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Electronic Notes in Theoretical Computer Science 155 (2006) 331–340

Electronic Notes in Theoretical Computer Science

www.elsevier.com/locate/entcs

Strongly Compact Closed Semantics

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Abstract

These lecture notes survey some joint work with Samson Abramsky as it was presented by me at Mathematical Foundations of Programming Semantics XXI in Birmingham (2005).[3] It concerns a categorical semantics for quantum mechanics introduced in [3,4]. We present this semantics with a particular focus on the connection between categorical diagrams and certain pictures involving typed squares, triangles, diamonds and lines. Along the way we unravel the structural components which come with strong compact closure. We provide pointers to related literature.

Keywords: Strong compact closure, semantics, quantum mechanics, quantum logic, quantum informatics.

1 Introduction

The starting point of these developments was the Logic of Entanglement [11,12,13], which emerged from an investigation by Abramsky and myself on the connection between Quantum Entanglement and Geometry of Interaction [2], and which provided a scheme to derive protocols such as Quantum Teleportation [8], Logic-gate Teleportation [18], Entanglement Swapping [33] and various related ones through the notion of Quantum Information-flow (see also [12,13]). In [3,4] Abramsky and I axiomatized this quantum information-flow

[1] The research which led to these results was supported by the EPSRC grant EP/C500032/1 on High-Level Methods in Quantum Computation and Quantum Information. We thank the organizers of MFPS XXI for the invitation to present this work.

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[3] I only present part of that talk since some material is already available elsewhere [15].

doi:10.1016/j.entcs.2005.11.062

in category theoretic terms which via the work of Kelly & Laplaza [24] and Joyal & Street [20] formally justifies the grapical calculus informally initiated in [11], subsequently refined in [3,12,4], surveyed in my lecture notes entitled Kindergarten Quantum Mechanics and connected up by Abramsky & Duncan to the so-called proof nets of Linear Logic in [5]. The most elaborate mathematical presentation on this issue can currently be found in Selinger's paper [30] — while at first sight his graphical calculus looks quite different from ours, they are in fact formally equivalent, being both transcriptions of strong compact closure (or as Selinger calls it, dagger compact closure). However, Selinger's language depicts the logic only, while our calculus carries two stories, since it also identifies the *operationally relevant entities* from a physical perspective. Actually, the use of graphical calculi for tensor calculus goes back to Penrose [27], initiating as applications the theories of Braids and Knots in Mathematical Physics. We mention independent work by Louis Kauffman [22] which provides a topological interpretation of quantum teleportation and hence relates to the Logic of Entanglement in [11], and we mention independent work by John Baez [6] which relates to the developments in [3,4] in the sense that he exposes similarities of the category of relations, the category of Hilbert spaces, and the compact closed category of cobordisms which plays an important role in Topological Quantum Field Theory. There also seem to be promising connections with Basil Hiley's recent work [19] on Dirac's 'standard ket' [17] in the context of quantum evolution. The connection between the different categorical structures of our interest is:

$$\text{Strong Compact Closure [3,4]} \Rightarrow \text{Compact Closure [23]} \Rightarrow *\text{-autonomy [7]}$$

where the latter is the semantics for the multiplicative fragment of Linear Logic [29]. Linear Logic itself is a logic in which one is not allowed to copy nor delete premisses, hence enabling one to take computational resources into account — in view of the No-Cloning [32] and No-Deleting [26] theorems it is not a surprise that the axiomatization of quantum information-flow comprises this resource-sensitive logicality. A very substantial contribution to our program was made by Peter Selinger who discovered the construction which turns any strongly compact closed category of pure states and pure operations into one of mixed states and completely positive maps [30]. At the same time, I discovered the preparation-state agreement axiom [14], and in currently ongoing work I identified another axiom which combines preparation-state agreement and the structural content of Selinger's construction. It has the potential to provide a categorical foundation for quantum information theory [16] — many quantum information theoretic fidelities and capacities (see [25] for a structured survey on some of these) can indeed be unified in our graphical calculus once this axiom is added. At the more abstract side of the spectrum we refer for Joyal, Street and Verity's abstract theory of partial trace to [21] and for the current state of the art on free constructions of traced and strongly compact closed categories to Abramsky's [1].

2 Strong compact closed tensors

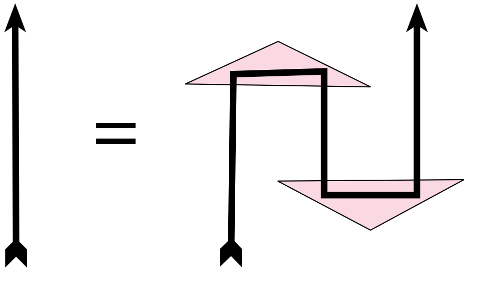
Following [4] a *strongly compact closed tensor* is a symmetric monoidal tensor together with

- a involution *dual* $A \mapsto A^*$ on objects,
- a contravariant strict $\otimes$-involution *adjoint* $f_{A\to B} \mapsto f^\dagger_{B\to A}$,
- and a *unit* $\eta_A : \mathrm{I} \to A^* \otimes A$ with $\eta_{A^*} = \sigma_{A^*,A} \circ \eta_A$ for each object

such that we have commutation of

$$\begin{array}{ccccc} A & \xleftarrow{\;\simeq\;} & \mathrm{I}\otimes A & \xleftarrow{\;\eta^\dagger_{A^*}\otimes 1_A\;} & (A\otimes A^*)\otimes A \\ {\scriptstyle 1_A}\uparrow & & & & \uparrow{\scriptstyle \simeq} \\ A & \xrightarrow[\simeq]{} & A\otimes \mathrm{I} & \xrightarrow[1_A\otimes\eta_A]{} & A\otimes(A^*\otimes A) \end{array}$$

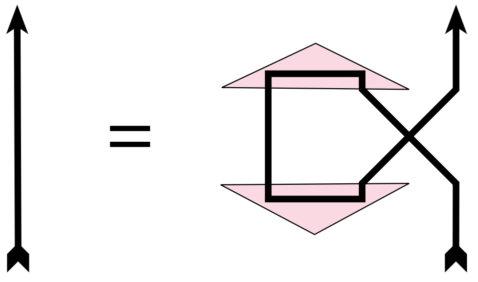
This diagram can equivalently be represented by the following picture:



where the left-hand-side of the equality stands for 1_A and in the right-hand-side the triangle at the bottom stands for η_A and the one at the top for η_{A^*}. For a more detailed introduction to these kind of pictures we refer the reader to another set of lecture notes [15]. The above diagram is also equivalent to

$$\begin{array}{ccccccc} A & \xleftarrow{\;\simeq\;} & \mathrm{I}\otimes A & \xleftarrow{\;\eta^\dagger_{A}\otimes 1_A\;} & (A^*\otimes A)\otimes A & \xleftarrow{\;\simeq\;} & A^*\otimes(A\otimes A) \\ {\scriptstyle 1_A}\uparrow & & & & & & \uparrow{\scriptstyle 1_{A^*}\otimes\sigma_{A,A}} \\ A & \xrightarrow[\simeq]{} & \mathrm{I}\otimes A & \xrightarrow[\eta_A\otimes 1_A]{} & (A^*\otimes A)\otimes A & \xrightarrow[\simeq]{} & A^*\otimes(A\otimes A) \end{array}$$

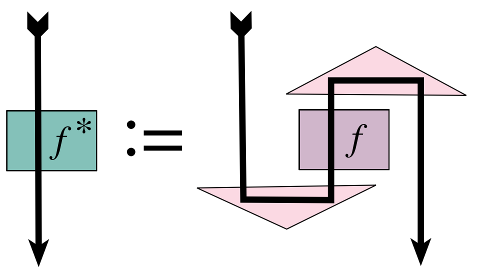
which has



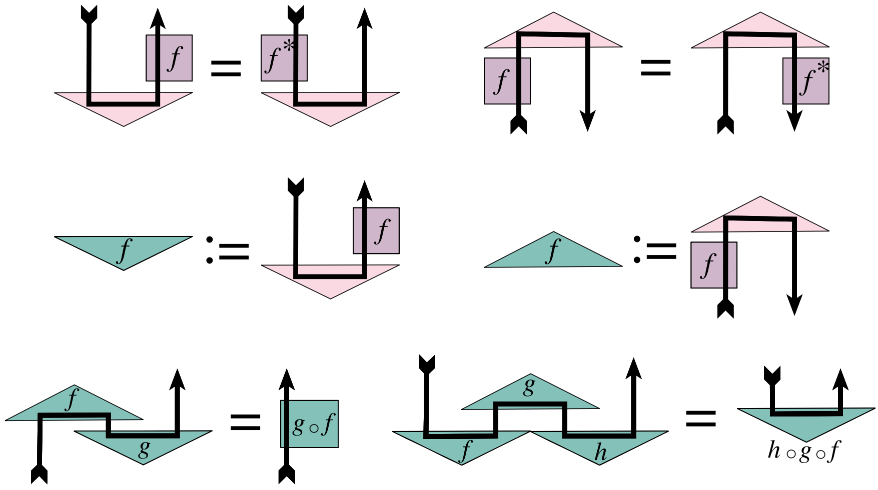
as its pictorial equivalent. A $*$-structure arises when setting:

$$
\begin{array}{ccccc}
A^* & \xleftarrow{\ \simeq\ } & A^* \otimes \mathrm{I} & \xleftarrow{1_{A^*} \otimes \eta_{B^*}^{\dagger}} & A^* \otimes B \otimes B^* \\
\uparrow f^* & & & & \uparrow 1_{A^*} \otimes f \otimes 1_{B^*} \\
B^* & \xrightarrow[\ \simeq\ ]{} & \mathrm{I} \otimes B^* & \xrightarrow[\eta_A \otimes 1_{B^*}]{} & A^* \otimes A \otimes B^*
\end{array}
$$

in a picture that is:



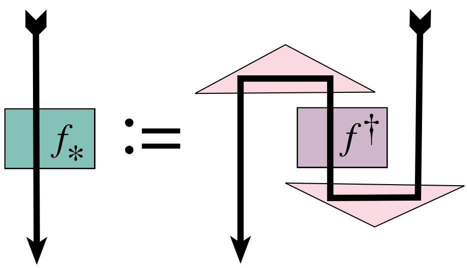
and this $*$-structure is not only $*$-*autonomous* in the sense of Barr [7] but also *compact closed* in the sense of Kelly [23]. As shown in [3,4] compact closure provides an intriguing compositional structure:



which subsumes the ‘seemingly acausal flow of information’ which was exposed in the Logic of Entanglement [11,12,13]. There is however a second $*$-structure which we call *lower star* [3,4]:

$$\begin{array}{ccccc}
B^* & \xleftarrow{\;\simeq\;} & B^*\otimes \mathrm{I} & \xleftarrow{\;1_{B^*}\otimes\eta^\dagger_{A^*}\;} & B^*\otimes A\otimes A^* \\
\uparrow{\scriptstyle f_*} & & & & \uparrow{\scriptstyle 1_{B^*}\otimes f^\dagger\otimes 1_{A^*}} \\
A^* & \xrightarrow[\;\simeq\;]{} & \mathrm{I}\otimes A^* & \xrightarrow[\;\eta_B\otimes 1_{A^*}\;]{} & B^*\otimes B\otimes A^*
\end{array}$$

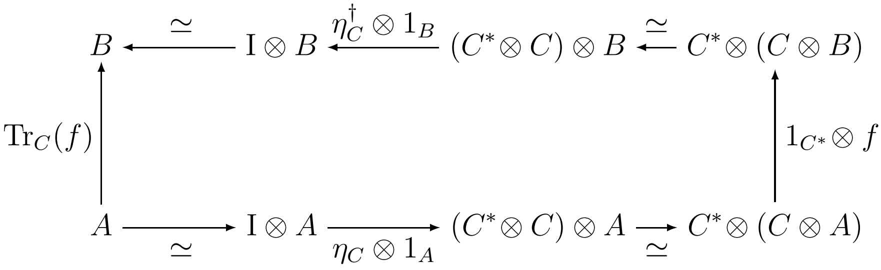
in a picture that is:



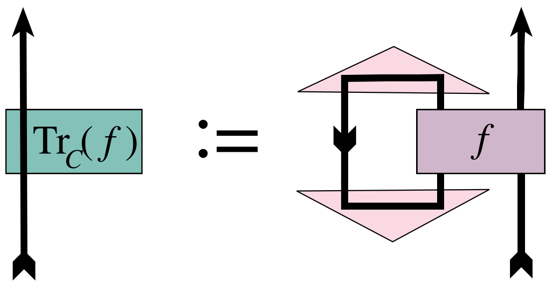
which constitutes a genuine extra piece of data as compared to ordinary compact closure. One could say that:

$$\frac{\text{strong compact closure}}{\text{compact closure}} \simeq \frac{\text{inner-product space}}{\text{vector space}}$$

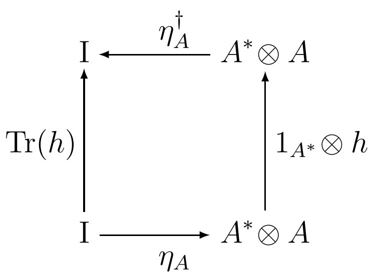
Compact closure (as always) also provides a trace structure in the sense of Joyal, Street and Verity [21]:



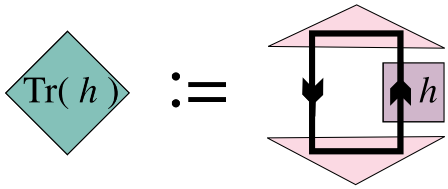
in a picture that is:



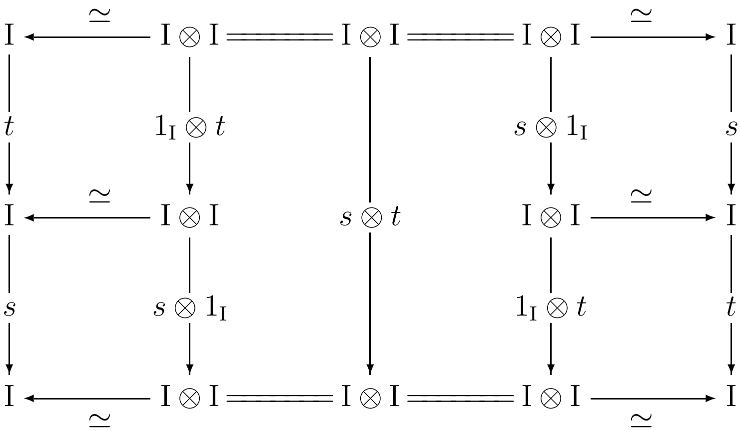
of which the corresponding full trace variant:



that is:



endows any morphism with a *scalar* , i.e. a morphism of type $\mathrm{I} \to \mathrm{I}$ which we want to interpret as a *probabilistic weight.* Recall here that in each monoidal category the endomorphism-monoid of scalars is always commutative [24]:



and allows to introduce scalar multiplication as:

$$s \bullet f := \lambda_B^{-1} \circ (s \otimes f) \circ \lambda_A : A \to B\,.$$

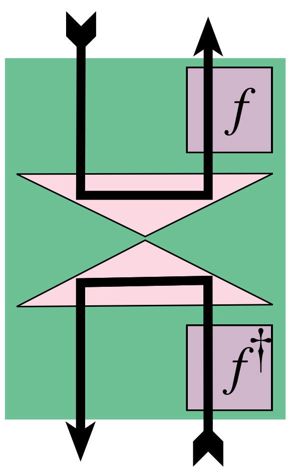
which satisfies

$$(s \bullet f) \circ (t \bullet g) = (s \circ t) \bullet (f \circ g) \qquad (s \bullet f) \otimes (t \bullet g) = (s \circ t) \bullet (f \otimes g)$$

i.e. scalars can freely propagate along composition and the tensor. The crucial connection with the quantum mechanical formalism is the fact that positive operations arises as morphisms of the form $g \circ g^\dagger : A \to A$ for some $g : C \to A$, projectors arise as morphisms of the form $g^\dagger \circ g = 1_A$ (which implies idempotence), we have non-degenerate projectors iff $C = \mathrm{I}$, and in particular, non-degenerate bipartite projectors always decompose as

$$\mathrm{P}_f := (1_{A_1^*} \otimes f) \circ \eta \circ \eta^\dagger \circ (1_{A_1^*} \otimes f^\dagger)$$

in a picture that is:



and it is this decomposition which allows trivial reconstruction of the quantum teleportation [8], logic-gate teleportation [18], entanglement swapping [33] protocols. As far as we know this decomposition for non-degenerate bipartite projectors was never observed before. Moreover, in most previous work on the axiomatization of quantum mechanics projectors were always conceived as primitive, non-decomposable. The *inner-product* of $\psi, \phi : \mathrm{I} \to A$ is a scalar

$$\langle \psi \mid \phi \rangle := \psi^\dagger \circ \phi : \mathrm{I} \to \mathrm{I}$$

cf. *Dirac's notation* [17]:

$$\texttt{bra} := \langle \psi | \qquad \texttt{ket} := | \phi \rangle \qquad \texttt{bra-ket} := \langle \psi \mid \phi \rangle$$

Adjointness implies

$$\langle f^\dagger \circ \psi \mid \phi \rangle = \langle \psi \mid f \circ \phi \rangle$$

Unitarity means $U^{-1} = U^\dagger : A \to B$ and implies

$$\langle U \circ \psi \mid U \circ \phi \rangle = \langle \psi \mid \phi \rangle \,.$$

In fact, the graphical calculus which comes with strong compact closure is a very substantial extension of Dirac's notation. Here's quantum mechanics:

- System of type A := Object A
- Composite of A and B := Tensor $A \otimes B$
- Process of type $A \to B$:= Morphism $f : A \to B$
- State of A := Element $\psi : \mathrm{I} \to A$
- Evolution of A := Unitary $U : A \to A$
- Measurement on A := Projectors $\{\mathrm{P}_i : A \to A\}_i$
  - · Data := $\nu \in \{i\}_i$
  - · Dynamics := $\psi \mapsto \mathrm{P}_\nu \circ \psi$
  - · Probability := $\psi^\dagger \circ \mathrm{P}_\nu \circ \psi : \mathrm{I} \to \mathrm{I}$

The only missing part is specification of what the *good families* of projectors are which define a measurement. Existing proposals which use *additive structures* can be found in [3] and [14]. A proposal which doesn't involve any additivity at all is currently under development with Dusko Pavlovic.

3 More of this kind

In [15] we continue this story purely relying on the graphical calculus. The corresponding formal developments can be found in the respective papers of Abramsky & myself, myself, and Selinger [3,4,14,30]. We in particular discuss in [14] how all this relates to developments started by Birkhoff and von Neumann [10], due to von Neumann's discomfort [9,28] with *his own* quantum formalism [31]. Selinger's paper [30] provides the key ingredient for a passage to the density matrix formalism of quantum mechanics.

References

[1] Abramsky, S. (2005) *Abstract scalars, loops, free traced and strongly compact closed categories.* In: *Proceedings of CALCO 2005*, pp. 1–31, Springer Lecture Notes in Computer Science **3629**.

[2] Abramsky, S. and Coecke, B. (2003) *Physical traces: quantum vs. classical information processing*. Electronic notes on Theoretical Computer Science **69** (special issue: Proceedings of Category Theory in Computer Science 2002). http://arXiv.org/cs/0207057

[3] Abramsky, S. and Coecke, B. (2004) *A categorical semantics of quantum protocols.* In: *Proceedings of the 19th Annual IEEE Symposium on Logic in Computer Science* (LiCS'04), IEEE Computer Science Press. An extended & improved version is available at http://arXiv.org/quant-ph/0402130

[4] Abramsky, S. and Coecke, B. (2005) *Abstract physical traces*. Theory and Applications of Categories **14**, 111–124. Available at www.tac.mta.ca/tac/ volumes/14/6/14-06abs.html

[5] Abramsky, S. and Duncan, R. W. (2004) *A categorical quantum logic*. In: *Proceedings of the 2nd Workshop on Quantum Programming Languages*, pp. 3–20, P. Selinger, Ed., TUCS General Publication.

[6] Baez, J. (2004) *Quantum quandaries: a category-theoretic perspective.* In: S. French et al. (Eds.) *Structural Foundations of Quantum Gravity*, Oxford University Press. http://arXiv.org/quant-ph/0404040

[7] Barr, M. (1979) *$*$-Autonomous Categories*. Lecture Notes in Mathematics **752**, Springer-Verlag.

[8] Bennet, C. H., Brassard, C., Crépeau, C., Jozsa, R., Peres, A. and Wooters, W. K. (1993) *Teleporting an unknown quantum state via dual classical and Einstein-Podolsky-Rosen channels.* Physical Review Letters **70**, 1895–1899.

[9] Birkhoff, G. (1958) *von Neumann and lattice theory.* Bulletin of the American Mathematical Society **64**, 50–56.

[10] Birkhoff, G. and von Neumann, J. (1936) *The logic of quantum mechanics.* Annals of Mathematics **37**, 823–843.

[11] Coecke, B. (2003) *The logic of entanglement. An invitation.* Oxford University Computing Laboratory Research Report nr. PRG-RR-03-12. An 8 page short version (which we recommend) is at http://arXiv.org/quant-ph/0402014, the full 160 page version is at http://www.comlab.ox.ac.uk/oucl/publications/tr/rr-03-12.html

[12] Coecke, B. (2004) *Quantum information flow, concretely, abstractly.* In: *Proceedings of the 2nd Workshop on Quantum Programming Languages*, pp. 57–74, P. Selinger, Ed., TUCS General Publication.

[13] Coecke, B. (2005) *Quantum information-flow, concretely, and axiomatically.* In: *Proceedings of Quantum Informatics 2004*, pp. 15–29, Y. I. Ozhigov, Ed., Proceedings of SPIE Vol. 5833. http://arXiv.org/quant-ph/0506132

[14] Coecke, B. (2005) *De-linearizing linearity: Projective quantum axiomatics from strong compact closure.* Electronic Notes in Theoretical Computer Science (special issue: Proceedings of the 3rd International Workshop on Quantum Programming Languages). http://arXiv.org/quant-ph/0506134

[15] Coecke, B. (2005) *Kindergarten quantum mechanics — lecture notes.* In: *Quantum Theory: Reconsiderations of the Foundations* III. A. Khrennikov, Ed., American Institute of Physics Press. http://arXiv.org/quant-ph/0510032

[16] Coecke, B. (2005) *On a categorical foundation for quantum information.* In preparation.

[17] Dirac, P. A. M. (1947) *The Principles of Quantum Mechanics* (third edition). Oxford University Press.

[18] Gottesman, D. and Chuang, I. L. (1999) *Quantum teleportation is a universal computational primitive*. Nature **402**, 390–393. http://arXiv.org/quant-ph/9908010

[19] Hiley, B. (2004) *Non-commutative geometry, a physicist's perspective.* Talk at the *Oxford Advanced Seminar on Informatic Structures* (OASIS), Oxford University Computing Laboratory, November 2004.

[20] Joyal, A. and Street, R. (1991) *The Geometry of tensor calculus* I. Advances in Mathematics **88**, 55–112.

[21] Joyal, A., Street, R. and Verity, D. (1996) *Traced monoidal categories*. Proceedings of the Cambridge Philosophical Society **119**, 447–468.

[22] Kauffman, L. H. (2004) *Teleportation topology.* Preprint. http://arXiv.org/quant-ph/0407224

[23] Kelly, G. M. (1972) *Many-variable functorial calculus* I. In: *Coherence in Categories*, pp.66–105, G. M. Kelly, M. Laplaza, G. Lewis and S. Mac Lane, Eds., Lecture notes in Mathematics **281**, Springer-Verlach.

[24] Kelly, G. M. and Laplaza, M. L. (1980) *Coherence for compact closed categories*. Journal of Pure and Applied Algebra **19**, 193–213.

[25] Kretschmann, D. and Werner, R. F. (2004) *Tema con variazions: quantum channel capacity.* New Journal of Physics **6**, nr. 26.

[26] Pati, A. K. and Braunstein, S. L. (2000) *Impossibility of deleting an unknown quantum state.* Nature **404**, 164–165.

[27] Penrose, R. (1971) *Applications of negative dimensional tensors*. In: *Combinatorial Mathematics and its Applications*, pp. 221–244, Academic Press.

[28] Rédei, M. (1997) *Why John von Neumann did not like the Hilbert space formalism of quantum mechanics (and what he liked instead)*. Studies in History and Philosophy of Modern Physics **27**, 493–510.

[29] Seely, R. A. G. (1998) *Linear logic, ∗-autonomous categories and cofree algebras*. Contemporary Mathematics **92**, 371–382.

[30] Selinger, P. (2005) *Dagger compact closed categories and completely positive maps*. Electronic Notes in Theoretical Computer Science (special issue: Proceedings of the 3rd International Workshop on Quantum Programming Languages).

[31] von Neumann, J. (1932) *Mathematische Grundlagen der Quantenmechanik*. Springer-Verlag. English translation (1955): *Mathematical Foundations of Quantum Mechanics*. Princeton University Press.

[32] Wootters, W. K. and Zurek, W. (1982) *A single quantum cannot be cloned*. Nature **299**, 802–803.

[33] Żukowski, M., Zeilinger, A., Horne, M. A. and Ekert, A. K. (1993) *'Event-ready-detectors' Bell experiment via entanglement swapping*. Physical Review Letters **71**, 4287–4290.