_2 \to 0$ as $x_2 \to \infty$) results in the ODE's at $x_1 = \infty$ and $x_2 = \infty$ discussed earlier. Similarly, assuming continuity of the first partial derivatives from the interior to the axis yields the axis ODE's. These ODE's form the boundary conditions of the game.

This system can only be solved numerically, for which we adapt the method in [62]. This method relies on the Open-Loop Stackelberg Equilibrium as a starting point for the numerical scheme, and so we describe its formulation and discuss its convergence in Section 3.4.2 after deriving the Open-Loop solution in Section 3.4.

Figure 2.3 shows the numerical solutions for the production strategies and value functions of both the leader and the follower. We confirm numerically that the price remains higher than player 2's shadow cost so that player 2 is not blockaded in the interior. The production rates for both players, and the follower value function each exhibit non-monotonicity in x_2 so that there are regions of the resource domain in which it is possible for player 2 (1) to produce less (more), and for the follower to receive less value, at a higher follower resource level.

These trends at finite x_1 are similar in nature to those that we have observed and discussed for $x_1 = \infty$. They appear to become more pronounced at higher levels of x_1, which is consistent with our explanation. We have argued that the leader's dip in production for small x_2 is driven by the bringing forward of monopoly profits at the point at which the follower is fully depleted. Given that these profits increase with x_1,



(a) Leader production rate, q_1

(b) Follower production rate, q_2

(c) Leader value function, v_1

(d) Follower value function, v_2

Figure 2.3: Feedback-Loop solution of the Stackelberg duopoly game

it is reasonable that this non-monotonicity becomes more pronounced at higher levels of x_1.

We are limited in the extent to which we can discuss these trends by our reliance on the method of numerical solution. In the following section, we introduce a variant of the model, incorporating a backstop technology, that allows us to derive an analytic approximation of the equilibrium production rates and value functions for low levels of backstop cost.

2.3 Introduction of a Backstop Technology

In this section we introduce a backstop technology of the same type as that used in Example 1.4.1.2. This technology is inexhaustible, and allows either player, at the point

at which they have exhausted their stock of the free-to-extract resource, to continue to produce on the market at cost c per unit produced. We recall from Section 1.4.1 that the inclusion of a backstop was used in the analysis of the analogous Cournot game in [62] to derive an analytic approximation to the solution of the game at low c, and we incorporate it into the Stackelberg game for the same purpose.

Apart from the inclusion of a backstop, we retain all previous assumptions in the model. Given this, player $i \in \{1, 2\}$ seeks to maximise their discounted lifetime profit

$$v_i = \int_0^\infty \mathrm{e}^{-rt} q_{i,t}(1 - q_{1,t} - q_{2,t} - c\mathbf{1}_{\{x_{i,t}=0\}})\mathrm{d}t,$$

where we denote usage of the backstop after depletion by the indicator function $\mathbf{1}_{\{x_{i,t}=0\}}$. As before, we consider the axis games before examining the interior game.

2.3.1 Boundary Conditions and Blockading

When both players are entirely depleted ($x_1 = x_2 = 0$), the game reverts to the inexhaustible form of the game discussed in Example 1.3.2.1, with constant costs of extraction to each player of c. By the same arguments, we therefore have

$$v_1 = \frac{(1-c)^2}{8r}, \qquad v_2 = \frac{(1-c)^2}{16r},$$
$$q_1 = \frac{1-c}{2}, \qquad q_2 = \frac{1-c}{4}.$$

Because both players pay the same extraction costs at the origin, blockading does not occur. Because of the asymmetric nature of Stackelberg competition, the case where $x_2 = 0$ and the follower is depleted needs to be considered separately from the case where $x_1 = 0$ and the leader is depleted.

2.3.1.1 $x_2 = 0$

For $x_2 = 0$, the follower has exhausted their supply of the free resource and can only produce from the costly resource. We use the axis game notation $V_i(x_1) = v_i(x_1, 0)$, $Q_i = q_i(x_1, 0)$. The Bellman equations are given by

$$rV_1 = \sup_{Q_1 \geq 0} \left\{ Q_1 \left(1 - Q_1 - Q_2 - \frac{\mathrm{d}V_1}{\mathrm{d}x_1} \right) \right\},$$
$$rV_2 = \sup_{Q_2 \geq 0} \left\{ Q_2 \left(1 - Q_1 - Q_2 - c \right) - Q_1 \frac{\mathrm{d}V_2}{\mathrm{d}x_1} \right\}.$$

Because the term $Q_2(\mathrm{d}V_2/\mathrm{d}x_2)$ does not enter player 2's maximisation, the maximisation follows the same steps as the inexhaustible case, but with costs $c_1 = \mathrm{d}V_1/\mathrm{d}x_1$, $c_2 = c$. We can therefore determine whether blockading occurs using the same conditions. We note that, because $\mathrm{d}V_1/\mathrm{d}x_1 < c$, it is possible for the leader to blockade the follower on the axis, but not for the follower to blockade the leader. We begin this section by deriving the values of $\mathrm{d}V_1/\mathrm{d}x_1$ for which blockading does or does not occur, and the resultant ODEs in both cases.

Proposition 2.3.1. *When*

$$\max(0, \frac{3c-1}{2}) \leq \frac{\mathrm{d}V_1}{\mathrm{d}x_1} \leq \frac{1}{2}(1+c), \tag{2.3.1}$$

both players produce and the equilibrium value functions and production strategies are

$$rV_1(x_1) = \frac{Q_1^2}{2}, \qquad rV_2(x_1) = Q_2^2 - Q_1\frac{\mathrm{d}V_2}{\mathrm{d}x_1}, \tag{2.3.2}$$
$$Q_1 = \frac{1}{2}\left(1 - 2\frac{\mathrm{d}V_1}{\mathrm{d}x_1} + c\right), \qquad Q_2 = \frac{1}{4}\left(1 - 3c + 2\frac{\mathrm{d}V_1}{\mathrm{d}x_1}\right).$$

Proof. As in the inexhaustible case, the condition (2.3.1) means that both players produce in equilibrium. Given this, the game only differs from the case $x_2 = \infty$ in the game without a backstop in the inclusion of a cost of extraction of c for player 2. Following the same steps results in production strategies:

$$Q_1 = \frac{1}{2}\left(1 - 2\frac{\mathrm{d}V_1}{\mathrm{d}x_1} + c\right), \qquad Q_2 = \frac{1}{4}\left(1 - 3c + 2\frac{\mathrm{d}V_1}{\mathrm{d}x_1}\right).$$

Substituting these into the Bellman equations gives the axis ODEs

$$rV_1 = \frac{1}{8}\left(1 - 2\frac{\mathrm{d}V_1}{\mathrm{d}x_1} + c\right)^2, \qquad rV_2 = \frac{1}{16}\left(1 - 3c + 2\frac{\mathrm{d}V_1}{\mathrm{d}x_1}\right)^2 - \frac{1}{2}\frac{\mathrm{d}V_2}{\mathrm{d}x_1}\left(1 - 2\frac{\mathrm{d}V_1}{\mathrm{d}x_1} + c\right).$$

□

Proposition 2.3.2. *For* $\mathrm{d}V_1/\mathrm{d}x < (3c-1)/2$*, the leader blockades the follower, and we have*

$$rV_1 = Q_1(1-Q_1) - Q_1\frac{\mathrm{d}V_1}{\mathrm{d}x_1}, \qquad rV_2 = -Q_1\frac{\mathrm{d}V_2}{\mathrm{d}x_1}.$$

The production strategy for the leader is given by

$$Q_1(x_1, 0) = \begin{cases} 1-c & \text{for } \max(0, 2c-1) \leq \mathrm{d}V_1/\mathrm{d}x_1 \leq \max(0, (3c-1)/2), \\ \frac{1}{2}\left(1 - \frac{\mathrm{d}V_1}{\mathrm{d}x_1}\right) & \text{for } 0 \leq \mathrm{d}V_1/\mathrm{d}x_1 \leq \max(0, 2c-1). \end{cases}$$

Proof. The case

$$\max(0, 2c-1) \leq \frac{\mathrm{d}V_1}{\mathrm{d}x_1} \leq \max(0, \frac{3c-1}{2}), \tag{2.3.3}$$

corresponds to the inexhaustible case where the leader produces at the artificially high rate of $1-c$ for the purpose of blockading the follower. Similarly, the case

$$0 \leq \frac{\mathrm{d}V_1}{\mathrm{d}x_1} \leq \max(0, 2c-1), \tag{2.3.4}$$

corresponds to the case where the leader's monopoly production rate $((1/2)(1-\mathrm{d}V_1/\mathrm{d}x))$ is sufficiently high to blockade the follower. Incorporating Q_1 as given above and $Q_2 = 0$ into the Bellman equations yields the required ODEs on the axis. □

We now solve explicitly for the player's value functions, and show that whether blockading occurs, and which type of blockading occurs, depends on the cost of production from the backstop and the remaining resource level of the leader. We consider the explicit solution of the game across three different ranges of c: $0 < c \leq 1/3$, $1/3 < c \leq 1/2$, and $1/2 < c \leq 1$.

Proposition 2.3.3. *For the linear pricing function $P(Q) = 1 - Q$, and for $c \leq 1/3$, there is no blockading for finite x_1 when the follower has exhausted their resources. For $c = 1/3$, blockading only occurs asymptotically as $x_1 \to \infty$. On $x_2 = 0$, the value function $V_1(x_1)$ for the leader is implicitly given by*

$$\frac{1}{1+c}\left(2\sqrt{2rV_1} - (1-c)\right) + \log\left(\frac{1+c-2\sqrt{2rV_1}}{2c}\right) = -\frac{2rx_1}{1+c}. \tag{2.3.5}$$

It is concave, increasing, and

$$\lim_{x_1 \to \infty} V_1(x_1) = \frac{(1+c)^2}{8r}.$$

The value function $V_2(x_1)$ of the follower is given by

$$\begin{aligned} V_2(x_1) =& \frac{1}{16r}(1-3c)^2 + \frac{1}{4r}\left(1 - c - \log(c)\,(3c-1)\right)\frac{\mathrm{d}V_1}{\mathrm{d}x_1} \\ &+ \frac{1}{4r}\frac{\mathrm{d}V_1}{\mathrm{d}x_1}\left((3c-1)\log\left(\frac{\mathrm{d}V_1}{\mathrm{d}x_1}\right) - \frac{\mathrm{d}V_1}{\mathrm{d}x_1}\right). \end{aligned} \tag{2.3.6}$$

Proof. Using Lemma 1.4.1 and substituting $\alpha = (1+c)/2$, $\beta = -2c/(1+c)$, $\kappa = 2r$ and $v_{1,0} = (1-c)^2/8r$ gives the equation

$$V_1(x_1) = \frac{(1+c)^2}{8r}\left[1 + \mathbf{W}\left(-\frac{2c}{1+c}\mathrm{e}^{-\frac{2(1+rx_1)}{1+c}}\right)\right]^2,$$

which can be rewritten in the form (2.3.5). To show that blockading does not occur, recall that (2.3.2) can be rewritten as

$$\frac{\mathrm{d}V_1}{\mathrm{d}x_1}(x_1, 0) = \frac{1}{2}\left(1 + c - 2\sqrt{2rV_1}\right).$$

Differentiating the above gives

$$\frac{\mathrm{d}^2V_1}{\mathrm{d}x_1^2} = -\frac{1}{2}\frac{\mathrm{d}V_1}{\mathrm{d}x_1}\sqrt{\frac{2r}{V_1}} < 0,$$

so that $V_1(x_1)$ is increasing and strictly concave in x_1. As $\mathrm{d}V_1/\mathrm{d}x_1(0) = c$, $0 < \mathrm{d}V_1/\mathrm{d}x_1 < c \leq 1/3$ for $c \leq 1/3$. This means that condition (2.3.1) is satisfied and there is no blockading. The ODE for V_2 can then be solved directly by using $\mathrm{d}V_1/\mathrm{d}x_1$ as the independent variable.

□

Proposition 2.3.4. *When $1/3 < c \leq 1/2$, blockading occurs for $x_1 \geq x_{b1}$, where*

$$x_{b1} = \frac{1}{2r}\left(c - 1 + (1 + c)\log\left(\frac{2c}{3c - 1}\right)\right).$$

The leader produces at an artificially high rate in order to blockade the follower from the market, so that

$$Q_1(x_1, 0) = 1 - c, \qquad x_1 \geq x_{b1}. \tag{2.3.7}$$

The value function $V_1(x_1)$ is given implicitly by (2.3.5) for $x_1 \leq x_{b1}$. For $x_1 > x_{b1}$, the leader's value function can be calculated explicitly as

$$V_1(x_1) = \frac{(1 - c)^2}{2r}\mathrm{e}^{-\frac{r(x_1 - x_{b1})}{1 - c}} + \frac{c}{r}(1 - c)\left(1 - \mathrm{e}^{-\frac{-r(x_1 - x_{b1})}{1 - c}}\right). \tag{2.3.8}$$

The value function for the follower for $x_1 > x_{b1}$ is given by

$$\mathrm{e}^{-r\frac{(x_1 - x_b)}{1 - c}}V_2(x_{b1}),$$

where $V_2(x_{b1})$ is given by substituting $\mathrm{d}V_1/\mathrm{d}x_1(x_{b1}, 0) = (3c - 1)/2$ into (2.3.6).

Proof. For $x_1 = x_{b1}$, $\mathrm{d}V_1/\mathrm{d}x_1 = (3c-1)/2$, $Q_2 = 0$, $Q_1 = 1-c$ and $V_1(x_{b1}) = ((1-c)^2)/2r$ by equation (2.3.2). It then follows from the concavity of V_1 that there is no blockading for $x_1 < x_{b1}$, and that $\mathrm{d}V_1/\mathrm{d}x_1 \leq (3c - 1)/2$ for $x_1 \geq x_{b1}$. Further, $c \leq 1/2$ implies that $\mathrm{d}V_1/\mathrm{d}x_1 \geq 0 \geq 2c - 1$ for all $x_1 > 0$, so that the game satisfies condition (2.3.3). We

therefore have that when $1/3 \leq c \leq 1/2$, the leader maintains production at $Q_1 = 1 - c$ for all $x_1 \geq x_{b1}$. Given that we know that player 1 produces at rate $1 - c$ for $x_1 \geq x_{b1}$, we can calculate the time T_{b1} at which the leader reaches x_{b1} as $(x_1 - x_{b1})/(1 - c)$. We can then calculate the value function for the leader with $x_1 \geq x_{b1}$ as

$$V_1(x_1) = \int_0^{T_{b1}} \mathrm{e}^{-rt} c(1-c)\mathrm{d}t + V_1(x_{b1})\mathrm{e}^{-rT_{b1}},$$

which results in equation (2.3.8). The equation for V_2 is found by multiplying the value function of the follower at time T_{b1} by a discounting factor. □

Proposition 2.3.5. *For $c > 1/2$ there are two phases of blockading. For $0 \leq x_1 \leq x_{b_1}$, no blockading occurs, $V_1(x_1)$ is given by* (2.3.5) *and $V_2(x_1)$ is given by* (2.3.6)*. For $0 < x_{b_1} < x_1 \leq x_{b2}$, where*

$$x_{b2} = x_{b1} + \frac{(1-c)}{r} \log\left(\frac{3c-1}{2(2c-1)}\right),$$

the follower is blockaded artificially by the leader, $V_1(x_1)$ and Q_1 are given by (2.3.8) *and* (2.3.7) *respectively. For $x_1 > x_{b2}$, the leader blockades the follower naturally using the monopoly production strategy, such that*

$$2\left(\sqrt{rV_1} - (1-c)\right) + \log\left(\frac{1 - 2\sqrt{rV_1}}{2c-1}\right) = -2r(x_1 - x_{b2}). \tag{2.3.9}$$

For $x_1 > x_{b2}$, the follower's value function is given by

$$V_2(x_1) = \frac{1}{2c-1}\frac{\mathrm{d}V_1}{\mathrm{d}x_1} V_2(x_{b1})\mathrm{e}^{-r\frac{(x_{b2}-x_{b1})}{1-c}},$$

Proof. At the lower blockade point x_{b1}, we still have $\mathrm{d}V_1/\mathrm{d}x_1 = (3c-1)/2$, so that it once again follows from the concavity of V_1 that condition (2.3.3) is met for some range of $x_1 > x_{b1}$, and in this range the leader blockades the follower by producing at the rate $1 - c$ as before.

Using the limiting result that $\mathrm{d}V_1/\mathrm{d}x_1 \to 0$ as $x_1 \to \infty$, there must exist some point $x_{b2} > x_{b1}$ at which

$$\frac{1}{2}\left(1 - \frac{\mathrm{d}V_1}{\mathrm{d}x_1}(x_{b2})\right) = 1 - c,$$

so that $\mathrm{d}V_1/\mathrm{d}x_1(x_{b2}) = 2c - 1$ and condition (2.3.4) is met, and the leader's natural monopoly strategy is sufficient to blockade the follower. Next note that, for values of x_1

below this point, while the leader is producing at an artificially high rate to sustain the blockade, we can calculate $\mathrm{d}V_1/\mathrm{d}x_1$ explicitly as

$$\frac{(3c-1)}{2}\mathrm{e}^{-r(x_1-x_{b1})/(1-c)}.$$

Hence $\mathrm{d}V_1/\mathrm{d}x_1(x_{b2}) = 2c-1$, so that the player's natural monopolist strategy is sufficient to meet the blockading criterion $Q_1 \geq 1-c$, for $x_1 \geq x_{b2}$. The equation (2.3.2) can then be solved with boundary condition $V_1(x_{b2}) = (1-c)^2$ to give (2.3.9). The ODE for V_2 can be solved directly by using $\mathrm{d}V_1/\mathrm{d}x_1$ as the independent variable, with boundary condition $V_2(\mathrm{d}V_1/\mathrm{d}x_1 = 2c-1) = V_2(x_{b1})\exp\left(-r(x_{b2}-x_{b1})/(1-c)\right)$. □

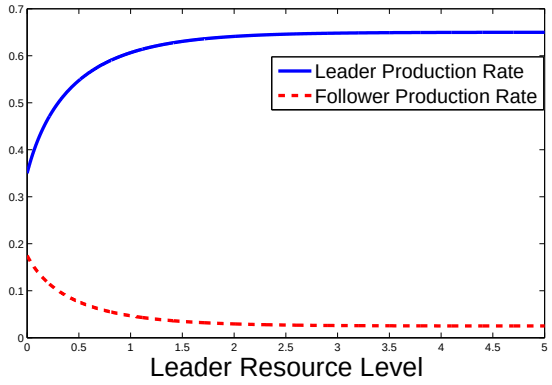
We show the value functions and production strategies for both the leader and follower below in Figure 2.4, for costs $c = 0.3$, $c = 0.4$, and $c = 0.55$ to demonstrate each type of behaviour. When $c = 0.3$, there is no blockading, and the game operates as normal. When $c = 0.4$, blockading occurs for large x_1 with the leader producing at the rate $1-c$ to sustain the blockade. When $c = 0.55$, blockading occurs for moderate x_1 with the leader producing at $1-c$, but for large x_1 the leader's shadow cost becomes sufficiently low that their monopoly rate of production overtakes it, and their production rises with larger x_1 in accordance with the monopolist strategy.

2.3.1.2 $x_1 = 0$

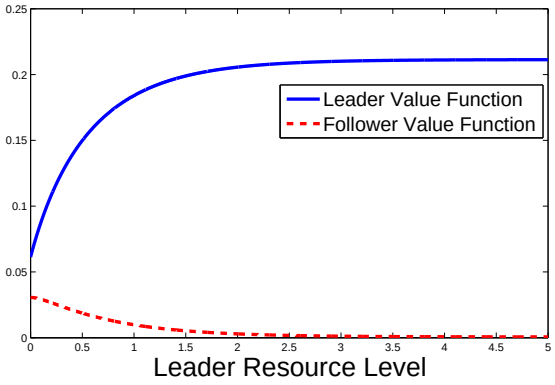
For $x_1 = 0$, the leader has exhausted their supply of the free resource and can only produce from the costly resource. We use the axis game notation $V_i(x_2) = v_i(0, x_2)$, $Q_i(x_2) = q_i(0, x_2)$. Using a similar approach as in the case $x_2 = 0$, it can be shown that the value functions and production strategies for values of x_2 and c at which both players are producing are given by

$$\begin{aligned} rV_1(x_2) &= Q_1(1-Q_1-Q_2-c) - Q_2\frac{\mathrm{d}V_1}{\mathrm{d}x_2}, \\ rV_2(x_2) &= Q_2^2, \\ Q_1 &= \frac{1}{2}\left(1-2c+\frac{\mathrm{d}V_1}{\mathrm{d}x_2}+\frac{\mathrm{d}V_2}{\mathrm{d}x_2}\right), \\ Q_2 &= \frac{1}{4}\left(1-3\frac{\mathrm{d}V_2}{\mathrm{d}x_2}-\frac{\mathrm{d}V_1}{\mathrm{d}x_2}+2c\right), \end{aligned}$$

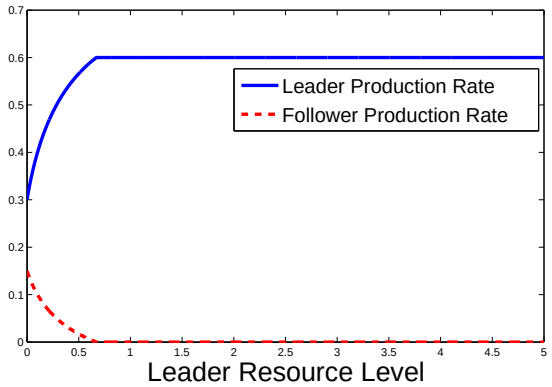
where $\mathrm{d}V_1/\mathrm{d}x_2 = \partial v_1/\partial x_2(0, x_2)$ and $\mathrm{d}V_2/\mathrm{d}x_2 = \partial v_2/\partial x_2(0, x_2)$. For each $c > 1/2$, there is a value x_2^b above which the leader is blockaded from the market. For these values, the value functions and strategies become



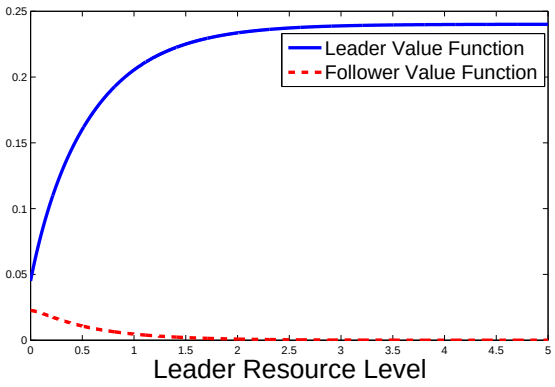
(a) Production rates for $c = 0.3$



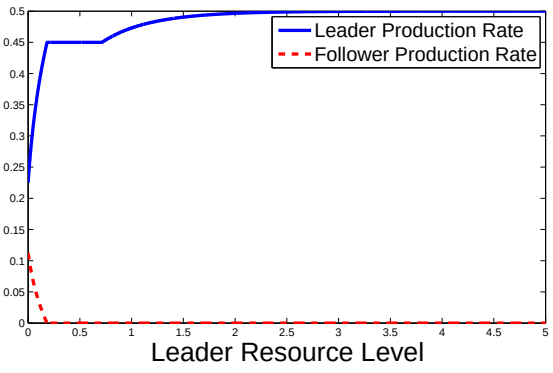
(b) Value functions for $c = 0.3$



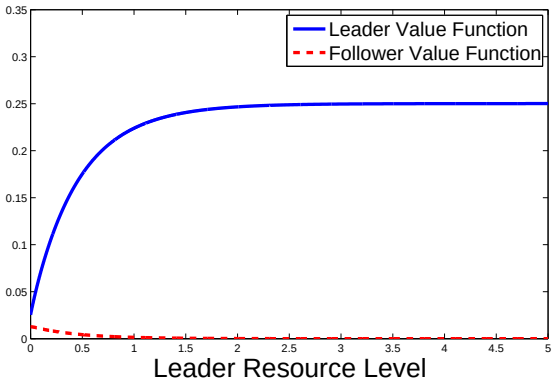
(c) Production rates for $c = 0.4$



(d) Value functions for $c = 0.4$



(e) Production rates for $c = 0.55$



(f) Value Functions for $c = 0.55$

Figure 2.4: Blockading in the axis game $x_2 = 0$

$$
\begin{aligned}
rV_1(x_2) &= -Q_2\frac{\mathrm{d}V_1}{\mathrm{d}x_2},\\
rV_2(x_2) &= Q_2^2,\\
Q_1(x_2) &= 0,\\
Q_2(x_2) &= \frac{1}{2}\left(1-\frac{\mathrm{d}V_2}{\mathrm{d}x_2}\right).
\end{aligned}
$$

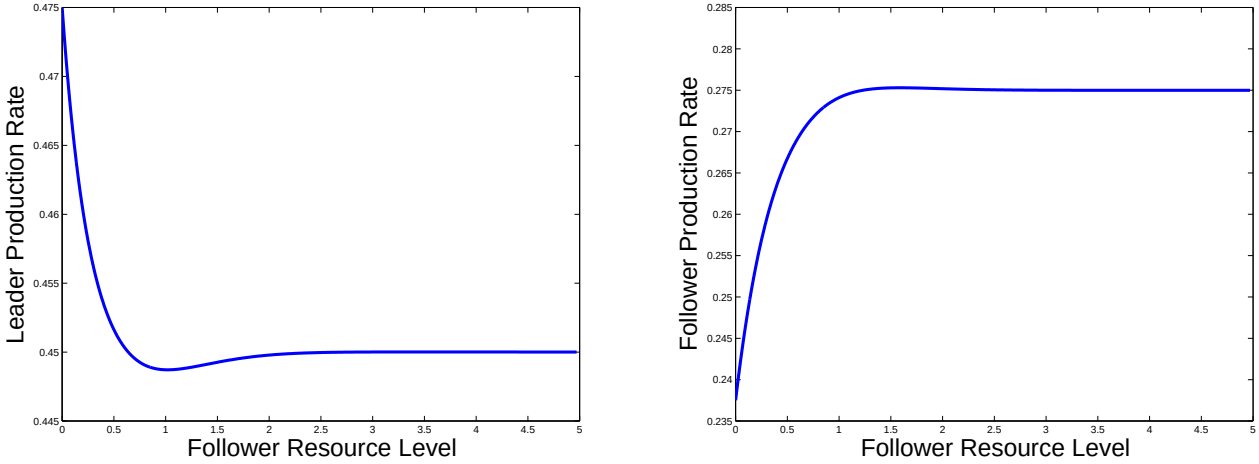
We note that the form taken by the ODEs without blockading is similar to those we encountered for $x_1 = \infty$ in the game without a backstop. The only differences are the inclusion of a cost of production for the leader at rate c, due to the fact that they are here producing from a backstop and not an infinite stockpile of the free resource, and the boundary conditions at $x_2 = 0$. The appearance of the cross derivative term $\mathrm{d}V_1/\mathrm{d}x_2$ in the strategies again complicates the solution of the problem to the point where the differential equations need to be solved numerically. We do this for x_2 increasing from zero, initially ignoring the constraint $q_1 \geq 0$. We identify the x_b at which blockading occurs as the first point at which $q_2 \leq 0$. Beyond this point, the same reasoning as held on the axis $x_2 = 0$ can be used to show that the value functions are given by

$$
\begin{aligned}
2(\sqrt{rV_2} - (1-c)) + \log\left(\frac{1-2\sqrt{rV_2}}{2c-1}\right) &= -2r(x_2 - x_b),\\
\frac{1}{2c-1}\frac{\mathrm{d}V_1}{\mathrm{d}x_2} &= V_2(0, x_b),
\end{aligned}
$$

where x_b is found numerically. Our numerical experiments indicate that blockading does not occur for $c \leq 1/2$, and that $x_b \to \infty$ as c approaches $1/2$ from above. The value functions and production strategies are shown for $c = 0.1$. We note that the non-monotonicity observed in the game without a backstop is again present. This is reasonable: although the leader is producing from a costly backstop in this form of the game, they still have the ability to continuously condition their strategy on the current reserve level of the follower.

2.3.1.3 $x_1 = \infty$, or $x_2 = \infty$

The ODEs for these cases are the same as those derived in the game without a backstop; the backstop only affects the boundary conditions. For $x_1 = \infty$, the players' value functions are functions of x_2 alone and satisfy (2.2.4)-(2.2.5). The boundary conditions



(a) Leader Production Strategy

(b) Follower Production Strategy

Figure 2.5: Production strategies on the axis game $x_1 = 0$ with $c = 0.1$

are given by the solution of the inexhaustible game derived in Example 1.3.2.1 with $c_1 = 0$, $c_2 = c$. These are:

$$v_1(0) = \begin{cases} \frac{1}{8r}(1+c)^2 & \text{for } 0 \le c \le \frac{1}{3}, \\ \frac{1}{r}c(1-c) & \text{for } \frac{1}{3} < c \le \frac{1}{2}, \\ \frac{1}{4r} & \text{for } \frac{1}{2} < c \le 1, \end{cases} \qquad v_2(0) = \begin{cases} \frac{1}{16r}(1-3c)^2 & \text{for } 0 \le c \le \frac{1}{3}, \\ 0 & \text{for } \frac{1}{3} < c \le 1. \end{cases}$$

For $x_2 = \infty$, the players' value functions are functions of x_1 alone and satisfy (2.2.6). The boundary conditions are given by the solution of the inexhaustible game with $c_1 = c$, $c_2 = 0$. These are:

$$v_1(0) = \begin{cases} \frac{1}{8r}(1-2c)^2 & \text{for } 0 \le c \le \frac{1}{2}, \\ 0 & \text{for } \frac{1}{2} < c \le 1, \end{cases} \qquad v_2(0) = \begin{cases} \frac{1}{16r}(1+2c)^2 & \text{for } 0 \le c \le \frac{1}{2}, \\ \frac{1}{4r} & \text{for } \frac{1}{2} < c \le 1. \end{cases}$$

2.3.2 The Interior Game

Having considered the game on the axis, we now consider the interior game. When one of the players runs out, the game reaches the axes, and the value functions and production strategies are given by the axis ODEs. These ODEs therefore form the boundary conditions of the game. Following the same steps as on the axes; taking the player 1 strategy q_1 as given, the value function for the follower, $v_2(\mathbf{x})$, is defined by the optimization problem

$$v_2 = \sup_{q_2 \geq 0} \left\{ \int_0^\infty \mathrm{e}^{-rt} q_{2,t}(1 - q_{1,t} - q_{2,t} - c\mathbf{1}_{\{x_{2,t}=0\}})\mathrm{d}t \right\},$$

where the term $c\mathbf{1}_{\{x_{2,t}=0\}}$ indicates cost of production from the backstop upon depletion of the free resource. We note that we are assuming without proof that the costly backstop technology is not employed until the depletion of the free resource. This is consistent with what is known to happen on the boundary. Numerical evidence also suggests that the shadow costs within the interior remain below the cost of production from the backstop and it is therefore not optimal to produce from the backstop. As this assumption means that the addition of the backstop technology only impacts on the boundary conditions of the game, the interior game with a backstop leads to the same PDEs as in the case without the backstop. We therefore have:

$$rv_1 = q_1(1-q_1-q_2) - q_1\frac{\partial v_1}{\partial x_1} - q_2\frac{\partial v_1}{\partial x_2}, \qquad rv_2 = q_2(1-q_1-q_2) - q_1\frac{\partial v_2}{\partial x_1} - q_2\frac{\partial v_2}{\partial x_2}, \quad (2.3.10)$$

$$q_1 = \frac{1}{2}\left(1 - 2\frac{\partial v_1}{\partial x_1} + \frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_2}\right), \qquad q_2 = \frac{1}{4}\left(1 - 3\frac{\partial v_2}{\partial x_2} - \frac{\partial v_1}{\partial x_2} + 2\frac{\partial v_1}{\partial x_1}\right). \quad (2.3.11)$$

Proposition 2.3.6. *If we assume that the production strategies and value functions are continuous up to the axis, then when $c < 1/3$, and blockading does not occur on the boundary $x_2 = 0$, the following Neumann conditions also follow:*

$$\frac{\partial v_1}{\partial x_2} = 0 \quad \textit{on } x_2 = 0, \quad (2.3.12)$$

$$\frac{\partial v_2}{\partial x_2} = c \quad \textit{on } x_2 = 0. \quad (2.3.13)$$

Proof. The Bellman equations derived in the interior and on the axis $x_2 = 0$, and the assumption of continuity of the value functions for all x_1, $x_2 \geq 0$ implies that

$$\lim_{x_2 \to 0} \left\{ q_1(1 - q_1 - q_2) - q_1\frac{\partial v_1}{\partial x_1} - q_2\frac{\partial v_1}{\partial x_2} \right\} = q_1(1 - q_1 - q_2) - q_1\frac{\mathrm{d}V_1}{\mathrm{d}x_1},$$

$$\lim_{x_2 \to 0} \left\{ q_2(1 - q_1 - q_2) - q_1\frac{\partial v_2}{\partial x_1} - q_2\frac{\partial v_2}{\partial x_2} \right\} = q_2(1 - q_1 - q_2 - c) - q_1\frac{\mathrm{d}V_2}{\mathrm{d}x_1}.$$

The result then follows immediately from our earlier assumption that both the production strategies and the first derivatives of the value functions are continuous up to the axis. □

When $c < 1/2$, and blockading does not occur on the boundary $x_1 = 0$, then the following Neumann boundary conditions follow by the same reasoning as above:

$$\frac{\partial v_1}{\partial x_1} = c \ \text{ on } x_1 = 0, \tag{2.3.14}$$

$$\frac{\partial v_2}{\partial x_1} = 0 \ \text{ on } x_1 = 0. \tag{2.3.15}$$

2.4 Small c Approximation

As we have shown above, for small values of c the game approaches the inexhaustible case, and as $c < 1/3$ both players participate at all resource levels on the boundary. We assume the same occurs in the interior, which is supported by numerical evidence. In this section, we follow a method introduced in [62] and construct an analytic approximation to the solution of the game for small c by constructing an asymptotic expansion. We look for an approximate solution of the system (2.3.10)-(2.3.11), with Neumann boundary conditions (2.3.12), (2.3.13), (2.3.14), and (2.3.15), of the form

$$v_1(\mathbf{x}) = \frac{1}{8r} + cv_1^{(1)}(\mathbf{x}) + o(c), \qquad v_2(\mathbf{x}) = \frac{1}{16r} + cv_2^{(1)}(\mathbf{x}) + o(c),$$

for some functions $v_1^{(1)}$, $v_2^{(1)}$ to be found. Inserting this expansion into the PDEs and Neumann boundary conditions results in the linearised system

$$\begin{aligned} rv_1^{(1)} &= \frac{1}{4}\frac{\partial v_2^{(1)}}{\partial x_2} - \frac{1}{2}\frac{\partial v_1^{(1)}}{\partial x_1} - \frac{1}{4}\frac{\partial v_1^{(1)}}{\partial x_2}, \\ rv_2^{(1)} &= \frac{1}{4}\frac{\partial v_1^{(1)}}{\partial x_1} - \frac{1}{8}\frac{\partial v_1^{(1)}}{\partial x_2} - \frac{1}{2}\frac{\partial v_2^{(1)}}{\partial x_1} - \frac{3}{8}\frac{\partial v_2^{(1)}}{\partial x_2}, \end{aligned}$$

$$\begin{aligned} \left.\frac{\partial v_1^{(1)}}{\partial x_1}\right|_{x_1=0} &= \left.\frac{\partial v_2^{(1)}}{\partial x_2}\right|_{x_2=0} = c, \\ \left.\frac{\partial v_1^{(1)}}{\partial x_2}\right|_{x_2=0} &= \left.\frac{\partial v_2^{(1)}}{\partial x_1}\right|_{x_1=0} = 0. \end{aligned}$$

Proposition 2.4.1. *The system admits the explicit solution*

$$v_1^{(1)}(x_1, x_2) = \frac{-\mathrm{e}^{-2x_1 r}}{2r} + \frac{\mathrm{e}^{-\frac{5rx_2}{2}}}{4r}\left(\frac{5}{\sqrt{7}}\sin\left(\frac{rx_2\sqrt{7}}{2}\right) + \cos\left(\frac{rx_2\sqrt{7}}{2}\right)\right),$$

$$v_2^{(1)}(x_1, x_2) = \mathrm{e}^{-2x_1 r}\left(\frac{x_1}{2} + \frac{1}{4r}\right) + \frac{\mathrm{e}^{-\frac{5rx_2}{2}}}{16r}\left(\frac{2\sqrt{7}}{7}\sin\left(\frac{rx_2\sqrt{7}}{2}\right) - 6\cos\left(\frac{rx_2\sqrt{7}}{2}\right)\right).$$

Proof. Making the additively separable *ansatz*

$$v_1^{(1)}(\mathbf{x}) = f_1(x_1) + g_1(x_2),$$
$$v_2^{(1)}(\mathbf{x}) = f_2(x_1) + g_2(x_2),$$

for some functions f_1, f_2, g_1, and g_2 allows the solution to follow immediately by substituting into (2.3.10)-(2.3.11). As an additional check, we can directly calculate the following limiting cases:

$$\begin{aligned}
\lim_{x_1\to\infty} v_1^{(1)}(x_1,0) &= \frac{1}{4r}, & \lim_{x_1\to\infty} v_2^{(1)}(x_1,0) &= -\frac{3}{8r},\\
\lim_{x_2\to\infty} v_1^{(1)}(0,x_2) &= -\frac{1}{2r}, & \lim_{x_2\to\infty} v_2^{(1)}(0,x_2) &= \frac{1}{4r},\\
\lim_{x_1,x_2\to\infty} v_1^{(1)}(x_1,x_2) &= 0, & \lim_{x_1,x_2\to\infty} v_2^{(1)}(x_1,x_2) &= 0,\\
v_1^{(1)}(0,0) &= -\frac{1}{4r}, & v_2^{(1)}(0,0) &= -\frac{1}{8r},
\end{aligned}$$

which in each case match the value obtained for $v_1^{(1)}$ and $v_2^{(1)}$ by calculating the value functions of the corresponding inexhaustible infinite horizon games explicitly and isolating the coefficients of terms of order c.

□

The production strategies are given by

$$q_1 = \frac{1}{2} + \frac{c}{2}\left(-2\mathrm{e}^{-2rx_1} - \mathrm{e}^{-\frac{5rx_2}{2}}\left(\frac{3\sqrt{7}}{7}\sin\left(\frac{rx_2\sqrt{7}}{7}\right) - \cos\left(\frac{rx_2\sqrt{7}}{7}\right)\right)\right) + o(c)$$

$$q_2 = \frac{1}{4} + \frac{c}{4}\left(2\mathrm{e}^{-2rx_1} + \mathrm{e}^{-\frac{5rx_2}{2}}\left(\frac{\sqrt{7}}{7}\sin\left(\frac{rx_2\sqrt{7}}{7}\right) - 3\cos\left(\frac{rx_2\sqrt{7}}{7}\right)\right)\right) + o(c)$$

2.4.1 Discussion of the Asymptotic Approximation

We plot $v_1^{(1)}$, $v_2^{(1)}$, $q_1^{(1)}$, and $q_2^{(1)}$ in Figure 2.6, for $0 \le x_1 \le 5$, $0 \le x_2 \le 5$. Because the solution we are linearising about at $c = 0$ corresponds to the inexhaustible case, we can use the asymptotic correction as a measure for the marginal impact of exhaustibility. We observe the following immediate effects:

- For each player, the effect of exhaustibility on both production and net present value is largely negative. In the case of the value functions, this is driven by the

impact of the additional cost of post-depletion extraction on the value functions. For the production rate, the additional post-depletion cost adds a scarcity value to the free resource, which drives the player to lower current production to delay depletion.

- Each player's production rate and net present value is higher under exhaustibility for regions in which their level of the free resource is significantly higher than that of the other player. That is because, while the player too is impacted by the greater cost of technology on exhaustion in the long term, the nearer-term impact is that of the other player having to pay higher extraction costs on depletion. As discussed previously, the effect of this is to lower the other player's production rate. This decrease in competition allows the player with larger reserves to profitably expand present production, and increases the net present value of their reserves.

- As x_1 and x_2 both become very large, the impact on both the value functions and production rates declines to zero. This is because the game approaches the inexhaustible case as the two players' reserves become larger.

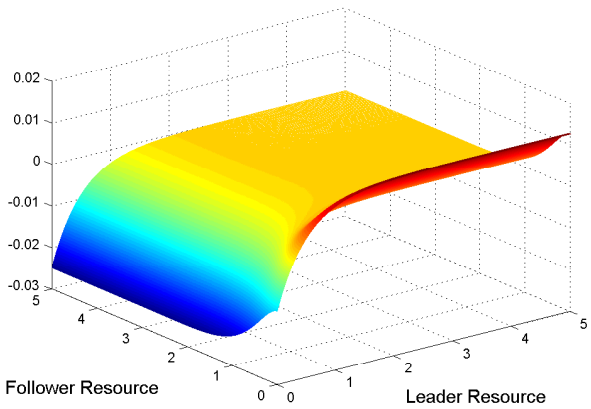
We see regions in which both players decrease their production, or one player increases production while the other decreases. These regions are shown below, labelled as:

- **I:** Production is lower for the leader, but higher for the follower;

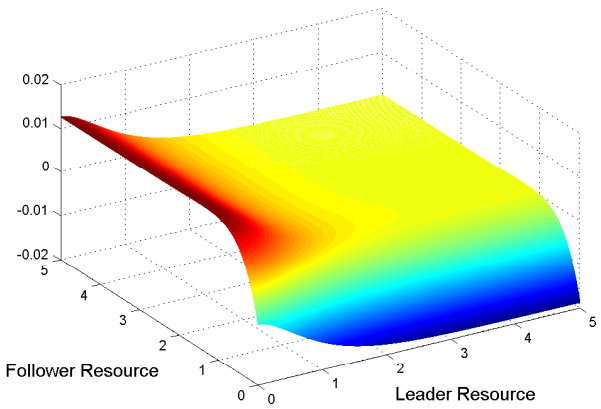
- **II:** Production is lower for both the leader and the follower;

- **III:** Production is higher for the leader, but lower for the follower.

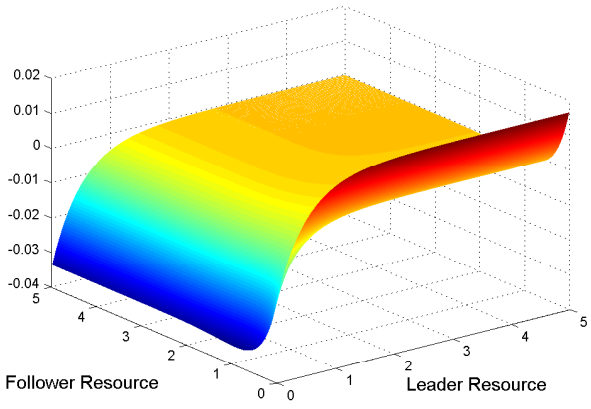
The asymmetry of the regions is an interesting feature of Stackelberg competition. It arises because the leader in the game has a tendency towards higher production rates, meaning that exhaustibility is more likely to result in a decrease in production in a relatively large area of the grid. In the case of the follower, exhaustibility increases production rates and value within a relatively large region because the economic cost of production from the expensive resource on depletion is outweighed by the benefit of the increase in current market price resulting from reduced production from the market-dominating leader.



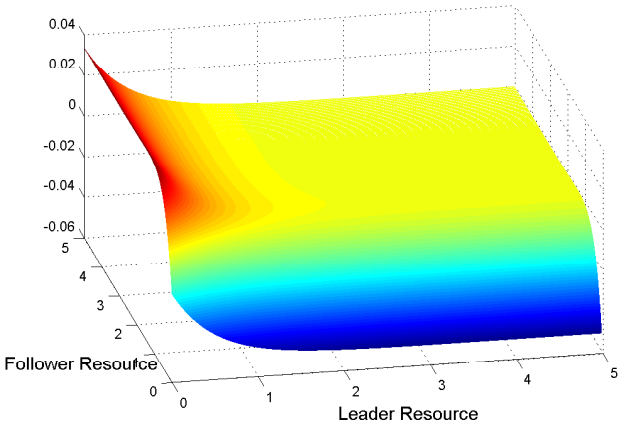
(a) $v_1^{(1)}$



(b) $v_2^{(1)}$

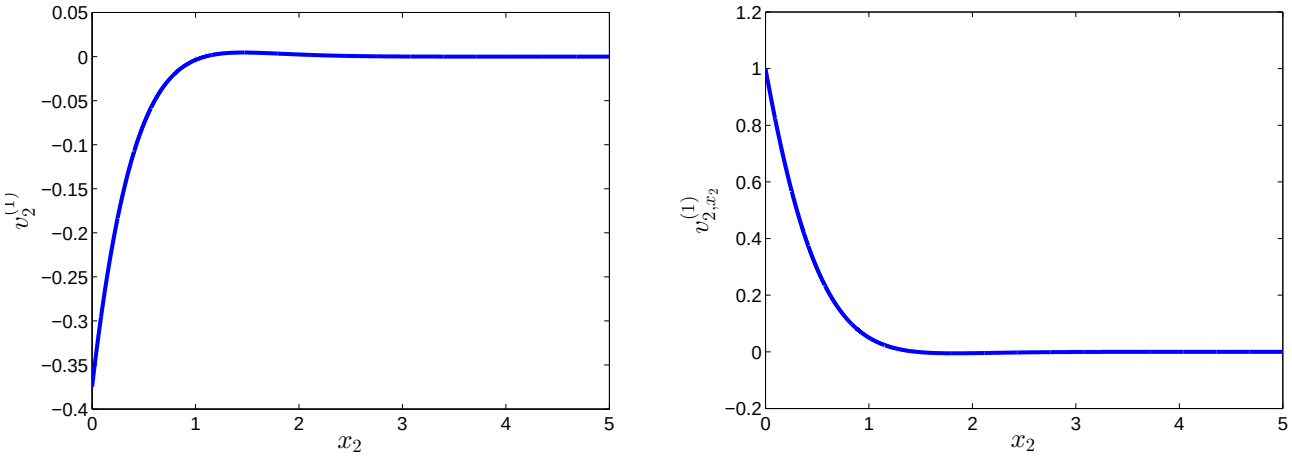


(c) $q_1^{(1)}$



(d) $q_2^{(1)}$

Figure 2.6: Asymptotic correction for small c



(a) $v_2^{(1)}$ (b) $v_{2,x_2}^{(1)}$

Figure 2.7: Player 2 asymptotic correction for $x_1 = \infty$

Finally, we highlight that the non-monotonicities we discussed in the one-dimensional case are present here as well. In Figure 2.7 we plot $v_2^{(1)}$ and $v_{2,x_2}^{(1)}$ for $x_1 = \infty$ to provide a clearer picture of the non-monotonicity of v_2 in x_2. This behaviour can also be confirmed analytically. Calculating the first derivative of $v_2^{(1)}$ in x_2 gives

$$v_{2,x_2}^{(1)} = \mathrm{e}^{-\frac{5rx_2}{2}} \left(\frac{\sqrt{7}}{7} \sin\left(\frac{rx_2\sqrt{7}}{2} \right) + \cos\left(\frac{rx_2\sqrt{7}}{2} \right) \right).$$

This equation has multiple turning points. Because $v_{2,x_2}^{(1)}$ is independent of x_1, the turning points are given by

$$x_2 = \frac{2}{r\sqrt{7}} \tan^{-1}\left(-1\sqrt{7}\right) + \frac{2n\pi}{r\sqrt{7}}, \qquad n \geq 1,$$

so that $v_2^{(1)}$ is non-monotonic in x_2.

Looking at the form of the analytic approximation for both the production and value functions supports several hypothesis we had developed in the game without the backstop:

- The first derivatives of the value functions in x_1 are given by

$$v_{1,x_1}^{(1)} = \mathrm{e}^{-2x_1 r},$$
$$v_{2,x_1}^{(1)} = -rx_1\mathrm{e}^{-2rx_1}.$$

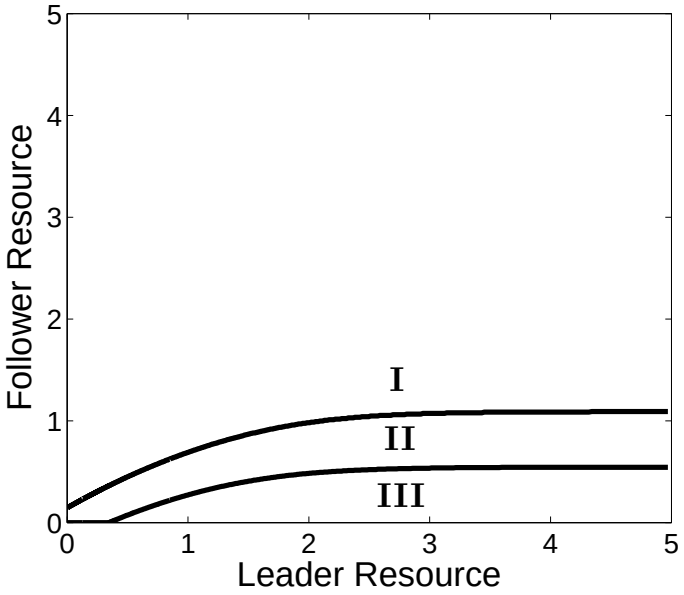


Figure 2.8: Sign of first order exhaustibility correction

Because $v^{(1)}_{1,x_1} > 0$, and $v^{(1)}_{2,x_1} < 0$ for all $x_1 > 0$, the non-monotonicity exists only in x_2, the resource level of the follower.

- The multiplication of the sin and cos functions by $\exp(-5rx_2/2)$ implies that the non-monotonicity becomes more pronounced as x_2 decreases.

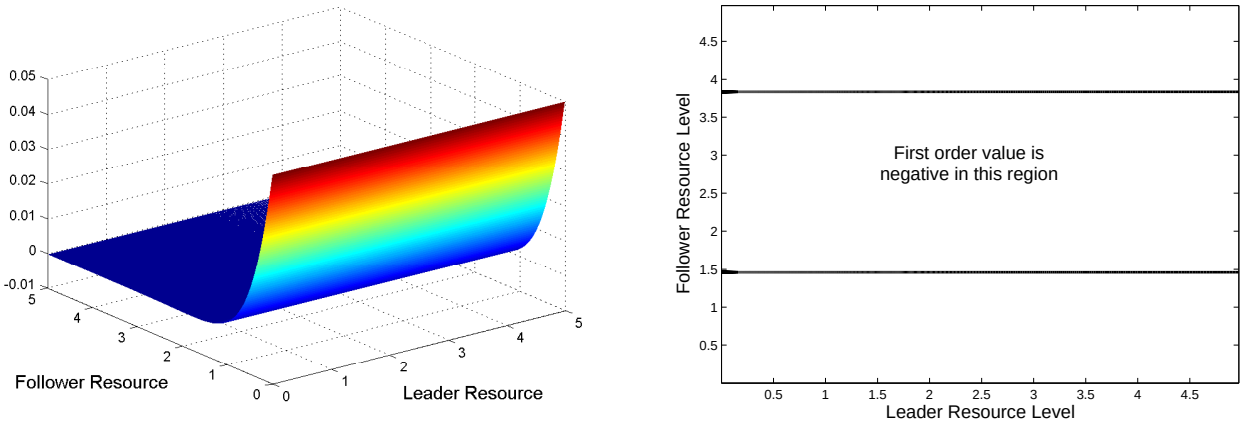
- The approximation is also consistent with the results of our earlier analysis of the two axis games with a backstop for low c.

We also note that the non-monotonicity w.r.t. x_2 as $x_1 \to \infty$ is less pronounced in magnitude than in the one-dimensional case of the game without a backstop. This is consistent with our hypothesis that the oscillations are due to the leader switching between a strategy of allowing the follower to deplete more rapidly for the purpose of then operating with less competition; and that of driving down the follower's production to increase revenue in the short term.

Intuitively, we would expect these variations to grow larger with the difference in the degree of competition faced by the leader from the follower, before and after the follower's total depletion of their free resource. The case here, in which there is a viable backstop that can be used after the follower's depletion represents a less extreme difference than

in the one-dimensional case, and we would therefore expect a less noticeable variation in the leader's production as a result.

Figure 2.9b plots changes in sign of $\partial v_2^{(1)}/\partial x_2$, and as we can see below, $\partial v_2^{(1)}/\partial x_2$ becomes negative in the middle region.



(a) $\partial v_2^{(1)}/\partial x_2$ (b) Sign changes of $\partial v_2^{(1)}/\partial x_2$

Figure 2.9: First-order approximation of player 2 cross-derivative

2.4.2 Numerical Solution at low c

The complexity of the equations (2.3.10)-(2.3.11) makes the numerical solution highly non-trivial. In particular, the existence of blockading where $1/3 < c < 1$ complicates the solution greatly. For low c, when both players produce at all points on the axes, and there is no blockading, we can construct a numerical solution, using the asymptotic expansion as a starting point. The method used is similar to that used in the Cournot game of [62]; we outline the key features below:

- We have solved for the steady state of the time-dependent stochastic analogue of the differential game, adding a small Laplacian term to the equations to eliminate grid-scale oscillations and obtain solutions with smooth first derivatives. This involves using the asymptotic expansion as a starting point and embedding the PDEs to be solved as the steady-state of a smoothed dynamical system for the

residual $R_i^\epsilon = v_i - (v_i^{(0)} + cv_i^{(1)})$. We then evolve to large time τ the system

$$\frac{\partial R_i^\epsilon}{\partial \tau} = q_i(1 - q_1 - q_2) - q_1 \frac{\partial v_i}{\partial x_1} - q_2 \frac{\partial v_i}{\partial x_2} - rv_i + \epsilon \Delta R_i^\epsilon \qquad i = 1, 2, \tag{2.4.1}$$

where q_1 and q_2 are given in (2.3.11) as before, and Δ is the Laplacian operator. We select the parameters ϵ and τ in line with [62]. We choose the diffusion coefficient ϵ to be of size $O(\delta x)$ for the purpose of adding stability to the scheme, and use a time step of size $\tau = 0.06$.

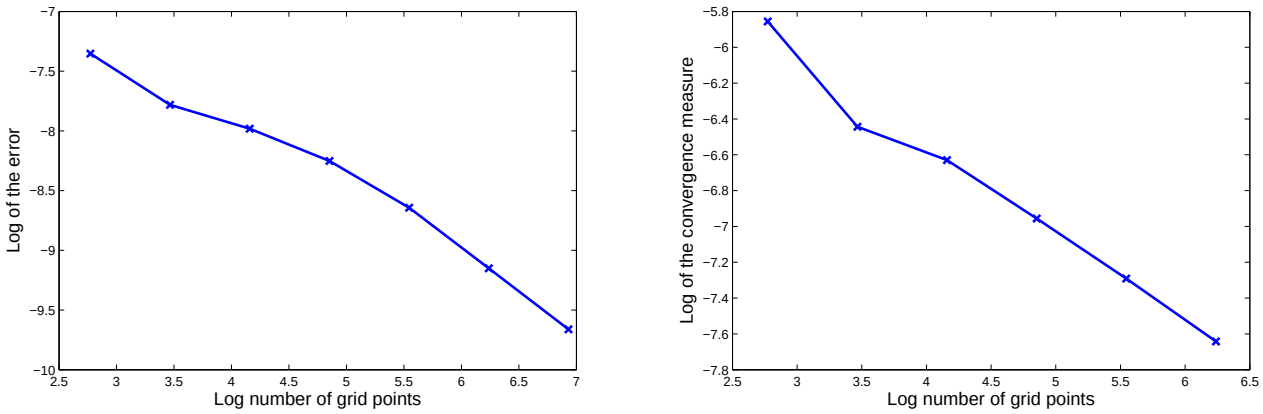
- We use a non-uniform grid to put greater resolution near the axes. We use central finite-differences, with second-order one-sided differences at the axes and far edges of the grid. To calculate the finite-differences, we transform the grid to a uniform one, and calculate the central finite-differences on the transformed grid. We use zero Neumann boundary conditions at the far edges of the computational grid to capture the asymptotic behaviour of the solutions as $x_1 \to \infty$, $x_2 \to \infty$. We note that this assumption requires the maximum values of rx_1 and rx_2 to be sufficiently high so as to make the condition appropriate.

- In tracking convergence within the numerical scheme, we define $E_i(j)$ as the ∞-norm of the residual at each point of the grid, given by substituting the scheme's j-th iteration values into the Bellman equation for player i, and $E(i) = \max(E_1(i), E_2(i))$. We also define $E_t(i) = (E(i) - E(i-1))/\tau E(i-1)$.

- The scheme runs to a minimum number of iterations, set to $10N$. The scheme terminates after either $40N$ iterations have completed, or until $|E_t(i)/E_i| < C$, where $C = 0.1c^2/(N-1)$ is a tolerance threshold.

Our numerical results indicate that the error decreases at rate $O(N^{-\frac{2}{3}})$ under our adjusted method.[4] We assess convergence for a grid size of $O(N)$, where N is the number of grid points in one direction of the grid, in two ways. Our primary measure of convergence is the value of the residual E on termination, which must vanish as $N \to \infty$. Figure 2.10a shows this error measure on a natural log scale. We also calculate the rate at which the solutions on successively refined grids converge, which we show on a natural log scale in Figure 2.10b. For each data point on a grid of size N, we calculate $v_{1,2N}$,

[4] This matches the convergence rate arising from the Cournot numerical scheme in [62].



(a) Log Error

(b) Convergence measure on a natural log scale

Figure 2.10: Convergence of the numerical method

the value function calculated with a grid size of $2N$ interpolated to fit the grid of size N, and $v_{1,N}$, the value function calculated with a grid size N. We then calculate

$$\text{Convergence} = \frac{\|v_{1,2N} - v_{1,N}\|_\infty}{\|v_{1,N}\|_\infty}, \tag{2.4.2}$$

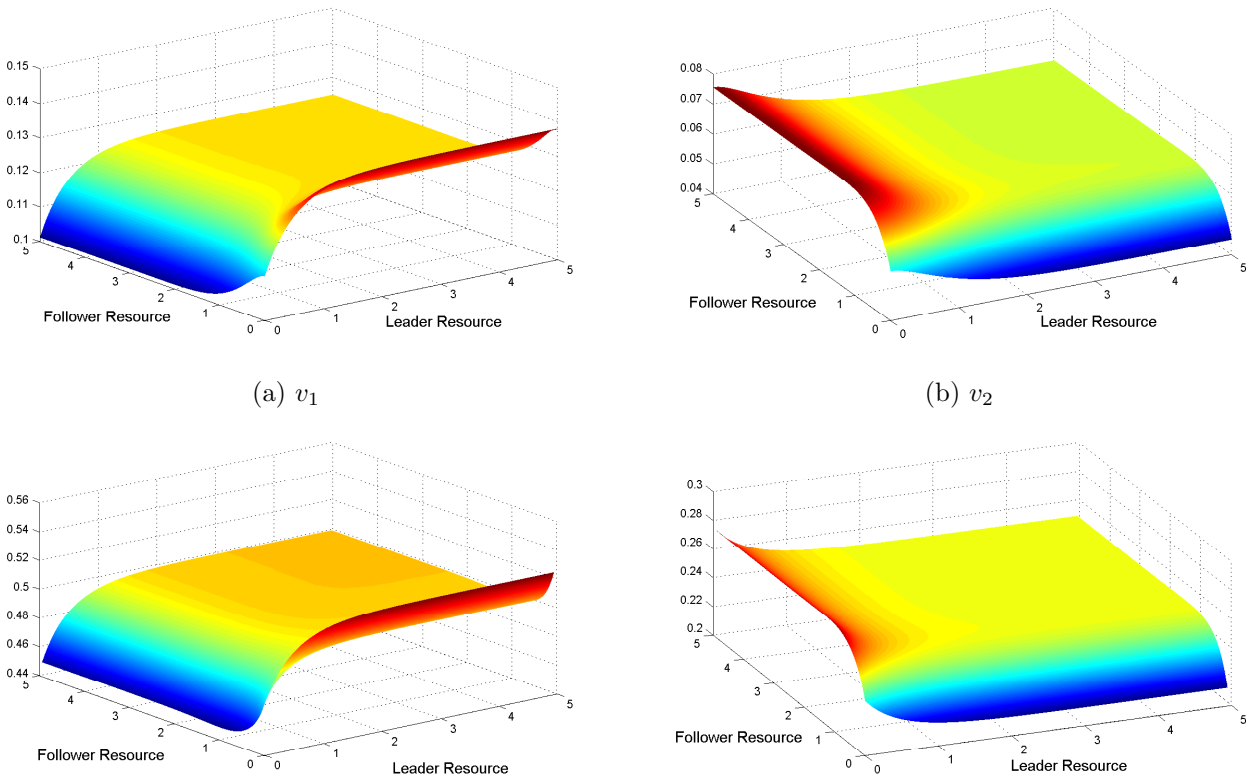
as a measure of relative convergence. This measure decreases at approximately the rate $O(N^{-\frac{1}{2}})$.[5]

2.4.3 Discussion of the Numerical Solution at low c

We show the numerical solution for $c = 0.1$ in Figures 2.11-2.12. Numerically, we solve for the value functions (v_1, v_2) and check that the shadow costs $(\partial v_1/\partial x_1, \partial v_2/\partial x_2)$ calculated from finite differencing do not exceed c. Analysis of the ODEs in Section 2.3.1.3 also confirms that blockading does not occur within the interior for $x_1 = \infty$ and $x_2 = \infty$. We confirm that the features discussed in the analysis of the asymptotic approximation are present here as well. We note in particular that the full numerical solution has even more noticeable non-monotonicity in x_2 for both v_1 and v_2.

We also make note of an interesting trend in the impact of exhaustibility on price. While the impact tends to be greatest for low resource levels, we can observe a small region close to the $x_2 = 0$ axis in which $\partial P/\partial x_2 > 0$, where P is the market price of the

[5]This measure is not used in [62], so we cannot comment on its value.



(a) v_1 (b) v_2

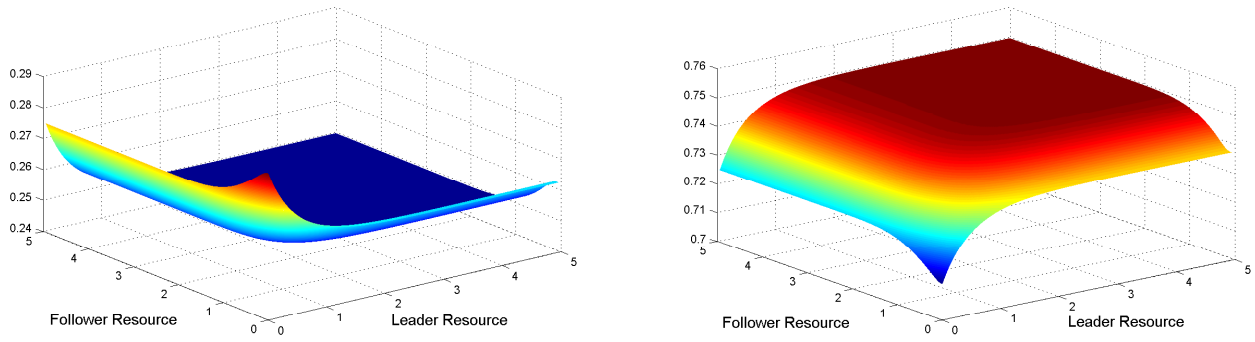
(c) q_1 (d) q_2

Figure 2.11: Numerical Solution for $c = 0.1$

resource. While it seems counterintuitive that the price could increase with an increase of the total level of available resources in the market, our hypothesis is that player 2's higher production as a result of having a greater resource stock is coupled with a disproportionately high decrease in production from the leader. This is consistent with our conjecture that the leader sets their production low to encourage the follower to set theirs higher, with the purpose of moving forward the point at which the follower is depleted and can only compete using the costly backstop.

2.5 Conclusion

We have derived analytic approximations to the solution of a two-player, two-dimensional Stackelberg resource extraction game with Markov perfect strategies, and used that to



(a) Price (b) Total Production

Figure 2.12: Equilibrium price and total production for $c = 0.1$

derive a numerical solution of the game. We have observed that the solution shows regions in which the value function to the follower is non-monotonic in its resource level, that is, that there are regions for which an increase in the follower's resource level results in a decrease in their value function. This is a highly counter-intuitive violation of the free disposal criterion, in that it implies that the player would be better off dumping the resource, were they given that opportunity. We have verified this result analytically in two ways: firstly, via analysis of the one-dimensional analogue of the game; secondly, for the game with a backstop, through the construction of a closed-form asymptotic approximation in c, of the game's value and production functions.

We argue that there is a parallel between the non-monotonicity found in this case, and that demonstrated in [14]. There, it is shown that members of a temporary cartel can receive less value from a higher quantity of resources if that makes competition after the dissolution of the cartel more aggressive. In other words, the decrease in value does not arise from the additional resource in itself, but from this inducing a value-eroding change in the behaviour of the competing players.

In this model, we have similarly argued that the non-monotonicities in the game are a result of the additional resource inducing a change in the strategy adopted toward the follower by the leader. For sufficiently low levels of reserve, it may be worthwhile for the leader to lower their production, if doing so means that the follower advances theirs and depletes their reserve faster, leaving the leader free to produce as a monopolist. When the follower is comparatively better stocked, discounting makes this consideration

less relevant, and the leader drives up production to capture more market share in the immediate term. This behaviour is highly singular and an ongoing research question is whether similar non-monotonicities can be found in simpler Stackelberg games.

In our analysis of the games on the boundary axes, we have also shown that the presence of a backstop technology can result in regions in which the leader drives up production in order to artificially blockade the follower. These regions result in phases of level production from the leader despite a decreasing resource level. The behaviour of trajectories in the interior as they approach these regions is unclear and is an area of future research.

Chapter 3

A Comparison of Open-Loop and Feedback-Loop Games under Cournot and Stackelberg competitions

3.1 Introduction

We discussed the importance of the period of commitment, and the divide in the literature between Open-Loop and Feedback-Loop models in Section 1.5. The two models imply different assumptions about the period of commitment in the game, in turn requiring the players to make different assumptions about the future behaviour of competing players in light of the available information. Moreover, we demonstrated in Sections 1.5.2.1 and 1.5.2.3 that the two models require different solution methods.

While the Open-Loop and Feedback-Loop versions of the isoelastic Cournot game in Examples 1.5.2.1 and 1.5.2.3 resulted in the same solution, it is shown in [44] that this can only be expected to hold when the equilibrium solutions for each player under both models are a function of their resource level alone. In other words, a player's production strategy must depend only on their own resource level, and be independent of the resource levels of their competitors. This property tends to be rare, and our result in Examples 1.5.2.1 and 1.5.2.3 is primarily a consequence of the singular properties of the isoelastic price function, as desribed in Section 1.2.2.2.

Where equilibrium solutions involve dependence on the resource levels of all players, the Open-Loop and Feedback-Loop solutions can generally be expected to differ. Com-

mon property resource games as introduced in Section 1.4.2.2, are analysed in [118], and it is shown that the period of commitment can have a highly significant impact on the production paths within the game. However, [118] relies heavily on the assumption of symmetry of the game's solution across the players to derive its conclusions.[1]

In separate pools games, introduced in Section 1.4.2.1, the determination of the information set is especially important. It not only influences the solution of the game, but also does so in a way that depends on the players' resource levels, thereby impacting the evolution of relative production and resource concentration in the market over time. In [44], a discrete separate pools Cournot model with linear demand is tested at several resource allocations to demonstrate that the Open-Loop and Feedback-Loop models differ, as a counter-example to the isoelastic case it also considers. We extend on this work by considering a continuous time model and a full resource grid with the aim of identifying a relationship between the difference in the Open-Loop and Feedback-Loop production and value functions, and the relative resource levels of the two players.

We specify and solve an Open-Loop duopoly game under both Cournot and Stackelberg competitions in closed form. We also solve each game numerically under a Feedback-Loop setting and examine the difference in player strategies and value functions under each extreme type of commitment period. We show that not only do the solutions exhibit differences, but also that the period of commitment impacts asymmetrically on the players when there is asymmetry in the game, whether in the initial allocation of resources across players, or in the presence of an asymmetric form of market competition. In doing so, we exploit the theory of papers such as [146] and [57] to derive closed-form expressions for the Open-Loop solutions of the games, and the work in [62] whose numerical techniques we use in calculating the Feedback-Loop solutions.

Under Cournot competition, we show that where there is significant asymmetry in the allocation of resources, the player with the larger pool is better off under the Feedback-Loop game and produces more, while the player with the smaller pool produces less and is worse off. Under Stackelberg competition, we show that the opposite holds of the production rates. Furthermore, while there is generally little difference in the value functions, both players are better off under the Feedback-Loop game for resource allocations in which the Stackelberg leader has a relatively high resource level compared

[1] The validity of the findings of [118] are further examined in Chapter 4.

to the follower. These are also the resource allocations in which the asymmetry of market influence implied by Stackelberg competition is most plausible.

In Section 3.2 we introduce the model in more detail. In Section 3.3 we examine the asymmetric impact of the period of commitment under Cournot competition. In Section 3.3.4 we consider the results of Section 3.3 in the limiting case in which one player has an infinite supply of the resource, and we can develop closed-form solutions for both types of game. In Section 3.4 we apply the analysis in Section 3.3 to Stackelberg competition, which is in itself an asymmetric form of market competition. We conclude in Section 3.5.

3.2 The Model

We assume a non-cooperating duopoly market. Cournot competition, defined in Section 1.3.1, is considered in Section 3.3 and Stackelberg competition, defined in Section 1.3.2, is considered in Section 3.4. We assume the linear demand curve of (1.3.1) throughout this chapter, so that $P = 1 - Q$. We use the formulation of Open-Loop and Feedback-Loop games provided in Section 1.5. Both types of models are discussed in this chapter; we use superscript notation o and f to denote variables under the Open-Loop and Feedback-Loop game respectively. For statements which apply to both models, we drop the superscript notation. We recall that the extraction of the resource means that $x_{i,t}$ depletes according to

$$\frac{\mathrm{d}x_{i,t}}{\mathrm{d}t} = -q_{i,t}, \qquad i = 1, 2, \tag{3.2.1}$$

with the constraint $x_{i,t} \geq 0$ arising from the exhaustibility of the resource. We assume that, given an initial resource allocation such that players 1 and 2 have x_1 and x_2 units respectively, player i seeks to maximise v_i, the discounted present value of their total profits over the life of the resource stock, and that each player applies the same constant discount rate r in their calculations.

This ultimately becomes an optimal control problem, solved for player i by the selection of the production rates, $q_{i,t}$, at time t. Depending on whether or not one or both players have run out, we note that the game at any point operates in one of the following cases:

- **Case 1:** both players have non-zero reserves and are able to produce simultaneously;

- **Case 2:** player 1 has non-zero reserves but player 2 does not, and only player 1 can produce to the market;
- **Case 3:** player 2 has non-zero reserves but player 1 does not, and only player 2 can produce to the market;
- **Case 4:** both players have zero reserves and neither can produce to the market.

It is clear that there is a directionality to these cases. Eventually, Case 1 leads to either Case 2 or 3, or exceptionally to Case 4, depending on whether one player runs out of their reserves first or they run out at the same time. Similarly, Cases 2 and 3 eventually lead to Case 4 at the point where the remaining player with non-zero reserves also runs out.

For each of these scenarios, the value function for each player is given by

$$v_i = \int_0^\infty \mathrm{e}^{-rt} q_{i,t}(1 - q_{1,t} - q_{2,t})\mathrm{d}t, \qquad \text{for } i = 1, 2. \tag{3.2.2}$$

However, the way in which the production rates incorporate time varies depending on the information set of the game. Under the Open-Loop game, both are functions of time alone so that $q^o_{i,t} = q^o_i(t)$. Under the Feedback-Loop structure, the strategies are functions of the resource allocation at time t and we have $q^f_{i,t} = q^f_i(x_{1,t}, x_{2,t})$. Regardless of the information set, the depletion of the resource with extraction means that the resource levels operate with dynamics given by (3.2.1), and their exhaustibility means that the constraint $x_{i,t} \geq 0$ holds at all times for $i = 1, 2$.

3.3 Cournot Game

3.3.1 Open-Loop Solution

Similar Open-Loop Cournot models to ours are solved in papers such as [96]. We note that swapping $x_{1,t}$ with $x_{2,t}$ is equivalent to swapping v_1 with v_2. By this symmetry, we need only consider the game for $x_1 \geq x_2 \geq 0$. The game moves from Case 1 to Case 2 to Case 4 if $x_1 > x_2$, or directly from Case 1 to Case 4 if $x_1 = x_2$. Assuming such an

initial resource allocation, the value functions are given by

$$\begin{aligned} v_1^o(x_{1,0}, x_{2,0}) = \sup_{q_1^o \geq 0} \int_0^{T_2} \mathrm{e}^{-rt} q_1^o(t)(1 - q_1^o(t) - q_2^o(t))\mathrm{d}t \\ + \int_{T_2}^{T_1} \mathrm{e}^{-rt} q_1^o(t)(1 - q_1^o(t))\mathrm{d}t, \end{aligned} \tag{3.3.1}$$

$$v_2^o(x_{1,0}, x_{2,0}) = \sup_{q_2^o \geq 0} \int_0^{T_2} \mathrm{e}^{-rt} q_2^o(t)(1 - q_1^o(t) - q_2^o(t))\mathrm{d}t, \tag{3.3.2}$$

with the dynamics of $x_{i,t}$ given by

$$\frac{\mathrm{d}x_{1,t}}{\mathrm{d}t} = -q_1^o(t), \tag{3.3.3}$$

$$\frac{\mathrm{d}x_{2,t}}{\mathrm{d}t} = -q_2^o(t), \tag{3.3.4}$$

where T_1 and T_2 are the finishing times of players 1 and 2 respectively. Throughout this thesis, we also use the notation $v_i(t, x_{1,0}, x_{2,0})$ to refer to the value to player i at time t from the game with the starting point of $(x_{1,0}, x_{2,0})$ at time 0, so that $v_i(x_{1,0}, x_{2,0}) = v_i(0, x_{1,0}, x_{2,0})$.

Proposition 3.3.1. *Where $x_1 \geq x_2 \geq 0$, player 2 is the first to exhaust their stock. The value functions and production strategies are given by*

$$q_1^o(t) = \begin{cases} \frac{1}{3}(1 - \frac{3}{2}\mathrm{e}^{-r(T_1-t)} + \frac{1}{2}\mathrm{e}^{-r(T_2-t)}) & \text{for} \quad t < T_2, \\ \frac{1}{2}(1 - \mathrm{e}^{-r(T_1-t)}) & \text{for} \quad T_2 \leq t < T_1, \\ 0 & \text{for} \quad t \geq T_1, \end{cases}$$

$$q_2^o(t) = \begin{cases} \frac{1}{3}(1 - \mathrm{e}^{-r(T_2-t)}) & \text{for} \quad t < T_2, \\ 0 & \text{for} \quad t \geq T_2, \end{cases}$$

$$v_1^o(t, x_{1,0}, x_{2,0}) = \begin{cases} \frac{1}{9r}\left(r(T_2-t)\mathrm{e}^{-r(T_2-t)} + 1 - \frac{3}{4}\mathrm{e}^{-r(T_2-t)} - \frac{1}{4}\mathrm{e}^{-2r(T_2-t)}\right. \\ \qquad \left. + \frac{9}{4}\mathrm{e}^{-2r(T_1-t)}\right) + \frac{1}{4r}\left(\mathrm{e}^{-r(T_2-t)} - 2\mathrm{e}^{-r(T_1-t)}\right) & \text{for} \quad t \leq T_2, \\ \frac{1}{4r}\left(1 - 2\mathrm{e}^{-r(T_1-t)} + \mathrm{e}^{-2r(T_1-t)}\right) & \text{for} \quad T_2 < t \leq T_1, \\ 0 & \text{for} \quad t > T_1, \end{cases}$$

$$v_2^o(t, x_{1,0}, x_{2,0}) = \begin{cases} \frac{1}{9r}\left(r(T_2-t)\left(\frac{3}{2}\mathrm{e}^{-r(T_1-t)} - \frac{1}{2}\mathrm{e}^{-r(T_2-t)}\right) - \frac{3}{2}\mathrm{e}^{-r(T_2-t)} + 1\right. \\ \left. + \frac{1}{2}\mathrm{e}^{-2r(T_2-t)} - \frac{3}{2}\left(\mathrm{e}^{-r(T_1-t)} - \mathrm{e}^{-r(T_1+T_2-2t)}\right)\right) & \text{for} \quad t \leq T_2, \\ 0 & \text{for} \quad t > T_2, \end{cases}$$

where $T_1 \geq T_2 \geq 0$ are given implicitly in terms of the initial values of x_1, x_2 by

$$rx_2 = \frac{1}{3}\left(rT_2 + \mathrm{e}^{-rT_2} - 1\right), \tag{3.3.5}$$

$$r(x_1 - x_2) = \frac{1}{2}\left(r(T_1 - T_2) - \mathrm{e}^{-rT_2} + \mathrm{e}^{-rT_1}\right). \tag{3.3.6}$$

Proof. We follow similar steps to Example 1.5.2.1, using Appendix A. Each player takes the production path of their competitor as given for the duration of the game and determines their best-response, resulting in two independent control problems. The equilibria are then given by solving these simultaneously. The Hamiltonians for the players are

$$H_1(t, x_1, x_2, q_1^o, \lambda) = \begin{cases} q_1^o(t)(1 - q_1^o(t) - q_2^o(t))\mathrm{e}^{-rt} + \lambda(t)q_1^o(t) & \text{for} \quad t < T_2, \\ q_1^o(t)(1 - q_1^o(t))\mathrm{e}^{-rt} + \lambda(t)q_1^o(t) & \text{for} \quad t \geq T_2, \end{cases}$$

$$H_2(t, x_1, x_2, q_2^o, \mu) = q_2^o(t)(1 - q_1^o(t) - q_2^o(t))\mathrm{e}^{-rt} + \mu(t)q_2^o(t) \qquad \text{for} \quad t < T_2,$$

where $\lambda(t)$ and $\mu(t)$ are adjoint functions for (3.3.3) and (3.3.4). The first-order conditions associated with the Hamiltonians are given by

$$1 - 2q_1^o(t) - q_2^o(t) = -\lambda \mathrm{e}^{rt} \qquad \text{for } t < T_2, \tag{3.3.7}$$

$$1 - 2q_2^o(t) - q_1^o(t) = -\mu \mathrm{e}^{rt} \qquad \text{for } t < T_2, \tag{3.3.8}$$

$$1 - 2q_1^o(t) = -\mathrm{e}^{rt}\lambda \qquad \text{for } T_2 \leq t < T_1, \tag{3.3.9}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = \frac{\mathrm{d}\mu}{\mathrm{d}t} = 0,$$

$$q_{2,T_2}^o = 0, \quad q_{1,T_1}^o = 0, \tag{3.3.10}$$

$$[q_1^o(t)]_{T_2^-}^{T_2^+} = 0, \tag{3.3.11}$$

where (3.3.11) follows from the continuity of the Hamiltonian at $t = T_2$. Now (3.3.9) and (3.3.10) give us

$$q_1^o(t) = \frac{1}{2}(1 - \mathrm{e}^{-r(T_1 - t)}) \text{ for } T_2 \leq t < T_1,$$

so that

$$\lambda = -\mathrm{e}^{-rT_1}.$$

Substituting the values of $q_1^o(t)$ and $q_2^o(t)$ at $t = T_2$ into (3.3.7) and (3.3.8) gives

$$\mu = -\frac{1}{2}(\mathrm{e}^{-rT_2} + \mathrm{e}^{-rT_1}),$$

which, on substituting back into (3.3.7) and (3.3.8) and solving simultaneously, gives $q_1^o(t)$ and $q_2^o(t)$ for $t < T_2$ as required. The value functions v_1^o and v_2^o can then be obtained by substituting $q_1^o(t)$ and $q_2^o(t)$ into (3.3.1)-(3.3.2), setting the lower limit in the integral to t, and integrating directly. T_1 and T_2 are found by substituting $q_1^o(t)$ and $q_2^o(t)$ into (3.3.3) and (3.3.4), and integrating from 0 to T_1 and 0 to T_2 respectively. It is clear that the RHS of (3.3.5)-(3.3.6) are monotonic in T_2 and T_1 respectively, and that for $x_1 \geq x_2 \geq 0$ they therefore uniquely define values with $T_1 \geq T_2 \geq 0$.

We note two special cases that merit separate discussion:

- When $x_1 = x_2$, Case 1 moves directly to Case 4 and the value functions and production strategies above hold with $T_1 = T_2$. This means that the game moves from both players producing directly into the state where neither player produces, and there is no intermediate period in which player 1 produces as a monopolist.
- When $x_2 = 0$, the value functions hold with $T_2 = 0$. It can be shown that the resulting production strategies and value functions for player 1 are identical to those in Example 1.4.1.2 which would result from solving the game as a monopoly problem for player 1.

□

We add that this solution method can be extended to an arbitrary number of players. We solve the game for N players in Appendix C.

3.3.2 Calculation of the Feedback-Loop Solution

The following PDEs are derived in [62] for the Feedback equilibrium production strategies and value functions in the Cournot form of the game where $x_1, x_2 > 0$:

$$rv_1^f = q_1^f(1 - q_1^f - q_2^f) - \frac{\partial v_1^f}{\partial x_1}q_1^f - \frac{\partial v_1^f}{\partial x_2}q_2^f,$$

$$rv_2^f = q_2^f(1 - q_1^f - q_2^f) - \frac{\partial v_2^f}{\partial x_1}q_1^f - \frac{\partial v_2^f}{\partial x_2}q_2^f,$$

$$q_1^f = \frac{1}{3}\left(1 - 2\frac{\partial v_1^f}{\partial x_1} + \frac{\partial v_2^f}{\partial x_2}\right), \tag{3.3.12}$$

$$q_2^f = \frac{1}{3}\left(1 - 2\frac{\partial v_2^f}{\partial x_2} + \frac{\partial v_1^f}{\partial x_1}\right). \tag{3.3.13}$$

The solution follows the theory in Appendix B and Section 1.5.2.3. We provide a brief sketch of the derivation of the above equations here; for more detail, see [62]. The strategies considered are Markov Perfect strategies $q^f_{1,t} = q_1(x_{1,t}, x_{2,t})$, $q^f_{2,t} = q_2(x_{1,t}, x_{2,t})$ such that for any starting point (x_1, x_2), at any time $t \geq 0$ in which both players have positive reserves remaining, the equilibrium value function for player i is given by

$$v^f_i = \int_t^\infty \mathrm{e}^{-r(s-t)} q^f_{i,s}(1 - q^f_{1,s} - q^f_{2,s})\mathrm{d}s.$$

In other words, the equilibrium strategies adopted at time $s = 0$ still hold at time $s = t$. Differentiating the above for each $i \in \{1, 2\}$ w.r.t. t and setting $t = 0$, followed by optimisation over the strategies, then gives the Bellman equations

$$rv^f_1 = \sup_{q^f_1 \geq 0} \left\{ q^f_1(1 - q^f_1 - q^f_2) - q^f_1 \frac{\partial v^f_1}{\partial x_1} - q^f_2 \frac{\partial v^f_1}{\partial x_2} \right\},$$

$$rv^f_2 = \sup_{q^f_2 \geq 0} \left\{ q^f_2(1 - q^f_1 - q^f_2) - q^f_1 \frac{\partial v^f_2}{\partial x_1} - q^f_2 \frac{\partial v^f_2}{\partial x_2} \right\}.$$

The theory of Appendix B and Section 1.5.2.3 is then applied to reduce each player's optimisation problem to a local optimisation one, and the problem of locating a Markov-perfect equilibrium to that of locating a local static Nash equilibrium, which yields

$$\frac{\partial v^f_1}{\partial x_1} = 1 - 2q^f_1 - q^f_2, \qquad \frac{\partial v^f_2}{\partial x_2} = 1 - 2q^f_2 - q^f_1,$$

which are solved to show

$$q^f_1 = \frac{1}{3}\left(1 - 2\frac{\partial v^f_1}{\partial x_1} + \frac{\partial v^f_2}{\partial x_2}\right), \qquad q^f_2 = \frac{1}{3}\left(1 - 2\frac{\partial v^f_2}{\partial x_2} + \frac{\partial v^f_1}{\partial x_1}\right), \tag{3.3.14}$$

as required.

The model in [62] introduces a backstop resource, of the kind used in Example 1.4.1.2. This means that the equations on the boundaries $x_1 = 0$ and $x_2 = 0$ in [62] differ to ours because they allow for competition on the axis. We do not incorporate a backstop, and so players cease production at the point at which they run out of resources. The axis games are therefore monopoly games. We show in Example 1.4.1.2 that for $x_2 = 0$, where we use boundary notation $v^f_i(x_1, 0) = V^f_i(x_1)$ and $q^f_i(x_1, 0) = Q^f_i(x_1)$, we have

$$V^f_1 = 0,$$

$$rV^f_2 = (Q^f_2)^2,$$

$$Q^f_2 = \frac{1}{2}\left(1 - \frac{\mathrm{d}V^f_2}{\mathrm{d}x_2}\right).$$

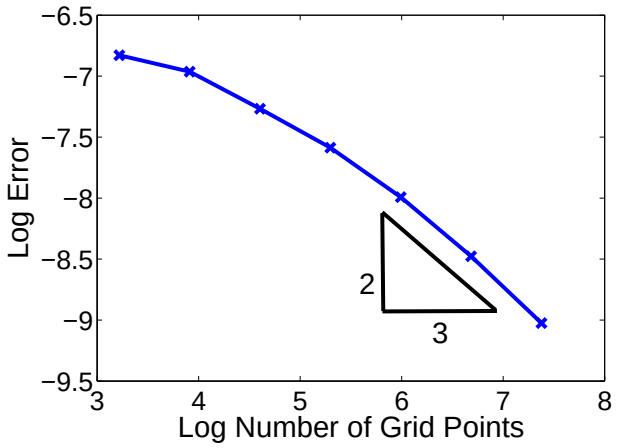
The value functions and production strategies for $x_1 = 0$ are functions of x_2 alone and are given by symmetry of the axis games. At the origin we have $V_1^f(0) = V_2^f(0) = 0$.

We note that the boundary ODEs can be solved explicitly by (1.4.7). The interior PDEs can only be solved numerically. The numerical method we use to solve this system of PDEs makes use of the result in [146] and [44] that although the Open-Loop solutions in either type of market competition are not equal to the Feedback-Loop solutions, it can be expected that they are similar.

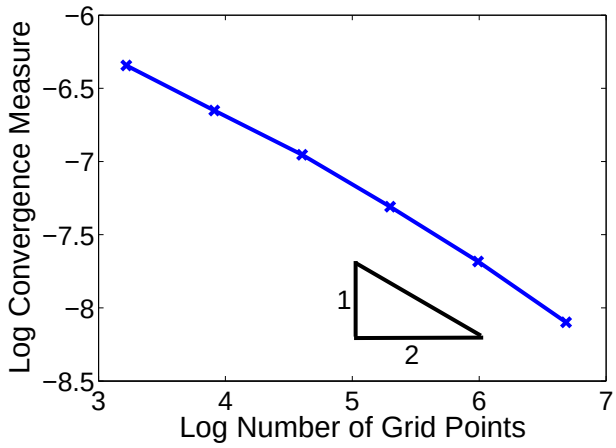
In calculating the Feedback-Loop solution, we therefore calculate the Open-Loop solution, and use this as an initial approximation for a scheme to calculate a correction function and determine the Feedback-Loop equilibrium strategies and value functions. The method we use is similar to that of [62] and the adapted version used in Section 2.4.2, but we use the Open-Loop solution as the initial approximation as opposed to an approximation based on asymptotic expansions. The method follows the same steps described in Section 2.4.2, but with the residual of (2.4.1) now given by $R_i^\epsilon = v_i^{f,\epsilon} - v_i^o$, and q_1^f, q_2^f now given by (3.3.14).

We use the same tests of convergence as in Section 2.4.2. We show the ∞-norm of R_i^ϵ on a natural log scale in Figure 3.2a, and the measure given by (2.4.2) in Figure 3.2b. Our numerical experiments indicates that the error decreases at the rate $O(N^{-2/3})$, in line with both the Stackelberg result, and the result for the Cournot game with a backstop in [62]. The convergence measure decreases at the rate $O(N^{-1/2})$, in line with the Stackelberg result.

Figure 3.3 shows the production strategies and value functions for each player for $r = 1$. The results are similar to that of [62], with the exception that the absence of production at the point the player has fully exhausted their resource creates a steeper shift in the production rates of each player as they approach the axes. In the case of the player approaching depletion, the production rate decreases more sharply because the absence of post-depletion production forces the player to ration their existing resource more carefully. The other player also increases their production rate more sharply, taking advantage of the space in the market created by the higher downward shift in the production rate of the player nearing depletion.

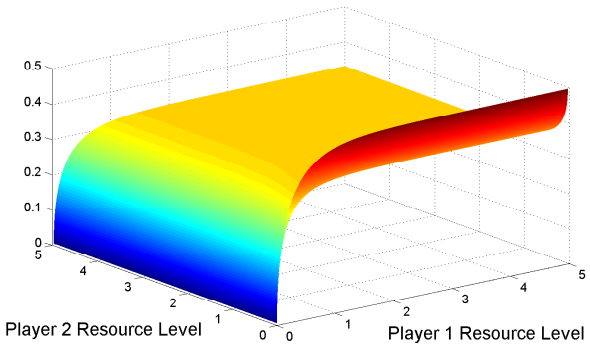


(a) Log of the residual error

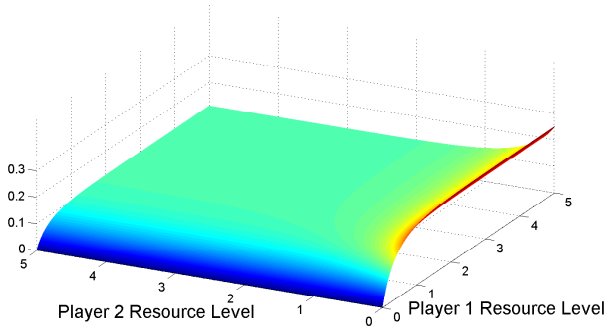


(b) Log of the convergence measure

Figure 3.2: Residual error and relative convergence of the numerical solution of the Feedback-Loop game

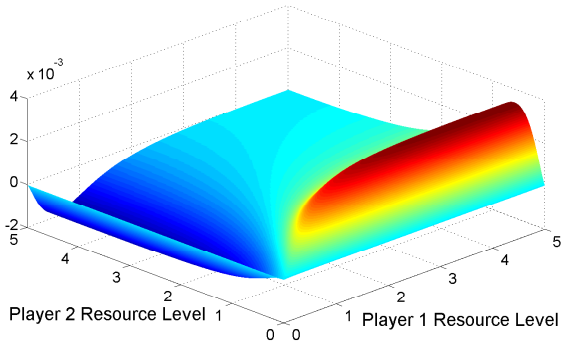


(a) Player 1 production rate, q_1^f

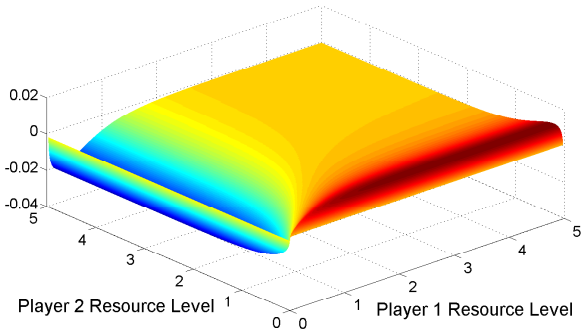


(b) Player 1 value function, v_1^f

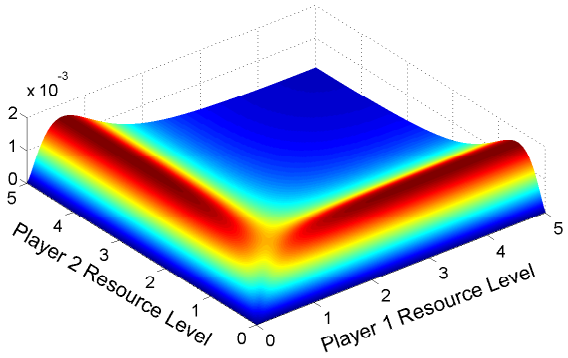
Figure 3.3: Feedback-Loop solution under Cournot competition for Player 1



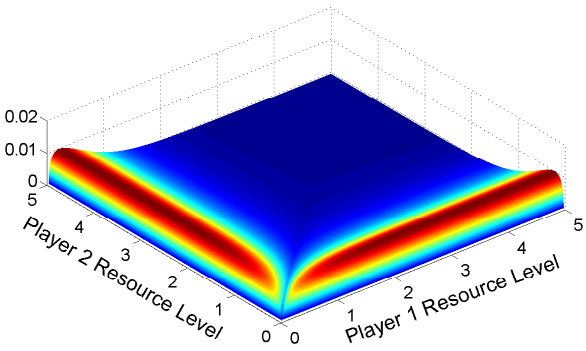
(a) $v_1^f - v_1^o$



(b) $q_1^f - q_1^o$



(c) $(v_1^f + v_2^f) - (v_1^o + v_2^o)$



(d) $(1 - q_1^f - q_2^f) - (1 - q_1^o - q_2^o)$

Figure 3.4: Feedback correction under Cournot competition

3.3.3 Discussion of the Feedback Correction

Figures 3.4a and 3.4b show the 'Feedback Correction function', i.e. the Feedback-Loop value minus the Open-Loop value, for player 1's value function and production strategy respectively. Those of player 2 can be deduced by symmetry. Figure 3.4c shows the corresponding difference in total value (the sum of the value functions), and Figure 3.4d shows the corresponding difference in price under each type of information set. In Appendix E, we show that the Open-Loop strategies in this model are time-consistent, so the differences calculated numerically for a two player game at inception also hold for a game that has been running for any length of time t. We can observe the following stylized facts in the data:

- On the axes, where we have monopoly games, both the value and production correction functions are zero.

- Each player's value function is higher under a Feedback-Loop game in the majority of the resource domain, while their production level is lower. In each of Figure 3.5a and Figure 3.5b we have divided the resource domain into three regions according to the sign of the value and production level corrections respectively:

  – In region **I** we have

  $$\begin{aligned} q^f_{1,t} > q^o_1(t), \quad q^f_{2,t} < q^o_2(t), \\ v^f_1 > v^o_1, \quad v^f_2 < v^o_2, \end{aligned}$$

  that is, player 1 both produces more and has a higher value function under Feedback-Loop than Open-Loop, and conversely for player 2.

  – In region **II** we have

  $$\begin{aligned} q^f_{1,t} < q^o_1(t), \quad q^f_{2,t} < q^o_2(t), \\ v^f_1 > v^o_1, \quad v^f_2 > v^o_2, \end{aligned}$$

  that is, both players produce less and have a higher value function under Feedback-Loop than Open-Loop.

  – In region **III** we have

  $$\begin{aligned} q^f_{1,t} < q^o_1(t), \quad q^f_{2,t} > q^o_2(t), \\ v^f_1 < v^o_1, \quad v^f_2 > v^o_2, \end{aligned}$$

  that is, player 1 both produces less and has a lower value function under Feedback-Loop than Open-Loop, and conversely for player 2.

We only show the signs of the correction functions for a truncated domain, because as x_1 and x_2 become large the correction functions become smaller than the numerical error of the scheme. This means we cannot comment on the sign of the correction functions as x_1 and x_2 become large. However, we note that while both the Open-Loop and Feedback-Loop production rates approach the same limit as $(x_1, x_2) \to (\infty, \infty)$, we show in Appendix D that they do not approach the limit at the same asymptotic rate.

- Outside region **II** in Figure 3.5b, the smaller player produces more and therefore runs out of their stock of resource more quickly under Open-Loop.

- The sum of the value functions of both players is greater everywhere in the grid under the Feedback-Loop model than under the Open-Loop model. Because this sum also represents the discounted value paid to the players by the market for the total resource stock, this also means that the market is forced to pay a greater economic value in total for the resource stock when the players are not forced to commit to their production trajectories at the beginning of the game.

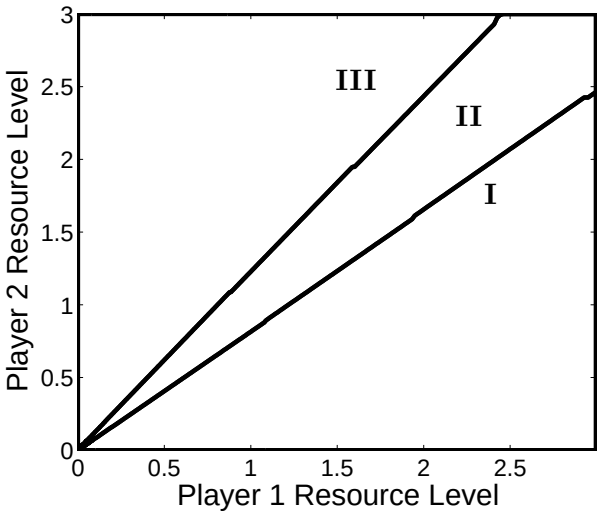
- The areas in which the peaks of the value correction functions for each player occur are very similar to those of the peaks in the corresponding production level correction, but the positive and negative sections are higher in magnitude at their peaks, and decline to zero more sharply in the production corrections than in the value corrections. They also occur closer to the axis.

- The price is larger under the Feedback-Loop competition than under Open-Loop, but for any given x_1, numerical results indicate a local minimum at $x_1 = x_2$. The opposite is true of the total quantity correction as it is the negative of the price correction.

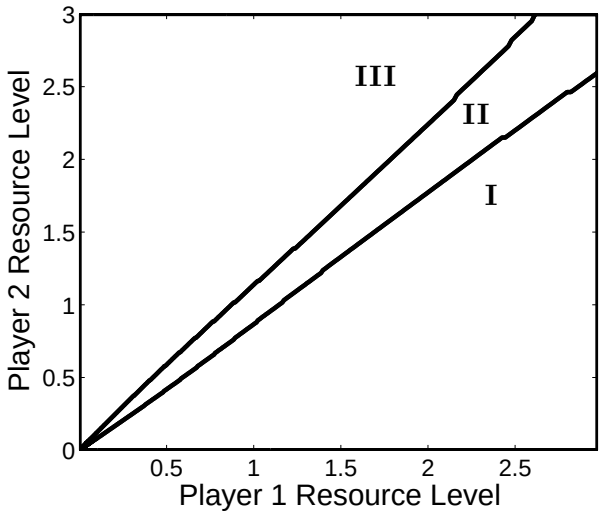
- The areas in which the production correction is negative for one player are similar to the areas in which the production correction for the competing player is positive, and yet the total production correction is negative everywhere in the grid. This indicates that in the large regions in which the production correction is positive for one player and negative for the other, the negative value is greater in magnitude than the positive one. The opposite holds for the value correction functions.

We discuss these numerically observed stylized facts below. For ease of discussion we refer to the player whose supply of reserve is small relative to the competing player as the small player, and the competing player as the large player.

- That price is always higher under the Feedback solution seems intuitive. Under a Feedback-Loop model, producers have a greater ability to respond to changes in the overall resource allocation. While we see that this is sometimes to the detriment of an individual producer, it seems reasonable that producers overall



(a) Regions showing signs of $v_1^f - v_1^o$ and $v_2^f - v_2^o$



(b) Regions showing signs of $q_1^f - q_1^o$ and $q_2^f - q_2^o$

Figure 3.5: The grid is divided into five distinct regions according to the signs of the production level and value corrections for each player, as region **II** in 3.5b lies entirely within that of 3.5a

are able to extract higher rents from the market. We also note that, because total production is therefore necessarily lower under the Feedback-Loop model, we see resource stocks exhausted more slowly than under the Open-Loop model.

- The relationship between the production correction and that of the value functions is intuitive. We would naturally expect the value function corrections to have similar directions as the production rates, but to be somewhat smoothed due to the inter-temporal discounting applied by each player. This also explains why we would not expect the regions in Figure 3.5a and 3.5b to be identical.

- The underlying cause of the direction and magnitude of the shifts in the production rate remains an open question. However, we can provide a conjecture resting on three observations:

  – the key feature that differentiates Open-Loop from Feedback-Loop competition is the ability of each player to continuously condition their production rate on not only their own current resource level, but that of the competing player as well;

  - the production rate of each player is increasing in the level of their own resource, and decreasing in that of the other player;
  - the sensitivity of the production rates of each player to each resource is highest when that resource level is low.

  Given this, we consider the value function. Each player gains from the ability to condition their strategies on the other player's resource level continuously, but suffers from the other player's ability to do the same. We argue that the fact that the optimal production rates of both players are significantly more sensitive to the resource level of the small player means that the benefit to the small player from being able to condition on the resource stock of the large player is more than offset by the effect of the large player being able to condition on their own resource stock. By this reasoning, we can expect the large player to be better off under a Feedback-Loop game, and the small player to be worse off.

- We extend this analysis to the production rates. At the point at which each player can condition on the reserve level of the other, we argue production carries with it an implicit cost, in addition to the shadow cost. The large player can now immediately respond to a decline in the stock of the small player, so we must consider the nature of this response. Given the relationship between reserve level and production, the large player will expect the production of the small player to be lower following the depletion of their reserve. This decrease in production creates space for the large player to increase their own production, which reduces the price the small player receives. While we would expect this implicit cost to exist for the large player as well, it will be significantly lower because the production rates are less sensitive to changes in the stock of the large player.

- We believe that this asymmetric cost induces an effect similar to that of any other kind of asymmetric extraction cost, incentivising the small player, for whom the cost is significantly greater, to reduce their production and spread out their extraction profile over a longer period. Conversely, the cost to the large player is small and therefore offset by the benefit of increasing production to fill the hole in the market created by the decrease in production of the smaller player. We also argue that this is why we see that, outside region **II** in Figure 3.5b, increases in

production from the large player only partially offset decreases in production from the small player, and total extraction is lower under the Feedback-Loop model everywhere.

- The fact that the magnitude of the total of the two quantity correction functions seems numerically to have a local minimum at $x_1 = x_2$ reinforces the claim that the primary divergence is driven by each player's ability to anticipate the future resource level of the other. This is because, along the line $x_1 = x_2$, both players adopt symmetric strategies under either form of the game and therefore trajectories move along the line $x_1 = x_2$.

- The fact that production rates tend to be higher for the large player and lower for the small player in the Feedback-Loop game means that we expect trajectories to move more towards the line $x_1 = x_2$ than under Open-Loop. In other words, in this model we expect the longer period of commitment of the Open-Loop game to correspond to a greater concentration of resources in the hands of the larger player over time.

We now move on to consider the limiting case as $x_1 \to \infty$. Here we can derive closed form expressions for the Feedback-Loop solution, as well as the Open-Loop solution, and thereby compare the two cases analytically. That of $x_2 \to \infty$ follows from symmetry.

3.3.4 Limiting Case: $x_1 \to \infty$

We now consider the limiting case of the game as $x_1 \to \infty$. The resource becomes inexhaustible for player 1, while player 2's resource remains exhaustible. Under this assumption, the game's solution becomes a function of x_2 alone. This simplifies the solution and allows for closed-form strategies and value functions to be calculated for both the Open-Loop and Feedback-Loop games, thereby allowing direct comparison across the two. Defining T as player 2's exhaustion time, the value functions for the players are given by

$$v_1 = \int_0^T \mathrm{e}^{-rt} q_{1,t}(1 - q_{1,t} - q_{2,t})\mathrm{d}t + \frac{1}{4r}\mathrm{e}^{-rT},$$
$$v_2 = \int_0^T \mathrm{e}^{-rt} q_{2,t}(1 - q_{1,t} - q_{2,t})\mathrm{d}t,$$

with the dynamics of x_2 given by

$$\frac{\mathrm{d}x_{2,t}}{\mathrm{d}t} = -q_{2,t}.$$

Given that we have assumed that only player 2 has a finite supply of the resource, there is only one exhaustion time in the game; that of player 2. This exhaustion time differs across the Feedback-Loop game and the Open-Loop game, so we use T_o and T_f respectively.

3.3.4.1 Open-Loop Solution

The Open-Loop Solution is found by taking $T_1 \to \infty$ as $x_1 \to \infty$ in the two player Open-Loop solution, and is given by

$$q_1^o(t) = \begin{cases} \frac{1}{3}(1 + \frac{1}{2}\mathrm{e}^{-r(T_o-t)}) & \text{for} \quad t < T_o, \\ \frac{1}{2} & \text{for} \quad t \geq T_o, \end{cases}$$

$$q_2^o(t) = \begin{cases} \frac{1}{3}(1 - \mathrm{e}^{-r(T_o-t)}) & \text{for} \quad t < T_o, \\ 0 & \text{for} \quad t \geq T_o, \end{cases}$$

$$v_1^o = \begin{cases} \frac{1}{9r}\left(r(T_o-t)\mathrm{e}^{-r(T_o-t)} + 1 - \frac{3}{4}\mathrm{e}^{-r(T_o-t)} - \frac{1}{4}\mathrm{e}^{-2r(T_o-t)}\right) + \frac{1}{4r}\mathrm{e}^{-r(T_o-t)} & \text{for} \quad t \leq T_o, \\ \frac{1}{4r} & \text{for} \quad t > T_o, \end{cases}$$

$$v_2^o = \begin{cases} \frac{1}{9r}\left(r(T_o-t)\frac{1}{2}\mathrm{e}^{-r(T_o-t)} - \frac{3}{2}\mathrm{e}^{-r(T_o-t)} + 1 + \frac{1}{2}\mathrm{e}^{-2r(T_o-t)}\right) & \text{for} \quad t \leq T_o, \\ 0 & \text{for} \quad t > T_o, \end{cases}$$

where

$$x_2 = \frac{1}{3r}\left[rT_o - (1 - \mathrm{e}^{-rT_o})\right].$$

When $x_2 = 0$, $T_o = 0$ and we have the monopoly solution with an infinite resource, $v_1^o = 1/4r$, $v_2^o = 0$.

3.3.4.2 Feedback-Loop Solution

The Feedback-Loop solution can be represented in the following parametrised form (see Appendix F for the derivation):

$$q^f_{1,t} = \begin{cases} \frac{1}{3}\left(1 + \frac{1}{2}\mathrm{e}^{-\frac{3r}{4}(T_f - t)}\right) & \text{for} \quad t < T_f, \\ \frac{1}{2} & \text{for} \quad t \geq T_f, \end{cases}$$

$$q^f_{2,t} = \begin{cases} \frac{1}{3}\left(1 - \mathrm{e}^{-\frac{3r}{4}(T_f - t)}\right) & \text{for} \quad t < T_f, \\ 0 & \text{for} \quad t \geq T_f, \end{cases}$$

$$v^f_1 = \begin{cases} \frac{1}{18r}\left(2 - 9\mathrm{e}^{-r(T_f - t)} + 8r\mathrm{e}^{-\frac{3r}{4}(T_f - t)} - \mathrm{e}^{-r\frac{3r}{2}(T_f - t)}\right) + \frac{1}{4r}\mathrm{e}^{-r(T_f - t)} & \text{for} \quad t \leq T_f, \\ \frac{1}{4r} & \text{for} \quad t > T_f, \end{cases}$$

$$v^f_2 = \begin{cases} \frac{1}{9r}\left(1 - 2\mathrm{e}^{-\frac{3r}{4}(T_f - t)} + \mathrm{e}^{-\frac{3r}{2}(T_f - t)}\right) & \text{for} \quad t \leq T_f, \\ 0 & \text{for} \quad t > T_f, \end{cases}$$

where

$$x_2 = \frac{1}{3r}\left[rT_f - \frac{4}{3}\left(1 - \mathrm{e}^{-\frac{3rT_f}{4}}\right)\right].$$

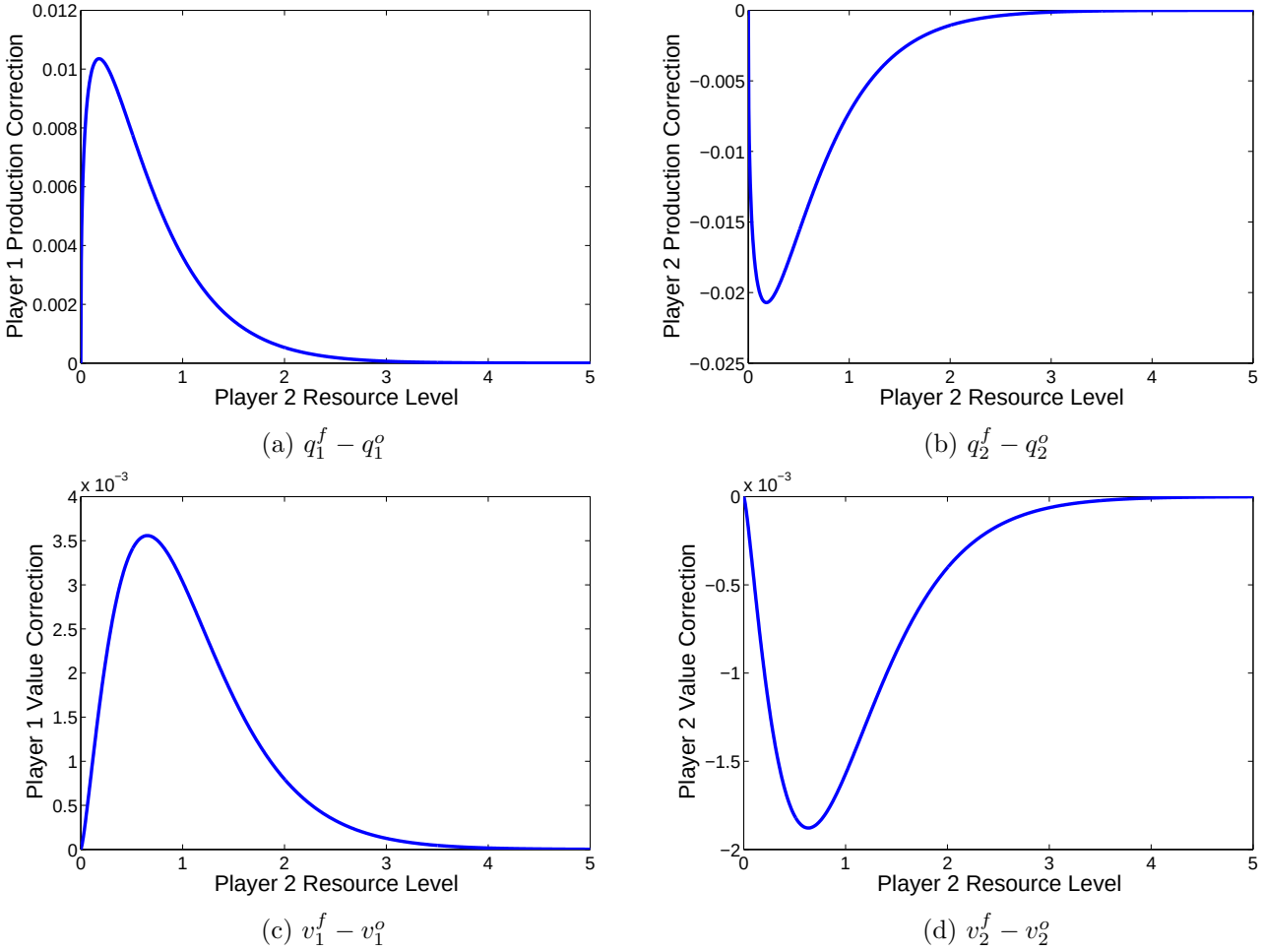
3.3.4.3 Analysis

Figures 3.6-3.7 show the correction function for both players' strategies and value functions, as well as the price and exhaustion times. The correction function is the Feedback-Loop value minus the Open-Loop value as before. In particular, we note that the observed differences are consistent with our previous numerical observations. Further, because we have closed-form solutions under both forms of the game, we can confirm analytically that the signs of the correction functions observed numerically hold for all x_2, and we prove these statements below.

Proposition 3.3.2. *$x_{2,t}$ is depleted more quickly under an Open-Loop game, so that $T_o < T_f$.*

Proof. Assume the same exhaustion time T under each information type, for games that exhaust $x^o_2(T)$ and $x^f_2(T)$ units respectively. The claim is equivalent to showing that $x^o_2 > x^f_2$. First note that we have

$$x^o_2(T) = \frac{1}{3r}\left[rT - (1 - \mathrm{e}^{-rT})\right],$$
$$x^f_2(T) = \frac{1}{3r}\left[rT - \frac{4}{3}(1 - \mathrm{e}^{-\frac{3}{4}rT})\right].$$



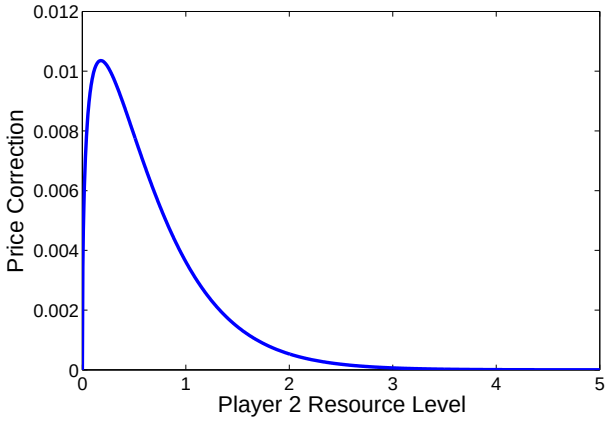
(a) $q_1^f - q_1^o$

(b) $q_2^f - q_2^o$

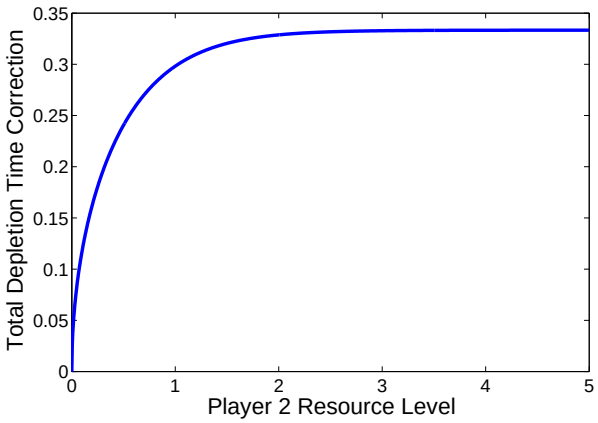
(c) $v_1^f - v_1^o$

(d) $v_2^f - v_2^o$

Figure 3.6: Feedback Correction for $x_1 = \infty$



(a) $(1 - q_1^f - q_2^f) - (1 - q_1^o - q_2^o)$



(b) $T_f - T_o$

Figure 3.7: Feedback Correction for $x_1 = \infty$

Let

$$\begin{aligned}\Delta_1(T) &= 3r(x_2^o(T) - x_2^f(T)) \\ &= \frac{1}{3}(1 - \mathrm{e}^{-\frac{3rT}{4}}) + \mathrm{e}^{-rT} - \mathrm{e}^{-\frac{3}{4}rT},\end{aligned}$$

noting that $\Delta_1(0) = 0$. Then

$$\frac{\mathrm{d}\Delta_1}{\mathrm{d}T} = r(\mathrm{e}^{-\frac{3}{4}rT} - \mathrm{e}^{-rT}) > 0 \qquad \text{for } T > 0,$$

Hence $\Delta_1(T) > 0$ for all $T > 0$. □

Proposition 3.3.3. *For all $x_2 > 0$, at $t = 0$ we have*

$$\begin{gathered} q_1^f > q_1^o, \quad q_2^f < q_2^o, \\ P^f > P^o, \\ \frac{3rT_f}{4} < T_o < T_f, \end{gathered}$$

where P^f, P^o are the market prices under the Feedback-Loop and Open-Loop games respectively.

Proof. First note that the above all follow if the sign of $\exp(-3rT_f/4) - \exp(-rT_o)$ is positive, which is only the case if $T_o > 3T_f/4$. Let $\tilde{x}_2$ represent the resource consumed

in an Open-Loop game with terminal time $3T_f/4$. As x_2 increases monotonically with T_o, the proof is complete if we can show that $\tilde{x}_2 - x_2 < 0$, where

$$\tilde{x}_2 = \frac{1}{3r}\left(\frac{3rT_f}{4} - \left(1 - \mathrm{e}^{-\frac{3rT_f}{4}}\right)\right),$$
$$x_2 = \frac{1}{3r}\left(rT_f - \frac{4}{3}\left(1 - \mathrm{e}^{-\frac{3rT_f}{4}}\right)\right).$$

Defining

$$\begin{aligned}\Delta_2 &= 3r(\tilde{x}_2 - x_2),\\ &= -\frac{r}{4}T_f + \frac{1}{3}(1 - \mathrm{e}^{-\frac{3rT_f}{4}}),\end{aligned}$$

we have

$$\frac{\mathrm{d}\Delta_2}{\mathrm{d}T_f} = -\frac{r}{4}(1 - \mathrm{e}^{-\frac{3rT_f}{4}}),$$

so that $\mathrm{d}\Delta_2/\mathrm{d}T_f < 0$ for all $T_f > 0$. Given that $\Delta_2(T_f = 0) = 0$ and Δ_2 is decreasing in T_f, we must have $\Delta_2 < 0$ for all $T_f > 0$, and we therefore also have $T_o > 3T_f/4$. □

Note the above result also implies that there exists a $t < T_o$ where the Open-Loop and Feedback strategies and prices cross (as at $t = T_o$ the sign of $\exp(-3r(T_f - t)/4) - \exp(-r(T_o - t))$ is reversed).

Proposition 3.3.4. *For all $x_2 > 0$,*

$$v_2^o > v_2^f.$$

Proof. We define

$$\begin{aligned}\Delta_3 &= 9r(v_2^o - v_2^f)\\ &= \frac{rT_o}{2}\mathrm{e}^{-rT_o} - \frac{3}{2}\mathrm{e}^{-rT_o} + \frac{1}{2}\mathrm{e}^{-2rT_o} + 2\mathrm{e}^{-\frac{3rT_f}{4}} - \mathrm{e}^{-\frac{3rT_f}{2}}\\ &= \frac{\mathrm{e}^{-rT_o}}{2}\left(rT_o - 1 + \mathrm{e}^{-rT_o}\right) + \left(\mathrm{e}^{-\frac{3rT_f}{4}} - \mathrm{e}^{-rT_o}\right) + \left(\mathrm{e}^{-\frac{3rT_f}{4}} - \mathrm{e}^{-\frac{3r}{2}}\right)\\ &= \left(\frac{3r\mathrm{e}^{-rT_o}}{2}x_2\right) + \left(\mathrm{e}^{-\frac{3rT_f}{4}} - \mathrm{e}^{-rT_o}\right) + \left(\mathrm{e}^{-\frac{3rT_f}{4}} - \mathrm{e}^{-\frac{3rT_f}{2}}\right).\end{aligned}$$

Using the earlier result that $T_o > 3T_f/4$, it is clear that $\Delta_3 > 0$ for all $x_2 > 0$. □

We note that this result aligns well with that of the interior of the game. Given that $x_1 = \infty$, player 2 is the smaller player and so is expected to be better off under the Open-Loop game.

3.4 Stackelberg Game

We now return to the Stackelberg competition of Chapter 2. We recall from Section 1.3.2 that Stackelberg competition is asymmetric. As before, we assume that player 1 is the leader in the competition, and player 2 is the follower, and we assume that there is no backstop available. We assume the same market demand curve, exhaustibility dynamics, and rate of discounting in this section as we did for the Cournot model.

3.4.1 Open-Loop Solution

We begin by considering the Open-Loop game. Here, where strategies are functions of time, the leader determines their production strategy for the duration of the game at the beginning, and then commits to it before the follower does the same. The solution is an Open-Loop Stackelberg Equilibrium, formally defined in Section 1.5.2.2.

As we see below, the game can be divided into two cases. Below the line $x_1 = 2x_2$, the follower is the first to run out of their stock, while above it the leader runs out before the follower. The line $x_1 = 2x_2$ is a boundary case of both of the above. Along this line, we show that $q^o_{1,t} = 2q^o_{2,t}$, so that trajectories travel along this line and finish at the origin.

That the trajectory leading into the origin differs from that of the Cournot game is indicative of the asymmetry of Stackelberg competition. The leader tends towards higher rates of production which allows them to "crowd out" the follower, but doing so means that they are depleted first for a larger region of the (x_1, x_2) plane.

3.4.1.1 Case 1: $x_1 \leq 2x_2$, player 1 is the first depleted

Assuming an initial resource allocation $2x_2 \geq x_1 \geq 0$, the value functions are given by

$$v^o_1 = \int_0^{T_1} \mathrm{e}^{-rt} q^o_{1,t}(1 - q^o_{1,t} - q^o_{2,t})\mathrm{d}t,$$
$$v^o_2 = \int_0^{T_1} \mathrm{e}^{-rt} q^o_{2,t}(1 - q^o_{2,t} - q^o_{1,t})\mathrm{d}t + \int_{T_1}^{T_2} \mathrm{e}^{-rt} q^o_{2,t}(1 - q^o_{2,t})\mathrm{d}t,$$

with the dynamics

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = -q^o_{1,t}, \qquad \frac{\mathrm{d}x_2}{\mathrm{d}t} = -q^o_{2,t}, \tag{3.4.1}$$

and the constraints $x_{1,t} \geq 0$, $x_{2,t} \geq 0$, and where T_1 and T_2 are the finishing times of players 1 and 2 respectively.

Proposition 3.4.1. *When $2x_2 \geq x_1 \geq 0$, player 1 is the first to run out of their reserves. The value functions and production strategies are given by*

$$q^o_{1,t} = \begin{cases} \frac{1}{2}\left(1 - \mathrm{e}^{-r(T_1-t)}\right) & \text{for } t \leq T_1, \\ 0 & \text{for } t > T_1, \end{cases}$$

$$q^o_{2,t} = \begin{cases} \frac{1}{4}\left(1 - 2\mathrm{e}^{-r(T_2-t)} + \mathrm{e}^{-r(T_1-t)}\right) & \text{for } t \leq T_1, \\ \frac{1}{2}\left(1 - \mathrm{e}^{-r(T_2-t)}\right) & \text{for } T_1 < t < T_2, \\ 0 & \text{for } t > T_2, \end{cases}$$

$$v^o_1 = \begin{cases} \dfrac{1}{8r}\Big(2r(T_1-t)\mathrm{e}^{-r(T_2-t)} + 1 - 2\mathrm{e}^{-r(T_1-t)} \\ \quad +\mathrm{e}^{-2r(T_1-t)} - 2\mathrm{e}^{-r(T_2-t)} + 2\mathrm{e}^{-r(T_1+T_2-2t)}\Big) & \text{for } t \leq T_1, \\ 0 & \text{for } t > T_1, \end{cases}$$

$$v^o_2 = \begin{cases} \dfrac{1}{16r}\left(2r(T_1-t)\mathrm{e}^{-r(T_1-t)} + 1 - \mathrm{e}^{-2r(T_1-t)}\right) & \\ \quad + \dfrac{1}{4r}(\mathrm{e}^{-r(T_1-t)} - 2\mathrm{e}^{-r(T_2-t)} + \mathrm{e}^{-2r(T_2-t)}) & \text{for } t < T_1, \\ \frac{1}{4r}(1 - 2\mathrm{e}^{-r(T_2-t)} + \mathrm{e}^{-2r(T_2-t)}) & \text{for } T_1 \leq t < T_2, \\ 0 & \text{for } t \geq T_2, \end{cases}$$

where $T_1 \leq T_2$ are given implicitly by

$$\frac{1}{2r}\left(rT_1 - 1 + \mathrm{e}^{-rT_1}\right) - x_1 = 0, \tag{3.4.2}$$

$$\frac{1}{2r}\left(rT_2 - 1 + \mathrm{e}^{-rT_2}\right) - \frac{1}{4r}\left(rT_1 - 1 + \mathrm{e}^{-rT_1}\right) - x_2 = 0, \tag{3.4.3}$$

Proof. Assume that there exist an initial resource allocation (x_1, x_2) such that player 1 is the first to exhaust their resource. Following the theory of Appendix A, the Hamiltonian for player 2 is given by

$$H_2 = \begin{cases} \mathrm{e}^{-rt} q^o_{2,t}(1 - q^o_{1,t} - q^o_{2,t}) + \mu q^o_{2,t} & \text{for } t \leq T_1, \\ \mathrm{e}^{-rt} q^o_{2,t}(1 - q^o_{2,t}) + \mu q^o_{2,t} & \text{for } T_1 < t < T_2, \end{cases}$$

which results in the first order conditions

$$q^o_{2,t} = \begin{cases} \frac{1}{2}(1 - q^o_{1,t} + \mu \mathrm{e}^{rt}) & \text{for } t \leq T_1, \\ \frac{1}{2}(1 - \mathrm{e}^{-r(T_2-t)}) & \text{for } T_1 < t < T_2, \end{cases}$$

$$\frac{\mathrm{d}\mu}{\mathrm{d}t} = 0,$$

$$\mu = q^o_{1,T_1}\mathrm{e}^{-rT_1} - \mathrm{e}^{-rT_2}.$$

The leader's Hamiltonian is given by

$$H_1 = \mathrm{e}^{-rt} q^o_{1,t}(1 - q^o_{1,t} - q^o_{2,t}) + \lambda_1 q^o_{1,t} + \lambda_2 q^o_{2,t}.$$

We note that the Hamiltonian for the leader has two adjoint variables, compared to one in the follower's. This is because the follower's strategy does not influence the leader's production, whereas the leader's strategy does influence the follower's production via the follower's best-response function. The leader therefore needs to consider the dynamics of both x_1 and x_2 in their optimisation, while the follower only needs to consider the dynamics of x_2. This differs from Cournot competition, in which neither player's strategy influences that of their competitor. Incorporating the follower response function gives

$$H_1 = \frac{1}{2}\mathrm{e}^{-rt} q^o_{1,t}(1 - q^o_{1,t} - \mu \mathrm{e}^{rt}) + \lambda_1 q^o_{1,t} + \frac{\lambda_2}{2}(1 - q^o_{1,t} + \mu \mathrm{e}^{rt}).$$

Combining the first-order conditions for this Hamiltonian with the follower response function gives

$$\begin{gathered} q^o_{1,t} = \frac{1}{2}(1 + 2\lambda_1 \mathrm{e}^{rt} - \mu \mathrm{e}^{rt} - \lambda_2 \mathrm{e}^{rt}), \\ q^o_{2,t} = \frac{1}{4}(1 + 3\mu \mathrm{e}^{rt} + \lambda_2 \mathrm{e}^{rt} - 2\lambda_1 \mathrm{e}^{rt}), \\ \frac{\mathrm{d}\lambda_1}{\mathrm{d}t} = \frac{\mathrm{d}\lambda_2}{\mathrm{d}t} = \frac{\mathrm{d}\mu}{\mathrm{d}t} = 0, \\ q^o_{1,T_1} = 0, \\ q^o_{2,T_1} = \frac{1}{2}\left(1 - \mathrm{e}^{-r(T_2 - T_1)}\right). \end{gathered}$$

We use these boundary conditions to determine the values of $2\lambda_1 - \lambda_2$, and μ, which gives us the production strategies $q^o_{1,t}$ and $q^o_{2,t}$. We determine the values of T_1 and T_2 by integrating (3.3.3) from 0 to T_1 and (3.3.4) from 0 to T_2. We find v^o_1 and v^o_2 by setting the lower limit of the integral in (3.2.2) to t and carrying out the integration for $i = 1, 2$. Two special cases merit comment:

- We can use (3.4.2)–(3.4.3) to show that $x_1 \to 2x_2$ as $T_1 - T_2 \to 0$. Hence there is a trajectory along the line $x_{1,t} = 2x_{2,t}$ on which $q^o_{1,t} = 2q^o_{2,t}$ for the duration of the game. This means that there is no intermediate period in which player 2 produces as a monopolist.

- When $x_1 = 0$, the value functions hold with $T_1 = 0$. It can be shown that the resulting production strategies and value functions for player 2 are identical to those that would result from solving the game as a monopoly problem for player 2, despite being the follower in the Stackelberg competition. This is because at the point at which player 1 runs out of their stock of resource they no longer produce, and player 2's best response is to produce as a Cournot monopolist would.

□

3.4.1.2 Case 2: $x_1 \geq 2x_2$, player 2 is the first depleted

Assuming an initial resource allocation $0 \leq 2x_2 \leq x_1$, the value functions are given by

$$v_1^o = \int_0^{T_2} \mathrm{e}^{-rt} q_{1,t}^o (1 - q_{1,t}^o - q_{2,t}^o)\mathrm{d}t + \int_{T_2}^{T_1} \mathrm{e}^{-rt} q_{1,t}^o (1 - q_{1,t}^o)\mathrm{d}t,$$

$$v_2^o = \int_0^{T_2} \mathrm{e}^{-rt} q_{2,t}^o (1 - q_{2,t}^o - q_{1,t}^o)\mathrm{d}t,$$

with the dynamics given by (3.4.1) and where T_1 and T_2 are the finishing times of players 1 and 2 respectively.

Proposition 3.4.2. *When $0 \leq 2x_2 \leq x_1$, player 2 is the first to deplete their reserves. The equilibrium value functions and production strategies are given by*

$$q_{1,t}^o = \frac{1}{2}\left(1 - \mathrm{e}^{-r(T_1 - t)}\right), \qquad q_{2,t}^o = \frac{1}{4}\left(1 - \mathrm{e}^{-r(T_2 - t)}\right),$$

$$v_1^o = \begin{cases} \frac{1}{8r}\Big(r(T_2 - t)(\mathrm{e}^{-r(T_2-t)} + \mathrm{e}^{-r(T_1-t)}) + 1 - \mathrm{e}^{-r(T_2-t)} & \\ \quad - \left(2\mathrm{e}^{-2r(T_1-t)} + \mathrm{e}^{-r(T_1+T_2-t)}\right)(\mathrm{e}^{rT_2} - 1)\Big) & \text{for } t < T_2, \\ \quad + \frac{1}{4r}(\mathrm{e}^{-r(T_2-t)} - 2\mathrm{e}^{-r(T_1-t)} + \mathrm{e}^{-r(2T_2-T_1)}) & \\ \frac{1}{4r}(1 - 2\mathrm{e}^{-r(T_1-t)} + \mathrm{e}^{-2r(T_1-t)}) & \text{for } T_2 \leq t < T_1, \\ 0 & \text{for } t \geq T_2, \end{cases}$$

$$v_2^o = \begin{cases} \frac{1}{16}\Big(2r(T_2 - t)\mathrm{e}^{-r(T_1-t)} + 1 - 2\mathrm{e}^{-r(T_2-t)} - 2\mathrm{e}^{-r(T_1-t)} & \\ \quad + 2\mathrm{e}^{-r(T_1+T_2-2t)} + \mathrm{e}^{-2r(T_2-t)}\Big) & \text{for } t < T_2, \\ 0 & \text{for } T_2 < t < T_1, \end{cases}$$

where $T_1 \geq T_2 \geq 0$ are given implicitly by

$$\frac{1}{2r}\left(rT_1 - 1 + \mathrm{e}^{-rT_1}\right) - x_1 = 0, \tag{3.4.4}$$

$$\frac{1}{4r}\left(rT_2 - 1 + \mathrm{e}^{-rT_2}\right) - x_2 = 0. \tag{3.4.5}$$

Proof. This case can be derived by assuming an initial resource allocation such that player 2 is the first to exhaust their stock of the resource. Carrying out the same steps as those in Case 1 then yields the required results. We again note two special cases:

- When $x_1 = 2x_2$, it can be shown that the value functions and production strategies above hold with $T_1 = T_2$ as before
- When $x_2 = 0$, the value functions hold with $T_2 = 0$. It can be shown that the resulting production strategies and value functions for player 1 are identical to those that would result from solving the game as a monopoly game for player 1.

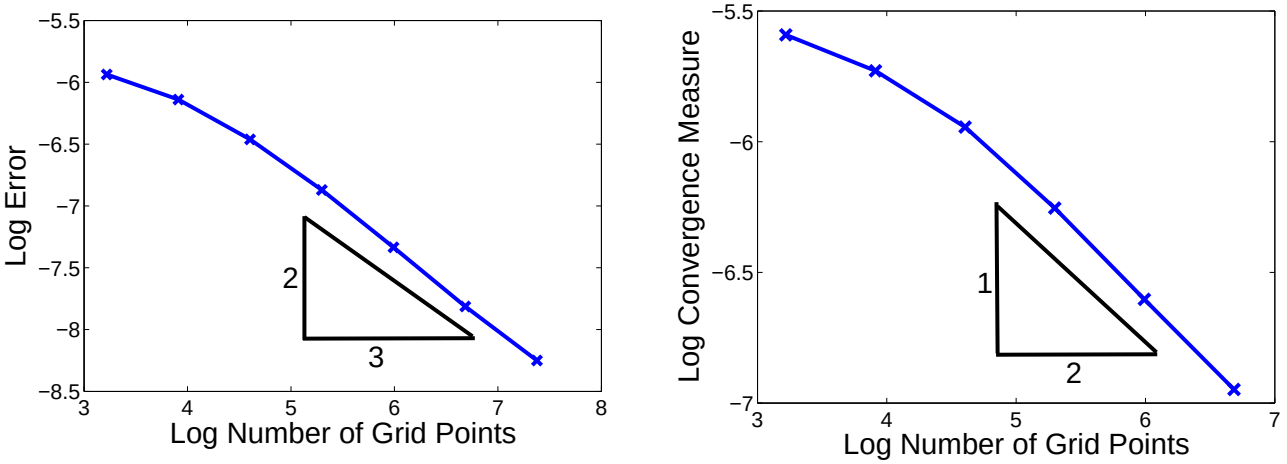
□

A comparison of (3.4.2)–(3.4.3) and (3.4.4)–(3.4.5) helps to illustrate the contrast between the price-taking behaviour of the follower and the price-making behaviour of the leader. In (3.4.3), we see the expression includes both T_1 and T_2, whereas in (3.4.5) it only includes T_2. In other words, the resource level of the leader is only relevant to the follower if the leader is depleted first. At the point at which the leader has sufficient reserves to outlast the follower, the level of those reserves is not a relevant consideration in the follower's strategy. Conversely, T_2 appears in neither (3.4.2) nor (3.4.4), so that the resource level of the follower is irrelevant to the determination of the leader's strategy regardless of whether or not the leader is the first to be depleted. As we later see, this is not the case in the Feedback-Loop game.

3.4.2 Calculation of the Feedback-Loop Solution

The PDEs which define the Feedback-Loop Stackelberg Equilibrium are derived in Section 2.2 and are given by (2.3.10), with the boundary conditions given by the solution of the monopoly game in Section 1.4.1.2 on each axis as before.

We use the same scheme as in the Cournot case, adapted to the Stackelberg PDEs and Open-Loop solution. Our numerical results indicate that our adjusted method results in convergence of the error at the rate $O(N^{-\frac{2}{3}})$, in line with the Cournot solution of the game both with and without a backstop. We plot the error measures of the numerical scheme in Figure 3.2. The production rates and value functions derived by the scheme are displayed in Figure 2.3, and discussed in Section 2.2.4. Consequently, we focus on the Feedback-Loop correction functions in our discussion below.



(a) Log of the residual error

(b) Log of the convergence measure

Figure 3.2: Convergence of the numerical solution of the Feedback-Loop Stackelberg game.

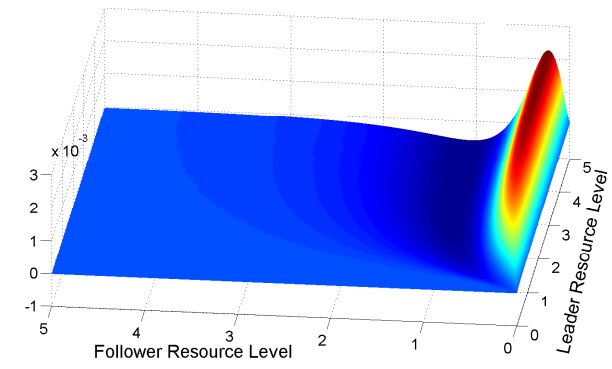
3.4.3 Discussion of the Feedback Correction

In Figures 3.3-3.4 we present the correction functions for the value functions and production strategies of each player, as well as the price and total value. As in the Cournot case, the corrections represent the value under the Feedback-Loop game minus the value under the Open-Loop game.

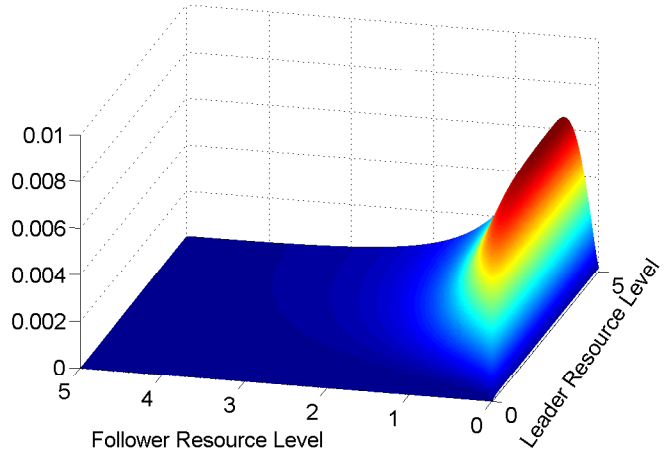
We note the following key features in the data, and focus on the region in which x_1 is considerably larger than x_2. This is both because this is the region which exhibits the most noteworthy results and because it is the region in which the leader's ability to pre-commit, required by Stackelberg competition, is most plausible in an economic sense.

- Both players' Feedback-Loop value functions are higher simultaneously, as opposed to the Cournot game in which, for the corresponding set of resources (x_1, x_2) in which one player was significantly better off, the other was significantly worse off; a consequence of symmetry.

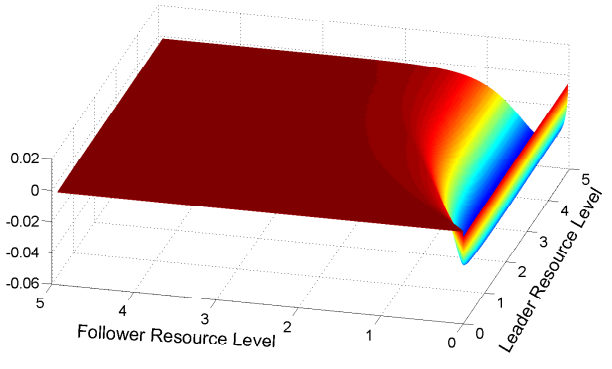
- There is generally very little difference in the production rates except where x_1 is considerably larger than x_2. Here, the leader's production rate is significantly lower under a Feedback-Loop game, while the follower's is higher. The difference



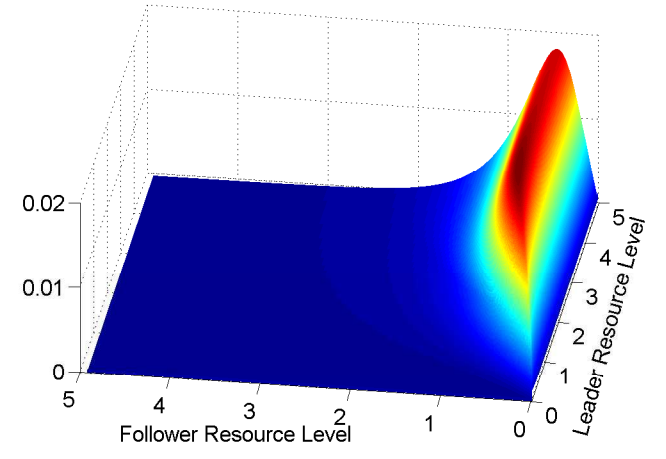
(a) $v_1^f - v_1^o$



(b) $v_2^f - v_2^o$

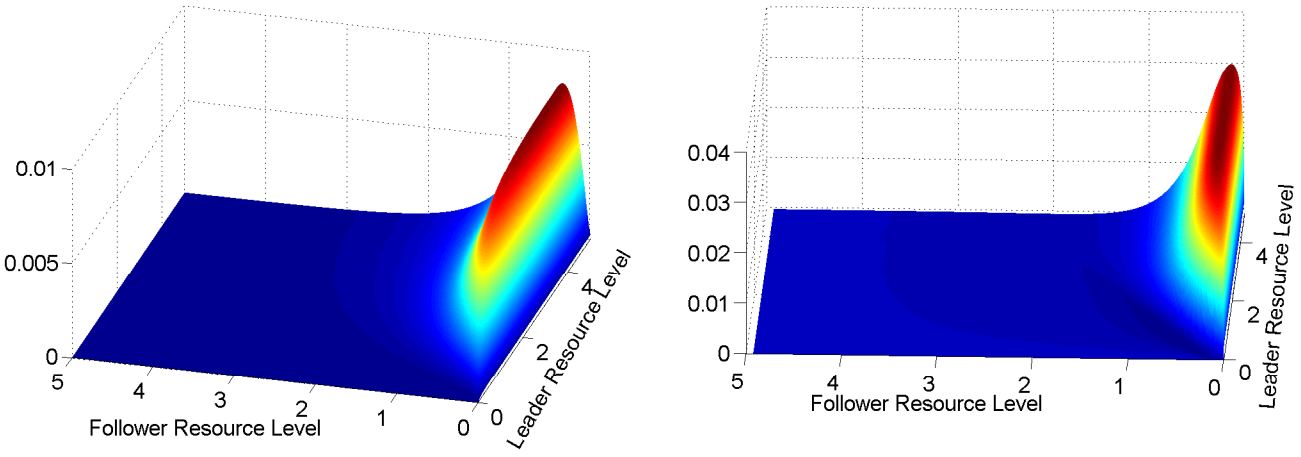


(c) $q_1^f - q_1^o$



(d) $q_2^f - q_2^o$

Figure 3.3: Feedback Correction under Stackelberg competition



(a) $(v_1^f + v_2^f) - (v_1^o + v_2^o)$ (b) $(1 - q_1^f - q_2^f) - (1 - q_1^o - q_2^o)$

Figure 3.4: Feedback Correction under Stackelberg competition

in production rates between the two games is more significant for the leader than the follower. This is the opposite of what happens under the Cournot model.

- That the price and total value are both higher in this region under the Feedback-Loop game follows straightforwardly from the previous two observations.

- The line $x_1 = x_2$ no longer seems to have any special significance, though this is to be expected due to the asymmetry introduced by Stackelberg competition. In the Open-Loop game, we have shown that trajectories approach the origin along the line $x_1 = 2x_2$. For the Feedback-Loop game, it can be shown analytically that $x_1 = 2x_2$ is a trajectory asymptotically close to the origin, and this is supported by the numerics, but numerical evidence indicates that it is not a trajectory for all x_1.

An additional observation of interest is that the non-monotonic behaviour in x_2 of the Feedback-Loop value functions and production strategies that we identified and discussed in Chapter 2 does not occur under the Open-Loop game. Here, the leader's production rate does not vary with respect to x_2, and the leader produces at the monopoly rate corresponding to their own resource level at all points in the resource grid.

This seems to support our hypothesis that the implicit directed-cooperation between the leader and the follower we have identified in the Feedback-Loop game is only a worthwhile strategy for the leader where they have the ability to continuously condition their production rate on the resource level of the follower. The fact that both players are better off under the Feedback-Loop game than the Open-Loop game, as opposed to the Cournot case, in which gains to one player generally corresponded to losses for the other player, further supports the argument that the non-monotonicity in the strategies is indicative of directed cooperation.

3.5 Conclusion

We have compared a duopoly model of resource extraction across two extremes of commitment, that of an Open-Loop game and a Feedback-Loop game, under both Cournot competition and Stackelberg competition. We have derived closed-form representations of the Open-Loop value functions and production strategies under each mode of competition. We then used them as the starting approximation for a numerical scheme to generate representations of the solutions under the corresponding Feedback-Loop game.

Under Cournot competition, we find that there is significant difference between the two games when the resource levels are far from equal. The larger player is better off and produces more under the Feedback-Loop than the Open-Loop game, while the smaller player is worse off and produces less. We propose that this is because the production rates are more sensitive to the resource level of the smaller player than the larger one, and so the players' ability to continuously condition their production rate on the resource level of their opponent, as well as their own, works to the advantage of the larger player and the detriment of the smaller, under the Feedback-Loop game.

We argued that, under a Feedback-Loop game, production carries with it an additional implicit cost, in that when (say) player 1 produces, player 2 can now immediately respond to the decrease in player 1's resource stock, and the fact that player 1's future production will decrease as a result of their decline in stock. Player 2 responds to the anticipated decline in player 1's production by increasing their own, which decreases the price that player 1 receives. Because this cost is much larger for the smaller player, it incentivises decreasing production in the present, creating space for the larger player to increase their own production. The production correction function of each player, and

the combined correction across the two players shows that we can expect resources to be extracted more quickly under the Open-Loop than under the Feedback-Loop game, and in a way that results in more concentration of resources in the hands of the larger player over time.

Under Stackelberg competition, we showed that the result is different. Focusing on the case in which the leader's supply is large compared to that of the follower, and the asymmetry implied by Stackelberg competition is relatively plausible, both players are generally better off under the Feedback-Loop game. Furthermore, the leader decreases their production while the follower increases theirs. We argued that this is because the Feedback-Loop game allows for a kind of directed cooperation in which the leader can induce the follower to produce more, and therefore exhaust their stock more rapidly, by decreasing their own production rate. At the point at which the follower is near depletion, the gains to the leader from being able to produce as a monopolist sooner make this worthwhile. However, because this cooperation is conditional on the follower's resource level, it does not occur under the Open-Loop game.

We note that we have only sought to highlight the asymmetric impact of the period of commitment using this example and there are a number of questions for future research. In particular, we have only considered the extremes of commitment. While the work on the common-property model of [118] indicates that we can expect a relatively smooth transition from the Open-Loop to the Feedback-Loop case for intermediate periods of commitment, it would be useful to confirm this directly with a model incorporating separate property pools. Secondly, considering nonlinear demand curves, or models which incorporate resource level dependent-extraction costs would be useful. Thirdly, given that we have seen noticeable differences in the impact of commitment between Stackelberg and Cournot competition, this study could also be extended to differentiated Bertrand competition, discussed in Section 1.3.4, in which players set the price rather than the quantity as the strategic variable. Finally, we have assumed the same period of commitment for both parties. The consideration of cases in which the period of commitment itself is asymmetric across the players is also an open question for future research.

Chapter 4

Analysis of Open-Loop Equilibria in a Common-Property Cournot Game

4.1 Introduction

This chapter addresses the issue of multiple equilibria in an Open-Loop game modelling the extraction of a common property resource by multiple players. Open-Loop games are formally defined in Section 1.5.2.1, and discussed further in Chapter 3. Common property games are defined and discussed in Section 1.4.2.2, though we provide a condensed summary below. While the model we address is specific, it is of a class that is widely used in the literature, and we expect our findings to be readily generalisable to other models.

Common property games are used to model a variety of common property resource extraction problems. For example, fisheries are analysed in [132], [29], and [13]; underground aquifers are analysed in [116]; forestries are analysed in [4].

We recall that common property resources are often prone to the *Tragedy of the Commons* posed in [61]. Firms extracting from an exhaustible common property resource pool often have an incentive to extract pre-emptively; resources conserved in the pool can be extracted by their competitors. For this reason, issues of sustainability and over-depletion feature heavily in the common property literature. Common property games can both be used to gauge the sustainability of resource pools under common property extraction, and to examine the impact and inform the construction of policies designed

to incentivise sustainable usage by extracting firms.[1]

Analysis of common property models is often more straightforward than for separate pools models. Consequently, these models have also been used to simplify the modelling of extractive industries in which resources are not held as common property. Where players in an extractive industry operate with separate pools, the dimension of the state-space increases with the number of players, which can complicate the analysis for multiple firms. With a common property resource pool, the state-space remains one-dimensional as the number of players increases, simplifying the analysis.

It is common for these models to be analysed in an Open-Loop setting, as this generally allows them to be dealt with as straightforward optimal control problems, often resulting in closed-form solutions. The terminal conditions are key in the solution of these control problems. It is sometimes the case that they yield multiple solutions for the terminal production rates of the players, and therefore multiple equilibria. Where this occurs, it is important that each of these equilibria be assessed independently.

A Stackelberg formulation of the Lancaster model of capitalism[2] is presented in [113], and critiqued in [37]. In it, the terminal condition for the workers' consumption rate is selected on the assumption that the rate is continuous at the termination point of the game. However, deriving the terminal conditions from the game's Hamiltonian instead shows that there are also infinitely many other valid solutions which exhibit a discontinuity at this point. Consequently, in [37] it is argued that the conclusion of [113] that there is a unique equilibrium is made incorrectly. Cases such as this demonstrate the importance of care in the calculation of terminal conditions, and caution against their selection on the basis of economic intuition alone.

Our work is based primarily on the N player, common property Cournot model analysed in [79]. This is a general model, and it and variants of it are used widely in the literature; see [118], [29], and [101] for examples. In it, two equilibria are derived; one of these involves production approaching zero as the resource pool approaches exhaustion, and the other involves production approaching a strictly positive point. Because production must necessarily be zero beyond the resource's exhaustion, the former of these is arguably more intuitively appealing by virtue of its continuity on termination. In some

[1] See [99], [116], and [91].

[2] A Lancaster game is a model of a capitalist economy as a differential game between workers and capitalists. See [86].

similar games with analogous dual equilibria, such as [118], this equilibrium has been adopted by assumption, arguably because of its intuitive appeal. Our work analyses this equilibrium in detail, and questions its suitability; in doing so, we also consider the suitability of Open-Loop models for common property extraction games in general.

In Section 4.2, we introduce our model. In Section 4.3 we derive both the Open-Loop equilibria of the game, and also derive the Feedback-Loop equilibrium to provide a point of comparison. We carry out the main body of our analysis of the Open-Loop equilibria in Section 4.4, and conclude in Section 4.5.

4.2 The Model

We make similar market assumptions as in the previous Chapters. We assume a non-cooperating Cournot oligopoly of N firms, with no possibility of entry by other firms. We assume that the market operates in continuous time. We assume a linear demand curve, as defined in (1.2.2), such that for a given total rate of supply of the resource Q, the price P is given by $P = \alpha - \beta Q$.

We assume that all players pay a common cost of extraction at the rate $0 < c < \alpha$. The rate of profit per quantity sold for each player is therefore given by $\alpha - \beta Q - c$. Recalling the argument of Section 1.3.1.1, we can adopt scalings such that we can assume $\alpha - c = 1$, $\beta = 1$ without loss of generality. Consequently, the extraction costs do not appear explicitly in the main body of our work. However, we highlight how our work still allows for symmetric extraction costs for two reasons; firstly so as to be consistent with the presentation of [79]; secondly because we discuss the implications of asymmetric extraction costs for one of our key results in Appendix G.[3]

We use $q_{i,t}$ to denote the production level for player i at time t, and Q_t to denote the total production across all players: $Q_t = \sum_{i=1}^{N} q_{i,t}$. We also define $Q_{-i,t} = \sum_{j=1, j\neq i}^{N} q_{j,t}$ as the sum of production levels across all players other than player i. We assume that all players draw from a common property resource pool, and use x_t to denote the remaining stock of this resource at time t. The extraction of the resource implies that x_t depletes with the dynamics

$$\frac{\mathrm{d}x_t}{\mathrm{d}t} = -Q_t.$$

[3]The scaling argument we have used above can only be used to eliminate one constant. Where the extraction costs are asymmetric, they cannot both be scaled away.

The symmetry of the game, and in particular, the assumption of common extraction costs, precludes equilibria in which one or more of the N players is blockaded. To see this, first note that a firm can only be blockaded if the market price is lower than the firm's cost of extraction. Because the players share the same cost of extraction, this would imply that all firms are blockaded at the same price level. Therefore, if a group of firms were to produce at a level high enough to blockade one or more players, then it is clear that their rate of cashflow would be negative, and they would be better off decreasing their production until either the price rises above c or their production rate reaches zero.

We consider the game both in an Open-Loop and Feedback-Loop setting, which are defined formally in Sections 1.5.2.1 and 1.5.2.3 respectively. The Feedback-Loop game has been analysed in detail in [79] and is considered here primarily as a point of comparison with the Open-Loop game. The value function for each player is given by

$$v_i = \int_0^\infty \mathrm{e}^{-rt} \left[q_{i,t} \left(1 - q_{i,t} - \sum_{j=1, j\neq i}^{N} q_{j,t} \right) \right] \mathrm{d}t, \qquad \text{for } i = 1, ..., N.$$

4.3 Derivation of Equilibria

4.3.1 Feedback-Loop Equilibrium

The Feedback-Loop equilibria for a general price curve are derived in [79]. We follow the same steps, but go into more detail and focus on the case of a linear demand curve. The strategies considered are Markov Perfect strategies $q_{i,t} = q_i(x_t)$, $i = 1, ..., N$ such that for any starting point x_0, at any time $t \geq 0$ at which $x_t > 0$, the value function for player i is given by

$$v_i = \int_t^\infty \mathrm{e}^{-r(s-t)} q_{i,s} \left(1 - q_{i,s} - \sum_{j=1, j\neq i}^{N} q_{j,s} \right) \mathrm{d}s,$$

with the boundary condition $v_i(0) = 0$. Differentiating the above for each $i = 1, ..., N$ w.r.t. t and setting $t = 0$, followed by optimisation over the strategies, then gives the Bellman equations

$$rv_i = \sup_{q_i \geq 0} \left\{ q_i \left(1 - q_i - \sum_{j=1, j\neq i}^{N} q_j \right) - \frac{\mathrm{d}v_i}{\mathrm{d}x} \sum_{i=1}^{N} q_i \right\}, \qquad i = 1, ..., N.$$

The Bellman principle arguments of Appendix B are then applied to reduce each player's optimisation problem to a local one, and the problem of locating a Markov-perfect equilibrium to that of locating a local static Nash equilibrium, which yields

$$\frac{\mathrm{d}v_i}{\mathrm{d}x} = 1 - 2q_i - \sum_{j=1, j\neq i}^{N} q_j, \qquad i = 1, ..., N.$$

We can then use symmetry to show that the production and value functions are equal, so that $q_i = q = Q/N$ and $v_i = v \ \forall i$, where

$$\begin{aligned} q &= \frac{1}{N+1}\left(1 - \frac{\mathrm{d}v}{\mathrm{d}x}\right), \\ rv &= q(NQ - (N-1)) \\ &= \frac{1}{N+1}\left(1 - \frac{\mathrm{d}v}{\mathrm{d}x}\right)\left(\frac{N^2}{N+1}\left(1 - \frac{\mathrm{d}v}{\mathrm{d}x}\right) - (N-1)\right). \end{aligned} \tag{4.3.1}$$

Substituting the boundary condition[4] $v(0) = 0$ in (4.3.1) results in a quadratic equation in $\mathrm{d}v/\mathrm{d}x$ with two real roots.[5] Solving this and using the relation

$$Q = \frac{N}{N+1}\left(1 - \frac{\mathrm{d}v}{\mathrm{d}x}\right), \tag{4.3.2}$$

gives the boundary conditions for total production:

$$Q(0) = 0, \qquad \text{or} \qquad Q(0) = \frac{N-1}{N}.$$

However, we can rule out the boundary condition $Q(0) = 0$. Multiplying (4.3.1) by N and substituting (4.3.2) gives

$$Nrv = Q(NQ - (N-1)).$$

Differentiating gives

$$Nr\frac{\mathrm{d}v}{\mathrm{d}x} = \frac{\mathrm{d}Q}{\mathrm{d}x}(2NQ - (N-1)). \tag{4.3.3}$$

Substituting $Q(0) = 0$ into (4.3.2) gives $\mathrm{d}v/\mathrm{d}x|_{x=0} = 1$. For $x = 0$, (4.3.3) can therefore be written as

$$\left.\frac{\mathrm{d}Q}{\mathrm{d}x}\right|_{x=0} = -\frac{rN}{N-1}, \tag{4.3.4}$$

[4]The boundary condition is an initial condition in x; because the Bellman equation is solved backwards in x, this results in a terminal condition for the game.

[5]These roots coincide for $N = 1$, so that there is a unique equilibrium in the monopoly case.

which is clearly negative, as $N > 1$ and $c \geq 0$.[6] If $Q(0) = 0$, then (4.3.4) implies that $Q < 0$ for x positive and close to zero. This violates the constraint that $Q \geq 0$ for all positive x and so $Q(0) = 0$ cannot be a valid initial condition. There is therefore a unique initial condition for Q (and therefore q_i), given by $Q(0) = (N-1)/N$.

4.3.2 Open-Loop Equilibria

Under an Open-Loop information structure, the players seek to find

$$v_i(x) = \max_{q_i > 0} \left\{ \int_0^\infty \mathrm{e}^{-rt} q_i(t) \left(1 - q_i(t) - \sum_{j=1, j\neq i}^{N} q_j(t) \right) \mathrm{d}t \right\}, \tag{4.3.5}$$

with the dynamics of x_t given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\sum_{i=1}^{N} q_i(t). \tag{4.3.6}$$

The Hamiltonian for the above control problem for player i is given by

$$H_i(t, x, q_i, \mu) = q_i \left(1 - q_i - \sum_{j=1, j\neq i}^{N} q_j \right) \mathrm{e}^{-rt} + \mu_i \sum_{j=1}^{N} q_j, \tag{4.3.7}$$

where $\mu = (\mu_1, ..., \mu_N)$ are the adjoint functions for (4.3.6).

Maximising the Hamiltonian in q_i and multiplying both sides by e^{rt} gives the first order conditions

$$1 - 2q_i - \sum_{j=1, j\neq i} q_j + \mu_i \mathrm{e}^{rt} = 0, \tag{4.3.8}$$

$$\frac{\mathrm{d}\mu_i}{\mathrm{d}t} = 0,$$

$$H_i(T, 0, q_i, \mu) = 0. \tag{4.3.9}$$

Summing (4.3.8) over i gives the following condition for total production Q:

$$N - (N+1)Q + \phi \mathrm{e}^{rt} = 0,$$

where

$$\phi = \sum_{i=1}^{N} \mu_i.$$

[6] This does not hold for $N = 1$, as the relation $v'' = -v'\sqrt{r/v}$ implies that $v'' \to \infty^+$ as $x \to 0$.

Solving for symmetric strategies gives

$$q_i(t) = \frac{1}{N+1}\left(1 + \frac{\phi}{N}\mathrm{e}^{rt}\right) \qquad i = 1, ..., N, \tag{4.3.10}$$

Multiplying both sides of (4.3.9) by e^{rT} and summing over i gives

$$0 = Q(T)(1 - Q(T)) + Q(T)\phi\mathrm{e}^{rT}. \tag{4.3.11}$$

Summing (4.3.10) over i, setting $t = T$ and substituting into (4.3.11), and solving for $Q(T)$ gives $Q(T) = (N-1)/N$ or $Q(T) = 0$ as before. These correspond to the following terminal conditions for player i (each of which corresponds to a different Open-Loop equilibrium):[7]

$$q_i(T) = \frac{(N-1)}{N^2}, \quad \text{or} \quad q_i(T) = 0.$$

These equilibria are similar in form to those derived under the Feedback-Loop game. However, under an Open-Loop structure, the strategy associated with $Q(T) = 0$ does not result in negative production. Both equilibria therefore need to be considered separately.

From here on, we refer to the equilibrium associated with the terminal condition $q_i(T) = (N-1)/N^2$ as $q^A = (q_1^A, ..., q_N^A)$, or Equilibrium A; and the equilibrium associated with the terminal condition $q_i(T) = 0$ as $q^B = (q_1^B, ..., q_N^B)$, or Equilibrium B. We extend the A and B superscript notation to the terminal times, value functions, and prices associated with each strategy as well. When comparing with the Feedback-Loop game, we use the notation superscript F to refer to values under the Feedback-Loop game. We derive a closed-form parametric representation for each of the Open-Loop equilibria in the following sections.

4.3.2.1 Equilibrium A: $q_i^A(T^A) = (N-1)/N^2$

Putting $t = T^A$ into (4.3.10), substituting into the terminal condition $Q(T^A) = (N-1)/N$, and solving for the constant ϕ^A gives

$$\phi^A = -\frac{\mathrm{e}^{-rT^A}}{N}.$$

Substituting this back into (4.3.10) gives

$$q_i^A(t) = \frac{1}{N^2(N+1)}\left(N^2 - \mathrm{e}^{-r(T^A - t)}\right).$$

[7] For $N = 1$ it is clear that the two terminal conditions coincide, resulting in a unique equilibrium, as occurred in the Feedback-Loop game.

We find T^A implicitly by the relation

$$\begin{aligned} x_0 &= N\int_0^{T^A} q_i^A(t)\mathrm{d}t \\ &= \frac{1}{rN(N+1)}\left[N^2rT^A - 1 + \mathrm{e}^{-rT^A}\right]. \end{aligned}$$

Then, v_i^A can be found by direct calculation of the integral (4.3.5):

$$\begin{aligned} v_i^A =& \frac{1}{rN^3(1+N)^2}\left[N(N-1)r(T^A-t)\mathrm{e}^{-r(T^A-t)} + N^3\left(1-\mathrm{e}^{-r(T^A-t)}\right)\right. \\ & \left. -\mathrm{e}^{-r(T^A-t)} + \mathrm{e}^{-2r(T^A-t)}\right]. \end{aligned}$$

4.3.2.2 Equilibrium B: $q_i(T^B) = 0$

Following the same steps as before with the terminal condition $Q(T^B) = 0$ gives

$$\begin{aligned} \phi^B &= -N\mathrm{e}^{-rT^B}, \\ q_i^B(t) &= \frac{1}{N+1}\left(1-\mathrm{e}^{-r(T^B-t)}\right). \end{aligned}$$

Here, T^B is defined implicitly by the relation

$$\begin{aligned} x_0 &= N\int_0^{T^B} q_i^B(t)\mathrm{d}t \\ &= \frac{N}{r(N+1)}\left(rT^B - 1 + \mathrm{e}^{-rT^B}\right). \end{aligned}$$

The value function is then found by evaluating (4.3.5), substituting in the known form of the production strategies to give

$$\begin{aligned} v_i^B &= \int_t^{T^B} \frac{\mathrm{e}^{-r(s-t)}}{(N+1)^2}\left(1-\mathrm{e}^{-r(T^B-s)}\right)\left(1+N\mathrm{e}^{-r(T^B-s)}\right)\mathrm{d}t \\ &= \frac{1}{r(N+1)^2}\left((N-1)r(T^B-t)\mathrm{e}^{-r(T^B-t)} + 1 - (N+1)\mathrm{e}^{-r(T^B-t)} + N\mathrm{e}^{-2r(T^B-t)}\right). \end{aligned} \tag{4.3.12}$$

We note that the production and value functions approach the same limit as $x \to \infty$, such that

$$\begin{aligned} \lim_{x\to\infty} q^A &= \frac{1}{N+1} = \lim_{x\to\infty} q^B, \\ \lim_{x\to\infty} v^A &= \frac{1}{r^2(N+1)^2} = \lim_{x\to\infty} v^B. \end{aligned}$$

As $N \to \infty$, total production across the players approaches 1 for both equilibria; the production rate and value function for an individual player approach 0.

4.3.3 Numerical Illustration and Discussion

The results for the production, value, and price functions are presented in Figure 4.2 for both Open-Loop equilibria and the Feedback-Loop equilibrium for $N = 3$, taking (without loss of generality) $r = 1$. In particular, we note that:

- The value functions and prices are higher for the Open-Loop equilibria than for the Feedback-Loop equilibrium for all x, while total production is lower. This result differs from that of the separate pools game in Chapter 3, in which there are regions in which a player's value function is higher under the Feedback-Loop game than the Open-Loop game, and indeed the total value and production are higher and lower respectively everywhere. This is because the Feedback-Loop production strategies are decreasing functions of the resource level, which create an incentive amongst the players to extract preemptively in the common property model.[8]

- The time at which the resource is completely depleted is higher under both Open-Loop equilibria than for the Feedback-Loop equilibrium.

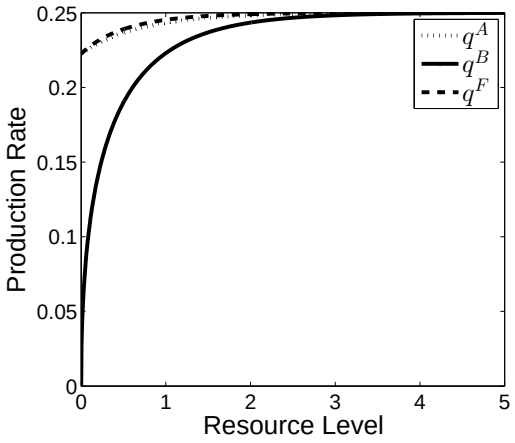
- Equilibrium A is situated between Equilibrium B and the Feedback-Loop Equilibrium for value, production, exhaustion time, and price, i.e.

$$q_i^F > q_i^A > q_i^B, \qquad P^F < P^A < P^B,$$
$$T^F < T^A < T^B, \qquad v_i^F < v_i^A < v_i^B.$$

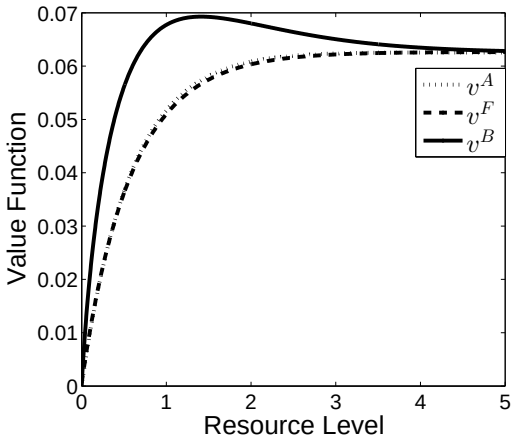
- There is significant non-monotonicity in x in v^B. We consider this in greater detail in the next section.

We note here that, although we have assumed symmetric extraction costs which can be scaled out of the problem, neither the existence of multiple equilibria nor the non-monotonicity of the value functions under Equilibrium B are contingent on this assumption. We demonstrate this in a two player setting with asymmetric costs in Appendix G.

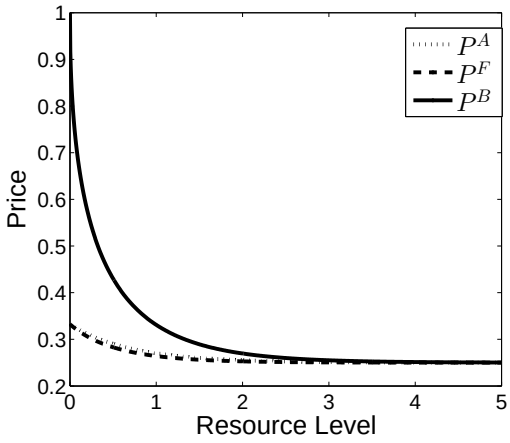
[8] See [49], [46], [79], and [119] for similar comparisons.



(a) Comparison of Production Rates



(b) Comparison of Value Functions



(c) Comparison of Price Levels

Figure 4.2: Comparison of Common Property Equilibria

4.4 Analysis of the Open-Loop Equilibria

The numerical results of Section 4.3.3 indicate significant non-monotonicity in the value functions for all players under Equilibrium B. This non-monotonicity is counter-intuitive: if v_i^B decreases with x, then all players receive less value, despite having a greater level of resource between them. Although we recall from Chapter 2 that such a result is not unheard of in the literature, there is little to explain it in this instance, and it should therefore be met with a degree of caution.

In this section, we confirm analytically that this non-monotonicity exists. We then consider the validity of this equilibrium. We show that its existence is contingent on the implicit exclusion of all strategies in which all players do not adopt the same terminal time T^B, and that the equilibrium is not robust to strategies outside this set. Finally, we discuss the causes of this non-monotonicity, and in particular, the assumptions adopted in the literature to ensure the existence of this equilibrium. We argue that the non-monotonicity in this equilibrium is caused by the assumption of a common anticipated exhaustion time implicitly introducing a cooperative mechanism into the game.

Although we have shown numerically that v_i^B is non-monotonic in x_0, we now prove this analytically for all N. We begin by proving the following lemma.

Lemma 4.4.1.

$$\frac{\mathrm{d}x}{\mathrm{d}T^B} > 0.$$

Proof. We have

$$x = \frac{N}{N+1}\left(rT^B - \left(1 - \mathrm{e}^{-rT^B}\right)\right),$$

so that

$$\frac{\mathrm{d}x}{\mathrm{d}T^B} = \frac{rN}{N+1}\left(1 - \mathrm{e}^{-rT^B}\right) > 0.$$

□

Proposition 4.4.2. *For every $N \geq 2$, $\exists \tilde{T} > 0$ such that $\mathrm{d}v_i^B/\mathrm{d}x < 0$, for all $T^B > \tilde{T}$.*

Proof. First note that because we have proved earlier that $\mathrm{d}x/\mathrm{d}T^B > 0$, we only need to prove that there exists a $\tilde{T}$ beyond which $\mathrm{d}v_i^B/\mathrm{d}T^B < 0$. Differentiating v_i^B w.r.t. T^B gives

$$\frac{\mathrm{d}v_i^B}{\mathrm{d}T^B} = \frac{1}{(N+1)^2}\mathrm{e}^{-rT^B}\left(2N(1 - \mathrm{e}^{-rT^B}) - r(N-1)T^B\right). \tag{4.4.1}$$

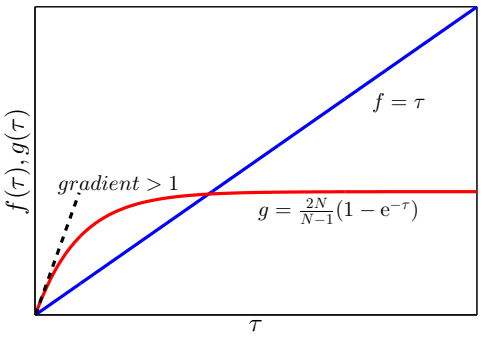


Figure 4.3: $\mathrm{d}v/\mathrm{d}T^B$ is negative after the intersection of f and g.

First note that for $N = 1$ it is clear that the right-hand side of (4.4.1) is strictly positive. For $N > 1$, we define $\tau = rT^B$, $f(\tau) = \tau$ and $g(\tau) = (1-e^{-\tau})(2N/(N-1))$ as sketched in Figure 4.3. The gradients of g and f at $\tau = 0$ are equal to $2N/(N-1)$ and 1 respectively, and we note that $2N/(N-1) = 2 + 2/(N-1) > 1$. As $g''(\tau) = -e^{-\tau}(2N/(N-1)) < 0$, and $\lim_{\tau\to\infty} g(\tau) = 2N/(N-1)$, f and g have a unique point of intersection for $\tau > 0$. Beyond this point, (4.4.1) is negative as required. □

4.4.1 Discussion of Equilibrium B

In this section we discuss the validity of Equilibrium B. We provide two criticisms. In the first of these, we introduce a small impurity into the game, and show analytically that only the game under Equilibrium A is robust to this impurity, even as it approaches zero. In the second, we show numerically that the strategies of q^B are not robust as a best-response to strategies in which other players select extraction profiles that result in the resource running out before T^B. We do this by the construction of an alternative candidate production path for one of the players, on the assumption that the remaining players adopt q^B, that results in a greater value for that player than they would gain by also adopting q^B.

4.4.1.1 Equilibrium B with a small perturbation

To investigate the plausibility of Equilibrium B, we introduce a small impurity into the original game. This is motivated by similar approaches adopted by [65] and [63] in other

games. On termination of the game, we now assume that the players receive a small terminal payment of size ϵ/rN. The players' value functions are now given by

$$v_i(x) = \max_{q_i>0} \left\{ \int_0^T \mathrm{e}^{-rt} q_i(t) \left(1 - q_i(t) - \sum_{j=1, j\neq i}^{N} q_j(t) \right) \mathrm{d}t + \mathrm{e}^{-rT} \frac{\epsilon}{rN} \right\},$$

with the constraint (4.3.6). Using the same variational method as before the game's Hamiltonian is again given by (4.3.7), and the terminal condition is now given by

$$q_i(T)(1 - q_i(T) - \sum_{j=1, j\neq i}^{N} q_j(T))\mathrm{e}^{-rT} + \mu_i Q(T) = \mathrm{e}^{-rT} \frac{\epsilon}{N} \qquad i = 1, ..., N.$$

Summing this condition over i and multiplying both sides by e^{rT} gives

$$Q(T)(1 - Q(T)) + \phi Q(T)\mathrm{e}^{rT} = \epsilon.$$

Making the substitution

$$\phi \mathrm{e}^{rT} = (N + 1)Q(T) - N,$$

and solving for $Q(T)$ gives the roots

$$Q(T) = \frac{1}{2N}((N - 1) \pm ((N - 1)^2 + 4N\epsilon)^{\frac{1}{2}}). \tag{4.4.2}$$

Taking $\epsilon \to 0$ shows that the two roots correspond to those of the game without the perturbation: taking the plus sign in (4.4.2) corresponds to Equilibrium A, while taking the minus sign corresponds to Equilibrium B. However, it is clear that, for $0 < \epsilon < 1$, the root corresponding to the minus sign in (4.4.2) is negative and is therefore inadmissible. The root corresponding to Equilibrium A in the unperturbed game remains a valid terminal condition for $\epsilon > 0$.

4.4.1.2 A Defection Strategy for Equilibrium B

In Section 4.3.2, we showed that if all players adopt q^B, then the game terminates at $t = T^B$ and each player extracts x/N of the resource over the duration of the game, receiving a value of v^B. In this Section, we show numerically that a player (player 1 without loss of generality) can select a strategy which increases their value beyond v^B by extracting more than x/N, and exhausting the resource before time T^B. In doing so, we show that q^B is not robust as a best-response to the full set of possible strategies that

the other players can adopt. In particular, it is dependent on the implicit assumption that all players anticipate the same terminal time of the game.

We assume an N player game, and that all players for $j > 1$ elect to adopt the strategy associated with Equilibrium B, q^B. We construct a strategy for player 1 that yields a higher level of value to them than that of adopting q_1^B, and deduce that by extension, strategy B is not a best response to itself.

Because we are allowing player 1 to consider strategies which result in the game ending before T^B, we require the additional assumption that players $2, ..., N$ cease production at the point at which $x_t = 0$. We incorporate this assumption via the inclusion of an indicator term $\mathbf{1}_{x_t>0}$ in q_j^B, such that it is now given by

$$q_j^B = \frac{1}{N+1}\left(1 - \mathrm{e}^{-r(T^B-t)}\right)\mathbf{1}_{x_t>0},$$

where

$$x_0 = \frac{N}{r(N+1)}\left(rT^B - \left(1 - \mathrm{e}^{-r(T^B-t)}\right)\right).$$

We assume that player 1 looks for a strategy q^C which results in extracting more of the resource than they would under q^B, resulting in a finishing time $T^C < T^B$. If player 1 takes the production of the remaining players as given, their value function is given by

$$\begin{aligned} v_1 &= \int_0^{T^C} \mathrm{e}^{-rt} q_{1,t}^C \left(1 - q_{1,t}^C - (N-1)q_t^B\right) \mathrm{d}t, \\ &= \int_0^{T^C} \mathrm{e}^{-rt} \frac{1}{N+1} q_{1,t}^C \left(\left(2 + (N-1)\mathrm{e}^{-r(T^B-t)}\right) - (N+1)q_{1,t}^C\right) \mathrm{d}t, \end{aligned}$$

with the dynamics of x_t given by

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -\left(q_{1,t}^C + (N-1)q_t^B\right). \tag{4.4.3}$$

The Hamiltonian for this problem is given by

$$\begin{aligned} H(t, x, q_1^C, \mu_1) =& \mathrm{e}^{-rt} \frac{1}{N+1} q_1^C \left(\left(2 + (N-1)\mathrm{e}^{-r(T^B-t)}\right) - (N+1)q_1^C\right) \\ &+ \mu_1 \left(q_1^C + \frac{N-1}{N+1}\left(1 - \mathrm{e}^{-r(T^B-t)}\right)\right), \end{aligned}$$

which gives the following first order conditions for q_1^C:

$$\begin{aligned} \left(2 + (N-1)\mathrm{e}^{-r(T^B-t)}\right) - 2(N+1)q_1^C + (N+1)\mu_1 \mathrm{e}^{rt} = 0, \\ \frac{\mathrm{d}\mu_1}{\mathrm{d}t} = 0, \end{aligned} \tag{4.4.4}$$

$$\begin{aligned}\mathrm{e}^{-rT^C}\frac{1}{N+1}q_1^C(T^C)\left(\left(2+(N-1)\mathrm{e}^{-r(T^B-T^C)}\right)-(N+1)q_1^C(T^C)\right)\\ +\mu_1\left(q_1^C(T^C)+\frac{N-1}{(N+1)}\left(1-\mathrm{e}^{-r(T^B-T^C)}\right)\right)=0,\end{aligned}\tag{4.4.5}$$

where T^C is the optimal exhaustion time for player 1. Setting $t = T^C$ in (4.4.4) and substituting into (4.4.5) for μ_1 gives

$$\begin{aligned}\left((N+1)\mu_1\mathrm{e}^{rT^C}\right)^2+\mu_1\mathrm{e}^{rT^C}\left(4N(N+1)-2(N^2-1)\mathrm{e}^{-r(T^B-T^C)}\right)\\ +\left(2+(N-1)\mathrm{e}^{-r(T^B-T^C)}\right)^2=0,\end{aligned}$$

which has roots

$$\mu_1\mathrm{e}^{rT^C}=\frac{1}{N+1}\left((N-1)\mathrm{e}^{-r(T^B-T^C)}-2N\pm 2\left((N^2-1)\left(1-\mathrm{e}^{-r(T^B-T^C)}\right)\right)^{\frac{1}{2}}\right).$$

Substituting these roots into (4.4.4) gives

$$q_1^C(T^C)=\frac{1}{2(N+1)}\left(-2(N-1)\left(1-\mathrm{e}^{-r(T^B-T^C)}\right)\pm 2\left((N^2-1)\left(1-\mathrm{e}^{-r(T^B-T^C)}\right)\right)^{\frac{1}{2}}\right).$$

However, taking the root of μ_1 with the negative sign results in $q_1^C(T^C) < 0$, and is therefore invalid. We define

$$\zeta=\frac{\mathrm{e}^{-rT^C}}{N+1}\left((N-1)\mathrm{e}^{-r(T^B-T^C)}-2N+2\left((N^2-1)\left(1-\mathrm{e}^{-r(T^B-T^C)}\right)\right)^{\frac{1}{2}}\right),$$

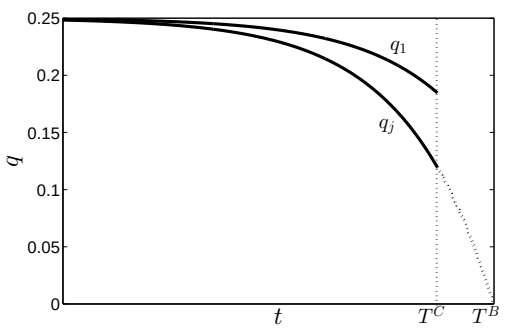
so that

$$q_1^C(t)=\frac{1}{2(N+1)}\left(2+(N-1)\mathrm{e}^{-r(T^B-t)}+(N+1)\zeta\mathrm{e}^{rt}\right).$$

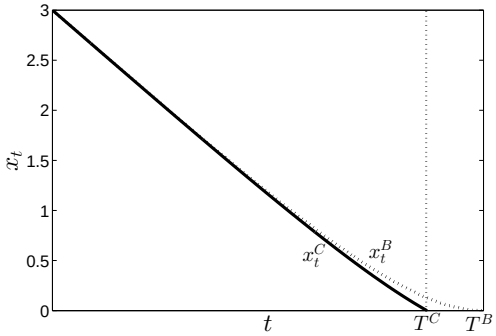
The finishing time, T^C is calculated by integrating the constraint (4.4.3) from 0 to T^C to give

$$x_0=\frac{1}{2r(N+1)}\left(2NrT^C+\left(\mathrm{e}^{rT^C}-1\right)\left((N+1)\zeta-(N-1)\mathrm{e}^{-rT^B}\right)\right).$$

We recall that $T^C < T^B$. Because the production trajectories of the remaining players extend beyond time $t = T^C$, their production drops to zero at this point according to our earlier assumption. We can also show that $q_1^C > q_j^B$ for $2 \le j \le N$, and for all $0 \le t \le T^C$. To see this, note that we can write the production trajectories in the form $q_1^C = m + k_1\mathrm{e}^{rt}$, $q_j^B = m + k_j\mathrm{e}^{rt}$. Therefore, if there were a time s such that $q_1^C(s) < q_j^B(s)$, it would imply that $q_1^C(t) < q_j^B(t)$ for all $t \ge 0$. This would imply $T^C > T^B$, a contradiction.



(a) Production Level



(b) Resource Level

Figure 4.4: Sample trajectories for $N = 3$, $x_0 = 3$, $r = 1$. The defection of player 1 means that the resource follows the path x_t^C, while the remaining players expect it to follow x_t^B. Player 1 produces along q_1. The remaining players produce along the solid portion of q_j until T^C, when the resource is depleted. Because $x_{T^C}^B > 0$, the remaining players anticipate being able to continue producing along the dotted portion of q_j, but are now unable to.

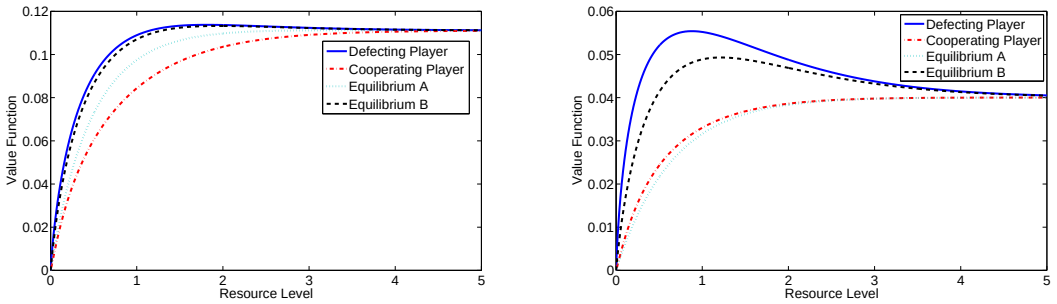
We illustrate this with the sample trajectories in Figure 4.4. In Figure 4.4b, we use x_t^C to reference the actual reserve level at time t, factoring in the increased production by player 1, and x_t^B to reference the level of the reserve anticipated by the players adhering to q^B. In Figure 4.4a we see that the increased production from player 1 results in depletion at time T^C, before T^B, and while $x_t^B > 0$. Given this, the remaining players are forced to stop producing prematurely, as opposed to following the dotted segment of q_j.

We calculate v_1^C and v_j^C directly by substituting q_1^C and q_j^B into

$$\begin{aligned} v_1^C &= \int_0^{T^C} \mathrm{e}^{-rt} q_1^C(t)(1 - q_1^C(t) - (N-1)q^B(t))\mathrm{d}t, \\ v_j^C &= \int_0^{T^C} \mathrm{e}^{-rt} q^B(t)(1 - q_1^C(t) - (N-1)q^B(t))\mathrm{d}t, \end{aligned} \tag{4.4.6}$$

and integrating to give

$$\begin{aligned} v_1^C =& \frac{1}{4r(N+1)^2}\Big[4r(N-1)T^c\mathrm{e}^{-rT^B} \\ &+4\left(1-\mathrm{e}^{-rT^C}\right)+\left(\mathrm{e}^{rT^C}-1\right)\left((N-1)^2\mathrm{e}^{-2rT^B}-(N+1)^2\zeta^2\right)\Big], \end{aligned}$$



(a) 2 players (b) 4 players

Figure 4.5: Robustness of Equilibrium B

$$v_j^C = \frac{1}{2r(N+1)^2}\left[rT^C\left((N-3)\mathrm{e}^{-rT^B} - (N+1)\zeta\right) + 2\left(1-\mathrm{e}^{-rT^C}\right)\right.$$
$$\left. - \left(\mathrm{e}^{rT^C}-1\right)\left((N-1)\mathrm{e}^{-2rT^B} - (N+1)\zeta\mathrm{e}^{-rT^B}\right)\right].$$

We have plotted these against v^A and v^B, respectively the value that all players would receive if q^A and q^B were uniformly adopted, in Figure 4.5a for $N = 2$ and in Figure 4.5b for $N = 4$.

For both $N = 2$ and $N = 4$, as well as all other values of N tested, the value player 1 receives by adopting q^C is greater than that of adopting q^B, so that q^B cannot of itself be a best response. For $N = 2$ we can also see that the value received by player 2 is not only lower than the value it would receive when both players adopt strategy B, but is also below the value it would receive when both players adopt strategy A. This remains the case for $N = 3$, but is not the case for $N > 4$. This is simply because the impact of one player 'defecting' from q^B on the other players is diluted by the assumption that the remaining $N - 1$ players adhere to it.

We conclude that, while all players might be better off if they all choose strategy B, each player can improve their own welfare by defecting from this strategy and depleting the resource more rapidly than the other players. In other words, classic prisoners' dilemma considerations make this strategy invalid as a Nash equilibrium strategy unless there are mechanisms in place for binding all players to it.

4.4.1.3 Discussion

It remains to be explained how the superiority of q^C is not picked up by the control theory approach used in Section 4.3.2, or in work such as [79]. We argue that Open-Loop common property games of this type implicitly assume that all players anticipate the same terminal time, and therefore do not require robustness to strategies such as q^C as a necessary condition for equilibrium.[9]

This is because any strategy such as q^C that depends on over-exploitation of the resource relative to an equilibrium necessarily results in the other players anticipating extracting after the resource is depleted. The fact that the production level of those players is forced to zero by the exhaustion of the reserve means that in allowing a strategy such as q^C we have implicitly made the production strategies of the other players dependent on the resource level itself. In doing so, we have violated the requirement that all of the strategies are strictly functions of time alone, and so the resulting strategies may not be true Open-Loop strategies.

We have explained how the viability of q^B is contingent on the implicit assumption of exclusion of strategies in which the players anticipate different finishing points of the game, but there are also works which make this assumption explicitly.[10] We argue that in either case, this assumption allows for implicit cooperation between the players, which we later argue results in the non-monotonicity of v^B, and therefore should not be made lightly. This is because ruling out strategies in which players anticipate different terminal times also then prevents the individual players from 'defecting' from q^B.

If all players other than player 1 adopt q^B, then if player 1 adopts a strategy anticipating a finishing time of T^B, the total amount they can extract is also fixed at x/N, and they are only left with the problem of allocating this extraction effectively over the time of the game. If they choose to over-exploit and select a strategy which anticipates extracting a greater total level of resources over the game, then this strategy will necessarily result in an anticipated terminal time $T < T_B$, and the strategies would

[9] The Open-Loop game is reformulated into a set of 'Feedback style' ODEs in [79]. The claim is made that this formulation avoids the assumption of common anticipated terminal times. We review the formulation and demonstrate that this claim does not hold in Appendix H.

[10] In [118] for example, it is assumed that there is an independent regulator which assesses the candidate Open-Loop production trajectories of the players and rejects them if they result in any player anticipating extraction once the resource is depleted.

be rejected by the earlier assumption.[11] By eliminating this opportunity for defection, this assumption can therefore induce cooperation.

We also argue that this possibility for cooperation offers an explanation for the non-monotonicity detected in v_i^B. Given that this cooperation is induced via the finishing time of the game, we can expect it to have a greater effect for lower values of x, when the game's duration is short and exhaustibility is an important consideration, than for higher values of x where exhaustibility becomes less relevant to the players' strategies and value functions. If the value gained from the cooperation implicit in q^B decreases as x increases, then this could offset the gains we would otherwise expect from having a greater level of resource, and the value function v_i^B decreases as x increases. This explanation bears some similarity to that offered for the result of [14], which we referenced in Chapter 2, in which the gains from temporary cartelisation are found to be non-monotonic in the size of the cartel's resource pool, for certain parameter values of the model.

4.5 Conclusion

We have considered an N player common property Cournot game of resource extraction under linear demand. We have confirmed the finding of [79] that solving this game via standard methods results in multiple equilibria. However, we have highlighted the method's implicit assumption that the players share a common anticipated exhaustion time for the pool. While this is clearly a necessary condition for equilibrium, it is not a sufficient condition, and we have shown that one of the equilibria can be disqualified when this assumption is relaxed.

In proving this, we allowed for a player to 'defect' from the terminal time implied by Equilibrium B, by choosing to extract more and bring forward the exhaustion of the resource. Doing so demonstrated that the player can improve their position by extracting more than they would under Equilibrium B, also diminishing the welfare of the other players. Insofar as each player can therefore gain from unilaterally defecting from Equilibrium B, it cannot be said to be a true equilibrium.

We draw a comparison between this game and a classic Prisoner's Dilemma problem. By adopting the terminal time implied by Equilibrium B, the player is implicitly cooperating with the other players for mutual gain, even though they can gain more by

[11] This is true of strategy C, as we showed that it results in player 1 extracting more than x/N over the duration of the game.

unilaterally 'defecting' by producing more. We argued that the semi-cooperative nature of the premiums extracted by the firms under Equilibrium B may also help to explain how it is possible for the firms' values to be non-monotonic in the level of the pool.

We note that our finding has only been confirmed numerically for a set of values of N. While we have closed form expressions for both v_1^B and v_1^C in (4.3.12) and (4.4.6) respectively, proving that $v_1^C \geq v_1^B$ as would be required is made non-trivial by the complexity of (4.4.6). However, we strongly suspect that the finding holds for all N, although this remains a conjecture. Finally, we note that the conclusions we have made are thus far specific to the case of linear demand. Extending these findings to more complex demand curves is a question for future research.

Chapter 5

Price-Caps and Joint Production Constraints in Cournot Competition

5.1 Introduction

Most Nash-Cournot models in the literature assume a smooth price curve. This assumption greatly simplifies the solution of static games,[1] insofar as it allows for the straightforward application of variational methods. Because the solutions of more complex dynamic models often follow from the analysis of the static game, smooth price curves are a key assumption for both static and dynamic market games.

Problematically, this assumption is sometimes not supported by the realities of a given market. There are many examples of Cournot type markets in which the smoothness of the price curve is disrupted by exogenous constraints. Some of these are imposed by specific actors, and others occur organically. Governments and other regulators often impose caps or tariffs on the highest price which can be set within a market as a check on the power of large producers. This is especially true of recently deregulated markets, such as electricity markets in the UK and US.

There are also supply-side constraints which do not directly change the price curve, but have a similar effect. Many industries rely on shared infrastructure with limited carrying capacity, which can create short-term limits on the total sales of the firms reliant on that infrastructure. Sometimes these constraints are managed by independent bodies which allocate that capacity in advance, converting a capacity constraint over a group

[1] See Example 1.3.1.1.

of firms into a set of individual production constraints. Examples include tendering processes for the allocation of train lines and airport runway capacity. In other industries, including privatised water and bandwidth markets, this does not occur, creating coordination problems around meeting the shared constraints.

An important example is that of privatised electricity markets, in which the sharing of transmission infrastructure across multiple electricity-generating firms means that firms are sometimes limited in their total sales capacity by the transmission capacity of the shared infrastructure. Other more complex supply-side constraints are known to exist. In Norway, regulators impose upper bounds on the market share of the largest producers in certain markets, which is in essence a production cap on the firm, the size of which is determined by the total production of the remaining firms.

Conditions such as these make the price curve or cost curve piece-wise continuous, so that variational methods can no longer readily be used to establish existence and uniqueness results for equilibria. The analysis of games with piecewise continuous demand or cost curves is a relatively nascent field of research. Existence of solutions where the price curve is piecewise continuous in total production is analysed in [68], while [109] analyses the existence of solutions where the cost curve is piecewise continuous in total production across all firms operating in the market. It is explained in [24] that the uniqueness of solutions can be precluded by these constraints. The implications of this result are analysed in more detail for price caps in [60].

This chapter extends the work of [60] on price caps in deriving uniqueness results for games with joint constraints on production, and games in which both types of constraint are applied together. We then use these results to analyse a game in which there is not an imposed price cap per se, but a linear tariff which is imposed if the price rises above a specified level. We show that the uniqueness results for the games with price and production caps can be used to analyse this form of game as well, and that much of the uniqueness-disrupting effects of price caps can still occur even when pricing over the cap is permitted but penalised. We solve the game via decomposition into tractable sub-games, a method which we believe can also be applied to other similar piecewise continuous games.

In Section 5.2 we introduce the market model and define the price and production caps we consider. In Section 5.3 we illustrate the impact of these caps on Cournot equilibria in a two-player game. In Section 5.4 we extend this analysis to a multiple

player setting. We introduce and analyse the penalty model in Section 5.5. In Section 5.6 we introduce and discuss a particularly significant area for future research, that of joint constraints under resource exhaustibility. In Section 5.7 we conclude.

5.2 Market Model

We assume an N player Cournot market for a commodity resource. We assume that the resource is sufficiently homogeneous that the market sets a common price across the output of the players. The players simultaneously determine their levels of output $(q_1, ..., q_N)$ with the common price being a function of total production, $Q = \sum_{i=1}^{N} q_i$. We assume a linear price curve as defined in (1.2.2) so that $P = \alpha - \beta Q$.

We recall that the scaling argument of Section 1.3.1.1 allows us to set $\alpha = 1$, $\beta = 1$ without loss of generality. We do so for the price curve in Sections 5.3 and 5.5, but use the unscaled form of the price curve in Section 5.4. This is firstly so as to be consistent with the results and presentation of [60], and secondly to aid our presentation of the results in Section 5.5.[2] We define the price and production caps in this Section using the unscaled price curve so as to be consistent with both the scaled and unscaled forms of later sections.

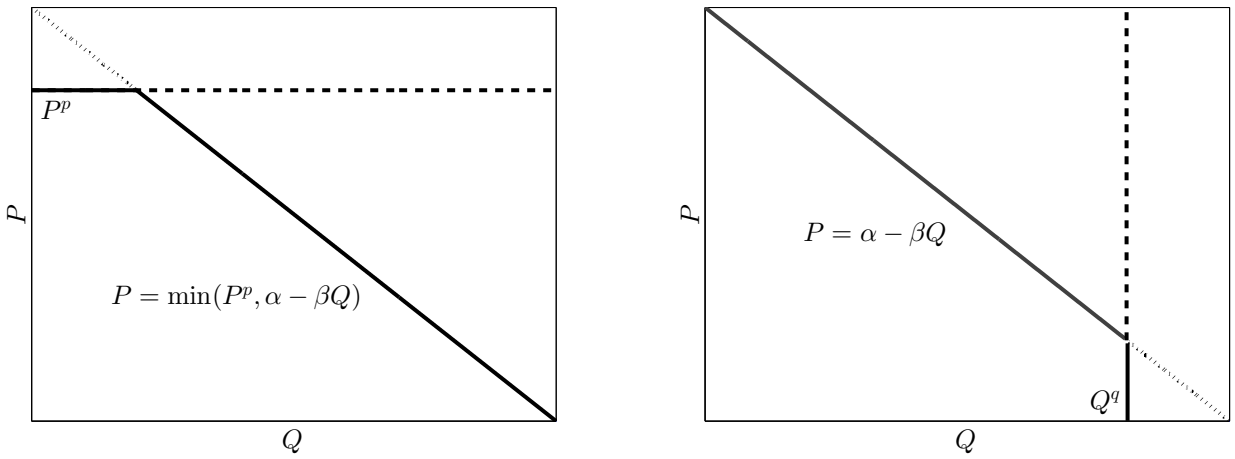
We assume that firm i pays extraction costs at rate c_i, and that the firms are indexed in ascending order of cost such that $0 \leq c_1 \leq c_2 \leq ... \leq c_N$. We next introduce price and production caps. In order to differentiate between them, we shall use the superscript notation p and q to denote caps on price and production respectively.

5.2.1 Price Caps

In the first class of models we consider, we assume an exogenous cap on the price the market can set. If we define the level of this cap by P^p, and the level of demand corresponding to this in the price curve (1.3.1) by $Q^p = (\alpha - P^p)/\beta$, then the price curve with the cap is given by

$$P(Q) = \min(\alpha - \beta Q, P^p).$$

[2] We show that the penalty of this section can be incorporated as a change in the values of the parameters α, β.



(a) Price Cap

(b) Production Cap

Figure 5.1: The Price Curve under caps on price and production.

We illustrate a price curve under a price cap in Figure 5.1a. Throughout this Chapter, we make the assumption

$$P^p > \max(c_1, ..., c_N), \tag{5.2.1}$$

so as to ensure that the price cap does not price any of the firms out of the market.

5.2.2 Production Caps

In the second class of models we consider, we assume a cap on the total level of supply within the market. This could arise as a result of either an exogenously determined policy, such as in [107], or a shared capacity constraint, such as in [68]. Such caps are common in privatised energy, broadband, and water markets in particular, as a result of the sharing of transmission infrastructure amongst multiple firms.[3]

We define the level of the cap on production by Q^q, and the price P^q associated with it by $P^q = \alpha - \beta Q^q$. We illustrate the price curve under a production cap in Figure 5.1b. The production cap is different to a price cap in that it does result in a floor on the market price, but does so via an explicit constraint on total production, as opposed to a change to the price curve.

[3]See [24].

5.3 Illustration: Two Players

To illustrate how price and production caps impact Cournot markets, we first consider a duopoly example of each. This restriction allows for the best-response functions of each player to be represented as one-dimensional functions, which can then be readily illustrated to demonstrate the impact of the caps on the equilibria of the game.

5.3.1 Price Cap

In addition to Assumption (5.2.1) we assume that the players' extraction costs are such that there is no blockading and both players participate.[4] The players' value functions are given by

$$v_1 = \int_0^\infty \mathrm{e}^{-rt} \left[q_1 \min(1 - q_1 - q_2, P^p) - c_1 q_1\right] \mathrm{d}t,$$
$$v_2 = \int_0^\infty \mathrm{e}^{-rt} \left[q_2 \min(1 - q_1 - q_2, P^p) - c_2 q_2\right] \mathrm{d}t.$$

Because we are assuming that the resource is inexhaustible, the players set production so as to maximise revenue at each point in time. The maximization problem therefore becomes a static one, with the players' equilibrium strategies (q_1^*, q_2^*) given by

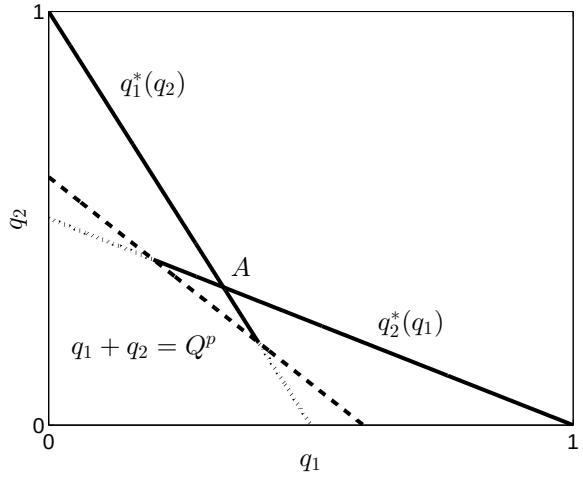
$$q_i^* = \arg\max_{q_i \geq 0} \left\{q_i \min(1 - q_i - q_{-i}^*, P^p) - c_i q_i\right\} \qquad i = 1, 2, \tag{5.3.1}$$

where q_{-i}^* is the equilibrium production of the competing player. Under the assumptions we have made, we can show that no equilibrium exists for which $q_1^* + q_2^* < Q^p$.

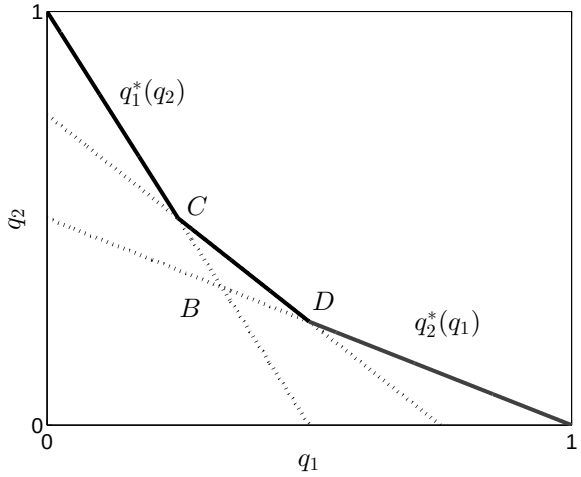
Suppose the contrary, and $q_1^* + q_2^* = Q^p - \delta$ for some $\delta > 0$. It is trivial to show that either player could gain more value by increasing production by δ, as doing so would increase their quantity sold while leaving the price constant at P^p. We therefore have that if (q_1^*, q_2^*) is a valid equilibrium, then $q_1^* + q_2^* \geq Q^p$. Consequently, we incorporate the cap as an explicit minimum constraint on total production. The best-response functions are found by solving equation (5.3.1) for each player with the constraint $0 \leq q_i \leq Q^p - q_{-i}$, and are therefore given by

$$q_1 = \max\left(\frac{1}{2}(1 - q_2 - c_1), Q^p - q_2, 0\right),$$
$$q_2 = \max\left(\frac{1}{2}(1 - q_1 - c_2), Q^p - q_1, 0\right).$$

[4] As we showed in Example 1.3.1.1, this is the case if $2c_2 - 1 \leq c_1 \leq \frac{1}{2}(1 + c_2)$.



(a) $(2 - c_1 - c_2)/3 \geq Q^p$



(b) $(2 - c_1 - c_2)/3 < Q^p$

Figure 5.2: Response functions and equilibria under price caps. In 5.2a, the best-response functions intersect above the constraint line $q_1 + q_2 = Q^p$; the price cap does not restrict production. In 5.2b, the unconstrained best-response functions intersect below the constraint line $q_1 + q_2 = Q^p$. The price cap induces a range of equilibria along CD.

We determine the equilibria of the game by finding the points at which these best-response functions intersect.

Proposition 5.3.1. *If $(2 - c_1 - c_2)/3 \geq Q^p$, then equilibrium production is given by*

$$q_1^* = \frac{1}{3}(1 - 2c_1 + c_2), \qquad q_2^* = \frac{1}{3}(1 - 2c_2 + c_1).$$

If $(2 - c_1 - c_2)/3 < Q^p$, then no unique Nash Equilibrium exists. Instead, there is a range of equilibria such that

$$1 - c_1 - Q^p \leq q_1^* \leq 2Q^p + c_2 - 1, \qquad q_2^* = Q^p - q_1^*.$$

Proof. Figure 5.2a shows the response functions when $(2 - c_1 - c_2)/3 \geq Q^p$. The players' best-response functions intersect at point A above the line $Q^p = q_1 + q_2$, and the unconstrained Nash Equilibrium strategy results in the market price falling short of the price cap. Figure 5.2b shows the response functions when $(2 - c_1 - c_2)/3 < Q^p$. The players' best-response functions intersect at point B below the line $Q^p = q_1 + q_2$ and the constraint $q_1 + q_2 \geq Q^p$, imposed by the price cap, becomes binding. The best-response

functions therefore overlap on the line segment CD, creating a range of equilibria. Points C and D are given by (q_1^C, q_2^C), and (q_1^D, q_2^D), where

$$1 - 2q_1^C - q_2^C - c_1 = 0, \qquad Q^p - q_1^C - q_2^C = 0,$$
$$1 - 2q_2^D - q_1^D - c_2 = 0, \qquad Q^p - q_1^D - q_2^D = 0,$$

so that

$$q_1^C = 1 - c_1 - Q^p, \qquad q_2^C = 2Q^p + c_1 - 1,$$
$$q_1^D = 2Q^p + c_2 - 1, \qquad q_2^D = 1 - c_2 - Q^p.$$

□

We note that R, the length of the range of possible values of both q_1^* and q_2^*, is given by

$$R = 3Q^p + c_1 + c_2 - 2,$$

so that R depends on the total extraction costs paid by the two players, but not on the difference between those costs. This is a result of our assumption of a linear price curve. Recalling that the unconstrained best-response for player i is given by $q_i = (1 - q_{-i} - c_i)$, a change in c_i results in an upwards or downwards translation of the best-response curve. Because the gradient of the line $q_1 + q_2 = Q^p$ representing the price cap is -1, the effect on R of increasing c_i is offset by decreasing c_{-i} by the same amount; R is determined by the total cost.

5.3.2 Production Cap

The value functions of the two players are given by

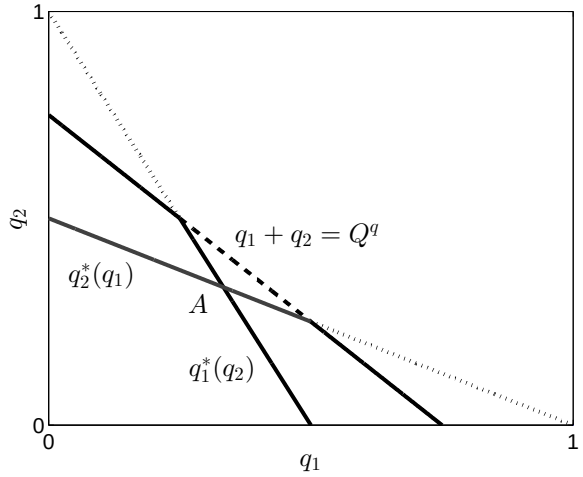
$$v_1 = \int_0^\infty \mathrm{e}^{-rt} q_1(1 - q_1 - q_2 - c_1)\mathrm{d}t,$$
$$v_2 = \int_0^\infty \mathrm{e}^{-rt} q_2(1 - q_1 - q_2 - c_2)\mathrm{d}t,$$

with the constraint

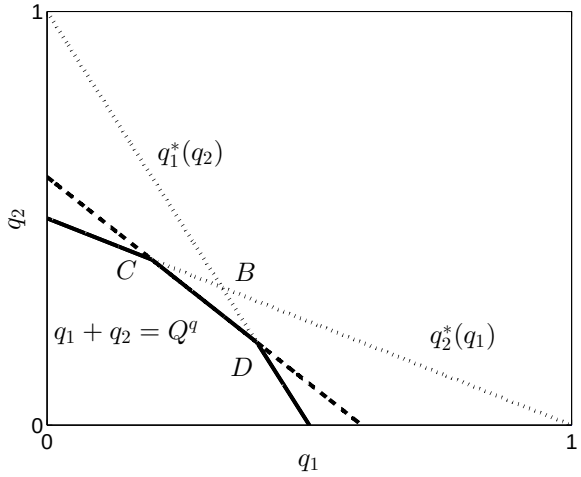
$$q_1 + q_2 \leq Q^q.$$

Carrying over the same market assumptions from the game with a price cap, the best-response functions of the two players are now given by

$$q_1 = \min\left(\frac{1}{2}(1 - q_2 - c_1), Q^q - q_2\right), \qquad q_2 = \min\left(\frac{1}{2}(1 - q_1 - c_2), Q^q - q_1\right).$$



(a) $(2 - c_1 - c_2)/3 \leq Q^q$



(b) $(2 - c_1 - c_2)/3 > Q^q$

Figure 5.3: Response functions and equilibria under common production caps. In 5.3a, the best-response functions intersect below the line $q_1 + q_2 = Q^q$; the production cap becomes irrelevant. In 5.3b, the unconstrained response functions intersect above the line $q_1 + q_2 = Q^q$. The production cap induces a range of equilibria along CD.

We determine the equilibria of the game by finding the points at which these best-response functions intersect.

Proposition 5.3.2. *If $(2 - c_1 - c_2)/3 \leq Q^q$, then production is given by*

$$q_1^* = \frac{1}{3}\left(1 - 2c_1 + c_2\right), \qquad q_2^* = \frac{1}{3}\left(1 - 2c_2 + c_1\right).$$

If $(2 - c_1 - c_2)/3 > Q^q$, then no unique Nash Equilibrium exists. Instead, there is a range of equilibria such that

$$2Q^q + c_2 - 1 \leq q_1^* \leq 1 - c_1 - Q^q, \qquad q_2^* = Q^q - q_1^*.$$

Proof. Figure 5.3a shows the response functions when $(2 - c_1 - c_2)/3 \leq Q^q$. The players' best-response functions intersect at point A below the line $Q^q = q_1 + q_2$. The unconstrained Nash Equilibrium strategy therefore results in total production below the cap Q^q. Figure 5.3b shows the response functions when $(2 - c_1 - c_2)/3 > Q^q$. The players' best-response functions intersect at point B above the line $Q^q = q_1 + q_2$ and the production cap becomes binding. The best-response functions therefore overlap on the line

segment CD, creating a range of equilibria. Points C and D are given by (q_1^C, q_2^C) and (q_1^D, q_2^D) where

$$1 - 2q_2^C - q_1^C - c_2 = 0, \qquad Q^q - q_1^C - q_2^C = 0,$$
$$1 - 2q_1^D - q_2^D - c_1 = 0, \qquad Q^q - q_1^D - q_2^D = 0,$$

so that

$$q_1^C = 2Q^q + c_2 - 1, \qquad q_2^C = 1 - c_2 - Q^q,$$
$$q_1^D = 1 - c_1 - Q^q, \qquad q_2^D = 2Q^q + c_1 - 1.$$

□

We note that R is still dependent on $c_1 + c_2$ only, but is now given by

$$R = 2 - 3Q^q - c_1 - c_2.$$

5.3.3 Cooperative Solutions Under Joint Constraints

In this section, we provide a brief digression on the impact of joint constraints on cooperative solutions, recalling the discussion of Section 1.1.3. For simplicity we assume the players extract at the same cost, which we can set to 0 without loss of generality by the scaling argument of Section 1.3.1.1. This allows us to apply the result of [147] which states that the Pareto frontier of such a game is that along which the total value across the two players is equal to the value a monopolist player would receive. In the absence of price or production caps, the monopoly solution, denoted by a superscript M, is given by

$$v^M = \frac{1}{4r}, \qquad q^M = \frac{1}{2}.$$

The Pareto Frontier in (v_1, v_2) is therefore given by the line $v_1 + v_2 = 1/4r$.

5.3.3.1 Price Caps

Under a price cap, if $P^p < 1/2$, then the price cap is binding on the monopoly solution. It is therefore given by $V^M = P^p Q^p / r$. Total production under the unconstrained Nash Equilibrium is given by $2/3$, resulting in a price of $1/3$. It is therefore clear that any price cap which is binding on the non-cooperative game will also be binding on the monopoly solution. If we assume such a price cap, then the Pareto frontier is given by the line

$v_1 + v_2 = P^p Q^p / r$. There is a range of equilibria, resulting in a range of value functions, given by

$$P^p(1 - Q^p) \leq r v_1 \leq P^p(2Q^p - 1), \qquad v_2 = \frac{1}{r} P^p Q^p - v_1.$$

It is clear that these points all lie on the line $r(v_1 + v_2) = P^p Q^p$, and therefore lie on the Pareto frontier. In Figure 5.4a, we show the Pareto frontier for $P^p = 0.3$ and $r = 1$. The range of equilibria is given by AB. Because this line lies on the Pareto frontier, the players can neither resolve the non-uniqueness problem, nor increase their total value by cooperating.

5.3.3.2 Production Caps

Under a production cap, it is possible to have $1/2 < Q^q < 2/3$, so that the cap is binding on the Nash Equilibrium, but not on the monopoly game. In this case, the range of equilibrium value functions is given by

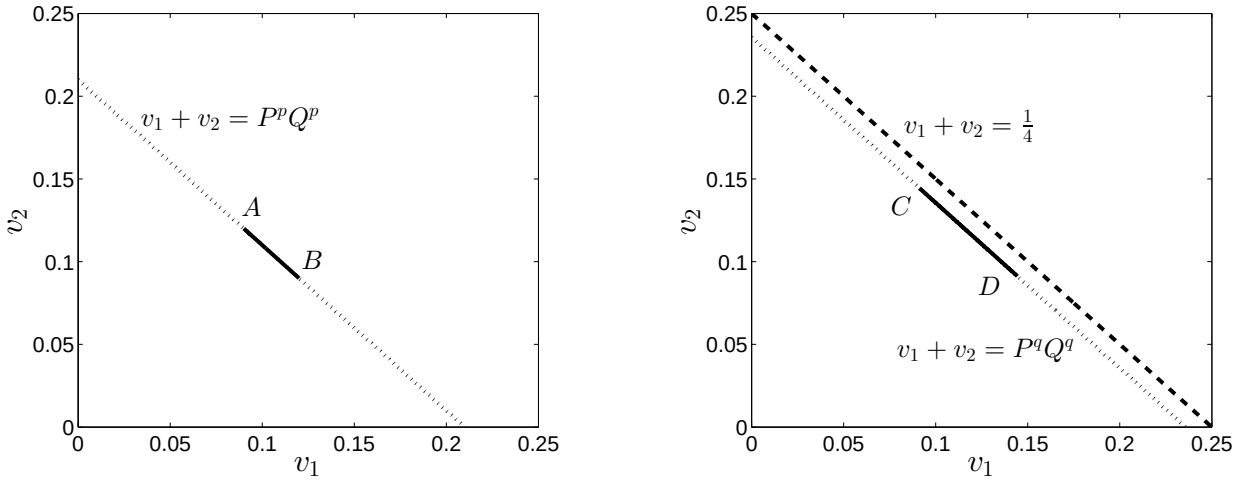
$$P^q(2Q^q - 1) \leq r v_1 \leq P^q(1 - Q^q), \qquad v_2 = \frac{1}{r} P^q Q^q - v_1,$$

which is illustrated in Figure 5.4b by CD for $Q^q = 0.62$. This line is parallel to the Pareto frontier but lies closer to the origin. While this means that the players are technically able to increase their total value by cooperating, it does not resolve the non-uniqueness problem. To see this, recall from Section 1.1.3.1 that most mechanisms for cooperative bargaining use the Nash Equilibrium (threat point) as a reference point. If there is a range of threat points, giving different feasible bargaining sets, then any bargaining mechanism would result in a range of potential outcomes.

The players would again be required either to be externally compelled to adopt a given threat point, or to adopt one by convention, for example, by adopting the midway point on the equilibrium range as the threat point.

We summarise the results of this section below:

- Both production and price caps can result in a range of equilibria, depending on whether total production under the uncapped equilibrium is too high or low respectively;
- The range of equilibria is given by the segment of the constraint line that coincides with both players' response functions;



(a) Price Cap

(b) Production Cap

Figure 5.4: Joint constraints and cooperative bargaining.

- The level of P^p at which multiple equilibria result, as well as the size of the resultant range of equilibria, increases with $c_1 + c_2$. The level of Q^q for which multiple equilibria result, as well as the size of the resultant range of equilibria, decreases as $c_1 + c_2$ increases.

- Price and production caps can disrupt cooperative bargaining mechanisms as well. Price caps leave no opportunity for cartelisation gains if they are binding on the non-cooperative game. It is possible for production caps to allow for cartelisation gains even when binding on the non-cooperative game, but the problem of non-uniqueness of the threat point remains.

5.4 Price and Production Caps with Many Players

In this section we consider both price caps and supply caps in a game with N players. Existence and uniqueness of equilibria for the game with price caps have been derived in [109] and [60] respectively, and so we simply present the results. We follow a similar process to [60] in analysing the uniqueness of equilibria for the game with production caps, and combine the results for price and production caps in our analysis of the game in which both constraints are imposed simultaneously.

In all cases, we consider the unscaled linear price curve given by (1.3.1) so as to be consistent with the presentation of [60]. Furthermore, we make the simplifying assumption that

$$\alpha - (N+1)c_N + \sum_{i=1}^{N} c_i \geq 0. \tag{5.4.1}$$

It is straightforward to show that making this assumption results in no blockading for the N player Cournot game without price or production caps.[5]

5.4.1 Price Caps

Once again, we assume a price cap P^p such that the players' value functions are given by

$$v_i = \int_0^\infty \mathrm{e}^{-rt} \left[q_i \min(\alpha - \beta(q_i - Q_{-i}), P^p) - c_i q_i \right] \mathrm{d}t, \qquad i = 1, ..., N,$$

where $Q_{-i} = \sum_{j=1, j\neq i}^{N} q_j$, α and β are scalar constants parametrising the price curve, and with the assumption

$$P^p > \max(c_1, ..., c_N).$$

The derivation of existence and uniqueness results for the game with price caps is carried out in [60], so we simply present the key results below.[6]

By the same logic as in the two player game, it can be shown that there are no equilibria in which $\sum_{i=1}^{N} q_i < Q^p$. Making this an explicit constraint on the set of admissible strategies of the game allows it to be reformulated in vector notation as the Linear Complementarity Problem

$$0 \leq \mathbf{q} \perp \beta(I + E)\mathbf{q} - \alpha \mathbf{1} + \mathbf{c} - \mu \geq 0, \tag{5.4.2}$$

$$0 \leq \mu \perp E\mathbf{q} - Q^p \mathbf{1} \geq 0, \tag{5.4.3}$$

where $\mathbf{q} = (q_1, ..., q_N)$, $\mu = (\mu_1, ..., \mu_N)$, and $\mathbf{c} = (c_1, ..., c_N)$, $\perp$ implies that the left- and right-hand side expressions are orthogonal, and where I and E are the $N \times N$ identity and unit matrices respectively. We have $\mathbf{q}^*$ is an equilibrium if and only if there exists a vector μ such that $(\mathbf{q}^*, \mu)$ is a solution of (5.4.2)-(5.4.3). It is shown in [60] that if $Q > Q^p$, then the equilibrium $\mathbf{q}^*$ is unique.[7] We call this equilibrium $\mathbf{q}^{\mathbf{A}}$, and

[5] This follows trivially from (5.4.4) and the assumed ordering of the c_i values, but a more detailed proof can be found in [88].

[6] We follow similar steps in deriving the analogous results for the game with production caps in Appendices I-K.

[7] The proof follows similar steps to the parallel result for production caps in Appendix J.

define $Q^A = \sum_{i=1}^N q_i^A$. This equilibrium is not derived in [60] but under the assumption $Q^A > Q^p$, we can derive it by straightforward differentiation. The equilibrium is given by

$$q_i^A = \frac{\alpha - (N+1)c_i + \sum_{j=1}^N c_j}{\beta(N+1)}, \qquad i = 1, ..., N. \tag{5.4.4}$$

This is the sole equilibrium of the game if the condition $Q^A > Q^p$ is met. This condition is given by

$$\alpha N - \sum_{i=1}^N c_i > (N+1)\beta Q^p. \tag{5.4.5}$$

Next we assume that $\sum_{i=1}^N q_i = Q^p$. It is shown in [60] that the solution set Ω_1 is given by

$$\Omega_1 = \left\{ q \in \mathbf{R}_+^N \left| \begin{array}{l} 0 \leq q_i, \ \dfrac{\alpha - \beta Q^p - c_i}{\beta} \leq q_i \qquad \forall i \in \{1, ..., N\} \\ \\ \displaystyle\sum_{i=1}^N q_i = Q^p \end{array} \right. \right\}. \tag{5.4.6}$$

If (5.4.5) is met then $\mathbf{q^A}$ is the sole equilibrium of G_1, otherwise the equilibrium set is given by Ω_1.

We expand on this result by providing an alternative geometric interpretation below in Section 5.4.1.1. The geometric interpretation of the production caps result in Section 5.4.2 follows similar lines and is therefore omitted from this thesis.

5.4.1.1 Geometric Illustration: Multiple Players

The constraint $\sum_{i=1}^N q_i = Q^p$, and the players' unconstrained best-response functions, given by $q_i = \max((\alpha - \beta Q_{-i} - c_i)/2\beta, 0)$ each represent hyperplanes of dimension $N-1$ in $(q_1, ..., q_N)$. Player i's best-response under the constraint is therefore given a piecewise affine $N-1$–dimensional manifold, such that it is equal to the constraint hyperplane for $0 \leq Q_{-i} \leq (2\beta Q^p + c_i - \alpha)/\beta$, after which it is given by the unconstrained best-response hyperplane.

The players' unconstrained best-response hyperplanes intersect at a unique point $\mathbf{q^A}$, given by (5.4.4). When this point is nearer to the origin than the constraint hyperplane, then it is is clearly barred by the price cap. The joint inter-section of the best-response functions under the price cap is given by an N sided polytope of dimension $N-1$. Each side is of dimension $N-2$ and the i-th side is given by the intersection of player i's best-response hyperplane with the constraint hyperplane.

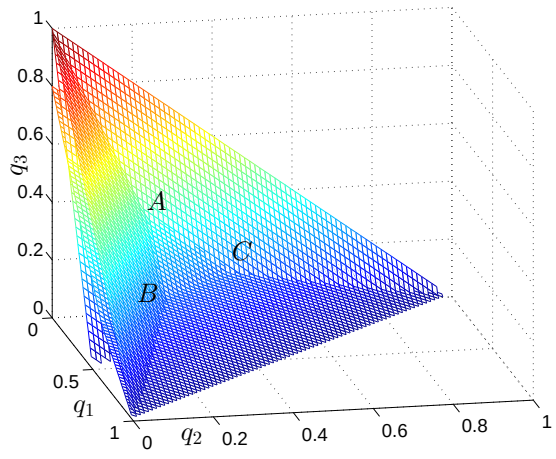


Figure 5.5: Graphical illustration of a three player game under a price cap, with $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. The price cap results in multiple equilibria, given by the triangle ABC.

We provide a graphical illustration for three players in Figure 5.5 for $\alpha = 1$, $\beta = 1$, $c_1 = 0$, $c_2 = 0$, $c_3 = 0$. The constraint and the unconstrained response functions for each player are given by 2 dimensional planes. The best-response function of player i under the constraint is given by $q_i^* = \max((1 - Q_{-i})/2, Q^p - Q_{-i}, 0)$. The intersection of the players' best-response functions is given by the area of the triangle ABC.

This also allows us to provide a geometrical interpretation of the significance of the parameters α, β, c, and Q^p. The values of α, β, and c determine the orientation and position of the players' best-response hyperplanes, while lowering or raising Q^p shifts the constraint hyperplane towards or away from the origin. Each of these clearly impacts whether or not $\mathbf{q^A}$ is precluded by the constraint hyperplane and the region of equilibria that result if it is.

5.4.2 Production Caps

The value function for each player is given by

$$v_i = \int_0^\infty \mathrm{e}^{-rt} \left[q_i(\alpha - \beta(q_i + Q_{-i})) - c_i q_i \right] \mathrm{d}t, \quad i = 1, ..., N, \tag{5.4.7}$$

with the constraint

$$Q \leq Q^q,$$

where $Q = \sum_{i=1}^{N} q_i$. The problem can be solved as before by reformulating it as a Linear Complementarity Problem

Proposition 5.4.1. $\mathbf{q}^*$ *is an equilibrium, if and only if there exists a vector* μ *such that* $(\mathbf{q}^*, \mu)$ *is a solution of*

$$0 \leq \mathbf{q}^* \perp \beta(I + E)\mathbf{q}^* - \alpha\mathbf{1} + \mathbf{c} + \mu \geq 0, \tag{5.4.8}$$

$$0 \leq \mu \perp Q^q\mathbf{1} - E\mathbf{q}^* \geq 0. \tag{5.4.9}$$

Proof. See Appendix I. □

In order to analyse the game's existence and uniqueness, we require the following result.

Proposition 5.4.2. *If there are two solutions* $\mathbf{q}^{(1)}$ *and* $\mathbf{q}^{(2)}$*, then* $\sum_{i=1}^{N} \mathbf{q}_i^{(1)} = \sum_{i=1}^{N} \mathbf{q}_i^{(2)}$.

Proof. See Appendix J. □

We can now consider the uniqueness of the game's equilibria

Proposition 5.4.3. *If* $\mathbf{q}^*$ *is an equilibrium of the game and total production is given by* $Q^* = \sum_{i=1}^{N} q_i^*$*, then there are 2 mutually exclusive possibilities:*

(a) $Q^* < Q^q$*, with a unique equilibrium, given by* $\mathbf{q}^{\mathbf{B}}$*:*

$$q_i^B = \frac{\alpha - (N+1)c_i + \sum_{j=1}^{N} c_j}{\beta(N+1)},$$

which occurs if and only if

$$\alpha N - \sum_{i=1}^{N} c_i < (N+1)\beta Q^p;$$

(b) $Q^* = Q^q$*, with the associated equilibrium set given by* Ω_2*:*

$$\Omega_2 = \left\{ q \in \mathbf{R}_+^N \;\middle|\; \begin{array}{c} 0 \leq q_i \leq \dfrac{\alpha - \beta Q^q - c_i}{\beta} \qquad \forall i \in \{1, ..., N\} \\ \\ \displaystyle\sum_{i=1}^{N} q_i = Q^q \end{array} \right\}. \tag{5.4.10}$$

Proof. See Appendix K. □

We recall that the assumption (5.4.1) is sufficient to guarantee that $q_i^B > 0$, $i = 1, ..., N$. As an aside we note that it can be trivially shown that relaxing this assumption and allowing blockading could lead to more firms producing in equilibrium as a result of the equilibrium-distorting effect of the imposition of a production cap.

To see this, recall that a production cap can also be formulated as a price minimum. If player i is blockaded in the absence of the production cap, it is because the uncapped equilibrium results in a price lower than c_i. Therefore, setting a cap so that $P^p > c_i$ will result in that player now being able to produce. In other words, a production cap can ensure that the price does not fall below the price at which players would be blockaded, even if they would be blockaded in the unconstrained equilibrium.

5.4.3 Simultaneous Caps on Price and Production

The theory used in Sections 5.4.1 and 5.4.2 can also be combined to analyse games in which a price cap and production cap exist simultaneously. We assume a price cap $P^p = \alpha - \beta Q^p$ and production cap $Q^q = (\alpha - P^q)/\beta$, such that $Q^p \leq Q^q$. The same argument used in the two player illustration can be used to show that there are no equilibria for which $Q < Q^p$, and so the price cap can be incorporated as an explicit constraint on production in the same way as the production cap. The value functions are given by

$$v_i = \int_0^\infty \mathrm{e}^{-rt} \left[q_i(\alpha - \beta(q_i + Q_{-i})) - c_i q_i \right] \mathrm{d}t,$$

with the constraint

$$Q^p \leq \sum_{i=1}^{N} q_i \leq Q^q.$$

Proposition 5.4.4. *Total production in equilibrium is unique, such that if there are two equilibria* $\mathbf{q}^{(1)}$ *and* $\mathbf{q}^{(2)}$*, then* $Q^{(1)} = Q^{(2)}$*. If* $\mathbf{q}^*$ *is an equilibrium, then there are* 2 *mutually exclusive possibilities:*

(a) $Q^p < Q^* < Q^q$*, and there is a unique equilibrium, given by* $\mathbf{q}^{\mathbf{C}}$*:*

$$q_i^C = \frac{\alpha - Nc_i + \sum_{j=1, j\neq i}^{N} c_j}{\beta(N+1)},$$

if

$$(N+1)\beta Q^p < \alpha N - \sum_{i=1}^{N} c_i < (N+1)\beta Q^q;$$

(b) otherwise $Q = Q^p$ or $Q = Q^q$, the equilibrium sets are given by (5.4.6) *and* (5.4.10) *respectively.*

Proof. Given in Appendix L. □

5.5 Penalty Model with Multiple Players

In this section, rather than a production cap or price cap, we assume that an external body imposes a penalty if the price rises above a certain level P^p. We once again define the quantity associated with this price as Q^p such that $P^p = 1 - Q^p$, where we return to the scaled form of the price curve. This can be thought of as a softer form of price cap, more along the lines of a windfall tax,[8] imposed only if prices are deemed to reach supernormal levels. We also show in Appendix M that some forms of market entry games can be reformulated as a problem of this type.

The results we derive can also be readily adjusted to analyse a game with a linear congestion tariff.[9] Congestion tariffs, in which companies incur an additional cost if production goes over a certain point, can be imposed by regulators in an attempt to manage demands on shared infrastructure. However, they can also arise organically within markets if there are additional costs in procuring key production inputs, such as specialised labour and equipment, once the market reaches a certain critical mass. Moreover, the method we use to solve this problem, that of decomposition of the game into tractable sub-games, is one which we believe can be applied to similar piece-wise continuous problems.

We assume that the penalty F_i, applied to player i is given by

$$F_i = \begin{cases} (A + Bq_i)(Q^p - Q) & \text{for } Q < Q^p, \\ 0 & \text{for } Q > Q^p. \end{cases}$$

The penalty model encompasses the case $A = 0$, where the penalty becomes a linear ad valorem penalty, and $B = 0$, where the penalty becomes a levy that is imposed on all players equally and irrespective of relative production.[10] Firm i's value function is as follows:

$$v = \int_0^\infty \mathrm{e}^{-rt} \left[q_i(1 - q_i - Q_{-i} - c_i) - F_i \right] \mathrm{d}t. \tag{5.5.1}$$

[8] See [19].

[9] See Appendix N for this formulation.

[10] The corresponding game for a congestion charge in Appendix N involves a cap on production, such that the penalty is applied if $Q > Q^q$.

We once again make the assumption (5.2.1) that $P^p > \max(c_1, ..., c_N)$. In order to avoid blockading in equilibrium, we make the additional simplifying assumption[11]

$$\min(1 - BQ^p + A, 1) - (N + 1)c_N + \sum_{i=1}^{N} c_i \geq 0.$$

We begin by deriving the best-response functions of an individual player, before considering existence and uniqueness of solutions.

5.5.1 Best-response Functions

Proposition 5.5.1. *The response function of player i to the sum of production by the competing players Q_{-i} is given by*

$$q_i = \begin{cases} \frac{1}{2(1-B)}(1 - BQ^p + A - (1-B)Q_{-i} - c_i) & \text{for } 0 < Q_{-i} \leq \frac{1}{1-B}(2Q^p - BQ^p - 1 - A + c_i), \\ Q^p - Q_{-i} & \text{for } \frac{1}{1-B}(2Q^p - BQ^p - 1 - A + c_i) < Q_{-i} \\ & \qquad < 2Q^p - 1 + c_i, \\ \frac{1}{2}(1 - Q_{-i} - c_i) & \text{for } 2Q^p - 1 + c_i \leq Q_{-i} \leq 1 - c_i, \\ 0 & \text{otherwise}, \end{cases} \tag{5.5.2}$$

which we have illustrated for player i in Figure 5.6.

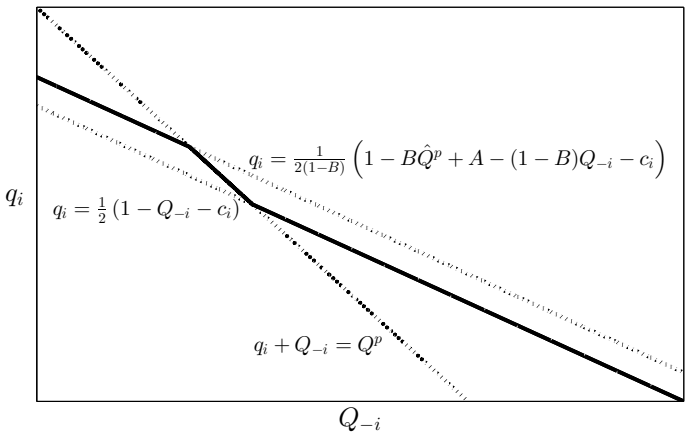


Figure 5.6: Best-response function for player i

[11] That this assumption precludes blockading follows trivially from (5.5.4) and (5.5.6).

Proof. To ease notation, we refer to the interval $0 \leq q_i + Q_{-i} \leq Q^p$ by L, and refer to the interval $q_i + Q_{-i} > Q^p$ by R. We also define

$$F_L(q_i) = q_i(1 - q_i - Q_{-i} - c_i) - (A + Bq_i)(Q^p - q_i - Q_{-i}),$$
$$F_R(q_i) = q_i(1 - q_i - Q_{-i} - c_i).$$

Player 1 therefore needs to maximise

$$F(q_i) = \begin{cases} F_L(q_i) & \text{for } 0 < q_i + Q_{-i} \leq Q^p, \\ F_R(q_i) & \text{for } q_i + Q_{-i} > Q^p. \end{cases}$$

The global maximum of F_L is reached at $q_i = (1 - BQ^p + A - (1 - B)Q_{-i} - c_i)/(2 - 2B)$. This point only lies within L for $0 < Q_{-i} < (2Q^p - BQ^p - 1 - A + c_i)/(1 - B)$, recalling that $(2Q^p - BQ^p - 1 - A + c_i)/(1 - B) > 0$ by assumption. The global maximum of F_R is reached at $q_i = (1 - Q_{-i} - c_i)/2$. This point only lies within R for $2Q^p - 1 + c_i < Q_{-i} < 1 - c_i$. For $Q_{-i} > 1 - c_i$, player i is blockaded and $q_i = 0$.

Under our assumption that $Q^p < 1 - c_N$, it can be shown that $(2Q^p - BQ^p - 1 - A + c_i)/(1 - B) < 2Q^p - 1 + c_i$. Within this interval, the global maximum of F_L lies in R, and the global maximum of F_R lies in L. The function F is therefore maximised at $q_i = Q^p - Q_{-i}$, which is an endpoint maximum for both F_L and F_R.

□

5.5.2 Solution via LCP Methods

We make use of theory in [109] and [60], which relies on reformulating the game as a linear complementarity problem for the analysis of existence and uniqueness of equilibria. However, the piecewise continuous nature of the penalty applied means that the Nash-Cournot game cannot be formulated into a linear complementarity problem directly. Instead, we consider two separate games, G_1 and G_2. The aim of this is to separate the set of admissible strategies by Q in such a way that the resulting sets, combined with the associated pay-off function values, allow for the use of LCP methods. We analyse G_1 and G_2 and use the solution sets of the two games to generate the solution set of the full game, G_F.

The set of admissible strategies for G_1, S_1 is given by

$$S_1 = \left\{ q \in \mathbf{R}_+^N \,|\, Q \geq Q^p \right\},$$

and each player has payoff

$$v_i^{(1)} = \int_0^\infty \mathrm{e}^{-rt} q_i(1 - q_i - Q_{-i} - c_i)\mathrm{d}t, \qquad i = 1, ..., N.$$

The set of admissible strategies for G_2, S_2 is given by

$$S_2 = \left\{ q \in \mathbf{R}_+^N \,|\, Q \leq Q^p \right\},$$

and each player has payoff $v_i^{(2)}$:

$$v_i^{(2)} = \int_0^\infty \mathrm{e}^{-rt} \left[q_i(1 - q_i - Q_{-i}) - c_i q_i - (A + Bq_i)(Q^p - q_i - Q_{-i}) \right] \mathrm{d}t, \qquad i = 1, ..., N. \tag{5.5.3}$$

Each of these games can be solved explicitly as a linear complementarity problem which we show allows for a full description of their solution sets.

5.5.3 Game G_1

The maximisation problem is given by

$$q_i^* = \arg\sup_{q_i^* > 0} \left\{ \int_0^\infty \mathrm{e}^{-rt} q_i(1 - q_i - Q_{-i} - c_i)\mathrm{d}t \right\},$$

with the constraint $Q \geq Q^p$. It is clear that this problem is therefore equivalent to that of the multiple player game with price caps with $\alpha = 1$ and $\beta = 1$. Using the results of Section 5.4.1, we therefore have that there is a unique equilibrium q^A:

$$q_i^A = \frac{1 - (N+1)c_i + \sum_{j=1}^N c_j}{(N+1)}, \tag{5.5.4}$$

if the condition $Q^A > Q^p$ is met. This condition is given by

$$N - \sum_{i=1}^N c_i > (N+1)Q^p. \tag{5.5.5}$$

If $\sum_{i=1}^N q_i = Q^p$ then there is no unique equilibrium, but a set of equilibria Ω_1:

$$\Omega_1 = \left\{ q \in \mathbf{R}_+^N \,\middle|\, \begin{array}{cc} 0 \leq q_i,\, 1 - Q^p - c_i \leq q_i & \forall i \in \{1, ..., N\}, \\ \sum_{i=1}^N q_i = Q^p & \end{array} \right\}.$$

5.5.4 Game G_2

If we ignore all terms in (5.5.3) with no dependence on q_i and rearrange, the optimal strategies are given by the equivalent maximisation problem

$$q_i^* = \arg\sup_{q_i^*>0} \left\{ \int_0^\infty \mathrm{e}^{-rt} q_i(1 - BQ^p + A - (1-B)(q_i + Q_{-i}) - c_i)\mathrm{d}t \right\},$$

with the constraint $Q \leq Q^p$. It is clear that this problem is equivalent to that of the multiple player game with production caps, with $\alpha = 1 - BQ^p + A$ and $\beta = 1 - B$. Using the results of Section 5.4.2, we therefore have that there is a unique equilibrium q^B, given by

$$q_i^B = \frac{1 - BQ^p + A - (N+1)c_i + \sum_{j=1}^N c_j}{(1-B)(N+1)}, \tag{5.5.6}$$

if the condition $Q^B < Q^p$ is met. This condition is given by

$$N(1 - BQ^p + A) - \sum_{i=1}^N c_i < (N+1)(1-B)Q^p. \tag{5.5.7}$$

If $\sum_{i=1}^N q_i = Q^p$ then there is no unique equilibrium, but a set of equilibria Ω_2 given by

$$\Omega_2 = \left\{ q \in \mathbf{R}_+^N \;\middle|\; \begin{array}{ll} 0 \leq q_i \leq \dfrac{1 + A - Q^p - c_i}{1 - B} & \forall i \in \{1, ..., N\}, \\ \displaystyle\sum_{i=1}^N q_i = Q^p & \end{array} \right\}.$$

5.5.5 Equilibria of the Full Game

We now construct the set of equilibria of the full game, G_F from the equilibrium sets of G_1 and G_2.

Proposition 5.5.2. *Any equilibrium of G_F must be an equilibrium in G_1 or G_2*

Proof. First note that if q^* is an equilibrium of G_F then $q^* \in S_1 \cup S_2$. If $q^* \in S_1$ and is an equilibrium then it is a best-response to all strategies $q' \in S_1 \cup S_2$, and is therefore also a best-response to all strategies $q' \in S_1$. Given that the payoff function for all players in games G_1 and G_F are the same $\forall q \in S_1$, we have that q^* is an equilibrium of G_1. By the same reasoning, $q^* \in S_2$ implies that q^* is an equilibrium of G_2. □

Proposition 5.5.3. *For $Q^* = Q^p$, the equilibrium set Ω_F in the full game is given by*

$$\Omega_F = \left\{ q \in \mathbf{R}_+^N \;\middle|\; \begin{array}{l} 1 - Q^p - c_i \leq q_i \leq \dfrac{1 - BQ^p + A - (1-B)Q^p - c_i}{(1-B)} \qquad \forall i \in \{1, ..., N\}, \\ \displaystyle\sum_{i=1}^{N} q_i = Q^p \end{array} \right\}.$$

Proof. Note that if q' is an equilibrium of the full game and $Q' = Q^p$, then it must be an equilibrium in both G_1 and G_2. Similarly, if q' is an equilibrium in both G_1 and G_2, then q' is a set of mutual best-responses to all strategies in both sets S_1 and S_2, and is therefore an equilibrium of the full game. The equilibrium set of the full game is therefore given by $\Omega_F = \Omega_1 \cap \Omega_2$, which is as given above. □

There are two equilibria remaining, q^A and q^B. We show that both q^A and q^B are unique equilibria.

Proposition 5.5.4. *Both q^A and q^B are unique equilibria.*

Proof. We begin by showing that q^A is a valid equilibrium if it is an equilibrium of G_1. First note that if we assume that all players apart from i play strategy q^A, then we have

$$Q_{-i} = \frac{(N-1)(1+c_i) - 2\sum_{j=1, j\neq i}^{N} c_j}{N+1}.$$

substituting this into the best-response function for player i given in (5.5.2) results in

$$q_i = \frac{1 - (N+1)c_i + \sum_{j=1}^{N} c_j}{N+1},$$

if (5.5.5) is met. Because (5.5.5) is required in order for q^A to be a valid equilibrium of G_1, then if q^A is an equilibrium of G_1 then it is an equilibrium of the full game. The process is similar for q^B. If all players apart from player i adopt strategy q^B then we have

$$Q_{-i} = \frac{(N-1)(1 - BQ^p + A + c_i) - 2\sum_{j=1, j\neq i}^{N} c_j}{(N+1)(1-B)},$$

Substituting this into (5.5.2) gives

$$q_i = \frac{1 - BQ^p A - (N+1)c_i + \sum_{j=1}^{N} c_j}{(N+1)(1-B)},$$

if (5.5.7) is met. Because (5.5.7) must be met in order for q^B to be a valid equilibrium of G_2, we have that q^B is an equilibrium of the full game if it is an equilibrium of G_2.

We have shown that q^A and q^B are both valid equilibria under certain conditions. In order to prove that they can only occur as unique equilibria, we show that (5.5.5) and (5.5.7) are mutually inconsistent. In order for both conditions to hold, we must have

$$\frac{N(1-BQ^p+A)-\sum_{i=1}^N c_i}{(N+1)(1-B)} < Q^p < \frac{N-\sum_{i=1}^N c_i}{N+1}. \tag{5.5.8}$$

It follows from Assumption A that if we define $\hat{\mathbf{c}} = \left(\sum_{i=1}^N c_i\right)/N$ we have

$$\begin{aligned}\frac{N(1-BQ^p+A-\hat{\mathbf{c}})}{(N+1)(1-B)} &> \frac{N(1-BQ^p-\hat{\mathbf{c}})}{(N+1)(1-B)} \\ &> \frac{N(1-B)(1-\hat{\mathbf{c}})}{(N+1)(1-B)} \\ &= \frac{N-\sum_{i=1}^N c_i}{N+1},\end{aligned}$$

so that (5.5.8) cannot hold. □

5.6 Joint Constraints and Exhaustibility

In this section, we identify and discuss an important problem for future research, that of incorporating joint constraints into multiple player games with resource exhaustibility. To our knowledge, this is a problem that is not only open, but as yet uncommented on in the literature.

For a one-player game with exhaustibility, it is a well known result that a price cap can influence the behaviour of the extracting firm, even if it is not immediately binding.[12] To illustrate this, we recall from Example 1.4.1.2 that the monopolist's shadow cost of extraction can be expected to increase over the life of the game. This results in the firm decreasing production as the resource level decreases, which increases the price. If the price cap is set in such a way as to limit these increases in some later stage of the game, this changes the game's boundary conditions. Doing so changes the solution of the game's Bellman equations and therefore also changes the shadow costs and optimum production rate *at each prior point of the game.*

[12] See [51].

In the same way, in a game with multiple competing firms,[13] even if a price cap would not immediately impact on the equilibrium production of either player, if it does so at a later stage of the game by virtue of the players' rising shadow costs, then this could result in multiple equilibria beyond that point. This is highly problematic, insofar as each equilibrium corresponds to a different set of boundary equations on the game's Bellman PDEs, and therefore a different set of production trajectories.

We illustrate this problem in more detail by example. We consider a differential Feedback-Loop duopoly game in which player 1 extracts from a finite but free resource, and produces from an infinite but costly to produce from backstop after it is depleted, while player 2 can only produce from the backstop.[14] We first consider the game in the absence of a price cap, and then discuss the implications of its inclusion.

We define x_t as the level of player 1's resource at time t and assume that player 2 pays extraction costs at the rate of c per unit production from the backstop, where $0 \leq c \leq 1/2$.[15] In the absence of a price cap, the value functions for the two players are given by

$$v_1 = \int_0^\infty \mathrm{e}^{-rt} q_1 \left(1 - q_1 - q_2 - c\mathbf{1}_{x_t=0}\right) \mathrm{d}t,$$
$$v_2 = \int_0^\infty \mathrm{e}^{-rt} q_2 \left(1 - q_1 - q_2 - c\right) \mathrm{d}t.$$

Following the theory in Appendix B, the game's solution can be found by solving the ODE system:

$$rv_1 = q_1(1 - q_1 - q_2) - q_1 v_1', \qquad rv_2 = q_2(1 - q_1 - q_2 - c) - q_2 v_2',$$

where $v_1' = \mathrm{d}v_1/\mathrm{d}x$, $v_2' = \mathrm{d}v_2/\mathrm{d}x$, and

$$q_1 = \frac{1}{3}(1 - 2v_1' + c), \qquad q_2 = \frac{1}{3}(1 - 2c + v_1'),$$

with boundary conditions $v_1(0) = (1/9)(1-c)^2$, $v_2(0) = (1/9)(1-c)^2$. The system is solved in [62], and the solution is given by

[13] Such as the Feedback-Loop game of Chapter 3, and [62].
[14] This game is analysed in detail in [62].
[15] This is sufficient to preclude blockading.

$$\frac{1}{1+c}\left(3\sqrt{rv_1}-(1-c)\right)+\log\left(\frac{1+c-3\sqrt{rv_1}}{2c}\right)=-\frac{9rx}{4(1+c)},$$
$$v_2=\frac{c(5c-2)}{3r}\left(\frac{v_1'}{c}\right)^{\frac{4}{3}}+\frac{1}{9r}\left((1-2c^2)+8(1-2c)v_1'-2(v_1')^2\right).$$

The price is given by $P=(1/3)(1+v_1'+c)$. As x_t goes from ∞ to 0, v_1' goes from 0 to c. The price therefore goes from $(1/3)(1+c)$ to $(1/3)(1+2c)$ as the resource is depleted.

To assess the impact of a price cap we consider $P^p=0.5$ for $c=1/3$. In the unconstrained game the price goes from 4/9 to 5/9 as the resource goes from ∞ to 0, so that the price cap is binding on the game in equilibrium for a range of x. The game can be viewed as a static game with player 1's cost given by its shadow cost v_1', and c. Therefore, we can expect multiple equilibria for all resource levels for which $v_1'>1/6$.

This means that there is a range of x for which there are multiple equilibria, lying on $1-v_1'-Q^p\leq q_1\leq 2Q^p+c-1$, $q_2=Q^p-q_1$, which corresponds to the line segment CD in Figure 5.2b. Different choices on this line segment lead to different solutions in this range, and indeed to different ranges of x in which $v_1'>1/6$ and the price cap binds. This therefore leads to different solutions for larger values of x as well.

We illustrate this in Figure 5.7 and Figure 5.8. Each of the solution curves adopt the common boundary conditions $v_1'(0)=c$, $v_2'(0)=0$. Solution 1 plots the value functions and production rates for the game in which player 1 produces at their upper limit, and player 2 produces at their lower limit. The production rates for the players are given by

$$q_1=2Q^p+c-1,\qquad q_2=1-Q^p-c,$$

and the ODEs for $v_1'>1/6$ are given by

$$rv_1=(2Q^p+c-1)\left(P^p-v_1'\right),$$
$$rv_2=(1-Q^p-c)^2-v_2'\left(2Q^p+c-1\right).$$

Solution 3 plots the value functions and production rates for the game in which player 1 produces at their lower limit, and player 2 produces at their upper limit. The production rates for the players are given by

$$q_1=1-v_1'-Q^p,\qquad q_2=2Q^p+v_1'-1,$$

and the ODEs for $v_1' > 1/6$ are given by

$$rv_1 = \left(1 - v_1' - Q^p\right)\left(P^p - v_1'\right),$$
$$rv_2 = \left(2Q^p + v_1' - 1\right)\left(P^p - c\right) - v_2'\left(1 - v_1' - Q^p\right).$$

Solution 2 plots the value functions and production rates for the game in which player 1 and player 2 both produce at the midway point of their admissible equilibrium production range (as an illustration of a solution-by-convention in the style of the egalitarian cooperative bargaining mechanism of Section 1.1.3.1). These are given by

$$q_1 = \frac{1}{2}\left(Q^p + c - v_1'\right), \qquad q_2 = \frac{1}{2}\left(Q^p + v_1' - c\right),$$

and the resultant ODEs for $v_1' > 1/6$ are given by

$$v_1 = \frac{1}{2}\left(Q^p + c - v_1'\right)\left(P^p - v_1'\right),$$
$$v_2 = \frac{1}{2}\left(Q^p - c + v_1'\right)\left(P^p - c\right) - \frac{1}{2}v_2'\left(Q^p + c - v_1'\right).$$

For all three solutions, we recall from above that the production rates and ODEs for the game for all x such that $v_1' \leq 1/6$ are given by

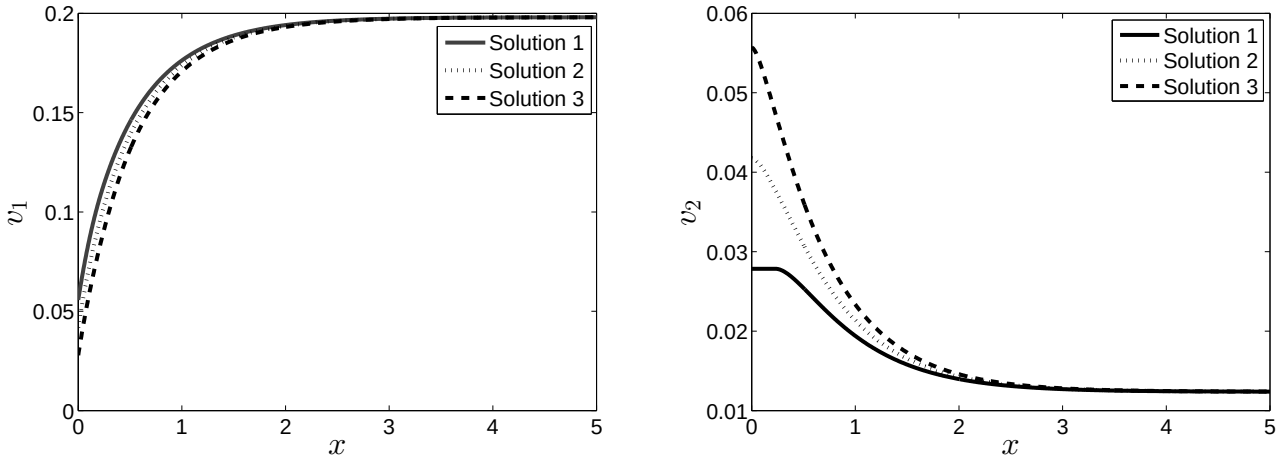
$$q_1 = \frac{1}{3}\left(1 - 2v_1' + c\right), \qquad q_2 = \frac{1}{3}\left(1 - 2c + v_1'\right),$$
$$v_1 = \frac{1}{9}\left(1 - 2v_1' + c\right)^2, \tag{5.6.1}$$
$$v_2 = \frac{1}{9}\left(1 - 2c + v_1'\right)^2 - \frac{1}{3}v_2'\left(1 - 2v_1' + c\right), \tag{5.6.2}$$

with boundary condition $v_1'(\hat{\mathbf{x}}) = 1/6$

We make several observations.

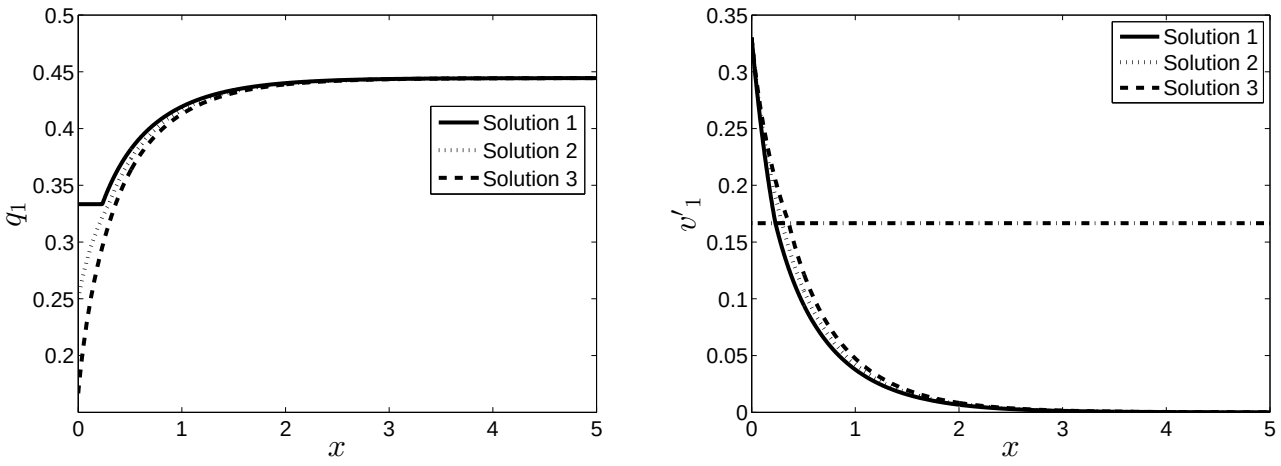
- If we define $\hat{\mathbf{x}}$ as the resource level at which $v_1' = 1/6$ in Figure 5.8b, then the range at which the price cap is binding is given by $0 \leq x < \hat{\mathbf{x}}$. For $x \geq \hat{\mathbf{x}}$, the price cap is no longer binding, and the game's ODEs switch to (5.6.1)-(5.6.2).

- $\hat{\mathbf{x}}$ varies across the solutions. This is because each solution involves a different split of the implicit production minimum, Q^p, which in turn results in different rates of profit to the players, and therefore a different path for the shadow cost for player 1. Consequently, $\hat{\mathbf{x}}$ also changes. This is a key difficulty with any method for splitting production which references the players' different costs; any such method influences the shadow cost faced by player 1, thereby shifting its reference point.



(a) Player 1 (b) Player 2

Figure 5.7: Value Functions



(a) Production (b) Shadow Cost

Figure 5.8: Player 1 Production and Shadow Cost

- The difference in solutions after the price cap is no longer binding is solely a function of the variation in $\hat{\mathbf{x}}$ across the solutions. The solutions from there on are translations of each other, so that $v_1(x + \hat{\mathbf{x}})$ is the same regardless of how the players produce in the region where $x < \hat{\mathbf{x}}$.

- Under Solution 1, q_1 is constant for $0 \leq x < \hat{\mathbf{x}}$. This is because the upper bound of the equilibrium values for q_1 only depends on c, and not v_1'. This means that the upper bound of the admissible equilibrium range does not change as x is depleted. The lower bound depends on v_1'. This changes with x, which is why q_1 is not constant under Solutions 2 and 3. This is also why v_2 is constant for $0 \leq x < \hat{\mathbf{x}}$ under Solution 1.

Because of these issues, resolving the non-uniqueness problem introduced by these joint constraints, even for simplified one-dimensional games, is highly non-trivial.

5.7 Conclusion

In this Chapter, we have considered static Cournot games with joint constraints on price and production. We have illustrated how these constraints can result in multiple equilibria in a 2 player setting. We discussed how this also complicates cooperative bargaining solutions, because the occurrence of multiple threat points result in there being no unique reference point from which to construct a bargaining solution. In an N player setting, we extended the uniqueness results under price caps in [60], to consider games with production caps, and games in which both production and price caps occur simultaneously.

We also considered piecewise linear charges that result in an additional cost to players if a joint condition on total production is satisfied. In doing so, we showed that, even where exceeding a production cap or price cap is penalised, and not prohibited outright, multiple equilibria can still result. Moreover, we believe that our method of solving this problem by decomposition into tractable subgames, for which standard LCP methods can be applied, is one that could also add value to the analysis of uniqueness results for similar piecewise continuous games.[16]

[16] Existence results are analysed for similar games in [109], but uniqueness is not considered.

This field of research is still relatively new, and there are a number of connected areas for future research. In particular, to our knowledge there is currently no existing research in the literature which considers the problem of games with joint constraints on production or price in a multiple-player setting with resource exhaustibility, which we discussed in Section 5.6. This is a highly non-trivial problem, and solving it may require the development of alternative equilibrium concepts.

We also note that we have only considered joint constraints in a Cournot setting. It follows trivially that the ordered decision making under Stackelberg competition means that the leader can unilaterally determine how joint constraints are met by the two players, via their announced strategy. This would mean that the imposition of the joint constraint would no longer result in multiple equilibria. We recall from Example 1.3.2.1 that, under certain circumstances, it can be profitable for a Stackelberg leader to strategically inflate their production in order to blockade the follower. A similar result that has arisen in our analysis of test problems is that it is possible for a production cap in a Stackelberg game to result in the leader producing at the level of the cap, even where total production in the unconstrained equilibrium lies below it. Given this, we believe that the study of strategic responses to joint constraints by Stackelberg leaders is an interesting area for future research.

Chapter 6

Conclusions

6.1 Conclusions

We have discussed our results in the concluding sections of each chapter, but we present a condensed summary below.

6.1.1 Chapter 2: Feedback-Loop Stackelberg Competition

Our conclusions from studying the Feedback-Loop Stackelberg model can be grouped into three strands. Firstly, we identified that, for certain regions in the (x_1, x_2) grid, v_{2,x_2} is negative. In other words, the value function of the follower is decreasing in its own resource level. This is a counter-intuitive result because it means that, were the follower able to dispose of some of their resource pool, then it is possible they would be better off doing so. Furthermore, this also translated to counter-intuitive non-monotonicity in the follower's value function, and in the production rates for both players. We identified this phenomenon in both the game with and without the backstop, in both one-dimensional and two-dimensional settings.

Our explanation of the result is consistent with those offered for similar kinds of non-monotonicity found in the resource extraction models of [14] and [15]. In these two cases, the free disposal criterion was not violated because the resource itself was of negative worth, but because having a higher resource level induced a change in the strategies of the competing firm(s), which decreased the players' value from the game more than the gain they received from the additional resource itself. In this model, we argued that such an effect is caused by the follower's additional resource level resulting

in the leader adopting a more aggressive production strategy to partially drive the seller out of the market.

A secondary result came from our analysis of the game with a backstop technology, on the axis $x_2 = 0$. In Cournot competition, blockading is only possible if one player's production as a monopolist is sufficient to reduce the price below the other player's cost of production. In Stackelberg competition, it is possible for the leader to blockade the follower, even where this is not the case, by pre-committing to production above their monopoly rate. In the axis game, we found that for $c > 1/3$, there are ranges of x_1 for which this occurs, resulting in periods of level production from the leader at the rate of $1 - c$, the minimum production level at which the follower is blockaded.

Finally, the small c approximation represents the corrections to the players' production and value functions arising from the introduction of a limited degree of exhaustibility into the game. This allowed us to briefly comment on the marginal impact of that exhaustibility on the players.

Firstly, we noted that, if a player has a sufficiently higher resource level than their competitor, then their production and value corrections would be positive. On one level, this appears to be a counterintuitive result, insofar as it indicates that the player is producing more, and benefiting from, the introduction of exhaustibility. However, this is simply a function of the fact that the benefit they receive from the smaller player running out and decreasing their production after switching to the backstop, arrives significantly sooner than the loss resulting from their own exhaustion. The discounting effect results in the former offsetting the latter in net present value terms.

We also noted that, for this to occur, the leader requires a significantly higher surplus over the follower than the follower does for the leader. This arose firstly from the fact that the follower's production is much lower than the leader's, resulting in a smaller increase in revenue for the leader once the follower is depleted, and secondly from the fact that the leader's higher production speeds forward their own exhaustion, reducing the discounting effect.

6.1.2 Chapter 3: Open-Loop vs Feedback-Loop

We discuss our findings for both the Cournot and Stackelberg models considered in Chapter 3. Under Cournot competition, we found that, when the asymmetry in the resource levels of the two players becomes sufficiently high, the larger player produces at

a higher rate and has a higher value function than the small player. This also means that trajectories move more rapidly towards the line $x_1 = x_2$ under the Feedback-Loop game, whereas under the Open-Loop game, resources become relatively more concentrated in the hands of the large player over time.

The reason for this is that the Open-Loop game requires each player to anticipate the other player's trajectory over the entire duration of the game, whereas under the Feedback-Loop game, they are able to condition their strategy on the resource level of the other player, as well as their own. This implicitly provides a relative advantage to the larger player. This is because the players' strategies become more sensitive to the level of a resource as it decreases. Consequently, the large player being able to condition their strategy on the resource level of the small player confers more of a benefit than the small player being able to condition their strategy on the resource level of the large player.

Under Stackelberg competition, we found that differences between the two approaches are only material where the leader has significantly higher reserves than the follower. Importantly, it is under this same condition that the asymmetric market power of the Stackelberg model is most likely to be justifiable from an economic point of view. In this region, both players have a higher value function under the Feedback-Loop game than the Open-Loop game, even though the leader produces less and the follower produces more. We argued that this further supports the claim we made in Chapter 2–that the non-monotonicities in the players' strategies in the Feedback-Loop Stackelberg competition are indicative of a form of directed cooperation, whereby the leader allows the nearly-depleted follower to produce more and "burn brightly before it goes", while the leader benefits from being able to produce as a monopolist sooner.

6.1.3 Chapter 4: Common Property Open-Loop

We confirmed the result of [79] that, while the derivation of the solution of the Feedback-Loop game yields only one equilibrium (Equilibrium F), that of the Open-Loop game yields two candidate equilibria (Equilibria A and B). Comparing these, we found that the players' value functions are lower under Equilibrium F than either Open-Loop equilibrium, while production is higher. The value and production functions for Equilibrium B are between A and F for all resource levels. This result differs from the separate pools result of Chapter 3, in which there are regions where a player's value function is higher

under the Feedback-Loop game than the Open-Loop game, and indeed the total value and production are higher and lower respectively everywhere. The difference lies in the fact that, under the common property game, the Markov property of the Feedback-Loop strategies gives the players an additional incentive to pre-emptively extract.

For the remainder of the Chapter, we focused our attention on Equilibrium B. Even though the trajectory associated with the corresponding solution of the Feedback-Loop boundary condition can be dismissed as an equilibrium, Equilibrium B is still arguably more intuitive than A, insofar as it involves the production rate approaching 0 as the resource level approaches 0. This means that the solution is continuous at $x = 0$, which we found also occurs for the corresponding separate pools models of Chapters 2 and 3. On the other hand, production under Equilibria A and F each approach strictly positive values as the resource level approaches 0, and then jump to 0 when the resource is fully depleted.

We argued that, despite this intuitive appeal, Equilibrium B is not a valid equilibrium. We highlighted the implicit assumption in the determination of the equilibria that the players anticipate the same terminal time for the game, and constructed a numeric counter-example of a strategy which allows a player to extract more than dictated by Equlibrium B and receive more value than v^B, at the same time bringing forward the termination of the game and decreasing the value of the other players. We supplemented this with an analytic argument in the style of [65] and [63], in which we showed that Equilibrium B is not a valid equilibrium of the game if a small perturbation is introduced, in the form of a terminal payment, even as this perturbation approaches 0.

6.1.4 Chapter 5: Price Caps and Joint Constraints

We provided a brief two player illustration, along the lines of [24], of the equilibrium-disrupting effects of joint constraints. If the price or production cap bars the Nash equilibrium solution, then multiple equilibria result and the players produce at the level dictated by the cap. We added that the existence of multiple equilibria in such cases also complicates cooperative bargaining solutions, as this results in multiple threat points, leaving no unique reference point from which to construct a bargaining solution.

We next considered games with multiple players, and extended the uniqueness results for games with price caps derived in [60], to games with production caps, and games in which both price and production caps are applied simultaneously. We also showed that

these results can be used to study games with linear windfall penalties, which apply if the price exceeds a certain level, or congestion charges, which come into effect if the total production exceeds a certain level.

For each type of penalty, we showed that, even though exceeding the cap is now permitted and simply penalised, it is possible for their imposition to result in multiple equilibria as before. Importantly, we argued that the approach applied in solving this game, that of decomposition into tractable sub-games, for which the price and production caps results of Section 5.4 could be applied, is a method which could also be used to study other games with similar piecewise continuous joint constraints.

6.2 Future Directions

We have already discussed possible future work which directly ties to the extensions put forward in this thesis within the concluding sections of their respective chapters. Here, we briefly discuss several other promising areas for future research within the field of exhaustible resource extraction games.

6.2.1 Backstop Adoption and Bargaining

While the focus of this thesis has been on the behaviour of resource extracting firms, there is also a wide pool of research which models interactions between resource-exporting countries (sellers), and the governments of resource-importing countries (buyers). This is particularly so for resource markets in which the country's production of the resource is often nationalised, such as oil.

The adoption of alternative backstop sources by buyers is clearly an important concern for sellers, as it could lead to a significant decrease in the size of their market. Many papers, such as [135], model this risk as a stochastic event, or one which occurs after a specified period of time, with the primary focus being the seller's optimal response to this (exogenously determined) event. However, several recently-published papers consider this issue from a different perspective, and model backstop adoption as a strategic option available to buyers. This approach involves consideration both of how the buyer should exercise their backstop option, and how the seller can strategically influence this exercise via their supply and pricing policies.

In [123] and [53], problems of this type are approached from discrete and dynamic game perspectives respectively. Both adopt models of a market with one resource buyer and one seller, in which the buyer has the ability to make an investment in a backstop which then becomes available after a specified period of time. The purpose of this lag is to represent the time taken to research and implement the backstop technology after the decision to invest in, and eventually adopt it, has been made. While a stochastic lag may seem more reasonable, a sample problem studied in [53] indicates that such an extension would not materially alter the model's result.

The results of the two papers are similar. Each shows that, of the variables relevant to the buyer's decision to invest, two can be controlled by the seller: the current supply of the resource made available to the buyer, and the level of the seller's resource remaining. As the seller's resource level decreases, deferring investment in the backstop becomes more costly for the buyer, insofar as there is less of the resource to meet the buyer's demand in the intermediate period between investment and adoption. This results in periods of the game for which the seller must maintain an increasing supply trajectory over time in order to compensate the buyer for maintaining their dependence on the resource by deferring their investment. This is a singular result, in that it defies the usual trend for optimal supply trajectories to be downward trending over time as a result of the discounting effect.

A number of possible extensions are considered in [53]. One which we wish to highlight is that of asymmetric information. In [53], an extension is studied in which only the seller knows the level of their resource. This is motivated by the fact that many of the world's key oil exporting countries, such as Saudi Arabia, do not allow independent audits of their reserve levels. Therefore, the only means by which the buyer can update their prior belief regarding the seller's resource level is via the signalling effect of the seller's supply decisions.

It is found that, for certain resource levels, it is profitable for a small seller to signal to the buyer that they are a large seller, by mimicking a large seller's supply strategy.[1] This is is harmful to the buyer in two ways: firstly because conveying this signal involves

[1]This results in what is called a *pooling equilibrium*, which means that, for some values, the seller's optimal supply signal does not uniquely match to the resource level. Pooling equilibria occur when the cost of sending a signal (in this case, the cost to the seller of overproducing to misrepresent their resource level) is lower than the benefit they gain from that signal being accepted and acted on (in this case, by the buyer deferring their investment decision).

the seller overproducing and depleting their resource more rapidly; secondly because it results in a persistent risk of a supply shock occurring, if the buyer defers their investment decision beyond the point at which there remains a sufficient quantity of the resource to cover their demand in the intermediate adoption period.

We identify several possible extensions to this theory which have not, to our knowledge, been explored. Firstly, relaxing the common restriction that the games involve only a single seller and buyer could be fruitful. The model of [53] showed that a buyer's investment decision was based on the total supply offered to them, and the total remaining quantity of that resource held for future sale. In a model which included multiple exporting countries, this decision would be dependent on their total production and resource holdings. Therefore, such an extension could indicate issues around the non-uniqueness of equilibria, similar to those relating to the joint constraints on production analysed in Chapter 5. We have confirmed this effect for test calculations carried out on highly simplified two-period models.

Perhaps even more promising is the consideration of multiple buyers, under the assumption of limited secondary trading between countries. In [53], at the point at which the buyer invests, the seller is then forced to sell as much of their resource to the buyer that can be sold without reducing their marginal rate of profit below 0, before the buyer's switch to the backstop is complete.[2] If another buyer were present in the market, this effect would be diminished.

We make an additional conjecture, applying to games in which the buyers make their decision to invest at different times, resulting in intermediate periods in which one player has exercised their investment option but not yet transitioned, while the other player still has their option active. In this case, the seller may have an incentive to retain additional stock for release to the buyer that has not yet made the investment, given that this active buyer's decision will depend on the remaining supply of the resource available to them. In other words, this could give the exporter a means of credibly committing to an extraction profile which is (coincidentally) punitive to the player which invests first, thereby possibly influencing that player's investment decision. We have investigated this problem, however, analysis is complicated by the game's intermediate period's dependence on both the seller's resource level, and the time remaining until the buyer that has already invested transitions to the backstop.

[2]This is because the seller's market disappears when the buyer switches to the backstop.

Furthermore, there is also potential for the inclusion of other actors. One such promising extension relates to the citizenries of resource exporting states. In [25], it is argued that some states, such as Saudi Arabia, face significant internal political pressure to keep oil exports above the game-theoretic optimal rate, and that this creates an incentive for the government to under-invest in its extraction, refinement, storage, and transport infrastructure. By doing this, the state can use this under-investment to create a notionally exogenous limit on their production, so that they can allocate their extraction over time more profitably, without being perceived to be underproducing by their citizenries. It would be interesting to see how such a consideration interacts with that of backstop switching from buyers, particularly in an asymmetric information setting.

Finally, we previously mentioned the model of asymmetric information in the form of the seller's resource level discussed in [123]. A related extension to this, raised in [53], could be the reverse case, in which the buyer has private information about their backstop option. We note that this information could relate to the details of the backstop option available to them, or to whether or not the buyer has chosen to invest. For such an extension to be fruitful, we add that any model considered would require an additional mechanism by which the buyer could provide a signal to the seller about their private information (in the same way that the seller uses their supply decisions in [53]). Possible signalling mechanisms include buyer-side taxation policies designed to adjust internal consumption and pricing preferences for the resource, or the buyer being able to make false announcements about their backstop option and stage of investment. We note that this extension may lead to further interesting results if studied in a multiple buyer setting, given our discussion of that extension above.

6.2.2 Green Paradoxes

We discussed Green Paradoxes in Section 1.4.2. There is a growing literature on the subject, but a great deal of it is dependent on several highly restricting assumptions.

Firstly, much of the literature assumes that firms operate as price takers in the market, and automatically produce along a Hotelling extraction path. While this assumption generally greatly simplifies calculations, such models are not well-equipped to examine possible linkages between future regulatory impositions and strategic produc-

tion responses from firms in the market. Instances of non-standard market elements resulting in non-Hotelling price paths abound in the literature.

The learning by doing model of [20], in which uncertainty in the size of the resource pool can be reduced through extraction, resulted in a U-shaped production path for certain parameter values. In [92], a bracket-based ad valorem taxation system is studied and found to result in periods of piecewise constant production. On green taxes, [41] analyses a game in which an extractor faces a future tax based on the environmental efficiency of extracting and refining their deposits. This model showed that the producer was likely to extract from their least efficient deposits first in order to beat the tax, even though they were also more costly to extract from. This not only resulted in a Green Paradox and non-Hotelling price path, but in the producer violating the commonly-held principle that firms prioritise extraction from resource pools with lower extraction costs.

Of the literature that does not adopt this assumption, most does so in either a highly simplified two period discrete time model, or dynamically in an Open-Loop setting. In the latter case, these models require that firms assume future tax paths as given and commit to their production trajectories for the entire duration of the game. This is problematic, because a number of important aspects of both explicit and implicit green taxes depend on variables other than time.

In the absence of strong externally binding commitments on governments to maintain announced green tax trajectories in the future, analysing the responses of extracting companies requires consideration of the effects of uncertainty implicit in these trajectories. This is not a question to which the Open-Loop approach is well suited, because it does not allow firms to respond, either to changes in current levels of green taxes, or to events and actions which should cause the firm to update their prior expectations relating to future green tax levels.

Outwith uncertainty in future tax policies, there are a number of important considerations for which the use of Open-Loop models is inappropriate. Firstly, the argument is made in [80] and [41] that optimal green tax mechanisms should also consider variables other than time, such as the current shortfall from emissions targets, or the environmental efficiency of producing from a given resource at a time. Analysis of such state-dependent policies require a Feedback-Loop style approach. Furthermore, in analysing interaction effects between green taxation policies and competitive behaviour, the draw-

backs of Open-Loop models with regards to modelling that competitive behaviour, as outlined in Chapter 3, still hold.

These considerations are even more important when analysing Green Paradoxes associated with the development of backstop technologies. Here, the key variable for firms is the path of the production costs of backstops. These can be expected to implicitly decrease over time, but the case has been made in papers such as [27] that this is better modelled via their explicit drivers.

The primary drivers of backstop costs are investments in research and development, as well as current production levels if the resource is one which benefits from learning by doing or economies of scale.[3] Furthermore, the relationship between backstop costs contains a degree of uncertainty, insofar as the gains from research and development are usually at least partly stochastic. Analysing the development of backstop costs as anything other than deterministic functions of time requires more complex models than can be dealt with in an Open-Loop setting. While analysis would be more complicated under a Feedback-Loop approach, we argue that it would be more appropriate for modelling purposes.

A second assumption, typical of most models which allow for firms to act in a price-making capacity, is that of a Cournot market. We discussed briefly in Section 5.7 how external joint constraints can be exploited for additional gain by a Stackelberg leader. One interesting question is firstly, whether the impact of green taxes on a market with asymmetric competition is also asymmetric, and secondly, whether or not firms in this markets react in a materially different way to their imposition than under a Cournot competition.

In [16], the connection between market structure and responses to green taxes is analysed via the example of the oil market. It is shown that, if the market is assumed to function as a cartel vs fringe model, with OPEC as the cartel, and other oil producers as a competitive price-taking fringe, then OPEC will respond to a typical carbon tax by lowering production to maintain the oil price. Because of this, the fringe producers will experience very little change in their wealth, while OPEC suffers most of the loss. If the market is assumed to be competitive, then non-OPEC producers experience the majority of the loss of wealth. This result also goes towards providing a possible explanation for the different attitudes of oil exporting countries to possible future green taxation. We

[3] Such as the example of the electric car industry studied in [27].

believe that extending this to consider other types of green taxes and other forms of competition could be similarly fruitful.

6.2.3 Extraction with Resource-Dependent Costs

The model of [122] analyses a Cournot duopoly game with separate pools, and with extraction costs which increase linearly as the resource level of the player decreases. Extraction costs are paid by player $i = 1, 2$ at the rate $C_i = c_{0,i} - c_i X_i$, where X_i is player i's total stock remaining, and $c_{i,0}$ and c_i are positive scalars. Under the assumption that the choke price ($\hat{\mathbf{p}}$) is less than $c_{i,0}$, then it is clear that the player will reach a point at which they cannot produce profitably, even though they have a strictly positive quantity of resource remaining. Defining $\hat{\mathbf{X}}_i$ so that $c_{i,0} - c_i \hat{\mathbf{X}}_i = \hat{\mathbf{p}}$, the authors then define $x_i = X_i - \hat{\mathbf{X}}_i$ as the firm's *economic* resource level, i.e. the portion of their resource pool which can be sold for a profit over the course of the game.

The motivation for this approach is that extraction often requires more advanced or labour intensive processes, and therefore becomes more expensive as the resource is depleted, to the extent that it is often not economically viable to extract the entirety of a resource pool. However, in [122], it is also shown that this model of extraction costs conveniently yields solutions in which the production strategies are linear in the economic resource levels, so that q_i can be given in the form $q_i = ax_i + bx_{-i}$ where a, b are positive scalars, and where x_{-i} is the economic resource level of the competing player. An interesting area for future research would be extending this approach to Feedback-Loop Stackelberg and differentiated Bertrand models to determine whether or not such a cost model has a similar effect in inducing linear strategies.

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Appendix A

The Pontryagin Maximum Principle

We rely heavily on the Pontryagin maximum principle (PMP) in deriving the Open-Loop solutions in this thesis. There are multiple formulations of the PMP, our statement of it comes from [1] Section 7.5 as it adheres most closely to the results of this thesis. The proof of the PMP is quite long, and it is possible to use the result without it. We therefore provide the statement of the result, but refer the reader to [26] for a formal proof of it.

We consider the following discounted infinite-horizon control problem

$$\max_{x(t),u(t)} J(x(t),u(t)) = \int_0^\infty \mathrm{e}^{-rt}\pi(x(t),u(t))\mathrm{d}t, \tag{A.0.1}$$

subject to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x(t),u(t)), \tag{A.0.2}$$

and

$$u(t) \in U(t),\, x(t) \in X(t) \text{ for all } t,\, x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq x_1, \tag{A.0.3}$$

where $U(t) \subset \mathbf{R}^m$, $X(t) \subset \mathbf{R}^n$ are the admissible sets for the values of the control and state variables respectively, and with $\pi : \mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^m \to \mathbf{R}$ and $f : \mathbf{R}\times\mathbf{R}^n\times\mathbf{R}^m \to \mathbf{R}^n$.

We also define

$$V(t_0,x_0) = \sup_{x(t),u(t)\in X(t)\times U(t)} \int_{t_0}^\infty \mathrm{e}^{-rt}\pi(x(t),u(t))\mathrm{d}t, \tag{A.0.4}$$

subject to (A.0.2)-(A.0.3).

Theorem A.0.1 (PMP: Discounted Infinite-Horizon Maximum Principle)**.** *Suppose the problem of maximising* (A.0.1) *subject to* (A.0.2)-(A.0.3)*, with* π *and* f *continuously differentiable, has an interior piecewise optimal control* $\hat{\mathbf{u}}(t) \in IntU(t)$ *with corresponding state variable* $\hat{\mathbf{x}}(t) \in IntX(t)$*. Let* $V(t, x(t))$ *be the value function defined in* (A.0.4)*. Suppose that* $V(t, \hat{\mathbf{x}}(t))$ *is differentiable in* x *and* t *for* t *sufficiently large, that* $\lim_{t\to\infty} V(t, \hat{\mathbf{x}}(t))$ *exists and is finite, and that* $\lim_{t\to\infty} \partial V(t, \hat{\mathbf{x}}(t))/\partial t = 0$*. Let* $H(t, x, u, \lambda)$ *be the current-value Hamiltonian, given by*

$$H(t, x, u, \lambda) = \pi(x(t), u(t)) + \mathrm{e}^{rt}\lambda(t) \cdot f(t, x(t), y(t)),$$

where $\lambda : [t_0, \infty) \to \mathbf{R}^n$ *is the adjoint function for* (A.0.2)*. Then, except at points of discontinuity of* $\hat{\mathbf{u}}(t)$*, the optimal control pair* $(\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t))$ *satisfies the following necessary conditions:*

$$H_u(t, \hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t), \lambda(t)) = 0 \qquad \text{for all } t \in \mathbf{R}^+, \tag{A.0.5}$$

$$\mathrm{e}^{rt}\lambda(t) = H_x(t, \hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t), \lambda(t)) \qquad \text{for all } t \in \mathbf{R}^+,$$

$$\frac{\mathrm{d}x}{\mathrm{d}t} = H_{\mathrm{e}^{rt}\lambda}(t, \hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t), \lambda(t)) \qquad \text{for all } t \in \mathbf{R}^+,\ x(0) = x_0 \text{ and } \lim_{t\to\infty} x(t) \geq 0, \tag{A.0.6}$$

and the transversality condition

$$\lim_{t\to\infty} \left[\mathrm{e}^{-rt} H(t, \hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t), \lambda(t))\right] = 0.$$

Under the concavity assumptions of Proposition A.0.2 below, this transversality condition can be strengthened to

$$\lim_{t\to\infty} [\lambda(t) \cdot \hat{\mathbf{x}}(t)] = 0.$$

Proof. See [26] □

Theorem A.0.2 (Sufficiency Conditions for the Infinite-Horizon PMP)**.** *Consider the problem of maximising* (A.0.1) *subject to* (A.0.2)-(A.0.3)*, with* f *and* π *continuously differentiable. Define* $H(t, x, u, \lambda)$ *as in* (A.0.1)*, and suppose that a solution* $\hat{\mathbf{u}}(t)$ *and the corresponding path of* $\hat{\mathbf{x}}(t)$ *satisfy* (A.0.5)-(A.0.6)*. Given the resulting costate variable* $\lambda(t)$*, define* $M(t, x, \lambda) = \max_{u(t)\in U(t)} H(t, x, u, \lambda)$*. Suppose that* $\lim_{t\to\infty} V(t, \hat{\mathbf{x}}(t))$ *exists and is finite (where* $V(t, x(t))$ *is defined in* (A.0.4)*), that for any admissible pair* $(x(t), u(t))$*,* $\lim_{t\to\infty}[\lambda(t) \cdot x(t)] \geq 0$*, and that* $X(t)$ *is convex and* $M(t, x, \lambda)$ *is concave in* $x \in X(t)$ *for all* t*. Then, the pair* $(\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t))$ *achieves the unique global maximum*

of (A.0.1). *Moreover, if* $M(t, x, \lambda)$ *is strictly concave in* x, $(\hat{\mathbf{x}}(t), \hat{\mathbf{u}}(t))$ *is the unique solution to* (A.0.1).

Proof. See [26] □

Appendix B

The HJB Equations

The Feedback-Loop problems we study in this thesis are of the form

$$V(t_0, x_0) = \sup_{u[t_0,\infty)\to U} \int_{t_0}^{\infty} \mathrm{e}^{-rt}\pi(x(t; t_0, x_0, u), u)\mathrm{d}t, \tag{B.0.1}$$

subject to

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, u), \tag{B.0.2}$$

with $x \in X$, and $u \in U$, where $U \subset \mathbf{R}^m$, $X \subset \mathbf{R}^n$ are the admissible sets for the values of the control and state variables respectively, and where $x(t; t_0, x_0, u)$ is the level of x at time t with initial data (t_0, x_0) and following control path u. The statement and proofs of the results below follow those in [42] Section 4.2, [48] Section 8.2, and [133], adapted for the form of (B.0.1)-(B.0.2). We make note of two additional results, which we make use of in the proofs below. Because neither π or f show any dependence on t, we have

$$V(t_0, x_0) = \mathrm{e}^{-rt_0}V(0, x_0). \tag{B.0.3}$$

Consequently, we define $V(x_0) = V(0, x_0)$ as the time invariant form of the value function. We note that it also follows from the time-invariant nature of f and π that

$$\frac{\partial V(t, x(t))}{\partial t} = -rV(t, x(t)). \tag{B.0.4}$$

Theorem B.0.3 (Dynamic Programming Principle)**.** *For any initial state $x_0 \in \mathbf{R}^n$ and for times t_0, t_1 such that $0 \le t_0 \le t_1$, we have*

$$V(t_0, x_0) = \sup_{u:[t_0,t_1]\to U} \left\{ V(t_1, x(t_1; t_0, x_0, u)) + \int_{t_0}^{t_1} \mathrm{e}^{-rt}\pi(x(t; t_0, x_0, u), u(t))\mathrm{d}t \right\}. \tag{B.0.5}$$

Proof. Defining J_0 as the right hand side of (B.0.5), and $x_1 = x(t_1; t_0, x_0, u)$, we begin by showing $V(t_0, x_0) \leq J_0$. Setting $\epsilon > 0$, we choose a control $u : [t_0, \infty) \to U$ such that

$$V(t_0, x_0) - \epsilon \leq J(u; t_0, x_0).$$

Given that $x(t; t_0, x_0, u) = x(t; t_1, x_1, u)$, we have

$$\begin{aligned} V(t_0, x_0) - \epsilon \leq J(u; t_0, x_0) &= J(u; t_1, x_1) + \int_{t_0}^{t_1} \mathrm{e}^{-rt} \pi(x(t; t_0, x_0, u), u(t)) \mathrm{d}t \\ &\leq V(t_1, x_1) + \int_{t_0}^{t_1} \mathrm{e}^{-rt} \pi(x(t; t_0, x_0, u), u(t)) \mathrm{d}t \leq J_0. \end{aligned}$$

Given that $\epsilon > 0$ can be set arbitrarily, we must therefore have that $V(t_0, x_0) \leq J_0$. To prove $V(t_0, x_0) \geq J_0$, we once again set an arbitrary $\epsilon > 0$. There exists a control $u_0 : [t_0, t_1] \to U$ such that

$$V(t_1, x(t_1, t_0, x_0, u_0)) + \int_{t_0}^{t_1} \mathrm{e}^{-rt} \pi(x(t; t_0, x_0, u_0), u(t)) \mathrm{d}t \geq J_0 - \epsilon. \tag{B.0.6}$$

We can then find a control $u_1 : [t_1, \infty) \to U$ such that

$$J(u_1, t_1, x_1) \geq V(t_1, x_1) - \epsilon, \tag{B.0.7}$$

where $x_1 = x(t_1; t_0, x_0, u)$. We define a new control $u : [t_0, \infty) \to U$ as the concatenation of u_0, u_1, such that

$$u(t) = \begin{cases} u_0(t) & \text{if } t \in [t_0, t_1], \\ u_1(t) & \text{if } t \in [t_1, \infty). \end{cases}$$

By (B.0.6) and (B.0.7) we have

$$V(t_0, x_0) \geq J(u, t_0, x_0) = J(u_1, t_1, x_1) + \int_{t_0}^{t_1} \mathrm{e}^{-rt} \pi(x(t; t_0, x_0, u_0), u_0(t)) \mathrm{d}t \geq J_0 - 2\epsilon.$$

Given that $\epsilon > 0$ can be set arbitrarily small, this proves the second inequality. Therefore, we have $V = J$. □

Theorem B.0.4 (HJB equation as a necessary condition of optimality)**.** *For the optimal control problem* (B.0.1)-(B.0.2)*, assuming that f, π are continuous functions and that U is compact, then on any open domain $\Omega \subset [0, \infty) \times \mathbf{R}^n$ where the function $V = V(t, x)$ is continuously differentiable, the following equation is satisfied:*

$$\max_{\omega \in U} \left\{ \mathrm{e}^{-rt} \pi(x, \omega) + f(x, \omega) \cdot \nabla V(t, x) \right\} = rV(t, x)$$

which can be rewritten, using the results (B.0.3)-(B.0.4)*, as the time-autonomous HJB equation:*

$$\max_{\omega\in U}\{\pi(x,\omega)+f(x,\omega)\cdot\nabla V(x)\}=rV(x).$$

Proof. Assume the contrary, and that at a point $(t_0,x_0)\in\Omega$ one has

$$\max_{\omega\in U}\left\{\mathrm{e}^{-rt_0}\pi(x,\omega)+f(x,\omega)\cdot\nabla V(t,x)\right\}-rV(t,x)>0. \tag{B.0.8}$$

By continuity, we can find the $w^*\in U$ and $\theta>0$ such that

$$\mathrm{e}^{-rt}\pi(x,\omega^*)+f(x,\omega^*)\cdot\nabla V_i(t,x)-rV(t,x)\geq\theta>0,$$

for all (t,x) in a neighbourhood of (t_0,x_0). We define the constant control $u^*(t)=\omega^*$, and let $t\to x^*(t)=x(t;t_0,x_0,u^*)$ be the corresponding trajectory. On a sufficiently small interval $[t_0,t_1]$, we have

$$\begin{aligned}V(t_1,x^*(t_1))&=V(t_0,x_0)+\int_{t_0}^{t_1}\left\{-rV(t,x^*(t))+\nabla V(t,x^*(t))\cdot f(t,x^*(t),u^*(t))\right\}\mathrm{d}t\\&\geq V(t_0,x_0)+\int_{t_0}^{t_1}\left\{\theta-\mathrm{e}^{-rt}\pi(x^*(t),u^*(t))\right\}\mathrm{d}t.\end{aligned}$$

Therefore

$$V(t_0,x_0)<V(t_1,x^*(t_1))+\int_{t_0}^{t_1}\mathrm{e}^{-rt}\pi(x^*(t),u^*(t))\mathrm{d}t,$$

which contradicts Theorem B.0.3. Therefore (B.0.8) cannot hold.

Next, assume that there exists a point $(t_0,x_0)\in\Omega$ for which holds

$$\max_{\omega\in U}\left\{\mathrm{e}^{-rt_0}\pi(x_0,\omega)+f(x_0,\omega)\cdot\nabla V(t_0,x_0)\right\}-rV(t_0,x_0)<0.$$

By continuity, it now follows

$$-rV(t,x)+f(x,\omega)\cdot\nabla V(t,x)+\mathrm{e}^{-rt}\pi(x,\omega)\leq-\theta<0,$$

for all $\omega\in U$ and all (t,x) in a neighbourhood of (t_0,x_0). If the interval $[t_0,t_1]$ is sufficiently small, for any control function $t\to u(t)\in U$ this implies

$$\begin{aligned}V(t_1,x(t_1))&=V(t_0,x_0)+\int_{t_0}^{t_1}\left\{-rV(t,x(t))+f(x(t),u(t))\cdot\nabla V(t,x(t))\right\}\mathrm{d}t\\&\leq V(t_0,x_0)-\int_{t_0}^{t_1}\left\{\mathrm{e}^{-rt}\pi(x(t),u(t))+\theta\right\}\mathrm{d}t.\end{aligned}$$

Therefore, for every admissible control $u(\cdot)$ we have

$$V(t_0, x_0) - \theta(t_1 - t_0) \geq V(t_1, x(t_1)) + \int_{t_0}^{t_1} \pi(t, x(t), u(t))\mathrm{d}t,$$

reaching again a contradiction with Theorem B.0.3. This completes the proof.

□

Theorem B.0.5 (HJB equation as a sufficient condition for optimality)**.** *Let* $W : [0, \infty) \times \mathbf{R}^n \to \mathbf{R}$ *be a* $\mathbf{C}^1$ *solution of the problem*

$$\max_{\omega \in U} \{\pi(x, \omega) + f(x, \omega) \cdot \nabla V(x)\} = rV(x).$$

Then W *coincides with the value function. In particular, for any given initial data* (t_0, x_0)*, a control* $u^*(\cdot)$ *that achieves the payoff* $J(u^*, t_0, x_0) = W(t_0, x_0)$ *is optimal.*

Proof. Let $V = V(t_0, x_0)$ be the value function for the optimal control problem (B.0.1)-(B.0.2). We begin by showing $V \leq W$. Given any initial data (t_0, x_0), consider any control $u : [t_0, \infty) \to U$, and let $t \to x(t) = x(t; t_0, x_0, u)$ be the corresponding trajectory. We claim that

$$J(u, t_0, x_0) = \int_0^{\infty} \mathrm{e}^{-rt}\pi(x(t), u(t))\mathrm{d}t \leq W(t_0, x_0).$$

Observe that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[W(t, x(t)) + \int_{t_0}^{t} \mathrm{e}^{-rs}\pi(x(s), u(s))\mathrm{d}s\right] \\
&\quad = -rW(t, x(t)) + f(x(t), u(t)) \cdot \nabla W(t, x(t)) + \mathrm{e}^{-rt}\pi(x(t), u(t)) \\
&\quad \leq -rW(t, x(t)) + \max_{\omega \in U}\left\{f(x(t), \omega) \cdot \nabla W(t, x(t)) + \mathrm{e}^{-rt}\pi(x(t), \omega)\right\} = 0.
\end{aligned}$$

Integrating over the time interval $[t_0, \infty)$ one obtains

$$W(t_0, x_0) \geq \int_{t_0}^{\infty} \mathrm{e}^{-rt}\pi(x(t), u(t))\mathrm{d}t = J(u, t_0, x_0). \tag{B.0.9}$$

Therefore

$$V(t_0, x_0) = \sup_{u(\cdot) \in U} J(u, t_0, x_0) \leq W(t_0, x_0).$$

By assumption, the control $u^*(\cdot)$ yields the payoff $J(u^*) = W(t_0, x_0)$. On the other hand, (B.0.9) shows that any other control $u(\cdot)$ yields a payoff $J(u) \leq W(t_0, x_0)$. Therefore u^* is optimal. In proving $W \leq V$, we assume that the map

$$(t, x) \to u^\star(t, x) = \arg\max_{u \in U}\left\{f(t, x, u) \cdot \nabla W(t, x) + \mathrm{e}^{-rt}\pi(x, u)\right\}, \tag{B.0.10}$$

is uniquely defined and continuous. In this case, given the initial condition (t, x_0), let $t \to x^*(t)$ be a solution to the Cauchy problem

$$\frac{\mathrm{d}x}{\mathrm{d}t} = f(x, u^\star(t, x)), \qquad x(t_0) = x_0.$$

By our assumption (B.0.10), the RHS of this ODE is continuous, and a solution exists. Defining $u^*(t) = u^\star(t, x^*(t))$, we have

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left[W(t, x^*(t)) + \int_{t_0}^{t} \mathrm{e}^{-rs}\pi(x^*(s), u^*(s))\mathrm{d}s\right] \\
&\quad = -rW(t, x^*(t)) + f(x^*(t), u^\star(t, x^*(t))) \cdot \nabla W(t, x^*(t)) + \mathrm{e}^{-rt}\pi(x^*(t), u^\star(t, x^*(t))) \\
&\quad = -rW(t, x^*) + \max_{\omega \in U}\left\{f(x^*(t), \omega) \cdot \nabla W(t, x^*(t)) + \mathrm{e}^{-rt}\pi(x^*, \omega)\right\} = 0.
\end{aligned}$$

Integrating over $[t_0, T)$ gives

$$\begin{aligned}
W(t_0, x_0) &= \int_{t_0}^{\infty} \mathrm{e}^{-rt}\pi(x^*(t), u^*(t))\mathrm{d}t \\
&= J(u^*, t_0, x_0) \leq V(t_0, x_0),
\end{aligned}$$

completing the proof. For a proof in the more general case that $u^\star$ is discontinuous, see [30].

□

Appendix C

A Separate Pools Open-Loop Cournot Game with N Players

In this section, we solve the N player analogue of the Open-Loop Cournot game of Section 3.3.1.[1] We assume that the players begin the game with separate resource pools, $x_1 \geq ... \geq x_N$, which deplete with the dynamics

$$\frac{\mathrm{d}x_{i,t}}{\mathrm{d}t} = -q_{i,t}. \tag{C.0.1}$$

The ordering of the resource pool sizes implies a similar ordering for their exhaustion times, so that $T_1 \geq ... \geq T_N$. The value function for player i is given by

$$v_i^o = \int_0^{T_i} \mathrm{e}^{-rt} q_{i,t}^o \left(1 - q_{i,t}^o - \sum_{j=1, j\neq i}^{N} q_j^o\right) \mathrm{d}t. \tag{C.0.2}$$

The game is solved in the same way as the duopoly case. We now have N Hamiltonian conditions, given by

$$H_i(t, x_1, ..., x_N, q_i^o, \lambda_i) = \mathrm{e}^{-rt} q_i^o \left(1 - q_i^o - \sum_{j=1, j\neq i}^{N} q_j^o\right) + \lambda_i q_i^o, \qquad i = 1, ..., N.$$

[1] We note that we have received assistance from Dr David Allwright in the calculations in this section.

These result in the first order conditions

$$1 - 2q_i^o - \sum_{j=1, j\neq i}^{N} q_j^o + \lambda_i \mathrm{e}^{rt} = 0, \tag{C.0.3}$$

$$\frac{\mathrm{d}\lambda_i}{\mathrm{d}t} = 0, \tag{C.0.4}$$

$$\mathrm{e}^{-rt} q_i^o(T_i) \left(1 - q_i^o(T_i) - \sum_{j=1, j\neq i}^{i-1} q_j^o(T_i) \right) = 0. \tag{C.0.5}$$

We note that the terminal condition for player i (C.0.5) includes only $q_1^o, ..., q_i^o$ because players $i+1, ..., N$ exhaust their stock before player i does. From (C.0.4), we can see that the adjoint functions are constant as in the duopoly case. We introduce the notation N_t to denote the number of players active at time t:

$$N_t = \begin{cases} N & \text{for } t \leq T_N, \\ i & \text{for } T_{i+1} < t \leq T_i, \\ 0 & \text{for } t > T_1. \end{cases}$$

Summing (C.0.3) over all active players gives

$$Q = \frac{1}{N_t + 1} \left(N_t + \sum_{i=1}^{N_t} \lambda_i \mathrm{e}^{rt} \right).$$

Substituting this into (C.0.3) gives

$$q_i = \frac{1}{N_t + 1} \left(1 + N_t \lambda_i \mathrm{e}^{rt} - \sum_{j=1, j\neq i}^{N_t} \lambda_j \mathrm{e}^{rt} \right). \tag{C.0.6}$$

We use this to solve for λ_i.

Proposition C.0.6.

$$-\lambda_i = \frac{1}{2}\mathrm{e}^{-rT_1} + \frac{1}{6}\mathrm{e}^{-rT_2} + ... + \frac{1}{i(i-1)}\mathrm{e}^{-rT_{i-1}} + \frac{1}{i}\mathrm{e}^{-rT_i}, \qquad 3 \leq i \leq N, \tag{C.0.7}$$

with $\lambda_1 = -\exp(-rT_1)$, *and* $\lambda_2 = -(\exp(-rT_1) + \exp(-rT_2))/2$.

Proof. We first note that, for each i, substituting (C.0.3) into (C.0.5) gives

$$q_i^2(T_i) = 0, \tag{C.0.8}$$

so that the terminal condition for each player is given by $q_i(T_i) = 0$. We confirm $\lambda_1 = -\exp(-rT_1)$ directly from setting $i = 1$ in (C.0.8) and solving for λ_1. Setting

$i = 2$ in (C.0.8), and substituting for λ_1, gives $\lambda_2 = -(\exp(-rT_1) + \exp(-rT_2))/2$. We calculate $\lambda_3 = -\exp(-rT_1)/2 - \exp(-rT_2)/6 - \exp(-rT_3)/3$ similarly. We can find the remaining adjoint variables by induction. We assume that (C.0.7) holds for $n \leq k$. Setting $i = k + 1$ in (C.0.8), and using (C.0.6) gives

$$\lambda_{k+1} = \frac{1}{k+1} \sum_{j=1}^{k} \lambda_j - \frac{1}{k+1} \mathrm{e}^{-rT_{k+1}}. \tag{C.0.9}$$

From (C.0.7), we have

$$\begin{aligned} -\sum_{j=1}^{k} \lambda_j &= \sum_{j=1}^{k} \left(\frac{1}{j} \mathrm{e}^{-rT_j} + \frac{(k-j)}{j(j+1)} \mathrm{e}^{-rT_j} \right) \\ &= \sum_{j=1}^{k} \frac{k+1}{j(j+1)} \mathrm{e}^{-rT_j}. \end{aligned}$$

Substituting this into (C.0.9) gives

$$-\lambda_{k+1} = \sum_{j=1}^{k} \frac{1}{j(j+1)} \mathrm{e}^{-rT_j} + \frac{1}{k+1} \mathrm{e}^{-rT_k},$$

so that (C.0.7) holds for $i = k + 1$ as required. □

Proposition C.0.7.

$$rT_i + \mathrm{e}^{-rT_i} - 1 = (i+1)rx_i + \sum_{j=i+1}^{N} rx_j \qquad i = 1, ..., N.$$

Proof. First note that substituting (C.0.7) into (C.0.6) and rearranging gives

$$q_i = \frac{1}{i+1} \left(1 - \mathrm{e}^{-r(T_i - t)} - \sum_{j=i+1}^{N} q_j \right).$$

Integrating the above w.r.t t from 0 to T_i, and substituting (C.0.1) gives the required result. □

We have a closed-form parametric expression for the production trajectories of each player. The value functions can be found by direct calculation of (C.0.2) as before. A key observation is that each player's production trajectory, and therefore exhaustion time, is dependent only on the size of those competing players' resource pools *which are smaller than their own.* This is a highly singular result and is an area for future research.

Appendix D

Asymptotic Analysis of Production Rates Under Cournot Competition

Here, we here show that the Open-Loop and Feedback-Loop production rates converge to the same point at different asymptotic rates as x_1 and x_2 approach infinity. We will show that by linearising the equations for each we can derive asymptotic approximations to the production rates under Open-Loop and Feedback-Loop and that these asymptotic approximations are not equal.

First order conditions (3.3.7) and (3.3.8) can be differentiated with respect to time using (3.3.3)-(3.3.4) and the relation

$$\begin{aligned}\frac{\mathrm{d}q_i^o(t)}{\mathrm{d}t} &= \frac{\partial q_i^o(t)}{\partial x_1}\frac{\mathrm{d}x_1}{\mathrm{d}t} + \frac{\partial q_i^o(t)}{\partial x_2}\frac{\mathrm{d}x_2}{\mathrm{d}t} \\ &= -\left(\frac{\partial q_i^o(t)}{\partial x_1}q_1^o(t) + \frac{\partial q_i^o(t)}{\partial x_2}q_2^o(t)\right).\end{aligned}$$

Using (3.3.7)-(3.3.8) to eliminate the adjoint functions, we can represent the production rates in the Open-Loop game via the system of partial differential equations

$$\begin{aligned}2\left(q_1^o\frac{\partial q_1^o}{\partial x_1} + q_2^o\frac{\partial q_1^o}{\partial x_2}\right) + \left(q_1^o\frac{\partial q_2^o}{\partial x_1} + q_2^o\frac{\partial q_2^o}{\partial x_2}\right) - r\left(1 - 2q_1^o - q_2^o\right) = 0 \\ 2\left(q_1^o\frac{\partial q_2^o}{\partial x_1} + q_2^o\frac{\partial q_2^o}{\partial x_2}\right) + \left(q_1^o\frac{\partial q_1^o}{\partial x_1} + q_2^o\frac{\partial q_1^o}{\partial x_2}\right) - r\left(1 - 2q_2^o - q_1^o\right) = 0,\end{aligned}$$

Linearising the system around the solution at $(x_1, x_2) = (\infty, \infty)$ in which $(q_1^o, q_2^o) =$

$(1/3, 1/3)$ results in the pde system

$$\frac{2}{3}\left(\frac{\partial q_1^o}{\partial x_1}+\frac{\partial q_1^o}{\partial x_2}\right)+\frac{1}{3}\left(\frac{\partial q_2^o}{\partial x_1}+\frac{\partial q_2^o}{\partial x_2}\right)+r\left(2q_1^o+q_2^o\right)=0$$
$$\frac{2}{3}\left(\frac{\partial q_2^o}{\partial x_1}+\frac{\partial q_2^o}{\partial x_2}\right)+\frac{1}{3}\left(\frac{\partial q_1^o}{\partial x_1}+\frac{\partial q_1^o}{\partial x_2}\right)+r\left(2q_2^o+q_1^o\right)=0,$$

which is solved by

$$q_1^o=\frac{1}{3}+C_1\mathrm{e}^{-3rx_1}+C_2\mathrm{e}^{-3rx_2}$$
$$q_2^o=\frac{1}{3}+C_2\mathrm{e}^{-3rx_1}+C_1\mathrm{e}^{-3rx_2}.$$

The PDEs for the Feedback-Loop game can be linearised in v_1^f and v_2^f around $(v_1^f(\infty,\infty), v_2^f(\infty,\infty)) = (1/9r, 1/9r)$ to give the system

$$rv=\frac{1}{9}\left(-4\frac{\partial v_1^f}{\partial x_1}+2\frac{\partial v_2^f}{\partial x_2}-3\frac{\partial v_1^f}{\partial x_2}\right)$$
$$rw=\frac{1}{9}\left(-4\frac{\partial v_2^f}{\partial x_2}+2\frac{\partial v_1^f}{\partial x_1}-3\frac{\partial v_2^f}{\partial x_1}\right).$$

Solving the above for v_1^f and v_2^f and using (3.3.12)-(3.3.13) allows us to calculate the corresponding asymptotic production strategies as

$$q_1^f=\frac{1}{3}+C_3\mathrm{e}^{-\frac{9rx_1}{4}}+C_4\mathrm{e}^{-\frac{9rx_2}{4}}$$
$$q_2^f=\frac{1}{3}+C_4\mathrm{e}^{-\frac{9rx_1}{4}}+C_3\mathrm{e}^{-\frac{9rx_2}{4}}.$$

Comparison of the exponential coefficients of x_1 and x_2 is sufficient to demonstrate that the strategies do not converge asymptotically, even though they both converge to the same limit.

Appendix E

Proof of Time Consistency of the Open Loop Strategies

We show here that strategies in the Open Loop Model between two players with separate resource pools are time consistent. We restrict the analysis to the case $x_1 \geq x_2$ as before. We assume that from a position (x_1, x_2), the game has run for time t_0 and the current resource allocation is $(\tilde{x}_1, \tilde{x}_2)$. Given that along the line $x_1 = x_2$ we have $q_1^o = q_2^o$, trajectories do not cross that line and we can safely assume that $\tilde{x}_1 \geq \tilde{x}_2$ if $x_1 \geq x_2$. We wish to show that the strategies the players adopt from this point on are identical to those they would choose to adopt if they were given the opportunity to recalculate their strategies based on $(\tilde{x}_1, \tilde{x}_2)$. Assume the players' strategies were calculated with terminal times T_1 and T_2 for players 1 and 2 respectively so that $T_1 \geq T_2$. The strategies from t_0 that the players are committed to are

$$q_1^o(t) = \frac{1}{3}(1 - \frac{3}{2}\mathrm{e}^{-r(T_1-t)} + \frac{1}{2}\mathrm{e}^{-r(T_2-t)})$$
$$q_2^o(t) = \frac{1}{3}(1 - \mathrm{e}^{-r(T_2-t)}),$$

where $t_0 < t < T_2$ (beyond T_2 the strategies revert to the monopoly strategy for player 1 while player 2 ceases production). By integrating the strategies w.r.t. t from t_0 on, we

can show that the following must hold:

$$\tilde{x}_2 = \frac{1}{3r}\left[r(T_2 - t_0) - (1 - \mathrm{e}^{-r(T_2 - t_0)})\right],$$
$$\tilde{x}_1 = \frac{1}{3r}\left[r(T_2 - t_0) + \frac{1}{2}(1 - \mathrm{e}^{-r(T_2 - t_0)}) - \frac{3}{2}(\mathrm{e}^{-r(T_1 - T_2)} - \mathrm{e}^{-r(T_1 - T_0)})\right]$$
$$+ \frac{1}{2r}\left[r(T_1 - T_2) + \mathrm{e}^{-r(T_1 - T_2)} - 1\right].$$

If we now assume that the players could recalculate their strategies from $(\tilde{x}_1, \tilde{x}_2)$ with terminal depletion times (τ_1, τ_2) then it is straightforward to show that these strategies are given by

$$\tilde{q}_1^o(t) = \frac{1}{3}(1 - \frac{3}{2}\mathrm{e}^{-r(\tau_1 - t)} + \frac{1}{2}\mathrm{e}^{-r(\tau_2 - t)}),$$
$$\tilde{q}_2^o(t) = \frac{1}{3}(1 - \mathrm{e}^{-r(\tau_2 - t)}),$$

where $0 < t < \tau_2$. τ_1 and τ_2 are given by

$$\tilde{x}_2 = \frac{1}{3r}\left[r\tau_2 - (1 - \mathrm{e}^{-r\tau_2})\right],$$
$$\tilde{x}_1 = \frac{1}{3r}\left[r\tau_2 + \frac{1}{2}(1 - \mathrm{e}^{-r\tau_2}) - \frac{3}{2}(\mathrm{e}^{-r(\tau_1 - \tau_2)} - \mathrm{e}^{-r\tau_1})\right]$$
$$+ \frac{1}{2r}\left[r(\tau_1 - \tau_2) + \mathrm{e}^{-r(\tau_1 - \tau_2)} - 1\right].$$

Comparison of the two expressions for $(\tilde{x}_1, \tilde{x}_2)$ shows that $\tau_1 = T_1 - t_0$ and $\tau_2 = T_2 - t_0$. Substituting these values into the expressions for $\tilde{q}_1^o(0)$ and $\tilde{q}_2^o(0)$ show that the strategies are equal to $q_1^o(t_0)$ and $q_2^o(t_0)$ as required, so the strategies are time-consistent.

Appendix F

Derivation of the Feedback-Loop Solution for $x_1 = \infty$

The Hamiltonians for the problem are given by

$$H_1(t, x_{1,t}, x_{2,t}, q^f_{1,t}, \lambda_t) = \begin{cases} q^f_{1,t}(1 - q^f_{1,t} - q^f_{2,t})\mathrm{e}^{-rt} + \lambda_t q^f_{2,t} & \text{for} \quad t < T_f \\ q^f_{1,t}(1 - q^f_{1,t})\mathrm{e}^{-rt} & \text{for} \quad t \geq T_f \end{cases}$$
$$H_2(t, x_{1,t}, x_{2,t}, q^f_{2,t}, \mu_t) = q^f_{2,t}(1 - q^f_{1,t} - q^f_{2,t})\mathrm{e}^{-rt} + \mu_t q^f_{2,t} \qquad \text{for} \quad t < T_f.$$

When $t > T_f, x_{2,t} = 0$ and player 2 ceases production. There is no longer the consideration of exhaustibility and the game reverts to a static maximisation problem for player 1 as a monopolist with zero costs. For $t < T_f$ we have first-order conditions

$$1 - 2q^f_{1,t} - q^f_{2,t} = 0 \tag{F.0.1}$$
$$1 - 2q^f_{2,t} - q^f_{1,t} + \mu_t \mathrm{e}^{rt} = 0 \tag{F.0.2}$$
$$\mathrm{e}^{rt}\frac{\mathrm{d}\mu}{\mathrm{d}t} = -q^f_{2,t}\frac{\mathrm{d}q^f_{1,t}}{\mathrm{d}x_{2,t}}, \tag{F.0.3}$$

with the continuity of the Hamiltonian across $t = T_f$ providing transversality condition

$$\mu_{T_f} = -\mathrm{e}^{-rT_f} \tag{F.0.4}$$

Using the chain rule and the exhaustibility constraint we have

$$\frac{\mathrm{d}q^f_{1,t}}{\mathrm{d}t} = -q^f_{2,t}\frac{\mathrm{d}q^f_{1,t}}{\mathrm{d}x_{2,t}},$$

which can be substituted into (F.0.3) to give

$$e^{rt}\dot{\mu}_t = \frac{dq^f_{1,t}}{dt}. \tag{F.0.5}$$

Next, we note that differentiating (F.0.2) with respect to t gives

$$\frac{dq^f_{2,t}}{dt} = -2\frac{dq^f_{1,t}}{dt}. \tag{F.0.6}$$

Differentiating (F.0.1) with respect to t and substituting (F.0.5) and (F.0.6) allows us to represent μ_t as the solution to the ode

$$4\dot{\mu}_t = -r\mu_t.$$

This can be solved with terminal condition (F.0.4) to give

$$e^{rt}\mu_t = -e^{-\frac{3r}{4}(T_f - t)},$$

which can be incorporated into (F.0.1) and (F.0.2) to give the production trajectories as

$$q^f_{1,t} = \begin{cases} \frac{1}{3}(1 + \frac{1}{2}e^{-\frac{3r}{4}(T_f - t)}) & \text{for } t < T_f \\ \frac{1}{2} & \text{for } t \geq T_f \end{cases}$$
$$q^f_{2,t} = \begin{cases} \frac{1}{3}(1 - e^{-\frac{3r}{4}(T_f - t)}) & \text{for } t < T_f \\ 0 & \text{for } t \geq T_f. \end{cases}$$

We can derive the parametric representation of T_f by integrating $q^f_{2,t}$ from 0 to T_f. The value functions v^f_1 and v^f_2 can be found by direct substitution of the production rates into the value integrals.

Appendix G

Two player game with Asymmetric Costs

Here, we briefly consider a two player Open-Loop game in which the two players pay different extraction costs c_1, c_2, where we assume $c_1 < c_2$. If we assume that the costs are such that both players participate at all resource levels,[1] then applying the same steps as in Section 4.3.2 with $N = 2$ shows that the optimal strategies are given by the first-order conditions

$$1 - 2q_1 - q_2 - c_1 + \mu_1 \mathrm{e}^{rt}(q_1 + q_2) = 0, \tag{G.0.1}$$

$$1 - 2q_2 - q_1 - c_2 + \mu_2 \mathrm{e}^{rt}(q_1 + q_2) = 0, \tag{G.0.2}$$

$$\frac{\mathrm{d}\mu_i}{\mathrm{d}t} = 0 \qquad i = 1, 2,$$

$$q_1(1 - q_1 - q_2 - c_1) + \mu_1 \mathrm{e}^{rT}(q_1 + q_2) = 0, \tag{G.0.3}$$

$$q_2(1 - q_1 - q_2 - c_2) + \mu_2 \mathrm{e}^{rT}(q_1 + q_2) = 0. \tag{G.0.4}$$

Solving (G.0.1) and (G.0.2) for q_1 and q_2 gives

$$q_1 = \frac{1}{3}\left(1 - 2(c_1 - \mu_1 \mathrm{e}^{rt}) + (c_2 - \mu_2 \mathrm{e}^{rt})\right),$$
$$q_2 = \frac{1}{3}\left(1 - 2(c_2 - \mu_2 \mathrm{e}^{rt}) + (c_1 - \mu_1 \mathrm{e}^{rt})\right).$$

Substituting into (G.0.3) and (G.0.4) and solving for $q_1(T)$ and $q_2(T)$ gives

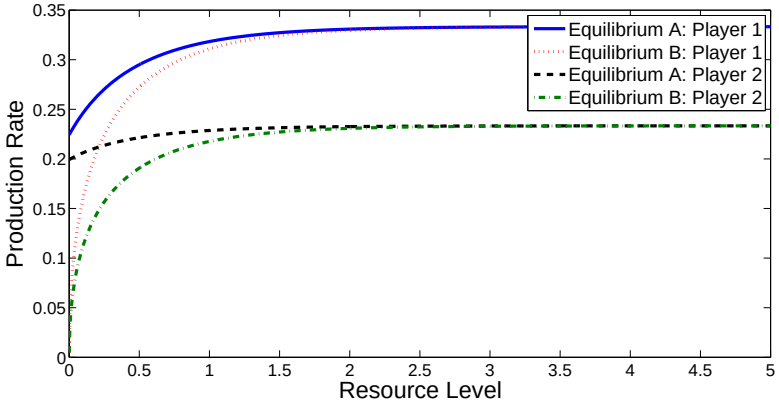
$$q_1^A(T) = \frac{(1 - c_1)^2(1 - c_2)}{(2 - c_1 - c_2)^2}, \qquad q_2^A = \frac{(1 - c_2)^2(1 - c_1)}{(2 - c_1 - c_2)^2},$$
$$q_1^B(T) = 0, \qquad q_2^B(T) = 0.$$

[1] The analysis in [62] indicates that this holds for all $(c_1, c_2) < 1/2$.

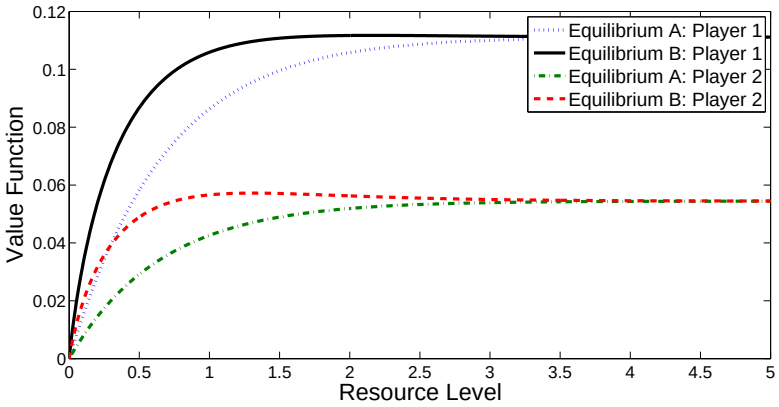
Solving for μ_1 and μ_2 gives

$$\mu_1^A = -\frac{(1-c_1)^3 \mathrm{e}^{-rT}}{(2-c_1-c_2)^2}, \qquad \mu_2^A = -\frac{(1-c_2)^3 \mathrm{e}^{-rT}}{(2-c_1-c_2)^2},$$
$$\mu_1^B = -(1-c_1)\mathrm{e}^{-rT}, \qquad \mu_2^B = -(1-c_2)\mathrm{e}^{-rT}.$$

T^A and T^B are found by substituting μ_1 and μ_2 into constraint (4.3.6), integrating over t, and then solving numerically. We use (4.3.5) to find (v_1^A, v_2^A) and (v_1^B, v_2^B). The value functions and production rates are shown in Figure G.1 for $c_1 = 0.1$ and $c_2 = 0.2$. We note from Figure G.1b that (v_1^B, v_2^B) demonstrates similar non-monotonicity to that present in Equilibrium B of the game with symmetric extraction costs. In other words, we have demonstrated by counterexample that neither the multiple equilibria of the Open-Loop game nor the non-monotonicity of the value functions under Equilibrium B are a consequence of the assumption of symmetric extraction costs.



(a) Production Strategies



(b) Value Functions

Figure G.1: The Open-Loop Game Under Asymmetric Costs

Appendix H

The Feedback Formulation of the Open-Loop Game

The formulation we use in this paper differs slightly from that of [79]. In [79], a substitution is made such that the control problem can be written as an ODE in x. Moreover, the claim is made that this formulation negates the need for the assumption that players adopt a common anticipated terminal time. Here, we highlight an error in this claim, thereby demonstrating that strategies under the 'Feedback-Loop' style formulation of the game also implicitly assume common terminal times. We alter the notation used in [79] so as to be consistent with our main body of work, but make no other changes in the presentation of their formulation.

The game is identical to ours, except that the price curve is given by an abstract function of total production, given by $P(Q)$, where P is strictly decreasing and thrice-differentiable in Q. The first steps of the solution method are identical to those we have used in Section 4.3.2, with the exception of the generalised price function P. The Hamiltonians for the control problem are given by

$$H = q_i(P(q_i + Q_{-i}))\mathrm{e}^{-rt} + \lambda_i(q_i + Q_{-i}).$$

The first order conditions are given by

$$(q_i P' + P)\mathrm{e}^{-rt} + \lambda_i = 0, \tag{H.0.1}$$

$$\frac{\mathrm{d}\lambda}{\mathrm{d}t} = 0, \tag{H.0.2}$$

where $P' = \mathrm{d}P/\mathrm{d}Q$. Symmetry of the solution allows (H.0.1) to be written as

$$QP' + NP + N\lambda_i \mathrm{e}^{rt} = 0. \tag{H.0.3}$$

Differentiating (H.0.3) w.r.t. t and using (H.0.1)-(H.0.2) to eliminate λ gives

$$r(QP' + NP) = \left[QP'' - (N+1)P'\right] \frac{\mathrm{d}Q}{\mathrm{d}t}.$$

Making the substitution

$$\begin{aligned} \frac{\mathrm{d}Q}{\mathrm{d}t} &= \frac{\mathrm{d}Q}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} \\ &= -Q\frac{\mathrm{d}Q}{\mathrm{d}x}, \end{aligned}$$

gives the ODE

$$\left[Q^2P'' + (N+1)QP'\right] \frac{\mathrm{d}Q}{\mathrm{d}x} + r\left[QP' + NP\right] = 0,$$

with the terminal condition now given by the initial condition in x

$$\left[Q(0)P'(Q(0)) + (N-1)P(Q(0))Q(0)\right] = 0.$$

The claim that this formulation does not assume a common terminal time rests upon the conversion of the terminal condition into an initial condition in x. However, we make the observation that, because this is an Open-Loop game, the $x(t)$ substitution made here refers to the trajectory of x under a *specific* set of production trajectories. In other words, the ODE represents the strategies of the players as functions of the path that x_t takes under a candidate equilibrium. They are not functions of x itself.

Therefore, this only holds when all players adhere to that equilibrium. If one of the players adopts a different strategy and produces more than they otherwise would have, the actual level of the resource pool will hit zero before the equilibrium path of x_t reaches zero, and the terminal condition does not come into effect. In this way, the strategies are still dependent on the assumption of a common anticipated terminal time via the implicit condition $x(T) = 0$.

Appendix I

Proof of Proposition 5.4.1

The rate of revenue for firm i is clearly concave in q_i at each point in time. By taking the negative of the objective function (5.4.7) we rewrite each firm's optimization problem as the convex minimisation problem

$$q_i = \arg\min \left\{ \beta q_i^2 - ((\alpha - c_i) - \beta Q_{-i}) q_i \right\}, \tag{I.0.1}$$

with the constraints

$$q_i \leq Q^q - Q_{-i}, \tag{I.0.2}$$

$$-q_i \leq 0. \tag{I.0.3}$$

Introducing Lagrange multipliers μ_i and ξ_i for the constraints (I.0.2) and (I.0.3), we have that q_i^* optimizes (I.0.1)-(I.0.3) if and only if there exist multipliers μ_i and ξ_i such that[1]

$$\beta q_i - \alpha + c_i + \beta Q + \mu_i - \xi_i = 0, \tag{I.0.4}$$

$$0 \leq \mu_i \perp Q^q - Q_{-i} - q_i \geq 0,$$

$$0 \leq \xi_i \perp q_i \geq 0. \tag{I.0.5}$$

Solving ξ_i from (I.0.4) and substituting into (I.0.5) gives the following equivalent linear complementarity problem:

$$0 \leq q_i^* \perp \beta q_i - \alpha + c_i + \beta Q + \mu_i \geq 0,$$

$$0 \leq \mu_i \perp Q^q - Q_{-i} - q_i \geq 0.$$

[1] See [33].

Stacking up the optimality conditions for the individual firms, we have that $\mathbf{q}^*$ is an equilibrium if and only if there exists a vector μ such that

$$0 \leq \mathbf{q}^* \perp \beta(I+E)\mathbf{q}^* - \alpha\mathbf{1} + \mathbf{c} + \mu \geq 0,$$
$$0 \leq \mu \perp Q^q\mathbf{1} - E\mathbf{q}^* \geq 0,$$

which can be written in the standard form

$$0 \leq \mathbf{x} \perp M\mathbf{x} + \mathbf{u} \geq 0, \tag{I.0.6}$$

where

$$\mathbf{x} = \begin{pmatrix} \mathbf{q}^* \\ \mu \end{pmatrix},$$
$$\mathbf{u} = \begin{pmatrix} -\alpha\mathbf{1} + \mathbf{c} \\ Q^q\mathbf{1} \end{pmatrix},$$
$$M = \begin{pmatrix} \beta(I+E) & I \\ -E & 0 \end{pmatrix}.$$

Appendix J

Proof of Proposition 5.4.2

The LCP (I.0.6) has the common constraint for all players, $\sum_{i=1}^{N} q_i^* \leq Q^q$. It is shown in [68] that common constraints such as these significantly complicate Linear Complementarity Problems and preclude a solution because the players may have different multiplier values associated with the constraint. Following [109] and [60], we consider a restricted multiplier formulation of the game. In this formulation, we require the multipliers of the common constraint (I.0.3) to be the same, replacing the vector μ with a scalar κ. The motivation for this is that uniqueness results are readily provable under this formulation. These are then be used to show uniqueness of results for all solutions of the original formulation (I.0.6) in which $Q < Q^q$. The restricted multiplier formulation is given by:

$$0 \leq \mathbf{y} \perp N\mathbf{y} + \mathbf{w} \geq 0, \tag{J.0.1}$$

where

$$\mathbf{y} = \begin{pmatrix} \mathbf{q} \\ \kappa \end{pmatrix},$$
$$\mathbf{w} = \begin{pmatrix} -\alpha\mathbf{1} + \mathbf{c} \\ Q^q \end{pmatrix},$$
$$N = \begin{pmatrix} \beta(I+E) & \mathbf{1} \\ -\mathbf{1}^T & 0 \end{pmatrix}.$$

The following Lemma can be used to establish successful termination of Lemke's method,[1] and thereby the existence of a solution.

[1] See [33].

Lemma J.0.8. *[[108], Lemma 2] Let M be a semimonotone matrix. If $\mathbf{d} > 0$ is such that for every $\tau > 0$, $SOL(\mathbf{q} + \tau\mathbf{d}; M)$ is bounded, then the $LCP(\mathbf{q}; M)$ has a solution that can be computed by Lemke's algorithm with $\mathbf{d}$ as the covering vector.*

Proposition J.0.9. *The Restricted Multiplier Formulation* (J.0.1) *has a solution which can be computed by Lemke's algorithm with $\mathbf{1}$ as the covering vector.*

Proof. We first need to show that N is semimonotone. We note that

$$N + N^T = \begin{pmatrix} 2\beta(I+E) & 0 \\ 0 & 0 \end{pmatrix}.$$

Given that $I + E$ is the sum of a positive definite and positive semi-definite matrix, it is therefore positive definite. This shows that $N + N^T$ is positive semidefinite, so that N is positive semidefinite and therefore semimonotone. Next, assume that there is a $\tau > 0$ such that $SOL(\mathbf{w} + \tau\mathbf{1}, N)$ contains a sequence of solutions $(\mathbf{q}^{(k)}, \kappa^{(k)})_{k=1}^{\infty}$ such that

$$0 \leq \mathbf{q}^k \perp \beta(I+E)\mathbf{q}^{(k)} - \alpha\mathbf{1} + \mathbf{c} + \kappa^{(k)}\mathbf{1} + \tau\mathbf{1} \geq 0,$$
$$0 \leq \kappa^{(k)} \perp Q^q - \mathbf{1}^T\mathbf{q}^{(k)} + \tau \geq 0.$$

First note that $\mathbf{q}^{(k)}$ is clearly bounded so it remains to show that $\kappa^{(k)}$ is bounded. Assuming that $\kappa^{(k)}$ is unbounded, and that $\kappa^{(k)} > 0$ for all k sufficiently large implies that

$$\sum_{i=1}^{N} q_i = Q^q + \tau,$$

by the second linear complementarity condition in (J.0.1). By the first condition, we therefore have

$$\beta(I+E)\mathbf{q}^{(k)} - \alpha\mathbf{1} + \mathbf{c} + \kappa^{(k)}\mathbf{1} + \tau\mathbf{1} = 0.$$

As $\mathbf{q}^{(k)}$ is bounded, we must therefore have $\kappa^{(k)}$ bounded, a contradiction. Therefore $\kappa^{(k)}$ is bounded. □

We now consider the uniqueness of solutions of (J.0.1). Because N is a positive semi-definite matrix, we have, for any two solutions $\mathbf{y}^{(1)} = (\mathbf{q}^{(1)}, \kappa^{(1)})$ and $\mathbf{y}^{(2)} = (\mathbf{q}^{(2)}, \kappa^{(2)})$ of the $LCP(\mathbf{w}, N)$

$$(N + N^T)(\mathbf{y}^{(1)} - \mathbf{y}^{(2)}) = 0.$$

Therefore, we have

$$2(\beta I + E)(\mathbf{q}^{(1)} - \mathbf{q}^{(2)}) = 0,$$

so that $\mathbf{q}^{(1)} = \mathbf{q}^{(2)}$ as $(\beta I + E)$ is positive definite. It therefore follows that even if (J.0.1) has multiple solutions, the equilibrium production $\mathbf{q}^*$ is unique. We now move on to consider the original formulation (I.0.6).

If $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are equilibria then there exist $\mu^{(1)}$ and $\mu^{(2)}$ such that $(\mathbf{q}^{(1)}, \mu^{(1)})$ and $(\mathbf{q}^{(2)}, \mu^{(2)})$ are solutions of the LCP problem (I.0.6). We assume without loss of generalisation that

$$Q^{(1)} = \sum_{i=1}^{N} q_i^{(1)} < \sum_{i=1}^{N} q_i^{(2)} = Q^{(2)}.$$

We begin by claiming $Q^{(2)} = Q^q$. If not, then $\mu^{(1)} = 0$ and $\mu^{(2)} = 0$ by (5.4.9), $(\mathbf{q}^{(1)}, 0)$ and $(\mathbf{q}^{(2)}, 0)$ satisfy the restricted multiplier formulation (J.0.1), and $\mathbf{q}^{(1)} = \mathbf{q}^{(2)}$, a contradiction. Therefore $Q^{(2)} = Q^q$. The assumption

$$\sum_{i=1}^{N} q_i^{(1)} < \sum_{i=1}^{N} q_i^{(2)},$$

implies that there must exist an i such that $0 \leq q_i^{(1)} < q_i^{(2)}$. By (5.4.8) we have

$$\beta q_i^{(1)} - \alpha + \beta \sum_{i=1}^{N} q_i^{(1)} + c_i + \mu_i^{(1)} \geq 0 \tag{J.0.2}$$

$$\beta q_i^{(2)} - \alpha + \beta \sum_{i=1}^{N} q_i^{(2)} + c_i + \mu_i^{(2)} = 0. \tag{J.0.3}$$

It follows from (5.4.9) that $\mu_i^{(1)} = 0 \leq \mu_i^{(2)}$. Subtracting (J.0.2) from (J.0.3) therefore gives

$$\beta(q_i^{(2)} - q_i^{(1)}) + \beta(Q^{(2)} - Q^{(1)}) + (\mu_i^{(2)} - \mu_i^{(1)}) = \beta(q_i^{(2)} + Q^{(2)} - q_i^{(1)} - Q^{(1)}) + \mu_i^{(2)} > 0,$$

a contradiction.

Appendix K

Proof of Proposition 5.4.3

First note that if $Q < Q^q$, then the equilibrium $\mathbf{q}^*$ is also a solution of the restricted multiplier formulation. Uniqueness therefore holds. Henceforth we shall call this equilibrium $\mathbf{q^B}$, and define $Q^B = \sum_{i=1}^N q_i^B$. Under the assumption $Q^B < Q^q$, the equilibrium can be derived by variational means and is given by

$$q_i^B = \frac{\alpha - (N+1)c_i + \sum_{j=1}^N c_j}{\beta(N+1)}.$$

This is the sole equilibrium if the condition $Q^B < Q^q$ is met. Rearranging this gives

$$\alpha N - \sum_{i=1}^N c_i < (N+1)\beta Q^q.$$

Next we assume that $\sum_{i=1}^N q_i = Q^q$, where solutions of (I.0.6) may not be solutions of the restricted LCP (J.0.1) and show that the equilibrium set is given by Ω_2, as defined in Proposition 5.4.3.

Taking an arbitrary equilibrium $\mathbf{q}'$, we show that $\mathbf{q}' \in \Omega_2$. First, a vector μ can clearly be constructed such that LCP is satisfied. By Proposition 5.4.2 and the assumption $Q^* = Q^q$, it is clear that $\sum_{i=1}^N q_i' = Q^p$. Next simply note that if $q_i > (\alpha - \beta Q^q - c_i)/\beta$, then we have

$$\beta q_i + \beta Q^q - \alpha + c_i + \mu_i > \mu_i > 0,$$

which violates (5.4.8). Therefore $\mathbf{q}' \in \Omega$. Next, we take an arbitrary $\mathbf{q}' \in \Omega$ and show that $\mathbf{q}'$ is an equilibrium of the game. We can construct μ' such that (5.4.8)-(5.4.9) are satisfied by setting

$$\mu_i' = \alpha - c_i - \beta(q_i + Q^q) \geq 0.$$

Because $(\mathbf{q}', \mu')$ satisfy the LCP (I.0.6), $\mathbf{q}'$ must be an equilibrium of the game. From this, it is clear that if $Q^B < Q^q$, then $\mathbf{q^B}$ is the sole equilibrium of G_2, otherwise the equilibrium set is given by Ω_2.

Appendix L

Proof of Proposition 5.4.4

The proof of this Proposition is similar to that of Propositions 5.4.1-5.4.3, but is presented for completeness. If we use λ_i and μ_i to represent the constraints imposed by the price and production caps respectively, then the optimization problem can be formulated as the Linear Complementarity Problem.[1]

$$0 \leq q_i \perp \beta q_i - \alpha + c_i + \beta Q + \mu_i - \lambda_i \geq 0,$$

$$0 \leq \lambda_i \perp Q_{-i} + q_i - Q^p \geq 0, \tag{L.0.1}$$

$$0 \leq \mu_i \perp Q^q - q_i - Q_{-i} \geq 0. \tag{L.0.2}$$

Combining the conditions for all players gives

$$0 \leq \mathbf{q} \perp \beta(I + E)\mathbf{q} - \alpha\mathbf{1} + \mathbf{c} + \mu - \lambda \geq 0, \tag{L.0.3}$$

$$0 \leq \lambda \perp \mathbf{1}^T\mathbf{q} - Q^p \geq 0, \tag{L.0.4}$$

$$0 \leq \mu \perp Q^q - \mathbf{1}^T\mathbf{q} \geq 0. \tag{L.0.5}$$

The conditions (L.0.1)-(L.0.2) are common across the players. In order to reach a solution, we therefore also consider the restricted multiplier formulation.

$$0 \leq \mathbf{q} \perp \beta(I + E)\mathbf{q} - \alpha\mathbf{1} + \mathbf{c} + (\kappa_2 - \kappa_1)\mathbf{1} \geq 0,$$

$$0 \leq \kappa_1 \perp \mathbf{1}^T\mathbf{q} - Q^p \geq 0,$$

$$0 \leq \kappa_2 \perp Q^q - \mathbf{1}^T\mathbf{q} \geq 0,$$

where κ_1 and κ_2 are scalars. This can be written in general form as

$$0 \leq \mathbf{y} \perp N\mathbf{y} + \mathbf{w} \geq 0, \tag{L.0.6}$$

[1] See [33].

where

$$\mathbf{y} = \begin{pmatrix} \mathbf{q} \\ \kappa_1 \\ \kappa_2 \end{pmatrix},$$
$$\mathbf{w} = \begin{pmatrix} -\alpha\mathbf{1} + \mathbf{c} \\ -Q^p \\ Q^q \end{pmatrix},$$
$$N = \begin{pmatrix} \beta(I+E) & -\mathbf{1} & \mathbf{1} \\ \mathbf{1}^T & 0 & 0 \\ -\mathbf{1}^T & 0 & 0 \end{pmatrix}.$$

The same argument as used in the game with a production cap can be used to show that the restricted multiplier formulation (L.0.6) leads to a solution, and that the solution is unique in production $\mathbf{q}$ if for every $\tau > 0$, $SOL(\mathbf{w} + \tau\mathbf{1}; N)$ is bounded. We prove this below

Proposition L.0.10. *For every $\tau > 0$, $SOL(\mathbf{w} + \tau\mathbf{1}; N)$ is bounded*

Proof. Assume that there is a $\tau > 0$ such that $SOL(\mathbf{w} + \tau\mathbf{1}, N)$ contains a sequence of solutions $(\mathbf{q}^{(k)}, \kappa_1^{(k)}, \kappa_2^{(k)})_{k=1}^{\infty}$ such that

$$0 \leq \mathbf{q}^{(k)} \perp \beta(I+E)\mathbf{q}^{(k)} - \alpha\mathbf{1} + \mathbf{c} + \kappa_2^{(k)}\mathbf{1} - \kappa_1^{(k)}\mathbf{1} + \tau\mathbf{1} \geq 0, \tag{L.0.7}$$
$$0 \leq \kappa_1^{(k)} \perp \mathbf{1}^T\mathbf{q}^{(k)} - Q^p + \tau\mathbf{1} \geq 0, \tag{L.0.8}$$
$$0 \leq \kappa_2^{(k)} \perp Q^q - \mathbf{1}^T\mathbf{q}^{(k)} + \tau\mathbf{1} \geq 0. \tag{L.0.9}$$

First note that $\mathbf{q}^{(k)}$ is clearly bounded so it remains to be shown that $(\kappa_1^{(k)}, \kappa_2^{(k)})$ is bounded. Assuming that $\kappa_1^{(k)}$ is unbounded, and that $\kappa_1^{(k)} > 0$ for all k sufficiently large implies that

$$\sum_{i=1}^{N} q_i^{(k)} = Q^p - \tau,$$

by (L.0.8). By (L.0.7) we therefore have

$$\beta(I+E)\mathbf{q}^{(k)} - \alpha\mathbf{1} + \mathbf{c} + \kappa_2^{(k)}\mathbf{1} - \kappa_1^{(k)}\mathbf{1} = 0.$$

As $\mathbf{q}^{(k)}$ is bounded, we must therefore also have that $\kappa_2^{(k)}$ is unbounded. By (L.0.9), this implies

$$\sum_{i=1}^{N} q_i^{(k)} = Q^q + \tau,$$

a contradiction as $Q^p \neq Q^q$. We therefore have that $\kappa_1^{(k)}$ is bounded, and the same argument with $\kappa_1^{(k)}$ and $\kappa_2^{(k)}$ reversed shows that $\kappa_2^{(k)}$ is bounded as well. □

In order to assess the uniqueness of equilibria, we need the following technical lemma.

Lemma L.0.11. *If* $\sum_{i=1}^N q_i = Q^p$, *then* $q_i > 0 \ \forall i$.

Proof. Assume otherwise and that for some i, $q_i = 0$. By (L.0.3) we have

$$\begin{aligned} 0 &\leq -\alpha + c_i + \beta \sum_{i=1}^N q_i + \mu_i, \\ &= -\alpha + \beta \hat{\mathbf{Q}}^p + c_i = c_i - P^p < 0, \end{aligned}$$

which violates the assumption that $P^p > \max(c_1, ..., c_N)$. Therefore $q_i > 0$ holds for all i. □

Proposition L.0.12. *If there are two solutions* $\mathbf{q}^{(1)}$ *and* $\mathbf{q}^{(2)}$, *then* $\sum_{i=1}^N q_i^{(1)} = \sum_{i=1}^N q_i^{(2)}$.

Proof. If $\mathbf{q}^{(1)}$ and $\mathbf{q}^{(2)}$ are equilibria, then there exist $(\lambda^{(1)}, \mu^{(1)})$ and $(\lambda^{(2)}, \mu^{(2)})$ such that $(\mathbf{q}^{(1)}, \lambda^{(1)}, \mu^{(1)})$ and $(\mathbf{q}^{(2)}, \lambda^{(2)}, \mu^{(2)})$ are solutions of the LCP problem (L.0.3)-(L.0.5). We begin by assuming

$$Q^{(1)} = \sum_{i=1}^N q_i^{(1)} > \sum_{i=1}^N q_i^{(2)} = Q^{(2)}.$$

We begin by claiming that at least one of $Q^{(2)} = Q^p$ or $Q^{(1)} = Q^q$ holds. If not, then $Q^p < Q^{(2)} < Q^{(1)} < Q^q$ as $Q^{(2)} = Q^q$ or $Q^{(1)} = Q^p$ would result in a contradiction. We therefore have $\lambda^{(1)} = 0$, $\lambda^{(2)} = 0$, $\mu^{(1)} = 0$, and $\mu^{(2)} = 0$ by (L.0.4)-(L.0.5). It then follows that $(\mathbf{q}^{(1)}, 0, 0)$ and $(\mathbf{q}^{(2)}, 0, 0)$ are solutions of the restricted multiplier formulation (L.0.6), and $\mathbf{q}^{(1)} = \mathbf{q}^{(2)}$ by the uniqueness of its solutions, a contradiction. We next consider the case $Q^{(2)} = Q^p$. Lemma L.0.11 and the assumption $Q^{(1)} > Q^{(2)}$ imply that there must exist an i such that $q_i^{(1)} > q_i^{(2)} > 0$. By (L.0.3) we therefore have

$$\beta q_i^{(1)} - \alpha + \beta \sum_{i=1}^N q_i^{(1)} + c_i + \mu_i^{(1)} - \lambda_i^{(1)} = 0$$

$$\beta q_i^{(2)} - \alpha + \beta \sum_{i=1}^N q_i^{(2)} + c_i + \mu_i^{(2)} - \lambda_i^{(2)} = 0.$$

The conditions (L.0.4)-(L.0.5) give us $\lambda_i^{(1)} = 0$ and $\mu_i^{(2)} = 0$, so that we have

$$\beta(q_i^{(1)} - q_i^{(2)}) + \beta(Q^{(1)} - Q^{(2)}) + (\mu_i^{(1)} - \mu_i^{(2)}) - (\lambda_i^{(1)} - \lambda_i^{(2)}) = \beta(q_i^{(1)} - q_i^{(2)}) + \beta(Q^{(1)} - Q^{(2)}) + \mu_i^{(1)} + \lambda_i^{(2)} > 0,$$

a contradiction, so that we cannot have $Q^p = Q^{(2)} < Q^{(1)} \leq Q^q$. A similar argument can be used to show that the case $Q^p \leq Q^{(2)} < Q^{(1)} = Q^q$ also results in a contradiction so that we must have $Q^{(1)} = Q^{(2)}$. □

We now assess the uniqueness of equilibria given that total production Q^* is known to be unique. If $Q^p < Q^* < Q^q$, then the equilibrium q^* is also a solution of the restricted multiplier formulation (L.0.6) and the equilibrium is unique. The same variational maximisation as used before can be used to show that the unique equilibrium, $\mathbf{q^C}$ is again given by $q_i^C = (\alpha - (N+1)c_i + \sum_{j=1}^N c_j)/(\beta(N+1))$. The condition $Q^p < Q^C < Q^q$ is given by

$$(N+1)\beta Q^p < \alpha N - \sum_{i=1}^N c_i < (N+1)\beta Q^q.$$

The results of Proposition 5.4.3 and Proposition 5.4.6 can be shown to apply to the cases where $Q^* = Q^p$ and Q^q respectively. Their equilibrium sets are therefore again given by (5.4.6) and (5.4.10) as required.

Appendix M

Market Entry Games as an Implicit Penalty Game

In this Appendix, we briefly illustrate how a market entry game can be reformulated as a penalty game of the form given in Section 5.5. We consider a static Cournot example with two incumbent firms and a potential entrant. We assume the two incumbent firms produce without cost while the entrant produces with costs at the rate of c per unit production. The incumbent firms first simultaneously determine their production, after which the entrant determines their production rate.[1] The value function for the players are given by

$$\begin{aligned} v_i &= q_i(1 - q_1 - q_2 - q_3), \qquad i = 1, 2, \\ v_3 &= q_3(1 - q_1 - q_2 - q_3 - c). \end{aligned}$$

Player 3 determines their production taking q_1 and q_2 as given, so we consider their best-response first. Player 3's best-response is given by

$$q_3 = \max\left(\frac{1}{2}(1 - q_1 - q_2 - c), 0\right).$$

Incorporating player 3's response function into the value functions for players 1 and 2 gives

$$v_i = q_i\left(1 - q_1 - q_2 - \frac{1}{2}(1 - c - q_1 - q_2)\mathbf{1}_{q_1+q_2<1-c}\right), \qquad i = 1, 2.$$

This is equivalent to the formulation given in Section 5.5, with $Q^p = 1 - c$, $A = 0$, and $B = 1/2$, and can therefore be solved in the same way.

[1] The game could therefore be viewed as a Stackelberg game with multiple leaders.

Appendix N

Uniqueness of Equilibria in a Game with Congestion Charges

Defining the congestion charge to player i by C_i:

$$C_i = \begin{cases} (A + Bq_i)(Q - Q^q) & \text{for } Q > Q^q, \\ 0 & \text{for } Q \leq Q^q, \end{cases}$$

the value function for player i is given by

$$v_i = \int_0^\infty \mathrm{e}^{-rt} \left[q_i(1 - q_i - Q_{-i} - c_i) - C_i \right] \mathrm{d}t.$$

The game can be solved via a similar decomposition introduced in Section 5.5 for the penalty game given in (5.5.1). Replacing Q^p with Q^q, and S_1 with S_2 in Section 5.5.2 and following the same steps allows us to show that total production is unique in equilibrium. If

$$\frac{N - \sum_{i=1}^N c_i}{N+1} \leq Q^q \leq \frac{N(1 + BQ^q - A) - \sum_{i=1}^N c_i}{(N+1)(1+B)},$$

then total production is given by $Q = Q^q$, and the equilibrium set is given by

$$\Omega_C = \left\{ q \in \mathbf{R}_+^N \;\middle|\; \begin{array}{l} \dfrac{1 + BQ^p - A - (1+B)Q^p - c_i}{(1+B)} \leq q_i \leq 1 - Q^p - c_i \qquad \forall i \in \{1, ..., N\} \\ \displaystyle\sum_{i=1}^N q_i = Q^p \end{array} \right\}.$$

If the condition

$$\frac{N(1 + BQ^q - A) - \sum_{i=1}^N c_i}{(N+1)(1+B)} > Q^q, \tag{N.0.1}$$

holds, then $Q > Q^q$ and there is a unique equilibrum, given by $\mathbf{q^A}$:

$$q_i^A = \frac{1 + BQ^q - A - (N+1)c_i + \sum_{j=1}^N c_j}{(1+B)(N+1)}.$$

Otherwise, the condition

$$\frac{N - \sum_{i=1}^N c_i}{N+1} < Q^q, \tag{N.0.2}$$

holds, and $Q < Q^q$. There is a unique equilibrium, given by $\mathbf{q^B}$:

$$q_i^B = \frac{1 - (N+1)c_i + \sum_{j=1}^N}{N+1}.$$

The assumption $P^q \geq \max(c_1, ..., c_N)$ implies that

$$\begin{aligned}
\frac{N(1 + BQ^q - A) - \sum_{i=1}^N c_i}{(N+1)(1+B)} &< \frac{N(1 + BQ^q) - \sum_{i=1}^N c_i}{(N+1)(1+B)} \\
&< \frac{N(1 - \hat{\mathbf{c}})(1+B)}{(N+1)(1+B)} \\
&= \frac{N - \sum_{i=1}^N c_i}{N+1},
\end{aligned}$$

so that the conditions (N.0.1) and (N.0.2) are incompatible, and $\mathbf{q^A}$ and $\mathbf{q^B}$ cannot occur simultaneously.