Asymptotic Structure of Banach Spaces

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The notion of asymptotic structure of an infinite dimensional Banach space was introduced by Maurey, Milman and Tomczak-Jaegermann. The asymptotic structure consists of those finite dimensional spaces which can be found everywhere 'at infinity'. These are defined as the spaces for which there is a winning strategy in a certain vector game. The above authors introduced the class of asymptotic ℓ_p spaces, which are the spaces having simplest possible asymptotic structure. Key examples of such spaces are Tsirelson's space and James' space. We prove some new properties of general asymptotic ℓ_p spaces and also compare the notion of asymptotic ℓ_2 with other notions of asymptotic Hilbert space behaviour such as weak Hilbert and asymptotically Hilbertian.

We study some properties of smooth functions defined on subsets of asymptotic ℓ_∞ spaces. Using these results we show that that an asymptotic ℓ_∞ space which has a suitably smooth norm is isomorphically polyhedral, and therefore admits an equivalent analytic norm. We give a sufficient condition for a generalized Orlicz space to be a stabilized asymptotic ℓ_∞ space, and hence obtain some new examples of asymptotic ℓ_∞ spaces. We also show that every generalized Orlicz space which is stabilized asymptotic ℓ_∞ is isomorphically polyhedral.

In 1991 Gowers and Maurey constructed the first example of a space which did not contain an unconditional basic sequence. In fact their example had a stronger property, namely that it was hereditarily indecomposable. The space they constructed was 'ℓ_1-like' in the sense that for any n successive vectors $x_1 < \ldots < x_n$, $\frac{1}{f(n)} \sum_{i=1}^n \|x_i\| \le \| \sum_{i=1}^n x_i \| \le \sum_{i=1}^n \|x_i\|$, where $f(n) = \log_2(n+1)$. We present an adaptation of this construction to obtain, for each $p \in (1,\infty)$, an hereditarily indecomposable Banach space, which is 'ℓ_p-like' in the sense described above.

We give some sufficient conditions on the set of types, $\mathcal{T}(X)$, for a Banach space X to contain almost isometric copies of ℓ_p (for some $p \in [1,\infty)$) or of c_0. These conditions involve compactness of certain subsets of $\mathcal{T}(X)$ in the strong topology. The proof of these results relies heavily on spreading model techniques. We give two examples of classes of spaces which satisfy these conditions. The first class of examples were introduced by Kalton, and have a structural property known as Property (M). The second class of examples are certain generalized Tsirelson spaces.

We introduce the class of stopping time Banach spaces which generalize a space introduced by Rosenthal and first studied by Bang and Odell. We look at subspaces of these spaces which are generated by sequences of independent random variables and we show that they are isomorphic to (generalized) Orlicz spaces. We deduce also that every Orlicz space, h_ϕ, embeds isomorphically in the stopping time Banach space of Rosenthal. We show also, by using a suitable independence condition, that stopping time Banach spaces also contain subspaces isomorphic to mixtures of Orlicz spaces.

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Contents

Chapter 1

Introduction

Contents

1.1 Motivation

During the last half century, mathematicians have sought to understand the structure of infinite dimensional Banach spaces. The hope was to find some kind of ‘nice’ subspaces in any given Banach space. There have been many suggestions about what ‘nice’ subspace should be taken to mean. One conjecture, was that every Banach space should contain a subspace isomorphic to ℓ_p, for some $p \in [1, \infty)$, or to c_0. Another weaker question that could be asked is whether every Banach space contains a subspace isomorphic to ℓ_1, c_0, or a reflexive subspace. A third possibility, related to the first two, is to ask whether every Banach space contains a subspace with an unconditional basis. In the last decade it has been shown that no such simple structural theory exists. Examples constructed by Gowers and Maurey have shown that there is a much more varied structure than was first expected. We will discuss these examples in Section 1.2.

However, the structure of finite dimensional subspaces of Banach spaces has been well understood. Many results on the behaviour of high dimensional subspaces have been found. The results usually demonstrate some kind of regularity in the properties of finite dimensional subspaces as their dimension increases to infinity. We will mention some results in this area in Section 1.1.1.

In a 1995 article (see [59]), Maurey, Milman and Tomczak-Jaegermann, used similar asymptotic methods to study infinite dimensional properties. Their idea was to disregard finite dimensional information, and study those finite dimensional subspaces that occur 'arbitrarily far away'. This leads to the concept of *asymptotic finite dimensional spaces*, which we discuss in Section 1.1.2.

1.1.1 Asymptotic theory of finite dimensional spaces

Asymptotic theory of finite dimensional spaces is part of what is known as the Local Theory of Banach spaces, which relates the structure of an infinite dimensional Banach space to the structure of its lattice of finite dimensional subspaces. There are several books which deal with this subject (see e.g. [60]). Many of the results in this area can be stated in terms of finite representability. We will define this concept precisely in Section 2.1.4, but roughly speaking we say that Z is finitely representable in X if every finite dimensional subspace of Z can be found in X. One of the first results of this kind is Dvoretzky's Theorem, which states that ℓ_2 is finitely representable in every Banach space. Another famous result is Krivine's Theorem (Theorem 2.1.5), which states that given a suitable sequence $(x_i)_{i=1}^{\infty}$ in a Banach space, there exists $p \in [1, \infty]$, such that for all $\varepsilon > 0$ and for any N, there are N successive block vectors with respect to $(x_i)_{i=1}^{\infty}$ which are $(1+\varepsilon)$-equivalent to the standard basis of ℓ_p^N.

Another notion which has been useful in the study of local theory is that of type and cotype. We say that a Banach space X has type p $(1 \leq p \leq 2)$ if there is a constant C, such that for every $x_1, \ldots, x_n \in X$ we have

$$\frac{1}{2^n} \sum_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \leq C \left(\sum_{i=1}^{n} \|x_i\|^p \right)^{\frac{1}{p}} .$$

Similarly, we say that a Banach space X has cotype q $(2 \leq q \leq \infty)$ if there is a constant C, such that for every $x_1, \ldots, x_n \in X$ we have

$$\frac{1}{2^n} \sum_{\varepsilon_i = \pm 1} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \geq C \left(\sum_{i=1}^{n} \|x_i\|^q \right)^{\frac{1}{q}} .$$

Notice that the type and cotype of a space are determined completely by the finite dimensional structure of the Banach space. One famous theorem using these ideas is the Maurey-Pisier theorem which we state.

Theorem 1.1.1 ([60, Theorem 13.2]) *Let X be an infinite dimensional Banach space. Define $p_X = \sup\{p : X \text{ has type } p\}$ and $q_X = \inf\{q : X \text{ has cotype } q\}$. Then ℓ_{p_X} and ℓ_{q_X} are finitely representable in X.*

1.1.2 Asymptotic structure of Banach spaces

Let X be an infinite dimensional Banach space. We want to look at finite dimensional subspaces which occur 'at infinity'. The picture that the reader should have in mind is that of a Banach space with a basis, and the asymptotic spaces as being those finite block subspaces which occur arbitrarily far down the basis.

We will define asymptotic spaces using a vector game between two players **V** and **S**. Let $\mathcal{B}(X)$ be a family of infinite dimensional subspaces of X satisfying the filtration condition

$$F, G \in \mathcal{B}(X) \Rightarrow \exists H \in \mathcal{B}(X) \text{ s.t. } H \subseteq F \cap G.$$

For example, we could take $\mathcal{B}(X)$ to be the finite co-dimensional subspaces of X, or if X has a basis we could take the family of tail subspaces. Throughout this thesis we will be looking at Banach spaces with a basis, so we will assume that the family $\mathcal{B}(X)$ is the collection of tail subspaces from now onwards.

The game is played as follows. **S** chooses a subspace $X_1 \in \mathcal{B}(X)$. **V** responds by choosing a normalized vector $x_1 \in X_1$. **S** then chooses a subspace $X_2 \subseteq X_1$ with $X_2 \in \mathcal{B}(X)$. **V** then responds by choosing a normalized vector $x_2 \in X_2$ such that $\{x_1, x_2\}$ is basic with basis constant ≤ 2. The game continues in this vein until the n^{th} turn, where **S** selects a subspace $X_n \in \mathcal{B}(X)$ with $X_n \subseteq X_{n-1}$, and **V** chooses a normalized vector $x_n \in X_n$ such that $\{x_1, \ldots, x_n\}$ is basic with basic constant ≤ 2.

Given a finite dimensional Banach space E with basis $(e_i)_{i=1}^n$ with basis constant ≤ 2, and $\varepsilon > 0$, we say that **V** wins the game for E and ε, if the vectors $\{x_1, \ldots, x_n\}$ are $(1+\varepsilon)$-equivalent to $(e_i)_{i=1}^n$. We will call E an asymptotic space of X, if **V** has a winning strategy for the game for E and every $\varepsilon > 0$. The collection of all n-dimensional asymptotic spaces for X will be denoted by $\{X\}_n$.

It is a consequence of Krivine's theorem (see Section 2.1.4 and [59, 1.6.3]) that there exists $p \in [1, \infty]$ such that for every n, ℓ_p^n (with its standard basis) is in $\{X\}_n$. Hence, the simplest possible asymptotic structure that can occur is where the basis $(e_i)_{i=1}^n$ of every $E \in \{X\}_n$ is C-equivalent to the standard basis of ℓ_p^n for some C independent of n. We will call such spaces asymptotic ℓ_p spaces. Later in Chapter 3, we will see an alternative description of these spaces in terms of block vectors which will be more useful for our purposes.

1.2 Distortion and hereditarily indecomposable spaces

1.2.1 Distortion

An infinite dimensional Banach space X is said to be λ-distortable, where $\lambda > 1$, if there exists an equivalent norm $|||\cdot|||$ on X such that for any infinite dimensional subspace Y of X,

$$\sup\left\{\frac{|||y|||}{|||z|||} : \|y\| = \|z\| = 1\right\} \geq \lambda.$$

A Banach space X is said to be distortable if it is λ-distortable for some $\lambda > 1$, and X is said to be arbitrarily distortable if it is λ-distortable for every $\lambda > 1$.

It is a classical result due to James (see [53, Proposition 2.e.3.]) that ℓ_1 and c_0 are not distortable. The first example of an arbitrarily distortable Banach space, Schlumprecht space, was constructed in [74], using a Tsirelson type construction. For a long time it was unknown whether ℓ_p spaces ($1 < p < \infty$) were distortable, or indeed whether the Hilbert space ℓ_2 was distortable. However, using Schlumprecht's space, it was shown in [64] (see also [57]) that every ℓ_p space ($1 < p < \infty$) is distortable, and in fact arbitrarily distortable. The reader is referred to [63] for a review of these results. Another longstanding open problem is the question of whether every distortable space is arbitrarily distortable. The primary candidate for a counterexample to this question is Tsirelson's space. It is known that Tsirelson's space is $(2 - \varepsilon)$-distortable for every $\varepsilon > 0$, but it is not known whether or not it is arbitrarily distortable. Much work has been done on the distortion of Tsirelson type spaces (see [68], [4] and [69]).

The question of distortability has connections with the asymptotic structure of the Banach space. In particular, we have the following theorem.

Theorem 1.2.1 ([61, Theorem 0.7]) *Let X be an infinite dimensional Banach space. If X does not contain an arbitrarily distortable subspace, then X contains a subspace which is asymptotic ℓ_p, for some $p \in [1, \infty]$.*

It is shown in [56] that every asymptotic ℓ_p space with an unconditional basis which does not contain ℓ_1^n uniformly is arbitrarily distortable. This combines with Theorem 1.2.1 to show that every Banach space with an unconditional basis which does not contain ℓ_1^n uniformly, contains an arbitrarily distortable subspace. A result from [75] also shows that a Banach space of type $p > 1$ always has an arbitrarily distortable subspace.

Asymptotic biorthogonal systems

It is difficult to show whether or not a given Banach space is distortable. In this section we describe a useful sufficient condition which guarantees that a space is arbitrarily distortable. We begin with a couple of definitions.

Definition 1.2.2 *Let X be a Banach space, and $A \subseteq S_X$. Then we say that A is an asymptotic set if $A \cap S_Y \neq \varnothing$ for every infinite dimensional (not necessarily closed) subspace Y of X.*

Definition 1.2.3 *Let X be a Banach space, and $(A_n)_{n=1}^{\infty}$, $(A_n^*)_{n=1}^{\infty}$ be sequences of subsets of the unit sphere of X, and the unit ball of X^* respectively. We will say that $(A_n)_{n=1}^{\infty}$, $(A_n^*)_{n=1}^{\infty}$ form an asymptotic biorthogonal system with constant δ (with $0 < \delta < \frac{1}{2}$) if the following conditions are satisfied;*

(i) For every n, A_n is an asymptotic set.

(ii) For every n, and every $x \in A_n$, there exists $x^* \in A_n^*$ such that $x^*(x) > 1 - \delta$.

(iii) If $n \neq m$ then $|x^*(x)| < \delta$ for every $x \in A_n$ and $x^* \in A_m^*$.

Notice the definition above is uninteresting if $\delta > \frac{1}{2}$, since one may take $A_n = S_X$ and $A_n^* = \frac{1}{2}B_{X^*}$ for every n in this case. However, whenever $\delta < \frac{1}{2}$, the existence of asymptotic biorthogonal systems is non-trivial. The existence of asymptotic biorthogonal systems for all $\delta > 0$ tells us in particular that the space is arbitrarily distortable.

Proposition 1.2.4 ([12, Proposition 13.28]) *If X admits an asymptotic biorthogonal system with constant δ for every $\delta \in (0, \frac{1}{2})$, then X is arbitrarily distortable.*

The proof of Proposition 1.2.4 follows by defining a new norm, for an arbitrary $\varepsilon > 0$, by

$$|||x||| = \varepsilon \, \|x\| + \sup\{|x^*(x)| : x^* \in A_1^*\}.$$

This norm is a $(1 + \varepsilon - \delta)/(\varepsilon + \delta)$ distortion of the original norm. This is enough since δ and ε can be chosen as close to zero as we wish.

The existence of asymptotic biorthogonal systems also has implications on properties of unconditional basic sequences in X, as the next theorem shows. This theorem reveals the important connection between distortability and the unconditional basic sequence problem.

Theorem 1.2.5 ([33, Theorem 2]) *Let X be a separable Banach space which admits an asymptotic biorthogonal system with $\delta < \frac{1}{36}$. Then there is an equivalent norm on X such that the unconditional constant of every basic sequence in X (with respect to the new norm) is at least $(36\delta)^{-\frac{1}{2}}$.*

1.2.2 Hereditarily indecomposable spaces

One of the first results proved about the structure of Banach spaces was the fact that every Banach space contains a basic sequence (see e.g. [53, Theorem 1.a.5]). It is therefore a natural question to ask whether every Banach space contains an *unconditional* basic sequence. This problem proved to be very difficult and was not solved until 1991, when Gowers and Maurey produced an example which contained no unconditional basic sequence (see [33]). In fact this example had a stronger property, namely that it is *hereditarily indecomposable.* We say that a Banach space X is hereditarily indecomposable or H.I. if no infinite dimensional subspace of X can be written as a topological direct sum $Y \oplus Z$, where Y and Z are infinite dimensional. An equivalent definition is that for any two infinite dimensional subspaces Y and Z of X and $\varepsilon > 0$ there exist $y \in Y$ and $z \in Z$ with $\|y\| = \|z\| = 1$ such that $\|y - z\| < \varepsilon$. It is clear that an H.I. space can not contain an unconditional basic sequence, since any space with an unconditional basis $(e_i)_{i=1}^{\infty}$ can be decomposed into $Y = \overline{\text{span}}(\{e_{2i}\}_{i=1}^{\infty})$ and $Z = \overline{\text{span}}(\{e_{2i-1}\}_{i=1}^{\infty})$.

One of the properties of H.I. spaces is that they have a 'small' space of operators. If X is a complex H.I. Banach space then every bounded linear operator $T : X \to X$ can be written in the form $T = \lambda I + S$, where $\lambda \in \mathbb{C}$ and S is strictly singular (see [33], [21], [32]). Recall that an operator is said to be strictly singular if it is not an isomorphism on any infinite dimensional subspace. Equivalently, $S : X \to X$ is strictly singular if for every $\varepsilon > 0$ and every infinite dimensional subspace Y of X, there exists $y \in Y$ with $\|Sy\| < \varepsilon \|y\|$. The strictly singular operators form an ideal in the space of operators ([53, Proposition 2.c.5]) which contains all the compact operators. In general however, not all strictly singular operators are compact. The fact that every operator can be decomposed in this way has been used in operator theory to investigate the behaviour of C_0-semigroups and C_0-groups on H.I. spaces (see e.g. [72]). In fact, H.I. spaces can be characterized by the property of having 'small' spaces of operators, as the following theorem shows.

Theorem 1.2.6 ([21]) *A complex Banach space X is hereditarily indecomposable if and only if every operator $T : Y \to X$ from a subspace Y of X into X is the sum of a multiple of the inclusion map and a strictly singular operator.*

Since the first example of an H.I. space was found, many others have been constructed. In [29], Gowers constructed an example of an H.I. space that has an asymptotically unconditional basis. This space contains arbitrarily long finite 2-unconditional sequences, but does not contain any unconditional basic sequence. An example of a uniformly convex H.I. space has been constructed by Ferenczi in [22]. More examples have been found by Gasparis in [26], who constructed a continuum of totally incomparable H.I. spaces. Also, an example of an H.I. space that is asymptotic ℓ_1 has been constructed (see [6]).

Gowers has also proved a dichotomy for Banach spaces in terms of H.I. subspaces (see [30] and [78]) which, roughly speaking, says that any Banach space contains either a 'nice' subspace, or a 'very bad' subspace.

Theorem 1.2.7 ([30, Theorem 2]) *Let X be a Banach space. Then at least one of the following two conditions holds;*

(i) X contains an unconditional basic sequence.

(ii) X has an H.I. subspace.

The proof of the above theorem by Gowers uses vector games and Ramsey theory results. Another proof has been given by Maurey in [58] which does not use the vector game approach.

Recent papers have shown that the class of H.I. spaces is quite large (see [8], [23]). For example, it is shown that every Banach space either contains ℓ_1 or contains a quotient of an H.I. space. Also, in [5], it is shown that if a Banach space is universal for all reflexive H.I. spaces then it is actually universal for all separable Banach spaces.

1.2.3 Homogeneous Banach spaces, the hyperplane problem and the c_0-ℓ_1-reflexive subspace problem

In this section we briefly discuss the three problems mentioned in the above title. These were all long-standing open problems which were solved as a result of the work of Gowers and Maurey on H.I. spaces.

A Banach space is said to be *homogeneous* if it is isomorphic to all of its closed subspaces. It was a longstanding open problem, posed by Banach, whether the only homogeneous Banach space is ℓ_2. This problem was recently solved by Gowers in [30] (see also [76]), who showed that the only homogeneous Banach space is indeed ℓ_2. The proof of this result hinges on Gowers' dichotomy (Theorem 1.2.7). If X is a homogeneous Banach space, then since all subspaces of X are isomorphic to X, the dichotomy tells us that either X is H.I. or X has an unconditional basis. It follows from a result in [46] that a homogeneous space with an unconditional basis is isomorphic to ℓ_2. This leaves the case of X being a homogeneous H.I. space. To exclude this possibility we need to know some results from operator theory.

Definition 1.2.8 *An operator* $T : X \to Y$ *is said to be a Fredholm operator if there exist subspaces* X_1, B *of* X *and* Y_1, C *of* Y, *such that* $X = X_1 \oplus B$, $Y = Y_1 \oplus C$, $T\restriction_B = 0$, $T\restriction_{X_1}$ *is an isomorphism onto* Y_1, *and* $\dim B < \infty$, $\dim C < \infty$. *The index,* $i(T)$, *of a Fredholm operator is then defined by* $i(T) = \dim B - \dim C$.

It is a classical result that if T is Fredholm and S is strictly singular, then $T + S$ is Fredholm with the same index as T (see [31, Lemma 18]). Therefore since every operator $T : X \to X$, can be written as $T = \lambda I + S$, where S is strictly singular, it follows that T is either strictly singular (if $\lambda = 0$), or Fredholm with index 0 (if $\lambda \neq 0$). In either case T can not be an isomorphism onto a proper subspace of X. We thus obtain the following theorem, which strongly contradicts the possibility of an H.I. space being homogeneous.

Theorem 1.2.9 ([30, Theorem 4]) *Let* X *be a hereditarily indecomposable Banach space. Then* X *is not isomorphic to any proper subspace of itself.*

The above theorem is also relevant to the so-called hyperplane problem. This asks whether every Banach space is isomorphic to its hyperplanes i.e. its closed subspaces of co-dimension 1. The theorem implies that an H.I. space fails the above property. Other examples are known of such spaces. For instance, the first known example of such a space had an unconditional basis and was constructed by Gowers in [28].

The final problem that we will mention that was solved as a result of the work of Gowers' and Maurey's solution of the unconditional basic sequence problem is the c_0-ℓ_1-reflexive subspace problem. This asks whether every Banach space contains either c_0, ℓ_1 or a reflexive subspace. It is well-known that a space with an unconditional basis is reflexive if and only if it does not contain ℓ_1 and c_0. Thus, the answer is positive for any space which possesses an unconditional basic sequence. However, in general, this need not be the case. In [27], Gowers modified the Gowers-Maurey construction of an H.I. space to obtain a space not containing either c_0, ℓ_1 or a reflexive subspace.

1.3 Stable Banach spaces

We call a sequence $(x_n)_{n=1}^{\infty} \subseteq X$ type determining if for every $x \in X$, $\lim_{n\to\infty} \|x + x_n\|$ exists. The limiting function will be called a type of X. We will discuss types later in greater depth (see Section 2.4 and Chapter 6). We define stable Banach spaces and weakly stable Banach spaces in terms of these sequences.

Definition 1.3.1 *A Banach space is said to be stable (resp. weakly stable) if whenever* $(x_n)_{n=1}^{\infty}$ *and* $(y_m)_{m=1}^{\infty}$ *are type determining sequences (resp. weakly convergent type determining sequences) then the following iterated limits exist and are equal*

$$\lim_{m\to\infty}\lim_{n\to\infty}\|x_n + y_m\| = \lim_{n\to\infty}\lim_{m\to\infty}\|x_n + y_m\|.$$

Stable Banach spaces were studied by Krivine and Maurey in [49]. The main result of this article was that every stable Banach space contains a subspace isomorphic to ℓ_p for some $p \in [1, \infty)$. It was later observed that the corresponding result is true for weakly stable Banach spaces, namely that every weakly stable Banach space contains a subspace which is isomorphic to either c_0 or to ℓ_p for some $p \in [1, \infty)$. The main tools in the proof of these results is the theory of types and spreading models. The general procedure is to prove the existence of a special type which generates an ℓ_p (or c_0) spreading model, and then to pull the spreading model back into our Banach space. We shall use similar methods in Chapter 6 to prove existence of ℓ_p subspaces under a different set of hypotheses.

1.4 Overview of this thesis

In Chapter 2 we will explain the notation which will be used throughout this thesis. We will then introduce some examples of Banach spaces which will be particularly useful for us. These will include Tsirelson's space, and various generalizations of it, James' space, and the classes of Orlicz and generalized Orlicz spaces. We will briefly describe the Brunel-Sucheston construction of spreading models, and give an alternative construction using ultrafilters. Finally, we will introduce the space of types and strong types on a Banach space, and describe the topologies and operations that can can be applied to these spaces.

Chapter 3 is devoted to asymptotic ℓ_p spaces. We define the notions of asymptotic ℓ_p and stabilized asymptotic ℓ_p spaces with respect to a given basis. Some key examples of such spaces are introduced. We review some known properties of these spaces including properties of the basis and duality results, and then go on to give some results that do not appear in the existing literature. These include a proof that asymptotic ℓ_p spaces have a stabilized asymptotic ℓ_p subspace, a result about finite representability of ℓ_p in subspaces of asymptotic ℓ_p spaces, and properties of the spreading models of these spaces. Finally, we describe the classes of weak Hilbert and asymptotically Hilbertian spaces and compare these notions with asymptotic ℓ_2 spaces.

In Chapter 4 we prove a renorming result for asymptotic ℓ_∞ spaces. We show that if the norm on the asymptotic ℓ_∞ space is suitably smooth, then we can find an equivalent analytic norm on the space. We then give a construction of some asymptotic ℓ_∞ spaces and show that they are different from the known examples of asymptotic ℓ_∞ spaces. The results of this chapter are motivated by the relationship between smooth bump functions and smooth norms for separable spaces. It is not clear how one could construct a smooth norm given a smooth bump function. Thus, some of these spaces may be candidates for spaces which have smooth bump functions but do not admit equivalent smooth norms.

In Chapter 5 we construct hereditarily indecomposable spaces which have upper ℓ_p estimates and a type of lower ℓ_p estimate. We do this by modifying the original construction of Gowers and Maurey. This construction should be contrasted with that of Ferenczi (see [22]), where a similar type of space is constructed using interpolation techniques.

Chapter 6 presents some results proving existence of almost isometric copies of ℓ_p spaces (or c_0), given some compactness conditions in the space of types or strong types. The proof proceeds by deducing existence of ℓ_p (or c_0) spreading models and then pulling the spreading model back into the Banach space. We then present some examples to which this result applies. One application is to a class of spaces studied by Kalton in [44]. We also show that the result applies to certain generalized Tsirelson spaces.

In Chapter 7 we introduce the class of spaces called stopping time Banach spaces. This class generalizes a space, S^d, studied by Bang and Odell in [9] and [10], although we use a more probabilistic description than was used in these papers. The norms in these spaces enjoy a backwards recursion property which proves very useful to us. This enables us to prove a correspondence between the block types on S^d and integrable martingales. We also look at certain subspaces of general stopping time Banach spaces, which are generated by independent random variables and prove that they are isomorphic to Orlicz spaces (or generalized Orlicz spaces). We are then able to show that S^d contains isomorphic copies of Orlicz spaces h_ϕ for any Orlicz function ϕ. We also introduce the so called mixtures of Orlicz spaces and show that they arise as subspaces of stopping time Banach spaces generated by sequences of random variables satisfying a ‘conditional independence’ condition.

Chapter 2

Preliminaries

Contents

2.1 Notation

2.1.1 Banach space notation

The notation that we will use is standard from the literature on this subject. All of the Banach spaces we consider will be over the field of real numbers unless we specify otherwise.

Given a Banach space X, we will denote the unit sphere by S_X, and the closed unit ball by B_X. Given a set $A \subset X$, we will denote the linear span of A by $\operatorname{span}(A)$, and the closed linear span by $\overline{\operatorname{span}}(A)$. In the case of a sequence (x_i), we will denote the closed linear span by $[x_i]$. We will denote by c_{00} the collection of finitely supported sequences of real numbers. Sometimes we will use the notation $\mathbb{R}^{(\mathbb{N})}$ when it is standard in existing literature to do so. Whenever we talk of subspaces of a Banach space, we will mean *closed* subspaces, and usually infinite dimensional subspaces unless we state otherwise.

If X, Y are two isomorphic Banach spaces then we will denote by $d(X, Y)$ the Banach-Mazur distance between X and Y.

We will say that two Banach spaces X and Y are *totally incomparable* if no infinite dimensional subspace of X is isomorphic to an infinite dimensional subspace of Y.

2.1.2 Bases and Basic sequences

Most of the spaces we consider will have a (Schauder) basis which will typically be denoted by $(e_i)_{i=1}^{\infty}$. This means that every $x \in X$ has a unique expansion of the form $x = \sum_{i=1}^{\infty} x_i e_i$. A sequence (x_n) in X is said to be a basic sequence if it forms a basis for $\overline{\operatorname{span}}(\{x_n : n \in \mathbb{N}\})$. Given a basis $(e_i)_{i=1}^{\infty}$, we define a sequence of projections by

$$P_n\left(\sum_{i=1}^{\infty} x_i e_i\right) = \sum_{i=1}^{n} x_i e_i.$$

It is straightforward to show that $\sup_n \|P_n\| = K < \infty$. We call this K the *basis constant.* If $K = 1$ then we say that the basis is *monotone.* We will say that the basis is *bimonotone* if for every $n \in \mathbb{N}$,

$$\|x\| \geq \max\left(\left\|\sum_{i=1}^{n} x_i e_i\right\|, \left\|\sum_{i=n+1}^{\infty} x_i e_i\right\|\right).$$

We will denote by e_n^* the n^{th} coordinate functional defined by $e_n^*(x) = x_n$. The sequence $(e_n^*)_{n=1}^{\infty}$ always forms a basic sequence in X^*. We call the basis $(e_i)_{i=1}^{\infty}$ *shrinking* if $(e_n^*)_{n=1}^{\infty}$ forms a basis for X^*. We will call the basis $(e_i)_{i=1}^{\infty}$ *boundedly complete* if $\sum_{i=1}^{\infty} x_i e_i$ converges in X whenever $\sup_n \|\sum_{i=1}^{n} x_i e_i\| < \infty$. It is well known ([53, Theorem 1.b.5]) that a space with a basis is reflexive if and only if the basis is both shrinking and boundedly complete. We say that two basic sequences $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ are *C-equivalent* if, for any scalars (a_i),

$$\frac{1}{C}\left\|\sum_{i=1}^{\infty} a_i y_i\right\| \leq \left\|\sum_{i=1}^{\infty} a_i x_i\right\| \leq C\left\|\sum_{i=1}^{\infty} a_i y_i\right\|.$$

We say that the basic sequences are *equivalent* if they are C-equivalent for some $C \geq 1$. An alternative definition is that $(x_i)_{i=1}^{\infty}$ and $(y_i)_{i=1}^{\infty}$ are equivalent provided $\sum_{i=1}^{\infty} a_i x_i$ converges in X if and only if $\sum_{i=1}^{\infty} a_i y_i$ converges.

Block bases and notation involving bases

Let X be a Banach space with a basis $(e_i)_{i=1}^{\infty}$. We define $\operatorname{supp}(x)$, the support of $x = \sum_i x_i e_i$, to be the set $\{i \in \mathbb{N} : x_i \neq 0\}$. Given a subset E of $\mathbb{N}$, we define $Ex = \sum_{i \in E} x_i e_i$. If $x \in X$ and $n \in \mathbb{N}$, then we will write $n \leq x$ if $n \leq \min \operatorname{supp}(x)$. If $x, y \in X$, then we will write $x < y$ if x is finitely supported and $\max \operatorname{supp}(x) < \min \operatorname{supp}(y)$. We say that x is a *block vector* with respect to the basis $(e_i)_{i=1}^{\infty}$ if $x = \sum_{i=p}^{q} x_i e_i$ for some finite p and q. A *block basis* is then a sequence of successive block vectors $x_1 < x_2 < \ldots$. A block basis is again a basic sequence. A *block subspace* is defined to be the closed linear span of a block basis.

Unconditional bases

We will say that a basis $(e_i)_{i=1}^{\infty}$ is C*-unconditional* if for every choice of signs $(\varepsilon_i) \in \{\pm 1\}^{\mathbb{N}}$

$$\left\| \sum_{i=1}^{\infty} \varepsilon_i a_i e_i \right\| \leq C \left\| \sum_{i=1}^{\infty} a_i e_i \right\| .$$

We will call the smallest C for which the above is valid the *unconditional basis constant.* It is useful to note that in this situation, if $\lambda \in \ell_\infty$, then

$$\left\| \sum_{i=1}^{\infty} \lambda_i x_i e_i \right\| \leq C \left\| \lambda \right\|_\infty \left\| \sum_{i=1}^{\infty} x_i e_i \right\| .$$

In particular, if a basis is 1-unconditional then for every choice of signs,

$$\left\| \sum_{i=1}^{\infty} a_i e_i \right\| = \left\| \sum_{i=1}^{\infty} \pm a_i e_i \right\| .$$

We will call a basis *conditional* if it is not C-unconditional for any C. If a space has an unconditional basis then we can show (see [53]) that the following hold;

(i) The basis is shrinking if and only if $\ell_1 \not\hookrightarrow X$.

(ii) The basis is boundedly complete if and only if $c_0 \not\hookrightarrow X$.

Hence a space with an unconditional basis is reflexive if and only if $\ell_1 \not\hookrightarrow X$ and $c_0 \not\hookrightarrow X$.

2.1.3 The p-Convexification of a Banach space

Suppose that X is a Banach space with a 1-unconditional basis $(e_i)_{i=1}^\infty$ and that $1 < p < \infty$. We define a new space $X^{(p)}$, the *p-convexification of* X, to be

$$X^{(p)} = \{(x_i)_{i=1}^\infty : x^p = \sum_{i=1}^\infty |x_i|^p e_i \in X\},$$

and we equip $X^{(p)}$ with the norm

$$\|x\|_{X^{(p)}} = \|x^p\|_X^{\frac{1}{p}} .$$

Notice that the sequence of coordinate vectors in $X^{(p)}$ forms a 1-unconditional basis for $X^{(p)}$. Also, it follows from the triangle inequality in X that $X^{(p)}$ satisfies an upper ℓ_p estimate. In other words, if $x_1 < \ldots < x_n$ are successive block vectors in $X^{(p)}$ then

$$\left\| \sum_{i=1}^n x_i \right\| \le \left(\sum_{i=1}^n \|x_i\|^p \right)^{\frac{1}{p}} .$$

This idea also extends in a natural way to Banach lattices (see [54]), but we will not use this here.

2.1.4 Finite representability and Krivine's Theorem

We say that a Banach space Z is finitely representable in a Banach space X if every finite dimensional subspace of Z can be found as a finite dimensional subspace of X. More precisely, we make the following definitions.

Definition 2.1.1 *We say that a Banach space Z is finitely representable in a Banach space X if and only if for every $C > 1$ and for all finite dimensional subspaces $E \le Z$ there exists a subspace $F \le X$ with $d(E, F) < C$.*

Definition 2.1.2 *We say that a Banach space Z is crudely finitely representable in a Banach space X if and only if there exists $C > 1$ such that for all finite dimensional subspaces $E \le Z$ there exists a subspace $F \le X$ with $d(E, F) < C$.*

Later, we will need to use the following result due to Krivine (proved in [48]).

Theorem 2.1.3 ([48, Section III]) *If ℓ_p is crudely finitely representable in X then ℓ_p is finitely representable in X.*

We will also need an alternative form of finite representability, namely that of *block finitely representability*, which we define below.

Definition 2.1.4 *A sequence $(y_j)_{j=1}^{\infty}$ in a Banach space X is said to be block finitely representable in a sequence $(x_i)_{i=1}^{\infty}$ in a Banach space Y, if for every $\varepsilon > 0$ and for every integer n, there exist vectors $(z_i)_{i=1}^{n}$ in Y which are successive block vectors with respect to $(x_i)_{i=1}^{\infty}$ such that for any scalars $(a_j)_{j=1}^{n}$,*

$$(1-\varepsilon)\left\|\sum_{j=1}^{n} a_j z_j\right\| \leq \left\|\sum_{j=1}^{n} a_j y_j\right\| \leq (1+\varepsilon)\left\|\sum_{j=1}^{n} a_j z_j\right\|.$$

The following result, Theorem 2.1.5, is a deep result of great importance in Banach space theory. This was proved in [48] (see also [51] for an alternative proof). There are various forms of this result. The following theorem is one of the simplest statements.

Theorem 2.1.5 (Krivine's Theorem) *Let $(x_i)_{i=1}^{\infty}$ be a basic sequence in a Banach space. Then the unit vector basis of c_0 or of some ℓ_p (for $1 \leq p < \infty$) is block finitely representable in $(x_i)_{i=1}^{\infty}$.*

This can be strengthened to give the following theorem, which appeared in this form in [51] and [73].

Theorem 2.1.6 *Let $C \geq 1$, $n \in \mathbb{N}$ and $\varepsilon > 0$. There exists $m = m(C, n, \varepsilon) \in \mathbb{N}$ such that if $(x_i)_{i=1}^{m}$ is a finite basic sequence in some Banach space with basis constant C then there exists $p \in [1, \infty]$ and a block basis $(y_i)_{i=1}^{n}$ of $(x_i)_{i=1}^{m}$ so that $(y_i)_{i=1}^{n}$ is $(1+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^n.*

Krivine's Theorem also has the following corollary (stated in this form in [11, Chapter 6, Section 1, Corollary 3])

Corollary 2.1.7 *Let $(x_k)_{k=1}^{\infty}$ be a bounded sequence in a Banach space, without any convergent subsequences. Then there exists $p \in [1, \infty]$, such that for each $n \geq 1$ and each $\varepsilon > 0$, there is a sequence $(z_k)_{k=1}^{\infty}$ of successive blocks with respect to $(x_k)_{k=1}^{\infty}$ (with each z_k having the same coefficients), such that any n of the z_k generate a subspace $(1+\varepsilon)$-isomorphic to ℓ_p^n.*

2.2 Some important examples of Banach spaces

2.2.1 Tsirelson's space

We will present here the Figiel-Johnson construction of Tsirelson's space T (see [15]). This is in fact the dual of the space which was originally constructed by Tsirelson in [77], which was the first example of a Banach space which does not contain any ℓ_p or c_0. This property is also satisfied by T. The key difference between Tsirelson's space and classical Banach spaces is that the norm on T is defined implicitly. The norm satisfies the implicit equation

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{\frac{1}{2}\sum_{i=1}^{n}\|E_i x\| : n \leq E_1 < \ldots < E_n \text{ intervals}\right\}\right).$$

We then define T to be the completion of c_{00} under this norm. T is a reflexive Banach space, and the coordinate vectors $(e_n)_{n=1}^{\infty}$ form a 1-unconditional basis for T.

There are many variants of Tsirelson's space. The simplest are the spaces T_θ where $0 < \theta < 1$. These are defined exactly as above except the norm satisfies the implicit equation

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{\theta\sum_{i=1}^{n}\|E_i x\| : n \leq E_1 < \ldots < E_n \text{ intervals}\right\}\right).$$

T_θ has many of the same properties as T, but surprisingly the spaces T_θ are in a sense very different from one another. If $\theta \neq \phi$, then T_θ is totally incomparable to T_ϕ (see [15, Theorem X.a.3.]).

Since T has a 1-unconditional basis, we may also form the p-convexification for any $p \in (1, \infty)$. The norm in the resulting space $T^{(p)}$ then satisfies the implicit equation

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{2^{-\frac{1}{p}}\left(\sum_{i=1}^{n}\|E_i x\|^p\right)^{\frac{1}{p}} : n \leq E_1 < \ldots < E_n \text{ intervals}\right\}\right).$$

2.2.2 Schlumprecht's space

Schlumprecht's space, S, first appeared in [74] and is defined in a manner very similar to Tsirelson's space. S has a 1-unconditional basis $(e_i)_{i=1}^{\infty}$ and the norm satisfies the implicit equation

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{\frac{1}{f(n)}\sum_{i=1}^{n}\|E_i x\| : n \in \mathbb{N},\, E_1 < \ldots < E_n \text{ intervals}\right\}\right),$$

where $f(n) = \log_2(n+1)$. This space is arbitrarily distortable and is the starting point for the Gowers-Maurey construction of an H.I. space.

2.2.3 More general Tsirelson type spaces

The construction of Tsirelson space and Schlumprecht's space have obvious similarities. We present here a class of spaces which generalizes both of these spaces. Suppose that $\mathcal{M}$ is a collection of finite subsets of $\mathbb{N}$. We say that a sequence $(E_i)_{i=1}^n$ is $\mathcal{M}$-admissible if there exists $\{k_1, \ldots, k_n\} \in \mathcal{M}$ such that

$$k_1 \leq E_1 < k_2 \leq \ldots < k_n \leq E_n.$$

Suppose that $\varnothing \neq I \subseteq \mathbb{N}$ (I may be either finite or infinite), and for each $k \in I$, $\mathcal{M}_k$ is a collection of finite subsets of $\mathbb{N}$, and that $\theta_k \in [0,1]$ for each $k \in I$. We can define a *mixed Tsirelson space* $T((\mathcal{M}_k, \theta_k)_{k \in I})$ to be the completion of c_{00} under a norm satisfying the implicit equation

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{\theta_k \sum_{i=1}^n \|E_i x\| : \ k \in I, \ (E_i)_{i=1}^n \ \mathcal{M}_k\text{-admissible}\right\}\right).$$

Note that this class of spaces includes both Tsirelson's space and Schlumprecht's space. We define the Schreier family, $\mathcal{S} = \{A \subseteq \mathbb{N} : |A| \leq \min A\}$, and $\mathcal{A}_k = \{A \subseteq \mathbb{N} : |A| \leq k\}$. Then $T(\mathcal{S}, \frac{1}{2})$ is Tsirelson's space, and $T((\mathcal{A}_k, (\log_2(k+1))^{-1})_{k=1}^\infty)$ is Schlumprecht's space.

We can further generalize this construction by taking a sequence of scalars $(\theta_{k,i})_{i=1}^\infty$ contained in $[0,1]$ for each $k \in I$ and obtain a space whose norm satisfies the implicit equation

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{\sum_{i=1}^n \theta_{k,i} \|E_i x\| : \ k \in I, \ (E_i)_{i=1}^n \ \mathcal{M}_k\text{-admissible}\right\}\right).$$

For example, if $\mathbf{a} = (a_1, \ldots, a_k)$ and each $a_i \in [0,1]$, then $T(\mathcal{A}_k, \mathbf{a})$ is the space whose norm is given implicitly by

$$\|x\| = \max\left(\|x\|_\infty, \sup\left\{\sum_{i=1}^k a_i \|E_i x\| : E_1 < E_2 < \ldots < E_k\right\}\right).$$

2.2.4 Generalized Tsirelson spaces

A more general form of Tsirelson space has been constructed in [13]. We will not give this definition in its full generality, since we will only use a special case of it in this thesis. This definition replaces the sequence of constants $(\theta_{k,i})$ in the above, with a sequence of norms (F_k) on c_{00}. For convenience we will describe this only in the case where a single norm is used, and just for a simple admissibility condition.

Definition 2.2.1 *Let f be a 1-unconditional normalized norm on $\mathbb{R}^k$. We define $T(\mathcal{A}_k, f)$, to be the completion of c_{00} under the norm which satisfies the implicit equation*

$$\|x\| = \|x\|_\infty \vee \sup\{f(\|E_1 x\|, \ldots, \|E_k x\|) : E_1 < \ldots < E_k\}.$$

As for the original Tsirelson space, the coordinate vectors (e_i) form a 1-unconditional basis of $T(\mathcal{A}_k, f)$.

Spaces of the form $T(\mathcal{A}_k, f)$ have been studied in [13]. In many situations they turn out to be isomorphic to either some ℓ_p space or to c_0. Necessary and sufficient conditions on the norm f are given for this to be the case. An example which is not isomorphic to any ℓ_p or to c_0 is also constructed.

2.2.5 James' space

We define James' space J to be the vector space of real sequences (a_i) which satisfy both;

(i) $\lim_{i\to\infty} a_i = 0$, and

(ii) $\sup_{n_1<\ldots<n_k} \sum_{i=1}^{k-1} (a_{n_{i+1}} - a_{n_i})^2 < \infty$.

We then equip J with the norm given by

$$\|(a_i)\| = \sup_{n_1<\ldots<n_k} \left(\sum_{i=1}^{k-1} (a_{n_{i+1}} - a_{n_i})^2 \right)^{\frac{1}{2}}.$$

With this norm J is a Banach space. The reason J was first introduced is that it is a space which is isomorphic to its second dual J^{**}, but is not reflexive. In fact J is of codimension 1 in J^{**}, so in some sense J is very close to being reflexive. We let $(e_i)_{i=1}^\infty$ be the sequence of coordinate vectors in J. It is straightforward to show that $(e_i)_{i=1}^\infty$ is a conditional basis for J.

2.2.6 Orlicz sequence spaces

Orlicz sequence spaces are a class of spaces which are natural generalizations of ℓ_p spaces for $1 \leq p < \infty$ and of c_0. Each Orlicz space will be determined by an Orlicz function. The Orlicz functions $\phi(t) = t^p$ will generate the ℓ_p spaces.

Definition 2.2.2 *We will call a function $\phi : [0, \infty) \to [0, \infty)$ an Orlicz function if $\phi(0) = 0$, ϕ is convex, continuous, non-decreasing and $\lim_{t\to\infty} \phi(t) = \infty$.*

Let $x = (x_i)_{i=1}^{\infty}$ be a sequence of real numbers. Then we define,

$$I_\phi(x) = \sum_{i=1}^{\infty} \phi(|x_i|).$$

The Orlicz space, ℓ_ϕ, is then defined by,

$$\ell_\phi = \left\{ x : I_\phi\left(\frac{x}{\rho}\right) < \infty \text{ for some } \rho > 0 \right\}.$$

We will also be interested in the subset, h_ϕ, of ℓ_ϕ given by,

$$h_\phi = \left\{ x : I_\phi\left(\frac{x}{\rho}\right) < \infty \text{ for all } \rho > 0 \right\}.$$

It is easy to show that h_ϕ is a closed subspace of ℓ_ϕ, and the sequence of coordinate vectors form a 1-unconditional basis for h_ϕ.

There are three standard norms on Orlicz spaces. The first norm $\|\cdot\|_\phi^O$ is known as the *Orlicz norm*, and is the one originally introduced by Orlicz. We define the Orlicz norm by,

$$\|x\|_\phi^O = \sup\{\sum_{i=1}^{\infty} x_i y_i : \sum_{i=1}^{\infty} \phi^*(|y_i|) \leq 1\},$$

where ϕ^* is the function complementary to ϕ, defined by

$$\phi^*(v) = \sup\{tv - \phi(t) : 0 < t < \infty\}.$$

The second norm is the *Luxemburg norm.* This is the norm that we will usually use on our Orlicz spaces. It is defined by

$$\|x\|_\phi = \inf\{\rho > 0 : I_\phi\left(\frac{x}{\rho}\right) \leq 1\}.$$

The third norm is the *Amemiya norm*, given by

$$\|x\|_\phi^A = \inf_{k>0} \frac{1}{k}(1 + I_\phi(kx)).$$

These three norms are all equivalent. In fact, in [43], it is shown that $\|x\|_\phi^A = \|x\|_\phi^O$ for every $x \in \ell_\phi$. It is also quite easy to verify that

$$\|x\|_\phi \leq \|x\|_\phi^A \leq 2\,\|x\|_\phi\,.$$

For further details concerning Orlicz spaces we refer the reader to [53] and [47].

2.2.7 Generalized Orlicz spaces

Suppose that $\Phi = (\phi_j)_{j=1}^{\infty}$ is a sequence of Orlicz functions. Then, we can define a generalized Orlicz space in an analogous way to Orlicz spaces. We define

$$\ell_\Phi = \left\{ (x_i) : \sum_{j=1}^{\infty} \phi_j \left(\frac{|x_j|}{\rho} \right) < \infty \text{ for some } \rho > 0 \right\},$$

and

$$h_\Phi = \left\{ (x_i) : \sum_{j=1}^{\infty} \phi_j \left(\frac{|x_j|}{\rho} \right) < \infty \text{ for all } \rho > 0 \right\}.$$

As in the Orlicz space case, h_Φ is a closed subspace of ℓ_Φ with the coordinate vectors forming a 1-unconditional basis. Again, there are various norms that can be put on a generalized Orlicz space. The usual norm is the analogue of the Luxemburg norm given by

$$\|x\|_\Phi = \inf\{\rho > 0 : \sum_{j=1}^{\infty} \phi_j \left(\frac{|x_j|}{\rho} \right) \leq 1\}.$$

We can also consider the analogue of the Amemiya norm, given by

$$\|x\|_\Phi^A = \inf_{k>0} \frac{1}{k}(1 + \sum_{j=1}^{\infty} \phi_j(kx_j)).$$

As for Orlicz spaces it is easy to check that $\|\cdot\|_\Phi$ and $\|\cdot\|_\Phi^A$ are equivalent norms on ℓ_Φ satisfying

$$\|x\|_\Phi \leq \|x\|_\Phi^A \leq 2 \|x\|_\Phi .$$

2.3 Spreading models of Banach spaces

We will explain briefly here the construction of spreading models of a Banach space. Given a separable Banach space X and a bounded sequence (x_n) (with suitable properties) we will define a new Banach space, which will be the completion of $X \oplus \mathbb{R}^{(\mathbb{N})}$, equipped with a norm which extends the original norm in X. We will denote the coordinate sequence in $\mathbb{R}^{(\mathbb{N})}$ by (e_i). We will then call the closed linear span of $\{e_i : i \in \mathbb{N}\}$ a *spreading model* of X. It will be immediate from our definition that the structure of this space will only depend on the asymptotic properties of the sequence (x_n), in the sense that if (y_n) is another sequence such that $\|x_n - y_n\| \to 0$, then the spreading model associated with (y_n) will be the same as that for (x_n).

2.3.1 Brunel-Sucheston construction of spreading models

The standard way of constructing spreading models is due to Brunel and Sucheston (see [11]). We will briefly explain this method, and then explain an alternative construction using ultrafilters. The key to the Brunel-Sucheston construction is the following lemma, which can be proved using Ramsey theory.

Lemma 2.3.1 ([11, Chapter 1, Proposition 1]) *Let X be a separable Banach space, and (x_n) be a bounded sequence in X. Then there exists a subsequence (x'_n) of (x_n), such that for any $k \in \mathbb{N}$ and any $x \in X$ and any scalars $a_1, \ldots, a_k$ the following limit exists*

$$L(x, \mathbf{a}) = \lim_{n_1 \to \infty} \ldots \lim_{n_k \to \infty} \left\| x + \sum_{i=1}^{k} a_i x'_{n_i} \right\| .$$

A subsequence (x'_n) satisfying the conclusion of Lemma 2.3.1 will be called a *Brunel-Sucheston sequence.*

The limit given by the preceding lemma allows us to define a semi-norm on $X \oplus \mathbb{R}^{(\mathbb{N})}$ by

$$\left\| x + \sum_{i=1}^{k} a_i e_i \right\| := L(x, \mathbf{a}).$$

Notice that this semi-norm extends the original norm on X. It is straightforward to see that this in fact defines a norm if and only if the sequence (x'_n) that we extracted does not converge (see [11, Chapter 1, Proposition 2]). In this situation we define the extension $\mathfrak{F}$ of X to be the completion of $X \oplus \mathbb{R}^{(\mathbb{N})}$ under the given norm. We call the closed linear span of the coordinate vectors in $\mathbb{R}^{(\mathbb{N})}$ a spreading model of X.

It is important to note that a bounded sequence (x_n) may contain many different Brunel-Sucheston subsequences which may give very different spreading models.

2.3.2 Construction of spreading models using ultrafilters

We will now describe an alternative construction of spreading models using ultrafilters. This method avoids the use of Ramsey theory.

If $\mathcal{U}$ is any ultrafilter on a Banach space X, and $f : X \to \mathbb{R}$ is a bounded function then we can define $\lim_{x \to \mathcal{U}} f(x)$, to be the real number λ such that for all $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that $|f(x) - \lambda| < \varepsilon$ for all $x \in U$. The existence of such a limit can be proved similarly to the proof of the Bolzano-Weierstrauss Theorem, and uniqueness is immediate from the definition.

We will be interested in limits of the form $\lim_{z\to\mathcal{U}} \|x+z\|$ where $x \in X$ is fixed. Since the function $z \mapsto x+z$ is unbounded, the above comment does not immediately apply. Indeed the above limit need not exist for an arbitrary ultrafilter $\mathcal{U}$. Consider

$$\mathcal{B} = \{X \setminus nB_X : n \in \mathbb{N}\}.$$

$\mathcal{B}$ forms a filter base (i.e. it consists of non-empty sets and is closed under finite intersections). Let $\mathcal{F}$ be the filter generated by $\mathcal{B}$, (i.e. $\mathcal{F} = \{A \subseteq X : \exists B \in \mathcal{B} \text{ s.t. } B \subseteq A\}$), and let $\mathcal{U}$ be any ultrafilter which contains $\mathcal{F}$. It is easy to see that every $U \in \mathcal{U}$ must be unbounded, and therefore it is clear that $\lim_{z\to\mathcal{U}} \|x+z\|$ is undefined for every $x \in X$.

Now suppose that $\mathcal{U}$ contains a bounded set U. Then consider the restriction of $z \mapsto x+z$ to the set U, which is a bounded function from U to $\mathbb{R}$. The ultrafilter $\mathcal{U}$ induces an ultrafilter $\mathcal{U}'$ on U by

$$\mathcal{U}' = \{U \cap V : V \in \mathcal{U}\} \subseteq \mathcal{U}.$$

By the above comment, $\lim_{z\to\mathcal{U}'} \|x+z\|$ exists, and it is immediate from the definitions that this is the same as $\lim_{z\to\mathcal{U}} \|x+z\|$.

Hence given an ultrafilter (that contains a bounded set) we can define for every $x \in X$ and scalars $a_1, \dots, a_k$ a semi-norm on $X \oplus \mathbb{R}^{(\mathbb{N})}$ by

$$\left\| x + \sum_{i=1}^{k} a_i e_i \right\| = \lim_{x_1\to\mathcal{U}} \dots \lim_{x_n\to\mathcal{U}} \|x + a_1x_1 + \dots + a_nx_n\|.$$

We will now show that this will in fact be a norm provided the ultrafilter $\mathcal{U}$ does not converge. This corresponds to the non-convergence of the Brunel-Sucheston sequence. To do this we need the following lemma about ultrafilters on complete metric spaces.

Lemma 2.3.2 *Let (X, d) be a complete metric space, and $\mathcal{U}$ be an ultrafilter on X which does not converge. Then there exists $\delta > 0$ such that* $\operatorname{diam}(U) \geq \delta$ *for all $U \in \mathcal{U}$.*

Proof. Suppose the result is false. Then for every $n \in \mathbb{N}$, there exists a set $U_n \in \mathcal{U}$ such that $\operatorname{diam}(U_n) < \frac{1}{n}$. We define for each n, the set $V_n = U_1 \cap \dots \cap U_n \in \mathcal{U}$. Since $V_n \subseteq U_n$, we have $\operatorname{diam}(V_n) \leq \operatorname{diam}(U_n) \leq \frac{1}{n}$. Now choose a sequence $x_n \in V_n$ for every n. Since (V_n) is a decreasing sequence of sets whose diameters converge to 0, (x_n) is a Cauchy sequence, and so converges to some point $x \in X$. Let $\varepsilon > 0$, and choose N such that $\mathrm{d}(x_n, x) < \frac{\varepsilon}{2}$ for every $n \geq N$ and $\frac{1}{N} < \frac{\varepsilon}{2}$. Then $V_N \subseteq B_{\frac{1}{N}}(x_N) \subseteq B_{\frac{\varepsilon}{2}}(x_N) \subseteq B_\varepsilon(x)$. Hence $B_\varepsilon(x) \in \mathcal{U}$. Therefore $\mathcal{U} \to x$, giving us a contradiction. $\square$

Proposition 2.3.3 *The semi-norm defined on $X \oplus \mathbb{R}^{(\mathbb{N})}$ by*

$$\left\| x + \sum_{i=1}^{k} a_i e_i \right\| = \lim_{x_1 \to \mathcal{U}} \ldots \lim_{x_n \to \mathcal{U}} \| x + a_1 x_1 + \ldots + a_n x_n \| ,$$

is a norm if and only if the ultrafilter $\mathcal{U}$ does not converge.

Proof. Suppose that $\mathcal{U} \to x$. Then $\lim_{z \to \mathcal{U}} \|x - z\| = 0$. Therefore $\|x - e_1\| = 0$, and $\|\cdot\|$ is not a norm on $X \oplus \mathbb{R}^{(\mathbb{N})}$.

Now suppose that $\mathcal{U}$ does not converge, and that $\left\| x + \sum_{i=1}^{k} a_i e_i \right\| = 0$. By Lemma 2.3.2, there is a $\delta > 0$ such that $\mathrm{diam}(U) > \delta$ for all $U \in \mathcal{U}$. For fixed $x_1, \ldots, x_{k-1}$, let $\lambda = \lim_{x_k \to \mathcal{U}} \left\| x + \sum_{i=1}^{k} a_i x_i \right\|$. For any $\varepsilon > 0$, there exists $U \in \mathcal{U}$ such that for all $y \in U$,

$$\left| \left\| x + \sum_{i=1}^{k-1} a_i x_i + a_k y \right\| - \lambda \right| < \varepsilon .$$

Choose two elements $y_1, y_2 \in U$ with $\|y_1 - y_2\| \geq \delta$. Then

$$\begin{aligned} \delta |a_k| \leq\ & |a_k| \, \|y_1 - y_2\| \\ =\ & \left\| (x + \sum_{i=1}^{k-1} a_i x_i + a_k y_1) - (x + \sum_{i=1}^{k-1} a_i x_i + a_k y_2) \right\| \\ \leq\ & 2\lambda + 2\varepsilon . \end{aligned}$$

Since ε is arbitrary, we have $\lambda \geq \frac{1}{2}\delta |a_k|$, and therefore $\left\| x + \sum_{i=1}^{k} a_i e_i \right\| \geq \frac{1}{2}\delta |a_k|$. Thus in our situation, $\left\| x + \sum_{i=1}^{k} a_i e_i \right\| = 0$ implies that $a_k = 0$. Iterating this gives that all of the scalars a_i are zero. Then $x = 0$ follows from the positive definiteness of the norm on X. Therefore $\|\cdot\|$ defines a norm as required. ☐

2.3.3 Properties of spreading models

The basic properties of spreading models can be found in [11]. The key properties we shall use are as follows;

(i) The sequence (e_i) is spreading, (or 1-subsymmetric over X) i.e. for every increasing sequence $n_1 < \ldots < n_k$ of integers,

$$\left\| x + \sum_{i=1}^{k} a_i e_i \right\| = \left\| x + \sum_{i=1}^{k} a_i e_{n_i} \right\| .$$

(ii) The extension $\mathcal{F}$ of X is finitely representable in X. In fact a stronger property holds. Indeed, if F is a finite dimensional subspace of X, $\varepsilon > 0$, and $k \in \mathbb{N}$, then there exists $x_1 \in X$ such that the mapping

$$x + \sum_{i=1}^{k} a_i e_i \mapsto x + c_1 x_1 + \sum_{i=2}^{k} a_i e_i,$$

from $F \oplus [e_i]_{i=1}^{k} \to F \oplus [x_1] \oplus [e_i]_{i=2}^{k}$ is a $(1+\varepsilon)$-isomorphism.

Spreading models have several applications in the theory of Banach spaces. For example, the non-existence of spreading models isomorphic to ℓ_1 has connections to various Banach-Saks type properties. Another application is stable Banach spaces. Spreading models techniques can be used to prove the existence of subspaces isomorphic to ℓ_p for some $p \in [1, \infty)$ (see [49]). The spreading models of certain Tsirelson type spaces have also been studied. One particularly striking example is that of a space similar to Schlumprecht's space, all of whose spreading models do not contain a subspace isomorphic to any ℓ_p or c_0. For this, and other examples and applications see [67], [65] and [66].

2.4 Types and strong types

The notions of type and strong type have been used in several papers (see e.g. [55], [42], [39], [62]). The idea is to associate to a Banach space X, the topological spaces $\mathcal{T}(X)$ and $\mathcal{S}(X)$ of types and strong types on X. There are various topologies that are of interest on these spaces. We then hope to deduce properties of our original Banach space X from topological properties of $\mathcal{T}(X)$ and $\mathcal{S}(X)$. For example, in [42], separability of a certain topology on $\mathcal{T}(X)$ is used to deduce that the Banach space X contains either ℓ_1 or a reflexive subspace.

2.4.1 Types

Let X be a Banach space, and let $x \in X$. Consider the map $\theta_x : X \to \mathbb{R}$ defined by

$$\theta_x(y) = \|x + y\| .$$

We will call this map a *degenerate type*. We consider the collection of these degenerate types, $\tilde{X} = \{\theta_x : x \in X\}$, as a subset of $\mathbb{R}^X = \prod_{y \in X} \mathbb{R}$ equipped with the Tychonoff product topology. We then define $\mathcal{T}(X)$, the set of *types*, to be the closure of $\tilde{X}$ in this topology. The topology induced on $\mathcal{T}(X)$ by $\mathbb{R}^X$ is that of pointwise convergence, which we will call the *weak topology.* Thus if $\theta \in \mathcal{T}(X)$, then there exists a bounded net (x_i) in X such that

$$\theta(y) = \lim_i \|x_i + y\| \text{ for all } y \in X.$$

Alternatively, if $\theta \in \mathcal{T}(X)$, then there is an ultrafilter $\mathcal{U}$ on X such that

$$\theta(y) = \lim_{x \to \mathcal{U}} \|x + y\| \text{ for all } y \in X.$$

If X is a separable Banach space then with the weak topology, $\mathcal{T}(X)$ is a separable metrizable space. Observe that each type is a Lipschitz function with Lipschitz constant 1 (this follows from the triangle inequality for the degenerate types, and is immediate for general types by taking pointwise limits). Also notice that $\mathcal{T}(X)$ is locally compact in the weak topology. In particular, for each $r \geq 0$, the sets

$$\{\theta \in \mathcal{T}(X) : \theta(0) \leq r\},$$

are weakly compact, since they are closed subsets of the compact set $\prod_{y \in X}[0, \|y\| + r]$. We will often write $\|\theta\|$ for $\theta(0)$.

There are various other topologies of interest on $\mathcal{T}(X)$. One that will be of use to us in this thesis is the so called *strong topology*, also called the topology of uniform convergence on bounded sets. This is induced by the pseudometrics

$$d_M(\theta, \phi) = \sup\{|\theta(y) - \phi(y)| : \|y\| \leq M\}.$$

In other words, a set $U \subseteq \mathcal{T}(X)$ is open in the strong topology if for every $\theta \in U$ there exists M and $\varepsilon > 0$ such that $\{\phi \in \mathcal{T}(X) : d_M(\theta, \phi) < \varepsilon\} \subseteq U$. The strong topology on $\mathcal{T}(X)$ is always metrizable. We can take the metric to be

$$d(\theta, \phi) = \sum_{m=1}^{\infty} \frac{1}{2^m} d_m(\theta, \phi).$$

The map $x \mapsto \theta_x$ is a bijection between X and $\tilde{X}$. It can also be shown to be a homeomorphism whenever $\tilde{X}$ is given the topology induced by either the weak or the strong topologies on $\mathcal{T}(X)$. Hence we may identify the elements of $\tilde{X}$ with the elements in X. Also it follows that if $\mathcal{T}(X)$ is strongly separable, then X must be separable. The converse is however not true. For example it is shown in [42] that $\mathcal{T}(c_0)$ is not strongly separable. In the same paper it is proved that if X is a Banach space with strongly separable types, then X contains either a reflexive subspace, or a subspace isomorphic to ℓ_1.

We have a natural 'scalar multiplication' on $\mathcal{T}(X)$. We will call this map a dilation. Given a scalar $\alpha \neq 0$, and a type θ, we define

$$(\alpha \cdot \theta)(x) = |\alpha|\theta(\alpha^{-1}x) \text{ for all } x \in X.$$

In the case where $\alpha = 0$ we set $(0 \cdot \theta) = \theta_0$. Notice that for degenerate types dilation corresponds to scalar multiplication,

$$(\alpha \cdot \theta_y)(x) = \|\alpha y + x\| = \theta_{\alpha y}(x),$$

and if θ is given by a net $(x_i)_{i\in I}$, then $\alpha \cdot \theta$ is given by the net $(\alpha x_i)_{i\in I}$.

2.4.2 Strong types

We now introduce the concept of a strong type on a Banach space. We construct these in a similar way to that of the constructing the types. We consider the following collection of real-valued functions defined on the types of X,

$$\bar{X} = \{\tau_x : x \in X\} \subseteq \mathbb{R}^{\mathcal{T}(X)},$$

where,

$$\tau_x(\theta) = \theta(x) \text{ for all } \theta \in \mathcal{T}(X).$$

We put the Tychonoff product topology on $\mathbb{R}^{\mathcal{T}(X)}$, and then we define the set of *strong types*, $\mathcal{S}(X)$, to be the closure of $\bar{X}$ in this topology. Hence if τ is a strong type on X, then there is a bounded net $(x_i)_{i\in I}$ in X such that

$$\tau(\theta) = \lim_{i\to\infty} \theta(x_i) \text{ for all } \theta \in \mathcal{T}(X).$$

Again, we have an equivalent formulation in terms of ultrafilters, so that if τ is a strong type on X, then there is an ultrafilter $\mathcal{U}$ on X such that

$$\tau(\theta) = \lim_{x\to\mathcal{U}} \theta(x) \text{ for all } \theta \in \mathcal{T}(X).$$

The topology on $\mathcal{S}(X)$ is again locally compact. To see this, let $\tau \in \mathcal{S}(X)$, and suppose that $\tau(\theta) = \lim_{x\to\mathcal{U}} \theta(x)$. Given $\varepsilon > 0$, we can find $U \in \mathcal{U}$ such that $|\tau(\theta_0) - \|x\|\,| < \varepsilon$ for all $x \in U$. For any $\theta \in \mathcal{T}(X)$, we can find $V \in \mathcal{U}$ such that $|\tau(\theta) - \theta(x)| < \varepsilon$ for all $x \in V$. Pick any $x \in U \cap V$. Then

$$\begin{aligned}\tau(\theta) &\leq \theta(x) + \varepsilon \\ &\leq \theta(0) + \|x\| + \varepsilon \\ &\leq \theta(0) + \tau(\theta_0) + 2\varepsilon.\end{aligned}$$

We will often write $\|\tau\|$ for $\tau(\theta_0)$. The above then shows that $\tau(\theta) \leq \|\tau\| + \|\theta\|$. In particular,

$$\{\tau \in \mathcal{S}(X) : \|\tau\| \leq M\} \subseteq \prod_{\theta \in \mathcal{T}(X)} [0, M + \|\theta\|].$$

Hence, by Tychonoff's theorem, $\{\tau \in \mathcal{S}(X) : \|\tau\| \leq M\}$ is compact in $\mathcal{S}(X)$.

We have a natural map $p : \mathcal{S}(X) \to \mathcal{T}(X)$ given by $(p(\tau))(x) = \tau(\theta_x)$. This is clearly a map into $\mathbb{R}^X$. Also, $(p(\tau_y))(x) = \tau_y(\theta_x) = \theta_x(y) = \|x + y\| = \theta_y(x)$. Thus p maps $\bar{X}$ onto $\tilde{X}$. p is also continuous. For if (τ_i) is a net converging to τ in $\mathcal{S}(X)$, then for any $x \in X$, $(p(\tau_i))(x) = \tau_i(\theta_x) \to \tau(\theta_x) = (p(\tau))(x)$, i.e. $p(\tau_i) \to p(\tau)$ in the weak topology of $\mathcal{T}(X)$. Hence p is indeed a map into $\mathcal{T}(X)$. We can also show that p is surjective, for if $\theta \in \mathcal{T}(X)$ is given by a bounded net $(x_i)_{i \in I}$, then by the local compactness of $\mathcal{S}(X)$ we can find a limit point, τ, of $\{\tau_{x_i} : i \in I\}$. It is straightforward to show that $p(\tau) = \theta$.

We again have a dilation defined on the strong types similar to that for types. For $\alpha \neq 0$ we set

$$(\alpha \cdot \tau)(\theta) = |\alpha| \tau(\alpha^{-1}\theta) \text{ for all } \theta \in \mathcal{T}(X),$$

and for $\alpha = 0$ we set $(0 \cdot \tau) = \tau_0$. Again, if τ is a strong type given by a net $(x_i)_{i \in I}$ then $\alpha \cdot \tau$ is given by the net $(\alpha x_i)_{i \in I}$.

We have another operation on the strong types which we will call convolution. We will construct this in a series of steps. Suppose $x \in X$ and $\theta \in \mathcal{T}(X)$. We define

$$(x * \theta)(y) = \theta(x + y), \text{ for all } y \in X.$$

Consider the map $\theta \mapsto x * \theta$ from $\mathcal{T}(X)$ into $\mathbb{R}^X$. It is easy to see that this map sends $\tilde{X}$ into $\tilde{X}$ (since $x * \theta_z = \theta_{x+z}$). We will show that it is also a continuous map, from which it follows that $x * \theta \in \mathcal{T}(X)$. For convenience let f denote the map $\theta \mapsto x * \theta$. Suppose that $U \subseteq \mathbb{R}^X$ is open (in the Tychonoff topology) and that $\theta \in f^{-1}(U)$. Since $x * \theta \in U$, there exists $\varepsilon > 0$ and $x_1, \ldots, x_n \in X$ such that

$$\{g \in \mathbb{R}^X : |g(x_i) - (x * \theta)(x_i)| < \varepsilon \text{ for } i = 1, \ldots, n\} \subseteq U.$$

Let $W = \{\phi \in \mathcal{T}(X) : |\phi(x + x_i) - \theta(x + x_i)| < \varepsilon \text{ for } i = 1, \ldots, n\}$, which is a weak neighbourhood of θ in $\mathcal{T}(X)$. It is immediate that $W \subseteq f^{-1}(U)$. Thus $f^{-1}(U)$ is open in $\mathcal{T}(X)$, and so f is a continuous map.

We now do the same for strong types. Given $\tau \in \mathcal{S}(X)$, we define a map $x * \tau$ in $\mathbb{R}^{\mathcal{T}(X)}$ by

$$(x * \tau)(\theta) = \tau(x * \theta).$$

By the same method as above, we see that the map $\tau \mapsto x * \tau$ is continuous from $\mathcal{S}(X)$ into $\mathbb{R}^{\mathcal{T}(X)}$, and maps $\bar{X}$ into $\bar{X}$. Hence $x * \tau \in \mathcal{S}(X)$.
Now suppose that $\tau \in \mathcal{S}(X)$ and $\theta \in \mathcal{T}(X)$. We define $\tau * \theta \in \mathbb{R}^X$ by

$$(\tau * \theta)(y) = \tau(y * \theta) \text{ for all } y \in X.$$

Notice that when $\tau = \tau_x$, we get $(\tau_x * \theta)(y) = \tau_x(y * \theta) = (y * \theta)(x) = \theta(x + y)$. Hence this definition agrees with our definition of $x * \theta$ (identifying x and τ_x). Also note that the mapping $\tau \mapsto \tau * \theta$ is a continuous map from $\mathcal{S}(X)$ to $\mathbb{R}^X$ which maps $\bar{X}$ into $\mathcal{T}(X)$. Hence $\tau * \theta \in \mathcal{T}(X)$.
Finally, if $\sigma, \tau \in \mathcal{S}(X)$ then we define

$$(\sigma * \tau)(\theta) = \sigma(\tau * \theta).$$

Again, the map $\sigma \mapsto \sigma * \tau$ is continuous from $\mathcal{S}(X)$ to $\mathbb{R}^{\mathcal{T}(X)}$, maps $\bar{X}$ into $\mathcal{S}(X)$, and so $\sigma * \tau \in \mathcal{S}(X)$.

Suppose that (x_i) and (y_j) are nets such that $\sigma(\theta) = \lim_i \theta(x_i)$ and $\tau(\theta) = \lim_j \theta(y_j)$ for every $\theta \in \mathcal{T}(X)$. Then

$$\begin{aligned}(\sigma * \tau)(\theta) &= \sigma(\tau * \theta)\\ &= \lim_i (\tau * \theta)(x_i)\\ &= \lim_i \tau(x_i * \theta)\\ &= \lim_i \lim_j (x_i * \theta)(y_j)\\ &= \lim_i \lim_j \theta(x_i + y_j).\end{aligned}$$

It is straightforward to show that convolution is associative i.e. if $\rho, \sigma, \tau \in \mathcal{S}(X)$ then

$$\rho * (\sigma * \tau) = (\rho * \sigma) * \tau.$$

It also follows from the definitions that

$$(\alpha \cdot \sigma) * (\alpha \cdot \tau) = \alpha \cdot (\sigma * \tau).$$

This convolution operation will prove to be a useful tool in Chapter 6. It should be noted that a similar convolution defined on the types has been useful in the study of stable Banach spaces (see Section 1.3).

Chapter 3

Asymptotic ℓ_p spaces

Contents

In this chapter we will introduce the class of asymptotic ℓ_p spaces and give some examples of such spaces. We will then review some basic properties of these spaces which are already known, and give some further simple properties. We will also investigate spreading models of asymptotic ℓ_p spaces. In the reflexive case, these turn out to be all isomorphic to ℓ_p. The non-reflexive case is not so simple, but we will show that the spreading models still retain some ℓ_p structure. Finally, we look at the relationships between asymptotic ℓ_2 spaces and some other forms of asymptotic Hilbert space behaviour that were introduced by Pisier.

3.1 Definition of asymptotic ℓ_p spaces

The notion of an asymptotic ℓ_p space first appeared in [61], where the collection of spaces that are now known as *stabilized asymptotic ℓ_p spaces* were introduced. It was shown here that every Banach space with bounded distortions has a subspace that is stabilized asymptotic ℓ_p for some $p \in [1, \infty]$. Later, in [59], a more general collection of spaces, known as *asymptotic ℓ_p spaces* was introduced. We will be interested only in the case of Banach spaces with a basis, where these notions are more easily visualized. Roughly speaking, in this situation, a space will be said to be asymptotic ℓ_p if for any N, any N normalized successive block vectors $(x_i)_{i=1}^N$ behave like the unit vector basis for ℓ_p^N provided the vectors are far enough down the basis and sufficiently spread out. More precisely we make the following definition.

Definition 3.1.1 *A Banach space X is said to be asymptotic ℓ_p with respect to a basis $(e_i)_{i=1}^\infty$, if there exists a constant $C \geq 1$ (the asymptotic constant), such that for every $N \in \mathbb{N}$ there is a function $F_N : \mathbb{N} \to \mathbb{N}$, so that whenever $x_1 < \ldots < x_N$ are successive normalized block vectors, with* $\operatorname{supp}(x_i) = [p_i, q_i]$ *with $p_1 \geq F_N(0)$ and $p_{i+1} \geq F_N(q_i)$ for $i = 1, \ldots, N-1$, then $(x_i)_{i=1}^N$ are C-equivalent to the unit vector basis of ℓ_p^N i.e. for any N-tuple of scalars $\mathbf{a} = (a_1, \ldots, a_N)$,*

$$\frac{1}{C} \left\|\mathbf{a}\right\|_p \leq \left\| \sum_{i=1}^N a_i x_i \right\| \leq C \left\|\mathbf{a}\right\|_p .$$

The notion of stabilized asymptotic ℓ_p is similar, except we only require the N block vectors (x_i) to start far enough down the sequence (depending on N). We do not require the vectors to be spread out as we did in the above definition.

Definition 3.1.2 *A Banach space X is said to be stabilized asymptotic ℓ_p with respect to a basis $(e_i)_{i=1}^\infty$, if there exists a constant $C \geq 1$, such that for every $N \in \mathbb{N}$, there exists $m \in \mathbb{N}$, so that whenever $m \leq x_1 < \ldots < x_N$ are successive normalized block vectors, then $(x_i)_{i=1}^N$ are C-equivalent to the unit vector basis of ℓ_p^N.*

3.2 Examples of asymptotic ℓ_p spaces

For $p \in [1, \infty)$, ℓ_p is a trivial example of an asymptotic ℓ_p space with respect to its usual basis. Similarly c_0 is an asymptotic ℓ_∞ space. We will introduce in this section some more interesting asymptotic ℓ_p spaces. These are spaces that have been much studied in their own rights for many years, and will be useful examples throughout this thesis.

3.2.1 Tsirelson's space

Consider first the usual Tsirelson's space T (see Section 2.2.1). It is immediate from the definition of the norm that if $x_1 < \ldots < x_n$ are n normalized block vectors (with respect to $(e_i)_{i=1}^{\infty}$) with $\min\operatorname{supp} x_1 \geq n$ then for any scalars $a_1, \ldots, a_n$,

$$\frac{1}{2}\sum_{i=1}^{n}|a_i| \leq \left\|\sum_{i=1}^{n} a_i x_i\right\| \leq \sum_{i=1}^{n}|a_i|.$$

In other words, Tsirelson's space is a stabilized asymptotic ℓ_1 space. The same argument applies to the spaces T_θ.

Now suppose that $p \in (1, \infty)$. Then the p-convexification $T^{(p)}$ of T is a stabilized asymptotic ℓ_p space. For if $x_1 < \ldots < x_n$ are n normalized block vectors (with respect to the standard basis) with $\min\operatorname{supp} x_1 \geq n$, then for any scalars $a_1, \ldots, a_n$,

$$2^{-\frac{1}{p}}\left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}} \leq \left\|\sum_{i=1}^{n} a_i x_i\right\| \leq \left(\sum_{i=1}^{n}|a_i|^p\right)^{\frac{1}{p}}.$$

3.2.2 James' space

James' space (see Section 2.2.5) is an asymptotic ℓ_2 space with respect to its standard basis. Indeed, if $x_1 < x_2 < \ldots < x_n$ are n normalized blocks with respect to this basis such that $\max\operatorname{supp} x_i + 1 < \min\operatorname{supp} x_{i+1}$ for $i = 1, \ldots, n-1$, then for any scalars $a_1, \ldots, a_n$ we have

$$\left(\sum_{i=1}^{n}|a_i|^2\right)^{\frac{1}{2}} \leq \left\|\sum_{i=1}^{n} a_i x_i\right\| \leq \sqrt{2}\left(\sum_{i=1}^{n}|a_i|^2\right)^{\frac{1}{2}}.$$

Note however that James' space is not a stabilized asymptotic ℓ_2 space, because we require a gap between each of the block vectors x_i.

3.3 Some known properties of asymptotic ℓ_p spaces

3.3.1 The basis of asymptotic ℓ_p spaces

In this section we will look at the basis of an asymptotic ℓ_p space. It should be noted that when we say that a space is asymptotic ℓ_p with respect to a given basis then the choice of basis is important. There are examples of spaces which are asymptotic ℓ_p with respect to one basis, but not with respect to another basis. We have the following proposition.

Proposition 3.3.1 *Let X be asymptotic ℓ_p with respect to a basis $(e_i)_{i=1}^\infty$. Then the following hold;*

(i) If $p > 1$, then the basis $(e_i)_{i=1}^\infty$ is shrinking.

(ii) If $p < \infty$, then the basis $(e_i)_{i=1}^\infty$ is boundedly complete provided either the basis is unconditional, or X is stabilized asymptotic ℓ_p with respect to $(e_i)_{i=1}^\infty$.

Proof. The proof of (i), and the proof of (ii) in the stabilized asymptotic ℓ_p case can be found in [59, 4.2]. The result in the unconditional case seems to be new, so we include the proof. Suppose that the basis is not boundedly complete. Then there is a sequence of scalars $(a_i)_{i=1}^\infty$ such that for any $n \in \mathbb{N}$, $\|\sum_{i=1}^n a_i e_i\| \leq 1$ but $\sum_{i=1}^n a_i e_i$ does not converge as $n \to \infty$. Since the sequence of partial sums is not Cauchy, there exists $\varepsilon > 0$ and a sequence of successive vectors $x_1 < x_2 < \ldots$ such that $\|x_i\| \geq \varepsilon$ for all i and each x_i is of the form $x_i = \sum_{j=p_i}^{q_i} a_j e_j$. Fix $N \in \mathbb{N}$. Choose $y_1 < y_2 < \ldots < y_N$ from the sequence (x_i) which are sufficiently spread out so that $(y_i / \|y_i\|)_{i=1}^N$ are C-equivalent to the unit vector basis of ℓ_p^N. Then using the unconditionality of the basis,

$$\frac{\varepsilon}{C} N^{\frac{1}{p}} \leq \left\| \sum_{i=1}^N \|y_i\| \frac{y_i}{\|y_i\|} \right\| = \left\| \sum_{i=1}^N y_i \right\| \leq K,$$

where K is the unconditional basis constant. This gives us a contradiction, since N is arbitrary and $p < \infty$. ☐

This immediately yields the following corollary.

Corollary 3.3.2 *If $1 < p < \infty$ and X is either stabilized asymptotic ℓ_p or asymptotic ℓ_p with respect to an unconditional basis, then X is reflexive.*

Remark 3.3.3 *If $p < \infty$ and X is an asymptotic ℓ_p space, then we will see later that X can not contain a subspace isomorphic to c_0. Hence, by [53, Theorem 1.c.10], if the basis of X is unconditional then this tells us immediately that the basis is boundedly complete.*

Observe that Proposition 3.3.1 can not be improved. For example asymptotic ℓ_1 spaces need not have a shrinking basis (e.g. ℓ_1), and asymptotic ℓ_∞ spaces need not have a boundedly complete basis (e.g. c_0). Even if $1 < p < \infty$, then the basis need not be boundedly complete. For example James' space is asymptotic ℓ_2 with respect to its standard basis, but the basis is not boundedly complete (the basis is shrinking, but J is not reflexive). Recall that the standard basis for James' space is conditional and that J is not stabilized asymptotic ℓ_2.

3.3.2 Duality of asymptotic ℓ_p spaces

It was proved in [59] that for $p \in (1, \infty)$, the dual of a *reflexive* asymptotic ℓ_p space is asymptotic ℓ_q, where $\frac{1}{p} + \frac{1}{q} = 1$. The proof of this result actually proves the following.

Theorem 3.3.4 ([40, Theorem 1]) *Let X be a Banach space with a shrinking basis $(e_i)_{i=1}^{\infty}$. If X^* is asymptotic ℓ_q with respect to $(e_i^*)_{i=1}^{\infty}$ then X is asymptotic ℓ_p with respect to $(e_i)_{i=1}^{\infty}$.*

It is worth remarking that it is not known whether the dual of an asymptotic ℓ_p space is always asymptotic ℓ_q even when $p \in (1, \infty)$. For example, J^*, the dual of James' space, is not asymptotic ℓ_2 with respect to the biorthogonal functionals $(e_i^*)_{i=1}^{\infty}$. However, J^* is known to be asymptotic ℓ_2. It is however certainly not true when $p = 1$. In [40], an example of an asymptotic ℓ_1 space is constructed whose dual is isomorphic to ℓ_1, and so is certainly not asymptotic ℓ_∞.

3.4 Further properties of asymptotic ℓ_p spaces

3.4.1 The relationship between asymptotic ℓ_p and stabilized asymptotic ℓ_p spaces

It is clear from the definitions that a stabilized asymptotic ℓ_p space is asymptotic ℓ_p. We show in this section, that an asymptotic ℓ_p space contains a block subspace which is stabilized asymptotic ℓ_p.

Theorem 3.4.1 *Let X be an asymptotic ℓ_p space with respect to a basis $(e_i)_{i=1}^{\infty}$. Then there is an increasing sequence of integers $(k_n)_{n=1}^{\infty}$ such that $[e_{k_n}]_{n=1}^{\infty}$ is a stabilized asymptotic ℓ_p space.*

Proof. We start with some observations about asymptotic ℓ_p spaces. Firstly it is clear that the functions F_N in the definition can be assumed to be increasing functions from $\mathbb{N}$ to $\mathbb{N}$. Also, for fixed $n \in \mathbb{N}$, we may as well suppose that $F_N(n)$ is strictly increasing as a function of N. (Otherwise, we can define $\widetilde{F_N}(n) = \max(F_1(n), F_2(n), \ldots, F_N(n))$, and the space will still be asymptotic ℓ_p with respect to the functions $\widetilde{F_N}$). Thus we may assume that if $N_1 \geq N_2$ and $n_1 \geq n_2$, then $F_{N_1}(n_1) \geq F_{N_2}(n_2)$.

We can now define the integers k_n. Let $k_2 = F_2(0)$, and $k_{n+1} = F_{n+1}(k_n)$ for $n \geq 2$. Use the notation that $k_1 = 0$ so that $k_n = F_n(k_{n-1})$ for all $n \geq 2$. We define $Y = [e_{k_n}]_{n \geq 2}$ and claim that any N normalized block vectors with respect to the natural basis in Y , starting after N are C-equivalent to the unit vector basis of ℓ_p^N. Let $N \geq 2$. Suppose that $k_N \leq y_1 <$

$y_2 < \ldots < y_N$ are normalized block vectors with $y_i = \sum_{j=p_i}^{q_i} a_j e_{k_j}$, $p_1 \geq N$ and $q_i + 1 \leq p_{i+1}$ for $i < N$. Then $k_{p_1} \geq k_N = F_N(k_{N-1}) \geq F_N(0)$. Also $k_{p_{i+1}} \geq k_{q_i+1} = F_{q_i+1}(k_{q_i}) \geq F_N(k_{q_i})$. Hence, by the definition of the asymptotic ℓ_p space X, $(y_i)_{i=1}^N$ is C-equivalent to the unit vector basis of ℓ_p^N. Thus Y is a stabilized asymptotic ℓ_p space. □

3.4.2 Finite representability of ℓ_p in subspaces of asymptotic ℓ_p spaces

In this section we will show that ℓ_p is finitely representable in every subspace of an asymptotic ℓ_p space. We begin with two easy lemmas.

Lemma 3.4.2 *Let X be a Banach space with a basis, and let P_n be the natural projections associated with the basis. Let Y be an infinite dimensional subspace of X. Then $Y \cap \ker(P_n)$ is infinite dimensional for all $n \in \mathbb{N}$.*

Proof. Suppose for a contradiction that $Y \cap \ker(P_n)$ were finite dimensional with basis $y_1, \ldots, y_k$. Since Y is infinite dimensional we can find $n+1$ vectors $z_1, \ldots, z_{n+1}$ in Y such that $y_1, \ldots, y_k, z_1, \ldots, z_{n+1}$ are linearly independent. Now consider $P_n(z_1), \ldots, P_n(z_{n+1})$, which are $n+1$ vectors in an n-dimensional space. Thus there exist scalars $(a_i)_{i=1}^{n+1}$, not all zero, such that $z = \sum_{i=1}^{n+1} a_i z_i$ lies in $\ker(P_n)$. Hence we obtain a $z \neq 0$ which is in $\ker(P_n) \cap Y$, but which is not in the span of $y_1, \ldots, y_k$. Thus $Y \cap \ker(P_n)$ is infinite dimensional. □

Lemma 3.4.3 *Fix $p \in [1, \infty]$. Let $\varepsilon > 0$, and $(x_i)_{i=1}^n$ be C-equivalent to the unit vector basis of ℓ_p^n. Then there exists $\eta = \eta(n, \varepsilon)$ such that whenever $(u_i)_{i=1}^n$ are vectors such that $\|x_i - u_i\| \leq \eta$ for all i, then $(u_i)_{i=1}^n$ are $(C+\varepsilon)$-equivalent to the unit vector basis of ℓ_p^n.*

Proof. We can take $\eta(n, \varepsilon) = n^{-1/q}\varepsilon / C(C+\varepsilon)$, where $\frac{1}{p} + \frac{1}{q} = 1$. □

We can now proceed with our proof that ℓ_p is finitely representable in every infinite dimensional subspace of an asymptotic ℓ_p space.

Theorem 3.4.4 *Let X be an asymptotic ℓ_p space with respect to its basis $(e_i)_{i=1}^\infty$, and let Y be an infinite dimensional subspace of X. Then ℓ_p is finitely representable in Y.*

Proof. By Theorem 2.1.3 we only have to prove that ℓ_p is crudely finitely representable in Y. Fix $N \in \mathbb{N}$, and let $p_1 = F_N(0)$. Then by Lemma 3.4.2, $Y \cap \ker P_{p_1}$ is infinite dimensional. Pick a normalized vector $y_1 \in Y \cap \ker P_{p_1}$. Then we can write $y_1 = \sum_{j=p_1}^\infty a_{j,1} e_j$. Let $u_1 = \sum_{j=p_1}^{q_1} a_{j,1} e_j / \left\| \sum_{j=p_1}^{q_1} a_{j,1} e_j \right\|$, where q_1 is chosen sufficiently large that $\|y_1 - u_1\| < \eta(N, 1)$. Now consider $Y \cap \ker P_{F_N(q_1)}$, which is infinite dimensional by Lemma 3.4.2. In particular, we can pick a normalized vector $y_2 = \sum_{j=F_N(q_1)}^{\infty} a_{j,2}e_j$. Restricting this vector to a block vector as before, and continuing in the obvious way, we get a sequence of vectors $y_1, \dots, y_N$ and corresponding normalized block vectors $u_1 < u_2 < \dots < u_N$ such that $\|y_i - u_i\| < \eta(N, 1)$ for each i, and also so that the vectors $(u_i)_{i=1}^N$ are sufficiently spread out to be C-equivalent to the unit vector basis of ℓ_p^N. By Lemma 3.4.3, we have that $y_1, \dots, y_N$ are $(C + 1)$-equivalent to the unit vector basis of ℓ_p^N. Writing $W = \operatorname{span}(y_1, \dots, y_N)$ we have a subspace $W \leq Y$ such that $d(W, \ell_p^N) \leq 1 + C$. We have shown that ℓ_p is crudely finitely representable in Y and we are done. $\square$

3.4.3 Subspaces of asymptotic ℓ_p spaces isomorphic to ℓ_q or c_0

In this section we show that for $p < \infty$, asymptotic ℓ_p spaces can not contain subspaces isomorphic to c_0 or to any ℓ_q, except possibly for $q = p$. We also show that asymptotic ℓ_∞ spaces can not contain subspaces isomorphic to ℓ_q for any $q \in [1, \infty)$.

Theorem 3.4.5 *Let X be an asymptotic ℓ_p space. Then the following hold;*

(i) If $p < \infty$ then $c_0 \not\hookrightarrow X$ and $\ell_q \not\hookrightarrow X$ for $q \neq p$.

(ii) If $p = \infty$ then $\ell_q \not\hookrightarrow X$ for all $q \in [1, \infty)$.

Proof. Suppose that $p > 1$. By Proposition 3.3.1 the basis of X is shrinking, and in particular X^* is separable. But if $\ell_1 \hookrightarrow X$ then X^* is non-separable. Therefore $\ell_1 \not\hookrightarrow X$.

To deal with the case of c_0 and ℓ_q for $1 < q < \infty$, we can consider the standard bases in these spaces. We consider the case of ℓ_q with $q \neq 1$ and $q \neq p$. The proof for the c_0 case is the same, putting $q = \infty$. If $\ell_q \hookrightarrow X$, then the coordinate sequence of ℓ_q gives us a weakly null sequence (x_i) in X which is M-equivalent to the usual basis of ℓ_q. By a moving hump argument we get a subsequence which is 2-equivalent to a sequence of successive block vectors. If we fix $N \in \mathbb{N}$ then we can choose N vectors, $y_1, \dots, y_N$ from this subsequence such that $(y_i / \|y_i\|)_{i=1}^N$ are $2C$-equivalent to the unit vector basis of ℓ_p^N, but also so that $(y_i)_{i=1}^N$ are M-equivalent to the unit vector basis of ℓ_q^N. By considering $\left\|\sum_{i=1}^N y_i\right\|$, we find that

$$N^{|1/p - 1/q|} \leq 2M^2C,$$

which gives a contradiction for sufficiently large N. $\square$

3.4.4 Spreading models of asymptotic ℓ_p spaces

Suppose that X is an asymptotic ℓ_p space, and that (x_n) is a normalized Brunel-Sucheston sequence in X which generates a spreading model. If (x_n) is a sequence of successive block vectors then it is clear that the spreading model will be isomorphic to ℓ_p. If (x_n) is a weakly null sequence, then by approximating these vectors by block vectors and passing to a subsequence, we again see that the spreading model is isomorphic to ℓ_p (or to c_0 if $p = \infty$). If (x_n) is a weakly convergent sequence converging to x, then by [11, Chapter 1, Section 5, Proposition 5], the spreading models generated by (x_n) and $(x_n - x)$ are isomorphic, so yet again we get spreading models isomorphic to ℓ_p or c_0. This immediately gives the following corollary.

Corollary 3.4.6 *If X is a reflexive asymptotic ℓ_p space, then every spreading model of X is isomorphic to ℓ_p if $1 \leq p < \infty$, and to c_0 if $p = \infty$.*

PROOF. By reflexivity, every bounded sequence in X has a weakly convergent subsequence. Since the spreading model generated by a subsequence of (x_n) is the same as that generated by (x_n), it follows from the discussion above that every spreading model is isomorphic to ℓ_p or c_0. ☐

For a non-reflexive space X, Corollary 3.4.6 need not hold. For example, James' space has spreading models that are isomorphic to James' space (see [11, Chapter 4]), but James' space is asymptotic ℓ_2. However, James' space does contain subspaces isomorphic to ℓ_2. We will show that every spreading model of an asymptotic ℓ_p will at least contain a subspace isomorphic to ℓ_p or to c_0 if $p = \infty$.

From the discussion above we only need to consider the case where (x_n) is a sequence which has no weakly convergent subsequences. By Rosenthal's ℓ_1 theorem, any bounded sequence either has a subsequence which is equivalent to the unit vector basis of ℓ_1 or has a weakly Cauchy subsequence. If (x_n) has a subsequence equivalent to the unit vector basis of ℓ_1, then the spreading model generated by (x_n) will be ℓ_1. In our settings of asymptotic ℓ_p spaces, by Theorem 3.4.5, this can only happen if $p = 1$. Therefore we need to investigate what happens when (x_n) is a normalized weakly Cauchy sequence which has no weakly convergent subsequences.

Let (x_n) be a normalized weakly Cauchy sequence with no weakly convergent subsequences, and suppose that X is an asymptotic ℓ_p space with $1 \leq p < \infty$. Since (x_n) is not norm convergent, we can find $\delta > 0$ such that for all N, there exists $n > m \geq N$ with $\|x_n - x_m\| \geq \delta$. Using this we obtain a sequence of differences $y_i = x_{a_i} - x_{b_i}$ with $\|y_i\| \geq \delta$ for every i and $a_1 < b_1 < a_2 < b_2 < \ldots$. $(y_i)_{i=1}^{\infty}$ is then a weakly null sequence. For a fixed k and scalars $a_1, \ldots, a_k$ we will consider $\left\|\sum_{i=1}^{k} a_i(e_{2i-1} - e_{2i})\right\|$. Given $\varepsilon > 0$, we can find an

N such that

$$\left| \left\| \sum_{i=1}^{k} a_i(e_{2i-1} - e_{2i}) \right\| - \left\| \sum_{i=1}^{k} a_i(x_{n_{2i-1}} - x_{n_{2i}}) \right\| \right| < \varepsilon,$$

whenever $N \leq n_1 < n_2 < \ldots < n_{2k}$. In particular, whenever $q_1 < \ldots < q_k$ and q_1 is sufficiently large, we have

$$\left| \left\| \sum_{i=1}^{k} a_i(e_{2i-1} - e_{2i}) \right\| - \left\| \sum_{i=1}^{k} a_i y_{q_i} \right\| \right| < \varepsilon.$$

Since $(y_i)_{i=1}^{\infty}$ is weakly null we can apply a moving hump argument, and we obtain (passing to a subsequence which for convenience we will again call $(y_i)_{i=1}^{\infty}$) a sequence of successive block vectors $(z_i)_{i=1}^{\infty}$ such that $\|y_i - z_i\| < \delta/2^i$ for each i. Provided we choose q_1 sufficiently large, we get for $q_1 < \ldots < q_k$,

$$\left| \left\| \sum_{i=1}^{k} a_i(e_{2i-1} - e_{2i}) \right\| - \left\| \sum_{i=1}^{k} a_i z_{q_i} \right\| \right| < 2\varepsilon,$$

Since X is asymptotic ℓ_p, provided we choose the q_i suitably spread out (and q_1 sufficiently large), the vectors $(z_{q_i}/\left\|z_{q_i}\right\|)_{i=1}^{k}$ will be C-equivalent to the unit vector basis for ℓ_p^k. Thus,

$$\frac{1}{C}\left(\sum_{i=1}^{k} |a_i|^p \left\|z_{q_i}\right\|^p\right)^{\frac{1}{p}} - 2\varepsilon \leq \left\| \sum_{i=1}^{k} a_i(e_{2i-1} - e_{2i}) \right\| \leq C\left(\sum_{i=1}^{k} |a_i|^p \left\|z_{q_i}\right\|^p\right)^{\frac{1}{p}} + 2\varepsilon.$$

Since $\|z_{q_i}\| \geq \delta/2$ and $\|z_{q_i}\| \leq 2 + \frac{\delta}{2}$ for every i, we obtain on letting $\varepsilon \downarrow 0$, that

$$\frac{\delta}{2C}\left(\sum_{i=1}^{k} |a_i|^p\right)^{\frac{1}{p}} \leq \left\| \sum_{i=1}^{k} a_i(e_{2i-1} - e_{2i}) \right\| \leq C\left(2 + \frac{\delta}{2}\right)\left(\sum_{i=1}^{k} |a_i|^p\right)^{\frac{1}{p}}.$$

The corresponding argument also works when $p = \infty$ to show that a spreading model of an asymptotic ℓ_∞ space always contains a subspace isomorphic to c_0.

3.5 Comparison between asymptotic ℓ_2 spaces and other notions of asymptotic structure

In this section we compare the class of asymptotic ℓ_2 spaces to other classes of spaces which are 'asymptotically' like Hilbert spaces. We will briefly introduce weak Hilbert spaces and asymptotically Hilbertian spaces, and show some of the relationships between these and asymptotic ℓ_2 spaces.

3.5.1 Weak Hilbert Spaces

It is well known that a Banach space is isomorphic to a Hilbert space if and only if it is both type 2 and cotype 2 (see [50]). In this section we describe two weaker properties, namely weak type 2 and weak cotype 2, and use them to define the class of weak Hilbert spaces. These spaces were first introduced by Pisier in [70].

We begin with some definitions. Let $T : X \to Y$ be an operator between two Banach spaces X and Y. We define the *approximation numbers* of T by

$$a_n(T) = \inf\{\|T - S\| : S : X \to Y, \operatorname{rank} S < n\}.$$

Suppose that E is a Banach space, $u : \ell_2^n \to E$ is a linear operator, and γ_n is the canonical Gaussian probability measure on $\mathbb{R}^n$. Then we define,

$$l(u) = \left(\int_{\mathbb{R}^n} \|u(x)\|^2 \, \mathrm{d}\gamma_n(x) \right)^{\frac{1}{2}}.$$

Finally, suppose that $v : E \to \ell_2^n$ is a linear operator. Then, we define

$$l^*(v) = \sup\{\sum_{i=1}^{n} < v^*(e_i), u(e_i) >: l(u) \leq 1\}.$$

We are now able to define what we mean by weak type 2 and weak cotype 2.

Definition 3.5.1 *Let X be a Banach space. We say that X is weak cotype 2 if there exists a constant C_1 such that for all n, and for all operators $u : \ell_2^n \to X$, we have,*

$$\sup_k k^{\frac{1}{2}} a_k(u) \leq C_1 \, l(u).$$

X is said to be weak type 2 if there exists a constant C_2 such that for all n, and all operators $v : X \to \ell_2^n$, we have,

$$\sup_k k^{\frac{1}{2}} a_k(v) \leq C_2 \, l^*(v).$$

It can be shown that type 2 implies weak type 2, and cotype 2 implies weak cotype 2 (see [71, Chapters 10,11]). In view of Kwapien's isomorphic characterization of Hilbert spaces, it is then natural to make the following definition.

Definition 3.5.2 *A Banach space X is called a weak Hilbert space if it is both weak type 2 and weak cotype 2.*

We will now mention some simple properties of weak Hilbert spaces. Many of these appeared in [70] (see also [71]). Suppose that X is a weak Hilbert space. Then all subspaces and quotients of X are also weak Hilbert. If Y is finitely representable in X, then Y is again weak Hilbert. Also, X has type p and cotype q for every $p < 2$ and every $q > 2$. Thus, in a sense, weak Hilbert spaces are close to Hilbert spaces. It is also known that X is weak Hilbert if and only if X^* is weak Hilbert. In fact, as for Hilbert spaces, weak Hilbert spaces are all reflexive and possess the approximation property.

A non-trivial example of a weak Hilbert space is $T^{(2)}$, the 2-convexified Tsirelson space. In fact, $T^{(2)}$ is of weak cotype 2 and type 2. However, it is not isomorphic to a Hilbert space, and does not even contain a subspace isomorphic to ℓ_2. Further examples of weak Hilbert spaces can be found in [2].

3.5.2 Property (H) and Asymptotically Hilbertian spaces

In this section we introduce two properties related to the weak Hilbert property. These arose naturally in W.B. Johnson's proof (see [71]) of the reflexivity of weak Hilbert spaces.

Definition 3.5.3 *We will say that the Banach space X possesses property (H) if for each $\lambda > 1$ there is a constant $K(\lambda)$ such that for any n and any normalized λ-unconditional basic sequence $x_1, \ldots, x_n$ we have*

$$K(\lambda)^{-1} n^{\frac{1}{2}} \leq \left\| \sum_{i=1}^{n} x_i \right\| \leq K(\lambda) n^{\frac{1}{2}}.$$

Definition 3.5.4 *A Banach space X is said to be asymptotically Hilbertian if there exists a constant K such that for every m there exists a finite co-dimensional subspace X_m of X such that every m-dimensional subspace of X_m is K-isomorphic to ℓ_2^m.*

These two properties are related to the weak Hilbert property as follows.

$$\text{Weak Hilbert} \quad \Rightarrow \quad \text{property (H)} \quad \Rightarrow \quad \text{asymptotically Hilbertian.}$$

It should be noted that asymptotically Hilbertian spaces need not have property (H) (see [71, Chapter 14]). However, there is no known example of a space with property (H) which is not weak Hilbert. The proof that weak Hilbert spaces are reflexive proceeds by showing that asymptotically Hilbertian spaces are reflexive. In contrast with weak Hilbert spaces, asymptotically Hilbertian spaces need not have the approximation property (see [14]). Thus, in particular, an asymptotically Hilbertian space may not have a basis.

3.5.3 Relationships between asymptotically Hilbertian spaces and asymptotic ℓ_2 spaces

There is a clear similarity between the definitions of asymptotically Hilbertian spaces and asymptotic ℓ_2 spaces. Indeed, the standard examples of weak Hilbert spaces or asymptotically Hilbertian spaces, such as $T^{(2)}$, are also asymptotic ℓ_2 spaces. It is natural therefore to explore the relationship between these two classes of spaces. Note that since asymptotically Hilbertian spaces need not have a basis, we need to be slightly careful about what we mean by such a space being asymptotic ℓ_2.

Let X be asymptotically Hilbertian and suppose we take $\mathcal{B}(X)$ to be the family of finite co-dimensional subspaces of X. Now consider the vector game described in Section 1.1.2. When the game is played (for any fixed $n \in \mathbb{N}$), the subspace player, **S**, can only improve his chances of winning by choosing at each stage a subspace contained in the finite co-dimensional subspace, X_n, given by the asymptotic Hilbertian property. Therefore, all of the asymptotic spaces, $\{X\}_n$, are K-isomorphic to ℓ_2^n. Hence, an asymptotically Hilbertian space is asymptotic ℓ_2 with respect to the family of finite co-dimensional subspaces.

Now suppose that X is an asymptotically Hilbertian space which has a basis. Since X is reflexive, the basis is shrinking, and therefore the asymptotic structure of X with respect to the finite co-dimensional subspaces and with respect to the tail subspaces is the same (see [59, 1.6.1]). Hence, X is asymptotic ℓ_2 in the sense of Definition 3.1.1. In fact, since any finite co-dimensional subspace contains a subspace which is almost isometric to a tail subspace, it follows that X is stabilized asymptotic ℓ_2.

The converse however is not true. An asymptotic ℓ_2 space need not be asymptotically Hilbertian (e.g. James' space which is asymptotic ℓ_2, but fails to be reflexive). Even a stabilized asymptotic ℓ_2 space, which is reflexive by Corollary 3.3.2, need not be asymptotically Hilbertian. In the remainder of this section we present an example of such a space.

Generalized Schreier families

The Schreier family $\mathcal{S}$, occurs naturally in the definition of Tsirelson's space, and is defined by

$$\mathcal{S} = \{A \subseteq \mathbb{N} : |A| \leq \min A\}.$$

The generalized Schreier families were first introduced in [1]. They have since appeared in various papers, including [6], [4] and [69], where some mixed Tsirelson spaces defined using these families have been studied. The generalized Schreier families are defined recursively by

$$\mathcal{F}_0 = \{\varnothing\} \cup \{\{n\} : n \in \mathbb{N}\},$$

and for $n \geq 0$,

$$\mathcal{F}_{n+1} = \{\varnothing\} \cup \{\cup_{i=1}^{k} A_i : k \in \mathbb{N},\, A_i \in \mathcal{F}_n,\, k \leq A_1 < \ldots < A_k\}.$$

The definition can also be extended to define $\mathcal{F}_\alpha$ for any countable ordinal α, but we shall not need to use this here. Note that $\mathcal{F}_1 = \mathcal{S}$ and that the families $(\mathcal{F}_n)_{n=1}^{\infty}$ are increasing.

The starting point for our example is a mixed Tsirelson space of the form $T[(\mathcal{F}_n, \theta_n)_{n=1}^{\infty}]$. Since $\mathcal{F}_1 = \mathcal{S}$ it follows that such a space is stabilized asymptotic ℓ_1 with respect to its standard basis. Also the basis is 1-unconditional and the space is reflexive. We will need to make use of the following result.

Theorem 3.5.5 ([7, Theorem 1.6]) *Let $(\theta_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $\theta_n \in (0, 1)$ for all n, $\theta_n \downarrow 0$, $\theta_{n+m} \geq \theta_n\theta_m$ for all $n, m \in \mathbb{N}$ and $\lim \theta_n^{1/n} = 1$. Let $X = T[(\mathcal{F}_n, \theta_n)_{n=1}^{\infty}]$. Then for all $\varepsilon > 0$, every infinite dimensional block subspace Y of X contains for every n, a sequence $(y_i)_{i=1}^{n}$ of disjointly supported vectors which are $(1 + \varepsilon)$-equivalent to the usual basis of ℓ_∞^n.*

Let $(\theta_n)_{n=1}^{\infty}$ be any sequence of real numbers satisfying the conditions of Theorem 3.5.5. Since the basis of $T[(\mathcal{F}_n, \theta_n)_{n=1}^{\infty}]$ is 1-unconditional we can form the 2-convexification $X = T^{(2)}[(\mathcal{F}_n, \theta_n)_{n=1}^{\infty}]$. It is then immediate that X is a stabilized asymptotic ℓ_2 space with respect to its natural basis, and the natural basis is again 1-unconditional. However, X is not an asymptotically Hilbertian space. Suppose for a contradiction that X were asymptotically Hilbertian. Then for any $n \in \mathbb{N}$, let Y be a finite co-dimensional subspace of X such that any n-dimensional subspace of Y is K-isomorphic to ℓ_2^n. Let $\varepsilon > 0$ be arbitrary. Then, Y contains a further subspace Z which is $(1+\varepsilon)$-isomorphic to a block subspace of X. Therefore, by Theorem 3.5.5, we can find vectors $(y_i)_{i=1}^{n}$ in Y (corresponding to disjointly supported vectors in Z) which are $(1 + 2\varepsilon)$-equivalent to the usual basis of ℓ_∞^n. However, by the choice of Y, $\operatorname{span}\{y_i : 1 \leq i \leq n\}$ is K-isomorphic to ℓ_2^n. This gives us the required contradiction for sufficiently large n. Hence, X is a stabilized asymptotic ℓ_2 space which is not asymptotically Hilbertian.

Chapter 4

Smooth renormings of asymptotic ℓ_∞ spaces

Contents

In this chapter we will prove a renorming result for asymptotic ℓ_∞ spaces. We will show that if an asymptotic ℓ_∞ has a sufficiently smooth norm then in fact it automatically admits an equivalent analytic norm. Recall that a norm $\|\cdot\|$ is said to be analytic if for each $x \neq 0$, there exists $\delta > 0$ such that if $\|h\| < \delta$, then

$$\|x+h\| = \|x\| + \sum_{n=1}^{\infty} p_n(h),$$

where p_n is a homogeneous polynomial of degree n, and convergence is uniform over all $\|h\| < \delta$.

We will begin by looking at suitably smooth functions on subsets of asymptotic ℓ_∞ spaces and proving that the derivative maps neighbourhoods of each point to a relatively compact set. To do this we will use methods from [36], [37] and [18]. We will then deduce from this that an asymptotic ℓ_∞ space with suitably smooth norm is isomorphically polyhedral. Then using results from [38], the existence of an equivalent analytic norm follows immediately. Finally, we will give a method of constructing some new asymptotic ℓ_∞ spaces, and show that they are isomorphically polyhedral.

Motivation

Recall that a bump function on a Banach space X is a function $\phi : X \to \mathbb{R}$ with bounded non-empty support. It is well known that for a separable space X, the existence of a $\mathscr{C}^1$-smooth bump function is equivalent to the existence of a $\mathscr{C}^1$-smooth norm. However for separable X and $k > 1$, the relationship between $\mathscr{C}^k$-smooth norms and $\mathscr{C}^k$-smooth bump functions is not known. It will follow from the results in this chapter that if X is an asymptotic ℓ_∞ space which has a $\mathscr{C}^2$-smooth norm, then there is a countable sequence $(K_n)_{n=1}^\infty$ of relatively compact sets in X^* which cover a boundary of X (see Definition 4.3.1). If we assume instead the existence of a $\mathscr{C}^2$-smooth bump function then the proof is still valid, except the set we obtain need no longer be a boundary for X. It is not at all clear how to overcome this problem. Therefore, it may be fruitful to look for a space admitting a $\mathscr{C}^2$-smooth bump function but no $\mathscr{C}^2$-smooth norm in the class of asymptotic ℓ_∞ spaces.

4.1 Strong sequential continuity

We begin by introducing the notion of *strong sequential continuity* of a function defined on a subset of a Banach space. This definition is the natural analogue of the one in [18], where the functions considered were defined on the whole of some abelian topological group.

Definition 4.1.1 *Let X and Y be Banach spaces, and $B, U, G \subseteq X$ such that $B + U \subseteq G$. Let $f : G \to Y$ be a function. We say that f is strongly sequentially continuous on B relative to U if for every sequence $(x_n) \subseteq B$ and every weakly null sequence $(h_i) \subseteq U$, one has*

$$\lim_{i\to\infty}\left(\liminf_{n\to\infty}\|f(x_n + h_i) - f(x_n)\|\right) = 0.$$

The usefulness of the above definition hinges on the following proposition which gives an alternative characterization of strong sequential continuity.

Proposition 4.1.2 *Let f, B, U and G be as above. Then the following are equivalent;*

(i) f is strongly sequentially continuous on B relative to U.

(ii) For any weakly null sequence $(h_i) \subseteq U$*, the functions*

$$f_n(x) = \inf\{\|f(x+h_i) - f(x)\| : i \leq n\}$$

converge uniformly to zero on B*.*

Proof. (i) $\Rightarrow$ (ii). Suppose that there is a weakly null sequence $(h_i) \subseteq U$ such that f_n does not converge uniformly to zero on B. Since the sequence (f_n) is a monotone decreasing sequence of functions, this implies that $\exists \varepsilon > 0$ such that $\forall n \in \mathbb{N}$, $\|f_n\restriction_B\|_\infty \geq \varepsilon$. Hence $\forall n \in \mathbb{N}$, there exists $x_n \in B$ such that $\|f(x_n + h_i) - f(x_n)\| \geq \varepsilon$ for all $i \leq n$. Hence for a fixed $i \in \mathbb{N}$, $\liminf_{n\to\infty} \|f(x_n + h_i) - f(x_n)\| \geq \varepsilon$. Thus

$$\lim_{i\to\infty}\left(\liminf_{n\to\infty} \|f(x_n + h_i) - f(x_n)\|\right)$$

can not be zero. Therefore f is not strongly sequentially continuous on B relative to U.

(ii) $\Rightarrow$ (i). Suppose that f is not strongly sequentially continuous on B relative to U. Then there is a sequence $(x_n) \subseteq B$, and a weakly null sequence $(h_i) \subseteq U$ such that

$$\lim_{i\to\infty}\left(\liminf_{n\to\infty} \|f(x_n + h_i) - f(x_n)\|\right)$$

either does not exist, or is non-zero. In either case, by passing to a subsequence of (h_i) we may assume that $\liminf_{n\to\infty} \|f(x_n + h_i) - f(x_n)\| > \varepsilon > 0$ for all $i \in \mathbb{N}$. Then by passing to a subsequence of (x_n) and a Cantor diagonalisation argument, we may suppose that $\|f(x_n + h_i) - f(x_n)\| > \varepsilon$ for all $n \geq i$. Thus $\|f_n\restriction_B\|_\infty \geq \varepsilon$ for all n, so that (f_n) does not converge to 0 uniformly on B. $\square$

We now show that a strongly sequentially continuous function will map certain weak Cauchy sequences to norm convergent sequences. This corresponds to the notion of local weak sequential continuity in [36] and [37].

Lemma 4.1.3 *Let* X*,* Y *be Banach spaces,* $V \subseteq B_X$ *be such that* $V + V - V \subseteq B_X$ *and* $f : B_X \to Y$ *be strongly sequentially continuous on* V *with respect to* $V - V$*. Then* f *maps weak Cauchy sequences* $(x_i) \subseteq V$ *to norm convergent sequences* $(f(x_i))$ *in* Y*.*

In particular, for any δ *such that* $0 < \delta \leq \frac{1}{3}$*, if* $f : B_X \to Y$ *is strongly sequentially continuous on* δB_X *with respect to* $2\delta B_X$*, then* f *maps weak Cauchy sequences* $(x_i) \subseteq \delta B_X$ *to norm convergent sequences* $(f(x_i))$ *in* Y*.*

PROOF. Suppose the result is false. Then there is a weak Cauchy sequence $(x_i) \subseteq V$ such that $(f(x_i))$ is not norm convergent. Let $\mathcal{S} = \{f(x_i) : i \in \mathbb{N}\}$. If $\mathcal{S}$ is relatively compact in Y, then $\mathcal{S}$ has at least two accumulation points, and so there exists $\varepsilon > 0$ and two subsequences (y_n), (z_m) of (x_i) such that $\|f(y_n) - f(z_m)\| \geq \varepsilon$ for all $n, m \in \mathbb{N}$. If $\mathcal{S}$ is not relatively compact in Y, then there exists an ε-separated sequence $(f(u_i))$ in $\mathcal{S}$. Taking $y_n = u_{2n}$ and $z_m = u_{2m+1}$, gives two sequences such that $\|f(y_n) - f(z_m)\| \geq \varepsilon$ for all $n, m \in \mathbb{N}$. Hence in either case for any $n, m, i \in \mathbb{N}$

$$\varepsilon \leq \|f(y_n) - f(z_m)\| \leq \|f(y_n) - f(y_n + (z_m - x_i))\| + \|f(z_m + (y_n - x_i)) - f(z_m)\| .$$

By Ramsey's theorem for triples we have two possibilities;

(i) $\forall i, n, m \in \mathbb{N}$, $m < i < n$, $\|f(y_n) - f(y_n + (z_m - x_i))\| \geq \varepsilon/2$.

(ii) $\forall i, n, m \in \mathbb{N}$, $m < i < n$, $\|f(z_m) - f(z_m + (y_n - x_i))\| \geq \varepsilon/2$.

If (i) holds then, in particular

$$\liminf_{n\to\infty} \|f(y_n) - f(y_n + (z_{i-1} - x_i))\| \geq \varepsilon/2,$$

for all $i \geq 1$. But $(z_{i-1} - x_i) \subseteq V - V$ is weakly null, and so this contradicts the fact that f is strongly sequentially continuous on V relative to $V - V$. If (ii) holds then we have in particular that $\|f(z_0) - f(z_0 + (y_{i+1} - x_i))\| \geq \varepsilon/2$ for all $i \in \mathbb{N}$. But this contradicts the fact that f is sequentially continuous at z_0 (with respect to sequences in $V - V$). Hence the result is proved. The final conclusion follows immediately from the first by taking $V = \delta B_X$. $\square$

4.2 Smooth functions on asymptotic ℓ_∞ spaces

We now turn to the particular case in which we are interested. We look at smooth functions f defined on the unit ball of an asymptotic ℓ_∞ space. To be more precise, we require that f is Fréchet differentiable with uniformly continuous derivative on B_X. We will first define precisely what we mean by this.

Definition 4.2.1 *Given a continuous function f from a metric space (X_1, d_1) into a metric space (X_2, d_2), we define the modulus of continuity to be the increasing function $\omega(\delta)$: $[0, \infty) \to [0, \infty]$ given by*

$$\omega(\delta) = \sup\{d_2(f(x_1), f(x_2)) \ : \ d_1(x_1, x_2) \leq \delta\} .$$

We say that f is uniformly continuous if

$$\lim_{\delta \to 0} \omega(\delta) = 0.$$

We now state the main result that we will need to use. The important point is that the minimum size of the n depends only upon the modulus of continuity ω.

Lemma 4.2.2 ([36, Lemma 5]) *Let $\varepsilon > 0$, f be a real function on $B_{c_0^m}$ with uniformly continuous derivative (with modulus of continuity $\omega(\delta)$) and such that $\sup_{B_{c_0^m}} \|f'\|_1 \leq \omega(2)$. Let $v \in B_{c_0^m}$ and $(u_i)_{i=1}^n$ be a block sequence such that $v + u_i \in B_{c_0^m}$. If n is large enough, then $\min_i |f(v+u_i) - f(v)| < \varepsilon$.*

We now show that if f is a suitably smooth real-valued function on the unit ball of an asymptotic ℓ_∞ space then f is strongly sequentially continuous on a smaller ball. It then follows that weak Cauchy sequences in a smaller ball must get mapped to norm convergent sequences.

Proposition 4.2.3 *Let X be an asymptotic ℓ_∞ space with asymptotic constant $C \geq 1$, and $f : B_X \to \mathbb{R}$ be Fréchet differentiable with uniformly continuous derivative on B_X. Let $\delta_1 > 0$, $\delta_2 > 0$ be such that $\delta_1 + C\delta_2 \leq 1$. Then f is strongly sequentially continuous on $\delta_1 B_X$ relative to $\delta_2 B_X$.*

PROOF. Suppose the result is false. Then there exists a weakly null sequence $(h_i) \subseteq \delta_2 B_X$ such that $f_n(x) = \inf\{|f(x+h_i) - f(x)| : i \leq n\}$ does not converge to 0 uniformly on $\delta_1 B_X$. Thus there is an $\varepsilon > 0$ such that $\|f_n\|_\infty \geq \varepsilon$ for all $n \in \mathbb{N}$. Notice that (h_i) does not converge to 0 in norm. By a perturbation argument and passing to a subsequence, we may assume that (h_i) are successive block vectors and $\|h_i\| = \delta$ (where $0 < \delta \leq \delta_2$) for all $i \in \mathbb{N}$. Now fix $N \in \mathbb{N}$. Choose $y_1 < y_2 < \ldots < y_N$ from the sequence h_i which are sufficiently spread out with respect to the asymptotic structure of X. Suppose that $y_N = h_k$. Then as $\|f_k\|_\infty \geq \varepsilon$, there exists $x \in \delta_1 B_X$ such that $|f(x+y_i) - f(x)| \geq \varepsilon$ for $i = 1, 2, \ldots, N$. But the y_i are $C\delta$ equivalent to the unit vector basis of c_0^N i.e. for any choice of scalars $(a_i)_{i=1}^N$,

$$\frac{\delta}{C} \max_{i=1,\ldots,n} |a_i| \leq \left\| \sum_{i=1}^N a_i y_i \right\| \leq \delta C \max_{i=1,\ldots,n} |a_i| \leq \delta_2 C \max_{i=1,\ldots,n} |a_i|.$$

Define a function $g : B_{c_0^N} \to \mathbb{R}$ by

$$g((b_1, \ldots, b_N)) = f\left(x + \sum_{i=1}^N b_i y_i\right).$$

Then g is differentiable with uniformly continuous derivative (with modulus of continuity $C\delta_2\omega_f$) on $B_{c_0^N}$ and $|g(e_i)-g(0)| \geq \varepsilon$ for $i = 1, \dots N$. For N sufficiently large this contradicts Lemma 4.2.2. □

Corollary 4.2.4 *Let X be an asymptotic ℓ_∞ space with asymptotic constant $C \geq 1$. Let $f : B_X \to \mathbb{R}$ be Fréchet differentiable with uniformly continuous derivative on B_X. Let $\delta > 0$ such that $\delta \leq \frac{1}{1+2C}$. Then f maps weak Cauchy sequences $(x_i) \subseteq \delta B_X$ into norm convergent sequences $(f(x_i))$ in Y.*

Proof. By Proposition 4.2.3 with $\delta_1 = \delta$ and $\delta_2 = 2\delta$, f is strongly sequentially continuous on δB_X relative to $2\delta B_X$. Since $C \geq 1$, we have that $\delta \leq \frac{1}{1+2C} \leq \frac{1}{3}$. Thus, by the last part of Lemma 4.1.3, f maps weak Cauchy sequences $(x_i) \subseteq \delta B_X$ into norm convergent sequences $(f(x_i))$ in $\mathbb{R}$. □

The following lemma relates weak sequential continuity with compactness properties of the derivative.

Lemma 4.2.5 ([37, Lemma 5]) *Let X be a Banach space such that $\ell_1 \not\hookrightarrow X$. Let U be an open, bounded and convex subset of X and let f be a real valued function with uniformly continuous derivative on U. Then the following are equivalent;*

(i) f is weakly sequentially continuous on U i.e. f maps weak Cauchy sequences in U to norm convergent sequences in $\mathbb{R}$.

(ii) $f'(U)$ is relatively compact in X^*.

We can combine Corollary 4.2.4 and Lemma 4.2.5 to give the following result.

Corollary 4.2.6 *Let X be an asymptotic ℓ_∞ space with asymptotic constant $C \geq 1$. Let $f : B_X \to \mathbb{R}$ be Fréchet differentiable with uniformly continuous derivative on B_X. Then $f'(\frac{1}{1+2C}B_X)$ is relatively compact in X^*.*

Proof. Let U be the open ball in X of radius $\frac{1}{1+2C}$ centred at 0, and let B be the closed ball of the same radius. Since X is an asymptotic ℓ_∞ space, $\ell_1 \not\hookrightarrow X$ by Theorem 3.4.5. Therefore, by Corollary 4.2.4 and Lemma 4.2.5, $f'(U)$ is relatively compact in X^*. As f' is continuous on B_X, we have

$$f'(B) = f'(\overline{U}) \subseteq \overline{f'(U)}.$$

Thus $f'(B)$ is relatively compact in X^*, and we are done. □

4.3 A renorming result for asymptotic ℓ_∞ spaces

We know that an asymptotic ℓ_∞ space has separable dual. Thus by [17, Theorem 5.3] we know that an asymptotic ℓ_∞ space always admits an equivalent Fréchet differentiable norm. In this section we address the question of higher order smoothness. To be more precise, we will show that if the norm on an asymptotic ℓ_∞ space X is suitably smooth, namely Fréchet differentiable (on $X \setminus \{0\}$ as is usual in renorming theory) with locally uniformly continuous derivative, then in fact there is an equivalent norm on X which is analytic. To do this we will use results and methods from [38], [35] and [16]. We start with some definitions.

Definition 4.3.1 *We will say that a set $B \subseteq X^*$ is a boundary for X, if for every $x \in X$, there is an $f \in B$ such that $\|x\| = |f(x)|$.*

Definition 4.3.2 *A Banach space X is isomorphically polyhedral if there is an equivalent polyhedral norm on X. A norm is said to be polyhedral if every finite dimensional section of the unit ball (with respect to the given norm) is a polytope.*

Isomorphically polyhedral spaces were studied first by Fonf in [24] and [25]. In particular, he proved that a separable isomorphically polyhedral Banach space is c_0-saturated and has a separable dual. Consequently, a separable isomorphically polyhedral space can not be reflexive.

We will start by showing that an asymptotic ℓ_∞ space X (with suitably smooth norm) admits an equivalent norm which has a countable boundary. To do this we will need the following theorem giving an equivalent condition for the existence of a norm with a countable boundary. This result was first proved by Fonf, but only published in Russian, and was later proved independently by Hájek.

Theorem 4.3.3 ([35, Theorem 1]) *Let $(X, \|\cdot\|)$ be a normed space. Then the following are equivalent;*

(i) X admits an equivalent norm having a countable boundary.

(ii) X admits an equivalent norm with a boundary B, such that there is a sequence $(K_n)_{n\in\mathbb{N}}$ of norm compact sets in X^* satisfying $B \subseteq \bigcup_{n\in\mathbb{N}} K_n$.

We are now able to prove the following proposition.

Proposition 4.3.4 *Let X be an asymptotic ℓ_∞ space whose norm $\|\cdot\|$ is Fréchet differentiable with locally uniformly continuous derivative. Then X admits an equivalent norm having a countable boundary.*

Proof. We verify condition (ii) in Theorem 4.3.3. We define a function $f : X \setminus \{0\} \to \mathbb{R}$ by $f(x) = \|x\|$, and let $\delta : X \setminus \{0\} \to X^*$ denote the derivative of f, which by assumption is locally uniformly continuous. Now notice that for any $x \in S_X$, the functional $\delta x \in S_X^*$ is a functional (in fact, the unique functional by Smŭlyan's Lemma) such that $(\delta x)(x) = 1$. Therefore B $:= \delta(S_X)$ is a boundary for X. Thus we want to show that B is covered by a countable union of norm compact sets in X^*. This will follow from Corollary 4.2.6. Indeed, for every $x \in S_X$, we obtain by Corollary 4.2.6 an open ball U_x centred at x whose image $\delta(U_x)$ is relatively compact in X^*. Since X is separable, by the Lindelöf property, we can obtain a countable sub-covering $(U_i)_{i\in\mathbb{N}}$ of S_X consisting of these balls. Then

$$B = \delta(S_X) \subseteq \delta\left(\bigcup_{i\in\mathbb{N}} U_i\right) = \bigcup_{i\in\mathbb{N}} \delta(U_i) \subseteq \bigcup_{i\in\mathbb{N}} \overline{\delta(U_i)}.$$

Thus condition (ii) of Theorem 4.3.3 is verified and therefore X admits an equivalent norm which has a countable boundary. ☐

The following result from [24] shows that the existence of a norm with countable boundary is equivalent to being isomorphic polyhedral.

Theorem 4.3.5 *Let X be a separable Banach space. Then X is isomorphically polyhedral if and only if X admits an equivalent norm with a countable boundary.*

The following theorem shows that an isomorphically polyhedral space admits an equivalent analytic norm.

Theorem 4.3.6 ([16, Theorem 3.1]) *Every separable isomorphically polyhedral Banach space X admits an equivalent analytic norm.*

In conclusion, we have proved the following theorem.

Theorem 4.3.7 *Let X be an asymptotic ℓ_∞ space whose norm $\|\cdot\|$ is Fréchet differentiable with locally uniformly continuous derivative. Then X admits an equivalent analytic norm.*

4.4 Generalized Orlicz spaces which are asymptotic ℓ_∞ spaces

In this section we will give a method of constructing new asymptotic ℓ_∞ spaces. We will then show that these spaces are all isomorphically polyhedral. It follows immediately that they admit an equivalent analytic norm. The known examples of asymptotic ℓ_∞ spaces are c_0 and T^* the dual of Tsirelson's space (and other such Tsirelson type spaces). It is clear that c_0 has a countable boundary, is isomorphically polyhedral, and so admits an equivalent analytic norm. There are also explicit constructions of such norms (see e.g. [19]). On the other hand, T^* is not isomorphically polyhedral since isomorphically polyhedral spaces are c_0-saturated. We will give a necessary and sufficient condition for the spaces we construct not to be isomorphic to c_0. As these spaces will be isomorphically polyhedral, they will also not be isomorphic to T^*.

The spaces we are going to construct are generalized Orlicz sequences spaces (see Section 2.2.7). We will suppose that the norm is the Luxemburg norm. For convenience we will suppose that all the Orlicz functions we use are normalized i.e. $\phi(1) = 1$. This condition guarantees that the coordinate vectors each have norm 1.

4.5 A sufficient condition for h_Φ to be an asymptotic ℓ_∞ space

Our first proposition in this section is a simple characterization of those generalized Orlicz spaces which are stabilized asymptotic ℓ_∞ spaces.

Proposition 4.5.1 *Let $\Phi = (\phi_j)_{j=1}^\infty$ be a sequence of normalized Orlicz functions. Then the following are equivalent;*

(i) h_Φ is stabilized asymptotic ℓ_∞ (with respect to its standard basis).

(ii) There exists $\lambda \geq 1$ such that for all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $N \leq p_1 \leq q_1 < p_2 \leq \ldots < p_n \leq q_n$, and $\sum_{j=p_i}^{q_i} \phi_j(|a_j|) \leq 1$ for $i = 1, \ldots, n$, then

$$\sum_{i=1}^{n} \sum_{j=p_i}^{q_i} \phi_j\left(\frac{|a_j|}{\lambda}\right) \leq 1.$$

(iii) There exists $\lambda \geq 1$ such that for all $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ such that whenever $N \leq p \leq q$ and $\sum_{j=p}^{q} \phi_j(|a_j|) \leq 1$, then

$$\sum_{j=p}^{q} \phi_j\left(\frac{|a_j|}{\lambda}\right) \leq \frac{1}{n}.$$

Proof. (i) $\Leftrightarrow$ (ii). By definition h_Φ is a stabilized asymptotic ℓ_∞ space if and only if there exists $\lambda \geq 1$ such that for every $n \in \mathbb{N}$, there exists $N \in \mathbb{N}$ so that any n successive normalized block vectors, $N \leq x_1 < \ldots < x_n$, satisfy

$$\left\| \sum_{i=1}^{n} b_i x_i \right\| \leq \lambda \max_i |b_i|.$$

The required lower bound is trivial by the 1-unconditionality of the basis in a generalized Orlicz space. The above condition is clearly equivalent by homogeneity to

$$\left\| \sum_{i=1}^{n} b_i x_i \right\| \leq \lambda,$$

whenever $|b_i| \leq 1$ for each i. This can be rewritten as

$$\left\| \sum_{i=1}^{n} y_i \right\| \leq \lambda,$$

whenever $N \leq y_1 < \ldots < y_n$ and $\|y_i\| \leq 1$ for each i. If we write out the block vectors as $y_i = \sum_{j=p_i}^{q_i} a_j e_j$, then the second condition in the statement of this proposition is precisely what we obtain when we rewrite the above condition using the definition of the norm in a generalized Orlicz space.
(ii) $\Leftrightarrow$ (iii). It is clear that (iii) $\Rightarrow$ (ii). We prove the converse statement. Suppose that (ii) holds but (iii) does not. Then there exists $n \in \mathbb{N}$ such that for any $M \in \mathbb{N}$ there exist $M \leq p \leq q$ and scalars (a_j) such that $\sum_{j=p}^{q} \phi_j(|a_j|) \leq 1$ and $\sum_{j=p}^{q} \phi_j\left(\frac{|a_j|}{\lambda}\right) > \frac{1}{n}$. Let $N \in \mathbb{N}$ be the natural number given by (ii) and use the above result n times to find $N \leq p_1 \leq q_1 < p_2 \leq \ldots < p_n \leq q_n$ and scalars (a_j), such that $\sum_{j=p_i}^{q_i} \phi_j(|a_j|) \leq 1$ and $\sum_{j=p_i}^{q_i} \phi_j\left(\frac{|a_j|}{\lambda}\right) > \frac{1}{n}$ for $i = 1, \ldots, n$. But then, $\sum_{i=1}^{n} \sum_{j=p_i}^{q_i} \phi_j\left(\frac{|a_j|}{\lambda}\right) > 1$, which contradicts (ii). $\square$

In fact, the stabilized property is not important in the above proposition. We show in the next proposition that the only possible asymptotic ℓ_∞ spaces in the class of generalized Orlicz spaces are *stabilized* asymptotic ℓ_∞.

Proposition 4.5.2 *If h_Φ is asymptotic ℓ_∞ (with respect to its standard basis), then h_Φ is actually stabilized asymptotic ℓ_∞ (with respect to its standard basis).*

PROOF. Suppose that h_Φ is asymptotic ℓ_∞ with asymptotic constant λ. We will show that (iii) holds in Proposition 4.5.1. Suppose for a contradiction that (iii) does not hold. Then, as before, there exists $n \in \mathbb{N}$ such that for any $M \in \mathbb{N}$ there exist $M \leq p \leq q$ and scalars (a_j) such that $\sum_{j=p}^{q} \phi_j(|a_j|) \leq 1$ and $\sum_{j=p}^{q} \phi_j\left(\frac{|a_j|}{\lambda}\right) > \frac{1}{n}$. Let F_n denote the function given in Definition 3.1.1 corresponding to n. Then, applying the above result n times, we find $(p_i)_{i=1}^{n}$, $(q_i)_{i=1}^{n}$ and scalars (a_i) such that $F_n(0) \leq p_1 \leq q_1 < p_2 \leq q_2 < p_n \ldots \leq q_n$ with $p_{i+1} \geq F_n(q_i)$ for each i and such that both $\sum_{j=p_i}^{q_i} \phi_j(|a_j|) \leq 1$ and $\sum_{j=p_i}^{q_i} \phi_j(|a_j|/\lambda) > \frac{1}{n}$ for each $i = 1, \ldots, n$. In particular $\sum_{i=1}^{n} \sum_{j=p_i}^{q_i} \phi_j(|a_j|/\lambda) > 1$. On the other hand, by the definition of the generalized Orlicz space norm, if we write $x_i = \sum_{j=p_i}^{q_i} a_j e_j$, then $\|x_i\| \leq 1$ for each i. By the choice of the p_i and q_i, $(x_i/\|x_i\|)_{i=1}^n$ is λ-equivalent to the usual basis of ℓ_∞^n. Therefore, $\|\sum_{i=1}^n x_i\| \leq \lambda$, which implies $\sum_{i=1}^n \sum_{j=p_i}^{q_i} \phi_j(|a_j|/\lambda) \leq 1$. This gives us the required contradiction. □

We define a sequence of real numbers $(\psi_\lambda(j))_{j=1}^\infty$ by

$$\psi_\lambda(j) = \inf\left\{\frac{\phi_j(\lambda t)}{\phi_j(t)} : t > 0\right\}.$$

Note that for any $j \in \mathbb{N}$ and any $t \geq 0$, $\psi_\lambda(j)\phi_j(t) \leq \phi_j(\lambda t)$. We are now able to give a simple sufficient condition on Φ for h_Φ to be a stabilized asymptotic ℓ_∞ space.

Proposition 4.5.3 *Let $\Phi = (\phi_j)_{j=1}^\infty$ be a sequence of normalized Orlicz functions. Suppose that for some $\lambda > 1$, $\psi_\lambda(j) \to \infty$ as $j \to \infty$. Then h_Φ is a stabilized asymptotic ℓ_∞ space.*

Proof. We show that (iii) holds in Proposition 4.5.1. Let $n \in \mathbb{N}$, and choose N sufficiently large that $\psi_\lambda(j) \geq n$ for every $j \geq N$. Suppose that $N \leq p \leq q$, and $\sum_{j=p}^q \phi_j(|a_j|) \leq 1$. Then,

$$\begin{aligned}\sum_{j=p}^q \phi_j\left(\frac{|a_j|}{\lambda}\right) &\leq \sum_{j=p}^q \frac{1}{\psi_\lambda(j)}\phi_j\left(|a_j|\right)\\ &\leq \frac{1}{n}\sum_{j=p}^q \phi_j(|a_j|)\\ &\leq \frac{1}{n}.\end{aligned}$$

Therefore, by Proposition 4.5.1, h_Φ is a stabilized asymptotic ℓ_∞ space. □

Example 4.5.4 *If we define $\alpha_j = \log_2(j+1)$, and we define $\phi_j(t) = t^{\alpha_j}$ for $t \geq 0$. Then the functions $(\phi_j)_{j=1}^\infty$ satisfy the conditions of Proposition 4.5.3 with $\lambda = 2$ and $\psi_2(j) = j+1$ for every $j \in \mathbb{N}$.*

4.6 A necessary and sufficient condition for $h_\Phi \sim c_0$

Proposition 4.6.1 *Suppose that $\Phi = (\phi_j)_{j=1}^\infty$ is a sequence of normalized Orlicz functions such that for some $t_0 > 0$,*

$$\sum_{j=1}^\infty \phi_j(t_0) \leq 1.$$

Then h_Φ is isomorphic to c_0.

PROOF. Suppose $x \notin c_0$. Then there exists an $\varepsilon > 0$ such that $|x_j| \geq \varepsilon$ infinitely often. Hence $\phi_j(|x_j|/\varepsilon) \geq \phi_j(1) = 1$ infinitely often, so that $\sum_j \phi_j(|x_j|/\varepsilon)$ does not converge. Therefore $x \notin h_\Phi$ i.e. $h_\Phi \subseteq c_0$.

Now suppose that $x \in c_0$. Let $\rho > 0$ be fixed. Then there exists a J such that $j \geq J$ implies that $|x_j| \leq \rho t_0$. Thus $\phi_j(|x_j|/\rho) \leq \phi_j(t_0)$ for $j \geq J$. Hence $\sum_j \phi_j(|x_j|/\rho)$ converges, and so $x \in h_\Phi$. Hence, as sets, $h_\Phi = c_0$. Now consider the identity map $I : h_\Phi \to c_0$. It is immediate from the definition of the norm that $\|x\| \geq \|x\|_\infty$ for all $x \in h_\Phi$, so that I is a continuous bijection. By the open mapping theorem, I is an isomorphism. □

Remark 4.6.2 *Notice that the condition $\sum_{j=1}^{\infty} \phi_j(t_0) \leq 1$ for some $t_0 > 0$ is equivalent to the existence of $t_1 > 0$ such that $\sum_{j=1}^{\infty} \phi_j(t_1) < \infty$. This follows from the monotonicity of the ϕ_j's, their continuity and since $\phi_j(0) = 0$. It is also clear that any such numbers t_0 and t_1 must be in $(0,1)$, since $\phi_j(1) = 1$.*

The converse of Proposition 4.6.1 is also true. To prove this we will need the following proposition, concerning the 'uniqueness' of the basis in c_0.

Proposition 4.6.3 ([53, Proposition 2.b.9]) *Every normalized unconditional basis in c_0 is equivalent to the unit vector basis of c_0.*

Proposition 4.6.4 *Suppose that $\Phi = (\phi_j)_{j=1}^{\infty}$ is a sequence of Orlicz functions such that for all $t \in (0,1)$,*

$$\sum_{j=1}^{\infty} \phi_j(t) > 1.$$

Then h_Φ is not isomorphic to c_0.

PROOF. Suppose that $\eta > 1$ is arbitrary. Then because $\sum_j \phi_j(\frac{1}{\eta}) > 1$, there exists $N \in \mathbb{N}$ such that $\sum_{j=1}^{N} \phi_j(\frac{1}{\eta}) > 1$. Hence $\left\|\sum_{i=1}^{N} e_i\right\| > \eta$. We will show that this can not happen if $h_\Phi \sim c_0$.

Suppose that $h_\Phi \sim c_0$, and let $T : h_\Phi \to c_0$ be an isomorphism such that for all $x \in h_\Phi$,

$$\frac{1}{M_1} \|x\| \leq \|Tx\|_\infty \leq M_1 \|x\|.$$

Define $y_i = Te_i / \|Te_i\|_\infty$. Then $(y_i)_{i=1}^{\infty}$ is a normalized unconditional basis for c_0, and hence by Proposition 4.6.3, is equivalent to the unit vector basis of c_0. Therefore there exists an M_2 such that for any scalars a_i,

$$\frac{1}{M_2} \sup_i |a_i| \leq \left\| \sum_{i=1}^{\infty} a_i y_i \right\|_\infty \leq M_2 \sup_i |a_i|.$$

Thus for any $n \in \mathbb{N}$,

$$\begin{aligned}\left\|\sum_{i=1}^n e_i\right\| &\leq M_1 \left\|\sum_{i=1}^n \|Te_i\|_\infty y_i\right\|_\infty \\ &\leq M_1 M_2 \max_i \|Te_i\|_\infty \\ &\leq M_1^2 M_2.\end{aligned}$$

Since η was arbitrary, we have a contradiction, and so $h_\Phi \not\sim c_0$. $\square$

4.7 New examples of asymptotic ℓ_∞ spaces

We can now construct some new examples of stabilized asymptotic ℓ_∞ spaces using the results of Section 4.5 and Section 4.6. These will be from the class of generalized Orlicz spaces and not isomorphic to c_0. However, they will contain subspaces isomorphic to c_0, and in fact, we will later see that they are c_0-saturated. In particular, this implies that they can not be isomorphic to T^*, the dual of Tsirelson's space.

We begin with a simple observation. Let $\Phi = (\phi_j)_{j=1}^\infty$ be a sequence of normalized Orlicz functions. Suppose $\lambda > 1$ and that $\psi_\lambda(j) \to \infty$ as $j \to \infty$. Then by Proposition 4.5.3, h_Φ is a stabilized asymptotic ℓ_∞ space. Now choose an increasing sequence $(j_k)_{k=1}^\infty$ of natural numbers, such that $\psi_\lambda(j_k) \geq k$ for every $k \in \mathbb{N}$. Then,

$$\phi_{j_k}\left(\frac{1}{\lambda^2}\right) \leq \frac{1}{\psi_\lambda(j_k)}\phi_{j_k}\left(\frac{1}{\lambda}\right) \leq \frac{1}{\psi_\lambda(j_k)^2}\phi_{j_k}(1) = \frac{1}{\psi_\lambda(j_k)^2} \leq \frac{1}{k^2}.$$

Thus, $\sum_{k=1}^\infty \phi_{j_k}\left(\frac{1}{\lambda^2}\right) < \infty$. Therefore by Proposition 4.6.1 (and Remark 4.6.2), it follows that $[e_{j_k}]_{k=1}^\infty$ is isomorphic to c_0. As we commented above, this implies immediately that h_Φ is not isomorphic to T^*.

It remains to show that we can find a sequence of Orlicz functions satisfying the above conditions, but such that h_Φ is not isomorphic to c_0. Consider Example 4.5.4, where $\phi_j(t) = t^{\alpha_j}$, with $\alpha_j = \log_2(j+1)$. These functions satisfy the conditions of Proposition 4.5.3 with $\lambda = 2$ and $\psi_2(j) = (j+1)$. It follows similarly to the above argument that for these ϕ_j, h_Φ is isomorphic to c_0. However we show now that it is easy to obtain from Φ a sequence of Orlicz functions Γ such that h_Γ is not isomorphic to c_0.

Suppose that Φ is any sequence of *non-degenerate* Orlicz functions such that for some $\lambda > 1$, $\psi_\lambda(j)$ converges to infinity as $j \to \infty$. We now choose a strictly increasing sequence of integers $(n_j)_{j=0}^\infty$ recursively as follows. We take $n_0 = 0$ and then iteratively choose n_j such that $(n_j - n_{j-1})\phi_j(1/j) > 1$. Now define $\gamma_k(t) = \phi_j(t)$ when $n_{j-1} < k \leq n_j$, and $\tilde{\psi}_\lambda(k) = \inf\{\gamma_k(\lambda t)/\gamma_k(t) : t > 0\}$. Then $\tilde{\psi}_\lambda(k) = \psi_\lambda(j)$ whenever $n_{j-1} < k \leq n_j$, and therefore, $\tilde{\psi}_\lambda(k) \to \infty$ as $k \to \infty$. Writing $\Gamma = (\gamma_k)_{k=1}^\infty$, Proposition 4.5.3 tells us that h_Γ is a stabilized asymptotic ℓ_∞ space. We now show that h_Γ is not isomorphic to c_0. Suppose that $0 < t < 1$. Choose an integer j such that $1/j < t$. Then

$$\sum_{k=1}^\infty \gamma_k(t) \geq \sum_{k=1}^\infty \gamma_k\left(\frac{1}{j}\right) \geq \sum_{k=n_{j-1}+1}^{n_j} \gamma_k\left(\frac{1}{j}\right) = (n_j - n_{j-1})\,\phi_j\left(\frac{1}{j}\right) > 1.$$

Thus, by Proposition 4.6.4, h_Γ is not isomorphic to c_0. In conclusion, we have shown that given any sequence Φ of non-degenerate Orlicz functions, satisfying the hypotheses of Proposition 4.5.3, we can extend Φ to a sequence Γ such that h_Γ is a stabilized asymptotic ℓ_∞ space which is not isomorphic to c_0 or T^*.

4.8 Isomorphically polyhedral generalized Orlicz spaces

In this section we show that if a generalized Orlicz space h_Φ is stabilized asymptotic ℓ_∞ then h_Φ is isomorphically polyhedral. As we have already mentioned, this implies in particular that these spaces admit equivalent analytic norms and are c_0-saturated.

We will need to make use of the following characterization of isomorphically polyhedral spaces given in [52].

Theorem 4.8.1 ([52, Theorem 3]) *Let (e_n) be a shrinking basis of a Banach space $(E, \|\cdot\|)$. Then the following are equivalent;*

(i) E is isomorphically polyhedral.

(ii) There exists an equivalent norm $|||\cdot|||$ on E such that (e_n) is a monotone basis with respect to $|||\cdot|||$, and for all $\sum_n a_n e_n \in E$, there exists $m \in \mathbb{N}$ such that

$$\left|\left|\left|\sum_{n=1}^\infty a_n e_n\right|\right|\right| = \left|\left|\left|\sum_{n=1}^m a_n e_n\right|\right|\right|.$$

Using this characterization, we will prove the result that we stated at the beginning of this section.

Theorem 4.8.2 *Let $\Phi = (\phi_j)_{j=1}^\infty$ be a sequence of Orlicz functions. If h_Φ is stabilized asymptotic ℓ_∞ then h_Φ is isomorphically polyhedral.*

Proof. Throughout this proof we will denote by $\lambda > 1$ the asymptotic constant of h_Φ. Observe that by Proposition 3.3.1 the basis of an asymptotic ℓ_∞ space is automatically shrinking, and so we may make use of Theorem 4.8.1.

Given $n \in \mathbb{N}$, let $f(n)$ be the natural number given by condition (iii) of Proposition 4.5.1. In other words, whenever $f(n) \leq p \leq q$ and $\sum_{j=p}^{q} \phi_j(|a_j|) \leq 1$, then $\sum_{j=p}^{q} \phi_j\left(|a_j|/\lambda\right) \leq \frac{1}{n}$. Without loss of generality we may suppose that $(f(n))_{n=1}^{\infty}$ is a strictly increasing sequence with $f(1) = 1$. We define also $g(m) = \sup\{n : f(n) \leq m\}$. Then $(g(m))_{m=1}^{\infty}$ is an increasing sequence of natural numbers converging to infinity as $m \to \infty$. It follows from these definitions that if $\sum_{j=m}^{m+k} \phi_j(|a_j|) \leq 1$, then $\sum_{j=m}^{m+k} \phi_j\left(|a_j|/\lambda\right) \leq 1/g(m)$. We now choose a sequence $(\eta_k)_{k=1}^{\infty}$ decreasing to 1 such that for each k,

$$\eta_k > \left(1 - \frac{1}{g(k+1)}\right)^{-1}.$$

(If $g(k+1) = 1$ (i.e. $k+1 < f(2)$), then we may take $\eta_k = 3$.)
We define a new norm $|||\cdot|||$ on h_Φ by,

$$|||(a_n)||| = \sup_k \eta_k \left\|(a_1, a_2, \ldots, a_k, 0, \ldots,)\right\|.$$

Notice that $\eta_k \left\|(a_1, a_2, \ldots, a_k, 0, \ldots,)\right\| \to \|(a_n)\|$ as $k \to \infty$. Thus it is clear that

$$\|(a_n)\| \leq |||(a_n)||| \leq \eta_1 \left\|(a_n)\right\|.$$

It is also easy to check that $(e_i)_{i=1}^{\infty}$ forms a monotone basis with respect to $|||\cdot|||$. We aim to show that $|||\cdot|||$ satisfies (ii) in Theorem 4.8.1.
Suppose that $(a_n) \in h_\Phi$. We claim that there exists a k such that

$$\|(a_n)\| \leq \eta_k \left\|(a_1, \ldots, a_k, 0, \ldots)\right\|.$$

Otherwise, for all k, $\|(a_n)\| > \eta_k \|(a_1, \ldots, a_k, 0, \ldots)\|$. By homogeneity, we may assume that $\|(a_n)\| = 1$. Hence $\sum_{j=1}^{\infty} \phi_j(|a_j|) = 1$ and $\sum_{j=1}^{k} \phi_j(\eta_k |a_j|) \leq 1$. Choose an integer $m \geq f(2)$ such that

$$\|(0, \ldots, 0, a_m, a_{m+1}, \ldots)\| \leq \lambda^{-1}.$$

This implies that $\sum_{j=m}^{\infty} \phi_j(\lambda |a_j|) \leq 1$, and therefore $\sum_{j=m}^{\infty} \phi_j(|a_j|) \leq 1/g(m)$. Thus,

$$\begin{aligned} 1 = \sum_{j=1}^{\infty} \phi_j(|a_j|) &= \sum_{j=1}^{m-1} \phi_j(|a_j|) + \sum_{j=m}^{\infty} \phi_j(|a_j|) \\ &\leq \frac{1}{\eta_{m-1}} \sum_{j=1}^{m-1} \phi_j(\eta_{m-1}|a_j|) + \frac{1}{g(m)} \\ &\leq \frac{1}{\eta_{m-1}} + \frac{1}{g(m)} \\ &< 1. \end{aligned}$$

This contradiction proves our claim. Thus there exists a k such that

$$\|(a_n)\| \leq \eta_k \|(a_1, \ldots, a_k, 0, \ldots)\| .$$

Since $\eta_j \|(a_1, a_2, \ldots, a_j, 0, \ldots,)\| \to \|(a_n)\|$ as $j \to \infty$, this tells us that the supremum in the definition of $|||\cdot|||$ is always attained. Suppose that it is attained at j for the vector (a_n). Then,

$$\begin{aligned} |||(a_n)||| &= \eta_j \|(a_1, a_2, \ldots, a_j, 0, \ldots)\| \\ &\leq |||(a_1, a_2, \ldots, a_j, 0, \ldots)||| \\ &\leq |||(a_n)||| . \end{aligned}$$

Thus $|||(a_n)||| = |||(a_1, a_2, \ldots, a_j, 0, \ldots)|||$ and by Theorem 4.8.1, h_Φ is isomorphically polyhedral. $\square$

Recall that the motivation for the work in this chapter was to investigate the question of whether there is an asymptotic ℓ_∞ space which admits a $\mathscr{C}^2$-smooth bump function but no $\mathscr{C}^2$-smooth norm. The results of this section shows that no generalized Orlicz space can possibly satisfy these conditions. Any such space which is asymptotic ℓ_∞ is isomorphically polyhedral and so, in particular, always admits an equivalent analytic norm.

Chapter 5

Hereditarily Indecomposable Banach Spaces with upper ℓ_p estimates

Contents

We say that a Banach space X with a basis has an upper ℓ_p estimate $(1 < p < \infty)$ with constant C, if for any successive block vectors $x_1 < x_2 < \ldots < x_n$,

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq C\left(\sum_{i=1}^{n}\|x_i\|^p\right)^{\frac{1}{p}}.$$

We will also be interested in spaces satisfying a type of lower ℓ_p estimate. Given a function $f : \mathbb{N} \to \mathbb{R}_+$, we will say that X has a lower (ℓ_p, f) estimate if for any vector x,

$$\|x\| \geq \sup\left\{\frac{1}{f(n)^{\frac{1}{p}}}\left(\sum_{i=1}^{n}\|E_i x\|^p\right)^{\frac{1}{p}} : n \in \mathbb{N},\ E_1 < E_2 < \ldots < E_n \text{ intervals}\right\}.$$

In this chapter we will construct a Banach space X which is hereditarily indecomposable, satisfies an upper ℓ_2 estimate and a lower (ℓ_2, f) estimate. A similar construction can be used to construct spaces with an upper ℓ_p estimate and a lower (ℓ_p, f) estimate for any $p \in (1, \infty)$. Spaces with similar properties have been constructed by Ferenczi in [22], using complex interpolation methods. The method we will use is based upon that of Gowers and Maurey in [33], where the first example of an hereditarily indecomposable space was constructed.

5.1 2-convexified Schlumprecht Space $S^{(2)}$

We will start by constructing an example of a space which satisfies an upper ℓ_2 estimate and a lower (ℓ_2, f) estimate, where $f(x) = \log_2(x+1)$. Let S denote Schlumprecht space which was introduced in [74] (see Section 2.2.2). Since S has a 1-unconditional basis, we may form its 2-convexification (see Section 2.1.3), which we will denote by $S^{(2)}$ i.e.

$$S^{(2)} = \{x = (x_i) : x^2 = (x_i^2) \in S\},$$

$$\|x\| = \left(\left\|x^2\right\|_S\right)^{\frac{1}{2}}.$$

Proposition 5.1.1 *For every vector $x \in S^{(2)}$,*

$$\|x\| = \|x\|_\infty \vee \sup\left\{\left(\frac{1}{f(n)}\sum_{i=1}^n \|E_i x\|^2\right)^{\frac{1}{2}} : n \geq 2,\ E_1 < \ldots < E_n \ \textit{intervals}\right\}.$$

Proof.

$$\begin{aligned}\|x\|^2 = \left\|x^2\right\|_S &= \left\|x^2\right\|_\infty \vee \sup_{n\geq 2}\left(\frac{1}{f(n)}\sum_{i=1}^n \left\|E_i(x^2)\right\|_S\right)\\ &= \|x\|_\infty^2 \vee \sup_{n\geq 2}\left(\frac{1}{f(n)}\sum_{i=1}^n \left\|(E_i x)^2)\right\|_S\right)\\ &= \|x\|_\infty^2 \vee \sup_{n\geq 2}\left(\frac{1}{f(n)}\sum_{i=1}^n \|E_i x\|^2\right).\end{aligned}$$

□

Proposition 5.1.2 $S^{(2)}$ *has an upper ℓ_2 estimate with constant* 1.

Proof. Let $x_1 < x_2 < \ldots < x_n$ be successive vectors in $S^{(2)}$. Then

$$\left\|\sum_{i=1}^n x_i\right\|^2 = \left\|\left(\sum_{i=1}^n x_i\right)^2\right\|_S = \left\|\sum_{i=1}^n x_i^2\right\|_S \leq \sum_{i=1}^n \|x_i^2\|_S = \sum_{i=1}^n \|x_i\|^2.$$

□

We now define a class of functions, $\mathcal{F}$, to be those functions $f : [1,\infty) \to [1,\infty)$ which satisfy the following conditions;

(i) $f(1) = 1$ and $f(x) < x$ for every $x > 1$.

(ii) f is strictly increasing and tends to infinity.

(iii) $\lim_{x\to\infty} x^{-q} f(x) = 0$ for every $q > 0$.

(iv) The function $x/f(x)$ is concave and non-decreasing.

(v) $f(xy) \leq f(x)f(y)$ for every $x, y \geq 1$.

For our purposes we will be considering the function $f(x) = \log_2(x+1)$ which is in $\mathcal{F}$. It is important to note that $\sqrt{f} \in \mathcal{F}$ too.

We also define a class of normed spaces, $\mathcal{X}$, to be the collection of spaces of the form $X = (c_{00}, \|\cdot\|)$ such that (e_i) form a normalized bimonotone basis for X, with an upper ℓ_2 estimate with constant 1, and a lower (ℓ_2, f) estimate for some $f \in \mathcal{F}$.

In our construction, certain special vectors, which we call ℓ_{2+}^N-averages, will be very important. Our first result is to show that every block subspace of a space in $\mathcal{X}$ contains a large number of these special vectors.

Definition 5.1.3 *Given a Banach space Y with a basis, $y \in Y$ is said to be an ℓ_{2+}^N-average with constant C, if $\|y\| = 1$, and there exist successive vectors $y_1 < \ldots < y_N$, such that $y = \sum_{i=1}^N y_i$ with $\|y_i\| \leq CN^{-\frac{1}{2}}$.*

We say that y is an ℓ_{2+}^N-vector, if there exist successive vectors $y_1 < \ldots < y_N$, such that $y = \sum_{i=1}^N y_i$ with $\|y_i\| \leq CN^{-\frac{1}{2}} \|y\|$.

Lemma 5.1.4 *Let $X \in \mathcal{X}$, $C > 1$, $n \in \mathbb{N}$, Y a block subspace of X. Then Y contains an ℓ_{2+}^n-average with constant C.*

PROOF. Suppose that $C > 1$, $n \in \mathbb{N}$ are given and the result were false. Fix $k \in \mathbb{N}$, and let $N = n^k$. Choose N successive normalized block vectors $x_1 < x_2 < \ldots < x_N$ in Y, and define $x = \sum_{i=1}^N x_i$. Now for $0 \leq i \leq k$, $1 \leq j \leq n^{k-i}$, define

$$x(i,j) = \sum_{t=(j-1)n^i+1}^{jn^i} x_t.$$

Then notice that for $1 \leq j \leq n^k$, we have $x(0,j) = x_j$, and $x(k,1) = x$. Further, each $x(i+1,j)$ is a sum of n different $x(i,r)$s. Since by hypothesis Y contains no ℓ_{2+}^n-average with constant C, by induction we get that

$$\|x(i,j)\| \leq C^{-i} n^{\frac{i}{2}}.$$

In particular, we get $\|x\| = \|x(k,1)\| \leq C^{-k} n^{\frac{k}{2}} = N^{-q} N^{\frac{1}{2}}$, where $C = n^q$ and $q > 0$. But since the norm of X satisfies a lower (ℓ_2, f) estimate we get

$$\|x\| \geq \left(\frac{N}{f(N)}\right)^{\frac{1}{2}}.$$

Thus we get that

$$N^{-q} f(N)^{\frac{1}{2}} \geq 1,$$

but this yields a contradiction if we choose k, and hence N, sufficiently large. □

We now obtain a simple, but useful norm inequality, which applies to ℓ_{2+}^N-vectors.

Lemma 5.1.5 *Let $X \in \mathcal{X}$, $M, N \in \mathbb{N}$, $C \geq 1$, x be an ℓ_{2+}^N-vector with constant C, and $E_1 < E_2 < \ldots < E_M$ be intervals. Then*

$$\left(\sum_{j=1}^{M} \|E_j x\|^2\right)^{\frac{1}{2}} \leq C\left(1 + \frac{2M}{N}\right)^{\frac{1}{2}} \|x\|.$$

Proof. Write $x = \sum_{i=1}^{N} x_i$ where $x_1 < \ldots < x_N$ and $\|x_i\| \leq C N^{-\frac{1}{2}} \|x\|$. Define

$$A_j = \{i : \operatorname{supp}(x_i) \subseteq E_j\},$$
$$B_j = \{i : \operatorname{supp}(x_i) \cap E_j \neq \varnothing\}.$$

By bimonotonicity, and from the upper ℓ_2 estimate we find

$$\|E_j x\| \leq \left\|\sum_{i \in B_j} x_i\right\| \leq \left(\sum_{i \in B_j} \|x_i\|^2\right)^{\frac{1}{2}} \leq C N^{-\frac{1}{2}} \|x\| \; |B_j|^{\frac{1}{2}}.$$

Thus,

$$\begin{aligned}
\left(\sum_{j=1}^{M} \|E_j x\|^2\right)^{\frac{1}{2}} &\leq C N^{-\frac{1}{2}} \|x\| \left(\sum_{j=1}^{M} |B_j|\right)^{\frac{1}{2}} \\
&\leq C N^{-\frac{1}{2}} \|x\| \left(\sum_{j=1}^{M} (|A_j| + 2)\right)^{\frac{1}{2}} \\
&\leq C N^{-\frac{1}{2}} \|x\| (N + 2M)^{\frac{1}{2}} \\
&= C\left(1 + \frac{2M}{N}\right)^{\frac{1}{2}} \|x\|.
\end{aligned}$$

□

We now look at certain finite sequences of ℓ_{2+}-vectors which we will call *rapidly increasing sequences*. Define a function $M_f : [1,\infty) \to [1,\infty)$ by $M_f(x) = f^{-1}(144x^4)$. This function is used to specify how rapidly the sequences increase.

Definition 5.1.6 *A sequence $x_1 < x_2 < \ldots < x_N$ is a rapidly increasing sequence of ℓ_{2+}-averages (RIS), for f of length N with constant $1+\varepsilon$, if each x_k is an $\ell_{2+}^{n_k}$-average with constant $1+\varepsilon$ for each k, where*

$$n_1 \geq \max\left(\frac{2(1+\varepsilon)}{\varepsilon' f'(1)} M_f\left(\frac{N}{\varepsilon'}\right), \frac{1+\varepsilon+\varepsilon'}{\varepsilon' N}\right),$$

$$\frac{(\varepsilon')^2}{4} f(n_k)^{\frac{1}{2}} \geq |\operatorname{supp}(x_{k-1})| \quad \textit{when } 1 < k \leq N.$$

We will also want to consider certain special elements of the dual space, which we will call (M,g)-functionals.

Definition 5.1.7 *A functional x^* is said to be an (M,g)-functional if $\|x^*\| \leq 1$ and there exist functionals $x_1^* < x_2^* < \ldots < x_M^*$ with $\|x_i^*\| \leq g(M)^{-\frac{1}{2}}$ and constants $(\alpha_j)_{j=1}^M$ with $\sum_{j=1}^M \alpha_j^2 \leq 1$ such that*

$$x^* = \sum_{j=1}^{M} \alpha_j x_j^*.$$

The remaining lemmas in this section are very important to our construction. They give us inequalities involving the norm of the sum of an RIS which will be vital to our construction of an H.I. space. Throughout the rest of this section, when $\varepsilon > 0$, we will write $\varepsilon' = \min(\varepsilon, 1)$.

Lemma 5.1.8 *Let $f, g \in \mathcal{F}$ with $g \geq f^{\frac{1}{2}}$, $X \in \mathcal{X}$ satisfy a lower (ℓ_2, f) estimate, $\varepsilon > 0$, $x_1, x_2, \ldots, x_N$ be an RIS in X for f with constant $1+\varepsilon$, $x = \sum_{i=1}^N x_i$, $M \geq M_f(\frac{N}{\varepsilon'})$, x^* an (M,g)-functional, and $E \subseteq \mathbb{N}$ an interval. Then*

$$|x^*(Ex)| \leq 1 + \varepsilon + \varepsilon'.$$

Proof. Notice first that if x^* is an (M,g)-functional, then so is Ex^*. Hence we may ignore the interval E in the statement of the lemma. Let n_i be maximal such that x_i is an $\ell_{2+}^{n_i}$-average with constant $1+\varepsilon$. Write $x^* = g(M)^{-\frac{1}{2}} \sum_{j=1}^M \alpha_j x_j^*$ where $\sum \alpha_j^2 \leq 1$ and $\|x_j^*\| \leq 1$ for every j. Finally, we write $E_j = \operatorname{supp}(x_j^*)$. We now estimate $x^*(x_i)$. Firstly, since $\|x^*\| \leq 1$, we have $|x^*(x_i)| \leq 1$. Also, as $\|x_j^*\| \leq 1$ for each j,

$$\begin{aligned} |x^*(x_i)| &= \left| g(M)^{-\frac{1}{2}} \sum_{j=1}^{M} \alpha_j x_j^*(x_i) \right| \\ &\leq g(M)^{-\frac{1}{2}} \left(\sum_{j=1}^{M} \alpha_j^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{M} |x_j^*(x_i)|^2 \right)^{\frac{1}{2}} \\ &\leq g(M)^{-\frac{1}{2}} \left(\sum_{j=1}^{M} \|E_j x_i\|^2 \right)^{\frac{1}{2}}. \end{aligned}$$

By applying the above estimate, and the lower (ℓ_2, f) estimate for the norm of x_i we can obtain,

$$|x^*(x_i)| \leq g(M)^{-\frac{1}{2}} f(|\operatorname{supp}(x_i)|)^{\frac{1}{2}} \leq f(M)^{-\frac{1}{4}} f(|\operatorname{supp}(x_i)|)^{\frac{1}{2}}.$$

Alternatively, by applying Lemma 5.1.5, we can show,

$$|x^*(x_i)| \leq g(M)^{-\frac{1}{2}}(1+\varepsilon)\left(1 + \frac{2M}{n_i}\right)^{\frac{1}{2}} \leq f(M)^{-\frac{1}{4}}(1+\varepsilon)\left(1 + \frac{2M}{n_i}\right)^{\frac{1}{2}}.$$

Let t be maximal such that $n_t \leq M$. If $i < t$ then,

$$f(|\operatorname{supp}(x_i)|) \leq 4^{i-t+1} f(|\operatorname{supp}(x_{t-1})|).$$

Also,

$$f(|\operatorname{supp}(x_{t-1})|) \leq \frac{(\varepsilon')^2}{4} f(n_t)^{\frac{1}{2}} \leq \frac{(\varepsilon')^2}{4} f(M)^{\frac{1}{2}}.$$

Putting all of these inequalities together gives,

$$\begin{aligned} |x^*(x)| &\leq \sum_{i=1}^{N} |x^*(x_i)| \\ &\leq \sum_{i=1}^{t-1} \frac{2^{i-t+1}}{f(M)^{\frac{1}{4}}} f(|\operatorname{supp}(x_{t-1})|)^{\frac{1}{2}} + 1 + \sum_{i=t+1}^{N} (1+\varepsilon) f(M)^{-\frac{1}{4}} \left(1 + \frac{2M}{n_i}\right)^{\frac{1}{2}} \\ &\leq \varepsilon' + 1 + (N-t)(1+\varepsilon) f(M)^{-\frac{1}{4}} \sqrt{3} \\ &\leq 1 + \varepsilon' + N(1+\varepsilon)\sqrt{3} f(M)^{-\frac{1}{4}}. \end{aligned}$$

But since $M \geq M_f(\frac{N}{\varepsilon'})$, we know that $f(M) \geq 144 \frac{N^4}{(\varepsilon')^4}$. Hence $f(M)^{\frac{1}{4}} \geq 2\sqrt{3}\frac{N}{\varepsilon'}$. Thus,

$$|x^*(x)| \leq 1 + \varepsilon' + N(1+\varepsilon)\frac{\sqrt{3}\varepsilon'}{2\sqrt{3}N} = 1 + \varepsilon' + (1+\varepsilon)\frac{\varepsilon'}{2} \leq 1 + \varepsilon + \varepsilon'.$$

□

Corollary 5.1.9 *Let ε, M, $x_1, \ldots, x_N$, x be as in Lemma 5.1.8, and $E_1 < E_2 < \ldots < E_M$ be intervals, and E be any interval. Then*

$$f(M)^{-\frac{1}{2}} \left(\sum_{i=1}^{M} \|E_i E x\|^2 \right)^{\frac{1}{2}} \leq 1 + \varepsilon + \varepsilon'.$$

PROOF. Let x_j^* be a support functional for $E_j E x$, and let $x^* = f(M)^{-\frac{1}{2}} \sum_{i=1}^{M} \alpha_i x_i^*$, where the α_i will be chosen later to satisfy $\sum \alpha_i^2 \leq 1$. Then x^* is an (M, f)-functional, and so by Lemma 5.1.8,

$$f(M)^{-\frac{1}{2}} \sum_{i=1}^{M} \alpha_i \|E_i E x\| = f(M)^{-\frac{1}{2}} \sum_{i=1}^{M} \alpha_i x_i^*(Ex) = x^*(Ex) \leq 1 + \varepsilon + \varepsilon'.$$

We complete the proof by choosing

$$\alpha_i = \frac{\|E_i E x\|}{\left(\sum_{j=1}^{M} \|E_j E x\|^2 \right)^{\frac{1}{2}}}.$$

□

To state the next result we need some more notation. Let $x_1 < \ldots < x_N$ be an RIS for f with constant $1 + \varepsilon$. For each i, suppose that x_i is an $\ell_{2+}^{n_i}$-average with constant $1 + \varepsilon$. Write $x_i = x_{i,1} + \ldots + x_{i,n_i}$, where $\|x_{i,j}\| \leq (1 + \varepsilon) n_i^{-\frac{1}{2}}$ for every j. Let E be an interval in $\mathbb{N}$, and let i_E and j_E be respectively the minimal i and maximal j such that E meets x_i and x_j. Let r_E and s_E denote the minimal r and maximal s such that E meets $x_{i_E,r}$ and $x_{j_E,s}$. We define $\lambda(E)$, the length of E, to be $j_E - i_E + s_E/n_{j_E} - r_E/n_{i_E}$. The length of E gives us a fractional measure of how many of the x_is are covered by E. It follows from the definitions that if $E_1 < \ldots < E_m$ and $E = \cup_{i=1}^{m} E_i$, then $\sum_{i=1}^{m} \lambda(E_i) \leq \lambda(E)$. We will use this in the proof of the next lemma.

Lemma 5.1.10 *Let $f, g \in \mathcal{F}$ with $g \geq f^{\frac{1}{2}}$, $X \in \mathcal{X}$ satisfy a lower (ℓ_2, f) estimate, $\varepsilon > 0$, $x_1 < x_2 < \ldots < x_N$ be an RIS for f with constant $1 + \varepsilon$, $x = \sum_{i=1}^{N} x_i$. Suppose that*

$$\|Ex\| \leq \sup\{|x^*(Ex)| : M \geq 2,\ x^* \text{ an } (M, g)\text{-functional}\},$$

for every interval E such that $\lambda(E) \geq 1$. Then

$$\|x\| \leq (1 + \varepsilon + \varepsilon') \left(\frac{N}{g(N)} \right)^{\frac{1}{2}}.$$

PROOF. By bimonotonicity and the upper ℓ_2 estimate in X we have

$$\begin{aligned}\|Ex\| &\leq \left\| \sum_{r=r_E}^{n_{i_E}} x_{i_E,r} + \sum_{i=i_E+1}^{j_E-1} x_i + \sum_{s=1}^{s_E} x_{j_E,s} \right\| \\ &\leq \left(\sum_{r=r_E}^{n_{i_E}} \|x_{i_E,r}\|^2 + \sum_{i=i_E+1}^{j_E-1} \|x_i\|^2 + \sum_{s=1}^{s_E} \|x_{j_E,s}\|^2 \right)^{\frac{1}{2}} \\ &\leq \left((n_{i_E} - r_E + 1)\frac{(1+\varepsilon)^2}{n_{i_E}} + j_E - i_E - 1 + s_E \frac{(1+\varepsilon)^2}{n_{j_E}} \right)^{\frac{1}{2}} \\ &\leq (1+\varepsilon)\left(\lambda(E) + \frac{1}{n_{i_E}} \right)^{\frac{1}{2}} \\ &\leq (1+\varepsilon)\left(\lambda(E) + \frac{1}{n_1} \right)^{\frac{1}{2}}.\end{aligned}$$

Hence, if $\lambda(E) \geq (1+\varepsilon)/(\varepsilon' n_1)$, then $\|Ex\| \leq (1+\varepsilon+\varepsilon')\lambda(E)^{\frac{1}{2}}$.
We now define the function

$$G(x) = \begin{cases} x & \text{if } 0 \leq x \leq 1, \\ x/g(x) & \text{if } x \geq 1. \end{cases}$$

Then G is concave and increasing on $\mathbb{R}_+$ and also satisfies $G(xy) \geq G(x)G(y)$ for any x, y.

We now claim that if E is an interval such that $\lambda(E) \geq (1+\varepsilon)/(\varepsilon' n_1)$, then $\|Ex\| \leq (1+\varepsilon+\varepsilon')G(\lambda(E))^{\frac{1}{2}}$. Notice that from the above, this is immediate provided $\lambda(E) \leq 1$.

Suppose that this result were false. Then let E be an interval of minimal length such that $\lambda(E) \geq (1+\varepsilon)/(\varepsilon' n_1)$ but the result does not hold. Then clearly $\lambda(E) \geq 1$. By hypothesis there exists an (M, g)-functional $x^* = g(M)^{-\frac{1}{2}} \sum_{i=1}^{M} \alpha_i x_i^*$, such that $\|Ex\| \leq |x^*(Ex)|$. Then by Lemma 5.1.8 the inequality would fail unless $M \leq M_f(N/\varepsilon')$.
Write $E_i = E \cap \operatorname{supp}(x_i^*)$. Then,

$$\|Ex\| \leq |x^*(Ex)| = g(M)^{-\frac{1}{2}} \left| \sum_{i=1}^{M} \alpha_i x_i^*(Ex) \right| \leq g(M)^{-\frac{1}{2}} \left(\sum_{i=1}^{M} \|E_i x\|^2 \right)^{\frac{1}{2}}.$$

Let $\lambda_i = \lambda(E_i)$. Then for each i, either $\lambda_i \leq (1+\varepsilon)/(\varepsilon' n_1)$, or by the minimality of E,

$$\|E_i x\| \leq (1+\varepsilon+\varepsilon')G(\lambda_i)^{\frac{1}{2}}.$$

We now define,

$$\begin{aligned} A &= \{i : \lambda_i \leq (1+\varepsilon)/(\varepsilon' n_1)\}, \\ B &= A^c. \end{aligned}$$

Let $k = |A|$, so that $|B| = M - k$.

Recall Jensen's inequality in the following form. If f is concave, and $\sum_i \alpha_i = 1$, then

$$f(\sum_i \alpha_i x_i) \geq \sum_i \alpha_i f(x_i).$$

Applying this result to the concave function G gives,

$$\sum_{i\in B} \|E_i x\|^2 \leq (1+\varepsilon+\varepsilon')^2 \sum_{i\in B} G(\lambda_i) \leq (1+\varepsilon+\varepsilon')^2 (M-k) G(\lambda/(M-k)).$$

Hence,

$$\begin{aligned}
\|Ex\|^2 &\leq g(M)^{-1}\Big(\sum_{i\in A}\|E_i x\|^2 + \sum_{i\in B}\|E_i x\|^2\Big)\\
&\leq g(M)^{-1}\Big(\sum_{i\in A}(1+\varepsilon)^2\Big(\lambda_i + \frac{1}{n_1}\Big) + (1+\varepsilon+\varepsilon')^2(M-k)G\Big(\frac{\lambda}{M-k}\Big)\Big)\\
&\leq (1+\varepsilon+\varepsilon')^2\Big((1-\frac{k}{M})G((1-\frac{k}{M})^{-1}\lambda) + k\frac{1+\varepsilon}{n_1\varepsilon'}\Big).
\end{aligned}$$

Let $G'(1)$ denote the right derivative of G at 1. We can define another concave function $H : \mathbb{R}_+ \to \mathbb{R}$ by,

$$H(x) = \begin{cases} G(x) & \text{for } x \geq 1,\\ G(1) + xG'(1) - G'(1) & \text{for } 0 \leq x \leq 1.\end{cases}$$

Hence for any $t \in [0,1]$ and $x \in \mathbb{R}_+$, we have

$$(1-t)H(x) + tH(0) \leq H((1-t)x).$$

Taking $0 \leq t < 1$, and $x = \lambda/(1-t)$ where $\lambda \geq 1$ tells us that

$$(1-t)G\left(\frac{\lambda}{1-t}\right) + t(G(1) - G'(1)) \leq G(\lambda),$$

for every $\lambda \geq 1$ and $0 \leq t < 1$. Also, by differentiating we get $G(1) - G'(1) = g'(1)$, and as $g \geq \sqrt{f}$, we have $g'(1) \geq \frac{1}{2}f'(1)$. From the definition of RIS we have $n_1 \geq 2(1+\varepsilon)/(\varepsilon' f'(1))M_f(N/\varepsilon')$, and hence

$$k\frac{1+\varepsilon}{n_1\varepsilon'} \leq \frac{k}{M}(1+\varepsilon)\frac{M_f(N/\varepsilon')}{\varepsilon' n_1} \leq \frac{k}{M}\frac{f'(1)}{2} \leq \frac{k}{M}g'(1).$$

Hence putting $t = k/M$,

$$\begin{aligned}
\|Ex\|^2 &\leq (1+\varepsilon+\varepsilon')^2\Big((1-\frac{k}{M})G((1-\frac{k}{M})^{-1}\lambda) + k\frac{1+\varepsilon}{n_1\varepsilon'}\Big)\\
&\leq (1+\varepsilon+\varepsilon')^2\big((1-t)G((1-t)^{-1}\lambda) + tg'(1)\big)\\
&\leq (1+\varepsilon+\varepsilon')^2 G(\lambda).
\end{aligned}$$

But this then contradicts our choice of interval E, so the claim is proved.

Since in the definition of RIS, $n_1 \geq (1+\varepsilon+\varepsilon')/(\varepsilon' N)$, we know that $N \geq (1+\varepsilon+\varepsilon')/(\varepsilon' n_1)$. Hence we may take any interval E which covers all of x, so that $\lambda(E) = N - \frac{1}{n_1} \geq (1+\varepsilon)/(\varepsilon' n_1)$. Then the above claim gives

$$\|x\| \leq (1+\varepsilon+\varepsilon')\left(\frac{N}{g(N)}\right)^{\frac{1}{2}}.$$

□

5.2 Construction of an asymptotic biorthogonal system in $S^{(2)}$

Let $\delta \in (0,1)$ and let $N_1 < N_2 < \ldots$ be a sequence of integers such that $f(N_1)/N_1 < \delta^2/4$, $f(N_1) > 16\delta^{-2}$, $N_j > M_f(2N_{j-1})$ and $N_j > 4N_{j-1}^2$.

We define A_k to be the set of norm-one vectors of the form $x = \sum_{i=1}^{N_k} x_i$, where $x_1, \ldots, x_{N_k}$ is a multiple of an RIS with constant $1+\delta/2$. From the definition of the norm in $S^{(2)}$, if we suppose this multiple to be α, then $|\alpha| \leq (\frac{f(N_k)}{N_k})^{\frac{1}{2}}$. We define A_k^* to be the set of (N_k, f)-functionals.

Suppose we have $y^* \in A_j^*$ and $x \in A_k$. If $j > k$, then we know that $N_j > M_f(2N_k)$. Thus by Lemma 5.1.8, with $\varepsilon = 1/2$, $M = N_j$,

$$|y^*(x)| \leq \left(\frac{f(N_k)}{N_k}\right)^{\frac{1}{2}}(1+\varepsilon+\varepsilon') = 2\left(\frac{f(N_k)}{N_k}\right)^{\frac{1}{2}} \leq 2\left(\frac{f(N_1)}{N_1}\right)^{\frac{1}{2}} < \delta.$$

If $j < k$, then by Lemma 5.1.10 with $g = f$, $\varepsilon = 1/2$, whenever $A \subseteq \{1, 2, \ldots, N_k\}$,

$$\left\|\sum_{i\in A} x_i\right\| \leq 2\left(\frac{f(N_k)|A|}{N_k f(|A|)}\right)^{\frac{1}{2}}.$$

But if $|A| \geq \sqrt{N_k}$, then this gives $\left\|\sum_{i\in A} x_i\right\| \leq 2\sqrt{2}(\frac{|A|}{N_k})^{\frac{1}{2}}$. Dividing x into $\sqrt{N_k}$ pieces gives that x is an $\ell_{2+}^{\sqrt{N_k}}$-average with constant $2\sqrt{2}$. But then Lemma 5.1.5 gives,

$$\begin{aligned}|y^*(x)| &\leq f(N_k)^{-\frac{1}{2}}\left(\sum_{i=1}^{N_k}\|E_i x\|^2\right)^{\frac{1}{2}} \\ &\leq f(N_k)^{-\frac{1}{2}} 2\sqrt{2}\left(1+\frac{2N_j}{\sqrt{N_k}}\right)^{\frac{1}{2}} \leq 4f(N_k)^{-\frac{1}{2}} < \delta.\end{aligned}$$

If $j = k$ then let x_i^* be a support functional for x_i. Then by Lemma 5.1.10 with $\varepsilon = \delta/2$, for each i we have

$$1 = \|x\| \leq (1+\delta)\left(\frac{N_k}{f(N_k)}\right)^{\frac{1}{2}}\|x_i\|.$$

Let $x^* = f(N_k)^{-\frac{1}{2}} \sum_{i=1}^{N_k} N_k^{-\frac{1}{2}} x_i^* \in A_k^*$. Then

$$\begin{aligned} x^*(x) &= f(N_k)^{-\frac{1}{2}} \sum_{i=1}^{N_k} N_k^{-\frac{1}{2}} x_i^*(x) \\ &= f(N_k)^{-\frac{1}{2}} \sum_{i=1}^{N_k} N_k^{-\frac{1}{2}} \|x_i\| \\ &\geq f(N_k)^{-\frac{1}{2}} \sum_{i=1}^{N_k} \frac{f(N_k)^{\frac{1}{2}}}{(1+\delta)N_k} \\ &= \frac{1}{1+\delta} \\ &> 1-\delta. \end{aligned}$$

Hence the sets A_k, A_k^* form an asymptotic biorthogonal system in $S^{(2)}$.

5.3 An hereditarily indecomposable space X satisfying an upper ℓ_2 estimate

Let $\mathcal{Q}$ be the set of scalar sequences with finite supports taking rational values of modulus at most one at each coordinate. Let $J \subseteq \mathbb{N}$ be written in increasing order as $\{j_1, j_2, j_3, \ldots\}$ such that $f(j_1) > 45^4$, and such that

$$n < m \ , \ n, m \in J \Rightarrow \log\log\log m \geq 8n^4.$$

Let $K = \{j_1, j_3, j_5, \ldots\}$ and $L = \{j_2, j_4, j_6, \ldots\}$.

Let σ be an injection from the set of finite sequences of successive vectors in $\mathcal{Q}$ to L such that if $z_1, z_2, \ldots, z_s$ is such a sequence, and we write $z = \sum_{i=1}^{s} z_i$ and $l = \sigma(z_1, \ldots, z_s)$, then $\frac{1}{400} f(l^{1/40})^{\frac{1}{2}} \geq |\operatorname{supp}(z)|$.

Let $Z = (c_{00}, \|\cdot\|)$. Then let $A_m^*(Z)$ be the set of functionals of the form $f(m)^{-1/2} \sum_{i=1}^{m} \alpha_i f_i$ where $\sum_{i=1}^{m} \alpha_i^2 \leq 1$, $\|f_i\|^* \leq 1$ for each i, and $f_1 < f_2 < \ldots < f_m$. If $k \in \mathbb{N}$, then let $\Gamma_k(Z)$ be the set of sequences $g_1, g_2, \ldots, g_k$ such that $g_i \in \mathcal{Q}$ for each i, $g_1 \in A_{j_{2k}}^*(Z)$ and $g_{i+1} \in A_{\sigma(g_1, g_2, \ldots, g_i)}^*(Z)$ for $i = 1, 2, \ldots, k-1$. We will call the elements of $\Gamma_k(Z)$ special sequences. We want to define special functionals in the same kind of way as in [33], but in order to get a space which satisfies an upper ℓ_2 estimate we need to be more careful. To explain this we will need to consider restrictions of functionals in $A_m^*(Z)$ to intervals.

Let z^* be in $A_m^*(Z)$. We can write z^* as

$$z^* = f(M)^{-\frac{1}{2}} \sum_{i=1}^{M} \alpha_i f_i,$$

where $\sum_{i=1}^{M} \alpha_i^2 \leq 1$ and $\|f_i\|^* \leq 1$. Let E be an interval contained in $\mathbb{N}$. Then

$$\begin{aligned} Ez^* &= f(M)^{-\frac{1}{2}} \sum_{i=1}^{M} \alpha_i E f_i \\ &= f(M)^{-\frac{1}{2}} \left(\sum_j \alpha_j^2 \|Ef_j\|^2 \right)^{\frac{1}{2}} \sum_{i=1}^{M} \frac{\alpha_i \|Ef_i\|}{\left(\sum_j \alpha_j^2 \|Ef_j\|^2\right)^{1/2}} \frac{Ef_i}{\|Ef_i\|}. \end{aligned}$$

This tells us that the functional $\dfrac{Ez^*}{\left(\sum_j \alpha_j^2 \|Ef_j\|^2\right)^{\frac{1}{2}}}$ is also in $A_m^*(Z)$. In the case where $\|\cdot\|$ also satisfies a lower (ℓ_2, f) estimate we also get

$$\|Ez^*\| \leq \left(\sum_j \alpha_j^2 \|Ef_j\|^2 \right)^{-\frac{1}{2}}.$$

To simplify the notation we will define

$$\beta(E) = \left(\sum_j \alpha_j^2 \|Ef_j\|^2 \right)^{\frac{1}{2}}.$$

Then the above results can be restated as $Ez^*/\beta(E)$ is in $A_m^*(Z)$ and $\|Ez^*\| \leq \beta(E)$ when the norm satisfies a lower (ℓ_2, f) estimate.

We are now in a position to define our special functionals. Given a sequence of successive functionals $g_1, \ldots, g_k \in \Gamma_k(Z)$ and an interval E of $\mathbb{N}$, we define

$$\begin{aligned} i(E) &= \min\{i : \ E \text{ meets } \operatorname{supp}(g_i)\}, \\ j(E) &= \max\{i : \ E \text{ meets } \operatorname{supp}(g_i)\}. \end{aligned}$$

Define $B_k^*(Z)$ to be the set of functionals of the form

$$\lambda f(k)^{-1/4} \left(\sum_{j=i(E)}^{j(E)} \beta_j(E)^2 \right)^{-\frac{1}{2}} \sum_{j=i(E)}^{j(E)} FEg_j,$$

such that $|\lambda| \leq 1$, $g_1, \ldots, g_k \in \Gamma_k(Z)$ and E, F are intervals contained in $\mathbb{N}$. We will call elements in $B_k^*(Z)$, special functionals of length k. A functional will be called a special functional if it is in $B^*_k(Z)$ for some $k \in K$. It should be noted that if we replace the λ in the definition above by 1 then it will not affect our construction but it is useful to include it for technical reasons that will become apparent later.

Lemma 5.3.1 *Let $\|\cdot\|$ be a bimonotone norm on X with an upper ℓ_2 estimate with constant 1. Then X^* satisfies a lower ℓ_2 estimate with constant 1 on successive blocks.*

Proof. Let $f_1 < \ldots < f_n$ be a sequence of successive functionals on X, and write $f = \sum_{i=1}^n f_i$. Since the norm on X is bimonotone we can find vectors $x_i \in S_X$ such that $\operatorname{supp} x_i \subseteq \operatorname{supp} f_i$ and $f_i(x_i) = \|f_i\|$. Then, for any constants $\alpha_i \geq 0$, applying the upper ℓ_2 estimate in X gives

$$\|f\| \geq \frac{f(\sum_{i=1}^n \alpha_i x_i)}{\|\sum_{i=1}^n \alpha_i x_i\|} = \frac{\sum_{i=1}^n \alpha_i \|f_i\|}{\|\sum_{i=1}^n \alpha_i x_i\|} \geq \frac{\sum_{i=1}^n \alpha_i \|f_i\|}{(\sum_{i=1}^n \alpha_i^2)^{1/2}}.$$

If we choose

$$\alpha_i = \frac{\|f_i\|}{(\sum_j \|f_j\|^2)^{\frac{1}{2}}},$$

then this shows that

$$\|f\| \geq \left(\sum_{i=1}^n \|f_i\|^2\right)^{\frac{1}{2}}.$$

□

Remark 5.3.2 *Suppose that E is an interval in $\mathbb{N}$ and that $E_1 < E_2 < \ldots < E_n$ are successive intervals contained in E. It follows from Lemma 5.3.1 that if a space has an upper ℓ_2 estimate then for any functional f,*

$$\|Ef\|^2 \geq \sum_{i=1}^n \|E_i f\|^2.$$

In particular, this means that

$$\beta(E)^2 \geq \sum_{i=1}^n \beta(E_i)^2.$$

Lemma 5.3.3 *Let $\|\cdot\|$ be a bimonotone norm on c_{00} which satisfies an upper ℓ_2 estimate with constant 1. Define a new norm $|||\cdot|||$ on c_{00} by*

$$|||x||| = \max\{\|x\|, \sup\{|z^*(x)| : z^* \in B^*_k((c_{00}, \|\cdot\|)), k \in K\}\}.$$

Then $|||\cdot|||$ defines a bimonotone norm on c_{00} which satisfies an upper ℓ_2 estimate with constant 1.

Proof. We show the bimonotone property to begin with. Since $\|\cdot\|$ is bimonotone it suffices to check that $|z^*(Fx)| \leq |||x|||$ whenever F is an interval in $\mathbb{N}$ and z^* is in some B_k^*. But this is immediate since by definition $Fz^* \in B_k^*$, and so $z^*(Fx) = Fz^*(x) \leq |||x|||$.

Now we show that the new norm satisfies the upper ℓ_2 estimate. Suppose that $x_1 < x_2 < \ldots < x_n$ are successive vectors in c_{00}, and write $x = \sum_{i=1}^n x_i$. Since $\|\cdot\|$ satisfies an upper ℓ_2 estimate, it suffices to prove that $|z^*(x)| \leq \left(\sum_{i=1}^n |||x_i|||^2\right)^{\frac{1}{2}}$ for every $z^* \in B_k^*$. Suppose that $z^* = f(k)^{-\frac{1}{4}} \left(\sum_{j=i(E)}^{j(E)} \beta_j(E)^2\right)^{-\frac{1}{2}} \sum_{j=i(E)}^{j(E)} FEz_j^*$. For each i, let E_i be the interval $E \cap \operatorname{Supp}(x_i)$. Then,

$$\begin{aligned}
|z^*(x)| &= \left| f(k)^{-\frac{1}{4}} \left(\sum_{j=i(E)}^{j(E)} \beta_j(E)^2\right)^{-\frac{1}{2}} \sum_{i=1}^{n} \sum_{j=i(E)}^{j(E)} FEz_j^*(x_i) \right| \\
&\leq f(k)^{-\frac{1}{4}} \left(\sum_{j=i(E)}^{j(E)} \beta_j(E)^2\right)^{-\frac{1}{2}} \sum_{i=1}^{n} \left| \sum_{j=i(E_i)}^{j(E_i)} FE_i z_j^*(x_i) \right| .
\end{aligned}$$

By the definition of $|||\cdot|||$ and B_k^* it follows that,

$$f(k)^{-\frac{1}{4}} \left|\left|\left| \sum_{j=i(E_i)}^{j(E_i)} FE_i z_j^* \right|\right|\right|^* \leq \left(\sum_{j=i(E_i)}^{j(E_i)} \beta_j(E_i)^2\right)^{\frac{1}{2}} .$$

Hence by Cauchy-Schwartz and Remark 5.3.2 we have,

$$\begin{aligned}
|z^*(x)| &\leq \left(\sum_{j=i(E)}^{j(E)} \beta_j(E)^2\right)^{-\frac{1}{2}} \sum_{i=1}^{n} \left(\sum_{j=i(E_i)}^{j(E_i)} \beta_j(E_i)^2\right)^{\frac{1}{2}} |||x_i||| \\
&\leq \left(\sum_{j=i(E)}^{j(E)} \beta_j(E)^2\right)^{-\frac{1}{2}} \left(\sum_{i=1}^{n} \sum_{j=i(E_i)}^{j(E_i)} \beta_j(E_i)^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^{n} |||x_i|||^2\right)^{\frac{1}{2}} \\
&\leq \left(\sum_{i=1}^{n} |||x_i|||^2\right)^{\frac{1}{2}} .
\end{aligned}$$

□

5.4 Recursive definition of X

In this section we define two sequences $(\|\cdot\|_n)_{n=1}^\infty$ and $(\|\cdot\|_n')_{n=0}^\infty$ of bimonotone norms on c_{00} which satisfy an upper ℓ_2 estimate with constant 1. We will denote by X_n and X_n' the space

c_{00} with the norms $\|\cdot\|_n$ and $\|\cdot\|_n'$ respectively. We will define these norms by applying Lemma 5.3.3 at each iteration. We begin by defining $\|x\|_0 = \|x\|_\infty$. Trivially, $\|\cdot\|_0$ is bimonotone and satisfies an upper ℓ_2 estimate with constant 1. Given a bimonotone norm $\|\cdot\|_n$ which satisfies an upper ℓ_2 estimate, we define

$$\|x\|_n' = \|x\|_n \vee \sup\left\{ f(N)^{-\frac{1}{2}} \left(\sum_{i=1}^{N} \|E_i x\|_n^2 \right)^{\frac{1}{2}} : E_1 < \ldots < E_N \right\}.$$

It follows easily from the corresponding properties of $\|\cdot\|_n$ that $\|\cdot\|_n'$ is bimonotone and satisfies an upper ℓ_2 estimate with constant 1. Hence by Lemma 5.3.3 we can define a new bimonotone norm satisfying an upper ℓ_2 estimate with constant 1 by

$$\|x\|_{n+1} = \max\{\|x\|_n', \sup\{|z^*(x)| : z^* \in B_k^*(X_n'), k \in K\}\}.$$

Therefore the norms we have constructed satisfy

$$\|x\|_n \leq \|x\|_n' \leq \|x\|_{n+1} \text{ for all } n \in \mathbb{N}.$$

It also follows from the upper ℓ_2 estimate that each of these norms satisfy that $\|x\|_n \leq \|x\|_{\ell_2}$. Hence we can finally define a norm on c_{00} by $\|x\| = \lim_{n\to\infty} \|x\|_n$. It follows from the properties of $\|\cdot\|_n$ that $\|\cdot\|$ is bimonotone and satisfies an upper ℓ_2 estimate with constant 1. Also, from the definition of $\|\cdot\|_n'$, it follows that $\|\cdot\|$ satisfies a lower (ℓ_2, f) estimate. We let X be the space c_{00} equipped with the norm $\|\cdot\|$, which by the comments above is in $\mathcal{X}$. It also follows from the increasing property of these norms that $B_k^*(X_n') \subseteq B_k^*(X_{n+1}')$. (This is where the scalar λ is important in the definition of $B_k^*(Z)$.)

We have an alternative implicit definition of the above norm in the usual way by

$$\begin{aligned} \|x\| = \|x\|_\infty \vee \sup\{f(n)^{-\frac{1}{2}} \left(\sum_{i=1}^{n} \|E_i x\|^2 \right)^{\frac{1}{2}} : n \geq 2, E_1 < \ldots < E_n \text{ intervals}\} \\ \vee \sup\{|z^*(x)| : k \in K, z^* \in B_k^*(X)\}. \end{aligned}$$

5.5 Properties of rapidly increasing sequences in X

The next construction will be vital to our study of the properties of X. We will use exactly the same notation as in [33]. Let $K_0 \subseteq K$, and define a function $\phi : [1,\infty) \to [1,\infty)$ by

$$\phi(x) = \begin{cases} f(x)^{1/2} & \text{if } x \in K_0, \\ f(x) & \text{otherwise.} \end{cases}$$

Let h be the submultiplicative hull of ϕ, and $H(x) = x/h(x)$, and let G be the concave envelope of H. Finally, let $g(x) = x/G(x)$. It is shown in [33] that $f^{1/2} \leq g \leq \phi \leq f$, and that $g \in \mathcal{F}$. The following lemma, again contained in [33], is vitally important.

Lemma 5.5.1 *If $N \in J \setminus K_0$ then $g(x) = f(x)$ for every x in the interval $[\log N, \exp N]$.*

Lemma 5.5.2 *Let $N \in L$, $n \in [\log N, \exp N]$, $\varepsilon > 0$, and $x_1, x_2, \ldots, x_n$ be an RIS with constant $1+\varepsilon$. Then*

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq (1+\varepsilon+\varepsilon')\left(\frac{n}{f(n)}\right)^{\frac{1}{2}}.$$

PROOF. Clearly the norm of X satisfies a lower (ℓ_2, f) estimate. Let g be the function given in the case when $K_0 = K$. Then every vector in X either has the supremum norm or is normed by an (M, g)-functional. Also, it is clear that a vector of the form Ex can not have the supremum norm whenever $\lambda(E) \geq 1$. Since $g \geq f^{1/2}$, we may apply Lemma 5.1.10 to give

$$\left\|\sum_{i=1}^{n} x_i\right\| \leq (1+\varepsilon+\varepsilon')\left(\frac{n}{g(n)}\right)^{\frac{1}{2}}.$$

Then Lemma 5.5.1 tells us that $g(n) = f(n)$, so the result is proved. □

Lemma 5.5.3 *Let $N \in L$, $0 < \varepsilon < \frac{1}{4}$, $M = N^{\varepsilon}$, and $x_1, x_2, \ldots, x_N$ be an RIS with constant $1+\varepsilon$. Then $x = \sum_{i=1}^{N} x_i$ is an ℓ_{2+}^{M}-vector with constant $1+4\varepsilon$.*

PROOF. Let $m = N/M = N^{1-\varepsilon}$. For $1 \leq j \leq M$, let $y_j = \sum_{i=(j-1)m+1}^{jm} x_i$. Then y_j is the sum of an RIS of length m with constant $1+\varepsilon$. By Lemma 5.5.2, we have that $\|y_j\| \leq (1+2\varepsilon)\left(\frac{m}{f(m)}\right)^{1/2}$. But $\left\|\sum_{j=1}^{M} y_j\right\| = \|x\| \geq \left(\frac{N}{f(N)}\right)^{1/2}$. Then,

$$\begin{aligned}
\|y_j\| &\leq (1+2\varepsilon)\left(\frac{mf(N)}{Nf(m)}\right)^{1/2}\|x\| \\
&= (1+2\varepsilon)\left(\frac{f(N)}{f(m)}\right)^{1/2} M^{-1/2}\|x\| \\
&\leq \frac{(1+2\varepsilon)}{(1-\varepsilon)^{1/2}} M^{-1/2}\|x\| \\
&\leq (1+4\varepsilon) M^{-1/2}\|x\|.
\end{aligned}$$

□

5.6 X is hereditarily indecomposable

We will now show that X is hereditarily indecomposable. Let Y, Z be two infinite dimensional subspaces with $Y \cap Z = \{0\}$. Let $\delta > 0$ and pick $k \in K$ with $\delta \geq \frac{24}{5} f(k)^{-1/4}$. Without loss of generality we may suppose that Y, Z are block subspaces. By Lemma 5.1.4 every block subspace of X contains, for every $\varepsilon > 0$ and $N \in \mathbb{N}$ an ℓ_{2+}^N-average with constant $1+\varepsilon$. Also, from the definition of the norm, each vector either has the supremum norm or satisfies

$$\|Ex\| \leq \sup\{|x^*(Ex)| : M \geq 2, x^* \text{ an } (M,g)\text{-functional}\}.$$

Let $x_1 \in Y$ be a normalized sum of an RIS of length $M_1 = j_{2k} \in L$ and constant $1 + \varepsilon/4$ where $\varepsilon = 1/10$ and $M_1^{\varepsilon/4} = N_1 \geq 4M_f(k/\varepsilon)/\varepsilon f'(1)$. Let the non-normalized RIS whose sum is x_1 be $x_{11}, \ldots, x_{1M_1}$. By Lemma 5.5.3, x_1 is an $\ell_{2+}^{N_1}$-average with constant $1+\varepsilon$. By Lemma 5.1.10, $1 = \|x_1\| \leq (1+\varepsilon)\left(\frac{M_1}{g(M_1)}\right)^{1/2} \|x_{11}\|$. For j between 1 and M_1 let x_{1j}^* be a support functional for x_{1j}. Let $(x_1^*)'$ be the (M_1,g)-functional $g(M_1)^{-1/2}M_1^{-1/2}\sum_{j=1}^{M_1} x_{1j}^*$. Then $(x_1^*)'(x_1) = (M_1/g(M_1))^{1/2}\|x_{11}\| \geq 1/(1+\varepsilon)$. Thus we may find an (M_1,g)-functional $x_1^* \in \mathcal{Q}$ such that $|x_1^*(x_1) - 1/2| \leq k^{-1}$ and $\operatorname{supp}(x_1) = \operatorname{supp}(x_1^*)$.

Now let $M_2 = \sigma(x_1^*)$, and pick a normalized RIS vector $x_2 \in Z$ of length M_2 with constant $1+\varepsilon/4$ such that $x_1 < x_2$. Then x_2 is an $\ell_{2+}^{N_2}$-average with constant $1+\varepsilon$, where $N_2 = M_2^{\varepsilon/4}$. As above, we can find an (M_2,g)-functional x_2^* such that $|x_2^*(x_2) - 1/2| \leq k^{-1}$ and $\operatorname{supp}(x_2^*) = \operatorname{supp}(x_2)$.

Continuing in this way, we obtain a pair of sequences $x_1, \ldots, x_k$ and $x_1^*, \ldots, x_k^*$ with the following properties. $x_i \in Y$ for odd i, and $x_i \in Z$ for even i. $\|x_i\| = 1$ for each i, and $\|x_i^*\| \leq 1$ for each i. Also $|x_i^*(x_i) - 1/2| \leq 1/k$ for every i. The choice of σ and our lower bound for N_1 ensures that $x_1, \ldots, x_k$ form an RIS of length k. Also, $x_1^*, \ldots, x_k^*$, form a special sequence of length k. Thus, from the implicit definition of the norm, we get

$$\left\|\sum_{i=1}^k x_i\right\| \geq f(k)^{-1/4}\left(\sum_{j=1}^k \beta_j(\mathbb{N})^2\right)^{-\frac{1}{2}} \sum_{i=1}^k x_i^*(x_i).$$

Since $\beta_j(\mathbb{N}) \leq 1$, this give us

$$\left\|\sum_{i=1}^k x_i\right\| \geq f(k)^{-1/4}k^{-1/2}\sum_{i=1}^k x_i^*(x_i) \geq f(k)^{-1/4}k^{-1/2}(k/2-1) \geq \frac{1}{4}\frac{k^{1/2}}{f(k)^{1/4}}.$$

We will be done provided we can find a suitable upper bound for

$$\left\|\sum_{i=1}^k (-1)^{i-1} x_i\right\|.$$

We will do this in Lemma 5.6.2. First we need another result which says that if x is a sum of an RIS then $\|Fx\|$ can not be too small provided F is large enough.

Lemma 5.6.1 *Let $x_1, \dots, x_k$ be an RIS where each x_i is an $\ell_{2+}^{n_i}$ average with constant $1+\varepsilon$ ($\varepsilon = \frac{1}{10}$), and let $x = \sum_{i=1}^k x_i$. Let F be an interval in $\mathbb{N}$ such that $\lambda(F) \geq 1$. Then $\|Fx\| > \frac{1}{5}$.*

Proof. Since $\lambda(F) \geq 1$, the interval F must cover at least one half of some x_i. Write this x_i as $\sum_{j=1}^{n_i} x_{i,j}$ where $\|x_{i,j}\| \leq (1+\varepsilon) n_i^{-\frac{1}{2}}$. Let r and s be respectively the minimal and maximal j such that F covers $x_{i,j}$. Then by bimonotonicity and the upper ℓ_2 estimate,

$$\begin{aligned}
\|Fx\| &\geq \left\| \sum_{j=r}^{s} x_{i,j} \right\| \\
&\geq \|x_i\| - \left\| \sum_{j<r,\ j>s} x_{i,j} \right\| \\
&\geq 1 - \left(\sum_{j<r,\ j>s} \|x_{i,j}\|^2 \right)^{\frac{1}{2}} \\
&\geq 1 - (1+\varepsilon) \left(\sum_{j<r,\ j>s} n_i^{-1} \right)^{\frac{1}{2}} \\
&\geq 1 - (1+\varepsilon) \left(\frac{n_i}{2} n_i^{-1} \right)^{\frac{1}{2}} \\
&= 1 - \frac{1+\varepsilon}{\sqrt{2}}.
\end{aligned}$$

Remembering that $\varepsilon = \frac{1}{10}$, this shows that $\|Fx\| > \frac{1}{5}$. □

Lemma 5.6.2 *Let $\varepsilon = \frac{1}{10}$. Then*

$$\left\| \sum_{i=1}^{k} (-1)^i x_i \right\| \leq (1+2\varepsilon) \left(\frac{k}{f(k)} \right)^{\frac{1}{2}}.$$

Proof. By Lemma 5.5.3, each x_i is an $\ell_{2+}^{N_i}$-average with constant $1+\varepsilon$, where $N_i = M_i^{\varepsilon/4}$. By the choice of σ, and the lower bound for N_1, $x_1, x_2, \dots, x_k$ is an RIS of length k with constant $1+\varepsilon$. Note that the same is also true for the alternating sequence $x_1, -x_2, x_3, \dots, (-1)^{k-1} x_k$.

We will prove that $\left\| \sum_{i=1}^k (-1)^i x_i \right\|$ is not normed by any special functional arising from a special sequence of length k. Let $z_1^*, z_2^*, \dots, z_k^*$ be any special sequence of length k, and E,

F be any intervals contained in $\mathbb{N}$. We will show that $|z^*(x)| \le \frac{1}{5}$ where,

$$z^* = f(k)^{-1/4} \left(\sum_{j=i(E)}^{j(E)} \beta_j(E)^2 \right)^{-\frac{1}{2}} \sum_{i=i(E)}^{j(E)} FEz_i^*$$

is a $(k, f^{1/2})$-functional and $x = \sum_{i=1}^{k} (-1)^i x_i$.

Let $t \le s$ be maximal such that $z_t^* = x_t^*$ or zero if no such t exists. If $i \ne j$ or one of i, j is greater than $t+1$, then since σ is injective, there exist L_1, $L_2 \in L$ with $L_1 \ne L_2$ such that z_i^* is an (L_1, f)-functional and x_j is the normalized sum of an RIS of length L_2 and also is an $\ell_{2+}^{L_2'}$-average with constant $1+\varepsilon$, where $L_2' = L_2^{\varepsilon/4}$. Since z_i^* is an (L_1, f)-functional, we know that $Ez_i^*/\beta_i(E)$ is also an (L_1, f)-functional.

If $L_1 < L_2$, then the definition of L gives $L_1 < L_2'$. Also, $L_1 \ge j_{2k}$ as L_1 appears in a special sequence of length k. Write $Ez_i^*/\beta_i(E) = f(L_1)^{-1/2} \sum_{r=1}^{L_1} \gamma_r f_r$. Then by Lemma 5.1.5, writing $F_r = F \cap \operatorname{supp} f_r$ we have

$$\begin{aligned}
\left| \frac{Ez_i^*}{\beta_i(E)} (Fx_j) \right| &= \frac{1}{f(L_1)^{\frac{1}{2}}} \sum_{r=1}^{L_1} \gamma_r f_r(Fx_j) \\
&\le \frac{1}{f(L_1)^{\frac{1}{2}}} \left(\sum_{r=1}^{L_1} \|F_r x_j\|^2 \right)^{\frac{1}{2}} \\
&\le \frac{1}{f(L_1)^{\frac{1}{2}}} (1+\varepsilon) \left(1 + 2\frac{L_1}{L_2'} \right)^{\frac{1}{2}} \\
&\le \frac{\sqrt{3}(1+\varepsilon)}{f(L_1)^{\frac{1}{2}}}.
\end{aligned}$$

From the definition of L, $f(l) \ge 4k^4$ whenever $l \ge j_{2k}$, so $|Ez_i^*(Fx_j)| < 1/k^2 \beta_i(E)$.

Suppose now that $L_1 > L_2$. By Lemma 5.1.8 (with $\varepsilon = 1$) applied to the non-normalized sum x_j' of the RIS we get $|((Ez_i^*)/\beta_i(E))(Fx_j')| \le 3$ (Note that the definition of L gives $M_f(L_2) < L_1$). Hence

$$|((Ez_i^*)/\beta_i(E))(Fx_j)| \le \frac{3}{\|x_i'\|} \le 3 \left(\frac{f(L_2)}{L_2} \right)^{\frac{1}{2}} \le \frac{1}{k^2}$$

by the choice of L. So again, we have $|Ez_i^*(Fx_j)| < 1/k^2 \beta_i(E)$.

We now estimate $|z^*(Fx)|$.

$$|z^*(Fx)| = \left| f(k)^{-1/4} \left(\sum_{j=i(E)}^{j(E)} \beta_j(E)^2 \right)^{-\frac{1}{2}} \sum_{i=i(E)}^{j(E)} \sum_{j=1}^{k} Ez_i^*(F(-1)^j x_j) \right|.$$

We split this double sum into three separate parts. The first part is

$$\sum_{i=t+2}^{j(E)} \sum_{j=1}^{k} Ez_i^*(F(-1)^j x_j).$$

By applying the above estimates this is at most

$$\sum_{i=t+2}^{j(E)} \frac{1}{k}\beta_i(E).$$

The second term corresponds to those given by the functional z_{t+1}^*. This is

$$\sum_{j=1}^{k} Ez_{t+1}^*(F(-1)^j x_j),$$

and its modulus is bounded above by

$$\frac{1}{k}\beta_{t+1}(E) + \left\|Ez_{t+1}^*\right\| \le 2\beta_{t+1}(E).$$

The final term consists of the remaining terms,

$$\sum_{i=i(E)}^{t} \sum_{j=1}^{k} Ez_i^*(F(-1)^j x_j).$$

For i in this range, $z_i^* = x_i^*$ and recall also that for each i, $\operatorname{supp} x_i = \operatorname{supp} x_i^*$. Choose r, s to be respectively the first and last x_i^* such that $E \cap F$ meets $\operatorname{supp} x_i = \operatorname{supp} x_i^*$ and $i(E) \le r \le s \le t$. Since $x_i^*(x_i)$ is within $\frac{1}{k}$ of $\frac{1}{2}$ we have the following estimate.

$$\begin{aligned}\left|\sum_{i=i(E)}^{t} \sum_{j=1}^{k} Ez_i^*(F(-1)^j x_j)\right| &= \left|\sum_{i=r}^{s}(-1)^i Ex_i^*(Fx_i)\right| \\ &\le \|Ex_r^*\| + \left|\sum_{i=r+1}^{s-1}(-1)^i x_i^*(x_i)\right| + \|Ex_s^*\| \\ &\le \beta_r(E) + 2 + \beta_s(E).\end{aligned}$$

Observe that if $r+1 \le i \le s-1$ then E covers all of the functional $z_i^* = x_i^*$. Hence

$$\beta_i(E) \ge \|Ex_i^*\| = \|x_i^*\| \ge \frac{1}{2} - \frac{1}{k} \ge \frac{1}{4}.$$

Hence,

$$2 \le 8 \sum_{i=r+1}^{s-1} \frac{\beta_i(E)}{s-r-1}.$$

Applying Cauchy-Schwartz to the whole sum tells us that

$$|z^*(Fx)| \leq 9f(k)^{-\frac{1}{4}} < \frac{1}{5} \text{ by the choice of } K.$$

Let ϕ' be the function

$$\phi'(x) = \begin{cases} (\log_2(x+1))^{1/2} & \text{if } x \in K,\, x \neq k, \\ \log_2(x+1) & \text{otherwise.} \end{cases}$$

Let g' be the function obtained from ϕ' by the construction before Lemma 5.5.1, in the case where $K_0 = K \setminus \{k\}$. Then Lemma 5.5.1 tells us that $g'(l) = f(l)$ for every $l \in L \cup \{k\}$.

By Lemma 5.6.1, for every interval F such that $\lambda(F) \geq 1$, we have $\|Fx\| > 1/5$. Hence we know that Fx can not be normed by a special functional arising from a special sequence of length k. Thus

$$\frac{1}{5} < \|Fx\| \leq \sup\{|x^*(Fx)| : M \geq 2,\, x^* \text{ an } (M, g')\text{-functional}\}$$

for any interval F such that $\lambda(F) \geq 1$. Thus by Lemma 5.1.10 we have that

$$\|x\| \leq (1+2\varepsilon)(k/g'(k))^{1/2} = (1+2\varepsilon)(k/f(k))^{1/2}.$$

$\square$

We can now complete our proof that X is hereditarily indecomposable. Let $y \in Y$ be the sum of the odd x_is and $z \in Z$ be the sum of the even x_is. Then

$$\|y+z\| = \left\| \sum_{i=1}^{k} x_i \right\| \geq \frac{1}{4} \frac{k^{1/2}}{f(k)^{1/4}},$$

and by Lemma 5.6.2,

$$\|y-z\| = \left\| \sum_{i=1}^{k} (-1)^{i-1} x_i \right\| \leq (1+2\varepsilon)(k/f(k))^{1/2}.$$

Therefore,

$$\begin{aligned} \|y+z\| &\geq \frac{1}{4} \frac{k^{1/2}}{f(k)^{1/4}} \\ &\geq \frac{1}{4(1+2\varepsilon)} f(k)^{1/4} \|y-z\| \\ &= \frac{5}{24} f(k)^{1/4} \|y-z\| \\ &\geq \delta^{-1} \|y-z\|. \end{aligned}$$

But as δ is arbitrary, this implies that the projection $y + z \mapsto y$ is not continuous. Indeed, suppose that the map $\tilde{y} + \tilde{z} \mapsto \tilde{y}$ were continuous from $Y + Z$ to Y. Then there exists a constant $K > 0$ such that $\|\tilde{y}\| \leq K \|\tilde{y} + \tilde{z}\|$ for all $\tilde{y} \in Y$, $\tilde{z} \in Z$. This implies also that $\|\tilde{z}\| \leq (K + 1) \|\tilde{y} + \tilde{z}\|$ for all $\tilde{y} \in Y$, $\tilde{z} \in Z$. Suppose first that the vectors y and z that we have constructed satisfy $\|z\| \leq \|y\|$. Then,

$$\frac{1}{K} \|y\| \leq \|y - z\| \leq \delta \|y + z\| \leq 2\delta \|y\| .$$

Otherwise, if $\|y\| \leq \|z\|$, then,

$$\frac{1}{K+1} \|z\| \leq \|y - z\| \leq \delta \|y + z\| \leq 2\delta \|z\| .$$

Hence, in either case $\delta \geq \frac{1}{2(K+1)}$, which gives us the required contradiction for sufficiently small δ. Therefore we have shown that X is hereditarily indecomposable.

Finally, it is easy to show that X is reflexive.

Proposition 5.6.3 *X is reflexive.*

PROOF. We show that the basis of X is shrinking and boundedly complete. Suppose that the basis is not shrinking. Then there is a norm one functional x^* and successive normalized block vectors $x_1 < x_2 < \dots$ such that $x^*(x_n) \geq \varepsilon$ for all $n \in \mathbb{N}$. But then for any $N \in \mathbb{N}$,

$$N\varepsilon \leq x^* \left(\sum_{i=1}^{N} x_i \right) \leq \left\| \sum_{i=1}^{N} x_i \right\| \leq \left(\sum_{i=1}^{N} \|x_i\|^2 \right)^{\frac{1}{2}} = N^{\frac{1}{2}},$$

which gives us a contradiction for sufficiently large N.

Now suppose that the basis is not boundedly complete. Then there is a sequence of scalars (a_i) such that $\sup_n \|\sum_{i=1}^{n} a_i e_i\| = M < \infty$ but $\sum_{i=1}^{\infty} a_i e_i$ does not converge. Thus there exist an $\varepsilon > 0$ and a sequence of block vectors $y_1 < y_2 < \dots$ where each y_i has the form $y_i = \sum_{j=p_i}^{q_i} a_j e_j$ such that $\|y_i\| \geq \varepsilon$. Then

$$M \geq \left\| \sum_{i=1}^{q_N} a_i e_i \right\| \geq \frac{1}{f(N)^{\frac{1}{2}}} \left(\sum_{j=1}^{N} \|y_j\|^2 \right)^{\frac{1}{2}} \geq \varepsilon \left(\frac{N}{f(N)} \right)^{\frac{1}{2}},$$

which gives us the required contradiction for large N. □

Chapter 6

Consequences of strong compactness in $\mathcal{T}(X)$

Contents

In this chapter we will consider the consequences of strong compactness of certain subsets of $\mathcal{T}(X)$, the set of types on X. We will show that in certain circumstances, we can deduce the existence of spreading models that are ℓ_p over X for some p (or c_0 over X) i.e. for any

$x \in X$ and any scalars $(a_i)_{i=1}^n$,

$$\left\| x + \sum_{i=1}^n a_i e_i \right\| = \left\| x + \left(\sum_{i=1}^n a_i^p \right)^{\frac{1}{p}} e_1 \right\| .$$

We will also show that, under the same conditions, X will then contain $(1+\varepsilon)$-isomorphic copies of ℓ_p or c_0. We then give some examples of spaces for which this compactness condition is true.

6.1 Existence of symmetric strong types

Definition 6.1.1 *A strong type τ is said to be symmetric if $\tau = (-1) \cdot \tau$.*

Our proof that symmetric strong types exist uses the Borsuk-Ulam Theorem, which can be found in any introductory book on algebraic topology (see e.g. [34, 26.15]).

Theorem 6.1.2 (Borsuk-Ulam Theorem) *If $n > m$, and $f : S^n \to \mathbb{R}^m$ is a map such that $f(-x) = -f(x)$ for all $x \in S^n$, then there exists $x_0 \in S^n$ such that $f(x_0) = 0$.*

We are now able to prove our existence result.

Lemma 6.1.3 *If X is an infinite dimensional Banach space, then $\mathcal{S}(X)$ contains a non-degenerate symmetric strong type.*

Proof. Notice first that the only degenerate symmetric strong type is τ_0. For if τ_y is symmetric, then in particular for any $x \in X$,

$$\|x + y\| = \tau_y(\theta_x) = \tau_y(-1 \cdot \theta_x) = \tau_y(\theta_{-x}) = \|x - y\| .$$

Taking $x = y$ tells us that $y = 0$.

Suppose that $F = \{\theta_1, \ldots, \theta_n\}$ is a finite subset of $\mathcal{T}(X)$, and consider the mapping $x \mapsto (\theta_j(x) - \theta_j(-x))_{j=1}^n$ from X to $\mathbb{R}^n$. By the Borsuk-Ulam Theorem, there exists $x_F \in S_X$ such that $\theta_j(x_F) = \theta_j(-x_F)$ for $j = 1, \ldots, n$. Then for every such F, $\tau_{x_F} \in \mathcal{S}_1(X) = \{\sigma \in \mathcal{S}(X) : \|\sigma\| \leq 1\}$, which is a compact subset of $\mathcal{S}(X)$. We define a filter $\mathcal{F}$ on $\mathcal{S}_1(X)$ by

$$\mathcal{F} = \{A \subseteq \mathcal{S}_1(X) : \exists F \subseteq \mathcal{T}(X) \text{ finite s.t. } \tau_{x_G} \in A \text{ when } G \supseteq F\}.$$

By the compactness of $\mathcal{S}_1(X)$, $\mathcal{F}$ has an accumulation point τ in $\mathcal{S}_1(X)$. We will show that τ is a symmetric strong type.

Suppose that $\theta \in \mathcal{T}(X)$ and $\varepsilon > 0$. Define a neighbourhood U of τ in $\mathcal{S}(X)$ by

$$U = \{\sigma \in \mathcal{S}(X) : |\sigma(\theta) - \tau(\theta)| < \varepsilon, |\sigma(-\theta) - \tau(-\theta)| < \varepsilon\},$$

and a member A of $\mathcal{F}$ by

$$A = \{\tau_{x_G} : G \supseteq \{\theta\}\}.$$

Since τ is an accumulation point of $\mathcal{F}$, $U \cap A \neq \varnothing$. Thus, there exists a finite subset G containing θ, such that $|\tau_{x_G}(\theta) - \tau(\theta)| < \varepsilon$ and $|\tau_{x_G}(-\theta) - \tau(-\theta)| < \varepsilon$. Notice that since $\theta \in G$, $\tau_{x_G}(-\theta) = \theta(-x_G) = \theta(x_G) = \tau_{x_G}(\theta)$. Hence

$$\begin{aligned}|\tau(\theta) - (-\tau)(\theta)| &= |\tau(\theta) - \tau(-\theta)| \\ &\leq |\tau(\theta) - \tau_{x_G}(\theta)| + |\tau_{x_G}(-\theta) - \tau(-\theta)| \\ &\leq 2\varepsilon.\end{aligned}$$

Since ε is arbitrary, this shows that $\tau = -1 \cdot \tau$. Also, since for every $F \subseteq \mathcal{T}(X)$, $\|\tau_{x_G}\| = \|x_G\| = 1$, an argument similar to the above one shows that $\|\tau\| = 1$. Thus, $\tau \neq \tau_0$, and so τ is non-degenerate. ☐

6.2 Spreading models induced by strong types

Let τ be a non-degenerate strong type given by $\tau(\theta) = \lim_{x \to \mathcal{U}} \theta(x)$ for every $\theta \in \mathcal{T}(X)$. Using τ, we can obtain a spreading model of X, by defining

$$\left\| x + \sum_{i=1}^{k} c_i e_i \right\| = (p(c_1\tau * \ldots * c_k\tau)(x)).$$

Observe that,

$$\begin{aligned}(p(c_1\tau * \ldots * c_k\tau))(x) &= (c_1\tau * \ldots * c_k\tau)(\theta_x) \\ &= (c_1\tau)((c_2\tau * \ldots * c_k\tau) * \theta_x) \\ &= \lim_{x_1 \to \mathcal{U}} ((c_2\tau * \ldots * c_k\tau) * \theta_x)(c_1 x_1) \\ &= \lim_{x_1 \to \mathcal{U}} (c_2\tau * \ldots * c_k\tau)(\theta_{x + c_1 x_1}) \\ &= \lim_{x_1 \to \mathcal{U}} (p(c_2\tau * \ldots * c_k\tau))(x + c_1 x_1).\end{aligned}$$

Iterating this, gives

$$\begin{aligned}(p(c_1\tau * \ldots * c_k\tau))(x) &= \lim_{x_1 \to \mathcal{U}} \ldots \lim_{x_{k-1} \to \mathcal{U}} (p(c_k\tau)) \left(x + \sum_{i=1}^{k-1} c_i x_i \right) \\ &= \lim_{x_1 \to \mathcal{U}} \ldots \lim_{x_k \to \mathcal{U}} \left\| x + \sum_{i=1}^{k} c_i x_i \right\|.\end{aligned}$$

We know that the above defines a spreading model provided the ultrafilter $\mathcal{U}$ does not converge. This follows from the fact that τ is non-degenerate, for if $\mathcal{U} \to x$, then it is clear that $\tau = \tau_x$.

Now suppose that in addition τ is symmetric. Then clearly for any scalars $c_1, \ldots, c_k$,

$$(\pm c_1 \tau) * \ldots * (\pm c_k \tau) = c_1 \tau * \ldots * c_k \tau.$$

In other words, the spreading model generated by τ is 1-unconditional over X i.e.

$$\left\| x + \sum_{i=1}^{k} \pm c_i e_i \right\| = \left\| x + \sum_{i=1}^{k} c_i e_i \right\|.$$

Hence Lemma 6.1.3 guarantees the existence of spreading models that are 1-unconditional over X.

6.3 Sufficient conditions allowing us to pull the spreading model back into X

Definition 6.3.1 *Given* $\tau \in \mathcal{S}(X)$*, we define* $K(\tau)$ *to be the closure in* $\mathcal{S}(X)$ *of*

$$\{c_1 \tau * \ldots * c_k \tau : k \in \mathbb{N} \text{ and } c_1, \ldots, c_k \in \mathbb{R}\}.$$

We then define, for any $M > 0$*,*

$$K^M(\tau) = \{\sigma \in K(\tau) : \|\sigma\| \leq M\}.$$

Remark 6.3.2 $K^1(\tau)$ *is a closed subset of the compact set* $\{\sigma \in \mathcal{S}(X) : \|\sigma\| \leq 1\}$*, and hence is itself compact. Since* $p : \mathcal{S}(X) \to \mathcal{T}(X)$ *is continuous, it follows that* $p(K^1(\tau))$ *is compact in the weak topology of* $\mathcal{T}(X)$*. In particular,* $p(K^1(\tau))$ *is closed in the weak topology, and therefore closed in the strong topology. In what follows therefore we shall consider the consequences if we have the stronger property that* $p(K^1(\tau))$ *is strongly compact.*

Our first lemma shows that strong compactness of $p(K^1(\tau))$ implies strong compactness of $p(K^M(\tau))$ for any $M > 0$.

Lemma 6.3.3 *If* $\tau \in \mathcal{S}(X)$ *is such that* $p(K^1(\tau))$ *is strongly compact, then* $p(K^M(\tau))$ *is strongly compact for all* $M > 0$*.*

PROOF. Let $q_M : \mathcal{S}(X) \to \mathcal{S}(X)$ denote the map $\sigma \mapsto M \cdot \sigma$, and $r_M : \mathcal{T}(X) \to \mathcal{T}(X)$ denote the map $\theta \mapsto M \cdot \theta$. Observe that $\|p(\sigma)\| = (p(\sigma))(0) = \sigma(\theta_0) = \|\sigma\|$ and that $\|\alpha \cdot \sigma\| = |\alpha| \, \|\sigma\|$. Thus it is clear that $K^M(\tau) = q_M(K^1(\tau))$, since q_M maps $K(\tau)$ into $K(\tau)$ and q_M is invertible. It is immediate from the definitions that $p(q_M(\sigma)) = r_M(p(\sigma))$. Hence $p(K^M(\tau)) = r_M(p(K^1(\tau)))$. To complete the proof it suffices to show that r_M is continuous from $\mathcal{T}(X)$ to itself when $\mathcal{T}(X)$ is equipped with the strong topology. Let $U \subseteq \mathcal{T}(X)$ be open in the strong topology, and let $\theta \in r_M^{-1}(U)$. Hence $M \cdot \theta \in U$, and so there exists N and $\varepsilon > 0$ such that

$$d_N(M \cdot \theta, \phi) < \varepsilon \Rightarrow \phi \in U.$$

Since $d_N(M \cdot \theta, M \cdot \psi) = M d_{N/M}(\theta, \psi)$, it is immediate that

$$d_{N/M}(\theta, \psi) < \frac{\varepsilon}{M} \Rightarrow M \cdot \psi \in U \Rightarrow \psi \in r_M^{-1}(U).$$

Therefore $r_M^{-1}(U)$ is open in the strong topology, and we are done. $\square$

Lemma 6.3.4 *For any x, $y \in X$, $x * \tau_y$ is a continuous function from $\mathcal{T}(X)$, equipped with its strong topology, to $\mathbb{R}$. Further, if $p(K^1(\tau))$ is strongly compact, then for any $M_1, M_2 > 0$, the set*

$$\left\{(x * \tau_y)\restriction_{p(K^M(\tau))} : \|x\| \le M_1 \text{ and } \|y\| \le M_2\right\},$$

is a relatively compact subset of $C(p(K^M(\tau))$, where we again have the strong topology on $p(K^M(\tau))$.

PROOF. Let $(\theta_i)_{i \in I}$ be a net contained in $\mathcal{T}(X)$ which converges to θ in the strong topology. Then in particular, (θ_i) converges to θ weakly. Hence

$$(x * \tau_y)(\theta_i) = \tau_y(x * \theta_i) = (x * \theta_i)(y) = \theta_i(x + y) \to \theta(x + y) = (x * \tau_y)(\theta).$$

Thus, $x * \tau_y$ is a continuous function on $\mathcal{T}(X)$ equipped with the strong topology.

If $p(K^1(\tau))$ is strongly compact then by Lemma 6.3.3, $p(K^M(\tau))$ is strongly compact and the strong topology is metrizable. Therefore, to prove relative compactness of a subset of $C(p(K^M(\tau))$ it suffices, by the Arzela-Ascoli Theorem, to show that the family is equicontinuous and uniformly bounded.

We start by showing equicontinuity. Recall that the strong topology has the metric $d(\theta, \phi) = \sum_{m=1}^{\infty} 2^{-m} d_m(\theta, \phi)$. Therefore whenever $\|x\| \le M_1$ and $\|y\| \le M_2$, letting N be an integer bigger than $M_1 + M_2$, we have

$$\begin{aligned}|(x * \tau_y)(\theta) - (x * \tau_y)(\phi)| &= |\theta(x+y) - \phi(x+y)| \\ &\leq d_N(\theta, \phi) \\ &\leq 2^N d(\theta, \phi).\end{aligned}$$

This completes the proof of the equicontinuity.
If $\theta \in p(K^M(\tau))$, then $\|\theta\| \leq M$. Hence,

$$|(x * \tau_y)(\theta)| = |\theta(x+y)| \leq \|x+y\| + \|\theta\| \leq M_1 + M_2 + M.$$

Hence $\Big\{(x * \tau_y)\restriction_{p(K^M(\tau))} \colon \|x\| \leq M_1 \text{ and } \|y\| \leq M_2\Big\}$ is a uniformly bounded family of functions as well as being equicontinuous. It follows by the Arzela-Ascoli theorem that this set is relatively compact in $C(p(K^M(\tau)))$. ☐

We will also need the following lemma due to Haydon and Maurey. We use the notation $\lambda \overset{1+\varepsilon}{\sim} \mu$ to mean $(1+\varepsilon)^{-1}\lambda \leq \mu \leq (1+\varepsilon)\lambda$.

Lemma 6.3.5 ([42, Lemma 1.1]) *Let U be a normed space, V a normed space containing U as a closed subspace, and v an element of $V \setminus U$. Given $\varepsilon > 0$, there exists M and $\eta > 0$ such that, if W is a normed space containing U, and $w \in W$ satisfies*

$$|\,\|w+z\| - \|v+z\|\,| \leq \eta \text{ for all } z \in U \text{ with } \|z\| \leq M,$$

then

$$\|w+z\| \overset{1+\varepsilon}{\sim} \|v+z\|$$

for all $z \in U$.

Lemma 6.3.6 *Suppose τ is a symmetric non-degenerate strong type such that $p(K^1(\tau))$ is strongly compact, and consider the spreading model given by τ. Then for every finite dimensional subspace F of X, and every $\varepsilon > 0$ there exists $x_1 \in X$ such that for all $x \in F$ and every k and every sequence of scalars $c_1, \ldots, c_k$,*

$$\left\| x + \sum_{i=1}^{k} c_i e_i \right\| \overset{1+\varepsilon}{\sim} \left\| x + c_1 x_1 + \sum_{i=2}^{k} c_i e_i \right\|.$$

Proof. We will apply Lemma 6.3.5 with

$$\begin{aligned} U &= F \oplus \operatorname{span}(\{e_n : n \geq 2\}), \\ V &= X \oplus \operatorname{span}(\{e_n : n \geq 1\}), \\ W &= X \oplus \operatorname{span}(\{e_n : n \geq 1\}). \end{aligned}$$

We take the vector $e_1 \in V \setminus U$. Thus we have M and $\eta > 0$ such that if $y \in W$ satisfies $|\,\|y+z\| - \|e_1+z\|\,| < \eta$ for all $z \in U$ with $\|z\| \leq M$, then $\|y+z\| \overset{1+\varepsilon}{\sim} \|e_1+z\|$ for all $z \in U$.

Suppose that $(y_i)_{i\in I}$ is a net such that $\tau(\theta) = \lim_{i\to\infty} \theta(y_i)$ for all $\theta \in \mathfrak{T}(X)$. We may assume that the net (y_i) is bounded, with say $\|y_i\| \leq K$ for all $i \in I$. By Lemma 6.3.4,

$$\left\{(x * \tau_y)\restriction_{p(K^{2M}(\tau))} : \|x\| \leq M \text{ and } \|y\| \leq K\right\}$$

is a relatively compact subset of $C(p(K^{2M}(\tau))$. For all $\theta \in \mathfrak{T}(X)$, $(x * \tau_{y_i})(\theta) \to (x * \tau)(\theta)$ as $i \to \infty$. Choose a finite $\frac{\eta}{3}$-net, $\mathcal{E}$, for MB_F. By the compactness in $C(p(K^{2M}(\tau))$, we can find $x_1 \in X$ (which will be one of the y_i) such that

$$|(x * \tau_{x_1})(\theta) - (x * \tau)(\theta)| < \frac{\eta}{3}$$

for all $x \in \mathcal{E}$ and all $\theta \in p(K^{2M}(\tau))$. It follows from this that

$$|(x * \tau_{x_1})(\theta) - (x * \tau)(\theta)| < \eta. \tag{6.1}$$

for all $x \in MB_F$ and all $\theta \in p(K^{2M}(\tau))$.

Suppose that $z \in U$ with $\|z\| \leq M$. We write $z = x + \sum_{i=2}^{k} c_i e_i$ with $x \in F$. Since τ is symmetric, the spreading model is 1-unconditional over X. Hence $\|x\| \leq M$ and $\left\|\sum_{i=2}^{k} c_i e_i\right\| \leq 2M$. Now,

$$\begin{aligned}
\|x_1 + z\| &= \left\| x + x_1 + \sum_{i=2}^{k} c_i e_i \right\| \\
&= (p(\overset{k}{\underset{2}{*}} c_i\tau))(x + x_1) \\
&= \tau_{x+x_1}(p(\overset{k}{\underset{2}{*}} c_i\tau)) \\
&= (x * \tau_{x_1})(p(\overset{k}{\underset{2}{*}} c_i\tau)),
\end{aligned}$$

and similarly,

$$\begin{aligned}
\|e_1 + z\| &= \left\| x + e_1 + \sum_{i=2}^{k} c_i e_i \right\| \\
&= \tau((\overset{k}{\underset{2}{*}} c_i\tau) * \theta_x) \\
&= \tau(x * p(\overset{k}{\underset{2}{*}} c_i\tau)) \\
&= (x * \tau)(p(\overset{k}{\underset{2}{*}} c_i\tau)).
\end{aligned}$$

Now $x_1 \in X \subseteq W$, $x \in MB_F$ and $p(\overset{k}{\underset{2}{*}}c_i\tau)) \in p(K^{2M}(\tau))$, and hence by (6.1) we have

$$|\,\|x_1+z\|-\|e_1+z\|\,| = \left|(x*\tau_{x_1})(p(\overset{k}{\underset{2}{*}}c_i\tau)) - (x*\tau)(p(\overset{k}{\underset{2}{*}}c_i\tau))\right| < \eta.$$

Thus by the result of Lemma 6.3.5,

$$\|x_1+z\| \overset{1+\varepsilon}{\sim} \|e_1+z\| \text{ for all } z \in U.$$

In other words for all $x \in F$, all k and scalars $c_2, \dots, c_k$,

$$\left\|x + x_1 + \sum_{i=2}^{k} c_i e_i\right\| \overset{1+\varepsilon}{\sim} \left\|x + e_1 + \sum_{i=2}^{k} c_i e_i\right\|.$$

Hence, for all $c_1, \dots, c_k$,

$$\left\|x + c_1 x_1 + \sum_{i=2}^{k} c_i e_i\right\| \overset{1+\varepsilon}{\sim} \left\|x + \sum_{i=1}^{k} c_i e_i\right\|.$$

This follows by homogeneity for $c_1 \neq 0$, and by continuity for $c_1 = 0$. This completes the proof of the lemma. □

Theorem 6.3.7 *Let τ be a symmetric non-degenerate strong type and suppose that $p(K^1(\tau))$ is strongly compact. Then for any $\varepsilon > 0$ there is a sequence (x_n) in X which is $(1+\varepsilon)$-equivalent to the sequence (e_n) in the spreading model determined by τ. Moreover, in the case where X is separable and (y_n) is a Brunel-Sucheston sequence for this spreading model, then (x_n) may be taken to be a subsequence of (y_m).*

Proof. This follows easily from Lemma 6.3.6. Choose a sequence (ε_i) of positive numbers such that $\prod_{i=1}^{\infty}(1+\varepsilon_i) \leq (1+\varepsilon)$. We define our sequence (x_n) in X recursively. We begin by choosing x_1 to be the vector given by Lemma 6.3.6 with $F = \{0\}$ and $\varepsilon = \varepsilon_1$. Then given $x_1, \dots, x_{n-1}$, we take x_n to be the vector given by Lemma 6.3.6 with $F = \text{span}(x_1, \dots, x_{n-1})$ and $\varepsilon = \varepsilon_n$. Then for any k and any constants $c_1, \dots, c_k$ we have,

$$\begin{aligned}
\left\|\sum_{i=1}^{k} c_i e_i\right\| &\overset{1+\varepsilon_1}{\sim} \left\|c_1 x_1 + \sum_{i=2}^{n} c_i e_i\right\| \\
&\overset{(1+\varepsilon_1)(1+\varepsilon_2)}{\sim} \left\|c_1 x_1 + c_2 x_2 + \sum_{i=3}^{n} c_i e_i\right\| \\
&\dots \\
&\overset{\prod_{i=1}^{k}(1+\varepsilon_i)}{\sim} \left\|\sum_{j=1}^{k} c_j x_j\right\|
\end{aligned}$$

Thus,

$$\left\| \sum_{i=1}^{k} c_i e_i \right\| \overset{1+\varepsilon}{\sim} \left\| \sum_{i=1}^{k} c_i x_i \right\|,$$

which completes our proof. If our spreading model is given by a Brunel-Sucheston sequence, then notice that in the proof of Lemma 6.3.6 we can choose the vectors x_i to be elements of the Brunel-Sucheston sequence. ☐

6.4 Sufficient conditions for existence of spreading models that are ℓ_p over X

Lemma 6.4.1 *Let τ be a non-degenerate strong type on X. For every $n \in \mathbb{N}$ and $\varepsilon > 0$ there exists a strong type $\sigma = (c_1 \cdot \tau) * \ldots * (c_l \cdot \tau)$ and $\lambda_1, \ldots, \lambda_n \in \mathbb{R}$ with the property that $\|\sigma\| = 1$ and*

$$\sup_{\theta \in \mathcal{T}(X)} |\underbrace{(\sigma * \ldots * \sigma)}_{m \textit{ terms}}(\theta) - (\lambda_m \cdot \sigma)(\theta)| \leq \varepsilon \textit{ for } m = 1, 2, \ldots, n.$$

Proof. Let $Y = \mathbb{R}^{(J)}$ where $J = \mathbb{Q} \cap (0,1)$. We define a norm on Y by

$$\left\| \sum_{j=1}^{m} c_j e_{q_j} \right\| = (c_1 \tau * \ldots * c_m \tau)(\theta_0),$$

where $q_1 < q_2 < \ldots < q_m$. We define operators $T_1, \ldots, T_n$ on Y by

$$T_k e_q = \sum_{j=0}^{k-1} e_{\left(\frac{q+j}{k}\right)}.$$

T_k is the operator on Y which takes the vector $x \in Y$ to the vector which repeats x k-times in the intervals $(0, \frac{1}{k}), \ldots, (\frac{k-1}{k}, 1)$. These operators commute, with $T_i T_j = T_{ij} = T_j T_i$, and they are each continuous, since if $y = \sum_{j=1}^{n} c_j e_{q_j}$, then

$$\begin{aligned} \|T_k y\| &= \left\| \sum_{r=0}^{k-1} \sum_{j=1}^{n} c_j e_{\left(\frac{q_j + r}{k}\right)} \right\| \\ &\leq \sum_{r=0}^{k-1} \left\| \sum_{j=1}^{n} c_j e_{\left(\frac{q_j + r}{k}\right)} \right\| \\ &= \sum_{r=0}^{k-1} \left\| \sum_{j=1}^{n} c_j e_{q_j} \right\| \\ &= k \|y\|. \end{aligned}$$

Hence we can find $\lambda_1, \ldots, \lambda_n \geq 0$, and a sequence $(y_m)_{m=1}^{\infty} \subseteq Y$ such that $\|y_m\| = 1$ and

$$\|T_k y_m - \lambda_k y_m\| \to 0 \text{ for } 1 \leq k \leq n.$$

Given $\varepsilon > 0$, choose m sufficiently large that $\|T_k y_m - \lambda_k y_m\| \leq \varepsilon$ for $1 \leq k \leq n$. Write $y_m = \sum_{i=1}^{l} c_i e_{q_i}$. If we transfer this into the language of spreading models, then this tells us that in the spreading model generated by τ,

$$\left\| \sum_{j=0}^{k-1} \sum_{i=1}^{l} c_i e_{jl+i} - \lambda_k \sum_{i=1}^{l} c_i e_{ki} \right\| \leq \varepsilon. \tag{6.2}$$

Now if $\theta \in \mathcal{T}(X)$, then notice that

$$\left(\mathop{*}_{j=0}^{k-1} \mathop{*}_{i=1}^{l} c_i \tau \right)(\theta) = \lim_{x_1 \to \mathcal{U}} \ldots \lim_{x_{kl} \to \mathcal{U}} \theta \left(\sum_{j=0}^{k-1} \sum_{i=1}^{l} c_i x_{jl+i} \right),$$

and,

$$\left(\mathop{*}_{i=1}^{l} c_i \tau \right)(\theta) = \lim_{x_1 \to \mathcal{U}} \ldots \lim_{x_{kl} \to \mathcal{U}} \theta \left(\sum_{i=1}^{l} c_i x_{ki} \right),$$

Also, by (6.2) we have

$$\lim_{x_1 \to \mathcal{U}} \ldots \lim_{x_{kl} \to \mathcal{U}} \left\| \sum_{j=0}^{k-1} \sum_{i=1}^{l} c_i x_{jl+i} - \lambda_k \sum_{i=1}^{l} c_i x_{ki} \right\| \leq \varepsilon.$$

Hence, given $\eta > 0$ we can find $x_1, \ldots, x_{kl} \in X$ such that

$$\left| \left(\mathop{*}_{j=0}^{k-1} \mathop{*}_{i=1}^{l} c_i \tau \right)(\theta) - \theta \left(\sum_{j=0}^{k-1} \sum_{i=1}^{l} c_i x_{jl+i} \right) \right| < \eta,$$

$$\left| \left(\lambda_k \cdot \mathop{*}_{i=1}^{l} c_i \tau \right)(\theta) - \theta \left(\sum_{i=1}^{l} \lambda_k c_i x_{ki} \right) \right| < \eta \text{ and,}$$

$$\left\| \sum_{j=0}^{k-1} \sum_{i=1}^{l} c_i x_{jl+i} - \lambda_k \sum_{i=1}^{l} c_i x_{ki} \right\| \leq \varepsilon + \eta.$$

Using these inequalities and the fact that θ is Lipschitz with Lipschitz constant 1, tells us that

$$\left| \left(\mathop{*}_{j=0}^{k-1} \mathop{*}_{i=1}^{l} c_i \tau \right)(\theta) - \left(\lambda_k \cdot \mathop{*}_{i=1}^{l} c_i \tau \right)(\theta) \right| < \varepsilon + 3\eta.$$

We take $\sigma = \overset{l}{\underset{i=1}{*}} c_i\tau$, which then satisfies for any $\theta \in \mathcal{T}(X)$

$$|\underbrace{(\sigma * \ldots * \sigma)}_{\text{m terms}}(\theta) - (\lambda_m \cdot \sigma)(\theta)| \leq \varepsilon \text{ for } m = 1, 2, \ldots, n.$$

The fact that $\|\sigma\| = 1$ follows from the condition that $\|y_m\| = 1$. ☐

Lemma 6.4.2 *$p(K(\tau)) \subseteq \mathcal{T}(X)$ is closed in the weak topology.*

Proof. For any M, $K^M(\tau) = \{\sigma \in K(\tau) : \|\sigma\| \leq M\}$ is compact in $\mathcal{S}(X)$, as a closed subset of the compact set $\{\sigma \in \mathcal{S}(X) : \|\sigma\| \leq M\}$. Also the map $p : \mathcal{S}(X) \to \mathcal{T}(X)$ is a continuous surjection when $\mathcal{T}(X)$ is given the weak topology. Thus $p(K^M(\tau))$ is compact in the weak topology of $\mathcal{T}(X)$, and so is closed in the weak topology.

Suppose that $(\theta^i)_{i \in I}$ is a net in $p(K(\tau))$ such that $\theta^i \to \theta$ weakly in $\mathcal{T}(X)$. Now we write $\theta^i = p(\tau^i)$ where $\tau^i \in K(\tau)$ for each $i \in I$. Now,

$$\left\|\tau^i\right\| = \tau^i(\theta_0) = p(\tau^i)(0) = \theta^i(0) \to \theta(0).$$

Thus, we may assume that $(\left\|\tau^i\right\|)_{i \in I}$ is bounded by M say. Then

$$\theta^i = p(\tau^i) \in p(K^M(\tau)).$$

Since $p(K^M(\tau))$ is closed in the weak topology, it follows that $\theta \in p(K^M(\tau)) \subseteq p(K(\tau))$. Thus $p(K(\tau))$ is closed in the weak topology of $\mathcal{T}(X)$. ☐

Lemma 6.4.3 *Suppose that $\theta, \theta_i \in \mathcal{T}(X)$ and $\tau, \tau_i \in \mathcal{S}(X)$ are such that $\tau_i \to \tau$ in $\mathcal{S}(X)$, $\|\tau_i\| \leq M$, and $\theta_i \to \theta$ strongly in $\mathcal{T}(X)$. Then $\tau_i * \theta_i \to \tau * \theta$ weakly in $\mathcal{T}(X)$.*

Proof. Let $x \in X$, and $\varepsilon > 0$ then

$$\begin{aligned} |(\tau_i * \theta_i)(x) - (\tau * \theta)(x)| &\leq |(\tau_i * \theta_i)(x) - (\tau_i * \theta)(x)| + |(\tau_i * \theta)(x) - (\tau * \theta)(x)| \\ &= |\tau_i(x * \theta_i) - \tau_i(x * \theta)| + |\tau_i(x * \theta) - \tau(x * \theta)|. \end{aligned}$$

Since $\tau_i \to \tau$ in $\mathcal{S}(X)$, $|\tau_i(x * \theta) - \tau(x * \theta)| < \varepsilon$ provided i is sufficiently large. Suppose that $\mathcal{U}_i$ is an ultrafilter such that $\tau_i(\phi) = \lim_{z \to \mathcal{U}_i} \phi(z)$ for all $\phi \in \mathcal{T}(X)$. Since $\|\tau_i\| \leq M$, there exists $U \in \mathcal{U}_i$ such that $\|z\| \leq M + 1$ for all $z \in U$. Choose an N such that $N \geq M + 1 + \|x\|$. Then

$$\begin{aligned} |\tau_i(x * \theta_i) - \tau_i(x * \theta)| &= |\lim_{z \to \mathcal{U}_1} (x * \theta_i)(z) - (x * \theta)(z)| \\ &= |\lim_{z \to \mathcal{U}_1} \theta_i(x + z) - \theta(x + z)| \\ &\leq d_N(\theta_i, \theta) \\ &\leq \varepsilon \text{ for } i \text{ sufficiently large.} \end{aligned}$$

Thus, for large enough i,

$$|(\tau_i * \theta_i)(x) - (\tau * \theta)(x)| \leq 2\varepsilon,$$

i.e. $(\tau_i * \theta_i)$ converges weakly to $(\tau * \theta)$ in $\mathcal{T}(X)$. ☐

We will need to use the following simple lemma from topology.

Lemma 6.4.4 *Suppose that A is a set, and $\mathcal{S}$ and $\mathcal{T}$ are topologies on A, such that $\mathcal{S} \subseteq \mathcal{T}$. Suppose that $(A, \mathcal{S})$ is Hausdorff and $(A, \mathcal{T})$ is compact. Then $\mathcal{S} = \mathcal{T}$.*

Lemma 6.4.5 *Let τ be a strong type on X and suppose that $p(K^1(\tau))$ is strongly compact. Then*

$$(\sigma_1, \ldots, \sigma_n) \mapsto p(\sigma_1 * \ldots * \sigma_n)$$

is continuous from $K(\tau)^n \to p(K(\tau))$, where $K(\tau)$ is equipped with its weak topology.

Proof. We will do this by induction on n. Notice that we need to prove that the above map is into $p(K(\tau))$. If $n = 1$, then the map $\sigma \mapsto p(\sigma)$ is a map that takes $K(\tau)$ into $p(K(\tau))$. It is continuous, since, if (σ_i) is a net converging to σ in $\mathcal{S}(X)$ then for any $x \in X$,

$$p(\sigma_i)(x) = \sigma_i(\theta_x) \to \sigma(\theta_x) = p(\sigma)(x).$$

Now suppose that the map $(\sigma_1, \ldots, \sigma_{n-1}) \mapsto p(\sigma_1 * \ldots * \sigma_{n-1})$ is continuous from $K(\tau)^{n-1} \to p(K(\tau))$. Let $(\sigma_1^i, \ldots, \sigma_n^i)_{i \in I}$ be a net in $K(\tau)^n$ converging to $(\sigma_1, \ldots, \sigma_n)$. Let $x \in X$, and observe that

$$\begin{aligned} p(\sigma_1^i * \ldots * \sigma_n^i)(x) &= (\sigma_1^i * \ldots * \sigma_n^i)(\theta_x) \\ &= \sigma_1^i((\sigma_2^i * \ldots * \sigma_n^i) * \theta_x) \\ &= \sigma_1^i((\sigma_2^i * \ldots * \sigma_n^i) * \theta_x) \\ &= (\sigma_1^i * ((\sigma_2^i * \ldots * \sigma_n^i) * \theta_x))(0). \end{aligned}$$

For any $y \in X$,

$$\begin{aligned} ((\sigma_2^i * \ldots * \sigma_n^i) * \theta_x)(y) &= (\sigma_2^i * \ldots * \sigma_n^i)(\theta_{x+y}) \\ &= p(\sigma_2^i * \ldots * \sigma_n^i)(x+y) \\ &\to p(\sigma_2 * \ldots * \sigma_n)(x+y) \text{ by inductive hypothesis} \\ &= ((\sigma_2 * \ldots * \sigma_n) * \theta_x)(y). \end{aligned} \tag{6.3}$$

We may suppose that $\|\sigma_1^i\| \leq M$ and $\|\sigma_2^i\| + \ldots + \|\sigma_n^i\| \leq M$ for every $i \in I$. By the inductive hypothesis, $p(\sigma_2^i * \ldots * \sigma_n^i) \in p(K^M(\tau))$ for every $i \in I$. Consider the weak and strong topologies on $p(K^M(\tau))$. The strong topology is finer than the weak topology, the weak topology is Hausdorff and the strong topology on $p(K^M(\tau))$ is compact. By Lemma 6.4.4, the weak and strong topologies on $p(K^M(\tau))$ coincide. Therefore $p(\sigma_2^i * \ldots * \sigma_n^i)$ converges to $p(\sigma_2 * \ldots * \sigma_n)$ in the strong topology. Hence by (6.3), $((\sigma_2^i * \ldots * \sigma_n^i) * \theta_x)$ actually converges to $((\sigma_2 * \ldots * \sigma_n) * \theta_x)$ in the strong topology of $\mathcal{T}(X)$. Thus, by Lemma 6.4.3, we have

$$(\sigma_1^i * ((\sigma_2^i * \ldots * \sigma_n^i) * \theta_x))(0) \to (\sigma_1 * ((\sigma_2 * \ldots * \sigma_n) * \theta_x))(0).$$

In other words,

$$p(\sigma_1^i * \ldots * \sigma_n^i)(x) \to p(\sigma_1 * \ldots * \sigma_n)(x).$$

This shows that the map $(\sigma_1, \ldots, \sigma_n) \mapsto p(\sigma_1 * \ldots * \sigma_n)$ is continuous from $K(\tau)^n$ to $\mathcal{T}(X)$. It is clear that if each σ_j has the form $c_1\tau * \ldots c_n\tau$, then $p(\sigma_1 * \ldots * \sigma_n) \in p(K(\tau))$. By the continuity, for general $\sigma_j \in K(\tau)$, the image is in the weak closure of $p(K(\tau))$. Hence by Lemma 6.4.2, the map is into $p(K(\tau))$. This completes our proof. □

Proposition 6.4.6 *Let τ be a non-degenerate strong type on X, and suppose that $p(K^1(\tau))$ is strongly compact. Then there exists a normalized strong type $\sigma \in K(\tau)$ with the property that for all $m \in \mathbb{N}$,*

$$p((\mathop{*}_{j=1}^{k} c_j\sigma) * \underbrace{\sigma * \ldots \sigma}_{\text{m terms}} * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma)) = p((\mathop{*}_{j=1}^{k} c_j\sigma) * (\lambda_m \cdot \sigma) * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma)).$$

Proof. To begin with, consider a fixed $n \in \mathbb{N}$. Then for each $\varepsilon > 0$, let σ_ε be the strong type given by Lemma 6.4.1. Then for any $m \leq n$,

$$p((\mathop{*}_{j=1}^{k} c_j\sigma_\varepsilon) * \underbrace{\sigma_\varepsilon * \ldots \sigma_\varepsilon}_{\text{m terms}} * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon))(x) = (\mathop{*}_{j=1}^{k} c_j\sigma_\varepsilon)(\underbrace{\sigma_\varepsilon * \ldots \sigma_\varepsilon}_{\text{m terms}} * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon) * \theta_x).$$

Using the notation that $a \overset{\varepsilon}{=} b$ if $|a-b| \leq \varepsilon$, it follows from the conclusion of Lemma 6.4.1 that for any $y \in X$,

$$\begin{aligned}(\underbrace{\sigma_\varepsilon * \ldots \sigma_\varepsilon}_{\text{m terms}} * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon) * \theta_x)(y) &= (\underbrace{\sigma_\varepsilon * \ldots \sigma_\varepsilon}_{\text{m terms}} * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon))(\theta_{x+y}) \\ &= (\underbrace{\sigma_\varepsilon * \ldots \sigma_\varepsilon}_{\text{m terms}})((\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon) * \theta_{x+y}) \\ &\overset{\varepsilon}{=} (\lambda_m \cdot \sigma_\varepsilon)((\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon) * \theta_{x+y}) \\ &= ((\lambda_m \cdot \sigma_\varepsilon) * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon))(\theta_{x+y}) \\ &= ((\lambda_m \cdot \sigma_\varepsilon) * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon) * \theta_x)(y).\end{aligned}$$

It is straightforward to show that if $\rho \in \mathcal{S}(X)$, and $\theta_1, \theta_2 \in \mathcal{T}(X)$ are such that $\theta_1(y) \stackrel{\varepsilon}{=} \theta_2(y)$ for all $y \in X$, then $\rho(\theta_1) \stackrel{\varepsilon}{=} \rho(\theta_2)$. It follows that

$$(\mathop{*}_{j=1}^{k} c_j\sigma_\varepsilon)((\underbrace{\sigma_\varepsilon * \ldots \sigma_\varepsilon}_{\text{m terms}} * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon)) * \theta_x) \stackrel{\varepsilon}{=} (\mathop{*}_{j=1}^{k} c_j\sigma_\varepsilon)((\lambda_m \cdot \sigma_\varepsilon) * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon) * \theta_x). \tag{6.4}$$

Notice also that,

$$p((\mathop{*}_{j=1}^{k} c_j\sigma_\varepsilon) * (\lambda_m \cdot \sigma_\varepsilon) * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon))(x) = (\mathop{*}_{j=1}^{k} c_j\sigma_\varepsilon)((\lambda_m \cdot \sigma_\varepsilon) * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon) * \theta_x).$$

Hence, (6.4) shows that

$$p((\mathop{*}_{j=1}^{k} c_j\sigma_\varepsilon) * \underbrace{\sigma_\varepsilon * \ldots \sigma_\varepsilon}_{\text{m terms}} * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon))(x) \stackrel{\varepsilon}{=} p((\mathop{*}_{j=1}^{k} c_j\sigma_\varepsilon) * (\lambda_m \cdot \sigma_\varepsilon) * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma_\varepsilon))(x). \tag{6.5}$$

Now consider $\{\sigma_{1/r} : r \in \mathbb{N}\}$. This set is contained in the compact set $\mathcal{S}_1(X) = \{\rho \in \mathcal{S}(X) : \|\rho\| \leq 1\}$. Let σ be any limit point of $\{\sigma_{1/r} : r \in \mathbb{N}\}$. Since $\left\|\sigma_{1/r}\right\| = 1$ for every $r \in \mathbb{N}$, $\|\sigma\| = 1$, and also as each $\sigma_{1/r} \in K(\tau)$, we also have $\sigma \in K(\tau)$. Using the continuity result from Lemma 6.4.5, it follows from (6.5) that for every $x \in X$ and every $m \leq n$,

$$p((\mathop{*}_{j=1}^{k} c_j\sigma) * \underbrace{\sigma * \ldots \sigma}_{\text{m terms}} * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma))(x) = p((\mathop{*}_{j=1}^{k} c_j\sigma) * (\lambda_m \cdot \sigma) * (\mathop{*}_{j=1}^{n} c_{k+j}\sigma))(x).$$

Now to complete the proof we do this for every n to obtain a normalized strong type σ_n satisfying the above equation for every $m \leq n$. We then take an accumulation point σ of $\{\sigma_n : n \geq 1\}$, and this is the required strong type. ☐

The result of Proposition 6.4.6, tells us that in the spreading model generated by σ, we have for every $x \in X$ and every k,

$$\left\| x + \sum_{i=1}^{k} c_i e_i + \sum_{i=1}^{m} e_{k+i} + \sum_{i=1}^{n} c_{k+i} e_{k+m+i} \right\| = \left\| x + \sum_{i=1}^{k} c_i e_i + \lambda_m e_{k+1} + \sum_{i=1}^{n} c_{k+i} e_{k+m+i} \right\|.$$

If we also suppose that σ is symmetric, then we can apply an argument due to Lemberg (see [51]), which actually only requires the above result for $m = 2, 3$, to prove that the spreading model is isometric to ℓ_p over X for some p, or isometric to c_0 over X.

Suppose that u, v are vectors in the spreading model with $\operatorname{supp} u < j < j+3^k-1 < \operatorname{supp} v$.

Then,

$$\begin{aligned}\left\|x+u+\sum_{i=1}^{2^k} e_{j+i}+v\right\| &= \left\|x+u+\lambda_2\sum_{i=1}^{2^{k-1}} e_{j+i}+v\right\| \\ &= \left\|x+u+\lambda_2^2\sum_{i=1}^{2^{k-2}} e_{j+i}+v\right\| \\ &= \dots \\ &= \left\|x+u+\lambda_2^k e_j+v\right\|.\end{aligned}$$

In particular, $\left\|\sum_{i=1}^{2^k} e_i\right\| = \lambda_2^k\left\|e_1\right\|$. Similarly, $\left\|\sum_{i=1}^{3^k} e_i\right\| = \lambda_3^k\left\|e_1\right\|$.

Since we are supposing that σ is a symmetric type, the spreading model generated by σ is 1-unconditional over X. Hence,

$$\begin{aligned}&2^k \geq 3^m \Rightarrow \lambda_2^k \geq \lambda_3^m, \\ &2^k \leq 3^m \Rightarrow \lambda_2^k \leq \lambda_3^m.\end{aligned} \tag{6.6}$$

We now consider two cases. Firstly, suppose that $\lambda_2 = 1$. Notice that this holds if and only if $\lambda_3 = 1$ too. Let $(a_i)_{i=1}^k$ be scalars, with $a = \max|a_i| = |a_I|$. Then by the 1-unconditionality and the fact that $\lambda_2 = 1$, it follows that

$$\begin{aligned}\left\|x+\sum_{i=1}^{k} a_i e_i\right\| &\leq \left\|x+a\sum_{i=1}^{k} e_i\right\| \\ &= \left\|x+ae_1\right\|.\end{aligned}$$

But on the other hand,

$$\begin{aligned}\left\|x+\sum_{i=1}^{k} a_i e_i\right\| &\geq \left\|x+ae_I\right\| \\ &= \left\|x+ae_1\right\|.\end{aligned}$$

Thus,

$$\left\|x+\sum_{i=1}^{k} a_i e_i\right\| = \left\|x+\max_i|a_i|e_1\right\|,$$

which means that the spreading model is isometric to c_0 over X.

The second case is when $\lambda_2 > 1$, which will hold if and only if $\lambda_3 > 1$. It follows from (6.6) that we can find a $p \in [1,\infty)$ such that $\lambda_2 = 2^{\frac{1}{p}}$ and $\lambda_3 = 3^{\frac{1}{p}}$. Let $k \in \mathbb{N}$ and $\varepsilon > 0$. By a result in [73], we can find $n, m, n', m' \in \mathbb{N}$ such that

$$2^n 3^{-m} \leq k < 2^{n'} 3^{-m'} \leq (1+\varepsilon)2^n 3^{-m}. \tag{6.7}$$

Let $M = \max(m, m')$, then using the 1-unconditionality, we find

$$\begin{aligned}\left\|x + (2^n 3^{M-m})^{\frac{1}{p}} e_1\right\| &= \left\|x + \lambda_2^n \lambda_3^{M-m} e_1\right\| \\ &= \left\|x + \sum_{i=1}^{2^n 3^{M-m}} e_i\right\| \\ &\leq \left\|x + \sum_{i=1}^{3^M k} e_i\right\| \\ &= \left\|x + 3^{\frac{M}{p}} \sum_{i=1}^{k} e_i\right\| \\ &\leq \left\|x + \sum_{i=1}^{2^{n'} 3^{M-m'}} e_i\right\| \\ &= \left\|x + (2^{n'} 3^{M-m'})^{\frac{1}{p}} e_1\right\| \\ &\leq \left\|x + ((1+\varepsilon) 2^n 3^{M-m})^{\frac{1}{p}} e_1\right\|.\end{aligned}$$

Letting $\varepsilon \downarrow 0$ tells us that

$$\left\|x + \sum_{i=1}^{k} e_i\right\| = \left\|x + k^{\frac{1}{p}} e_1\right\|.$$

Now suppose that α_i are scalars with $\alpha_i = (2^{n_i} 3^{-m_i})^{\frac{1}{p}}$ for $i = 1, \ldots, k$. Let $M = \max_i m_i$, and $\beta_i = \alpha_i 3^{\frac{M}{p}}$. Then,

$$\begin{aligned}\left\|x + \sum_{i=1}^{k} \beta_i e_i\right\| &= \left\|x + \sum_{i=1}^{\sum_j 2^{n_j} 3^{M-m_j}} e_i\right\| \\ &= \left\|x + \left(\sum_i \beta_i^p\right)^{\frac{1}{p}} e_1\right\|.\end{aligned}$$

It follows that for any $x \in X$,

$$\left\|x + \sum_{i=1}^{k} \alpha_i e_i\right\| = \left\|x + \left(\sum_{i=1}^{k} \alpha_i^p\right)^{\frac{1}{p}} e_1\right\|.$$

Since we can approximate any scalars by numbers of the form $2^n 3^{-m}$ as in (6.7), this implies that the spreading model is isometric to ℓ_p over X.

6.5 Strong compactness of certain subsets of $\mathcal{T}(X)$ implies X contains $(1+\varepsilon)$-isomorphic copies of ℓ_p or c_0

Theorem 6.5.1 *Let X be a Banach space. Suppose that there exists a non-degenerate strong type τ on X with $p(K^1(\tau))$ strongly compact. Then there is a spreading model that is ℓ_p over X for some $p \in [1,\infty)$ or c_0 over X, and either there exists $p \in [1,\infty)$ such that $\ell_p \overset{1+\varepsilon}{\hookrightarrow} X$ for all $\varepsilon > 0$, or $c_0 \overset{1+\varepsilon}{\hookrightarrow} X$ for all $\varepsilon > 0$.*

Proof. Our first step is to show that there is a non-degenerate symmetric strong type in $K(\tau)$. We do this in a similar way to the proof of Lemma 6.1.3. Let $K = \{\sigma \in \mathcal{S}(X) : \|\sigma\| \leq 1\}$, which is a compact subset of $\mathcal{S}(X)$. For each n, let E_n denote the subspace of the spreading model spanned by $e_1, \ldots, e_n$. Given a finite subset $F = \{\theta_1, \ldots, \theta_n\}$ of $\mathcal{T}(X)$, we define a map $\phi : S_{E_{n+1}} \to \mathbb{R}^n$, by

$$\phi(\alpha) = (\tau_\alpha(\theta_i) - \tau_{-\alpha}(\theta_i))_{i=1}^n,$$

where

$$\alpha = \alpha_1 e_1 + \ldots + \alpha_{n+1} e_{n+1},$$

and

$$\tau_\alpha = \alpha_1 \tau * \ldots * \alpha_{n+1} \tau.$$

Then the Borsuk-Ulam Theorem implies that there is an $\alpha_F \in S_{E_{n+1}}$ such that $\phi(\alpha_F) = 0$. In other words, $\tau_{\alpha_F}(\theta_i) = (-\tau_{\alpha_F})(\theta_i)$ for $i = 1, \ldots, n$. Note that

$$\|\tau_{\alpha_F}\| = \|p(\tau_{\alpha_F})\| = \|\alpha_1 e_1 + \ldots + \alpha_{n+1} e_{n+1}\| = 1.$$

Hence each $\tau_{\alpha_F} \in K$, and so taking an accumulation point τ' as in the proof of Lemma 6.1.3, gives a symmetric strong type. Since each τ_{α_F} is contained in the weak closed set $K(\tau)$, we also have $\tau' \in K(\tau)$. By Lemma 6.4.5, we see that for any scalars $c_1, \ldots, c_n$, $p(c_1\tau' * \ldots * c_n\tau') \in p(K(\tau))$ and thus by continuity we have that $p(K(\tau')) \subseteq p(K(\tau))$. Hence $p(K^1(\tau')) \subseteq p(K^1(\tau))$, and since $p(K^1(\tau'))$ is weakly closed and thus closed in the strong topology, $p(K^1(\tau'))$ is strongly compact. Thus we may apply Proposition 6.4.6 to obtain a normalized strong type σ in $K(\tau')$, which will automatically be symmetric since τ' is symmetric. We can then apply the Lemberg argument to the spreading model generated by σ and show that it is isometric to c_0 or to some ℓ_p over X. Similarly to the above, $p(K^1(\sigma))$ is strongly compact and so we can apply Theorem 6.3.7 to obtain $(1+\varepsilon)$-isomorphic copies of c_0 or ℓ_p in X. □

From a practical point of view, Theorem 6.5.1 is not of much use. It is difficult to describe the strong types on a Banach space, and thus it is difficult to find a strong type τ such that $p(K^1(\tau))$ is strongly compact. We now want to show that a compactness condition on a more easily identified subset of $\mathcal{T}(X)$ is also sufficient. The subset we will be interested in is the so called weakly null types, which we will define below. This will be more useful, since in some cases we can find a simple description of the weakly null types.

Definition 6.5.2 *We define* $\mathcal{T}_0(X)$ *to be the set consisting of limit points in* $\mathcal{T}(X)$ *of sequences* (θ_{x_n}) *with* (x_n) *weakly null. We further define* $\mathcal{T}_0^1(X)$ *to denote the subset of* $\mathcal{T}_0(X)$*, where the weakly null sequence* (x_n) *satisfies* $\|x_n\| \leq 1$ *for every* n*. We will call an element of* $\mathcal{T}_0(X)$ *a weakly null type.*

The condition we will be interested in is strong compactness of $\mathcal{T}_0^1(X)$. If this condition holds, then notice that in particular, $\mathcal{T}_0^1(X)$ is weakly compact, and therefore weakly closed. In general this need not be the case, but we have the following result.

Proposition 6.5.3 *Let* X *be a Banach space with* X^* *separable. Then* $\mathcal{T}_0(X)$ *is closed in the weak topology on* $\mathcal{T}(X)$*. In particular, the result holds if* X *is a separable reflexive space.*

Proof. Suppose that $(\theta_i) \subseteq \mathcal{T}_0(X)$ such that $\theta_i \to \theta$ in $\mathcal{T}(X)$. We may suppose that for every $x \in X$

$$\theta_i(x) = \lim_{n\to\infty} \theta_{x_{i,n}}(x),$$

where $(x_{i,n})_{n=1}^{\infty}$ is weakly null. Since X and X^* are both separable we may choose dense sequences $(u_r) \subseteq X$ and $(f_s) \subseteq X^*$. By a Cantor diagonalisation argument, we may assume that

$$|f_s(x_{i,n})| < \frac{1}{2^i} \text{ for every } i \text{ and } n \geq s, \tag{6.8}$$

and

$$|\theta_i(u_r) - \theta_{x_{i,n}}(u_r)| < \frac{1}{2^i} \text{ for every } i \text{ and } n \geq r. \tag{6.9}$$

Consider the sequence $y_n = x_{n,n}$. Using (6.8) we can show that (y_n) is weakly null. Indeed, given $f \in X^*$ and $\varepsilon > 0$, we choose s such that $\|f - f_s\| < \frac{\varepsilon}{2}$. Then for $n \geq s$,

$$\begin{aligned} |f(y_n)| &\leq \frac{\varepsilon}{2}\|y_n\| + |f_s(y_n)| \\ &\leq \frac{\varepsilon}{2}\|y_n\| + \frac{1}{2^n}. \end{aligned} \tag{6.10}$$

Since $\lim_{i\to\infty}\lim_{n\to\infty}\|x_{i,n}\| = \theta(0)$, we may assume that the sequences $(x_{i,n})$ are uniformly bounded. Hence (y_n) is bounded, so (6.10) tells us that (y_n) is weakly null.

We now show that θ is a limit point of $(\theta_{y_n})_{n=1}^{\infty}$ and hence $\theta \in \mathcal{T}_0(X)$. Let U be a neighbourhood of θ. Without loss of generality we may suppose that

$$U = \{\phi : |\phi(z_t) - \theta(z_t)| < \varepsilon \text{ for } t = 1, \ldots, k\ \},$$

and that each z_t is an element in the sequence (u_r). Choose S sufficiently large that $\{z_1, \ldots, z_k\} \subseteq \{u_1, \ldots, u_S\}$. Then for $n \geq S$ and every i, we have by (6.9) that

$$|\theta_i(z_t) - \theta_{x_{i,n}}(z_t)| < \frac{1}{2^i}.$$

Since $\theta_i \to \theta$, there exists I such that for $i \geq I$,

$$|\theta_i(z_t) - \theta(z_t)| < \frac{\varepsilon}{2} \text{ for } t = 1, \ldots, k.$$

Hence, for $n \geq \max(S, I)$ we have

$$|\theta(z_t) - \theta_{y_n}(z_t)| < \frac{\varepsilon}{2} + \frac{1}{2^n} \text{ for } t = 1, \ldots, k.$$

Thus, for sufficiently large n, we have $\theta_{y_n} \in U$ which completes our proof. $\square$

In general, the result of Proposition 6.5.3 need not hold. The essential difficulty in the proof is the selection of a diagonal weakly null subsequence. This can not always be done (for example in spaces of the form $\ell_1(\ell_p)$). In view of this result, we will be interested in what we can deduce from strong compactness of $\overline{\mathcal{T}}_0^1(X)$, the weak closure of $\mathcal{T}_0^1(X)$.

The next proposition gives a simple, but useful, way of deciding which weakly null types are in $\mathcal{T}_0^1(X)$ and $\overline{\mathcal{T}}_0^1(X)$.

Proposition 6.5.4 $\mathcal{T}_0^1(X) = \{\theta \in \mathcal{T}_0(X) : \|\theta\| \leq 1\}$. *If we denote the weak closure of* $\mathcal{T}_0^1(X)$ *by* $\overline{\mathcal{T}}_0^1(X)$ *(and similarly for* $\mathcal{T}_0(X)$*), then* $\overline{\mathcal{T}}_0^1(X) = \left\{\theta \in \overline{\mathcal{T}}_0(X) : \|\theta\| \leq 1\right\}$.

Proof. We prove the first claim, the proof of the second being similar. It is clear that every element in $\mathcal{T}_0^1(X)$ has norm at most 1. Conversely, suppose that θ is a limit of (θ_{x_n}), where (x_n) is weakly null and $\|\theta\| \leq 1$. Since X is separable, we may assume by passing to a subsequence that $\theta(x) = \lim_{n\to\infty} \theta_{x_n}(x)$ for every $x \in X$. Hence $\lim_{n\to\infty} \|x_n\| = \|\theta\| \leq 1$. If $\lim_{n\to\infty} \|x_n\| < 1$, then it is clear that $\theta \in \mathcal{T}_0^1(X)$. Now suppose that $\lim_{n\to\infty} \|x_n\| = 1$. Consider $y_n = x_n / \|x_n\|$. Then for any $x \in X$,

$$\begin{aligned} |\theta_{y_n}(x) - \theta_{x_n}(x)| &= |\ \|x + y_n\| - \|x + x_n\|\ | \\ &\leq \|y_n - x_n\| \\ &= \|x_n\| \left|\frac{1}{\|x_n\|} - 1\right| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence, θ is the limit of the sequence θ_{y_n}, and the sequence (y_n) is a normalized weakly null sequence. Thus, $\theta \in \mathcal{T}_0^1(X)$. ☐

We are now nearly ready to prove our main result for this section. Before we do this, we will need prove the following two lemmas, which will be used in the proof of our result.

Lemma 6.5.5 *If* τ *is an accumulation point of the sequence* (τ_{z_n}) *in* $\mathcal{S}(X)$*, then* $p(\tau)$ *is an accumulation point of* (θ_{z_n}) *in* $\mathcal{T}(X)$*.*

Proof. Let U be an open set in $\mathcal{T}(X)$ such that $p(\tau) \in U$, and $N \in \mathbb{N}$. Since $p : \mathcal{S}(X) \to \mathcal{T}(X)$ is continuous, it follows that $p^{-1}(U)$ is open in $\mathcal{S}(X)$, and $\tau \in p^{-1}(U)$. Since τ is an accumulation point of the sequence (τ_{z_n}), there exists $n \geq N$ such that $\tau_{z_n} \in p^{-1}(U)$. Hence $\theta_{z_n} = p(\tau_{z_n}) \in U$. Thus, $p(\tau)$ is an accumulation point of (θ_{z_n}). ☐

Lemma 6.5.6 *Suppose that* $\sigma, \tau \in \mathcal{S}(X)$ *are such that* τ *is a limit point of* (τ_{y_n}) *in* $\mathcal{S}(X)$*, and* $p(\sigma)$ *is a limit point of* (θ_{z_n}) *in* $\mathcal{T}(X)$*. Then,* $p(\tau * \sigma)$ *is a limit point in* $\mathcal{T}(X)$ *of* $\{\theta_{y_n+z_m} : m > n\}$*.*

Proof. For any $x \in X$,

$$\begin{aligned} p(\tau * \sigma)(x) &= (\tau * \sigma)(\theta_x) \\ &= \tau(\sigma * \theta_x). \end{aligned}$$

Similarly,

$$(\tau * p(\sigma))(x) = \tau(x * p(\sigma)).$$

For any $y \in X$,

$$\begin{aligned} (\sigma * \theta_x)(y) &= \sigma(y * \theta_x) \\ &= \sigma(\theta_{x+y}) \\ &= p(\sigma)(x + y) \\ &= (x * p(\sigma))(y). \end{aligned}$$

Thus,

$$p(\tau * \sigma) = \tau * p(\sigma).$$

Suppose that U is an open set in $\mathcal{T}(X)$ such that $p(\tau * \sigma) \in U$. Without loss of generality we may suppose that $U = \{\theta \in \mathcal{T}(X) : |\theta(x_i) - p(\tau * \sigma)(x_i)| < \varepsilon \text{ for } i = 1, \ldots, k\}$. Let $N \in \mathbb{N}$. Then, since τ is a limit point of (τ_{y_n}) in $\mathcal{S}(X)$, there exists $n \geq N$ such that for $i = 1, \ldots, k$,

$$|\tau(x_i * p(\sigma)) - \tau_{y_n}(x_i * p(\sigma))| < \frac{\varepsilon}{2}.$$

In other words, for $i = 1, \ldots, k$,

$$|(\tau * p(\sigma))(x_i) - p(\sigma)(x_i + y_n)| < \frac{\varepsilon}{2}.$$

Since $p(\sigma)$ is a limit point of (θ_{z_n}) in $\mathcal{T}(X)$, there exists $m > n$ such that for $i = 1, \ldots, k$,

$$|p(\sigma)(x_i + y_n) - \theta_{z_m}(x_i + y_n)| < \frac{\varepsilon}{2}.$$

Hence, for $i = 1, \ldots, k$,

$$|p(\tau * \sigma)(x_i) - \theta_{y_n + z_m}(x_i)| < \varepsilon.$$

Hence $\theta_{y_n+z_m} \in U$ which completes our proof. □

Corollary 6.5.7 *If X is separable and the set $\overline{\mathcal{T}}_0^1(X)$ is strongly compact, then there exists $p \in [1, \infty)$ such that $\ell_p \overset{1+\varepsilon}{\hookrightarrow} X$, or $c_0 \overset{1+\varepsilon}{\hookrightarrow} X$.*

Proof. Suppose that $\ell_1 \not\hookrightarrow X$. Since ℓ_1 is not distortable this is equivalent to the condition that $\ell_1 \overset{1+\varepsilon}{\not\hookrightarrow} X$ for some $\varepsilon > 0$. By Rosenthal's ℓ_1 theorem, every bounded sequence in X has a subsequence that is weakly Cauchy. Suppose that we take a Brunel-Sucheston sequence (x_n) in X, then by passing to a subsequence we may assume that (x_n) is weakly Cauchy. Now consider the vectors $y_{n,m} = x_n - x_m$ where $n < m$. These form a weakly null sequence which we will denote by (z_n). Let τ be any accumulation point in $\mathcal{S}(X)$ of (τ_{z_n}).

We will now show by induction that for every k, $p(c_1\tau * \ldots * c_k\tau)$ is a limit point in $\mathcal{T}(X)$ of

$$\{\theta_{c_1 z_{n_1} + \ldots + c_k z_{n_k}} : n_1 < \ldots < n_k\}.$$

The case $k = 1$ follows from Lemma 6.5.5 since $c_1\tau$ is an accumulation point of $(\tau_{c_1 z_{n_1}})$. Given the result for $k-1$, we can deduce the result for k from Lemma 6.5.6. Note that $c_1\tau$ is a limit point of $(\tau_{c_1 z_{n_1}})$ in $\mathcal{S}(X)$, and by the inductive hypothesis $p(c_2\tau * \ldots * c_k\tau)$ is a limit point of $\{\theta_{c_2 z_{n_2} + \ldots + c_k z_{n_k}} : n_2 < \ldots < n_k\}$. Thus the result follows by Lemma 6.5.6.

It is immediate from this that $p(c_1\tau * \ldots * c_k\tau) \in \mathcal{T}_0(X)$ for any k and any scalars c_i. By definition $\overline{\mathcal{T}}_0(X)$ is closed in the weak topology. Hence, since p is continuous from $\mathcal{S}(X)$ into $\mathcal{T}(X)$, it follows that $p(K(\tau)) \subseteq \overline{\mathcal{T}}_0(X)$. Also, p preserves norms, so it follows from Proposition 6.5.4 that $p(K^1(\tau)) \subseteq \overline{\mathcal{T}}_0^1(X)$. $K^1(\tau)$ is compact in $\mathcal{S}(X)$ and p continuous, so $p(K^1(\tau))$ is weakly compact in $\mathcal{T}(X)$. Therefore $p(K^1(\tau))$ is weakly closed and therefore closed in the strong topology. Thus by the strong compactness of $\overline{\mathcal{T}}_0^1(X)$, it follows that $p(K^1(\tau))$ is strongly compact. Note that τ is non-degenerate because for any $x \in X$, $p(\tau)(x) = \|x + e_1 - e_2\|$. This shows that $p(\tau)$ is non-degenerate, and hence τ is too. The result then follows by Theorem 6.5.1. □

In fact we can prove a slightly stronger result.

Corollary 6.5.8 *Let X be a separable Banach space such that $\overline{\mathcal{T}}_0^1(X)$ is strongly compact. Then if Y is a subspace of X, there exists $p \in [1, \infty)$ such that $\ell_p \overset{1+\varepsilon}{\hookrightarrow} Y$, or $c_0 \overset{1+\varepsilon}{\hookrightarrow} Y$, for every $\varepsilon > 0$.*

Proof. Let $\theta \in \mathcal{T}_0^1(Y)$. Then there exists a weakly null sequence (y_n) contained in Y such that $\theta(y) = \lim_{n\to\infty} \|y + y_n\|$ for every $y \in Y$. The sequence (y_n) is also weakly null in the larger space X, so we may pass to a subsequence (y_n') such that $\tilde{\theta}(x) = \lim_{n\to\infty} \|x + y_n'\|$ exists for every $x \in X$. Hence every $\theta \in \mathcal{T}_0^1(Y)$ is a restriction to Y of an element $\tilde{\theta} \in \mathcal{T}_0^1(X)$. Since every element of $\overline{\mathcal{T}}_0^1(Y)$ is a pointwise limit of a sequence in $\mathcal{T}_0^1(Y)$, it follows that every $\theta \in \overline{\mathcal{T}}_0^1(Y)$ is a restriction to Y of an element $\tilde{\theta} \in \overline{\mathcal{T}}_0^1(X)$.

Thus, given a sequence $(\theta_i) \subseteq \overline{\mathcal{T}}_0^1(Y)$, we consider the sequence $(\tilde{\theta}_i) \subseteq \overline{\mathcal{T}}_0^1(X)$. By strong compactness of $\overline{\mathcal{T}}_0^1(X)$ there exists a subsequence $(\tilde{\theta}_{n_i})$ which converges uniformly on every bounded subset of X. Hence, in particular, (θ_{n_i}) converges uniformly on every bounded subset of Y. Thus $\overline{\mathcal{T}}_0^1(Y)$ is strongly compact and the result follows from Corollary 6.5.7. ☐

6.6 A class of spaces such that $\overline{\mathcal{T}}_0^1(X)$ is strongly compact

In this section we introduce a class of spaces for which $\overline{\mathcal{T}}_0^1(X)$ is strongly compact. These spaces have a structural property known as property (M). This property was introduced by Kalton in [44], where it arose in the classification of spaces, X, such that the compact operators $\mathcal{K}(X)$ form an M-ideal in $\mathfrak{L}(X)$. For further properties of spaces with property (M), we refer the reader to [20], [45] and [3].

Definition 6.6.1 *Let X be a Banach space. We say that X has property (M), if whenever $u, v \in X$ satisfy $\|u\| = \|v\|$, and (x_n) is a weakly null sequence then*

$$\limsup_{n\to\infty} \|u + x_n\| = \limsup_{n\to\infty} \|v + x_n\| .$$

Notice that property (M) is an isometric invariant, but not an isomorphic invariant. It is shown in Kalton's paper that every generalized Orlicz space can be equivalently renormed so as to have property (M).

Proposition 6.6.2 *Let X be a separable Banach space with property (M). If $\theta \in \overline{\mathcal{T}}_0(X)$, then there exists a 1-Lipschitz function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\theta(x) = f(\|x\|) \text{ for all } x \in X.$$

Proof. Suppose to start with that $\theta \in \mathcal{T}_0(X)$. Since X is separable, we may suppose that $\theta(x) = \lim_{n\to\infty} \|x + x_n\|$ for every $x \in X$, where (x_n) is a weakly null sequence. Since X has property (M), it follows immediately that if $x, y \in X$ satisfy $\|x\| = \|y\|$, then $\theta(x) = \theta(y)$. If $\theta \in \overline{\mathcal{T}}_0(X)$, then θ is a pointwise limit of a sequence $(\theta_i) \subseteq \mathcal{T}_0(X)$, and therefore $\theta(x) = \theta(y)$ whenever $\|x\| = \|y\|$. Hence there exists a function $f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\theta(x) = f(\|x\|)$ for all $x \in X$. Fix a vector $x \in S_X$, then for any scalars $\alpha, \beta \geq 0$, using the fact that types are 1-Lipschitz, we have

$$|f(\alpha) - f(\beta)| = |\theta(\alpha x) - \theta(\beta x)| \leq \|\alpha x - \beta x\| = |\alpha - \beta|.$$

$\square$

Proposition 6.6.3 *If X is a separable Banach space with property (M) then $\overline{\mathcal{T}}_0^1(X)$ is strongly compact.*

Proof. By Proposition 6.6.2, we can associate with $\overline{\mathcal{T}}_0^1(X)$ a family $\mathcal{S}$ of 1-Lipschitz functions $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $0 \leq f(0) \leq 1$. Fix any $M \in \mathbb{R}$, and consider $\mathcal{S}_M = \{f\restriction_{[0,M]} : f \in \mathcal{S}\} \subseteq C([0, M])$. $[0, M]$ is a compact metric space, and since each of the functions in $\mathcal{S}_M$ is 1-Lipschitz, $\mathcal{S}_M$ is an equicontinuous family. Also, for any $f \in \mathcal{S}$ and any $\alpha \in [0, M]$, we have $|f(\alpha)| \leq f(0)+\alpha \leq 1+M$, so the family $\mathcal{S}_M$ is also uniformly bounded. By the Arzela-Ascoli theorem therefore, $\mathcal{S}_M$ is a relatively compact subset of $C([0, M])$.

We will now show that $\overline{\mathcal{T}}_0^1(X)$ is strongly compact. Since the strong topology is always metrizable it suffices to show that it is sequentially strongly compact. Let (θ_i) be a sequence in $\overline{\mathcal{T}}_0^1(X)$. Let f_i be the Lipschitz function associated to θ_i in Proposition 6.6.2. By the compactness result proved above, given any $M \in \mathbb{R}$ we can pass to a subsequence of (f_i) such that the subsequence converges uniformly on $[0, M]$. Repeating this for every $M \in \mathbb{N}$ and passing to a diagonal subsequence gives us a subsequence (g_i) of (f_i) such that (g_i) converges pointwise to some function g, with convergence being uniform on each interval $[0, M]$. Hence if we let (ϕ_i) be the subsequence of (θ_i) corresponding to (g_i), then for any $x \in X$ we have

$$\phi_i(x) = g_i(\|x\|) \to g(\|x\|),$$

and the convergence is uniform over $x \in MB_X$ for any $M \in \mathbb{N}$. Hence $\phi_i(x)$ converges in the strong topology, and since $\overline{\mathcal{T}}_0^1(X)$ is in fact strongly closed, we have shown that $\overline{\mathcal{T}}_0^1(X)$ is strongly compact. $\square$

Corollary 6.6.4 *If X is a separable Banach space with property (M), then either $\ell_p \overset{1+\varepsilon}{\hookrightarrow} X$ for some $p \in [1, \infty)$, or $c_0 \overset{1+\varepsilon}{\hookrightarrow} X$.*

Proof. This follows immediately from Proposition 6.6.3 and Corollary 6.5.7. □

Notice that property (M) is hereditary i.e. if X has property (M) and Y is a subspace of X then Y also property (M). Thus Corollary 6.6.4 immediately yields the following corollary. Alternatively, this follows from Corollary 6.5.8. This result has been proved previously using a different method (see [44, Proposition 8]).

Corollary 6.6.5 *Let X be a separable Banach space with property (M). Then for any subspace Y of X, there either exists a $p \in [1, \infty)$ such that $\ell_p \overset{1+\varepsilon}{\hookrightarrow} Y$ for every $\varepsilon > 0$, or $c_0 \overset{1+\varepsilon}{\hookrightarrow} Y$ for every $\varepsilon > 0$.*

6.7 An application of strong compactness of types to Tsirelson type spaces

We will consider here spaces of the form $X = T(\mathcal{A}_2, (a, b))$ where $a, b < 1$. These spaces are defined to be the completion of $(c_{00}, \|\cdot\|)$, where the norm $\|\cdot\|$ satisfies the implicit relation

$$\|x\| = \|x\|_\infty \vee \sup\{a \|E_1 x\| + b \|E_2 x\| : E_1 < E_2\}.$$

The natural basis of such a space is 1-unconditional and so we lose nothing by the additional assumption that the sets E_1 and E_2 are adjacent intervals. We will begin by obtaining a description of the weakly null types which will allow us to prove that the set $\overline{\mathcal{T}}^1_0(X)$ is strongly compact. In fact, we will prove a result which is stronger than we require.

6.7.1 Notation

We first introduce the uniform metric on $\mathcal{T}(X)$. Given two types $\theta_1, \theta_2 \in \mathcal{T}(X)$, we define

$$\rho(\theta_1, \theta_2) = \sup\{|\theta_1(x) - \theta_2(x)| : x \in X\}.$$

$\rho(\theta_1, \theta_2)$ is always finite since every type is given by a bounded net $(x_i)_{i \in I}$. For example, if X is separable, then every type is given by a bounded sequence. Hence if θ_1 and θ_2 are limits of $(\theta_{x_i})_{i=1}^\infty$ and $(\theta_{y_i})_{i=1}^\infty$ respectively, where $\|x_i\| \le M$ and $\|y_i\| \le M$ for all $i \in \mathbb{N}$, then for any $x \in X$, $|\theta_1(x) - \theta_2(x)| \le 2M$ since,

$$|\ \|x + x_i\| - \|x + y_i\|\ | \le \|x_i - y_i\| \le 2M.$$

Given a vector $x \in X$, we will need a collection of functions to describe how the norm of x is obtained. The implicit definition of the norm leads naturally to the use of dyadic trees of intervals (see [41]). The approach we use here is similar, but slightly different, following the method in [62], in which the types of Tsirelson's space are studied. Our situation is on the one hand simpler because we do not have an admissibility condition, but on the other hand we have the added complication of the scalars a, b.

If $\mathbf{n} = (n_1, \ldots, n_k) \in \{0,1\}^k$, we will call a finite sequence $\{E_i\}_{i=1}^k$ of (possibly empty) subsets of $\mathbb{N}$ an $\mathbf{n}$-partition if the following properties are satisfied;

(i) $E_i = \varnothing$ if $n_i = 0$.

(ii) If $S = \{i : 1 \leq i \leq k,\ n_i = 1\}$ and $S = \{s_1, \ldots, s_r\}$ when written in increasing order, then $E_{s_r} < E_{s_{r-1}} < \ldots < E_2 < E_1$.

Given $j \in \{1, \ldots, k\}$, we define

$$n(j) = |\{i < j : n_i = 0\}| .$$

We also define $l(j) = j - 1 - n(j)$ and $r(j) = 1 + n(j)$. Finally, we set

$$f_{\mathbf{n}}(x) = \sup \left\{ \sum_{j=1}^{k} a^{l(j)} b^{r(j)} \|E_j x\| : \{E_i\}_{i=1}^k \text{ is an } \mathbf{n}\text{-partition} \right\},$$

and

$$f_{\varnothing}(x) = \|x\| .$$

Remark 6.7.1 *If* $x \in X$ *and* $\mathbf{n} \in \{0,1\}^k$*, then* $\|x\| \geq f_{\mathbf{n}}(x)$*. For example, if we take* $\mathbf{n} = (0,1,0,1)$*, and if* $(E_i)_{i=1}^4$ *is an* $\mathbf{n}$*-partition, then* $E_1 = E_3 = \varnothing$ *and* $E_4 < E_2$*. Given a subset* E *of* $\mathbb{N}$*, we will denote the interval* $[0, \min E - 1]$ *by* $\overleftarrow{E}$*. Then,*

$$\begin{aligned} \|x\| &\geq a \left\| (\overleftarrow{E_4}) x \right\| + b \, \|(E_4 \cup E_2)x\| \\ &\geq b \, \|(E_4 \cup E_2)x\| \\ &\geq b(a \, \|E_4 x\| + b \, \|E_2 x\|) \\ &= b^2 \, \|E_2 x\| + ab \, \|E_4 x\| \\ &= \sum_{j=1}^{4} a^{l(j)} b^{r(j)} \, \|E_j x\| . \end{aligned}$$

The following lemma is the most important part of our example. It will allow us to identify the weakly null types on X with a suitable metric space, and will provide us with a useful estimate for $\rho(\theta, \phi)$ for $\theta, \phi \in \mathfrak{T}_0^1(X)$.

Lemma 6.7.2 *Let* $u, x, y \in X$ *with* $\max \operatorname{supp} u < \min(\operatorname{supp} x \cup \operatorname{supp} y)$, *and put* $\gamma = \max(a, b)$. *Then for any* $k \in \mathbb{N}$,

$$| \, \|u + x\| - \|u + y\| \, | \leq \max\Big\{|f_{\mathbf{n}}(x) - f_{\mathbf{n}}(y)| : \mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}\Big\} + \gamma^k(\|x\| + \|y\|).$$

Proof. We will show that

$$\|u + x\| \leq \|u + y\| + \max\Big\{|f_{\mathbf{n}}(x) - f_{\mathbf{n}}(y)| : \mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}\Big\} + \gamma^k(\|x\| + \|y\|).$$

Consider $\|u + x\|$. Suppose first of all that $\|u + x\| = \|u + x\|_\infty$. Then either $\|u + x\| = \|u\|_\infty$ or $\|u + x\| = \|x\|_\infty$. In the first case,

$$\|u + x\| = \|u\|_\infty \leq \|u\| \leq \|u + y\|,$$

and in the second,

$$\|u + x\| = \|x\|_\infty \leq \|x\| \leq \|y\| + | \, \|x\| - \|y\| \, | \leq \|u + y\| + |f_\varnothing(x) - f_\varnothing(y)|.$$

Otherwise, $\|u + x\| = a \|F_1(u + x)\| + b \|F_2(u + x)\|$. By 1-unconditionality of the norm, we may suppose that $F_1 < F_2$ are consecutive intervals which cover all of u and x, and that F_1 meets u and F_2 meets x. Now, at most one of the intervals F_1 and F_2 meets both u and x. If F_2 meets only x, then put $n_1 = 1$, otherwise we put $n_1 = 0$. We now look at the interval F_i which possibly meets both u and x and consider $\|F_i(u + x)\|$. We continue this process, until after a finite number of steps, we obtain a sequence of intervals $E_1 > E_2 > \ldots > E_t$, with $E_i = \varnothing$ if $n_i = 0$, which do not meet both u and x, and $\mathbf{n} = (n_1, \ldots, n_t)$. This gives us,

$$\|u + x\| = U + \sum_{j=1}^{t} a^{l(j)} b^{r(j)} \|E_j x\|, \tag{6.11}$$

where U is the contribution from u.

Fix $k \in \mathbb{N}$. Suppose that $t \leq k$, and set $\mathbf{p} = (n_1, \ldots, n_t, \underbrace{0, \ldots, 0}_{k-t})$. It follows from (6.11), that $\|u + x\| \leq U + f_{\mathbf{p}}(x)$. Now suppose that $k < t$, and we define $\mathbf{p} = (n_1, \ldots, n_k)$, and $\mathbf{q} = (n_{k+1}, \ldots, n_t)$. Let A denote the number of zeros in $\mathbf{p}$. Then

$$\begin{aligned}
\sum_{j=k+1}^{t} a^{l(j)} b^{r(j)} \|E_j x\| &= a^{k-A} b^A \sum_{j=k+1}^{t} a^{l(j)+A-k} b^{r(j)-A} \|E_j(x)\| \\
&\leq a^{k-A} b^A f_{\mathbf{q}}(x) \\
&\leq a^{k-A} b^A \|x\| \text{ by Remark 6.7.1.} \\
&\leq \gamma^k \|x\|.
\end{aligned}$$

Hence, by (6.11), it follows that

$$\begin{aligned}\|u+x\| &\le U + \sum_{j=1}^{k} a^{l(j)} b^{r(j)} \|E_j x\| + \gamma^k \|x\| \\ &\le U + f_{\mathbf{p}}(x) + \gamma^k \|x\| .\end{aligned}$$

It also follows from the definition of the norm, and the fact that $\operatorname{supp} u < \operatorname{supp} y$, that

$$U + f_{\mathbf{p}}(y) \le \|u+y\| .$$

Hence,

$$\begin{aligned}\|u+x\| &\le U + f_{\mathbf{p}}(x) + \gamma^k \|x\| \\ &\le \|u+y\| + f_{\mathbf{p}}(x) - f_{\mathbf{p}}(y) + \gamma^k \|x\| .\end{aligned}$$

This gives the required result. □

6.7.2 A description of the weakly null types on X

Define $\Delta = \bigcup_{k=1}^{\infty} \{0,1\}^k \cup \{\varnothing\}$, and define a map $X \to \mathbb{R}^{\Delta}$ by $x \mapsto \omega_x$ where $\omega_x = (f_{\mathbf{n}}(x))_{\mathbf{n}\in\Delta}$. We equip $\mathbb{R}^{\Delta}$ with the Tychonoff product topology and we then define $\mathcal{A}_0$ to be the set of limits in $\mathbb{R}^{\Delta}$ of sequences (ω_{x_n}) where (x_n) is weakly null, and $\overline{\mathcal{A}_0}$ to be the closure of $\mathcal{A}_0$ in $\mathbb{R}^{\Delta}$. $\mathbb{R}^{\Delta}$ is a complete separable metric space and therefore $\overline{\mathcal{A}_0}$ is also a complete separable metric space. Given $\mathbf{n} \in \Delta$, we let $f_{\mathbf{n}}$ denote the coordinate function defined on $\mathbb{R}^{\Delta}$. Hence, each $f_{\mathbf{n}}$ is a continuous function on $\mathbb{R}^{\Delta}$. Also, since $f_{\mathbf{n}}(x_n) \le \|x_n\| = f_{\varnothing}(x_n)$ for every n, we get that $f_{\mathbf{n}}(\omega) \le f_{\varnothing}(\omega)$ for each $\omega \in \mathcal{A}_0$. We will now show using Lemma 6.7.2 that we can identify $\mathcal{A}_0$ with $\mathcal{T}_0(X)$, and $\overline{\mathcal{A}_0}$ with $\overline{\mathcal{T}}_0(X)$.

Suppose that (x_n) is a weakly null sequence such that $\omega = \lim \omega_{x_n}$. Let $\varepsilon > 0$, and suppose that $\|x_n\| \le M$ for all n. Choose k sufficiently large that $2M\gamma^k \le \frac{\varepsilon}{2}$. Now there exists N such that $n, m \ge N$ implies that

$$|f_{\mathbf{n}}(x_n) - f_{\mathbf{n}}(x_m)| < \frac{\varepsilon}{2} \text{ for all } \mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}.$$

Hence, Lemma 6.7.2 (and a moving hump argument), tells us that

$$|\ \|u+x_n\| - \|u+x_m\|\ | < \varepsilon \text{ for all } u \in X.$$

Hence $\theta(u) = \lim_{n\to\infty} \|u+x_n\|$ exists for all $u \in X$, and defines a type $\theta \in \mathcal{T}_0(X)$. Conversely, if $\theta(u) = \lim_{n\to\infty} \theta_{x_n}(u)$ for all $u \in X$, where x_n is weakly null, then it is a consequence of Tychonoff's theorem, that we can find a subsequence (x_n') such that $\omega = \lim_{n\to\infty} \omega_{x_n'}$ exists in $\mathbb{R}^{\Delta}$. Hence we can identify $\mathcal{A}_0$ with $\mathcal{T}_0(X)$.

Suppose that $\theta, \phi \in \mathcal{T}_0(X)$, with $\theta = \lim \omega_{x_n}$ and $\phi = \lim \omega_{y_n}$, where (x_n) and (y_n) are weakly null sequences. Then for any $k \in \mathbb{N}$, and any $u \in X$, writing $\Delta_k = \{0,1\}^k \cup \{\varnothing\}$,

$$\begin{aligned} |\theta(u) - \phi(u)| &= \lim_{m\to\infty} |\ \|u + x_m\| - \|u + y_m\|\ | \\ &\leq \lim_{m\to\infty} \Big(\max\{|f_{\mathbf{n}}(x_m) - f_{\mathbf{n}}(y_m)| : \mathbf{n} \in \Delta_k\} + \gamma^k(\|x_m\| + \|y_m\|) \Big) \\ &= \max\Big\{ \lim_{m\to\infty} |f_{\mathbf{n}}(x_m) - f_{\mathbf{n}}(y_m)| : \mathbf{n} \in \Delta_k \Big\} + \lim_{m\to\infty} \gamma^k(\|x_m\| + \|y_m\|) \\ &= \max\{|f_{\mathbf{n}}(\theta) - f_{\mathbf{n}}(\phi)| : \mathbf{n} \in \Delta_k\} + \gamma^k(\theta(0) + \phi(0)). \end{aligned} \tag{6.12}$$

Now we consider $\overline{\mathcal{A}_0}$. Let ω be an element in $\overline{\mathcal{A}_0}$. Then there exists a sequence $(\omega_i) \subseteq \mathcal{A}_0$ such that $\omega_i \to \omega$ in $\mathbb{R}^\Delta$. Let θ_i denote the weakly null type identified with ω_i. Then for any $x \in X$,

$$|\theta_i(x) - \theta_j(x)| \leq \max\{|f_{\mathbf{n}}(\theta_i) - f_{\mathbf{n}}(\theta_j)| : \mathbf{n} \in \Delta_k\} + \gamma^k(\theta(0) + \phi(0)).$$

Since (ω_i) converges in $\mathbb{R}^\Delta$ to ω, this guarantees that $\theta(x) = \lim_{i\to\infty} \theta_i(x)$ exists for every $x \in X$ and so defines $\theta \in \overline{\mathcal{T}}_0(X)$. Conversely, if $\theta \in \overline{\mathcal{T}}_0(X)$ and $\theta(x) = \lim_{i\to\infty} \theta_i(x)$ for each $x \in X$, where $\theta_i \in \mathcal{T}_0(X)$, let ω_i denote the element of $\mathcal{A}_0$ corresponding to θ_i. Then for any $\mathbf{n} \in \Delta$, we have $f_{\mathbf{n}}(\omega_i) \leq f_{\varnothing}(\omega_i) = \theta_i(0)$. Since $\theta_i(0) \to \theta(0)$, Tychonoff's theorem gives us a subsequence such that $(\omega_{i_r})_{r=1}^\infty$ converges in $\mathbb{R}^\Delta$. Therefore, we have a correspondence between $\overline{\mathcal{T}}_0(X)$ and $\overline{\mathcal{A}_0}$.

Suppose that $\theta, \phi \in \overline{\mathcal{T}}_0(X)$, then by taking pointwise limits in (6.12) we find that,

$$\begin{aligned} \rho(\theta, \phi) &= \sup_{u\in X} |\theta(u) - \phi(u)| \\ &\leq \max\Big\{ |f_{\mathbf{n}}(\theta) - f_{\mathbf{n}}(\phi)| : \mathbf{n} \in \{0,1\}^k \cup \{\varnothing\} \Big\} + \gamma^k(\theta(0) + \phi(0)). \end{aligned} \tag{6.13}$$

We now define a sequence (ρ_k) of pseudometrics on $\overline{\mathcal{T}}_0(X)$ by

$$\rho_k(\theta, \phi) = \max\Big\{ |f_{\mathbf{n}}(\theta) - f_{\mathbf{n}}(\phi)| : \mathbf{n} \in \{0,1\}^k \cup \{\varnothing\} \Big\}.$$

Then (6.13) can be rewritten as

$$\rho(\theta, \phi) \leq \rho_k(\theta, \phi) + \gamma^k(\theta(0) + \phi(0)).$$

6.7.3 Strong compactness of $\overline{\mathcal{T}}_0^1(X)$

Suppose that $(\theta_i) \subseteq \overline{\mathcal{T}}_0^1(X)$. Then for any $\mathbf{n} \in \Delta$, $0 \leq f_{\mathbf{n}}(\theta_i) \leq f_{\varnothing}(\theta_i) \leq 1$. Hence we can pass to a subsequence (θ_i^1) of (θ_i) such that $f_{\mathbf{n}}(\theta_i^1)$ converges for every $\mathbf{n} \in \{0,1\}^1 \cup \{\varnothing\}$. By choosing this subsequence suitably, we may suppose that

$$|f_{\mathbf{n}}(\theta_i^1) - f_{\mathbf{n}}(\theta_j^1)| < \frac{1}{2^i},$$

whenever $i < j$ for every $\mathbf{n} \in \{0,1\}^1 \cup \{\varnothing\}$.

We then iterate this procedure, so that for each $k \in \mathbb{N}$ we have a subsequence (θ_i^k) of (θ_i^{k-1}) such that

$$|f_{\mathbf{n}}(\theta_i^k) - f_{\mathbf{n}}(\theta_j^k)| < \frac{1}{2^i},$$

whenever $i < j$ for every $\mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}$.

We now pass to a diagonal subsequence (ψ_i) where $\psi_i = \theta_i^i$. Suppose that $\mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}$, and that $k \leq i < j$. Then there exists $r < s$ in $\mathbb{N}$ with $i \leq r$ such that $\psi_i = \theta_i^i = \theta_r^k$ and $\psi_j = \theta_j^j = \theta_s^k$. Hence

$$|f_{\mathbf{n}}(\psi_i) - f_{\mathbf{n}}(\psi_j)| = |f_{\mathbf{n}}(\theta_r^k) - f_{\mathbf{n}}(\theta_s^k)| < \frac{1}{2^r} \leq \frac{1}{2^i}. \tag{6.14}$$

Suppose that $\varepsilon > 0$. Choose k sufficiently large that $2\gamma^k < \frac{\varepsilon}{2}$ and $2^{-k} < \frac{\varepsilon}{2}$. If $j > i \geq k$, then by (6.14),

$$\rho_k(\psi_i, \psi_j) < \frac{1}{2^i} \leq \frac{1}{2^k} < \frac{\varepsilon}{2}.$$

Thus,

$$\begin{aligned} \rho(\psi_i, \psi_j) &\leq \rho_k(\psi_i, \psi_j) + \gamma^k(\psi_i(0) + \psi_j(0)) \\ &\leq \frac{\varepsilon}{2} + 2\gamma^k < \varepsilon. \end{aligned}$$

Hence, every sequence in $\overline{\mathcal{T}}_0^1(X)$ admits a subsequence which converges uniformly over all of X. Since $\overline{\mathcal{T}}_0^1(X)$ is closed under pointwise limits, the limiting function is also in $\overline{\mathcal{T}}_0^1(X)$. Also, since the convergence is uniform over all of X, it follows in particular, that this subsequence converges in the strong topology on $\mathcal{T}(X)$. Since the strong topology is metrizable, this shows that $\overline{\mathcal{T}}_0^1(X)$ is strongly compact.

6.8 Strong compactness of types for spaces $T(\mathcal{A}_2, f)$

In this section we extend the result of Section 6.7 to more general Tsirelson type spaces of the form $T(\mathcal{A}_2, f)$, for a suitable norm f on $\mathbb{R}^2$. We take $f(\cdot,\cdot)$ to be a normalized 1-unconditional norm on $\mathbb{R}^2$ of the form

$$f(\alpha, \beta) = \max\left(\|(\alpha,\beta)\|_\infty, g(\alpha,\beta)\right),$$

where $g(\cdot,\cdot)$ is also a 1-unconditional norm on $\mathbb{R}^2$ such that $g(1,0) \leq \gamma$ and $g(0,1) \leq \gamma$, where $\gamma < 1$. The shaded area in Figure 6.1 denotes the part of the unit ball of $f(\cdot,\cdot)$ in the quadrant $x \geq 0$, $y \geq 0$. Notice that the unit sphere has corners where it meets the lines

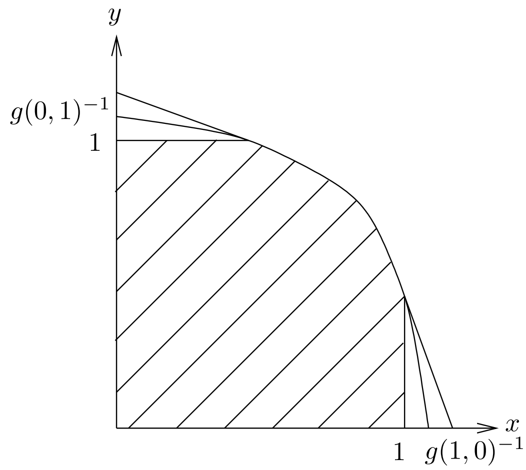

Figure 6.1: The positive quadrant of the unit ball for $f(\cdot,\cdot)$ in $\mathbb{R}^2$

$x = 1$ and $y = 1$, so that the tangents at these points meet the x and y axes at some point greater than 1.

The norm on $X = T(\mathcal{A}_2, f)$ satisfies the implicit relation

$$\|x\| = \|x\|_\infty \vee \sup\{f(\|E_1 x\|, \|E_2 x\|) : E_1 < E_2\},$$

and the standard basis of this space forms a 1-unconditional basis.

We will introduce a collection of functions $g_{\mathbf{n}}$, where $\mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}$ for some k, which will describe how the norms of vectors in X are obtained. We define $g_\varnothing(\alpha_1, \alpha_2) = g(\alpha_1, \alpha_2)$, and then given $\mathbf{n} \in \{0,1\}^k$ we define recursively

$$g_{(0,\mathbf{n})}(\alpha_1, \ldots, \alpha_{k+3}) = g(\alpha_1, g_{\mathbf{n}}(\alpha_2, \ldots, \alpha_{k+3})),$$
$$g_{(1,\mathbf{n})}(\alpha_1, \ldots, \alpha_{k+3}) = g(g_{\mathbf{n}}(\alpha_1, \ldots, \alpha_{k+2}), \alpha_{k+3}).$$

Similarly, we define a collection of functions $f_{\mathbf{n}}$, replacing g by f in the above definitions.

We begin by looking at some simple properties of these functions $g_{\mathbf{n}}$ and $f_{\mathbf{n}}$. We state these results for $g_{\mathbf{n}}$, but the same results clearly hold for $f_{\mathbf{n}}$.

Lemma 6.8.1 *If* $\mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}$*, and* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R}^{k+2}$*, then*

$$|g_{\mathbf{n}}(\boldsymbol{\alpha}) - g_{\mathbf{n}}(\boldsymbol{\beta})| \le g_{\mathbf{n}}(\boldsymbol{\alpha} - \boldsymbol{\beta}).$$

PROOF. The case where $\mathbf{n} = \varnothing$ is immediate from the triangle inequality applied to the norm $g(\cdot,\cdot)$. In general we prove the result by induction on the length of $\mathbf{n}$. Suppose the result is true for every $\mathbf{n} \in \{0,1\}^k$. If $\mathbf{m} \in \{0,1\}^{k+1}$ is of the form $(0, \mathbf{n})$ (the argument for $(1, \mathbf{n})$ is similar), then applying the triangle inequality for the norm g gives,

$$\begin{aligned}|g_{\mathbf{m}}(\boldsymbol{\alpha}) - g_{\mathbf{m}}(\boldsymbol{\beta})| &= |g(\alpha_1, g_{\mathbf{n}}(\alpha_2, \ldots, \alpha_{k+3})) - g(\beta_1, g_{\mathbf{n}}(\beta_2, \ldots, \beta_{k+3}))| \\ &\leq g(\alpha_1 - \beta_1, g_{\mathbf{n}}(\alpha_2, \ldots, \alpha_{k+3}) - g_{\mathbf{n}}(\beta_2, \ldots, \beta_{k+3})).\end{aligned}$$

But by inductive hypothesis,

$$|g_{\mathbf{n}}(\alpha_2, \ldots, \alpha_{k+3}) - g_{\mathbf{n}}(\beta_2, \ldots, \beta_{k+3})| \leq g_{\mathbf{n}}(\alpha_2 - \beta_2, \ldots, \alpha_{k+3} - \beta_{k+3}).$$

Using the 1-unconditionality of the norm g gives us that

$$\begin{aligned}|g_{\mathbf{m}}(\boldsymbol{\alpha}) - g_{\mathbf{m}}(\boldsymbol{\beta})| &\leq g(\alpha_1 - \beta_1, g_{\mathbf{n}}(\alpha_2 - \beta_2, \ldots, \alpha_{k+3} - \beta_{k+3})) \\ &= g_{\mathbf{m}}(\boldsymbol{\alpha} - \boldsymbol{\beta}).\end{aligned}$$

□

Lemma 6.8.2 *If* $\mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}$, *and* $\boldsymbol{\alpha} \in \mathbb{R}^{k+2}$, *then* $g_{\mathbf{n}}(\boldsymbol{\alpha}) \leq f_{\mathbf{n}}(\boldsymbol{\alpha})$.

Proof. This follows by induction on k. For $\mathbf{n} = \varnothing$, this is immediate from the definition of f in terms of g. Suppose the result is true for every $\mathbf{n} \in \{0,1\}^k$. If $\mathbf{m} \in \{0,1\}^{k+1}$ is of the form $(0, \mathbf{n})$ (again the case of $(1, \mathbf{n})$ is similar), then

$$\begin{aligned}g_{\mathbf{m}}(\boldsymbol{\alpha}) &= g(\alpha_1, g_{\mathbf{n}}(\alpha_2, \ldots, \alpha_{k+3})) \\ &\leq f(\alpha_1, g_{\mathbf{n}}(\alpha_2, \ldots, \alpha_{k+3})) \\ &\leq f(\alpha_1, f_{\mathbf{n}}(\alpha_2, \ldots, \alpha_{k+3})) \\ &= f_{\mathbf{m}}(\boldsymbol{\alpha}).\end{aligned}$$

□

Given $\mathbf{n} \in \{0,1\}^k$, we define $r(\mathbf{n})$ to be the number of zeros in $\mathbf{n}$.

Lemma 6.8.3 *Given* $\mathbf{n} \in \{0,1\}^k$ *and* $M \in \mathbb{N}$, *there exists a finite subset* $A_{\mathbf{n},M}$ *of* $\mathbb{R}^{r(\mathbf{n})}$ *such that whenever* $\boldsymbol{\alpha} \in \mathbb{R}^{r(\mathbf{n})}$ *satisfies* $f_{\mathbf{n}}(\boldsymbol{\alpha}, \mathbf{0}) \leq M$, *there exists* $\boldsymbol{\alpha}' \in A_{\mathbf{n},M}$ *such that for every* $\boldsymbol{\gamma} \in \mathbb{R}^{k+2-r(\mathbf{n})}$

$$|f_{\mathbf{n}}(\boldsymbol{\alpha}, \boldsymbol{\gamma}) - f_{\mathbf{n}}(\boldsymbol{\alpha}', \boldsymbol{\gamma})| < \frac{1}{k}.$$

Proof. This follows by a compactness argument. Let $A = \{\boldsymbol{\alpha} \in \mathbb{R}^{r(\mathbf{n})} : f_{\mathbf{n}}(\boldsymbol{\alpha})\} \leq M$. Note that any coordinate of an element in A is at most M. Hence given a sequence of elements of A, we can choose a subsequence so that the corresponding sequence of coordinates all converge. It follows by continuity that the limit lies in A. Thus A is compact. Now we choose $A_{\mathbf{n},M}$ to be a finite η-net for A. Provided we choose η sufficiently small, Lemma 6.8.1 implies that,

$$|f_{\mathbf{n}}(\boldsymbol{\alpha},\boldsymbol{\gamma}) - f_{\mathbf{n}}(\boldsymbol{\beta},\boldsymbol{\gamma})| \leq f_{\mathbf{n}}(\boldsymbol{\alpha}-\boldsymbol{\beta},\mathbf{0}) < \frac{1}{k}.$$

□

Definition 6.8.4 *We define a collection of functions $G_{\mathbf{n},\boldsymbol{\alpha}} : X \to \mathbb{R}$ for $\mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}$, and $\boldsymbol{\alpha} \in \mathbb{R}^{r(\mathbf{n})}$, by*

$$G_{\mathbf{n},\boldsymbol{\alpha}}(x) = \sup\left\{ f_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\|E_1 x\|,\ldots,\left\|E_{k-r(\mathbf{n})}x\right\|) : E_1 < \ldots < E_{k-r(\mathbf{n})} \right\},$$

whenever $\mathbf{n} \in \{0,1\}^k$, and we also define

$$G_{\varnothing,\varnothing}(x) = \|x\|.$$

Lemma 6.8.5 *Suppose that $u, x, y \in X$ are such that* $\max \operatorname{supp} u < \min(\operatorname{supp} x \cup \operatorname{supp} y)$ *with $\|u\| \leq M$. Then for every $k \in \mathbb{N}$,*

$$\begin{aligned} |\,\|u+x\| - \|u+y\|\,| \leq \sup\{&|G_{\mathbf{n},\boldsymbol{\alpha}}(x) - G_{\mathbf{n},\boldsymbol{\alpha}}(y)| : \mathbf{n} \in \{0,1\}^k \cup \{\varnothing\}, \boldsymbol{\alpha} \in A_{\mathbf{n},M}\} \\ &+ \frac{2}{k} + \gamma^k(\|u\|+\|x\|+\|y\|). \end{aligned}$$

Proof. Consider $\|u+x\|$. Suppose first of all that $\|u+x\| = \|u+x\|_\infty$. If $\|u+x\| = \|u\|_\infty$, then

$$\|u+x\| = \|u\|_\infty \leq \|u\| \leq \|u+y\|.$$

Else, $\|u+x\| = \|x\|_\infty$, and hence

$$\|u+x\| = \|x\|_\infty \leq \|x\| \leq \|y\| + |\,\|x\| - \|y\|\,| \leq \|u+y\| + |G_{\varnothing,\varnothing}(x) - G_{\varnothing,\varnothing}(y)|.$$

The second case is where $\|u+x\| = f(\|E_1(u+x)\|, \|E_2(u+x)\|)$ for some intervals $E_1 < E_2$. If $\|u+x\| = \|E_1(u+x)\|$ or $\|u+x\| = \|E_2(u+x)\|$ then we may consider $\|E_1(u+x)\|$ or $\|E_2(u+x)\|$ instead. Hence we may suppose that $\|u+x\| = g(\|E_1(u+x)\|, \|E_2(u+x)\|)$. At most one of the intervals E_1 and E_2 meets both u and x. If E_1 meets only u then we let $n_1 = 0$, otherwise we set $n_1 = 1$. We continue in this way, splitting the interval which meet both u and x. Suppose we continue down to k levels, then we get $\mathbf{n} \in \{0,1\}^k$ and intervals $F_1 < \ldots F_{r(\mathbf{n})} < G_1 < G_2 < E_1 < \ldots < E_{k-r(\mathbf{n})}$ such that

$$\|u+x\| = g_{\mathbf{n}}(\|F_1 u\|,\ldots,\left\|F_{r(\mathbf{n})}u\right\|, \|G_1(u+x)\|, \|G_2(u+x)\|, \|E_1 x\|,\ldots,\left\|E_{k-r(\mathbf{n})}\right\|).$$

The intervals G_1 and G_2 are the two obtained from the last splitting, and as usual at most one of these meets both u and x. For convenience we will write $\boldsymbol{\alpha}$ for $(\|F_1u\|,\ldots,\|F_{r(\mathbf{n})}u\|)$, and $\boldsymbol{\beta}$ for $(\|E_1x\|,\ldots,\|E_{k-r(\mathbf{n})}\|)$. By Lemma 6.8.1,

$$\begin{aligned}\|u+x\| &= g_{\mathbf{n}}(\boldsymbol{\alpha},\|G_1(u+x)\|,\|G_2(u+x)\|,\boldsymbol{\beta})\\ &\le g_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\boldsymbol{\beta})+g_{\mathbf{n}}(\mathbf{0},\|G_1(u+x)\|,\|G_2(u+x)\|,\mathbf{0})\\ &\le g_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\boldsymbol{\beta})+\gamma^k g(\|G_1(u+x)\|,\|G_2(u+x)\|)\\ &\le g_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\boldsymbol{\beta})+\gamma^k(\|u\|+\|x\|)\\ &\le f_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\boldsymbol{\beta})+\gamma^k(\|u\|+\|x\|).\end{aligned}$$

Notice that if the above method of norming $u+x$ ends in fewer than the k stages then since $f(1,0)=f(0,1)=1$ we can extend $\mathbf{n}$ by zeros and the above expression still holds with the $\boldsymbol{\alpha}$ extended by zeros.

Since $u<y$, we automatically get from the implicit definition of the norm that

$$\|u+y\|\ge f_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\boldsymbol{\gamma}),$$

where $\boldsymbol{\gamma}=(\|\tilde{E}_1y\|,\ldots,\|\tilde{E}_{k-r(\mathbf{n})}y\|)$ and $\tilde{E}_1<\ldots<\tilde{E}_{k-r(\mathbf{n})}$.

Hence,

$$\|u+x\|-\|u+y\|\le f_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\boldsymbol{\beta})-f_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\boldsymbol{\gamma})+\gamma^k(\|u\|+\|x\|).$$

Note that in particular

$$M\ge\|u\|\ge f_{\mathbf{n}}(\boldsymbol{\alpha},0,0,\mathbf{0}),$$

and hence we may choose $\boldsymbol{\alpha}'\in A_{\mathbf{n},M}$ satisfying the conclusion of Lemma 6.8.3. Hence,

$$\|u+x\|-\|u+y\|\le f_{\mathbf{n}}(\boldsymbol{\alpha}',0,0,\boldsymbol{\beta})-f_{\mathbf{n}}(\boldsymbol{\alpha}',0,0,\boldsymbol{\gamma})+\frac{2}{k}+\gamma^k(\|u\|+\|x\|).$$

It follows that

$$\|u+x\|-\|u+y\|\le |G_{\mathbf{n},\boldsymbol{\alpha}'}(x)-G_{\mathbf{n},\boldsymbol{\alpha}'}(y)|+\frac{2}{k}+\gamma^k(\|u\|+\|x\|).$$

This proves our result using the symmetry between x and y. □

6.8.1 A description of the weakly null types on $T(\mathcal{A}_2,f)$

We define,

$$\Delta=\{(\varnothing,\varnothing)\}\cup\bigcup_{k=1}^{\infty}\bigcup_{M=1}^{\infty}\{(\mathbf{n},\boldsymbol{\alpha}):\mathbf{n}\in\{0,1\}^k,\boldsymbol{\alpha}\in A_{\mathbf{n},M}\}.$$

Given $k, M \in \mathbb{N}$ we will also write $\Delta_{k,M}$ to denote the finite set

$$\{(\varnothing,\varnothing)\} \cup \{(\mathbf{n},\boldsymbol{\alpha}) : \mathbf{n} \in \{0,1\}^k, \boldsymbol{\alpha} \in A_{\mathbf{n},M}\}.$$

Δ is countable, and hence $\mathbb{R}^\Delta$, when equipped with the Tychonoff topology, is metrizable. We define a map from X into $\mathbb{R}^\Delta$, by $x \mapsto \omega_x = (G_\delta(x))_{\delta\in\Delta}$. We then define $\mathcal{A}_0(X)$ to be the set of limits in $\mathbb{R}^\Delta$ of sequences (ω_{x_n}) where (x_n) is a weakly null sequence in X, and let $\overline{\mathcal{A}_0}(X)$ denote the closure of $\mathcal{A}_0(X)$ in $\mathbb{R}^\Delta$. We can identify $\mathcal{A}_0(X)$ with the weakly null types on X. Indeed, suppose that ω is the limit of the sequence (ω_{x_n}) where (x_n) is weakly null. Let K be such that $\|x_n\| \le K$ for all n. Choose any $M \in \mathbb{N}$ and let $\varepsilon > 0$ be arbitrary. Choose k sufficiently large that $\frac{2}{k} + \gamma^k(M+2K) < \frac{\varepsilon}{2}$. Then whenever $\|u\| \le M$, by Lemma 6.8.5, we have

$$|\,\|u+x_n\| - \|u+x_m\|\,| \le \sup\{|G_\delta(x_n) - G_\delta(x_m)| : \delta \in \Delta_{k,M}\} + \frac{\varepsilon}{2}.$$

Since (ω_{x_n}) converges in $\mathbb{R}^\Delta$, we find $N = N(\varepsilon)$ such that $|\,\|u+x_n\| - \|u+x_m\|\,| < \varepsilon$ whenever $n, m \ge N$. Thus the limit $\lim_{n\to\infty}\|u+x_n\|$ exists for every $u \in X$, and so defines a weakly null type (and in fact this limit exists uniformly over bounded subsets of X).

Conversely, suppose (x_n) is a weakly null sequence which generates a weakly null type. We may assume that $\|x_n\| \le K$ for all $n \in \mathbb{N}$ and some K. If $\mathbf{n} \in \{0,1\}^k$ and $\boldsymbol{\alpha} \in A_{\mathbf{n},M}$, then $G_{\mathbf{n},\boldsymbol{\alpha}}(x_n) \le M + K$ by Lemma 6.8.1, and $G_{\varnothing,\varnothing}(x_n) \le K$. Hence, by Tychonoff's theorem, there is a subsequence (x'_n) of (x_n) such that the (ω_{x_n}) converges in $\mathbb{R}^\Delta$.

Given $\delta \in \Delta$ we will denote by G_δ the coordinate function on $\mathbb{R}^\Delta$. These functions are continuous from $\mathbb{R}^\Delta$ to $\mathbb{R}$.

Now suppose that θ, ϕ are in $\mathcal{T}_0(X)$ and that under the above identification we have $\theta = \lim \omega_{x_n}$ and $\phi = \lim \omega_{y_n}$, where (x_n) and (y_n) are weakly null. Then for each $k \in \mathbb{N}$ and $\|u\| \le M$, we have by Lemma 6.8.5 that,

$$\begin{aligned}\lim_{n\to\infty} |\,\|u+x_n\| - \|u+y_n\|\,| \le \lim_{n\to\infty} &\sup\{|G_\delta(x_n) - G_\delta(y_n)| : \delta \in \Delta_{k,M}\} \\ &+ \frac{2}{k} + \lim_{n\to\infty} \gamma^k(\|u\| + \|x_n\| + \|y_n\|).\end{aligned}$$

Thus whenever $\|u\| \le M$ we have

$$|\theta(u) - \phi(u)| \le \sup\{|G_\delta(\theta) - G_\delta(\phi)| : \delta \in \Delta_{k,M}\} + \frac{2}{k} + \gamma^k(\|u\| + \theta(0) + \phi(0)).$$

We can also identify $\overline{\mathcal{A}_0}(X)$ with $\overline{\mathcal{T}}_0(X)$. Suppose that $\omega \in \overline{\mathcal{A}_0}(X)$ is the limit of the sequence $(\omega_i)_{i=1}^\infty \subseteq \mathcal{A}_0(X)$. Let θ_i be the weakly null type corresponding to ω_i. Then if

$\|u\| \leq M$,

$$|\theta_i(u) - \theta_j(u)| \leq \sup\{|G_\delta(\theta_i) - G_\delta(\theta_j)| : \delta \in \Delta_{k,M}\} + \frac{2}{k} + \gamma^k(\|u\| + \theta_i(0) + \theta_j(0)).$$

Since, ω is a limit point of $(\omega_i)_{i=1}^\infty$, it follows that $\theta(u) = \lim_{i\to\infty} \theta_i(u)$ exists for every u and defines a type $\theta \in \overline{\mathcal{T}}_0(X)$. Conversely, if $\theta(u) = \lim_{i\to\infty} \theta_i(u)$ for every $u \in X$, where θ_i is a weakly null type, let ω_i be the element in $\mathcal{A}_0(X)$ corresponding to θ_i. Then for every $\mathbf{n} \in \{0,1\}^k$ and every $\boldsymbol{\alpha} \in A_{\mathbf{n},M}$, $G_{(\mathbf{n},\boldsymbol{\alpha})}(\omega_i) \leq M + \theta_i(0)$. Hence, by Tychonoff's theorem, we can pass to a subsequence of (ω_i) which converges in $\mathbb{R}^\Delta$. We identify the limit ω of this subsequence with θ.

Suppose that $\theta, \phi \in \overline{\mathcal{T}}_0(X)$. Then it follows from the above that

$$d_M(\theta, \phi) \leq \rho_{k,M}(\theta, \phi) + \frac{2}{k} + \gamma^k(M + \theta(0) + \phi(0)),$$

where d_M is the one of the pseudometrics that defines the strong topology on $\mathcal{T}(X)$, and

$$\rho_{k,M}(\theta, \phi) = \sup\{|G_\delta(\theta) - G_\delta(\phi)| : \delta \in \Delta_{k,M}\}.$$

6.8.2 Strong compactness of $\overline{\mathcal{T}}_0^1(X)$

Let (θ_i) be a sequence contained in $\overline{\mathcal{T}}_0^1(X)$. Observe that $G_\delta(\theta_i) \leq M+1$ whenever $\delta \in \Delta_{k,M}$. Since $\Delta_{1,1}$ is a finite set, we can pass to a subsequence (θ_i^1) such that $\rho_{1,1}(\theta_i^1, \theta_j^1) < \frac{1}{2}$ for all i, j. Iterating this procedure we can find a subsequence (θ_i^k) of (θ_i^{k-1}) such that $\rho_{k,k}(\theta_i^k, \theta_j^k) < \frac{1}{2^k}$ for all i, j. Passing to a diagonal subsequence, gives us a sequence (ϕ_i) of (θ_i) such that $\rho_{k,k}(\phi_i, \phi_j) < \frac{1}{2^k}$ whenever $k \leq i < j$. So for this subsequence we have

$$d_k(\phi_i, \phi_j) \leq \frac{1}{2^k} + \frac{2}{k} + \gamma^k(k+2),$$

whenever $k \leq i < j$.

So if $\varepsilon > 0$, and $M \in \mathbb{N}$ is given, then choose k sufficiently large that $\frac{1}{2^k} + \frac{2}{k} + \gamma^k(k+2) < \varepsilon$ and such that $k \geq M$. Then, whenever $k \leq i < j$, we have

$$d_M(\phi_i, \phi_j) \leq d_k(\phi_i, \phi_j) < \varepsilon.$$

Thus, our subsequence converges uniformly on bounded subsets of X. Hence $\overline{\mathcal{T}}_0^1(X)$ is strongly compact.

Chapter 7

Stopping time Banach spaces

Contents

The idea of a stopping time Banach space was first introduced by Rosenthal in an unpublished article. Some properties of this space were studied by Bang and Odell in [9] and [10]. The spaces which we will define in this chapter are generalizations of Rosenthal's space.

7.1 Definition and simple properties of stopping time Banach spaces

Definition of stopping time Banach spaces

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space, and let $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots$ be an increasing sequence of σ-subalgebras of $\mathscr{F}$.

Definition 7.1.1 *We say that a sequence of random variables,* $(X_n)_{n=0}^{\infty}$*, is adapted (with respect to the filtration* $(\mathscr{F}_n)_{n=0}^{\infty}$*) if for each* n*,* X_n *is* $\mathscr{F}_n$*-measurable.*

Definition 7.1.2 *We say that a random variable* T*, taking values in* $\mathbb{N}\cup\{\infty\}$*, is a stopping time if for each* n*,* $\{\omega \in \Omega : T(\omega) = n\} \in \mathscr{F}_n$*. We denote by* $\mathscr{T}$*, the collection of stopping times.*

Suppose that $X = (X_0, X_1, \ldots, X_n, 0, \ldots)$ is a finitely supported adapted process. We define

$$\|X\|_S = \sup_{T\in\mathscr{T}} \mathbb{E}(|X_T|).$$

The stopping time Banach space, S, is then defined to be the completion of the finitely supported adapted processes under this norm.

The dyadic stopping time Banach space S^d

The dyadic stopping time Banach space is the one studied by Bang and Odell in [9] and [10]. The description given in these papers is different to the one that we will give but it is easy to see that they are equivalent. Let $\Omega = [0,1]$, $\mathscr{F}$ be the Borel subsets of $[0,1]$, and let $\mathbb{P}$ be Lebesgue measure on $[0,1]$. We define a filtration by taking $\mathscr{F}_n^d$ to be the σ-algebra generated by

$$\{[(j-1)/2^n, j/2^n) : 1 \le j \le 2^n\}.$$

A function is then $\mathscr{F}_n^d$-measurable if and only if it is constant valued on each of the intervals $[(j-1)/2^n, j/2^n)$. The stopping time Banach space defined by this filtration will be called the dyadic stopping time Banach space, and will be denoted by S^d. This is the simplest example of such a space because each of the random variables can only take finitely many values.

Notation

We will introduce some notation to describe finitely supported processes. Given an $\mathscr{F}_i$-measurable random variable X_i, we will denote the adapted process $(0, \ldots, 0, X_i, 0, \ldots)$, where the X_i occurs in the i^{th} position, by $X_i e_i$. So for example, we may represent an adapted process $(X_0, X_1, \ldots, X_n, 0, \ldots)$ by $\sum_{i=0}^{n} X_i e_i$.

We shall also frequently be using conditional expectations, $\mathbb{E}(X\,|\,\mathscr{F}_i)$, of random variables conditional on our filtration. To simplify our working, we shall often denote $\mathbb{E}(X\,|\,\mathscr{F}_i)$ by $\mathbb{E}_i(X)$.

7.1.1 Backwards recursion property of the stopping time norm

In this section we prove an important property of the norm in any stopping time Banach space, S. This will enable us to calculate the norms of finitely supported vectors in a systematic way.

Lemma 7.1.3 *Let* $X = (X_0, X_1, \dots, X_n, X_{n+1}, 0, \dots)$ *be a finitely supported adapted process, and* $\tilde{X} = (X_0, X_1, \dots, , X_{n-1}, |X_n| \vee \mathbb{E}_n(|X_{n+1}|), 0, 0, \dots)$*. Then* $\|X\|_S = \|\tilde{X}\|_S$*.*

Proof. Let T be a stopping time, and for each i, let $B_i = \{\omega \in \Omega : T(\omega) = i\}$. Let $B = \{\omega : T(\omega) > n\} \in \mathscr{F}_n$. Then

$$\begin{aligned}\mathbb{E}(|X_T|) &= \sum_{i=0}^{n+1} \int_{B_i} |X_i| \; \mathrm{d}\mathbb{P} \\ &\leq \sum_{i=0}^{n} \int_{B_i} |X_i| \; \mathrm{d}\mathbb{P} + \int_B |X_{n+1}| \; \mathrm{d}\mathbb{P}.\end{aligned}$$

By the definition of conditional expectation, since $B \in \mathscr{F}_n$,

$$\int_B |X_{n+1}| \; \mathrm{d}\mathbb{P} = \int_B \mathbb{E}_n(|X_{n+1}|) \; \mathrm{d}\mathbb{P}.$$

Define a new stopping time

$$T'(\omega) = \begin{cases} T(\omega) & \text{if } \omega \in B_0 \cup \dots \cup B_{n-1}, \\ n & \text{if } \omega \in B_n \cup B. \end{cases}$$

Then,

$$\begin{aligned}\mathbb{E}(|X_T|) &\leq \sum_{i=0}^{n} \int_{B_i} |X_i| \; \mathrm{d}\mathbb{P} + \int_B \mathbb{E}_n(|X_{n+1}|) \; \mathrm{d}\mathbb{P} \\ &= \sum_{i=0}^{n-1} \int_{B_i} |X_i| \; \mathrm{d}\mathbb{P} + \int_{B_n} |X_n| \; \mathrm{d}\mathbb{P} + \int_B \mathbb{E}_n(|X_{n+1}|) \; \mathrm{d}\mathbb{P} \\ &\leq \sum_{i=0}^{n-1} \int_{B_i} |X_i| \; \mathrm{d}\mathbb{P} + \int_{B_n \cup B} (|X_n| \vee \mathbb{E}_n(|X_{n+1}|)) \; \mathrm{d}\mathbb{P} \\ &= \mathbb{E}(|\tilde{X}_{T'}|).\end{aligned}$$

This shows that $\|X\|_S \leq \|\tilde{X}\|_S$. We now establish the opposite inequality in a similar way. Let T be any stopping time, and as before we let $B_i = \{\omega \in \Omega : T(\omega) = i\}$ and $C = \{\omega : T(\omega) > n-1\} \in \mathscr{F}_{n-1} \subseteq \mathscr{F}_n$. Let $D = \{\omega : |X_n| < \mathbb{E}_n(|X_{n+1}|)\} \in \mathscr{F}_n$. Note that $C \cap D$ and $C \cap D^c$ are therefore in $\mathscr{F}_n$. Define a new stopping time

$$T'(\omega) = \begin{cases} T(\omega) & \text{if } \omega \in B_0 \cup \dots \cup B_{n-1}, \\ n & \text{if } \omega \in C \cap D^c, \\ n+1 & \text{if } \omega \in C \cap D. \end{cases}$$

Then,

$$\begin{aligned}
\mathbb{E}(|\tilde{X}_T|) &= \sum_{i=0}^{n-1} \int_{B_i} |X_i| \,\mathrm{d}\mathbb{P} + \int_{B_n} (|X_n| \vee \mathbb{E}_n(|X_{n+1}|)) \,\mathrm{d}\mathbb{P} \\
&\leq \sum_{i=0}^{n-1} \int_{B_i} |X_i| \,\mathrm{d}\mathbb{P} + \int_{C} (|X_n| \vee \mathbb{E}_n(|X_{n+1}|)) \,\mathrm{d}\mathbb{P} \\
&= \sum_{i=0}^{n-1} \int_{B_i} |X_i| \,\mathrm{d}\mathbb{P} + \int_{C\cap D^c} |X_n| \,\mathrm{d}\mathbb{P} + \int_{C\cap D} \mathbb{E}_n(|X_{n+1}|) \,\mathrm{d}\mathbb{P} \\
&= \sum_{i=0}^{n-1} \int_{B_i} |X_i| \,\mathrm{d}\mathbb{P} + \int_{C\cap D^c} |X_n| \,\mathrm{d}\mathbb{P} + \int_{C\cap D} |X_{n+1}| \,\mathrm{d}\mathbb{P} \\
&= \mathbb{E}(|X_{T'}|).
\end{aligned}$$

Therefore, we have established that $\|\tilde{X}\|_S \leq \|X\|_S$, which proves the lemma. □

We now introduce some notation to simplify the statement of the above lemma. Given a finitely supported adapted process, $X = (X_0, X_1, \ldots, X_n, 0, \ldots)$, we can define (by 'backwards recursion'),

$$\sigma_n(X) = |X_n|,$$

$$\sigma_j(X) = |X_j| \vee \mathbb{E}_j(\sigma_{j+1}(X)) \text{ for } j < n.$$

Then by induction, using Lemma 7.1.3, we can prove the following corollary.

Corollary 7.1.4 *If $X = (X_0, X_1, \ldots, X_n, 0, \ldots)$ is a finitely supported adapted process then for each $k \leq n$,*

$$\|X\|_S = \|(X_0, X_1, \ldots, X_{k-1}, \sigma_k(X), 0, \ldots)\|_S .$$

In particular,

$$\|X\|_S = \mathbb{E}(\sigma_0(X)) = \|\sigma_0(X)\|_{L_1(\Omega)} .$$

7.2 Block types on S^d

In this section we give a characterization of the sequences of successive block vectors which determine types on S^d. Our first lemma appears in [9], though we rewrite the proof in terms of our probabilistic description of S^d.

Lemma 7.2.1 *Let $(Y^k)_{k=1}^\infty$ be a sequence of successive block vectors in S^d. $(Y^k)_{k=1}^\infty$ is type determining if and only if for every $n \in \mathbb{N}$ and every $E \in \mathscr{F}_n$, $(\mathbb{E}(\sigma_n(Y^k); E))_{k=1}^\infty$ is convergent. Further if $(Z^k)_{k=1}^\infty$ is another type determining sequence of successive blocks then $(Z^k)_{k=1}^\infty$ determines the same type as $(Y^k)_{k=1}^\infty$ if and only if $\lim_k \mathbb{E}(\sigma_n(Y^k); E)$ and $\lim_k \mathbb{E}(\sigma_n(Z^k); E)$ are the same for every n and every $E \in \mathscr{F}_n^d$.*

Proof. ($\Leftarrow$) Let X be a finitely supported adapted sequence with $X \leq n$. Then for sufficiently large k, $\left\|X + Y^k\right\|_{S^d} = \left\|(X_0, X_1, \ldots, X_n, \sigma_{n+1}(Y^k), 0, \ldots)\right\|_{S^d}$. Since $\sigma_{n+1}(Y^k)$ is $\mathscr{F}^d_{n+1}$-measurable, if we write $E_j = [(j-1)/2^{n+1}, j/2^{n+1})$ then

$$\sigma_{n+1}(Y^k) = \sum_{j=1}^{2^{n+1}} \frac{\mathbb{E}(\sigma_{n+1}(Y^k); E_j)}{\mathbb{P}(E_j)} \mathbb{1}_{E_j}.$$

Let $\lambda(n, E)$ denote $\lim_{k\to\infty} \mathbb{E}(\sigma_n(Y^k); E)$, then

$$\begin{aligned}\left\|X + Y^k\right\|_{S^d} &= \left\|(X_0, X_1, \ldots, X_n, \sum_{j=1}^{2^{n+1}} \frac{\mathbb{E}(\sigma_{n+1}(Y^k); E_j)}{\mathbb{P}(E_j)} \mathbb{1}_{E_j}, 0, \ldots)\right\|_{S^d} \\ &\to \left\|(X_0, X_1, \ldots, X_n, \sum_{j=1}^{2^{n+1}} \frac{\lambda(n+1, E_j)}{\mathbb{P}(E_j)} \mathbb{1}_{E_j}, 0, \ldots)\right\|_{S^d}\end{aligned}$$

Hence, we have shown that $(Y^k)_{k=1}^\infty$ determines a type.

($\Rightarrow$) Suppose that $(Y^k)_{k=1}^\infty$ defines a type θ, but there exists some n and an $E \in \mathscr{F}^d_n$ such that $\mathbb{E}(\sigma_n(Y^k); E)$ does not converge as $k \to \infty$. Without loss of generality we may suppose that E is an atom in the σ-algebra $\mathscr{F}^d_n$. Since $(Y^k)_{k=1}^\infty$ determines a type, $\lim_{k\to\infty} \left\|Y^k\right\|_{S^d}$ exists, and so by homogeneity we may suppose that $\|Y_k\|_{S^d} \leq 1$ for all k and $\lim_{k\to\infty} \left\|Y^k\right\|_{S^d} = 1$. Since $(\mathbb{E}(\sigma_n(Y^k); E))_{k=1}^\infty$ is a bounded non-convergent sequence, we may find two sequences $(n_k)_{k=1}^\infty$ and $(m_k)_{k=1}^\infty$ of integers such that

$$r = \lim_k \mathbb{E}(\sigma_n(Y^{n_k}); E) \neq \lim_k \mathbb{E}(\sigma_n(Y^{m_k}); E) = s.$$

Let $X_n = \frac{1}{\mathbb{P}(E)} \mathbb{1}_E$, and $X = X_n e_n$. Then,

$$\begin{aligned}\theta(X) &= \lim_k \|X + Y^{n_k}\|_{S^d} \\ &= \lim_k \mathbb{E}(X_n \vee \sigma_n(Y^{n_k})).\end{aligned}$$

Notice that $\int_E \sigma_n(Y^k) \mathrm{d}\mathbb{P} \leq \left\|Y^k\right\|_{S^d} \leq 1$, and therefore $\sigma_n(Y^{n_k}) \leq \frac{1}{\mathbb{P}(E)}$ on E. Thus,

$$\begin{aligned}\theta(X) &= \lim_k \mathbb{E}(\mathbb{1}_E X_n + \mathbb{1}_{E^c} \sigma_n(Y^{n_k})) \\ &= \lim_k \left(\mathbb{E}(X_n; E) + \mathbb{E}(\sigma_n(Y^{n_k})) - \mathbb{E}(\sigma_n(Y^{n_k}); E)\right) \\ &= 2 - r.\end{aligned}$$

An identical argument shows that $\theta(X) = 2 - s$, which gives the desired contradiction.

For the final claim, notice that the proof of ($\Leftarrow$) shows that the value taken by a type only depends on the values of $\lambda(n, E)$. This immediately implies the claim. □

The preceding theorem has the following immediate corollary, which gives us a correspondence between the block types on S^d and integrable martingales.

Corollary 7.2.2 *There is a correspondence between block types on S^d and integrable martingales (with respect to $(\mathscr{F}_n^d)_{n=0}^\infty$).*

Proof. Let $(Y^k)_{k=1}^\infty$ be a block sequence determining a type, and let $W_n = \lim_k \sigma_n(Y^k) \in L_1(\mathscr{F}_n)$. Then for large k, $\sigma_n(Y^k) = \mathbb{E}_n(\sigma_{n+1}(Y^k))$. Letting k tend to infinity implies that $W_n = \mathbb{E}_n(W_{n+1})$. Therefore $(W_n)_{n=0}^\infty$ forms a martingale.

Conversely, given a martingale $(W_n)_{n=0}^\infty$, let $Z^k = W_k e_k$. Then $\sigma_n(Z^k) = W_n$ for $k \geq n$. Hence, by Lemma 7.2.1, $(Z^k)_{k=1}^\infty$ determines a type, which is the same as that determined by the block sequence $(Y^k)_{k=1}^\infty$. $\square$

The backwards recursion property of the stopping time norm should be contrasted with the behaviour of the norm in spaces with Property (M). Roughly speaking, if $x < y$ are successive block vectors in a space with Property (M), then $\|x+y\|$ does not depend on the specific form of x. In a stopping time Banach space, however, $\|x+y\|$ does not depend on the specific form of y (by backwards recursion). The results of Chapter 6 applied to spaces with Property (M). Unfortunately, we can not apply the same results to stopping time Banach spaces because as the next proposition shows $\overline{\mathcal{T}}_0^1(S^d)$ fails to be strongly compact.

Proposition 7.2.3 *$\overline{\mathcal{T}}_0^1(S^d)$ is not strongly separable, and therefore not strongly compact.*

Proof. Let $\Delta = \{0,1\}^{\mathbb{N}}$. Given $\boldsymbol{\alpha} \in \Delta$ and $k \geq 1$, we define

$$l_k(\boldsymbol{\alpha}) = \sum_{i=1}^k \frac{1}{2^i}\alpha_i,$$

and,

$$r_k(\boldsymbol{\alpha}) = l_k(\boldsymbol{\alpha}) + \frac{1}{2^k}.$$

We define $X^k(\boldsymbol{\alpha}) = 2^k \mathbb{1}_{[l_k(\boldsymbol{\alpha}), r_k(\boldsymbol{\alpha}))}$. Note that $X^k(\boldsymbol{\alpha})$ is $\mathscr{F}_k^d$-measurable with $\mathbb{E}(X^k(\boldsymbol{\alpha})) = 1$. Also the support of $X^k(\boldsymbol{\alpha})$ is an atom of $\mathscr{F}_k^d$, with the support of $X^j(\boldsymbol{\alpha})$ contained in that of $X^k(\boldsymbol{\alpha})$ whenever $j \geq k$. Thus, $\mathbb{E}_k(X^j(\boldsymbol{\alpha})) = X^k(\boldsymbol{\alpha})$ whenever $j \geq k$. It follows from Corollary 7.2.2 that $(X^k(\boldsymbol{\alpha})e_k)_{k=1}^\infty$ determines a block type, $\theta_{\boldsymbol{\alpha}}$, on S^d. In fact, it is easy to show that $(X^k(\boldsymbol{\alpha})e_k)_{k=1}^\infty$ is 1-equivalent to the usual basis of c_0, and so in particular is weakly null. Therefore, $\theta_{\boldsymbol{\alpha}}$ is in $\mathcal{T}_0^1(S^d)$.

Now suppose that $\boldsymbol{\alpha}$, $\boldsymbol{\beta} \in \Delta$ with $\boldsymbol{\alpha} \neq \boldsymbol{\beta}$. Let k be the first coordinate where $\alpha_k \neq \beta_k$. Then, since $X^k(\boldsymbol{\beta})$ and $X^n(\boldsymbol{\alpha})$ have disjoint supports whenever $n \geq k$,

$$\begin{aligned}\theta_{\boldsymbol{\alpha}}(X^k(\boldsymbol{\beta})e_k) &= \lim_{n\to\infty} \left\| X^k(\boldsymbol{\beta})e_k + X^n(\boldsymbol{\alpha})e_n \right\|_{S^d} \\ &= \left\| X^k(\boldsymbol{\beta})e_k + X^k(\boldsymbol{\alpha})e_k \right\|_{S^d} \\ &= 2.\end{aligned}$$

However,

$$\begin{aligned}\theta_{\boldsymbol{\beta}}(X^k(\boldsymbol{\beta})e_k) &= \lim_{n\to\infty} \left\| X^k(\boldsymbol{\beta})e_k + X^n(\boldsymbol{\beta})e_n \right\|_{S^d} \\ &= 1.\end{aligned}$$

Therefore, $\sup\{|\theta_{\boldsymbol{\alpha}}(x) - \theta_{\boldsymbol{\beta}}(x)| : \|x\|_{S^d} \leq 1\} \geq 1$. Recalling the definition of the metric for the strong topology (see Section 2.4.1), this shows that

$$d(\theta_{\boldsymbol{\alpha}}, \theta_{\boldsymbol{\beta}}) \geq \frac{1}{2} \text{ whenever } \boldsymbol{\alpha}, \boldsymbol{\beta} \in \Delta,\ \boldsymbol{\alpha} \neq \boldsymbol{\beta}.$$

Since Δ is uncountable, this shows that $\overline{\mathcal{T}}^1_0(S^d)$ is not strongly separable. Also, a compact metric space must be separable, and therefore $\overline{\mathcal{T}}^1_0(S^d)$ is not strongly compact. $\square$

7.3 Subspaces of S arising from independent random variables

Let $X_0, X_1, \ldots$ be a sequence of random variables with $X_i \geq 0$ and $\mathbb{E}(X_i) = 1$ such that each X_i is independent of $\mathscr{F}_{i-1}$ (i.e. the value of X_i is independent of the information available at time $i-1$). Let F_i denote the cumulative distribution function of X_i. Then, $\mathbb{E}_j(X_i) = \mathbb{E}(X_i) = 1$ almost surely whenever $j < i$, since X_i is independent of $\mathscr{F}_j$. In this situation we can apply Lemma 7.1.3 to obtain a simple expression for the norm. For $u, v \in \mathbb{R}$, we define functions $\Phi^i : \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+$ by

$$\Phi^i(u, v) = \mathbb{E}(|u|X_i \vee |v|).$$

It is important to note that Φ^i only depends on the distribution function of X_i. Indeed for any $u, v \geq 0$,

$$\begin{aligned}\Phi^i(u, v) &= \mathbb{E}(uX_i \vee v) \\ &= v + \mathbb{E}((uX_i - v)^+) \\ &= v + \int_0^\infty \mathbb{P}(uX_i - v > t)\ \mathrm{d}t \\ &= v + u\int_{\frac{v}{u}}^\infty 1 - F_i(s)\ \mathrm{d}s.\end{aligned}$$

Since $1 = \mathbb{E}(X_i) = \int_0^\infty 1 - F_i(s)\ \mathrm{d}s$, it follows that

$$\Phi^i(u,v) = u + u\int_0^{\frac{v}{u}} F_i(s)\ \mathrm{d}s. \tag{7.1}$$

We can then apply Lemma 7.1.3, to obtain,

$$\begin{aligned}\left\|\sum_{i=0}^{n}\alpha_i X_i e_i\right\|_S &= \left\|\sum_{i=0}^{n-2}\alpha_i X_i e_i + (\alpha_{n-1}X_{n-1} \vee \alpha_n \mathbb{E}_{n-1}(X_n))e_{n-1}\right\|_S \\ &= \left\|\sum_{i=0}^{n-2}\alpha_i X_i e_i + (\alpha_{n-1}X_{n-1} \vee \alpha_n)e_{n-1}\right\|_S\end{aligned}$$

Since X_{n-1} is independent of $\mathscr{F}_{n-2}$ it follows that $(\alpha_{n-1}X_{n-1} \vee \alpha_n)$ is also independent of $\mathscr{F}_{n-2}$. Therefore applying Lemma 7.1.3 again gives,

$$\begin{aligned}\left\|\sum_{i=0}^{n}\alpha_i X_i e_i\right\|_S &= \left\|\sum_{i=0}^{n-3}\alpha_i X_i e_i + (\alpha_{n-2}X_{n-2} \vee \mathbb{E}_{n-2}(\alpha_{n-1}X_{n-1} \vee \alpha_n))e_{n-2}\right\|_S \\ &= \left\|\sum_{i=0}^{n-3}\alpha_i X_i e_i + (\alpha_{n-2}X_{n-2} \vee \Phi^{n-1}(\alpha_{n-1},\alpha_n))e_{n-2}\right\|_S\end{aligned}$$

Continuing this process we find that,

$$\begin{aligned}\left\|\sum_{i=0}^{n}\alpha_i X_i e_i\right\|_S &= \left\|(\alpha_0 X_0 \vee \Phi^1(\alpha_1, \Phi^2(\ldots, \Phi^{n-1}(\alpha_{n-1},\alpha_n)\ldots))e_1\right\|_S \\ &= \Phi^0(\alpha_0, \Phi^1(\alpha_1, \Phi^2(\ldots, \Phi^{n-1}(\alpha_{n-1},\alpha_n)))\ldots).\end{aligned}$$

Therefore if we recursively define

$$\begin{gathered}(\Phi^0 * \Phi^1)(\alpha_0,\alpha_1,\alpha_2) = \Phi^0(\alpha_0, \Phi^1(\alpha_1,\alpha_2)), \\ (\Phi^0 * \Phi^1 * \ldots \Phi^{n+1})(\alpha_0,\alpha_1,\ldots,\alpha_{n+2}) = \Phi^0(\alpha_0, (\Phi^1 * \ldots \Phi^{n+1})(\alpha_1,\ldots,\alpha_{n+2})),\end{gathered}$$

then we can express the norm of finite linear combinations of $\{X_i e_i\}_{i=1}^\infty$ by

$$\left\|\sum_{i=0}^{n}\alpha_i X_i e_i\right\|_S = (\Phi^0 * \ldots * \Phi^{n-1})(\alpha_0,\ldots,\alpha_n).$$

In the situation where the X_i are also identically distributed then we get a simpler expression for the norm. If $\Phi(\cdot,\cdot)$ is the function Φ^i defined above (which is independent of i in this case), then

$$\left\|\sum_{i=0}^{n}\alpha_i X_i e_i\right\|_S = \Phi_n(\alpha_0,\ldots,\alpha_n),$$

where the function $\Phi_n = \underbrace{\Phi * \ldots * \Phi}_{n}$.

Remark 7.3.1 *Suppose that $n_0 < n_1 < n_2 < \dots$ is a strictly increasing sequence of natural numbers and that $(X_i)_{i=0}^{\infty}$ is a sequence of non-negative random variables such that $\mathbb{E}(X_i) = 1$, X_i is $\mathscr{F}_{n_i}$-measurable and independent of $\mathscr{F}_{n_{i-1}}$. We may regard $\sum_{i=0}^{n} \alpha_i X_i e_{n_i}$ as being an element in the stopping time space with respect to the filtration $(\mathscr{F}_{n_i})_{i=0}^{\infty}$. Therefore, the result of the previous section still applies giving,*

$$\left\| \sum_{i=0}^{n} \alpha_i X_i e_{n_i} \right\|_S = (\Phi^0 * \dots * \Phi^{n-1})(\alpha_0, \dots, \alpha_n).$$

Example 7.3.2 *In this example we consider a simple subspace of S^d given by a sequence of independent identically distributed random variables.*

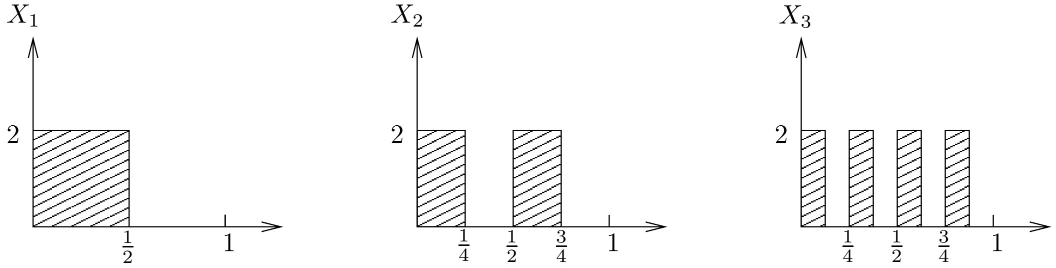

Figure 7.1: A sequence of independent random variables

Let $X_1, X_2, \dots$ be the sequence of random variables represented in Figure 7.1. It is easy to check that each X_i is independent of $\mathscr{F}_{i-1}$. The X_i each satisfy $\mathbb{E}(X_i) = 1$ and are identically distributed, with cumulative distribution function,

$$F(x) = \begin{cases} \frac{1}{2} & \text{if } 0 \leq x < 2, \\ 1 & \text{if } x \geq 2. \end{cases}$$

Hence, we can calculate $\Phi(u, v)$, giving for $u, v \geq 0$,

$$\Phi(u, v) = \begin{cases} u + \frac{1}{2}v & \text{if } v < 2u, \\ v & \text{if } v \geq 2u. \end{cases}$$

It is trivial that for any n,

$$\left\| \sum_{i=1}^{n} \alpha_i X_i e_i \right\|_{S^d} \geq \max_i |\alpha_i|.$$

In fact, we can prove that

$$\left\| \sum_{i=1}^{n} \alpha_i X_i e_i \right\|_{S^d} \leq 2 \max_i |\alpha_i|.$$

This will follow if we prove the stronger claim, that for every n,

$$\Phi_{n-1}(\alpha_1, \dots, \alpha_n) \leq \left(2 - \frac{1}{2^{n-1}}\right) \max_i |\alpha_i|.$$

We prove this by induction on n. The case $n = 1$ is trivial, and the case $n = 2$ is easy since,

$$\Phi_1(\alpha_1, \alpha_2) \leq \max\left(|\alpha_1| + \frac{1}{2}|\alpha_2|, |\alpha_2|\right) \leq \frac{3}{2} \max_i |\alpha_i|.$$

Given the result for n,

$$\begin{aligned}
\Phi_n(\alpha_1, \dots, \alpha_{n+1}) &= \Phi(\alpha_1, \Phi_{n-1}(\alpha_2, \dots, \alpha_{n+1})) \\
&\leq \Phi\left(\alpha_1, \left(2 - \frac{1}{2^{n-1}}\right) \max_{i \geq 2} |\alpha_i|\right) \\
&\leq \max\left(|\alpha_1| + \left(1 - \frac{1}{2^n}\right) \max_{i \geq 2} |\alpha_i|, \left(2 - \frac{1}{2^{n-1}}\right) \max_{i \geq 2} |\alpha_i|\right) \\
&\leq \left(2 - \frac{1}{2^n}\right) \max_{1 \leq i \leq n} |\alpha_i|.
\end{aligned}$$

This proves our claim. Hence we have shown that $[X_i e_i]_{i=1}^{\infty}$ is isomorphic to c_0.

7.4 Isomorphic structure of subspaces generated by independent random variables

In this section we show that for any stopping time Banach space, S, the closed linear span of $\{X_i e_i\}_{i=0}^{\infty}$, where X_i are identically distributed with $\mathbb{E}(X_i) = 1$ and each X_i independent of $\mathscr{F}_{i-1}$, is in fact isomorphic to an Orlicz space. Let $\Phi(\cdot, \cdot)$ be the function introduced in Section 7.3, which is a norm on $\mathbb{R}^2$. Define a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\phi(t) = \Phi(t, 1) - 1.$$

It follows from the properties of a norm and $\Phi(0, 1) = 1$ that ϕ is an Orlicz function. We will show that $[X_i e_i]_{i=0}^{\infty}$ is isomorphic to h_ϕ. For instance, in Example 7.3.2, we get

$$\phi(t) = \begin{cases} 0 & \text{if } t < \frac{1}{2}, \\ t - \frac{1}{2} & \text{if } t \geq \frac{1}{2}. \end{cases}$$

Thus, in this case, ϕ is a degenerate Orlicz function. It is straightforward to show that the Orlicz space, h_ϕ, is isomorphic to c_0, which agrees with the result of Example 7.3.2.

Lower Orlicz norm estimates

In this section we show that we can bound $\|\sum_{i=0}^{n} \alpha_i X_i e_i\|_S$ below by the Luxemburg norm in ℓ_ϕ. Let $\alpha_0, \dots, \alpha_n$ be any scalars with $\alpha_n \neq 0$, and for convenience let

$$\boldsymbol{\alpha} \cdot X = \sum_{i=0}^{n} \alpha_i X_i e_i.$$

Then,

$$\begin{aligned} \Phi_n(\alpha_0, \dots, \alpha_n) &= \Phi(\alpha_0, \Phi_{n-1}(\alpha_1, \dots, \alpha_n)) \\ &= \Phi_{n-1}(\alpha_1, \dots, \alpha_n)\ \Phi\left(\frac{\alpha_0}{\Phi_{n-1}(\alpha_1, \dots, \alpha_n)}, 1\right) \\ &= \Phi_{n-1}(\alpha_1, \dots, \alpha_n)\ \left(1 + \phi\left(\frac{|\alpha_0|}{\Phi_{n-1}(\alpha_1, \dots, \alpha_n)}\right)\right) \\ &= \Phi_{n-1}(\alpha_1, \dots, \alpha_n) + \Phi_{n-1}(\alpha_1, \dots, \alpha_n)\phi\left(\frac{|\alpha_0|}{\Phi_{n-1}(\alpha_1, \dots, \alpha_n)}\right). \end{aligned} \tag{7.2}$$

Hence, (7.2) tells us that

$$\|\boldsymbol{\alpha} \cdot X\|_S = \Phi_{n-1}(\alpha_1, \dots, \alpha_n) + \Phi_{n-1}(\alpha_1, \dots, \alpha_n)\phi\left(\frac{|\alpha_0|}{\Phi_{n-1}(\alpha_1, \dots, \alpha_n)}\right). \tag{7.3}$$

We repeat this procedure $n-1$ times, by substituting (7.2) (with appropriate n) for the first term in (7.3), giving

$$\begin{aligned} \|\boldsymbol{\alpha} \cdot X\|_S &= \Phi(\alpha_{n-1}, \alpha_n) + \sum_{i=0}^{n-2} \Phi_{n-i-1}(\alpha_{i+1}, \dots, \alpha_n)\phi\left(\frac{|\alpha_i|}{\Phi_{n-i-1}(\alpha_{i+1}, \dots, \alpha_n)}\right) \\ &= |\alpha_n| + \sum_{i=0}^{n-1} \Phi_{n-i-1}(\alpha_{i+1}, \dots, \alpha_n)\phi\left(\frac{|\alpha_i|}{\Phi_{n-i-1}(\alpha_{i+1}, \dots, \alpha_n)}\right). \end{aligned}$$

We rearrange this into the form,

$$1 = \frac{|\alpha_n|}{\|\boldsymbol{\alpha} \cdot X\|_S} + \sum_{i=0}^{n-1} \frac{\Phi_{n-i-1}(\alpha_{i+1}, \dots, \alpha_n)}{\|\boldsymbol{\alpha} \cdot X\|_S}\phi\left(\frac{|\alpha_i|}{\Phi_{n-i-1}(\alpha_{i+1}, \dots, \alpha_n)}\right). \tag{7.4}$$

Now, since ϕ is convex and $\phi(0) = 0$, it follows that for any $x \geq 0$ and any $\lambda \in [0, 1]$, $\phi(\lambda x) = \phi(\lambda x + (1 - \lambda)0) \leq \lambda\phi(x)$. In particular, since $\phi(1) = \Phi(1, 1) - 1 \leq 1$, $\phi(\lambda) \leq \lambda$ whenever $\lambda \in [0, 1]$. In particular,

$$\frac{|\alpha_n|}{\|\boldsymbol{\alpha} \cdot X\|_S} \geq \phi\left(\frac{|\alpha_n|}{\|\boldsymbol{\alpha} \cdot X\|_S}\right), \tag{7.5}$$

and for $0 \le i \le n-1$,

$$\frac{\Phi_{n-i-1}(\alpha_{i+1}, \ldots, \alpha_n)}{\|\boldsymbol{\alpha} \cdot X\|_S} \phi \left(\frac{|\alpha_i|}{\Phi_{n-i-1}(\alpha_{i+1}, \ldots, \alpha_n)} \right) \ge \phi \left(\frac{|\alpha_i|}{\|\boldsymbol{\alpha} \cdot X\|_S} \right). \tag{7.6}$$

Therefore, combining (7.5) and (7.6) with (7.4) shows that

$$\sum_{i=0}^{n} \phi \left(\frac{|\alpha_i|}{\|\boldsymbol{\alpha} \cdot X\|_S} \right) \le 1,$$

Thus, we have shown that $\|\boldsymbol{\alpha}\|_\phi \le \|\boldsymbol{\alpha} \cdot X\|_S$.

Upper Orlicz norm estimates

In this section we show that we can bound $\|\sum_{i=0}^{n} \alpha_i X_i e_i\|_S$ above by the Amemiya norm of $(\alpha_0, \ldots, \alpha_n, 0, \ldots)$. Clearly,

$$\left\| \sum_{i=0}^{n} \alpha_i X_i e_i \right\|_S = \sup_{T \in \mathscr{T}} \mathbb{E}(|\alpha_T| X_T) \le \mathbb{E}(\max_{0 \le i \le n} |\alpha_i| X_i).$$

Let $t > 0$ be arbitrary. Then,

$$\begin{aligned} \mathbb{E}\left(\max_{0 \le i \le n} |\alpha_i| X_i \right) &\le t + \sum_{j=0}^{n} \mathbb{E}((|\alpha_j| X_j - t)^+) \\ &= t + \sum_{j=0}^{n} (\mathbb{E}(|\alpha_j| X_j \vee t) - t) \\ &= t + t \sum_{j=0}^{n} \left(\mathbb{E}\left(\frac{|\alpha_j|}{t} X_j \vee 1 \right) - 1 \right) \\ &= t + t \sum_{j=0}^{n} \left(\Phi\left(\frac{|\alpha_j|}{t}, 1 \right) - 1 \right) \\ &= t + t \sum_{j=0}^{n} \phi\left(\frac{|\alpha_j|}{t} \right). \end{aligned}$$

Therefore,

$$\begin{aligned} \left\| \sum_{i=0}^{n} \alpha_i X_i e_i \right\|_S &\le \inf_{t > 0} \left(t + t \sum_{j=0}^{n} \phi\left(\frac{|\alpha_j|}{t} \right) \right) \\ &= \inf_{k > 0} \frac{1}{k} \left(1 + \sum_{j=0}^{n} \phi(k |\alpha_j|) \right) \\ &= \|\boldsymbol{\alpha}\|_\phi^A. \end{aligned}$$

We have shown that

$$\|\boldsymbol{\alpha}\|_\phi \leq \left\| \sum_{i=0}^{n} \alpha_i X_i e_i \right\|_S \leq \|\boldsymbol{\alpha}\|_\phi^A \leq 2 \|\boldsymbol{\alpha}\|_\phi .$$

Therefore, $[X_i e_i]_{i=0}^\infty$ is isomorphic to h_ϕ.

Remark 7.4.1 *If instead of the above situation we know that the X_i are each independent of $\mathscr{F}_{i-1}$, but not necessarily identically distributed then the above proof, with suitable modifications, shows that $[X_i e_i]_{i=0}^\infty$ is isomorphic to the generalized Orlicz space h_Ψ, where $\Psi = (\phi_i)_{i=0}^\infty$, with $\phi_i(t) = \Phi^i(t,1) - 1$.*

Remark 7.4.2 *Suppose that $n_0 < n_1 < n_2 < \dots$ is a strictly increasing sequence of natural numbers and that $(X_i)_{i=0}^\infty$ is a sequence of non-negative random variables such that $\mathbb{E}(X_i) = 1$, X_i is $\mathscr{F}_{n_i}$-measurable and independent of $\mathscr{F}_{n_{i-1}}$. Then for the same reasons as we mentioned in Remark 7.3.1, the results of this section still hold. In other words, $[X_i e_{n_i}]_{i=0}^\infty$ is isomorphic to an Orlicz space if the X_i are identically distributed, and to a generalized Orlicz space otherwise.*

7.5 Subspaces of S^d almost isometric to subspaces of S

In this section we prove a theorem which is a useful tool for finding subspaces of S^d. We show that for an arbitrary stopping time Banach space, S, the subspaces of S described in Section 7.3 also occur almost isometrically in S^d.

Theorem 7.5.1 *Let S be a stopping time Banach space with respect to some filtration $(\mathscr{F}_i)_{i=0}^\infty$. Let $(X_i)_{i=0}^\infty$ be a sequence of non-negative random variables such that X_i is $\mathscr{F}_i$-measurable and independent of $\mathscr{F}_{i-1}$ and $\mathbb{E}(X_i) = 1$. Then for every $\varepsilon > 0$ there exists a subspace of S^d which is $(1+\varepsilon)$-isomorphic to $[X_i e_i]_{i=0}^\infty$.*

Proof. Let $\varepsilon > 0$ and choose a sequence of positive numbers $(\varepsilon_n)_{n=0}^\infty$, converging to zero sufficiently rapidly that

$$\prod_{i=0}^\infty (1+\varepsilon_i) \leq 1 + \varepsilon \text{ and } \prod_{i=0}^\infty (1-\varepsilon_i) \geq \frac{1}{1+\varepsilon}.$$

Let F_i be the distribution function of X_i. Since $\mathbb{E}(X_i) = 1$ and $X_i \geq 0$, $1 - F_i(x) \in L_1(0,\infty)$ with L_1-norm equal to one. By approximating by step functions, if we choose n_0 sufficiently large then we can find an $\mathscr{F}_{n_0}^d$-measurable random variable $Y_0 \geq 0$, with distribution function G_0 such that

$$\|(1 - F_0(x)) - (1 - G_0(x))\|_{L_1(0,\infty)} < \varepsilon_0.$$

(see Figure 7.2. The shaded area represents the function $1 - G_0(x)$).

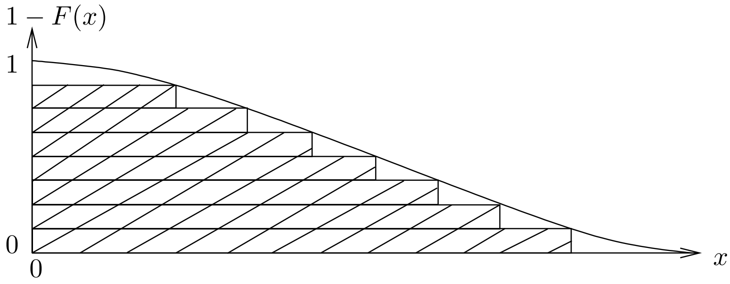

Figure 7.2: Approximating $1 - F(x)$

By re-scaling we may suppose that $\mathbb{E}(Y_0) = 1$. We then repeat this argument for X_1, which yields us a $\mathscr{F}^d_{n_1}$-measurable random variable Y_1, which we can also choose to be independent of $\mathscr{F}^d_{n_0}$, with distribution function G_1 such that

$$\|(1 - F_1(x)) - (1 - G_1(x))\|_{L_1(0,\infty)} < \varepsilon_1.$$

Continuing in this way we construct a sequence of random variables $Y_i \geq 0$ with $\mathbb{E}(Y_i) = 1$ and Y_i $\mathscr{F}^d_{n_i}$-measurable, independent of $\mathscr{F}^d_{n_{i-1}}$ and

$$\|(1 - F_i(x)) - (1 - G_i(x))\|_{L_1(0,\infty)} < \varepsilon_i.$$

Let $\Phi^k(u,v) = \mathbb{E}(\max(|u|X_k, |v|))$ and $\Psi^k(u,v) = \mathbb{E}(\max(|u|Y_k, |v|))$. Then for each k, and any $u, v \geq 0$,

$$\begin{aligned}
|\Phi^k(u,v) - \Psi^k(u,v)| &= \left| \left(v + u \int_{\frac{v}{u}}^{\infty} (1 - F_k(s))\mathrm{d}s \right) - \left(v + u \int_{\frac{v}{u}}^{\infty} (1 - G_k(s))\mathrm{d}s \right) \right| \\
&= u \left| \int_{\frac{v}{u}}^{\infty} (1 - F_k(s)) - (1 - G_k(s))\mathrm{d}s \right| \\
&\leq \varepsilon_k u \\
&\leq \varepsilon_k \Psi^k(u,v).
\end{aligned}$$

Therefore, for any $u, v \in \mathbb{R}$ and any $k \in \mathbb{N}$,

$$(1 - \varepsilon_k)\Psi^k(u,v) \leq \Phi^k(u,v) \leq (1 + \varepsilon_k)\Psi^k(u,v).$$

Using the right-hand inequality twice shows that

$$\begin{aligned}(\Phi^{n-2} * \Phi^{n-1})(\alpha_{n-2}, \alpha_{n-1}, \alpha_n) &= \Phi^{n-2}(\alpha_{n-2}, \Phi^{n-1}(\alpha_{n-1}, \alpha_n)) \\ &\leq (1+\varepsilon_{n-2})\Psi^{n-2}(\alpha_{n-2}, \Phi^{n-1}(\alpha_{n-1}, \alpha_n)) \\ &\leq (1+\varepsilon_{n-2})\Psi^{n-2}(\alpha_{n-2}, (1+\varepsilon_{n-1})\Psi^{n-1}(\alpha_{n-1}, \alpha_n)) \\ &\leq (1+\varepsilon_{n-2})(1+\varepsilon_{n-1})\Psi^{n-2}(\alpha_{n-2}, \Psi^{n-1}(\alpha_{n-1}, \alpha_n)) \\ &= (1+\varepsilon_{n-2})(1+\varepsilon_{n-1})(\Psi^{n-2} * \Psi^{n-1})(\alpha_{n-2}, \alpha_{n-1}, \alpha_n).\end{aligned}$$

Iterating this procedure gives,

$$\begin{aligned}(\Phi^0 * \ldots * \Phi^{n-1})(\alpha_0, \ldots, \alpha_n) &\leq \prod_{i=0}^{n-1}(1+\varepsilon_i)\,(\Psi^0 * \ldots * \Psi^{n-1})(\alpha_0, \ldots, \alpha_n) \\ &\leq (1+\varepsilon)\,(\Psi^0 * \ldots * \Psi^{n-1})(\alpha_0, \ldots, \alpha_n).\end{aligned} \tag{7.7}$$

A similar argument also shows,

$$\begin{aligned}(\Phi^0 * \ldots * \Phi^{n-1})(\alpha_0, \ldots, \alpha_n) &\geq \prod_{i=0}^{n-1}(1-\varepsilon_i)\,(\Psi^0 * \ldots * \Psi^{n-1})(\alpha_0, \ldots, \alpha_n) \\ &\geq \frac{1}{1+\varepsilon}\,(\Psi^0 * \ldots * \Psi^{n-1})(\alpha_0, \ldots, \alpha_n).\end{aligned} \tag{7.8}$$

Hence, by the results of Section 7.3, (7.7) and (7.8) tell us that

$$\frac{1}{1+\varepsilon}\left\|\sum_{i=0}^{n}\alpha_i Y_i e_{n_i}\right\|_{S^d} \leq \left\|\sum_{i=0}^{n}\alpha_i X_i e_i\right\|_S \leq (1+\varepsilon)\left\|\sum_{i=0}^{n}\alpha_i Y_i e_{n_i}\right\|_{S^d}.$$

Therefore, $[X_i e_i]_{i=0}^{\infty}$ is $(1+\varepsilon)$-isomorphic to $[Y_i e_{n_i}]_{i=0}^{\infty}$. ☐

7.6 Subspaces of S^d almost isometric to ℓ_p

It is easy to find subspaces of S^d that are isometric to ℓ_1 and c_0 (see e.g. Section 7.7). In fact, it is possible to find subspaces that are almost isometric to ℓ_p for any $p \in (1, \infty)$. This result is mentioned in [10] as a result of Schechtmann. The proof does not seem to appear in the literature, so we include a proof, which is presumably the same method used by Schechtmann.

Fix $p \in (1, \infty)$ and suppose that X is a random variable which has cumulative distribution function given by

$$F(x) = \begin{cases} x^{p-1}(1+x^p)^{\frac{1}{p}-1} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases} \tag{7.9}$$

By differentiating, we find that the density function of this random variable is given by

$$f(x) = \begin{cases} (p-1)x^{p-2}(1+x^p)^{\frac{1}{p}-2} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Then,

$$\begin{aligned} \mathbb{E}(X) = \int_0^\infty t f(t) \mathrm{d}t &= (p-1)\int_0^\infty t^{p-1}(1+t^p)^{\frac{1}{p}-2}\mathrm{d}t \\ &= \frac{p-1}{p}\int_0^\infty \frac{\mathrm{d}u}{(1+u)^{2-\frac{1}{p}}} \\ &= \frac{p-1}{p}\left[\frac{(1+u)^{\frac{1}{p}-1}}{\frac{1}{p}-1}\right]_{u=0}^\infty \\ &= 1. \end{aligned}$$

For this distribution function F we can explicitly calculate Φ. For $u, v \geq 0$, we have

$$\begin{aligned} \Phi(u,v) &= u + u\int_0^{\frac{v}{u}} s^{p-1}(1+s^p)^{\frac{1}{p}-1}\mathrm{d}s \\ &= u + u\left[(1+s^p)^{\frac{1}{p}}\right]_{s=0}^{\frac{v}{u}} \\ &= u(1 + \left(1 + \left(\frac{v}{u}\right)^p\right)^{\frac{1}{p}} - 1) \\ &= (u^p + v^p)^{\frac{1}{p}}. \end{aligned} \tag{7.10}$$

It is well known that we can construct a sequence $X_0, X_1, \dots$ of independent random variables on a suitable probability space $(\Omega, \mathscr{F}, \mathbb{P})$, each having (7.9) as its distribution function. We define a filtration by $\mathscr{F}_i = \sigma(X_0, \dots, X_i)$. Then each X_i is $\mathscr{F}_i$-measurable and independent of $\mathscr{F}_{i-1}$. Let S be the stopping time Banach space associated with this filtration. It follows easily from (7.10) that

$$\left\|\sum_{i=0}^n \alpha_i X_i e_i\right\|_S = \left(\sum_{i=0}^n |a_i|^p\right)^{\frac{1}{p}},$$

so that $[X_i e_i]_{i=0}^\infty$ is isometric to ℓ_p. Hence, by Theorem 7.5.1, S^d contains almost isometric copies of ℓ_p.

7.7 Orlicz subspaces of S^d

In Section 7.4, we showed that subspaces arising from sequences of independent identically distributed random variables are isomorphic to an Orlicz space, h_ϕ. It is therefore natural to ask for which Orlicz functions, ϕ, can we embed h_ϕ in S^d. We shall address this question in this section.

Properties of our Orlicz functions

In Section 7.4, the Orlicz functions which arose were of the form

$$\phi(t) = \Phi(t,1) - 1,$$

where $\Phi(t,1) = \mathbb{E}(\max(tX,1))$, and X is a non-negative random variable with $\mathbb{E}(X) = 1$. We obtain the following elementary estimate,

$$1 + (t-1)^+ = \max(t,1) \leq \Phi(t,1) \leq t+1.$$

Therefore,

$$(t-1)^+ \leq \phi(t) \leq t \text{ for all } t \geq 0.$$

In particular, this implies that $\phi'_-(t) \leq \phi'_+(t) \leq 1$ for all $t \geq 0$, where $\phi'_\pm$ denote the left and right derivatives of ϕ at t. Also, for any $t \geq 1$,

$$1 - \frac{1}{t} \leq \frac{\phi(t)}{t} \leq 1,$$

and therefore $\lim_{t\to\infty} \frac{\phi(t)}{t} = 1$. Also, $\frac{\phi(t)}{t} = \mathbb{E}(\max(X, \frac{1}{t})) - \frac{1}{t} = \mathbb{E}((X - \frac{1}{t})^+)$. Note that $(X - \frac{1}{t})^+ \to 0$ pointwise as $t \to 0$, and $|(X - \frac{1}{t})^+| \leq X$. Therefore, by the dominated convergence theorem,

$$\frac{\phi(t)}{t} = \mathbb{E}\left(\left(X - \frac{1}{t}\right)^+\right) \to 0 \text{ as } t \to 0.$$

Recovering the distribution from the Orlicz function

The Orlicz function ϕ is completely determined by the distribution of X. Indeed, if F is the distribution function of X, then by (7.1),

$$\phi(t) = \Phi(t,1) - 1 = t + t\int_0^{\frac{1}{t}} F(s)\ \mathrm{d}s - 1.$$

Rearranging the above expression shows that,

$$\frac{\phi(t) - t + 1}{t} = \int_0^{\frac{1}{t}} F(s)\ \mathrm{d}s.$$

Since F is right-continuous, we can take the left-derivative of both sides, and rearrange to obtain

$$F\left(\frac{1}{t}\right) = \phi(t) - t\phi'_-(t) + 1. \tag{7.11}$$

In the remainder of this section we show that if ϕ is an Orlicz function satisfying suitable conditions, then (7.11) defines a cumulative distribution function of some random variable which in turn induces the Orlicz function ϕ. We will deduce from this a sufficient condition on ϕ, for h_ϕ to embed in S^d.

Lemma 7.7.1 *Let ϕ be an Orlicz function such that $(t-1)^+ \leq \phi(t) \leq t$ for all $t \geq 0$ and such that $\lim_{t\to 0} \frac{\phi(t)}{t} = 0$. Define $F : (0,\infty) \to \mathbb{R}$ by*

$$F(s) = \phi\left(\frac{1}{s}\right) - \frac{1}{s}\phi'_-\left(\frac{1}{s}\right) + 1.$$

Then F is the distribution function of some non-negative random variable X with $\mathbb{E}(X) = 1$.

Proof. The conditions on ϕ ensure that $\phi'_-(t) \leq 1$ for all t. Hence,

$$\phi(t) - t\phi'_-(t) \geq \phi(t) - t \geq (t-1)^+ - t \geq -1.$$

Thus, $F(s) \geq 0$ for all $s > 0$. It is also immediate that $\lim_{s\to\infty} F(s) = 1$. F is right continuous since ϕ'_- is left continuous (see [47, Lemma 1.2]). Thus, F will be a cumulative distribution function of some non-negative random variable provided we show that F is increasing, or equivalently that $\psi : t \mapsto \phi(t) - t\phi'_-(t) + 1$ is decreasing. Let $0 \leq u < v$. Then

$$\psi(u) - \psi(v) = \phi(u) - \phi(v) + v\phi'_-(v) - u\phi'_-(u).$$

Since ϕ is convex,

$$\frac{\phi(v) - \phi(u)}{v - u} \leq \phi'_-(v).$$

Therefore,

$$\begin{aligned}\psi(u) - \psi(v) &\geq (u-v)\phi'_-(v) + v\phi'_-(v) - u\phi'_-(u)\\ &= u(\phi'_-(v) - \phi'_-(u)).\end{aligned}$$

Since $t \mapsto \phi'_-(t)$ is increasing, this shows that $\psi(u) \geq \psi(v)$. Therefore F is a distribution function for some non-negative random variable X.

Let $\varepsilon > 0$. By [47, Theorem 1.1], $\phi(u) = \int_0^u \phi'_-(t)\mathrm{d}t$ for all $u \geq 0$. Therefore,

$$\begin{aligned}
\int_\varepsilon^{\frac{1}{\varepsilon}} 1 - F(s)\mathrm{d}s &= \int_\varepsilon^{\frac{1}{\varepsilon}} \left(-\phi\left(\frac{1}{s}\right) + \frac{1}{s}\phi'_-\left(\frac{1}{s}\right)\right)\mathrm{d}s \\
&= -\int_\varepsilon^{\frac{1}{\varepsilon}}\int_0^{\frac{1}{s}} \phi'_-\left(t\right)\mathrm{d}t\,\mathrm{d}s + \int_\varepsilon^{\frac{1}{\varepsilon}} \frac{1}{s}\phi'_-\left(\frac{1}{s}\right)\mathrm{d}s \\
&= -\int_0^{\varepsilon}\int_\varepsilon^{\frac{1}{\varepsilon}} \phi'_-\left(t\right)\mathrm{d}s\,\mathrm{d}t - \int_\varepsilon^{\frac{1}{\varepsilon}}\int_\varepsilon^{\frac{1}{t}} \phi'_-\left(t\right)\mathrm{d}s\,\mathrm{d}t + \int_\varepsilon^{\frac{1}{\varepsilon}} \frac{1}{s}\phi'_-\left(\frac{1}{s}\right)\mathrm{d}s \\
&= (\varepsilon - \frac{1}{\varepsilon})\phi(\varepsilon) + \varepsilon(\phi(1/\varepsilon) - \phi(\varepsilon)) \\
&= \varepsilon\phi(1/\varepsilon) - \frac{\phi(\varepsilon)}{\varepsilon} \\
&\to 1 \text{ as } \varepsilon \to 0.
\end{aligned}$$

Since, $1 - F(s) \geq 0$ for all s, we can apply the monotone convergence theorem, to obtain,

$$\mathbb{E}(X) = \int_0^\infty 1 - F(s)\ \mathrm{d}s = 1.$$

$\square$

Lemma 7.7.2 *Let ϕ be an Orlicz function satisfying the conditions of Lemma 7.7.1, then for every $t > 0$,*

$$\phi(t) = t + t\int_0^{\frac{1}{t}} F(s)\mathrm{d}s - 1.$$

Proof. Let $\varepsilon > 0$. Then for any $t > 0$

$$\begin{aligned}
t + t\int_\varepsilon^{\frac{1}{t}} F(s)\ \mathrm{d}s - 1 &= t - 1 + t\int_\varepsilon^{\frac{1}{t}} \left(\phi\left(\frac{1}{s}\right) - \frac{1}{s}\phi'_-\left(\frac{1}{s}\right) + 1\right)\ \mathrm{d}s \\
&= t(1-\varepsilon) + t\int_\varepsilon^{\frac{1}{t}}\int_0^{\frac{1}{s}} \phi'_-(p)\,\mathrm{d}p\,\mathrm{d}s - t\int_\varepsilon^{\frac{1}{t}} \frac{1}{s}\phi'_-\left(\frac{1}{s}\right)\mathrm{d}s \\
&= t(1-\varepsilon) + t\int_0^{t}\int_\varepsilon^{\frac{1}{t}} \phi'_-(p)\mathrm{d}s\,\mathrm{d}p + t\int_t^{\frac{1}{\varepsilon}}\int_\varepsilon^{\frac{1}{p}} \phi'_-(p)\,\mathrm{d}s\,\mathrm{d}p \\
&\qquad - t\int_\varepsilon^{\frac{1}{t}} \frac{1}{s}\phi'_-\left(\frac{1}{s}\right)\mathrm{d}s \\
&= t(1-\varepsilon) + t\left(\frac{1}{t} - \varepsilon\right)\phi(t) - \varepsilon t(\phi(1/\varepsilon) - \phi(t)) \\
&= t - \varepsilon t + \phi(t) - t\varepsilon\phi(1/\varepsilon) \\
&\to \phi(t) \text{ as } \varepsilon \to 0.
\end{aligned}$$

Applying the monotone convergence theorem shows that

$$\phi(t) = t + t\int_0^{\frac{1}{t}} F(s)\ \mathrm{d}s - 1.$$

□

Corollary 7.7.3 *Let ϕ be an Orlicz function such that $(t-1)^+ \leq \phi(t) \leq t$ for all $t \geq 0$ and such that $\lim_{t\to 0}\frac{\phi(t)}{t} = 0$. Then the Orlicz space, h_ϕ, embeds isomorphically into S^d.*

Proof. Let F be the cumulative distribution function obtained in Lemma 7.7.1. As in Section 7.6, we can construct (on a suitable probability space) a sequence $(X_i)_{i=0}^\infty$ of independent random variables each having distribution function F. Define a filtration by $\mathscr{F}_i = \sigma(X_0, \dots, X_i)$, and let S be the corresponding stopping time Banach space. Then by Lemma 7.7.2, and the result in Section 7.4, it follows that $[X_ie_i]_{i=0}^\infty$ is 2-isomorphic to h_ϕ. Thus by Theorem 7.5.1 we can find subspaces of S^d that are $(2+\varepsilon)$-isomorphic to h_ϕ for every $\varepsilon > 0$. □

Embedding Orlicz spaces in S^d

In fact, we can show that every Orlicz space, h_ϕ, embeds isomorphically in S^d. This will follow from the fact that h_ϕ is determined up to isomorphism by the behaviour of ϕ on any neighbourhood of 0. More precisely, we have the following proposition.

Proposition 7.7.4 ([53, Proposition 4.a.5]) *Let ϕ_1 and ϕ_2 be two Orlicz functions. Then the following assertions are equivalent;*

- *(i)* $\ell_{\phi_1} = \ell_{\phi_2}$ *(i.e. both spaces consist of the same sequences) and the identity mapping is an isomorphism between* ℓ_{ϕ_1} *and* ℓ_{ϕ_2}.
- *(ii) The unit vector bases of* h_{ϕ_1} *and* h_{ϕ_2} *are equivalent.*
- *(iii)* ϕ_1 *and* ϕ_2 *are equivalent at zero i.e. there exist constants* $k > 0$, $K > 0$ *and* $t_0 > 0$ *such that, for all* $0 \leq t \leq t_0$, *we have*

$$K^{-1}\phi_2(k^{-1}t) \leq \phi_1(t) \leq K\phi_2(kt).$$

Let ϕ be any Orlicz function. The function $t \mapsto \frac{\phi(t)}{t}$ is increasing and for every $t \geq 0$,

$$\phi'_+(0) \leq \frac{\phi(t)}{t} \leq \phi'_-(t). \tag{7.12}$$

We now consider two cases. First, suppose that $\lim_{t\to 0}\frac{\phi(t)}{t}=0$. It follows from (7.12) that $\phi'_+(0)=0$. Recall that $t\mapsto\phi'_+(t)$ is an increasing right-continuous function, and therefore there exists $t_0>0$ such that $\phi'_-(t)\le\phi'_+(t)\le 1$ for all $t\le t_0$. Without loss of generality we may suppose that $t_0\le 1$. Define

$$\psi(t)=\begin{cases}\phi(t) & \text{if } t\le t_0,\\ t-t_0+\phi(t_0) & \text{if } t>t_0.\end{cases}$$

Then ψ is an Orlicz function which satisfies the conditions of Corollary 7.7.3, and therefore h_ψ embeds isomorphically into S^d. However, ψ and ϕ coincide on $[0,t_0]$, and so by Proposition 7.7.4, h_ϕ is isomorphic to h_ψ. Hence we have shown that h_ϕ embeds isomorphically into S^d.

The remaining case is when $\lim_{t\to 0}\frac{\phi(t)}{t}=\delta>0$. Then, there exists a $t_0>0$ such that

$$\delta t\le\phi(t)\le 2\delta t \text{ whenever } t\in[0,t_0].$$

Thus, ϕ is equivalent at zero to the Orlicz function $\psi: t\mapsto t$. Hence, by Proposition 7.7.4, h_ϕ is isomorphic to $h_\psi=\ell_1$. It is straightforward to find subspaces of S^d that are isomorphic to ℓ_1. Indeed, if we take $X_1, X_2, \ldots$ to be the sequence of random variables in Figure 7.3, then $[X_ie_i]_{i=1}^\infty$ is isometric to ℓ_1.

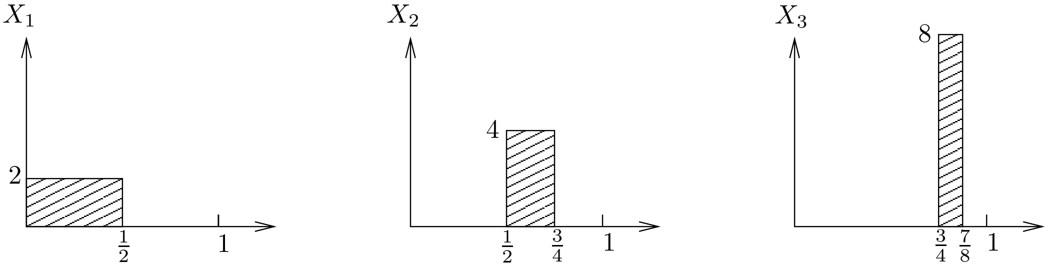

Figure 7.3: A sequence of random variables generating ℓ_1

For any $n\in\mathbb{N}$, we define a stopping time T by

$$T=\sum_{i=1}^{n} i\mathbb{1}_{[1-\frac{1}{2^{i-1}},1-\frac{1}{2^i})}+(n+1)\mathbb{1}_{[1-\frac{1}{2^n},1)}.$$

Then,

$$\begin{aligned}\left\|\sum_{i=1}^{n} a_i X_i e_i\right\|_{S^d} &\geq \mathbb{E}(|a_T X_T|)\\ &= \mathbb{E}\left(\sum_{i=1}^{n} |a_i| X_i \mathbb{1}_{[1-\frac{1}{2^{i-1}},1-\frac{1}{2^i})}\right)\\ &= \sum_{i=1}^{n} |a_i|.\end{aligned}$$

The opposite inequality is immediate from the triangle inequality, and therefore for any $n \in \mathbb{N}$ and any scalars (a_i),

$$\left\|\sum_{i=1}^{n} a_i X_i e_i\right\|_{S^d} = \sum_{i=1}^{n} |a_i|.$$

In summary, we have proved the following result.

Theorem 7.7.5 *Let ϕ be any Orlicz function. Then S^d has a subspace which is isomorphic to h_ϕ.*

7.8 Mixtures of Orlicz spaces

Suppose that $\mathscr{F}_0 \subseteq \mathscr{F}_1 \subseteq \mathscr{F}_2 \subseteq \ldots$ is an increasing sequence of σ-subalgebras of $\mathscr{F}$, and that $\omega \mapsto \phi_w$ is an $\mathscr{F}_0$-measurable mapping such that ϕ_w is an Orlicz function for each $\omega \in \Omega$. We define a mixture of Orlicz spaces to be the completion of the finitely non-zero sequences of real numbers $\mathbf{a} = (a_i)_{i=1}^{\infty}$ under the norm,

$$\|\mathbf{a}\| = \int_{\omega\in\Omega} \|\mathbf{a}\|_{\phi_\omega}\, \mathrm{d}\mathbb{P}.$$

In this section we construct subspaces of a stopping time Banach space which are isomorphic to mixtures of Orlicz spaces. The existence of these subspaces will follow from the results of Section 7.4, and a conditional backwards recursion property corresponding to that in Lemma 7.1.3. It should be noted that this construction works in an arbitrary stopping time Banach space, but it is only of interest in the case where $\mathscr{F}_0$ is not discrete. In the discrete case, these spaces are all isomorphic to c_0.

Lemma 7.8.1 *Let $X = (0, X_1, X_2, \ldots, X_n, X_{n+1}, 0, \ldots)$ be an adapted process, and $\tilde{X} = (0, X_1, X_2, \ldots, , X_{n-1}, |X_n| \vee \mathbb{E}_n(|X_{n+1}|), 0, 0, \ldots)$. Then,*

$$\sup_{T\in\mathscr{T}} \mathbb{E}_0(|X_T|) = \sup_{T\in\mathscr{T}} \mathbb{E}_0(|\tilde{X}_T|) \text{ almost surely.}$$

Proof. Let T be any stopping time, and define $B_i = \{\omega : T(\omega) = i\} \in \mathscr{F}_i$ and $B = \{\omega : T(\omega) > n\} \in \mathscr{F}_n$. Then,

$$\begin{aligned} \mathbb{E}_0(|X_T|) &= \sum_{i=1}^{n+1} \mathbb{E}_0(\mathbb{1}_{B_i}|X_i|) \\ &\leq \sum_{i=1}^{n-1} \mathbb{E}_0(\mathbb{1}_{B_i}|X_i|) + \mathbb{E}_0(\mathbb{1}_{B_n}|X_n| + \mathbb{1}_B|X_{n+1}|) \qquad (7.13) \\ &\leq \sum_{i=1}^{n-1} \mathbb{E}_0(\mathbb{1}_{B_i}|X_i|) + \mathbb{E}_0(\mathbb{1}_{B_n \cup B}(|X_n| \vee \mathbb{E}_n(|X_{n+1}|))). \end{aligned}$$

The last step follows since

$$\begin{aligned} \mathbb{E}_0(\mathbb{1}_B \mathbb{E}_n(|X_{n+1}|)) &= \mathbb{E}_0(\mathbb{E}_n(\mathbb{1}_B(|X_{n+1}|))) \text{ as } B \in \mathscr{F}_n \\ &= \mathbb{E}_0(\mathbb{1}_B|X_{n+1}|). \end{aligned}$$

Therefore, (7.13) shows that $\mathbb{E}_0(|X_T|) \leq \mathbb{E}_0(|\tilde{X}_{T'}|)$ almost surely, where T' is the stopping time defined by

$$T'(\omega) = \begin{cases} T(\omega) & \omega \in B_1 \cup \ldots \cup B_{n-1}, \\ n & \omega \in B_n \cup B. \end{cases}$$

We now prove the opposite inequality in a similar way. Let T be any stopping time, and define $B_i = \{\omega : T(\omega) = i\} \in \mathscr{F}_i$, $C = \{\omega : T(\omega) \geq n\} \in \mathscr{F}_{n-1} \subseteq \mathscr{F}_n$ and $D = \{\omega : |X_n|(\omega) \geq \mathbb{E}_n(|X_{n+1}|)(\omega)\} \in \mathscr{F}_n$. Then

$$\begin{aligned} \mathbb{E}_0(|\tilde{X}_T|) &= \sum_{i=1}^{n-1} \mathbb{E}_0(\mathbb{1}_{B_i}|X_i|) + \mathbb{E}_0(\mathbb{1}_{B_n}(|X_n| \vee \mathbb{E}_n(|X_{n+1}|))) \\ &\leq \sum_{i=1}^{n-1} \mathbb{E}_0(\mathbb{1}_{B_i}|X_i|) + \mathbb{E}_0(\mathbb{1}_C(|X_n| \vee \mathbb{E}_n(|X_{n+1}|))) \qquad (7.14) \\ &= \sum_{i=1}^{n-1} \mathbb{E}_0(\mathbb{1}_{B_i}|X_i|) + \mathbb{E}_0(\mathbb{1}_{C\cap D}|X_n| + \mathbb{1}_{C\cap D^c}\mathbb{E}_n(|X_{n+1}|)). \end{aligned}$$

Now,

$$\begin{aligned} \mathbb{E}_0(\mathbb{1}_{C\cap D^c}\mathbb{E}_n(|X_{n+1}|)) &= \mathbb{E}_0(\mathbb{1}_{C\cap D^c}\mathbb{E}_n(|X_{n+1}|)) \\ &= \mathbb{E}_0(\mathbb{E}_n(\mathbb{1}_{C\cap D^c}|X_{n+1}|)) \text{ as } C \cap D^c \in \mathscr{F}_n \qquad (7.15) \\ &= \mathbb{E}_0(\mathbb{1}_{C\cap D^c}|X_{n+1}|). \end{aligned}$$

Hence, substituting (7.15) into (7.14) shows that, $\mathbb{E}_0(|\tilde{X}_T|) \leq \mathbb{E}_0(|X_{T'}|)$, where T' is the stopping time defined by

$$T'(\omega) = \begin{cases} T(\omega) & \omega \in B_1 \cup \ldots \cup B_{n-1}, \\ n & \omega \in C \cap D, \\ n+1 & \omega \in C \cap D^c. \end{cases}$$

Combining these two inequalities gives us the result that, almost surely,

$$\sup_{T \in \mathscr{T}} \mathbb{E}_0(|X_T|) = \sup_{T \in \mathscr{T}} \mathbb{E}_0(|\tilde{X}_T|).$$

□

Lemma 7.8.1 allows us to apply backwards recursion as we did in Corollary 7.1.4. Using this result repeatedly shows that, almost surely,

$$\sup_{T \in \mathscr{T}} \mathbb{E}_0(|X_T|) = \sup_{T \in \mathscr{T}} \mathbb{E}_0(|Y_T|),$$

where $Y = (0, \sigma_1(X), 0, 0, \ldots)$. Therefore,

$$\sup_{T \in \mathscr{T}} \mathbb{E}_0(|X_T|) = \mathbb{E}_0(\sigma_1(X)),$$

and thus, taking expectations,

$$\begin{aligned} \mathbb{E}(\sup_{T \in \mathscr{T}} \mathbb{E}_0(|X_T|)) &= \mathbb{E}(\mathbb{E}_0(\sigma_1(X))) \\ &= \mathbb{E}(\sigma_1(X)) \qquad (7.16) \\ &= \|X\|_S . \end{aligned}$$

7.8.1 Independence conditional on $\mathscr{F}_0$

In Section 7.4, we showed that if $(X_i)_{i=1}^\infty$ were a sequence of random variables such that each X_i is independent of $\mathscr{F}_{i-1}$ then $[X_i e_i]_{i=1}^\infty$ was isomorphic to an Orlicz space. Mixtures of Orlicz spaces will arise in a similar way, with a different notion of independence. We introduce this in the next definition. Recall that two random variables X and Y are independent if and only if $\mathbb{E}(f(X)g(Y)) = \mathbb{E}(f(X))\mathbb{E}(g(Y))$, whenever f and g are measurable functions for which these expectations exist. This motivates the following definition of conditional independence.

Definition 7.8.2 *Suppose $\mathscr{F}_0 \subseteq \mathscr{G} \subseteq \mathscr{H}$ are σ-algebras, and that X is a $\mathscr{H}$-measurable random variable. Then we say that X is independent of $\mathscr{G}$ conditional on $\mathscr{F}_0$, if for every measurable function $f : \mathbb{R} \to \mathbb{R}$ such that $f(X) \in L_1(\mathscr{H})$ and every $Y \in L_\infty(\mathscr{G})$,*

$$\mathbb{E}_0(f(X)Y) = \mathbb{E}_0(f(X))\mathbb{E}_0(Y).$$

A well known result in probability is that if a random variable X is independent of a σ-algebra $\mathscr{G}$, then $\mathbb{E}(X|\mathscr{G}) = \mathbb{E}(X)$ almost surely. Our next lemma is the corresponding result for conditional independence.

Lemma 7.8.3 *Let $\mathscr{F}_0 \subseteq \mathscr{G}$ be σ-algebras, and suppose that X satisfies*

$$\mathbb{E}_0(X\mathbb{1}_G) = \mathbb{E}_0(X)\mathbb{E}_0(\mathbb{1}_G) \textit{ for all } G \in \mathscr{G}.$$

Then,

$$\mathbb{E}(X|\mathscr{G}) = \mathbb{E}_0(X) \textit{ almost surely.}$$

In particular, this result holds if X is independent of $\mathscr{G}$ conditional on $\mathscr{F}_0$.

Proof. Let $G \in \mathscr{G}$. Then,

$$\begin{aligned}\mathbb{E}_0(X\mathbb{1}_G) &= \mathbb{E}_0(X)\mathbb{E}_0(\mathbb{1}_G) \quad \text{(by conditional independence)}\\ &= \mathbb{E}_0(\mathbb{E}_0(X)\mathbb{1}_G).\end{aligned}$$

Taking expectations implies,

$$\int_G X \mathrm{d}\mathbb{P} = \mathbb{E}(X\mathbb{1}_G) = \mathbb{E}(\mathbb{E}_0(X)\mathbb{1}_G) = \int_G \mathbb{E}_0(X)\mathrm{d}\mathbb{P}.$$

$\mathbb{E}_0(X)$ is $\mathscr{F}_0$-measurable and so in particular is $\mathscr{G}$-measurable, and satisfies the defining condition of $\mathbb{E}(X|\mathscr{G})$. Therefore, $\mathbb{E}(X|\mathscr{G}) = \mathbb{E}_0(X)$ almost surely. ☐

7.8.2 Mixtures of Orlicz spaces

Suppose that $X_1, X_2, \dots$ is a sequence of non-negative random variables satisfying the following conditions;

(i) X_i is $\mathscr{F}_i$-measurable for each $i \geq 1$.

(ii) X_i is independent of $\mathscr{F}_{i-1}$ conditional on $\mathscr{F}_0$, for each $i \geq 1$.

(iii) Conditional on $\mathscr{F}_0$, $X_1, X_2, \dots$ are identically distributed with expectation 1.

By (iii), we mean that there is a $\mathscr{F}_0$-measurable mapping $\omega \mapsto F_\omega$, such that each F_ω is a distribution function, and the conditional distribution of each X_i given $\mathscr{F}_0$ is given by F i.e. $\mathbb{P}(X_i \leq x|\mathscr{F}_0)(\omega) = F_\omega(x)$ for each $i \geq 1$. We shall suppress the dependence of F on ω throughout the remainder of this section. As in Section 7.3, the norm in $[X_i e_i]_{i=1}^\infty$ will be determined by a function Φ which we define by

$$\Phi(u, v) = \mathbb{E}_0(|u|X_1 \vee |v|).$$

Notice that in this new situation however $\Phi(u, v)$ is an $\mathscr{F}_0$-measurable random variable instead of a real number. Φ is determined totally by the random distribution F. Indeed, for $u, v \geq 0$,

$$\begin{aligned}\Phi(u,v) = \mathbb{E}_0(uX_1 \vee v) &= \mathbb{E}_0(v + (uX_1 - v)^+) \\ &= v + \int_0^\infty \mathbb{P}((uX_1 - v) > t \big| \mathscr{F}_0)\mathrm{d}t \\ &= v + u\int_{\frac{v}{u}}^\infty (1 - F(s))\mathrm{d}s.\end{aligned}$$

It is clear from this formula, and (iii), that for any $k \geq 1$, $\Phi(u, v) = \mathbb{E}_0(|u|X_k \vee |v|)$. Also, taking $u = 1$ and $v = 0$, shows that $\mathbb{E}_0(X_k) = \int_0^\infty (1 - F(s))\mathrm{d}s = 1$. Therefore, if $j < k$, then by Lemma 7.8.3, $\mathbb{E}_j(X_k) = \mathbb{E}_0(X_k) = 1$ almost surely. We will use these facts later. Before we can show that $[X_i e_i]_{i=1}^\infty$ is isomorphic to a mixture of Orlicz spaces we need some technical results which we will prove in the following two lemmas.

Lemma 7.8.4 *Suppose that* $Y \geq 0$ *is* $\mathscr{F}_0$*-measurable. Then for any* $k \geq 1$*,*

$$\mathbb{E}_0(aX_k \vee Y) = \Phi(a, Y).$$

Proof. Suppose first that Y is simple. Then $Y = \sum_{i=1}^n b_i \mathbb{1}_{A_i}$, where the A_i are pairwise disjoint $\mathscr{F}_0$-measurable sets. Without loss of generality we may assume that $\bigcup_{i=1}^n A_i = \Omega$. Then,

$$\begin{aligned}\mathbb{E}_0(aX_k \vee Y) &= \mathbb{E}_0(\sum_{i=1}^n \mathbb{1}_{A_i}(aX_k \vee b_i)) \\ &= \sum_{i=1}^n \mathbb{1}_{A_i}\mathbb{E}_0(aX_k \vee b_i) \\ &= \sum_{i=1}^n \mathbb{1}_{A_i}\Phi(a, b_i) \\ &= \Phi(a, Y).\end{aligned}$$

Now suppose that Y is an arbitrary non-negative element of $L_1(\mathscr{F}_0)$. Choose an increasing sequence $(Y_n)_{n=1}^\infty$ of non-negative simple functions converging to Y almost surely. Then $(aX_k \vee Y_n)_{n=1}^\infty$ is pointwise increasing and converges to $aX_k \vee Y$ almost surely. Also,

$$\mathbb{E}(aX_k \vee Y_n) \leq a\mathbb{E}(X_k) + \mathbb{E}(Y_n) \leq a\mathbb{E}(X_k) + \mathbb{E}(Y) < \infty.$$

Hence, by the monotone convergence theorem, $aX_k \vee Y_n \to aX_k \vee Y$ in $L_1(\mathscr{F})$. The mapping $X \mapsto \mathbb{E}_0(X)$ is continuous from $L_1(\mathscr{F})$ to $L_1(\mathscr{F}_0)$, and therefore $\mathbb{E}_0(aX_k \vee Y_n) \to \mathbb{E}_0(aX_k \vee Y)$

in $L_1(\mathscr{F}_0)$. By passing to a subsequence, we may also assume that $\mathbb{E}_0(aX_k \vee Y_n) \to \mathbb{E}_0(aX_k \vee Y)$ pointwise almost surely.

On the other hand,

$$\Phi(a, Y_n)(\omega) = Y_n(\omega) + a \int_{\frac{Y_n(\omega)}{a}}^{\infty} (1 - F_\omega(u))\mathrm{d}u.$$

$Y_n(\omega) \to Y(\omega)$ almost surely, and for almost every $u \in [0, \infty)$,

$$\mathbb{1}_{[Y_n(\omega)/a,\infty)}(u)(1 - F_\omega(u)) \to \mathbb{1}_{[Y(\omega)/a,\infty)}(u)(1 - F_\omega(u)).$$

Also for every u,

$$|\mathbb{1}_{[Y_n(\omega)/a,\infty)}(u)(1 - F_\omega(u))| \leq (1 - F_\omega(u)) \in L_1(0, \infty).$$

Therefore, by the dominated convergence theorem,

$$\int_{\frac{Y_n(\omega)}{a}}^{\infty} (1 - F_\omega(u))\mathrm{d}u \to \int_{\frac{Y(\omega)}{a}}^{\infty} (1 - F_\omega(u))\mathrm{d}u.$$

Thus, we have shown that $\Phi(a, Y_n) \to \Phi(a, Y)$ almost surely.

Since each Y_n is simple, we know from the first part of this proof that $\mathbb{E}_0(aX_k \vee Y_n) = \Phi(a, Y_n)$. Taking pointwise limits on both sides shows that $\mathbb{E}_0(aX_k \vee Y) = \Phi(a, Y)$ almost surely, completing our proof. ☐

Lemma 7.8.5 *Suppose that $\mathscr{F}_0 \subseteq \mathscr{G} \subseteq \mathscr{H}$ are σ-algebras. Let $Y \geq 0$ be $\mathscr{F}_0$-measurable, and X be $\mathscr{H}$-measurable and independent of $\mathscr{G}$ conditional on $\mathscr{F}_0$. Then,*

$$\mathbb{E}(X \vee Y \big| \mathscr{G}) = \mathbb{E}_0(X \vee Y).$$

Proof. The proof follows a similar method to that of Lemma 7.8.4. We will show that $\mathbb{E}_0((X \vee Y)\mathbb{1}_G) = \mathbb{E}_0(X \vee Y)\mathbb{E}_0(\mathbb{1}_G)$ for all $G \in \mathscr{G}$. The result then follows immediately by Lemma 7.8.3.

Fix $G \in \mathscr{G}$. Suppose first that Y is simple, with $Y = \sum_{i=1}^n b_i \mathbb{1}_{A_i}$, where the A_i are pairwise disjoint $\mathscr{F}_0$-measurable sets and $\bigcup_{i=1}^n A_i = \Omega$. Then,

$$\begin{aligned}\mathbb{E}_0((X \vee Y)\mathbb{1}_G) &= \mathbb{E}_0((\sum_{i=1}^n (X \vee b_i)\mathbb{1}_{A_i})\mathbb{1}_G) \\ &= \mathbb{E}_0(\sum_{i=1}^n (X \vee b_i)\mathbb{1}_{A_i}\mathbb{1}_G) \\ &= \sum_{i=1}^n \mathbb{1}_{A_i}\mathbb{E}_0((X \vee b_i)\mathbb{1}_G) \\ &= \sum_{i=1}^n \mathbb{1}_{A_i}\mathbb{E}_0(X \vee b_i)\mathbb{E}_0(\mathbb{1}_G) \\ &= \mathbb{E}_0(X \vee Y)\mathbb{E}_0(\mathbb{1}_G).\end{aligned}$$

Now suppose that Y is an arbitrary non-negative element of $L_1(\mathscr{F}_0)$. Let $(Y_n)_{n=1}^\infty$ be a pointwise increasing sequence of simple functions converging almost surely to Y. Then $((X \vee Y_n)\mathbb{1}_G)_{n=1}^\infty$ is a pointwise increasing sequence converging to $(X \vee Y)\mathbb{1}_G$ almost surely. By the monotone convergence theorem, $(X \vee Y_n)\mathbb{1}_G \to (X \vee Y)\mathbb{1}_G$ in $L_1(\mathscr{H})$. Since the map $Z \mapsto \mathbb{E}_0(Z)$ is continuous from $L_1(\mathscr{H})$ to $L_1(\mathscr{F}_0)$, it follows that $\mathbb{E}_0((X \vee Y_n)\mathbb{1}_G) \to \mathbb{E}_0((X \vee Y)\mathbb{1}_G)$ in $L_1(\mathscr{F}_0)$. By passing to a subsequence we may assume that we have pointwise convergence almost everywhere. The same argument (with $G = \Omega$) shows that $\mathbb{E}_0((X \vee Y_n)) \to \mathbb{E}_0((X \vee Y))$ in $L_1(\mathscr{F}_0)$ and again we may assume pointwise convergence almost surely. Since each Y_n is simple, $\mathbb{E}_0((X \vee Y_n)\mathbb{1}_G) = \mathbb{E}_0(X \vee Y_n)\mathbb{E}_0(\mathbb{1}_G)$, and therefore letting $n \to \infty$ shows that $\mathbb{E}_0((X \vee Y)\mathbb{1}_G) = \mathbb{E}_0(X \vee Y)\mathbb{E}_0(\mathbb{1}_G)$ almost surely. ☐

We are now able to show that $[X_i e_i]_{i=1}^\infty$ is isomorphic to a mixture of Orlicz spaces. Our next proposition gives a formula for $\mathbb{E}_0(\sigma_1(\boldsymbol{\alpha} \cdot X))$ in terms of a convolution of Φ.

Proposition 7.8.6 *Let $\alpha_1, \ldots, \alpha_n$ be a finite sequence of scalars, and $\boldsymbol{\alpha} \cdot X = \sum_{i=1}^n \alpha_i X_i e_i$. Then,*

$$\mathbb{E}_0(\sigma_1(\boldsymbol{\alpha} \cdot X)) = (\underbrace{\Phi * \ldots * \Phi}_{n-1})(\alpha_1, \ldots, \alpha_n).$$

Proof. We calculate $\sigma_1(\boldsymbol{\alpha} \cdot X)$ using the recursive definition of the σ_i. We suppose for convenience that $\alpha_i \geq 0$ for all i. Recall that $\sigma_n(\boldsymbol{\alpha} \cdot X) = \alpha_n X_n$. Then,

$$\begin{aligned}\sigma_{n-1}(\boldsymbol{\alpha} \cdot X) &= \alpha_{n-1}X_{n-1} \vee \mathbb{E}_{n-1}(\sigma_n(\boldsymbol{\alpha} \cdot X)) \\ &= \alpha_{n-1}X_{n-1} \vee \alpha_n \mathbb{E}_{n-1}(X_n) \\ &= \alpha_{n-1}X_{n-1} \vee \alpha_n.\end{aligned}$$

Substituting the above equation into the definition of $\sigma_{n-2}(\boldsymbol{\alpha}\cdot X)$ gives,

$$\begin{aligned}\sigma_{n-2}(\boldsymbol{\alpha}\cdot X) &= \alpha_{n-2}X_{n-2}\vee\mathbb{E}_{n-2}(\sigma_{n-1}(\boldsymbol{\alpha}\cdot X))\\ &= \alpha_{n-2}X_{n-2}\vee\mathbb{E}_{n-2}(\alpha_{n-1}X_{n-1}\vee\alpha_n)\\ &= \alpha_{n-2}X_{n-2}\vee\mathbb{E}_{0}(\alpha_{n-1}X_{n-1}\vee\alpha_n) \text{ by Lemma 7.8.5}\\ &= \alpha_{n-2}X_{n-2}\vee\Phi(\alpha_{n-1},\alpha_n).\end{aligned}$$

Similarly,

$$\begin{aligned}\sigma_{n-3}(\boldsymbol{\alpha}\cdot X) &= \alpha_{n-3}X_{n-3}\vee\mathbb{E}_{n-3}(\sigma_{n-2}(\boldsymbol{\alpha}\cdot X))\\ &= \alpha_{n-3}X_{n-3}\vee\mathbb{E}_{n-3}(\alpha_{n-2}X_{n-2}\vee\Phi(\alpha_{n-1},\alpha_n))\\ &= \alpha_{n-3}X_{n-3}\vee\mathbb{E}_{0}(\alpha_{n-2}X_{n-2}\vee\Phi(\alpha_{n-1},\alpha_n)) \text{ by Lemma 7.8.5}\\ &= \alpha_{n-3}X_{n-3}\vee\Phi(\alpha_{n-2},\Phi(\alpha_{n-1},\alpha_n)) \text{ by Lemma 7.8.4.}\end{aligned}$$

Continuing this process, ultimately gives

$$\sigma_1(\boldsymbol{\alpha}\cdot X) = \alpha_1X_1\vee\Phi(\alpha_2,\Phi(\alpha_3,\ldots,\Phi(\alpha_{n-1},\alpha_n)\ldots)).$$

Therefore, by Lemma 7.8.4, $\mathbb{E}_0(\sigma_1(\boldsymbol{\alpha}\cdot X)) = \Phi(\alpha_1,\Phi(\alpha_2,\ldots,\Phi(\alpha_{n-1},\alpha_n)\ldots))$. □

By (7.16),

$$\left\|\sum_{i=1}^{n}\alpha_iX_ie_i\right\|_S = \mathbb{E}(\mathbb{E}_0(\sigma_1(\boldsymbol{\alpha}\cdot X))),$$

and Proposition 7.8.6 implies that,

$$\mathbb{E}_0(\sigma_1(\boldsymbol{\alpha}\cdot X))(\omega) = (\underbrace{\Phi_\omega * \ldots * \Phi_\omega}_{n-1})(\alpha_1,\ldots,\alpha_n).$$

It follows immediately from the results in Section 7.4 that, $\mathbb{E}_0(\sigma_1(\boldsymbol{\alpha}\cdot X))(\omega)$ is 2-equivalent to $\|\boldsymbol{\alpha}\|_{\phi_\omega}$, where ϕ_ω is the Orlicz function defined by

$$\phi_\omega(t) = \Phi_\omega(t,1) - 1.$$

Hence, taking expectations shows that,

$$\int_\Omega \|\boldsymbol{\alpha}\|_{\phi_\omega}\,\mathrm{d}\mathbb{P} \le \|\boldsymbol{\alpha}\cdot X\| \le 2\int_\Omega \|\boldsymbol{\alpha}\|_{\phi_\omega}\,\mathrm{d}\mathbb{P}.$$

Thus, we have shown that $[X_ie_i]_{i=1}^{\infty}$ is isomorphic to a mixture of Orlicz spaces.

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