Thesis on Behavioural Asset Pricing and Portfolio Choice

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Abstract

This thesis presents three papers in the field of behavioural financial economics and financial econometrics.

The first paper, entitled “Optimistic versus Pessimistic–Optimal Judgemental Bias with Reference Point”develops a model of reference-dependent assessment of subjective beliefs in which the loss-averse people optimally choose expectation as the reference point to balance the current felicity from the optimistic anticipation and the future disappointment from the realization. The choice of over-optimism or over-pessimism depends on the real chance of success and optimistic decision makers prefer receiving early information. In the portfolio choice problem, pessimistic investors tend to trade conservatively. However, they might trade aggressively if they are sophisticated enough to recognise the biases as low expectation can reduce their fear of loss.

The second paper, entitled “Information and Dynamic Trading with Gambler’s Fallacy”, develops a multi-period stock trading model in which there are two types of investors–“rational” type and “gambler’s fallacy” type, both observe the public signal about the fundamental value each period. The rational investor holds correct beliefs on the stochastic process of the signal, whereas the gambler’s fallacy investor falsely believes that the sequence of signals should exhibit systematic reversals. Both types of investors trade against each other to speculate the future price changes, based on their inferences about the fundamental value. This paper explores the competitive equilibrium, in which both types of investors have model consistent expectations adjusted for the heterogeneity in their beliefs about the signal generating process. The thesis examines the dynamics of prices, returns, optimal portfolios and trading volumes in reaction to the information flow. Consistent with empirical evidence, the market in this model exhibits short-run momentum and long-run reversal. It is also demonstrated that the equilibrium price is more close to the valuation of the gambler’s fallacy type.

Finally, the third empirical paper entitled "Regime Switching in Financial Market and Portfolio Choice" considers a variety of regime switching models with time varying transition probabilities for the joint distribution of stocks and bonds returns. Paper results support a two-regime univariate model for stocks with ISM and P/E ratio as leading predictors for the transition probabilities and support the fixed transition probability model for univariate distribution of bond returns. Under joint distribution assumption, the model selects a three regime model with ISM, unemployment rate and P/E ratio as predictors for the time varying transition probabilities. Even though both fixed and time varying transition probability models identify three regimes in the financial market, however, the time varying transition probability model provides better out of sample predictions, based on the regime-dependent portfolio performance.

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Chapter 1

Introduction

This thesis presents three papers in the field of behavioural financial economics and financial econometrics.

1.1 Optimistic versus Pessimistic–Optimal Judgemental Bias with Reference Point

The first paper, entitled "Optimistic versus Pessimistic–Optimal Judgemental Bias with Reference Point" models a decision maker, choosing between lotteries, who may create bias in his anticipation of the welfare (from the outcome of a lottery) by distorting his subjective beliefs (degrees of optimism and pessimism). The extent to which the decision maker creates bias in his beliefs optimistically (say) is tempered by an awareness that the greater the bias the greater will be the chances of disappointment. The disappointment is modelled by adopting the notion that the decision maker is loss averse with a reference point (endogenously) determined by the expectation given by the distorted belief. More precisely, the utility function mapping lotteries to a welfare index, is the (weighted) sum of two components. The first component is the expected utility, evaluated as per standard expected utility theory except that the expectation is evaluated using, possibly distorted, subjective beliefs. The second component, is expectation of a reference dependent (loss-averse) utility, evaluating the expected deviation from the reference point, given by the first component, where the expectation is computed using the objective probability. Finally, the judgemental bias is optimal in the sense that the decision maker chooses the distorted belief so as to maximize the welfare index. In terms of closely related literature, this form of utility functional is, essentially, a "marriage" of the optimal belief idea of a model in which agents choose the optimal subjective beliefs and optimal expectations as in develops a model of reference-dependent assessment of subjective beliefs in which loss-averse people optimally choose the expectation as the reference point to balance the current felicity from the optimistic anticipation and the future disappointment from the realization. The choice of over-optimism or over-pessimism depends on the real chance of success and optimistic decision makers prefer receiving early information. In the portfolio choice problem, pessimistic investors tend to trade conservatively, however, they might trade aggressively if they are sophisticated enough to recognise the biases since low expectation can reduce their fear of loss.

The total utility in this paper contains two parts: the first part is similar to the anticipation utility in Brunnermeier and Parker (2005), and Brunnermeier et al. (2007), while the second part is similar to the loss averse utility with an endogenous reference (Koszegi and Rabin (2006)) point. The two components are linked or, the marriage is effected, by making the biased anticipation in the first component and the reference point in the second component. The components together model the agents' inner conflicts in making predictions and decisions. They may enjoy being ambitious while hate to be disappointed; enjoy feeling lucky while hate to be hopeless. Optimal beliefs (and associated optimistic and pessimistic attitudes) are then chosen by weighing the current anticipation and future utility, depending on the probabilistic properties of lotteries, the future discount rates and the intensity of loss aversion.

Section 3 enriches the model by introducing two types of agents, "naive" and "sophisticated", according to their different cognitive processes: sophisticated agents recognise their biases in making decisions, while naive ones do not. The section establishes characterizations of the distinctive behaviour of the two types of agents – naive agents show preference to skewness and spreading in distribution, while sophisticated agents behave similar to rational ones, except overweighing the low-ranking outcomes.

Section 4 explores the implications of the model in the investment problem, where the "optimally biased" investor with concave utility function chooses the optimal portfolio consisting of a risky asset and a risk-free asset. Following the assumption made in section 3, it is assumed that two types of investors exist – the naive ones who maximise the expected return and the sophisticated ones who maximise the total utility, including both the anticipatory and prospective gain-loss utilities. It is shown that the naive investor and the sophisticated investor and adopt opposite investment strategies, even though they form their subjective beliefs through the same cognitive process. An equilibrium is further derived in the market with identical investors to look into the implications on asset pricing.

The results in section 4 allow us to conclude that naive optimistic agents trade aggressively, while pessimistic ones trade conservatively. Sophisticated agents can bear more (less) risks than rational ones when low-ranking returns are good (bad) relative to risk-free return, because loss aversion dictates a higher weight on bad returns. Moreover, sophisticated pessimistic investors can bear excess risks than optimistic ones because low anticipation reduces their fear for loss. Further, the analysis explores the pricing implication in a market with identical investors and short-sale constraints. Price of the risky asset in a naive market decreases as the market becomes pessimistic but exhibits a U-shape in a sophisticated market implying that a market with the moderate confidence level has the highest equity premium.

1.2 Information and Dynamic Trading with Gambler's Fallacy

The second paper, entitled "Information and Dynamic Trading with Gambler's Fallacy", develops a multi-period rational expectation model of stock trading in which rational investors trade against investors with gamblers fallacy – the mistaken belief that random sequences should exhibit systematic reversals. In this model, non-myopic investors observe a sequence of public signals concerning the underlying value of the stock. Based on the public information, the rational traders and fallacy traders form heterogeneous value predictions and trade against each other competitively in the market to speculate the future price changes.

Section 1 describes the details of the model setting. The rational investors believe signals are generated following a Gaussian Process, while the gambler's fallacy type believes that signals should exhibit systematic reversals. To be more specific, the signals are assumed to be the fundamental value of the stock plus a Gaussian error. The inference problems of the investors are formulated as state space models and Kalman filter is used to model their beliefs updating process. It is shown that compared to the rational traders, the gambler's fallacy traders fail to update to new information surficiently. The Lemma 1 further gives the steady state of the system.

In this paper, the investors are non-myopic and dynamically choose their optimal holdings of the stock to maximise their utility from consumption. It is assumed that they have exponential utility function and their optimal holding problems can be solved by dynamic optimisation with the proper guess of the value function. Based on their inference and optimisation problem, this paper solves the dynamic competitive "rational" expectation equilibrium, where the rational traders' beliefs on the price process are consistent with the real equilibrium price. The thesis proves the existence of a linear equilibrium and provides the format of the equilibrium price in Proposition 2. It is shown that the equilibrium price is a linear function of expectations of the state variables of the economy—the liquidation value and the cumulated effects of the past luck.

Section 3 presents the baseline model—a market with identical investors, either all rational, or all fallacy investors, and writes out the expectations, volatilities and autocorrelations in this market. In a single type market, the price is a pure reflection of the value expectation of that type.

Section 4 presents the equilibrium analysis by first showing that the optimal portfolio is composed of two parts – a mean-variance efficient portfolio and a hedging portfolio, reflecting the additional position due to the non-zero autocorrelations in returns. Furthermore, the effect of a signal shock is analysed at period t on the following equilibriums. The analytical conclusions are consistent with previous empirical findings, where the returns exhibit short-term continuations and long-term reversals due to the efforts of the gambler's fallacy traders. Basically, the gambler's fallacy traders fail to update their value estimations to new information sufficiently in the short-run, leading to under-reactions. The lack of observations in reversals makes them over-adjust their value estimations later on, leading to over-reactions. In the long-run, the price returns to fundamentals as past signals' effect recedes. The positions of both types are also analysed over time and it is shown that the rational investors act as if they have hot-hand fallacy in the price process as they are buyers of the stock when the price increases, but turn into sellers when the price decreases. The analysis on the market dominance proved that the gambler's fallacy traders become more powerful in determining the price than the rational traders as time goes on. This is because of two reasons – 1. the gambler's fallacy makes gambler's fallacy traders over-confident in their estimations and trade more aggressively 2. the existence of the gamblers' fallacy traders makes market near-term returns positively correlated, creating additional risks for rational traders. Finally, it is shown that if the trading time horizon is long enough, the gambler's fallacy traders' beliefs will always control the equilibrium price.

With regards to asset pricing implication, it is shown that the gambler's fallacy leads to momentum and reversals. Besides, gambler's fallacy traders' beliefs are more powerful in a better informed market. The market volatilities are lower when there are gambler's fallacy traders.

1.3 Regime Switching in Financial Market and Portfolio Choice

Finally, the third empirical paper entitled "Regime Switching in Financial Market and Portfolio Choice"considers a variety of regime switching models with time varying transition probabilities for the joint distribution of stocks and bonds returns.

This paper is an empirical paper, which lies in framework of the Markov Regime Switching models. It is different from previous efforts in this field, which either assume fixed transition probabilities or use single time series for the analysis of financial market regimes. This paper uses both equity and bond to identify the regimes and also uses ISM (the Institute of Supply Management Manufacturing Purchasing Manager Index, which is an indicator of the economic health of the manufacturing sector), P/E ratio (price to book ratio) and unemployment rate as the leading indicators to forecast the time varying transition probabilities.

Section 2 presents the univariate Markov regime switching model for individual time series of stock and equity, starting with the univariate Fixed Transition Probability (FTP) model. Model results suggest a two-regime (bullish and bearish market) AR(0) process for stocks and 3-regime AR(0) process for bonds. The FTP bond market model separates the periods before and after 1980s into two regimes with the third regime being very volatile regime for the bond market.

With regards to the Univariate Time Varying Transition Probability (UTVTP) Model, the study shows that 2-regime model with P/E ratio and ISM as the predictors for the transition probability matrix for the stock market is preferred to the FTP model, while for the bond market, even though the FTP model achieves lower AIC, the regimes identified by the TVTP model with AR(1) and ISM and PE ratio as leading indicators are more meaningful. To be more specific, in the stock market, the first regime is still the good regime with higher returns and lower volatility, while the second regime is the bad one with lower returns and higher volatility. Model indicates that the transition probability to stay in the good regime is insignificant (decreasing) in ISM and increasing in PE, while the transition probability to stay in a bad regime is decreasing in ISM and insignificant (increasing) in PE. For the bond market in the TVTP model with ISM, UEM and PE all together as leading indicators, the first regime represents the stable economy periods in which the bond price drops and volatilities are low. The second regime achieves the highest returns in bond and a moderate volatility. The third regime reflects the volatile periods in bonds market.

Section 3 extends the univariate model to bivariate model settings by first exploring the results of a bivariate FTP model, using the joint distribution of stocks and bonds. BFTP model supports the 3-regime assumption (stable, transition, crisis) with AR(0) for stocks and AR(1) for bonds.

Section 4 formally presents the Bivariate TVTP model. The BTVTP models still supports the 3-regime AR(0)-AR(1) assumption but with improved identifications on all 3 regimes. The BTVTP also achieves better AIC, as compared to the BFTP model. Furthermore, all three leading predictors (ISM, P/E ratio, unemployment rate) are significant. ISM and P/E have similar effects – higher values increases the chance to move into a crisis from stable state and also increases the chance to move into a stable state from a recovery state. On the other hand, as employment gets worse, the financial markets are more likely to move out of crisis or stable state.

Finally, section 5 examine the out-of-sample regime-dependent portfolio performance using the BFTP and BTVTP models in section 3 and 4. An expected-mean-expected-variance portfolio is built using filtered probabilities. Both BFTP and TVTP models outperform the SP500 benchmark. Besides, the dynamically optimised TVTP portfolio provides significantly better returns as compared to its fixed allocation benchmark, while FTP strategy does not. Using time varying transition probabilities improves the model ability to capture early signals of a new regime as compared to a classic FTP regime switching model.

Chapter 2

Optimistic versus Pessimistic–Optimal Judgemental Bias with Reference Point

Abstract

This paper develops a model of reference-dependent assessment of subjective beliefs in which the loss-averse people optimally choose the expectation as the reference point to balance the current felicity from the optimistic anticipation and the future disappointment from the realisation. The choice of over-optimism or over-pessimism depends on the real chance of success. This paper explores the preference of information timing based on their subjective beliefs. In the portfolio choice problem, the paper shows that "naive" pessimistic investors who underestimate future profits trade more conservatively than their rational counter-parts and "sophisticated" pessimistic investors who are non-myopic in decision making trade more aggressively as their low expectation reduces the fear of loss.

2.1 Introduction

Neoclassical economics uses a natural simplification of human behaviour as governed by limitless cognitive ability applied to a handful of perceptible goals and untangled by emotions (Stewart, 1998). One fundamental proposition from this presumption is rational expectation (Muth, 1961) simply declaring that the agents' predictions of the future value of economically relevant variables are not systematically wrong and all the errors are random. Equivalently, the agents' expectation is consistent with the true statistical mean. Nevertheless, there are many evidences of deviations from rational expectations, with the most prominent one showing that individuals err in their probability assessments and not in random, but systematic. Particularly, when the task is very difficult with rare positive events, people often exhibit over-pessimism and overestimate the probability of bad outcomes (Kruger, 1999; Windschitl, Kruger and Simms, 2003; Kruger and Burrus, 2004). On the other hand, when the task is easy and the probability of success is reasonably high, people tend to exhibit over-optimism and overestimate the probabilities of good outcomes (Fischhoff, Slovic and Lichtenstein, 1977; Svenson, 1981; Hoffrage, 2004)[1]. In short, subjective belief of a good outcome increases with its real chance but always contains systematic biases.

These systematic biases provide strong evidences against the prevailing assumptions that people are either completely rational or completely irrational and erratic in making predictions toward the future (De Long et al., 1990; Friedman and Mezzetti, 2005). Specifically, people are capable of capturing the probabilities of success and failure to a reasonable extent of accuracy, like rational individuals[2]; but when it comes to forming subjective beliefs, they deceive themselves and choose biased beliefs as if they are irrational. This is because utilities are not only from an accurate prediction.

Based on these observations, this paper builds a model in which loss-averse agents choose their optimal subjective beliefs to form their optimal expectations and make decisions accordingly. Their optimal subjective beliefs are determined by maximising two parts of utilities–the utility from their anticipation and the utility from realisation. The second part depends on the first part as the realisation utility is reference dependent with the anticipation level of the reference point. A higher anticipation utility will be punished by a lower realisation utility and vice versa. The decisions on investments based on these subjective beliefs are further examined. Investors choose their optimal holdings of a stock

[1] See Moore and Healy (2008) for a review.

[2] In situations with ambiguity, this statement is still valid (see Camerer and Weber (1992) for a review). Although agents facing ambiguity are unable to capture the true distribution, they can apprehend a family of such distributions and a probability distribution over the family. Their best guess works as a good proxy of the objective distribution.

today to maximise their utilities. Two types of investors are considered in this paper–the "naive" type and the "sophisticated" type. Even though both types form their beliefs by optimising the total utility from both anticipation and realisation, the "naive" type fails to realise their biases in beliefs and choose their action only based on their expected wealth (anticipation), while the "sophisticated" type make decisions by taking into account their realisation utilities as well.

The framework of this model is most close to two papers. Brunnermeier and Parker (2005) and Brunnermeier, Gollier and Parker (2007)[3] build a model of optimal subjective beliefs, where utility of agents contain two parts, the anticipation utility and the realisation utility. Agents choose the optimal beliefs and make an investment decision simultaneously by balancing the imaginary utility from anticipation and the worse-off utility from realisation by a distorting behaviour. Their investors, whether over-optimistic or over-pessimistic, always invest in the same direction as the rational investors but take more aggressive positions as they find the cost from distorting behaviour is only second order compare to the benefits of their anticipation utility which gives a first order utility. Aggressive positions, which generate higher anticipation utility, are always preferred by their agents. Similar to their model, the agents in this paper also consider two parts of utilities in forming their optimal beliefs. Different from their models, individuals are reference-dependent and loss-averse. It is shown that loss-aversion makes the cost from taking a biased action significant as the realisation utility is directly deducted by the anticipation. The kink point in the utility function due loss-aversion gives the first order effects in our model. Therefore, investors in this paper no longer take aggressive positions all the time. Whether they are more aggressive or conservative, depends on their optimistic and pessimistic attitudes towards future.

The second paper in this model is close to the study of (Koszegi and Rabin, 2006). Koszegi and Rabin (2006) build a model where the utility of agents also contained two parts. The first part is a classic reference independent consumption utility and the second part is a reference-dependent gain-loss utility. Their consumption utility is similar to the anticipation utility in this paper. They point out that the expected consumption utility serves naturally as the reference point for the gain-loss utility. They discussed the situation in which people can receive news about their future realisations. The news introduces fluctuations to their gain-loss utility and people make decisions based on the timing of the news. They show that people are indifferent about the timing of the information when they are indifferent on the gain-loss utility in different periods. This paper's gain-loss realisation

[3]Spiegler (2008) discussed the problems (violation of IIA) in revealed-preference approach and information acquisition raised by "utility-from-belief" models.

utility is similar to the formatting in Koszegi and Rabin (2006). Different from their paper, the beliefs are no longer objective, but are optimally selected. This paper also discusses the news timing application under the subjective beliefs and show that people welcome early news when optimistic but prefer staying uninformed when pessimistic. This is because biased beliefs also make them falsely believe the news is good or bad.

Overall, this paper is innovative in the sense that it combines both loss-aversion and subjective beliefs optimisation. Individuals manipulate their beliefs to trade off the benefit of an optimistic future with the cost of a painful loss. In particular, my model contains three main elements:

First, utilities consist of two parts: anticipatory utility and future gain-loss utility. Previous studies suggest that a forward-looking decision maker cares about both the utilities (Loewenstein, 1987; Caplin and Leahy, 2001; Benabou and Tirole, 2004; Bernheim and Thomadsen, 2005)[4]. Decision makers weigh anticipatory utility against the future utility depending upon the specific circumstances such as the time to payoff date and how significant they value the lottery.

Second, utility is reference dependent. It is assumed that the future utility is reference dependent (Kahneman and Tversky, 1979; Koszegi and Rabin, 2006, 2009) and the anticipation serves naturally as the reference point (Koszegi and Rabin, 2006). Anticipatory and future utilities are therefore linked together by the reference point, which is constituted by an agent's subjective beliefs. For simplicity, anticipatory utility is assumed to be reference independent and it takes the value of ordinary expectation.

Third, individuals are loss averse. Previous studies without loss-aversion usually led to optimistic judgemental biases (Brunnermeier and Parker, 2005; Brunnermeier, Gollier and Parker, 2007; Bernheim and Thomadsen, 2005). In this model, over-pessimism is also possible, because loss due to optimism can be more heavily felt than the felicity from a good anticipation.

All these three factors together result in inner conflicts for the decision makers while making predictions. They always enjoy being ambitious, while hate to be disappointed; enjoy feeling lucky, while hate to be hopeless. Optimal beliefs (and associated optimistic and pessimistic attitudes) are then chosen by weighing current anticipation and future utility, depending on the probabilistic properties of the lotteries, the future discount rate and the intensity of loss aversion.

Psychological theories and evidence support this intuition. Contemporary psychology acknowledges that the human behaviour is influenced simultaneously by conscious (control

[4]Caplin and Leahy (2004); Eliaz and Spiegler (2006) pointed out the problems in reveal preference approach brought by forward-looking utility.

System 1) and non-conscious (automatic, System 1) processes (Chaiken and Trope, 1999). Neuropsychology using fMRI technique has provided evidence that the ventral medial prefonrtal cortex competes with the prefrontal cortex for the control of the response to the problems given in the belief-bias test (Goel and Dolan, 2003). These studies indicate that individuals are more capable than previously assumed in capturing the real probabilistic pattern of stochastic events, but can subconsciously deceive themselves into biased beliefs. To quote Kahneman (2011), the process through which people choose subjective beliefs is the result of an uneasy interaction between two systems; "System 2 articulates judgements and makes choices, but it often endorses or rationalizes ideas and feelings that were generated by System 1.". Even though the real biological process in distorting beliefs is not fully deciphered, this view provides a reasonable interpretation that human mind is operated as a dual and conflicting system—pursuing mental pleasures like idealists while staying close to reality as physicalists.

Moreover, previous experimental studies pointed out that subjective beliefs can change from optimism to pessimism without any additional information (Shepperd, Ouellette and Fernandez, 1996; Taylor and Shepperd, 1998; Sweeny, Carroll and Shepperd, 2006; Mayraz, 2011*b*) For example, Shepperd, Ouellette and Fernandez (1996) conducted an experiment in which students estimated their exam scores a month before the exam, then again several times after completing the exam but prior to receiving feedbacks. As the date of feedback neared, students turned their optimistic forecasts into pessimistic forecasts. The switch in attitudes implies that people can manipulate beliefs for their interests–when the threat of loss becomes more relevant, pessimism is preferred over optimism as it placates the anxiousness of loss. In Mayraz (2011*b*), price predictions of financial assets based on the same historical charts are significantly higher for subjects gaining from high prices than those gaining from low prices. Asymmetry in predictions is attributed to the different choices of anticipations as reference points.

This paper proceeds as follows. After presenting the relevant literatures in Section 1, Section 2 presents the agent's formal problem and model predictions on the choice of optimal beliefs, starting with the general multiple-state model followed by the two-state example. Whether the subjective beliefs are biased up or down depend upon whether the real chance of success exceeds the cut-off probability which is uniquely determined by intensity of loss aversion and future discount rate. The paper further discusses a short application on information timing preference following the work in Koszegi and Rabin (2009) (KR2009). It is shown that as the confidence level decreases, biased agents abandon their preference to early information in favour of staying uninformed, while rational agents are indifferent. Section 3 explores the basic premise of the model—risk attitudes over lotteries. The paper divides the agents into two types—“naive” and “sophisticated” , according to their different cognitive processes: sophisticated agents recognise their biases in making decisions while naive ones do not. This paper shows that the naive agents favour skewness and spreading in distribution while sophisticated agents overweighing low-ranking outcomes but otherwise, behave similarly to rational ones . Section 4 applies the model to the portfolio choice problem. Following previous categorisation of agents, it is concluded that naive optimistic agents trade aggressively, while pessimistic ones trade conservatively. Sophisticated agents can bear more (less) risks than rational ones when low-ranking returns[5] are good (bad) relative to risk-free return, because loss aversion dictates a higher weight on bad returns. Moreover, sophisticated pessimistic investors can take extra risks than optimistic ones because low anticipation makes them fearless of loss. Furthermore, this paper explores the pricing implication in a market with identical investors and short-sale constraint. Intuitively, the price in the naive market decreases as investors become more pessimistic. However, the equilibrium price takes an U-shape in the sophisticated market. The risky asset achieves higher risk-premium when the market is neither over-optimistic nor over-pessimistic. Finally, Section 5 disccusses the model’s scope and limitations.

2.2 Relevant Literature

Related literatures are divided into three groups: 1.Optimal Beliefs 2.Reference-Dependent Utilities 3.Biased Beliefs in Asset Pricing.

2.2.1 Optimal Beliefs

Past literatures on distorted optimal beliefs is built on the assumption that people choose subjective biased beliefs departing from the real probabilities. Previous studies can be divided into three branches: 1. focusing on anticipatory utilities of forward-looking decision makers[6]; 2. focusing on cognitive dissonance in which people hold inconsistent beliefs to comfort their past experiences (Rabin, 1994; Epstein and Kopylov, 2006)[7];3. focusing on biased beliefs arising from self-signalling with imperfect memory (Benabou and Tirole, 2002,

[5]Low-ranking returns refer to the bad returns of a risky asset.

[6]Akerlof and Dickens (1982) proposed a model in which workers in hazardous professions choose their subjective beliefs of an accident to balance their anticipatory feelings of danger and money spent on safety equipments.

[7]For example, Epstein and Kopylov (2006) built a axiomatic model in which agents adjust their beliefs after taking an action so as to be more optimistic about the possible consequences.

2004; Bernheim and Thomadsen, 2005)[8]. This review focuses on the first branch that is most relevant to the thesis.

Brunnermeier and Parker (2005)(BP2005) built a structural model and gave two applications– choice between a risky and a risk-free asset and a consumption-saving problem with stochastic income. The underlying intuition is straightforward–agents with anticipatory utilities are willing to hold optimistic biased beliefs to achieve higher current felicity. They trade off the current felicity from a higher expectation with the cost of making a sub-optimal decision and thus realise worse outcomes due to the biased beliefs. Agents balance these effects and choose their optimal beliefs to maximize the average utility. They concluded that a small optimistic bias in beliefs typically leads to a first-order gain in anticipatory utility, while only induces second-order cost from the poor decision. Further applying this model to the financial market, BP achieved two conclusions– 1. Investors always overestimate the return of their investment, which encourages them to long or short too much of the risky asset compared to what would maximize their objective expected utility[9]. 2. Investors tend to invest in an asset with high level of positive skewness even if the asset earns a negative average excess return. In the consumption-saving application, BP further concluded that agents are both over-confident and over-optimistic.

Spiegler (2008) criticized BP's model in two perspectives–1. BP's model fails the rudimentary revealed preference test since the IIA (Irrelevance of Independent Alternatives) is violated. The reason is that subjective beliefs in BPs model are derived directly from payoffs of the lottery in the choice set[10]; 2. BP's model cannot capture the preference for biased information sources. Intuitively, by assuming people have desires to attain self-serving beliefs, BP's model should also provide explanations of people's preference for information sources, which can distort the beliefs indirectly. However, Spiegler proved that in BP's model, the decision maker is never averse to information as the support of subjective beliefs are not updated to signals containing uncertainty. The rationale for this criticism lies in BP's assumption that the action, rather than beliefs is the most fundamental choice variable. Subjective beliefs are inertial to new information as long as the punishment from the

[8]For example, Bernheim and Thomadsen (2005) developed multiself-consistency game where decision makers have both anticipatory utility and imperfect memory. In order to benefit from anticipation, agents with imperfect recall prefer staying uninformed and exhibit over-optimism.

[9]Specifically, they enter into over-possession of the risky asset with average excess return greater than 0, and hold inadequate risky asset with average excess return smaller than 0.

[10]Spiegler (2008) constructed 3 lotteries in states $s_1 ... s_n$ with payoffs in the matrix:

action/state	s_1	s_2	...	s_n
l_f	0	0	0	0
l_r	1	$-k$	$-k$	$-k$
$l_{r'}$	m	$-n$	$-n$	$-n$

,

where $k, m, n > 0, m > 1$ and k satisfies: $c_{BP}\{l_f, l_r\} = c\{l_f, l_r\}$. Based on BP's model, subjective belief to s_1 is $q_1 = 1$. When the choice set becomes $\{l_f, l_r, l_{r'}\}$, $q_1 = 1$ leads to $l_{r'} \succ l_r$ and $c_{BP}\{l_f, l_r, l_{r'}\} = l_{r'}$. However, with n big enough, $l_f \succ l_{r'}$,leading to a violation of IIA.

sub-optimal action stays the same. With the assistance of loss aversion, decision makers in this model are able to choose subjective beliefs directly and they exhibit different tastes for information depending upon their optimistic and pessimistic attitudes. Experimental evidence also fails to justify BP‘s conclusions. Mayraz (2011*b*) gives a further critique on BP2005– BP‘s model assumes that the biased beliefs are costly for a decision maker in material terms due to poor decisions. This means that the magnitude of the bias depends on the incentives for accuracy, and the bias can only be substantial when incentives for accuracy are weak. However, this assumption is rejected by experimental data in Mayraz (2011*b*). It is observed that the biases are independent of material costs from poor decisions[11]. An extension to Mayraz's intuition is that in situations where the decision maker has no control over final outcomes, there is no punishment for sub-optimal actions and optimal-belief holders in BP will distort beliefs towards optimistic as much as they wish. Finally, Mayraz developed an axiomatic model in which choice of beliefs is payoff dependent; agents believe what they want to be true (Mayraz, 2011*a*). Agents derive their optimistic and pessimistic attitudes based on the attitude parameter exogenous to the model.

The model in this paper mainly takes the framework of BP with some modifications. Similar to BP, decision makers in this model are forward-looking and the introduction of anticipatory utility to the ordinary economic model brings the tendency of optimism. Unlike BP, decision makers are loss averse and this behavioural element provides first-order counter force to anticipation, making pessimism preferable to optimism when loss aversion is strong enough. In situations where agents do not discount future, pessimism is always more beneficial since the intense hurt from loss by setting a high anticipation always exceeds the happiness from gain brought by that anticipation.

2.2.2 Reference-dependent Utilities

The prospect theory first proposed by Daniel Kahneman and Amos Tversky in their 1979 paper pointed out that the evaluation of outcomes is compared to a reference point; the aversion to loss is significantly more intensive than the felicity from gain with a diminishing sensitivity to changes in an outcome as it moves farther away from the reference point; subjective probability of a prospect is non-linear in the true probability—in particular, people overweigh small probabilities and under-weigh high probabilities. Koszegi and Rabin (2006) built a model with a separation of the reference-independent "consumption utility" and the "gain-loss utility" based on the essential intuition of Kahneman and Tversky (1979).

[11]Subjects observed the historical price chart of a financial asset, and received both an accuracy bonus for predicting the price at some future point, and an unconditional award that was either increasing or decreasing in this price. The statistical test indicates that the magnitude of the bias is independent of the amount paid for accurate predictions.

The reference point in their model is simply people's rational expectations determined by the "personal equilibrium" –an equilibrium in which expectation is consistent with the optimal choice under that expectation. With application to the consumer behaviour, they observed that the price a person wishes to pay for a commodity increases with the expected price conditional on purchase and the expected probability of purchase. Another application lies in the within-day labour-supply decision. A worker is likely to continue working only if they receive income less than their expectation. Based on their 2006 model, Koszegi and Rabin (2009)(KR2009) developed a rational dynamic model in which people are loss averse over changes in rational expectation about present and future consumption. They concluded that when agents are more sensitive to news about upcoming consumption than to news about distant consumption, then 1. agents prefer receiving early information rather than later; 2. agents boost consumption immediately but delay cuts; 3. Agents feel piecemeal information undesirable due to the diminishing marginal utility of loss. This model employs KR's setting of utility function–utility is composed of separable reference-independent "consumption utility" and reference-dependent "gain-loss utility". Different from their model, however, agents are no longer rational and can optimally manipulate their beliefs. It should be noted that Eliaz and Spiegler (2006) commented that "the model fails to account for a variety of realistic prior-dependent attitudes to information, which intuitively seems to originate from anticipatory feelings"[12]. Finally, Macera (2011) explored the time-path of subjective probability assessment. She built a two-state model in which an agent experiences gains and losses from the changes in anticipation and waits T periods for the realisation of the outcome. In each period, the agent makes assessment of her likelihood of success to maximize the intertemporal utility. One major conclusion is that the optimism decreases as the payoff date gets close because the threat of disappointment becomes significant. With application to the design of bonuses, she found that the decreasing path of optimism leads to a strong preference to bonuses. Thus optimal bonuses granted with a periodicity strengthen the motivation and restrain the payoffs in a reasonable size. Different from Macera (2011), this model explores the choice of subjective beliefs in the multi-state setting, where ex-post dissonance is excluded from the discussion.

2.2.3 Biased Beliefs in Asset Pricing

Barberis and Huang (2008) studied the asset pricing implications of the cumulative prospect theory in Tversky and Kahneman (1992), focusing on the probability weighting component.

[12] For instance, a patient who wants to have full knowledge of her medical condition when she is quite sure that she is in good health, yet does not want to know the whole truth when she is not so sure (Eliaz and Spiegler, 2006).

They proved that the CAPM still holds in a one-period equilibrium setting with normally distributed security payoffs and homogeneous investors evaluating the risk based on cumulative prospect theory. Moreover, they introduced an additional small, independent and positively skewed security into the economy and derived an equilibrium for homogeneous cumulative prospect theory investors. As the investors overweigh the tails of the portfolio and skewed security's return, they concluded that the skewed security can be overpriced and can earn a negative average excess return. A further extension to BP2005 is made in Brunnermeier, Gollier and Parker (2007) in which they built a general equilibrium model with complete markets. They showed that when investors hold optimal beliefs like in BP2005, portfolio choice and security prices match six observed patterns– 1. investors are not perfectly diversified due to biased beliefs; 2. the cost of biased beliefs puts limits on biases and make the utility cost not explosive; 3. Investors only over-invest in one Arrow-Debreu security and smooth the consumptions across all the other states because of the complementarity between believing a state more likely and purchasing more of the asset that pays off in that state; 4. Identical investors can have heterogeneous optimal portfolio choices since different households have various of states to be optimistic about; 5. investors tend to over-invest in the most skewed asset because the low-price and low-probability states are the cheapest states to buy consumption in. Thus, over-optimism about these states distorts consumption the least in the rest of the states; 6. more skewed assets provide lower returns because of the higher demand for them.

The structure of subjective beliefs in this model is compatible to the cumulative prospect theory, which serves as the foundation of Barberis and Huang (2008). Furthermore, the preference to pessimism may lead to different conclusions to Brunnermeier, Gollier and Parker (2007) in Arrow-Debreu asset pricing model. Though this thesis mainly focuses on the fundamental discussions of people's risk attitudes, light can be shed on asset pricing in markets with optimistic or pessimistic attitudes.

2.3 The Model

2.3.1 The Utility Function

Consider a multi-state model with two periods and one agent. There is a lottery with contingent payoffs and the associated distribution, known by the agent. It is assumed the agent has no control over the true distribution or the final outcomes but can deceive by manipulating the subjective beliefs. The first period corresponds to the time when the agent builds the beliefs and anticipation and the second period is the payoff realization period.

The agent at t=1 has an imminent utility from the anticipation of the future payoffs as well as a prospective gain-loss utility due to the difference between the payoffs actually realised at t=2 and the anticipation at t=1. The agent chooses the optimal subjective beliefs by considering both the anticipatory utility and the future prospective gain-loss utility. It is further assumed that the subjective beliefs remain constant over two periods, meaning that regression is not allowed once beliefs are formed. The model is formally described as following:

$\mathcal{S} = \{1, ..., S\}$, $S \geq 2$, is the set of states of nature and $Z_s = Z_1, ..., Z_S$ are the corresponding material payoffs, where $0 \leq Z_1 \leq Z_2 \leq ... \leq Z_S$. The objective probability associated with the state s is p_s, for $s = 1, ..., S$. Following the basic assumptions made in Koszegi and Rabin (2009), it is assumed that an agent's utility is separable and consists of a "consumption utility" and a "universal gain-loss utility" [13]; consumption utility takes the form of $u(Z_s)$, with $u' > 0, u'' \leq 0, u(0) = 0$, and the "gain-loss utility" takes the form of $\mu(x) = x$ for $x \geq 0$ and $\mu(x) = \lambda x \, (\lambda > 1)$ for $x < 0$.[14] For simplicity, $u(Z_s)$ is denoted by u_s and $u_1 \leq u_2 \leq ... \leq u_S$ by assumption. The anticipatory utility formed by the agent's subjective beliefs about the consumption utility realised in the second period is $\sum\limits_{s \in \mathcal{S}} q_s u_s$, where q_s is the subjective beliefs assigned to state s for $s = 1, ..., S$. The prospective gain-loss utility comes from the difference between the subjective beliefs and the objective ones is $\sum\limits_{s \in \mathcal{S}} p_s \mu(u_s - \sum\limits_{s \in \mathcal{S}} q_s u_s)$. The agent at t=1 chooses the optimal subjective beliefs to solve the following maximization problem:

$$\underset{\{q_s\}_{s \in \mathcal{S}}}{Max} \; U = \sum_{s \in \mathcal{S}} q_s u_s + \eta \{ \sum_{s \in \mathcal{S}} p_s \mu(u_s - \sum_{s \in \mathcal{S}} q_s u_s) \} \tag{2.1}$$

$$s.t. \quad \sum_{s \in \mathcal{S}} q_s = 1$$

[13]In Koszegi and Rabin (2009), the decision maker's period-t instantaneous utility u_t depends on the consumption in period t, and the changes in period t to beliefs about contemporaneous and future consumption:

$$u_t = m(c_t) + \sum_{\tau = t}^{T} \gamma_{t,\tau} N(F_{t,\tau} | F_{t-1,\tau}).$$

$m(c_t)$ represents the "consumption utility" and can be thought of as the classical reference-independent utility. $N(F_{t,\tau}|F_{t-1,\tau}) = \int_0^1 \mu(m(c_{F_{t,\tau}}(p)) - m(c_{F_{t-1,\tau}}(p)))dp$ represents the "gain-loss utility" and is derived from the changes in beliefs over future outcomes between periods, where $c_F(p)$ is the consumption level at percentile p and $\mu(\cdot)$ the universal gain-loss utility function. In this model, their model is simplified into two periods and it is assumed that there is no contemporaneous gain-loss utility at $t = 1$. Thus anticipation takes the value of expectation. At t=1, there is a prospective gain-loss utility due to the deviation of subjective beliefs from objective ones. It is assumed that $\gamma = 1$ since the effect of γ can be included in the effect of η under our linear setting of gain-loss utility function.

[14]In Appendix B, the case with a more general assumption $\mu'(-x) = \lambda(x)\mu'(x)$ is discussed, where $\lambda(x) > 1$, $x > 0$, and $\underset{x \to 0}{lim} \lambda(x) = 1$, $\mu''(x) \leq 0$. The conclusions are not different from the linear assumption case.

where $0 < \eta \leq 1$ is the weight on "gain-loss utility" with the weight on "consumption utility" normalized to 1. The upper bound of η is set to 1 to meet the revealed preference requirement.[15]. The weight on gain-loss utility can also be viewed as the discount rate of future utility. Overall, the target function in our model contains two parts: a traditional gain-loss utility with an endogenous reference point and an additional standard expected utility which we defined it as the anticipatory utility.

2.3.2 Optimal Beliefs

This section presents the fundamental properties of optimal beliefs in 2.1. All the proofs are given in Appendix A.

Proposition 1 *(Over-optimistic versus Over-pessimistic)*
Optimal beliefs defined by problem 2.1 feature the following properties:
(i) The probabilities of high(low) rank outcomes are over-estimated(under-estimated) if the objective probability of getting an outcome better than the objective expectation is high(low). That is, an agent is over-optimistic(over-pessimistic):

$$\sum_{s\in\mathcal{S}} q_s u_s > (<) \sum_{s\in\mathcal{S}} p_s u_s,$$

if $P_+^0 > (<) P^*$, *where* $P = \sum_{A^0} p_s with p_s the objective probability of state s$, $A^0 = \{s \in \mathcal{S} : u_s - \sum_{s\in\mathcal{S}} p_s u_s \geq 0\}$.

(ii) An agent is less likely to be over-optimistic if the agent is more loss averse and cares more about future utility, that is, P^* *is a probability that is uniquely determined by* λ *and* η. P^* *is increasing in* λ *and* η.
(iii)The optimal set of subjective beliefs $\{q_s\}$ *satisfies* $P_+ = P^*$ *and is not unique.*

Lemma 1 *(Biased Beliefs are Preferred to Rational Unbiased Beliefs):*
If $\lambda \geq \dfrac{1}{\eta}$, *then for* $s = 1, ..., S, S \geq 2$, *there exists at least one* $q_s \neq p_s$, *s.t.* $U_{BS} \geq U_{RE}$, *where* U_{BS} *and* U_{RE} *are the utilities under biased and rational beliefs, respectively.*

[15]Specifically, this assumption is required to meet the revealed preference between these two lotteries: $(0, 0)$ and $(1, 0)$ with probabilities $(p, 1-p)$. Obviously, $(1, 0) \succeq (0, 0)$ for any value of p. Therefore, $p + \eta(1-p) - \eta\lambda p \geq 0$. Then $\eta \leq \frac{-p}{1-p-\lambda p}$. For $p = \frac{1}{\lambda}$, it can be derived that $\eta \leq 1$.

Lemma 2 *(Beliefs Tradeoff among Different States):*
An agent prefers moving a small probability ε from a bad(good) state to a good(bad) state if the objective probability of getting an outcome better than the expectation is high(low):
$\forall k, l \in \mathcal{S}$ *with* $k > l$, *if* $P_+ > (<) P^*$, *then,*

$$U(q_k + \varepsilon, q_l - \varepsilon, q^-) > U(q_k, q_l, q^-),$$

where $\varepsilon > (<) 0$ *is a small number;* $P_+ = \sum_A p_s$, $A = \{s \in \mathcal{S} : \ u_s - \sum_{s \in \mathcal{S}} q_s u_s \geq 0\}$;
$P^* = \dfrac{\eta\lambda - 1}{\eta(\lambda - 1)}$.

Proposition 1 first formally defines over-optimistic and over-pessimistic in this paper. Agents are over-optimistic when the expectation under subjective probabilities is higher than the expectation under objective probabilities.

Lemma 1 further shows that an agent who cares about both anticipatory utility and prospective gain-loss utility are always either over-optimistic or over-pessimistic. The model rationalizes the existence of biased beliefs, when agents have no control power over the realisation of a gamble. Proposition 1 and Lemma 2 are closely related to each other. Lemma 2 describes the dynamic process of adjusting subjective probabilities among different states, while Proposition 1 illustrates the properties of subjective beliefs and expectations in the stable state.

A shared factor in the two propositions is the cut-off probability P^* which is uniquely determined by an agent's preference parameters– the intensity of loss-aversion λ and the weight on gain-loss utility η. For any lottery, an agent determines whether to further increase or decrease the subjective expectation by comparing the total chance of getting a gain with P^*. The cut-off rule implies that despite distortions in beliefs, there is a reasonable correspondence between true probabilities and optimal beliefs– the higher probabilities of good outcomes provides stronger reason to be optimistic.

Intuitively, people tend to be over-pessimistic and set a low anticipation to reduce the potential painful feeling of loss when the bad outcomes are more likely. On the contrary, when the chance for good outcomes is high, people tend to be over-optimistic and overestimate the chance of good outcomes, because the threat of loss is relatively weak and high anticipation is more beneficial. The intuition here is consistent with empirical evidence in the previous studies described in Sector 1.

In the optimised state, beliefs and expectations are adjusted to a level to make the total probability from "gain" states equal to the cut-off P^*. Whether the subjective expectation is above or below the rational expectation is determined by the real probability distribution of the lottery. In particular, the subjective expectation is higher(lower) than the rational level if the total probability of gains under rational expectation is smaller(greater) than P^*.

The second fold of proposition 1 is also very intuitive; a person who is more loss averse and cares more about the future utility tends to have low confidence. This intuition is directly reflected in the cut-off P^*; higher P^* leaves lesser room for optimism and those agents with high P^* turn into optimistic only if the lottery promises even higher chance of success. This result can be related to the empirical facts of Shepperd, Ouellette and Fernandez (1996). Taylor and Shepperd (1998) state that the level of optimism decreases as the realisation date comes closer, in which case, P^* gradually increases to 1 as the discount factor η approaches 1.

Finally, the optimal sets of beliefs are not unique. The cut-off rule only determines whether an agent is over-optimistic or over-pessimistic on average. Beliefs assigned to distribution tails can be opposite to the average trend, that is an optimistic agent can underestimate the chances of extremely good outcomes. Therefore, the structure of subjective beliefs is compatible with the Cumulative Prospect Theory proposed by Tversky and Kahneman (1992).

2.3.3 A Short Application: Information Timing Preference

The model above describes the process of self-deception—people directly choose their confidence level. Both casual observations and experimental evidence suggests that self-confidence has a lot to do with the information seeking behaviour. Following the work of KR2009 in which they studied rational agents' timing preference of information, this section extends the model to the preference of agents with optimistic and pessimistic biases.

This section follows the assumptions made above on utility functions. Now, the agent may receive an early signal $i \in I = \{1, 2, ..., S\}$ at t=1 about the future payoff and the signal is always correct, which means that the payoff realised at t=2 will be no different from what the agent has learned at t=1. Or, an agent can refuse to observe the early signal and wait until payoff is realised at t=2.
The agent's expected total utility from observing an early signal is:

$$U_{early} = \sum_{s\in\mathcal{S}} q_s u_s + \eta \cdot \sum_{s\in\mathcal{S}} q_s \mu(u_s - \sum_{s\in\mathcal{S}} q_s u_s) \tag{2.2}$$

The first term is the anticipatory utility stated as before, while the second term captures the expected prospective gain-loss utility in period 1 for an agent holding biased subjective beliefs $\{q_s\}$. As the signal is correct, no further gain-loss utility occurs in the second period. Compared with the previous situation, without early signal, the following propositions are proved to be valid (Details of the proof can be found in Appendix A):

Proposition 2 *(Information Timing Preference)*
An agent prefers (not) to receive early information about the payoff if the objective probability of getting an outcome better than the expectation based on objective beliefs is high(low). That is,

$$U_{early} > (<) U_{wait},$$

when $P_+^0 > (<) P^*$, where U_{wait} is the same total utility as defined at the beginning of this section. $P_+^0 = \sum_{A^0} p_s$, $A^0 = \{s \in \mathcal{S} : u_s - \sum_{s\in\mathcal{S}} p_s u_s \geq 0\}$ and $P^* = \dfrac{\eta\lambda - 1}{\eta(\lambda - 1)}$.

Proposition 2 states that if $P_+^0 > P^*$, an agent strictly prefers to receive the information early; if $P_+^0 = P^*$, agent is indifferent; if $P_+^0 < P^*$, agent prefers to stay uninformed. Intuitively, an optimistic agent tends to seek early information because it is believed that a good signal is more likely. Instead, early information is undesirable for a pessimistic agent as the agent is unwilling to expose to bad results early ex ante. Pessimistic agents prefer the gain-loss utility coming in the period of realisation as they overestimate the chance of loss in advance.

Furthermore, our conclusion extends the conclusion of KR2009. In KR2009, they proved that a rational agent weighing equally on "prospective gain-loss utilities" in both periods is indifferent between the early and later information[16]. Equation (2.2) is the respective utility function from observing early signal for an agent with constant weight on "prospective gain-loss utility"over periods [17]. Their intuition is that a rational agent is unbiased in the probabilities with which an early signal will move beliefs up and down. When the sense of loss for immediate and non-immediate outcomes is equally aversive, the rational agent is indifferent between early and late information. On the contrary, with the freedom in choosing subjective beliefs, agents exhibit a preference on information timing depending not only on their discount factor over future, but also on the distribution of the lottery they are playing with.

[16]Koszegi and Rabin (2009) built a multi-period model in which γ_{t_1,t_2} represents the strength of the concern in period t_1 for "prospective gain-loss utility" in period t_2. The prospective gain-loss utility stems from changes in beliefs between last period and this period in beliefs regarding future outcomes. Since there are only two periods in this model, γ_{t_1,t_2} is degenerated to γ since $\gamma_{t_1,t_1} = 1$ by their assumption.

[17]Formally, (2.2) can be written as

$$\sum_{s\in\mathcal{S}} q_s u_s + \eta \cdot \gamma \sum_{s\in\mathcal{S}} q_s \mu(u_s - \sum_{s\in\mathcal{S}} q_s u_s)$$

. When $\gamma = 1$, it becomes (2.2)

2.3.4 An Example

This section works out a simple two-state example, to serve as an antidote to the abstractness of the previous section. In this example, there are two possible outcomes, $x = 0, 1$ and $u(0) = 0, u(1) = 1$ with objective probability p and $1-p$, respectively. By holding subjective beliefs q and $1-q$, the anticipatory utility at t=1 is,

$$U_A = E_q u(x) = qu(1) + (1-q)u(0)$$

The prospective gain-loss utility from realised outcomes at t=1 is,

$$U_R = E_p \mu(u(x) - U_A) = p\mu(1-q) + (1-p)\mu(-q)$$

Total utility under subjective beliefs is,

$$U_{BS} = \eta p + (1-\eta\lambda)q + (\eta\lambda - \eta)pq;$$

Instead, an unbiased agent has:

$$U_{RE} = \eta p + (1-\eta\lambda)p + (\eta\lambda - \eta)p^2$$

It is easy to derive from here that the cut-off $P^* = \frac{\eta\lambda - 1}{\eta(\lambda - 1)}$, which is increasing in η and λ. An agent chooses $q > p$, thus is over-optimistic if and only if $p > P^*$. Otherwise, the agent chooses $q < p$ and stays in over-pessimistic if $p < P^*$. At $p = P^*$, $U_{RE} = U_{BS}$ for any q. Different from the multiple-state case, the discrete two-state setting has U_{BS} maximized at $q = 1$ for $p > P^*$, and at $q = 0$ for $p < P^*$ since p rarely equals P^*.

The agent's timing preference of information is further considered. The agent may receive an early signal $i = \{0, 1\}$ about the future outcomes at $t = 1$ and the signal is always correct. For an agent holding optimal biased beliefs, total utility by observing the early signal is,

$$U_{early} = q + \gamma\eta q(1-q) - \gamma\eta\lambda q(1-q),$$

where γ is the weight on prospective gain-loss utility following the assumption of KR2009. At $t = 1$, the agent holding biased beliefs believes that with probability q, she is going to observe $i = 1$—leading to an anticipatory utility $u(1)$ with certainty and a prospective gain compared to her prior q, and, with probability $1 - q$, the agent is going to observe $i = 0$—leading to an anticipatory utility $u(0)$ with certainty and a prospective loss. At t=2, as the agent has already updated the reference point to the right level, there will be no more gain-loss utilities in this period.

Instead, if the agent does not observe the signal, the total utility is the same as before:

$$U_{wait} = q + \eta p(1 - q) - \lambda\eta q(1 - p).$$

Hence, when $\gamma = 1$ as in KR2009, observing the signal generates strictly more expected utility than not observing it if and only if

$$(q - p)(1 - q) > \lambda(p - q)q. \tag{2.3}$$

Since only one side of eq. (2.3) can be greater than 0,
$U_{early} > U_{wait}$ if and only if $q > p$, that is $p > P^*$. Otherwise, if $q < P^*$, then $U_{early} < U_{wait}$, the agent will prefer staying uninformed.

This conclusion is intuitive. People being over-optimistic over payoffs are also over-optimistic in believing that they will get good news. Thus early news provides additional utility to their anticipation. In the real world, over-optimistic people are more likely to search for information than over-pessimistic people. This conclusion contains and further extends KR's conclusion on timing preference of information. KR2009 proved that people will be indifferent to the timing of information, when the sense of loss is exactly as aversive in period 1 as in period 2, that is, $\gamma = 1$. Our model repeats their conclusion when agents are rational. However, with biases in beliefs, individuals will have preferences over timing of information even if they have equal sense on prospective and realised gain-loss utilities, and the preferences of early and later information depends on their loss-aversion attitudes, weights on anticipation against realisation and the real chance of a good outcome.

For the case $\gamma < 1$, it is easy to prove that for people holding $q > p$, $U_{early} > U_{wait}$ and early information is good; whereas for $q < p$, U_{early} can be greater, equal to or smaller than U_{wait} for some value of q. Compared with the case in which $\gamma = 1$, there is a greater chance that people will prefer early information. KR's explanation is applicable here as the sense of loss for non-immediate outcomes is not as intense, so the agent is better off by receiving the information early. Similar analysis can be applied to $q < p$.

2.4 Risk Attitudes

This section explores some significant implications of the most fundamental premise of the model—risk attitudes under subjective beliefs with application to the choice between two independent lotteries. Section 3 has analysed the discrete multi-state case. The following study extends the previous conclusions by looking at the case of continuous distributions. The new assumption makes no changes to conclusions in section 3 while it avoids jumps of subjective expectation when moving beliefs from one state to another[18]. To better understand the effects of reference point on risk attitude, this section simply assumes that the consumption utility takes the linear format $u(x) = x$, in which case, agents are risk-neutral in the absence of gain-loss utility. The lottery has continuous distribution, which can be symmetrical or skewed. Agents are further categorised into two types—“naive” and “sophisticated” based on their different cognitive processes. Both types form their subjective beliefs as previously stated in section 3. However, when gambling, the “sophisticated” type recognises their cognitive biases and make decisions accordingly, while the “naive” type fails and behaves like EU maximizer without gain-loss utility. Another rationale for this separation is due to the discussion given by BP2005, in which they explored the behavioural implications of the “naive” type under our definition. In their portfolio choice application, the agent forms optimal subjective beliefs by maximizing the total intertemporal utility, while chooses optimal investment strategy only to maximize current anticipation. One straightforward argument is that agent can choose actions consistent with beliefs in maximizing the total intertemporal utility—leading to the discussion of sophisticated agents in our model[19]. Based on the continuity assumption and classification of agents, this section explores the risk attitudes implications of the model described in section 3.

2.4.1 Naive Agent

This section begins by defining the lotteries with continuous probability density function (Assumption 1) and the corresponding continuous objective beliefs (Assumption 2) followed by the formal definition of “naive”agents (Definition 1).

Assumption 1 *Z_A and Z_B are the contingent payoffs of two independent lotteries with continuous probability distribution functions $f_A(\cdot)$ and $f_B(\cdot)$ respectively.*

[18]For example, under discrete multi-state case, a small increase in subjective belief of a high rank state may or may not change some other states from gain to loss, while the gain-loss switch always happens to at least one state under continuous distributions.

[19]Consistent actions are chosen simultaneously with beliefs in the sophisticated case. Since there are only two periods in our model, we argue that agents can aim at long-term interests maximization in choosing actions.

Assumption 2 *An agent evaluates the payoffs of two lotteries separately and has optimal subjective beliefs* $g_A(\cdot)$ *and* $g_B(\cdot)$ *which are solutions to the following problem:*

$$\underset{g_i(\cdot)}{max} E_{g_i}(Z) + E_{f_i}\eta\mu[Z - E_{g_i}(Z)], i = A, B.$$

Definition 1 *(Naive Agent) For any two lotteries A, B satisfying Assumption 1, a naive agent is the one with optimal beliefs described by Assumption 2 and prefers the lottery with a higher value of:*

$$\underset{g_i(\cdot)}{Max} E_{g_i(\cdot)}(Z), i = A, B,$$

Definition 1 defines the "naive "type by describing their decision making behaviour. The naive agent, in making choice on a lottery, only considers the expected utility from the lottery just as a traditional expected utility maximiser in a classic framework, even though they form their subjective beliefs based on both the anticipation utility and the gain-loss realisation utility. By making this assumption, we are assuming the naive agent take the subjective beliefs as given in making decisions and fail to realise that biases exist in their beliefs. Another interpretation is that the naive agents are myopic and only care about their anticipation utility in taking actions. In this section, we assume the actions are discontinuous–that is, agents can only choose one of the lottery but not a combination of them. Besides, for simplicity, utility functions are assumed to be risk-neutral and linear. In the next section, continuous actions will be introduced. However, throughout the paper, the naive agents are always defined to make decisions based only on anticipation utilities.

Proposition 3 *(Two Symmetrically Distributed Lotteries)*
Suppose two lotteries A and B satisfy the assumptions 1-2 and the following conditions:
(i) $E_{f_A(\cdot)}(Z) = E_{f_B(\cdot)}(Z)$, *for* $f_A(\cdot) \neq f_B(\cdot)$*;*
(ii) Z_A *and* Z_B *are both symmetrically distributed;*
(iii) Z_A *and* Z_B *satisfy the single-crossing property, that is, if* $F_A(\cdot)$ *and* $F_B(\cdot)$ *are the cumulative distribution functions for* Z_A *and* Z_B*, there exists* z *such that* $F_A(x) < F_B(x)$ *for* $x < z$ *and* $F_A(x) > F_B(x)$ *for* $x > z$*.*
A naive agent defined in Definition 1 has the following preference over two lotteries:

$$Lottery\ A \succ Lottery\ B\ (Lottery\ B \succ Lottery\ A)$$

with $E_{g_A(\cdot)}(Z) > E_{g_B(\cdot)}(Z) (E_{g_B(\cdot)}(Z) > E_{g_A(\cdot)}(Z))$ *if the cut-off probability* $P^* \geq 1 - F(z) (P^* \leq 1 - F(z))$

Proposition 3 says that a naive agent with optimal beliefs is risk-seeking if optimistic and risk-averse if pessimistic.

Intuitively, low P^* implies weak loss-aversion and less valued future. Loss in the future is therefore bearable in this case. Higher risk lottery provides better chances of gains on the right tail of the distribution, which is treated as stronger evidence of a promising payoff by an optimistic agent—leading to further up-biases. The positive prospect from over-estimated chances of gains can significantly increase their anticipatory utility, while the negative prospect from more painful feeling of loss are countered by the further under-estimation of probabilities in the bad outcomes.

For symmetrical distributions with single-crossing property, the risky lottery gives a higher cumulative probability beyond the crossing point, meaning that the risky assets promise a higher chance of gain over a certain level of expectation. Therefore, a naive agent with a low cut-off P^* is more up-biased in lotteries with fat tails.

This proposition implies that a mean-preserved spreading is desirable for an optimistic agent and undesirable for a pessimistic one even if agents are risk-neutral under traditional economics definition. Based on the proof of Proposition 3 and the intuitions described above, the following lemmas are derived:

Lemma 3 *(Optimistic and pessimistic risk attitude)*
For the group of symmetric distributions, a mean-preserving spreading is desirable for an optimistic agent and undesirable for a pessimistic agent.

Lemma 4 *(Ranking of subjective expectations)*
For any distribution, subjective expectation is non-increasing in P^*.

Furthermore, a more general version of Proposition 3 is considered by relaxing the requirement on symmetry:

Proposition 4 *(Two Lotteries: The General Case)*
Consider two independent lotteries A and B with continuous and differentiable distribution functions $f_A(\cdot) \neq f_B(\cdot)$ and $E_{f_A(\cdot)}(Z) = E_{f_B(\cdot)}(Z)$. For an agent with cut-off probability $P^* = \dfrac{\eta\lambda - 1}{\eta(\lambda - 1)}$, we have:

(i) If $P^0_{+A} > (<) P^*, P^0_{+B} < (>) P^*$, then
$E_{g^*_A}(Z) > E_{g^*_B}(Z)$ $(E_{g^*_A}(Z) < E_{g^*_B}(Z))$;

(ii) If $P^0_{+A} > (<) P^*, P^0_{+B} > (<) P^*$, then
$E_{g^*_A}(Z) > E_{g^*_B}(Z)$ $(E_{g^*_A}(Z) < E_{g^*_B}(Z))$
iff $\int_a^{+\infty} [f_A(Z) - f_B(Z)]\mathrm{d}Z > 0$ $\left(\int_b^{+\infty} [f_B(Z) - f_A(Z)]\mathrm{d}Z > 0\right)$,

where a, b, P^0_{+A}, P^0_{+B} *are*
$P^* = \int\limits_a^{+\infty} f_A(Z)\mathrm{d}Z = \int\limits_b^{+\infty} f_B(Z)dZ,\ P^0_{+A} = \int\limits_{E_{f_A}}^{+\infty} f_A(Z)\mathrm{d}Z,\ P^0_{+B} = \int\limits_{E_{f_B}}^{+\infty} f_B(Z)\mathrm{d}Z.$

Notice that a and b in Proposition 4 are the subjective expectations under optimal beliefs. Since optimal expectations are set at the level to ensure the cumulative probability above it equals P^*, a and b are also indicators of the distribution skewness. For any given P^*, a higher value of a means a fatter "right tail" of the distribution and therefore a negative skewness[20]. As long as the mean remains the same, an optimistic agent has a decreasing preference to lotteries as the skewness of distribution changes from negative to positive.

2.4.2 Sophisticated Agent

Definition 2 *(Sophisticated Agent) For any two lotteries A and B satisfying Assumption 1, a sophisticated agent is the one with optimal beliefs described by Assumption 2 and prefers the lottery with higher value of*

$$\underset{g_i(\cdot)}{Max} E_{g_i}(Z) + E_{f_i}\eta\mu[Z - E_{g_i}(Z)], i = A, B,$$

Definition 2 formally defines the "sophisticated" agents by defining their decision making problem. In contrast to the "naive" type, who only considers the expected (anticipation) utility in decision marking, the " sophisticated" type considers total of the anticipation utility and the gain-loss realisation utility.

Proposition 5 *(Choice between Two Lotteries: Sophisticated Case)*
If assumptions 1,2 hold, then for any two risky lotteries with $f_A(\cdot) \neq f_B(\cdot)$*, a sophisticated agent strictly prefers lottery A to B iff*

$$\int_{E_{g_A}}^{+\infty} f_A(Z)Z\mathrm{d}Z + \lambda \int_{-\infty}^{E_{g_A}} f_A(Z)Z\mathrm{d}Z > \int_{E_{g_B}}^{+\infty} f_B(Z)Z\mathrm{d}Z + \lambda \int_{-\infty}^{E_{g_B}} f_B(Z)Z\mathrm{d}Z$$

Specifically, for two risky lotteries with equal objective expectation, i.e., $E_{f_{A(\cdot)}}(Z) = E_{f_{B(\cdot)}}(Z)$*, a sophisticated agent prefers lottery A to B iff,*

$$\int_{-\infty}^{E_{g_A}} f_A(Z)Z\mathrm{d}Z > \int_{-\infty}^{E_{g_B}} f_B(Z)Z\mathrm{d}Z$$

[20]The "right tail" does not only refer to the tail of a distribution. It represents the area under the p.d.f. from the subjective expectation controlled by P^* to the right limit.

Proposition 5 shows that for any two lotteries with the same statistical mean, the lottery that gives higher value in the loss region are preferred by the "sophisticated"type. In general, assuming optimism and the same statistical mean for both lotteries, from the perspective of the "sophisticated" type, the more negative-skewed or more spreading the objective probability distribution is (which leads to more optimistic beliefs), and the higher the values of the bad cases (lower exposure to losses), the better the lottery is.

As we can see from Proposition 5, the objective function of a sophisticated agent is just the expectation under rational beliefs but weighing the loss region with λ. Without loss aversion, i.e., $\lambda = 1$, the objective function of an sophisticated agent is simplified to the ordinary rational expectation.

Proposition 5 indicates that a sophisticated agent behaves similar to a rational agent who maximizes the expectation with unbiased beliefs. There is no surprise in this conclusion since a sophisticated agent considers both the anticipation and the deterioration in realization from the "reference effect" . Here, the "reference effect" denotes the deduction in realized utility arising from comparing an outcome with the anticipation(reference point). We notice that the direct effects from subjective beliefs, i.e., anticipatory utility, is eliminated by the "reference effect" at optimal beliefs as is proven in Appendix A. Intuitively, if the anticipation is too high, then the "reference effect" provides too many chances of loss. Since the agent is loss averse, decrease in total utility due to more states of loss exceeds the increase in anticipatory utility. Therefore, lower anticipation is beneficial. On the other hand, with low anticipation, lesser chances of loss makes the punishment from the "reference effect" insignificant. Since people discount the future utility, benefits from higher anticipation will exceed the decrease in future realised utility. At optimal beliefs, illusions from subjective beliefs on anticipation and reference point cancels out each other, leaving only the truth of the contingency.

The objective function here deviates from the rational expectation only in the loss region. This difference comes directly from the assumption of loss aversion since loss averse agent overweighs the bad outcomes below their expectation.

2.5 Application: Portfolio Choice

This section further explores the decision making problem under biases beliefs with concave utility function and continuous actions. Specifically, this section explores the model implications in the investment problem—a biased investor with concave utility function chooses the optimal portfolio consisting of a risky asset and a risk-free asset. Following the assumption made in the previous section, it is assumed that there exists two types of investors: the naive ones who maximize the expected returns and the sophisticated ones who maximize the total utility, including both the anticipatory and prospective gain-loss utilities. We points out that the naive investor and the sophisticated investor can adopt opposite investment strategies even though they form their subjective beliefs through the same cognitive process. An equilibrium price is further derived in a market with identical investors to look into the implications on asset pricing.

2.5.1 Choice Between One Risk-free Asset and One Risky Asset

The last section considers the discrete actions of choosing two lotteries. This section extends our previous analysis to continuous choices with concave utility function. For simplicity, a similar problem as in BP2005 is analysed, in which an agent allocates the wealth between a risk-free asset a risky asset.

Assumption 3 *The return of an asset has a continuous distribution function $f(\cdot)$ defined over $(-\infty, +\infty)$. The investor holds subjective belief $g(\cdot)$ of the asset's return, which is also a continuous p.d.f. on $(-\infty, +\infty)$.*

There are two assets in the market– a risk-free asset with return R_f; and a risky asset with gross return $R_R = R_f + R_m = R_f + E_{f(\cdot)}(R)$, where R_m is the gross excess return, R the realised excess return. Investors have unlimited access to the risk-free asset and the price of the risk-free asset is 0. The consumption utility takes the same assumption as in section 3– $u'(x) > 0, u''(x) \leq 0$. There are two periods. At t=1, the agent forms the subjective optimal beliefs $g(R)$ about payoffs of the risky asset and allocates the unit endowment between these two assets. In the second period, the payoffs of the assets are realised.

Following our previous categorisation of "naive" and "sophisticated" agents, we assume that the naive type has different objective functions in choosing optimal beliefs and optimal α. The choice of portfolio is "rational" based on their biased beliefs. The sophisticated type chooses optimal subjective beliefs and optimal α allocated to the risky asset simultaneously. To be more specific, at t=1, for any given optimal beliefs $g(R)$, the naive agent chooses the portfolio share, α^{BS} allocated to the risky asset to maximize the expected return:

$$\underset{\alpha}{Max} \int_{-\infty}^{+\infty} g(R)u(R_f + \alpha R)\mathrm{d}R.$$

Given the optimal choice of α^{BS}, the naive agent chooses the subjective beliefs $g(\cdot)$ to solve:

$$\underset{g(R)}{argmax}\ E_{g(R)}u(R_f + \alpha^{BS}R) + E_{f(R)}\eta\mu[u(R_f + \alpha^{BS}R) - E_{g(R)}u(R_f + \alpha^{BS}R)]$$

Instead, a sophisticated agent chooses her optimal portfolio share α and optimal beliefs $g(R)$ simultaneously to solve:

$$\underset{g(R),\,\alpha}{argmax}\; E_{g(R)}u(R_f + \alpha R) + E_{f(R)}\eta\mu[u(R_f + \alpha R) - E_{g(R)}u(R_f + \alpha R)]$$

Before presenting the portfolio choice results for both types, we first gives the model details in Brunnermeier and Parker (2005) for comparison purpose. Brunnermeier and Parker (2005) built a discrete model in which an agent has optimal beliefs determined by

$$\underset{\{q_s\}}{argmax}\; \sum_{s=1}^{S} q_s u(R_f + \alpha^{BS} R_s) + \sum_{s=1}^{S} p_s u(R_f + \alpha^{BS} R),$$

where α^{BS} is the solution to $\underset{\alpha}{max}\; \sum_{s=1}^{S} q_s u(R_f + \alpha R_s)$. This is a special case of the naive agent in our model when $\lambda = 1$ and $\eta = \frac{1}{2}$, that is agent is not loss-averse and evaluates the gain-loss utility equally as the anticipation utility.

BP2005 concluded that optimal-belief holders always trade more aggressively than the rational agents since the cost from distorted portfolio choice is second order, while the benefit from a higher anticipation is first order. That is, $\alpha^{BS} < \alpha^{RE} < 0$ or $0 < \alpha^{RE} < \alpha^{BS}$ where α^{RE} is the rational allocation.

The following proposition indicates that reference-dependent loss-averse investors no longer trade aggressively all the time. Even though the cost of distorted portfolio is still second order, biased beliefs not only generate first-order benefits from higher anticipation, but also give first-order punishment from the "reference effect". Therefore, agents in our model adopt aggressive or conservative trading strategy depending upon their optimistic and pessimistic attitudes.

2.5.1.1 Naive Agent

Proposition 6 *(Risk Taking due to Optimism and Pessimism: Naive Case):*
An optimistic investor with low P^* invests more aggressively than a rational investor or in the opposite direction; a pessimistic investor with high P^* invests in the same direction as the rational investor, but more conservatively.
If $E(R) > 0$, then $\alpha^{RE} > 0$,

$$\alpha^{OP} > \alpha^{RE} > 0 \text{ or } \alpha^{OP} < 0 < \alpha^{RE};\; 0 < \alpha^{PE} < \alpha^{RE}.$$

If $E(R) < 0$, *then* $\alpha^{RE} < 0$,

$$\alpha^{OP} < \alpha^{RE} < 0 \text{ or } \alpha^{OP} > 0 > \alpha^{RE};\ \alpha^{RE} < \alpha^{PE} < 0.$$

Similar to BP2005, the optimal α^{BS} under biased beliefs is always different from α^{RE} since biased beliefs ensure higher total welfare as proved in Proposition 1. Besides, the optimistic investor trades in the same direction, but more aggressively than a rational agent since the agent overestimates the chance of good returns[21]; or enters into a position opposite to the rational strategy. As stated at the beginning of this section, the investor in BP's model can be described as an optimistic agent with $P^* = 0$ in our model and the behavioural implication on an optimistic agent here can also be explained by BP's intuition: opposite trading happens when the asset is skewed enough in the opposite direction of the mean payoff.

Different from BP's conclusion, the investor in our model no longer always holds "optimistic" beliefs and invests aggressively. Instead, a pessimistic investor characterised by a high P^* invests in the same direction, but more conservatively than the rational counterpart[22]. Agents in BP's model always trade aggressively because the additional anticipation is a pure generator of felicity[23]. Although optimism and pessimism are both punished for distorted portfolio choices, the former dominates the latter because up-biases provide extra felicity from the higher anticipation, whereas down-biases do not.

Moreover, an agent with a low P^* in our model tends to overestimate the chance of good returns[24] no matter whether they have a short or long position. However, attitudes can also be strategy-dependent—the same person in a long position can exhibit an attitude opposite to that when in a short position. For example, facing a negative skewed asset with $E(R) < 0$, an investor enters into a short position will be pessimistic since the long tail on the left gives few chances of gains; the same investor in the long position instead can be optimistic, since the fat tail on the right gives high chance of gain. The choice of attitudes depends on the investment position, as well as the skewness of the risky asset.

[21]Returns are good or bad conditional on the long-short position. A positive return is good conditional on long and is bad conditional on short.

[22]"Optimistic" and "pessimistic" are defined differently from Brunnermeier and Parker (2005). BP's "pessimistic" investor assigns higher probabilities to negative returns while shorting the asset. "Pessimistic" defined in this model means "conservative" : pessimistic investors assign lower probabilities to negative returns while still shorting.

[23]Mathematically, BP's model requires $\alpha^{BS} > \alpha^{RE} > 0$ because their first order maximization problem requires this condition. The envelope condition is $(u_{s''} - u_{s'})\Delta\hat{\pi} + \eta\Sigma p_s u'(R_f + \alpha^{BS} R_s) R_s \Delta\alpha = 0$. When $\alpha^{RE} > \alpha^{BS} > 0$, $\Sigma p_s u'(R_f + \alpha^{BS} R_s) R_s > 0$, then bias in beliefs serves as a utility pump thus gives no limits to bias.

[24]Whether a return is good or bad is conditional on the position of the asset, i.e., a positive return is good conditional on long and bad conditional on short.

Finally, we point out that our conclusion partially depends on the construction of the optimization problem. By assuming that the agent takes beliefs as given when making portfolio choice and fails to recognise the biases, it is implicitly indicated, in a non-strict way, that the agent in our model chooses α after choosing the beliefs. Therefore, the deviation of α from rational level, like BP said, only puts on second order cost, whereas changes in beliefs introduce first order increase in anticipation and hence dominates the total effects. However, an agent can behave more complicatedly than the described naive type—when choosing the optimal portfolio, the agent realises the biases in beliefs and makes the investment decisions accordingly. In the following part, the behaviour of a sophisticated agent in the portfolio choice problem is further discussed.

2.5.1.2 Sophisticated Agent

Proposition 7 *(Risk Taking due to Optimism and Pessimism– Sophisticated Case):*

(i) Sophisticated investors invest more aggressively than the rational investors if the average return, conditional on loss is good enough. In other cases, sophisticated investors invest either more conservatively or in the opposite direction to the rational strategy. If $E(R) > (<)0$ *and* $\int_{-BS} f(R)u'(R_f + \alpha_{RE}R)R\mathrm{dR} > (<)0$*, then,*

$$\alpha^{BS} > \alpha^{RE} > 0(\alpha^{BS} < \alpha^{RE} < 0).$$

If $E(R) > (<)0$ *and* $\int_{-BS} f(R)u'(R_f + \alpha_{RE}R)R\mathrm{dR} < (>)0$*, then*

$$\alpha^{BS} < \alpha^{RE}(\alpha^{BS} > \alpha^{RE})$$

where α^{BS} *and* α^{RE} *are the optimal allocation of wealth on risky assets and "-BS" indicates the region of loss under biased beliefs taking* α_{RE} *as given.*

(ii) α^{BS} *decreases in* P^* *if* $R_{CE} > 0$*;*
α^{BS} *increases in* P^* *if* $R_{CE} < 0$*,*
where $E_{g(\cdot)}u(R_f + \alpha R) = u(R_f + \alpha R_{CE})$ *and* $R_f + R_{CE}$ *is certainty equivalent .*

The behaviour of sophisticated agents is more complicated as described by proposition 7. Like our analysis in section 4, a sophisticated agent considers both anticipation and the "reference effect" in making the portfolio choice. At optimal beliefs, these two effects cancel out each other, leaving only rational-like expected utility. Unlike naive agents, the biased beliefs no longer determine actions directly through expectation, but they still affect actions through the choice of reference point, which determines the sensitivity towards future payoffs. Basically, optimism, by introducing in more losses, also makes an agent more sensitive towards outcomes, while pessimism, by reducing the fear of loss, makes an agent less sensitive to outcomes.

The first part of Proposition 7 describes the difference in portfolio choice strategies between a sophisticated and a rational agent, which lies in whether they overweigh the returns in loss region. The intuition is simple; compared with a rational agent, a loss-averse agent cares more about low-rank outcomes. Therefore, when low-rank returns are good on average, loss-averse agents are happier than the rational ones and more willing to take excess risks. On the other hand, bad returns on the low-rank outcomes are more painful for loss-averse agents and thus lead to conservative or opposite trading strategy.

The second part of Proposition 7 examines the effects of optimistic and pessimistic attitudes on investment strategies. To understand the intuition more clearly, consider the following lemma directly derived from Proposition 7 part (ii).

Lemma 5 *Optimistic agents invest more aggressively than pessimistic agents when the return is perceived to be good;*
Pessimistic agents invest more aggressively than optimistic agents when the return is perceived to be bad. That is,
for $\alpha^{BS} > (<) 0$ *and* $R_{CE} > 0$*, we have,*

$$\alpha^{OP} > \alpha^{PE} > 0 (\alpha^{PE} < \alpha^{OP} < 0);$$

for $\alpha^{BS} > (<) 0$ *and* $R_{CE} < 0$*, we have,*

$$\alpha^{PE} > \alpha^{OP} > 0 (\alpha^{OP} < \alpha^{PE} < 0).$$

To better understand the intuition, consider the following example. Suppose there is an investor in the long position of a security and the security gives good returns on average in the future, which is known by the investor. An optimistic investor who takes large chance of future loss into consideration cares more about the returns due to the fear of loss. Since loss is the source of intensive feelings as compared to gains, more attention to the asset gives stronger feeling of returns. Hence, when returns are good in general, an optimistic investor is happier than a pessimistic one and is willing to hold more risky asset.

On the contrary, pessimism reduces the fear for loss and makes the agent less sensitive, and thus the agent cares less about future outcomes . Therefore, when the asset is a bad one, pessimism and numbness make the bad outcomes more tolerable and the pessimistic agents can bear extra risks than optimistic ones.

Compared with the bad-return situation, when the asset gives good return as in the first case, even though staying numb can make pessimistic ones avoid intensive feelings from bad outcomes, there is less chance to feel bad as the asset is a good one in general. Therefore, optimism is more beneficial because it intensifies the happiness and enables optimistic investors to bear more risks than pessimistic ones.

2.5.2 Equilibrium

In this section, we place the portfolio choice problem in an exchange economy with two assets, a free risk-free one with $R_f = 0$ and price $\pi_f = 0$ and a risky one with stochastic excess return R and price π. The distribution of R is publicly known. We assume the short-sale constraint binds, thus the proportion of wealth allocated to the risky asset α satisfies $0 \leq \alpha \leq 1$. The assumption is further simplified by making utility function take the linear format $u(x) = x$.

The simplest candidate equilibrium is a homogeneous holdings equilibrium–an equilibrium in which investors are identical to each other and hold the same portfolio. As it is proven later, the optimal portfolios for both naive and sophisticated agents are either only risky asset or only risk-free asset under the linear utility function assumption. Hence, the equilibrium price, which is constructed by such kind of portfolio choice must ensure investors are indifferent between two assets. The simplest case is a market with only rational EU maximising investors. The equilibrium price is set to make $R_f = \eta E_f - \pi_{RE}$ and the equilibrium price of the risky asset at t=1 is,

$$\pi^*_{RE} = \eta E_f(R).$$

2.5.2.1 Naive

The optimisation problem of the naive agent is now described by equaltions 2.4 and 2.5

$$g^*(R) = \underset{g(R)}{argmax} \int_{-\infty}^{+\infty} g(R)(R_f + \alpha^* R)\mathrm{dR} + \int_{-\infty}^{+\infty} f(R)\eta[R_f + \alpha^* R - \int_{-\infty}^{+\infty} g(r)(R_f + \alpha^* r)\mathrm{dr}]\mathrm{dR} \quad (2.4)$$

$$\alpha^* = \underset{\alpha \in [0,1]}{argmax} \int_{-\infty}^{+\infty} g^*(R)(R_f + \alpha R)\mathrm{dR}, \quad (2.5)$$

Solutions to the above optimisation problem with $R_f = 0$ are the following:

$$\alpha^* = 1 \text{ if } \quad E_{g^*}(R) \geq 0;$$
$$\alpha^* = 0 \text{ if } \quad E_{g^*}(R) \leq 0.$$

Given that in the equilibrium, the price of the risky asset π_N at $t = 1$ must satisfy $R_f = \eta E_{g^*}(R) - \pi_N$, the equilibrium price in a market with pure naive investors is

$$\pi_N^* = \eta E_{g^*}(R)$$

2.5.2.2 Sophisticated

Now consider the security pricing if all the investors are sophisticated. The optimization problem is defined by equation 2.6,

$$\max_{g(R),\alpha} \int_{-\infty}^{+\infty} g(R)\left[R_f + \alpha R\right] \mathrm{dR} + \int_{-\infty}^{+\infty} f(R)\eta \left\{ R_f + \alpha R - \int_{-\infty}^{+\infty} g(r)\left[R_f + \alpha r \mathrm{dr}\right] \right\} \mathrm{dR} \tag{2.6}$$

From the first order condition with respect to α, we have,

$$\alpha = 1 \text{ if } \eta \int_{-\infty}^{+\infty} f(R)R\mathrm{dR} + \eta(\lambda - 1) \int_{-\infty}^{E_{g^*}} f(R)R\mathrm{dR} - \pi_S > R_f;$$
$$\alpha = 0 \text{ if } \eta \int_{-\infty}^{+\infty} f(R)R\mathrm{dR} + \eta(\lambda - 1) \int_{-\infty}^{E_{g^*}} f(R)R\mathrm{dR} - \pi_S < R_f.$$

In equilibrium, the price of the risky asset π_S must satisfy

$$\eta \int_{-\infty}^{+\infty} f(R)R\mathrm{dR} + \eta(\lambda - 1) \int_{-\infty}^{E_{g^*}} f(R)R\mathrm{dR} - \pi_S = R_f$$

Therefore, the equilibrium price for a market with identical sophisticated agents is:

$$\pi_S^* = \eta E_f(R) + \eta(\lambda - 1) \int_{-\infty}^{E_{g^*}} f(R)R\mathrm{dR}.$$

2.5.2.3 Equilibrium

To gain further intuition on the equilibrium, consider a normal distributed asset with $R \sim N(1, 1)$. λ is set to 2.25 as suggested by experimental evidence . Variation in η changes P^* from 0 to 1[25]. The result is shown in Figure2.1.

[25] λ and η can change simultaneously, however, for calibration purpose, we set lambda at 2.25.

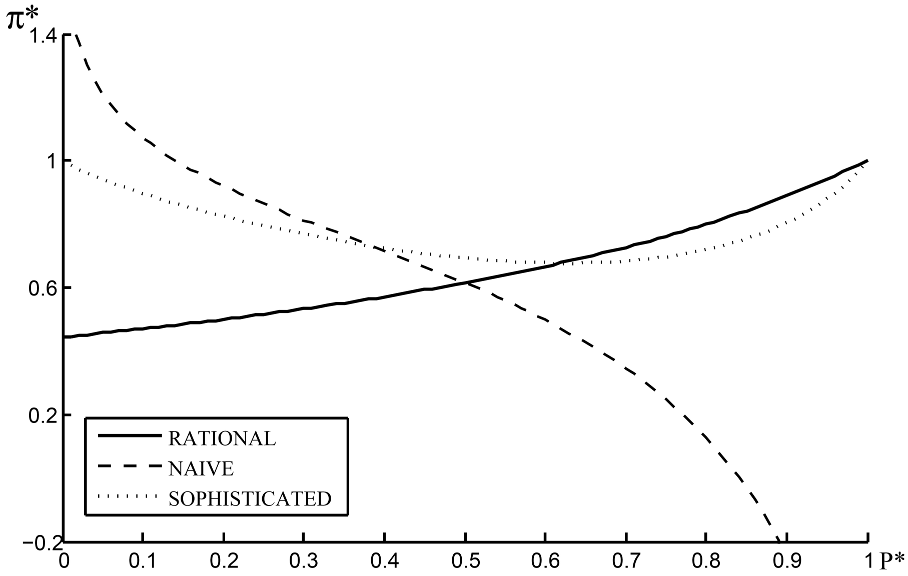


Figure 2.1: EQUILIBRIUM PRICE π^* AS A FUNCTION OF P^*

The solid line in Figure 2.1 plots the rational price as a function of P^*. Since P^* is increasing in η, the equilibrium price increases to 1 as the discount rate η goes to 1. The dash line plots the equilibrium price against P^* in a market with naive investors. Consistent with the prediction of Proposition 6 and since the short-sale constraint binds , the equilibrium price of the risky asset decreases as investors become more pessimistic. To be more specific, the "naive" price is higher than the rational level when the market is dominated by optimistic investors because the up-biased beliefs urge the investors to hold more risky asset. Instead, the "naive" price falls below rational level when investors are pessimistic. Pessimistic investors require higher equity premium to compensate for their stronger aversion to loss.

The dotted line plots the "sophisticated" equilibrium price as a function of P^*. Compared with the "naive" market, the sophisticated price exhibits a U-shaped pattern rather than strictly decreasing in P^*. The equilibrium price is upper bounded by the rational expectation reflecting our conclusion that sophisticated agents act similarly to rational agents. Moreover, consistent with the prediction of Proposition7, both the optimistic and the "very" pessimistic agents price the assets higher than moderate pessimistic agents. With short-sale constraint, optimistic agents for this symmetrically distributed asset are those with $P^* < 0.5$ while pessimistic agents have $P^* > 0.5$. Since $\mu = 1$, $R_{CE}^{OP} = E_g(R) > 0$ and utility increases as agents become more optimistic, so does the price. The average return captured by the optimistic agents is always greater than 0, therefore, the asset is a good one and the higher reference point intensifies the happiness for good returns.

On the contrary, for "extremely" pessimistic agents (in this case, $P^* > 0.63$), they have $R_{CE}^{PE} = E_g(R) < 0$ and the risky asset becomes more appealing as agents turn more pessimistic. As the return of the asset is not good enough for those very pessimistic agents to be up-biased, it is subjectively captured to be bad. Further pessimism reduces sensitivity towards bad returns and increases tolerance of risks.

Finally, "moderately" pessimistic agents who have $0.5 < P^* < 0.63$ in this example, correctly capture the goodness of the asset, that is they have $R_{CE}^{PE} = E_g(R) > 0$. They are pessimistic because the asset fails to give sufficient chances of good returns to persuade them to be optimistic. By being more pessimistic, they reduce the deserved happiness from good outcomes instead of reducing the painfulness from bad outcomes. Therefore, they are the most "risk-averse" ones among these three groups.

Furthermore, rational price is lower than "sophisticated" price when investors are optimistic and exceeds the latter as investors become more pessimistic. Consistent with the first part of Proposition 7, since $\mu > 0$, the average return of the loss region increases from negative to positive as confidence increases. Consequently, intensive feelings on bad returns push down the sophisticated price, when market is pessimistic and boosts it up when the market is optimistic.

2.6 Conclusion

This thesis develops a model of optimal judgemental biases with reference point. The model gives an insight into the rationale of over-optimism and over-pessimism by applying two behavioural assumptions—reference dependent utility and loss aversion into an intertemporal model. Contrary to the previous literatures on reference dependent utility models in which it is usually assumed that the utility is derived from beliefs, in our model, beliefs are optimally determined by the inverse process through utility maximization. Our model setting internalizes the over and under confidence commonly observed without employing the ad hoc parameter controlling optimistic and pessimistic attitudes. Applying the model to the preference of information timing sheds lights on the information-seeking behaviour of individuals holding biased beliefs; another example in the context of portfolio choice shows that pessimism can lead to conservative trading but can also encourage risk-taking investment strategy.

As part of the optimal beliefs literature, the model built in this thesis is still open to the critiques of Spiegler (2008) on the violation of IIA, since beliefs are payoff dependent. Further amendments might require preferences to be choice-set dependent—the same element in different choice bundles need to be viewed as different subjects. Moreover, the set of optimal beliefs in our model are not unique. Although the results are compatible with cumulative prospect theory and other theories on ambiguity, our model cannot provide explanations to phenomenon like "curse of knowledge"[26]. One possible solution to this problem is to further assume a reference-dependent anticipation. Reference-dependent anticipation assumption can be valid in the circumstances when agents have fresh recently experience [27]. Calibration results indicate that with concave utility function, model with reference point may better fit the observations.

Future work may include modelling of preference on information containing uncertainty. Agents in our model are motivated to choose subjective beliefs deviated from rational ones due to the fear of loss and the appeal of goodwill. Biased beliefs regarding information containing uncertainty is a step further. However, Bayes' rule can hardly hold here since beliefs are utility-serving. A simple rule of updating beliefs with economics intuitions is worth exploring. Finally, the equilibrium described in this thesis is incomplete. A more general equilibrium can be constructed based on both the current homogeneous assumption as well as an alternative one that investors holding heterogeneous beliefs trade with each other in the market.

[26]The curse of knowledge is a cognitive bias according to which better-informed agents may have the disadvantage that they lose some ability to understand less-informed agents. As such added information may convey some disutility, the curse of knowledge implies that the well-informed party, in this model, with a great chance of getting good results, are more likely to be pessimistic instead of optimistic as predicted.

[27]Another example to support this assumption is an amateur and a professional chess players having different levels of felicities from the same anticipation.

2.7 Mathematical Appendix A

Lemma 1: (Biased Beliefs are Preferred to Rational Unbiased Beliefs)

$$U = \sum_{s\in\mathcal{S}} q_s u_s + \eta\{\sum_{s\in\mathcal{S}} p_s \mu(u_s - \sum_{s\in\mathcal{S}} q_s u_s)\}$$

For k=1,....,S

$$\begin{aligned}\frac{\partial U}{\partial q_k} &= u_k\{1 - \eta \sum_{s\in\mathcal{S}} p_s \mu'(u_s - \sum_{s\in\mathcal{S}} q_s u_s)\} \\ &= u_k\{1 - \eta P_+ - \eta\lambda(1 - P_+)\}\end{aligned}$$

where $P_+ = \sum_A p_s,\ A = \{s \in \mathcal{S}:\ u_s - \sum_{s\in\mathcal{S}} q_s u_s \geq 0\}$.

For $\frac{\partial U}{\partial q_k}|_{p_1,\ldots p_S} \neq 0$ (for at least one k), the conclusion holds obviously.

Consider the case that $\frac{\partial U}{\partial q_k}|_{p_1,\ldots p_S} = 0,\ \forall k \in \mathcal{S}$.

$\frac{\partial U}{\partial q_k}|_{p_1,\ldots p_S} = 0$ holds for all k, iff $P_+ = \sum_{A^0} p_s = \frac{\eta\lambda - 1}{\eta(\lambda - 1)},\ A^0 = \{s \in \mathcal{S}:\ u_s - \sum_{s\in\mathcal{S}} p_s u_s \geq 0\}$.

Therefore, we have

$$\begin{aligned}U_{BS} &= \sum_{s\in\mathcal{S}} q_s u_s + \eta\{\sum_{s\in A} p_s u_s - (\sum_{s\in A} p_s)\cdot \sum_{s\in\mathcal{S}} q_s u_s + \lambda \cdot \sum_{s\in A} p_s u_s - \lambda \cdot (\sum_{s\in\overline{A}} p_s)\cdot \sum_{s\in\mathcal{S}} q_s u_s)\} \\ &= \eta\{\sum_{s\in A} p_s u_s + \lambda \cdot \sum_{s\in} p_s u_s\}\end{aligned}$$

The total utility is independent of q_s and $U_{BS} \equiv U_{RE}$. *Q.E.D.*

Lemma 2: (Beliefs Tradeoff among Different States)

Consider the following maximization problem

$$\max_{q_s, s=1,\ldots S} \sum_{s\in S} q_s u_s + \eta\{\sum_{s\in S} p_s \mu(u_s - \sum_{s\in S} q_s u_s)\}$$

$$s.t. \sum_{s\in\mathcal{S}} q_s = 1.$$

From $\frac{\partial U}{\partial q_k} = u_k\{1 - \eta P_+ - \eta\lambda(1 - P_+)\}$, we see that, for $u_k > 0$, U is increasing in q_k iff

$P_+ > P^*$ and is decreasing in q_k iff $P_+ < P^*$, where $P^* = \dfrac{\eta\lambda - 1}{\eta(\lambda - 1)}$.
The constraint $\sum\limits_{s\in\mathcal{S}} q_s = 1$ requires that any increase in q_k must come together with decrease(s) in subjective probability(ies) of other states. For simplification, consider the case that q_k and q_l $(k > l)$ change together. Suppose $P_+ > P^*$ and all the other subjective beliefs are given, since $u_k > u_l$ for all $k > l$, $\dfrac{\partial U}{\partial q_k} > \dfrac{\partial U}{\partial q_l}$. A small increase in q_k with a small decrease in q_l will increase the total utility. Similar analysis for the case $P_+ < P^*$. Our proof replicates but simplifies the proof using Kahn-Tucker condition. *Q.E.D.*

Proposition 1:(Over-optimistic versus Over-pessimistic)
From the proof of Lemma 2, we know that for $P_+ \neq P^*$, further biases are always desirable. Therefore, at optimal beliefs, $P_+ = P^* = \dfrac{\eta\lambda - 1}{\eta(\lambda - 1)}$. Since the optimal P_+ is uniquely determined by η and λ, and the objective probabilities are exogenous, the set of u_s above the expectation is also uniquely determined. Even though the value of $\sum\limits_{s\in\mathcal{S}} q_s u_s$ is given, there still exist multiple combinations of $\{q_s\}$. As long as sets of $\{q_s\}$ generate the required value of $\sum\limits_{s\in\mathcal{S}} q_s u_s$, they will all achieve the same value of total utility.

Further more, directly from proof of Lemma 2, we see that if $P_+^0 > (<) P^*$, the decision maker will be up-biased(down-biased) in the upper-rank outcomes and down-biased(up-biased) in the lower-rank outcomes. Therefore, $\sum\limits_{s\in\mathcal{S}} q_s u_s > (<) \sum\limits_{s\in\mathcal{S}} p_s u_s$ iff $P_+^0 > (<) P^*$. *Q.E.D.*

Proposition 2: (Information Timing Preference)
Suppose an agent holds optimal subjective beliefs $\{q_s\}_{s\in\mathcal{S}}$.
If $i = k$, the agent knows that Z_k will happen at t=2. Then, at t=1, $U_k^A = u_k + \eta\mu(u_k - \sum\limits_{s\in\mathcal{S}} q_s u_s)$; at t=2, $U_k^R = 1 \times \eta\mu(u_k - u_k) + 0 \times \sum\limits_{s\in\mathcal{S}/k} \eta\mu(u_s - \sum\limits_{s\in\mathcal{S}} q_s u_s) = 0$. The total utility from observing $i = k$ is,

$$U_k = U_k^A + U_k^R = u_k + \eta\mu(u_k - \sum_{s\in\mathcal{S}} q_s u_s),$$

The agent's prospective utility from getting information in advance is,

$$U_{early} = \sum_{s\in\mathcal{S}} q_s u_s + \eta\{\sum_{s\in\mathcal{S}} q_s \mu(u_s - \sum_{s\in\mathcal{S}} q_s u_s)\}.$$

Utility without early information is as before,

$$U_{wait} \quad = \quad \sum_{s\in\mathcal{S}} q_s u_s + \eta\{\sum_{s\in\mathcal{S}} p_s \mu(u_s - \sum_{s\in\mathcal{S}} q_s u_s)\}$$

Early information is strictly preferred iff $U_{early} > U_{wait}$ holds. Since

$$U_{early} - U_{wait} = \eta \cdot \{\sum_{s\in A}(q_s - p_s)\cdot(u_s - \sum_{s\in\mathcal{S}} q_s u_s) + \sum_{s\in\overline{A}}(q_s - p_s)\cdot\lambda(u_s - \sum_{s\in\mathcal{S}} q_s u_s)\},$$

$U_{early} > U_{wait}$ iff

$$\sum_{s\in A}(q_s - p_s)\cdot(u_s - \sum_{s\in\mathcal{S}} q_s u_s) > \lambda \sum_{s\in\overline{A}}(p_s - q_s)\cdot(u_s - \sum_{s\in\mathcal{S}} q_s u_s). \tag{2.7}$$

For $s \in A$, where $A = \{s \in \mathcal{S}:\ u(Z_s) - \sum_{s\in\mathcal{S}} q_s u(Z_s) \geq 0\}$, $(u_s - \sum_{s\in\mathcal{S}} q_s u_s) \geq 0$ by definition; and for $s \in \overline{A}$, $(u_s - \sum_{s\in\mathcal{S}} q_s u_s) < 0$.
If $P^0_+ > (<) P^*$, by Proposition 3, $q_s - p_s \geq (\leq) 0$ if $s \in A$ while $p_s - q_s \geq (\leq) 0$ if $s \in \overline{A}$, strict inequality holds for at least one s in each subset. Therefore, only one of $\sum_{s\in A}(q_s - p_s)\cdot(u_s - \sum_{s\in\mathcal{S}} q_s u_s) > 0$ and $\sum_{s\in\overline{A}}(p_s - q_s)\cdot(u_s - \sum_{s\in\mathcal{S}} q_s u_s) > 0$ can hold.
When $P^0_+ > P^*$, LHS of 2.7 is greater than 0, and $U_{early} > U_{wait}$; when $P_+ < P^*$, RHS of 2.7 is greater than 0, and $U_{early} < U_{wait}$. *Q.E.D.*

Proposition 3,4: (Two Symmetrically Distributed Lotteries & Two Lotteries: The General Case)
We prove the general case in Proposition 4. Proposition 3 can be easily derived from Proposition 4.
We start with part(ii) in Proposition 6 and prove the case $P^0_{+A} > P^*$, $P^0_{+B} < P^*$. The case $P^0_{+A} < P^*$, $P^0_{+B} > P^*$ holds by symmetry.
From Proposition 1, if $P^0_{+A} > P^*$ and $P^0_{+B} < P^*$, then an agent is on average over-optimistic about the payoff of the lottery A, and over-pessimistic about the payoff of the lottery B. Therefore, we have,

$$E_{g^*_{B(\cdot)}}(Z_B) < E_{f_{B(\cdot)}}(Z_B) = E_{f_{A(\cdot)}}(Z_A) < E_{g^*_{A(\cdot)}}(Z_A)$$

For a naive agent, lottery A is strictly preferred to lottery B.

Consider part(i), in which case either both P^0_{+A} and P^0_{+B} are greater than P^* or both smaller than P^*. Notice that optimal belief $g_i^*(\cdot)$ ensures that $\int\limits_a^{+\infty} f_A(Z)\mathrm{d}Z = P^*$, $\int\limits_b^{+\infty} f_B(Z)dZ = P^*$, where $a = min\{Z_A : \; Z_A - \int\limits_{-\infty}^{+\infty} g_A^*(Z_A)Z_A\mathrm{d}Z_A \geq 0\} = E_{g_A^*}(Z_A)$,

and $b = min\{Z_B : \; Z_B - \int\limits_{-\infty}^{+\infty} g_B^*(Z_B)Z_B\mathrm{d}Z_B \geq 0\} = E_{g_B^*}(Z_B)$.

Hence, if $a > b$, $E_{g_A^*}(Z_A) > E_{g_B^*}(Z_B)$ and $\int\limits_a^{+\infty} [f_A(Z) - f_B(Z)]\mathrm{d}Z > 0$.

Similar proof for the case $a < b$. It is easy to derive Proposition 3 from here. *Q.E.D.*

Proposition 5:(Choice between Two Lotteries: Sophisticated Case)

The objective function $U = E_g(Z) + \eta E_f\mu[Z - E_g(Z)]$ can be reformed as,

$$\begin{aligned} U &= \int_{E_g(Z)}^{+\infty} f(Z)Z\mathrm{d}Z + \lambda \int_{-\infty}^{E_g(Z)} f(Z)Z\mathrm{d}Z \\ &= \eta E_f(Z) + (1-\eta)E_g(Z) + (\lambda - 1)\eta \int\limits_{loss} f(Z)E_g(Z)\mathrm{d}Z + (\lambda - 1)\eta \int\limits_{loss} f(Z)Z\mathrm{d}Z. \end{aligned}$$

At optimal beliefs, we have $\int\limits_{loss} f(Z)\mathrm{d}Z = 1 - P^* = \dfrac{1-\eta}{\eta(\lambda-1)}$. Substitute $\dfrac{1-\eta}{\eta(\lambda-1)}$ back into the reformed objective function, we have,

$$\eta E_f(Z) + (\lambda - 1)\eta \int\limits_{loss} f(Z)Z\mathrm{dZ}$$

Since $E_{f_A}(Z_A) = E_{f_B}(Z_B)$, our conclusions hold obviously. *Q.E.D.*

Proposition 6:(Risk Taking due to Optimism and Pessimism: Naive Case)

The agent aims to maximize $\int u(R_f + \alpha R)\mathrm{dF(R)}$. The objective function is concave in α as $\int u''(R_f + \alpha R)R^2\mathrm{dF(R)} \leq 0$. For $0 \leq \alpha \leq 1$, if α is optimal, it must satisfy the Kuhn-Tucker first order condition:

$$\phi(\alpha) = \int u'(R_f + \alpha R)R\mathrm{dF(R)}\{ \begin{array}{ll} \leq 0 & if\ \alpha < 1 \\ \geq 0 & if\ \alpha > 0 \end{array}.$$

Notice that $\int RdF(R) > 0$ implies $\phi(0) > 0$. Hence, $\alpha = 0$ cannot satisfy the first order condition. We conclude that the optimal portfolio has $\alpha > 0$. Same analogy can be applied in the case $E(R) < 0$.

The proof below is presented in the discrete multi-state case to make mathematical expressions clear. All steps are applicable to the continuous distribution.
The problem of a biased agent to choose α^{BS} for given $\{q_s\}_{s\in\mathcal{S}}$ is

$$\underset{\alpha}{Max}\sum_{s\in\mathcal{S}} q_s u(R_f + \alpha R_s).$$

FOC of this problem is

$$\sum_{s\in\mathcal{S}} q_s u'(R_f + \alpha^{BS} R_s) = 0,$$

where α^{BS} is the optimal allocation of wealth to the risky asset under biased beliefs.
We examine the agent's FOC for optimal α^*. Consider moving $\mathrm{d}\hat{\omega}$ from state s' to state s'' with $R''_s > R'_s$, we have:

$$(u'(R_f + \alpha^* R''_s)R''_s - u'(R_f + \alpha^* R'_s)R'_s)\mathrm{d}\hat{\omega} + \sum_{s\in\mathcal{S}} \hat{q}_s u''(R_f + \alpha^* R_s)R_s^2 \mathrm{d}\alpha^* = 0$$

$$\frac{\mathrm{d}\alpha^*}{\mathrm{d}\hat{\omega}} = \frac{u'(\cdot)R'_s - u'(\cdot)R''_s}{\sum_{s\in\mathcal{S}} q_s u''(R_f + \alpha^* R_s)R_s^2} > 0.$$

Therefore, optimal α^* is increasing in the subjective probabilities putting on upper-ranking outcomes. From Proposition 2, an optimistic agent is up-biased because she overestimates the probabilities of good outcomes and underestimates the probabilities of bad outcomes. For $\alpha^{BS} > 0$, better outcomes refer to the states with higher positive returns, while for $\alpha^{BS} < 0$, better outcomes refer to the states with lower negative returns. For $\alpha^{RE} > 0$ and $\alpha^{BS} > 0$, an optimistic agent is one who overestimates the probabilities on positive returns. To bring down the biased beliefs from optimistic level to rational level, α must decrease. Therefore, we have $\alpha^{OP} > \alpha^{RE} > 0$. Instead, a pessimistic agent who overestimates the probabilities on the low returns needs to increase α to get beliefs back to the rational level. Hence, we have $0 < \alpha^{PE} < \alpha^{RE}$. For $\alpha^{RE} > 0$ and $\alpha^{BS} < 0$, probabilities on low returns are overestimated by an optimistic agent and α must increase to get beliefs back to the rational level. For this reason, we have $\alpha^{OP} < 0 < \alpha^{RE}$. For a pessimistic agent with $\alpha^{PE} < 0$, $\alpha^{RE} < \alpha^{PE} < 0$ leads to a contradiction to the assumption that $\alpha^{RE} > 0$. Our proof for $\alpha^{RE} > 0$ is completed. Similar analysis can be applied in the case $\alpha^{RE} < 0$. *Q.E.D.*

Proposition 7:(Risk Taking due to Optimism and Pessimism: Sophisticated Case)

The problem of choosing optimal α^{BS} for given optimal beliefs $\{q_s^*\}_{s\in\mathcal{S}}$ is,

$$\underset{\alpha}{Max}U = \sum_{s\in\mathcal{S}} q_s^* u(R_f + \alpha R_s) + \eta \sum_{s\in\mathcal{S}} p_s \mu[u(R_f + \alpha R_s) - \sum_{s\in\mathcal{S}} q_s^* u(R_f + \alpha R_s)]$$

We proved in Proposition 6 that if $E(R) > (<)0$ then $\alpha^{RE} > (<)0$. The first order condition with respect to α at optimal beliefs is

$$\frac{\partial U}{\partial \alpha} |_{\alpha, q_s = q_s^*} = \eta \sum_{s\in\mathcal{S}} p_s u'(\cdot) R_s + \eta(\lambda - 1) \sum_{-BS} p_s u'(\cdot) R_s = 0$$

Instead, the first order condition with respect to α for a rational agent is

$$\frac{\partial U}{\partial \alpha} |_{\alpha, q_s = p_s} = \sum_{s\in\mathcal{S}} p_s u'(\cdot) R_s = 0.$$

Since $\frac{\partial^2 U}{\partial \alpha^2} = \sum q_s u''(\cdot) R_s^2 < 0$, if $\alpha^{RE} > 0$ and $\eta(\lambda-1) \sum_{-BS} p_s u'(\cdot) R_s > 0$, then

$$\frac{\partial U}{\partial \alpha} |_{\alpha = \alpha^{RE}, q_s = q_s^*} > 0.$$

We must have $\alpha^{BS} > \alpha^{RE} > 0$.
Instead, when $\alpha^{RE} > 0$ and $\eta(\lambda-1) \sum_{-BS} p_s u'(\cdot) R_s < 0$, since

$$\frac{\partial U}{\partial \alpha} |_{\alpha = \alpha^{RE}, q_s = q_s^*} < 0,$$

we must have $\alpha^{BS} < \alpha^{RE}$ and the proof for the case $\alpha^{RE} > 0$ is completed. Conclusions for $\alpha^{RE} < 0$ can be proved by the same logic.

Next, we need to prove the second part of the proposition. First we assume that at optimal beliefs, the subjective expectation is $E_{q_s^*} u(R_f + \alpha R_s)$. Consider moving $\mathrm{d}\hat{\omega} > 0$ from state s' to s'' with $R_{s''} > R_{s'}$. Suppose with no change in α, the new subjective expectation is $E_{q_s^*} u(R_f + \alpha R_s) + \Delta$, where $\Delta = \mathrm{d}\hat{\omega}(u(R_f + \alpha R_{s''}) - u(R_f + \alpha R_{s'}))$. Suppose Δ is small enough such that only one state $\tilde{s}$ moves from gain to loss due to the increase in expectation. This assumption is always satisfied under the continuous distribution assumption. We examine the FOC for the optimal α:

$$\eta(\lambda-1) p_{\tilde{s}} (\mathrm{d}\hat{\omega}) u'(R_f + \alpha R_{\tilde{s}}) R_{\tilde{s}} + \eta(\lambda - 1) \sum_{-BS} p_s u''(R_f + \alpha R_s) R_s^2 \} \mathrm{d}\alpha = 0$$

Since $\eta \sum\limits_{s\in S} p_s u''(R_f+\alpha R_s)R_s^2 + \eta(\lambda-1)\sum\limits_{-BS} p_s u''(R_f+\alpha R_s)R_s^2 < 0$, for $\mathrm{d}\alpha > 0$, we must have $\eta(\lambda-1)p_{\tilde{s}}u'(R_f+\alpha R_{\tilde{s}})R_{\tilde{s}} > 0$ and $R_{\tilde{s}} > 0$.

Instead, if $R_{\tilde{s}} < 0$, we need the down-bias $-\mathrm{d}\hat{\omega}$ to move $\eta(\lambda-1)p_{\tilde{s}}u'(R_f+\alpha R_{\tilde{s}})R_{\tilde{s}}$ out of the loss region. Notice that the certain state moving into or out of the loss region is the one on the margin of the gain and loss regions. Therefore,

$$u(R_f+\alpha R_{\tilde{s}}) = E_{q_s^*}u(R_f+\alpha R_s).$$

With a small change in denotation, we can get $R_{\tilde{s}} = R_{CE}$ in our proposition.

We proved that if $R_{CE} > 0$, α is increasing in $\mathrm{d}\hat{\omega}$; if $R_{CE} < 0$, α is decreasing in $\mathrm{d}\hat{\omega}$. The final conclusion and Lemma 3 can be easily derived from here. *Q.E.D.*

2.8 Mathematical Appendix B

Appendix B presents the model in section 3 by relaxing the linear restriction on the gain-loss utility $\mu(x)$.
Specifically, we assume

$$\mu'(-x) = \lambda(x)\mu'(x),$$

where $\lambda(x) > 1, x > 0$, and $\lim_{x\to 0} \lambda(x) = 1$, $\mu''(x) \le 0$, $\lambda'(x) \le 0$. Conclusions under the new assumption are similar to those in section 3.
All the optimal conditions below are derived from the FOC: $\frac{\partial U}{\partial q_s} = 0$, where

$$U = \sum_{s\in\mathcal{S}} q_s u(Z_s) + \eta \sum_{s\in\mathcal{S}} p_s \mu[u(Z_s) - \sum_{s\in\mathcal{S}} q_s u(Z_s)].$$

Previous conclusions under linear assumption in Proposition 3 are:
For $\mu(x) = x \quad (x > 0), \mu(x) = \lambda x \quad (x < 0)$,

(i) Linear: Two-state Case
$P^* = \frac{\eta\lambda-1}{\eta(\lambda-1)} = 1 + \frac{\eta-1}{\eta(\lambda-1)}$ is independent of q.
Optimal q is: $q^* = 0$, if $p < P^*$; $q^* = 1$, if $p > P^*$.

(ii) Linear: S-state Case
Optimal $\{q_s^*\}_{s\in\mathcal{S}}$ satisfy: $P_+ = P^* = \frac{\eta\lambda-1}{\eta(\lambda-1)}$, where $P_+ = \sum_A p_s, A = \{s \in \mathcal{S} : u(Z_s) - \sum_{s\in\mathcal{S}} q_s u(Z_s) \ge 0\}$, and P^* is independent of q_s. Therefore, there exist more than one optimal set of $\{q_s^*\}_{s\in\mathcal{S}}$, and they all achieve a certain value of $\sum_{s\in\mathcal{S}} q_s^* u(Z_s)$(for given$\{u(Z_s)\}_s$), which is uniquely determined by $P^* = \frac{\eta\lambda-1}{\eta(\lambda-1)}$ for given $\{p_s\}_{s\in\mathcal{S}}$. (Conclusions from Proposition 3).

Next, we discuss the model under new assumptions.
We assume $\mu'(-x) = \lambda(x)\mu'(x)$, where $\lambda(x) > 1, x > 0$, and $\lim_{x\to 0} \lambda(x) = 1$, $\mu''(x) \le 0$, $\lambda'(x) \le 0$.

(i) General: Two-state Case
Suppose we have two states with $u(1) = 1, u(0) = 0$.
From FOC, we can derive $p = 1 + \frac{\eta\mu'(1-q)-1}{\eta[\mu'(q)\lambda(q)-\mu'(1-q)]}$
Since $0 < p < 1$ and $\frac{\eta\mu'(1-q)-1}{\eta[\mu'(q)\lambda(q)-\mu'(1-q)]} < 0$, the denominator and the numerator must have opposite signs.
For $\lambda'(x) \le 0$, the cut-off value $\frac{\eta\mu'(1-q)-1}{\eta[\mu'(q)\lambda(q)-\mu'(1-q)]}$ is negative and decreasing in q. Therefore, for $p > 1 + \frac{\eta\mu'(1-p)-1}{\eta[\mu'(p)\lambda(p)-\mu'(1-p)]}$, the optimal $q^* = 1$, otherwise, optimal

$q^* = 0$.

To be more specific, let $\mu'(x) = \beta$, then

$$p = \frac{\eta\beta\lambda(q) - 1}{\eta\beta[\lambda(q) - 1]} = 1 + \frac{\eta\beta - 1}{\eta\beta[\lambda(q) - 1]}.$$

For $0 < p < 1$, we must have $\eta\beta < 1$.

If $\lambda'(q) \leq 0$, then $\frac{\eta\beta\lambda(q)-1}{\eta\beta[\lambda(q)-1]}$ is decreasing in q. Optimal $q^* = 1$ if $p > \frac{\eta\beta\lambda(p)-1}{\eta\beta[\lambda(p)-1]}$ and $q^* = 0$ otherwise.

(ii) General: S-state Case

From FOC, we have

$$\eta\{\sum_{Gain} p_s\mu'(u_s - \sum_S q_s u_s) + \sum_{Loss} p_s\mu'(\sum_S q_s u_s - u_s)\lambda(\sum_S q_s u_s - u_s)\} = 1.$$

Specifically, for $\mu'(\cdot) \equiv \beta$, $\sum\limits_{Loss} p_s[\lambda(\sum\limits_S q_s u_s - u_s) - 1] = \frac{1-\eta\beta}{\eta\beta}$.

Since the $LHS > 0$, we have $\eta\beta < 1$.

For $\lambda'(\cdot) \leq 0$, if $\sum\limits_{Loss} p_s[\lambda(\sum\limits_S q_s u_s - u_s) - 1] < \frac{1-\eta\beta}{\eta\beta}$, then the total utility is increasing in $\sum\limits_S q_s u_s$. An increase in $\sum\limits_S q_s u_s$ will decrease the value of $\sum\limits_{Loss} p_s[\lambda(\sum\limits_S q_s u_s - u_s) - 1]$(if the range of loss remains the same), making it further below $\frac{1-\eta\beta}{\eta\beta}$. Therefore, optimal sets of $\{q_s^*\}$ are those satisfy

$$\sum_{Loss} p_s[\lambda(\sum_S q_s^* u_s - u_s) - 1] = \frac{1-\eta\beta}{\eta\beta},$$

where $Loss = \{s \in \mathcal{S} : \; u_s - \sum\limits_{s\in\mathcal{S}} q_s^* u_s < 0\}$.

Chapter 3

Information and Dynamic Trading with Gambler's Fallacy

Abstract

In this paper, we develop a multi-period stock trading model in which there are two types of investors–"rational" type and "gambler's fallacy" type, both observing a public signal about the fundamental value in each period. The rational type holds correct beliefs on the stochastic process of the signal, whereas the gambler's fallacy type mistakenly believes the sequence of signals should exhibit systematic reversals. Two types trade against each other to speculate the future price changes based on their inferences about the fundamental value. This paper explores a competitive equilibrium in which both types have model consistent expectations, adjusted for the heterogeneity in their beliefs about the signal generating process. We examine the dynamics of prices, returns, optimal portfolios and trading volumes in reaction to the information flow. Consistent with empirical evidence, the market in our model exhibits short-run momentum and long-run reversal. We also demonstrate that the equilibrium price is closer to the valuation of the gambler's fallacy type.

3.1 Introduction

A challenge to the efficient market hypothesis is that the individuals often over- or under-react to public information, leading to momentum and momentum reversal phenomena in the market. That is, on one hand, returns exhibit continuation in the short- to medium-term, while on the other hand, there exists a long-term reversal. Momentum and reversal have been documented extensively in previous empirical studies on a wide variety of assets. DeBondt and Thaler (1985) documented long-term reversal, showing that future stock returns with a horizon up to 5 years can be predicted by returns of the past 3-5 years. Jegadeesh and Titman (1993) showed that short to mid term momentum in individual US stocks–returns over 3-12 months can be predicted by the returns in the past 3-12 months. Fama and French (1992, 1998) found that this effect exists in other countries as well. Momentum is documented for industry component stocks returns (Moskowitz and Grinblatt, 1999), country indices (Asness, Liew and Stevens, 1997), currencies (Bhojraj and Swaminathan, 2006), bonds (Asness, Moskowitz and Pedersen, 2013), and commodities (Gorton, Hayashi and Rouwenhorst, 2013). These discoveries are against the classic microstructure models, such as Competitive Rational Expectations Equilibrium (CREE) (Muth, 1961; Grossman, 1976), non-myopic CREE (Brown and Jennings, 1989) and multi-period CREE (He and Wang, 1995)[1] since all these models only consider rational expectations and fundamental factors in predicting returns. This paper, instead, explains momentum and reversal based on "gambler's fallacy"–a common cognitive bias of market participants.

The gambler's fallacy first proposed by Tversky and Kahneman (1971) is closely related to momentum and reversal in the market. The gambler's fallacy is the erroneous belief that a certain random event is less likely to happen following an event or a series of events. Gambler's fallacy arises out of a belief in a "law of small numbers". According to the fallacy, "streaks"must eventually even out in order to be representative (Burns and Corpus, 2004). Tversky and Kahneman (1974) pointed out that the gambler's fallacy is a cognitive bias produced by a psychological heuristic called the representativeness heuristic, which states that people evaluate the probability of a certain event by assessing how similar it is to the events they have experienced before, and how similar the events surrounding those two processes are. Numerous early studies confirmed this general pattern of findings[2]. Experimental studies also showed that the gambler's fallacy widely exists in lottery games (Clotfelter and Cook, 1993), casino (Croson and Sundali, 2005), coin flipping (Bar-Hillel and Wagenaar, 1991). In particular, Andreassen and Kraus (1990) found that in the presence

[1] See Brunnermeier (2003) for a review.
[2] See Lee (1971) for a review.

of modest stock price fluctuations, subjects exhibit gambler's fallacy in presence of a short trend.

The purpose of this paper is to link this phenomenon to momentum and reversal observed in the market. We develops a multi-period stock trading model in which there are two types of investors–the rational type and the gambler's fallacy type, both observing a public signal about the fundamental value in each period. The rational type holds correct beliefs on the stochastic process of the signal, while the gambler's fallacy type mistakenly believes the sequence of signals should exhibit systematic reversals. Two types trade against each other to speculate the future price changes based on their inferences about the fundamental value. This paper explores a competitive equilibrium in which both types have model consistent expectations adjusted for the heterogeneity in their beliefs about the signal generating process. We examine the dynamics of prices, returns, optimal portfolios and trading volumes in reaction to the information flow.

Our model results are consistent with empirical evidence– market exhibits short-run momentum and long-run reversal following some information shocks. To see the intuitive connection between gambler's fallacy and momentum reversal, a market with only gambler's fallacy investors is considered first. In face of an information shock, the gambler's fallacy leads to under-reaction in stock valuation initially. However, the missing observations in a near term information reversal makes gambler's fallacy traders over-react by falsely over-adjusting their valuations. The over-reaction finally disappears and market returns to fundamental since information accumulates over time. As investors turns from under- to over-reaction and finally to fundamental, the market exhibits momentum and reversal accordingly. This intuition is also explained in Rabin and Vayanos (2010) as presented later on.

This study fully explores the market equilibrium in which there exists both rational and gambler's fallacy traders. Interestingly, the arbitrage opportunities still exist even when there are large proportion of rational investors. This is because, the gambler's fallacy, as a rule of thumb, makes investors over-confident in their forecasting and trade aggressively.

In general, the market dominance switches from rational to gambler's fallacy as it turns from momentum to reversal. The long-term reversal (near-term momentum) comes with negative (positive) correlations in today and future returns, encouraging (discouraging) rational traders taking risky positions.

A closely related cognitive bias of gambler's fallacy is the hot-hand fallacy—the fallacious belief that a person who has experienced success with a random event has a greater chance of further success in additional attempts. In this paper, by assuming gambler's fallacy alone, it is identified that the rational traders act as having hot-hand fallacy in prices. Due to the existence of gambler's fallacy, the momentum trading strategy becomes optimal. Rational traders tend to further buy the stock when the price increases, while short the stock when the past price decreases.

Furthermore, our model also provides a potential explanation to low market volatility. We find that the market price is less volatile as compared to a pure rational market in the presence of gambler's fallacy since mean-reversal believers update their forecast slowly.

The paper proceeds as follows. Section II formally presents the model, starting with the two types' inference problems of the public information and followed by the investors' optimisation problem.

By rewriting the inference problem into state space framework, we are able to analyse the recursive process as well as the steady state using Kalman filter. At the end of Section II, we solve the belief-consistent linear competitive equilibrium; a price taker's conjectured price process based on the beliefs must clear the market and must also coincide with the real price process given that the beliefs are correct about the economy. To better understand the the role of the fallacy traders, Section III explores the benchmark model—an economy with only rational traders. Section IV further discusses the properties of equilibrium in the mixed type market. We examine the composition of optimal portfolios followed by the study of market dynamics after an information shock. It is also discussed that the market dominance in the equilibrium. Our model shows that the market has a tendency of switching from rational to gambler's fallacy regime as the time goes on, in the sense that the price is mainly determined by the valuation of the corresponding type. Finally, we give the model implications on asset pricing, including the momentum and reversal (autocorrelation analysis), the volatility, and the optimal trading strategy. Section VII summarises major conclusions and further remarks the model's scope and limitations.

3.2 Relevant Literature

3.2.1 Momentum and Reversal

Previous work on momentum and reversal can be generally classified into three groups.

Daniel, Hirshleifer and Subrahmanyam (1998, 2001); Gervais and Odean (2001); Kausar and Taffler (2006) attempted to explain over-reaction and momentum using over-confidence or self-attribution. The reversal occurs because the price returns to fundamental in the long term.

The second group appeals to imperfect information. Hong and Stein (1999) suggested the under- and over-reaction is caused by the slow diffusion of information across population and the trend-chasing behaviour. Brav and Heaton (2002) used a model with uncertainty about model parameters, such as the asset value's mean, and rational Bayesian learning to explain predictable return patterns. In Brav and Heaton (2002), over-reactions and under-reactions are generated from the specific nature of model uncertainty. Hong, Kubik and Stein (2005) showed that incorrect and over-simplified models in forecasting leads to persistent errors and momentum.

The third group focuses on gambler's fallacy and hot-hand fallacy. Previous theoretical studies modelling hot-hand fallacy and gambler's fallacy under finance context include Barberis et al. (2013), Barberis, Shleifer and Vishny (1998), and Rabin and Vayanos (2010). Barberis, Shleifer and Vishny (1998) presented a model with one representative investor who extrapolates future earnings from random sequences. Their paper assumes two regimes directly—the reversal regime and the streak regime, corresponding to the gambler's fallacy and hot-hand fallacy biases. A more recent study by Barberis et al. (2013) developed a consumption-based asset pricing model in which the rational investors trading against investors form beliefs about future prices changes in the stock market by extrapolating past price changes—similar to hot-hand fallacy. Simulation results showed that their model successfully derived the momentum reversal but not momentum due to the existence of rational investors and the assumption that the extrapolators' demand for the risky asset depends on the most recent price change. Rabin and Vayanos (2010) developed an infinite time model of gambler's fallacy in which fallacy makers observing a sequence of signals exhibit gambler's fallacy in predictions when the streak is short and exhibit hot-hand fallacy when the streak is long. Without exploring the details, they discussed the momentum and momentum reversal in a market with one representative investor having gambler's fallacy in one stock's dividends series generated by an i.i.d. normal process. Finally, a related empirical paper from Hvidkjaer (2006) showed that small traders are buyers of loser momentum stocks. They turn into net sellers in these stocks subsequently, suggesting that by under-reacting to negative information, they may create momentum.

This paper, based on the third group, distinguishes itself from previous studies in the following ways. First, beliefs in our model are heterogeneous among the population. We attempt to incorporate both types of fallacies as well as the rational expectations while most of the previous literatures focussed on a single representative agent. Our assumption on gambler's fallacy is similar to Rabin and Vayanos (2010) but we further examines the equilibrium in an aggregate stock market with heterogeneous investors trading against each other. The setting of this paper allows to explore full implications of the equilibrium, including prices, returns, optimal holdings and trading volumes in a market with gambler's fallacy.

3.2.2 Competitive Rational Expectations Equilibrium

The equilibrium in this paper is an information-based, multi-period competitive equilibrium in which investors optimise their portfolio by making inferences about the underlying value of the asset. We explores an information-based competitive equilibrium in our model. The construction of the belief-consistent equilibrium in this paper is closely related to the Competitive Rational Expectations Equilibrium (Competitive REE). Due to the wide range of works in the field, we only summarise the ones closest to this paper. Muth (1961) was the first to formalise the idea of rational expectations equilibrium price. He adopted a new equilibrium concept that adds this to the usual requirements– the expected future price that agents use in determining their own current demands (and a price that their own current aggregate demands affect) is in fact the correct expectation of that price. Grossman (1976) described the first Competitive REE model with a closed form REE solution. In his model, information about the liquidation value of a single risky asset is dispersed among many traders. Each trader receives a noisy signal about the true payoff and are price takers in the market. By proposing a linear price conjecture and solving the agent's optimisation problem under the CARA-Normal framework, they derived a consistent equilibrium price. The equilibrium in their model is fully revealing and the market is strong-from efficient. Brown and Jennings (1989) extended Grossman's work to a two period non-myopic Competitive REE. The model setting is close to Grossman (1976) but there are two periods, and investors get private signals in both the periods. The investors intend to optimise the final wealth in the third period which is determined by the optimal portfolio in previous two periods. They derived the optimal stock holdings by backward induction and showed that technical analysis has value since a joint estimation using price conjectures in both periods enhances information revelation. He and Wang (1995) further analysed a multi-period version of Brown and Jennings (1989). They focused on the relationship between the pattern of trading volume and the nature of information flow. They also analysed the link between volume and price volatility. They found that the high volume generated by exogenous private or public information is accompanied by high volatility in prices, whereas the high volume generated by endogenous information (like prices) is not accompanied by high volatility. Our model takes a similar structure of the Competitive REE in He and Wang (1995), but investors are differentiated in their beliefs about the signal generating process and information is merely public. Some of the other related work with similar structures include Grundy and McNichols (1989) and Singleton (1986)[3].

The model in this paper is different from previous information-based models in that investors are no longer identical and rational in interpreting signals. In this paper, we

[3]See Brunnermeier (2003) for a review.

proposes a new dynamic belief-consistent competitive equilibrium, which is different from a typical CREE.

With all these assumptions above, our paper presents a potential asset pricing model addressing the short term momentum and long term reversal, simultaneously. The study also shows that conservatism can lead to aggressive trading behaviour due to gambler's fallacy—similar to the results in Chen (2012) but with different intuition.

3.3 The Model

In this section, we present a multi-period linear competitive equilibrium model with two types of investors; each holding a different belief on the information generating process concerning the underlying value of the stock. The "*rational*" type correctly believes signals are generated following a Gaussian process while the "gambler's fallacy" type (falsely) believes signals should exhibit systematic reversals. Both of them are Bayesian and update their beliefs about the state of the economy on the basis of past information. They know the existence and beliefs of the other type and competitively trade against each other by optimising their portfolios to maximise the future wealth. Our model is similar to previous information-based research (He and Wang, 1995; Brown and Jennings, 1989; Grossman, 1976) except the existence of the gambler's fallacy investors. The model is described in detail further below.

3.3.1 Information Structure

We consider an economy with T periods $1, ..., T$ and two assets—a risk-free and a risky asset (a stock) available for trading in the market. The risk-free asset is of perfectly elastic supply with a non-negative constant rate of return, r. The stock has a fixed supply Θ and each share of the stock pays a liquidation value V at the final date T. The unconditional distribution of V is normal with mean μ and variance σ_V^2. The realisation value of V is randomly drawn at the beginning of period $t = 1$ but is unobservable until the final period T. However, the investors can observe a signal about the liquidation value in each period.

In each period t, $t = 1, ..., T-1$, all investors observe a common signal y_t about the stock's liquidation value V formed by:

$$y_t = V + \xi_t \tag{3.1}$$

where $\xi_t \sim \mathcal{N}(0, \sigma_\xi^2)$ is the noise term. For simplicity, we assume ξ_t is *i.i.d.*. The equilibrium price P_t is also observable to all investors. Since the signals are public information and are the same for all investors, the investors' information set as follows:

$$\mathcal{F}_t^i = \{\mathcal{F}_0^i,\ \underline{P_t},\ \underline{y_t}\} = \{\mathcal{F}_0,\ \underline{y_t}\} = \mathcal{F}_t, \qquad i \in \mathcal{I},$$

where $\mathcal{F}_0^i$ represents the prior information as given by the prior distribution, and $\underline{P_t}$, $\underline{y_t}$ represents the realisations of stochastic processes $\{P_t\}$ and $\{y_t\}$ up to and including t. i represents the i'th type of the agent where $\mathcal{I} = \{R, G\}$ with R the rational type and G the gambler's fallacy type. We assume all investors share the prior beliefs $\mathcal{F}_0^i = \mathcal{F}_0$ and therefore, the information set at any date t is the same across the population. Since all the information are public information, $\underline{P_t}$ contains no more information than the signals observed, and therefore, it can be omitted from the information set. The assumption of symmetric information allows to better understand the nature of the trading behaviour in a market with gambler's fallacy. With only public information, the equilibrium price contains nothing more than the signals observed and the information set can be written as $\mathcal{F}_t = \{\mathcal{F}_0,\ \underline{y_t}\}$. We further assume the $r = 0$ and $\Theta = 0$ for simplicity, that is the risk-free rate is 0 and the supply of the stock is 0 and constant over time. In more general case, the net supply of the stock Θ should be positive. Since our results under the assumption $\Theta > 0$ is similar to when $\Theta = 0$, only $\Theta = 0$ is considered in the main content with the details of $\Theta > 0$ given in the Appendix. The information structure in our model is simple and standard with only public information. The inference problem is formally presented in the next subsection.

3.3.2 Inference Problem

Before introducing the equilibrium, it is essential to formally define the rational and gambler's fallacy types' beliefs and present the inference problems of the investors.

3.3.2.1 Beliefs of Two Types of Agents

There are two types of investors in the market, the rational type, and the gambler's fallacy type. The rational type correctly captures the signal generating process and believes the difference between signals in each period is merely due to random errors. The investors with gambler's fallacy, on the other hand, have fallacious beliefs that the sequence of the shocks on the signals is not *i.i.d.*, but exhibit systematic reversals.

Denoting the rational type with R and gambler's fallacy type with G, the beliefs of two types are presented as below. Consistent with equation (3.1), the rational investors believe the stochastic process of signals is:

$$y_t^R = V + \xi_t^R, \tag{3.2}$$

where $\xi_t^R \sim N(0, \sigma_\xi^2)$ is consistent with the real distribution of the signals in the economy. This assumption is the same as in a classic Competitive Rational Expectations Equilibrium (CREE) model[4].

Instead, the beliefs of the investors with gambler's fallacy are:

$$y_t^G = V + \kappa_t, \tag{3.3}$$

$$\kappa_t = \xi_t^G - \beta \sum_{k=0}^{t-1} \delta^k \kappa_{t-1-k}, \tag{3.4}$$

where $\xi_t^G \sim N(0, \sigma_\xi^2)$, $\beta, \delta \in [0, 1)$.

Equation(3.4) describes gambler's fallacy types' beliefs on the departure of a signal from the fundamental value over time. Similar to the rational agents' beliefs, fallacy agents also believe the signal y_t^G is the fundamental value V plus an error term κ_t. But instead of believing the error is randomly drawn from a Gaussian distribution, they believe the error κ_t is negatively correlated with its previous realisations $\sum_{k=0}^{t-1} \delta^k \kappa_{t-1-k}$, in which $0 < \delta < 1$ is the discount factor, indicating the power of a past signal decaying over time. The gambler's fallacy is captured by $\beta > 0$, indicating that the agent believes past positive signals tend to induce negative signals in the future. This assumption on Gambler's fallacy is similar to Rabin and Vayanos (2010) but has a finite time horizon[5]. Similar to their paper, we define $\epsilon_t \equiv \sum_{k=0}^{t-1} \delta^k \kappa_{t-1-k}$, which represents the luckiness of signals realised in history.

3.3.2.2 Inference Problem

The inference problems for both types can be reorganised into standard state-space models. We first introduce the following notations before presenting the updating processes in Proposition 8.

For the stochastic process, Z_t, we define the expectations of type i on Z_{t+k} conditional on $\mathcal{F}_t$ as $Z^i_{t+k|t} \equiv E_i[Z_{t+k}|\mathcal{F}_t]$. It defines the variance of type i on Z_{t+k} conditional on $\mathcal{F}_t$ as $\Sigma^i_{Z,t+k|t} \equiv Var_i[Z_{t+k}|\mathcal{F}_t]$. With these notations, Proposition 8 is given below:

[4] The beliefs of the rational type is the same as in Grossman (1976). In Grossman (1976), each individual observes a private signal $y_t = V + \xi_t$, and believes this is the real signal generating process.

[5] In Rabin and Vayanos (2010), the term $\beta \sum_{k=0}^{t-1} \delta^k \kappa_{t-1-k}$ is replaced by $\beta \sum_{k=0}^{+\infty} \delta^k \kappa_{t-1-k}$.

Proposition 8 *(Inference Process)*
$V^R_{t+1|t}$ and $S_{t+1|t} = [V^G_{t+1|t},\ \epsilon_{t+1|t}]'$ are state variables for rational and gambler's fallacy investors, and are given recursively by

$$V^R_{t+1|t} = V^R_{t|t-1} + K^R_t v^R_t, \tag{3.5}$$

$$S_{t+1|t} = \Upsilon S_{t|t-1} + K_t v^G_t. \tag{3.6}$$

The conditional variances are given recursively by

$$\Sigma^R_{V,t+1|t} = \Sigma^R_{V,t|t-1}(1 - K^R_t),$$

$$\Sigma^G_{S,t+1|t} = \Upsilon \Sigma^G_{S,t|t-1} \Upsilon^{\mathbf{T}} - [Y \Sigma^G_{S,t|t-1} Y^{\mathbf{T}} + H - YLY^{\mathbf{T}}] K_t K_t^{\mathbf{T}} + L,$$

where $v^R_t = y_t - V^R_{t|t-1}$ and $v^G_t = y_t - YS_{t|t-1}$ are signal shocks and K^R_t and $K_t = [K^G_t,\ k_t]'$ are the Kalman Gains. Other variables are matrices of constants given in Appendix.

Proposition 8 formulates the inference processes as Kalman Filtering problems. For each type, the state variables are unobserved values to be estimated based on the observation of signals. The signal shock simply evaluates the difference between the realisation of a signal and its previous estimation. Kalman Gains measures investors' sensitivities to new information. Figure 3.2 plots the Kalman Gains K^R_t, K^G_t, k_t on the valuations $V^R_{t|t-1}$, $V^G_{t|t-1}$ and the "lucky streaks" $\epsilon_{t|t-1}$[6].

The solid line and the dash line plot K^R_t and K^G_t respectively—the rational and fallacy investors' sensitivities in updating the fundamental values. It can be seen that $K^G_t < K^R_t$ for $t \leq T-1$, implying that the gambler's fallacy type fails to update the value estimations, sufficiently compared to the rational type. The belief of frequent reversal in signals makes them believe that the signal shocks contain more noise, thus the fallacy type does not value the contribution of a shock on the prediction of the hidden states as much.

Furthermore, both K^R_t and K^G_t decrease over time. Reactions to shocks get attenuated as time goes on. Since the real fundamental is constant over the trading periods, both types become better informed, i.e. the estimated variance of the fundamental value $\Sigma^i_{V,t|t-1}$ decreases over time. As they become more confident in their value estimations, new information shocks are less valued.

Finally, the Kalman Gain k_t on the "lucky streak" $\epsilon_{t+1|t}$ increases over time. The variance of the lucky streak never decays to 0. As people update their value estimations

[6] We plot $t = 5$ to $T-1$. The first four period are eliminated from the figure to remove the initial effects.

less, fallacy investors attribute the shocks almost entirely to the historical "lucky streak" Ȧn intuitive example is, with the full understanding of a fair coin, the gambler will believe that any unexpected sequences of heads or tails are simply due to luck, rather than an unfair coin. Our model focuses on the finite time horizon but here we still present the steady state when $T \to +\infty$ in Lemma 6 to support our statement that both types become finally certain about the state variable.[7] Lemma 6 below gives the details of the steady state.

Lemma 6 *(Steady State)*
In the infinite horizon where $T \to \infty$,

$$\Sigma_V^R = \lim_{t\to\infty} \Sigma_{V,t+1|t}^R = 0, \qquad \Sigma_S^G = \lim_{t\to\infty} \Sigma_{S,t+1|t}^G = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_\xi^2 \end{bmatrix};$$

$$K^R = \lim_{t\to\infty} K_t^R = 0 \qquad K^G = \lim_{t\to\infty} K_t^G = 0, \qquad k = \lim_{t\to\infty} k_t = \frac{\delta - \beta}{-\beta + (\beta+1)\sigma_\xi^2}.$$

Lemma 6 gives the steady state the system will approach when there are infinite periods for people to adjust their beliefs. Since $K^R = 0$ and $K^G = 0$, consistent with our previous analysis, each type will hold a stable estimation of the underlying value and no longer adjust it with respect to any new information in the limit. The gambler's fallacy investors attribute new signal shocks to the effects of noise contained in the previous information, while rational investors simply believe the shocks are merely random errors.

3.3.3 Preferences

The liquidation value determines investors' wealth level and thus the utility they could achieve from their investments. Investors' optimal investment plan, therefore, depends on the inference problem described above. To be more specific, investors adjust their holdings of the risky asset in every period to maximise their future wealth based on their estimations of the liquidation value. We examine the competitive equilibrium, in which all market participants are price takers. The fixed supply and the total demands of the risky asset from all investors determine the equilibrium price in each period. Investors are assumed to be risk-averse, non-myopic and have correct higher order beliefs about each other. The only bias in the market is the gambler's fallacy on the information flow. Investors are assumed to consume all the wealth in and only in the final period. This paper focuses on the linear

[7] In general, the Riccati Equation cannot be analytically solved. For example, in Rabin and Vayanos (2010), they presented the Riccati Equation without giving the explicit solutions. However, by assuming a constant value, the variance estimation approaches 0, allowing us to solve the problem completely.

equilibrium where the equilibrium price is a linear function of economy state variables. It is assumed that a linear equilibrium exists and we completes the proof by deriving it afterwards.

Formally, the optimisation problem of a type $i \in \{R, G\}$ investor at t is to choose the holding of the risky asset X_t^i to maximise the expected utility $\mathcal{U}(\cdot)$ from the final wealth of the following form

$$\underset{X_t^i}{\text{Max}} E_i[-e^{-\lambda W_T^i} | \mathcal{F}_t] \tag{3.7}$$

The utility function $\mathcal{U}(\cdot)$ takes the exponential form with λ the Arrow-Pratt risk aversion coefficient. $\mathcal{F}_t$ is the information set at date t. W_T^i is the wealth of type i in the final period T. We assume the investor consumes all the wealth at T.

Let P_t represents the real equilibrium price and $Q_t = P_t - P_{t-1}$ the real excess return on one share of the stock. Based on information set $\mathcal{F}_t$, both P_{t+k} and Q_{t+k} are random process for $k = 1,\ 2,\ 3, ..., T-t-1$. We denote P_t^i and $Q_t^i = P_t^i - P_{t-1}^i$ as the corresponding random process of the equilibrium price and excess return based on the beliefs of type i[8]. Since the investors are assumed to be price takers, the following condition on W_t^i for type i in the optimisation problem (3.7) is presented.

$$W_{t+1}^i = W_t^i + X_t^i Q_{t+1}^i \tag{3.8}$$

with $W_T^i = W_{T-1}^i + X_{T-1}^i \left(V - P_{T-1}^i\right)$.

With complete descriptions of the information structure and preferences given in subsection 3.3.2 and 3.3.3, we will formally present the competitive equilibrium in the next section.

3.3.4 Equilibrium

Previous models with symmetric information usually discuss the Competitive Rational Expectations Equilibrium (CREE). In a classic static CREE, each trader conjectures a price function from the information sets to the price space as given. Furthermore, each trader believes that the trading does not impact the equilibrium price. Given each trader's optimal demand, the market clearing condition has to be satisfied. Rationality dictates that the actual price process must coincide with the conjectured one. Under the dynamic setting, the conjectured price function is replaced by a conjectured price process and must

[8]The equilibrium price process is determined by investor's expectations of future values, which are determined by their beliefs of the signals. Therefore, the random processes of the real prices are different from the perspective of each type since they hold different views of the signal generating process.

be consistent with the real price process function as long as traders holds correct beliefs on the information process. In our model, instead, only a fraction of the population holds correct beliefs. We therefore explore a belief-consistent competitive equilibrium, where the equilibrium is achieved when the real price process function is consistent with conjectured price process for those with correct beliefs on the signal process. This is expressed formally in Definition 3.

Definition 3 *(Competitive Equilibrium)*
In an economy where investor $i \in \mathcal{I}$ holds subjective beliefs y_t^i on a public signal process y_t, the competitive equilibrium of the economy is $\left\{(X_t^i,\ P_t)\right\}_{t=1}^{t=T-1}$, where X_t^i is the optimal portfolio:

$$X_t^i = \underset{x_t^i}{argmax} E_i \mathcal{U}\left(W_{t-1} + x_t^i(P_{t+1} - P_t) + \sum_{j=t+1}^{T-1} X_j^i (P_{j+1} - P_j)|\mathcal{F}_t\right);$$

and P_t is the equilibrium price process clearing the market:

$$\sum_{i \in \mathcal{I}} X_t^i = 0.$$

Note that $\mathcal{F}_t = \{\mathcal{F}_0,\ y_1, ..., y_t\}$ is public information at date t and $\mathcal{U}(\cdot)$ is the CARA utility function defined as before. Furthermore, since the real information process y_t is different from subjective beliefs y_t^i, we use E_i to represent the expectations under the subjective beliefs of type i. To be specific, E_R is the expectation under the priors on signals y_t described by Eq. (3.2), while E_G is the expectation under the priors described by Eq. (3.3)-(3.4).

Since the real process of information y_t is only consistent with the rational trader's beliefs y_t^R, the price process P_t is different from the G type's conjectured price process P_t^G. However, the equilibrium is still "rational" since both types in the market hold the "correct" conjectured price given that their beliefs on information is correct. Furthermore, the price must clear the market, that is $\sum_{\mathcal{I}} X_t^i = 0$. Starting from the last period, the optimal holdings of the asset in each period can be solved recursively using backward induction.

The equilibrium in our model depends on the hierarchy of expectations. The first order expectations of two types are different because of their different beliefs[9]. Since y_t^i is public knowledge to both types, each type has the correct higher order expectation about the other, i.e., type i's expectation of type j's expectation is the same as type j's expectation[10].

[9] Expectations of the true states of the economy, e.g., $E_R[S_{t+k}|\mathcal{F}_t] \neq E_G[S_{t+k}|\mathcal{F}_t]$.
[10] Expectations of the other type's expectations, e.g., $E_R[E_G[S_{t+k}|\mathcal{F}_t]|\mathcal{F}_t] = E_G[S_{t+k}|\mathcal{F}_t]$.

Therefore, the equilibrium price can be simplified and expressed as a linear function of the first order expectations.

We define the first order expectations $V^R_{t+1|t}$, $V^G_{t+1|t}$, and $\epsilon_{t+1|t}$ to be the state variables and define Ψ_t as the state vector of the economy at t, that is $\Psi_t = [1,\ V^R_{t+1|t},\ V^G_{t+1|t},\ \epsilon_{t+1|t}]'$. Constant 1 is set as the first dimension of the state vector to represent a constant effect. In a linear equilibrium, the realisation of the equilibrium price P_t at every period t is a linear function of the state vector of the economy and we have our Proposition 9 below followed by its proofs.

Proposition 9 *(Linear Equilibrium)*
Given the economy defined as in Section 3.3.1 and two type of investors with beliefs defined in Eq.(3.2)-(3.4)*, there exists a competitive equilibrium in which the price of the risky asset is a linear function of the state variables of the economy:*

$$P_t = \alpha_t \Psi_t.$$

where $\alpha_t = [\alpha_{0,t},\ \alpha_{1,t},\ \alpha_{2,t},\ \alpha_{3,t}]$ *is a constant vector recursively determined by the system.*

To prove Proposition 9, we follow the standard steps in deriving a CREE as in Grossman (1976), He and Wang (1995)– 1. Propose a conjectured linear equilibrium price; 2. Solve the optimisation problem for each investor and derive the optimal portfolio for each period; 3. Aggregate the optimal demands across the population and apply the market clearing conditions to show that the equilibrium price derived is consistent with the proposed equilibrium in the step 1. Details of the proof are given in appendix.

In the next two sections, we fully explore the properties of the equilibrium and the impacts of the information flow on the market. We start by looking into a single type market with only rational traders and complete our analysis of the mixed market with both types later.

3.4 Rational Benchmark

Before formally discussing the model's implications in detail, a close study of the baseline model in which there are only rational traders is necessary. By comparing the mixed market with the benchmark, we can better understand the roles of rational and irrational traders in the equilibrium. It is assumed that the economy is correctly captured by the rational traders' beliefs—signal errors ξ_t are independent over periods. The benchmark case is merely a dynamic version of the CREE. We define the equilibrium price and dollar returns in a pure rational market as $P_{R,t}$ and $Q_{R,t}$.

It is easily seen that with identical beliefs, the optimisation problem is similar to the case in a myopic world[11]:

$$X_t^R = \frac{E_R\left(P_{R,t+1}|\mathcal{F}_t\right) - P_{R,t}}{\lambda Var_R(P_{R,t+1}|\mathcal{F}_t)}.$$

Therefore, the following proposition is presented:

Proposition 10 *(Expectation, Volatility and Autocovariance)*
i) trade theorem holds: $X_t^R = \dfrac{V_{t+1|t}^R - P_{R,t}}{\lambda Var(P_{R,t+1}|F_t)} = 0$; *ii)* $P_{R,t}$ *is a martingale process, that is conditional information set* $\mathcal{F}_t$*:*

$$E_R\left[P_{R,t+k}|\mathcal{F}_t\right] = V_{t+1|t}^R = P_{R,t}, \quad 0 < k < T - t;$$

iii) $Q_{R,t+k}$ *is martingale difference process, that is conditional information set* $\mathcal{F}_t$*:*

$$E_R\left[Q_{R,t+k}|\mathcal{F}_t\right] = 0, \quad 0 < k < T - t;$$

iv) Dollar returns are uncorrelated over periods:

$$Corr_R\left(Q_{R,t_2},\ Q_{R,t_1}\right) = 0, \quad 0 < t_1,\ t_2 < T,\ t_1 \neq t_2.$$

Proposition 10 shows that in the rational market with identical beliefs and public information, no trade happens and the optimal holdings are 0. The equilibrium price is a martingale sequence and always equals the predicted fundamental value based on the current information. In the case of no information, predictions for future prices simply take the mean of the fundamental value, that is $E_R\left[P_{R,t}|\mathcal{F}_0\right] = \mu$. Since the price sequence is martingale, the expected return is 0. The dollar returns are uncorrelated because rational investors sufficiently update their beliefs to new signals in every period. This can be seen from $Corr_R\left(v_{t_1}^R,\ v_{t_2}^R\right) = 0$. As a shock arrives, the price will immediately adjust to the right level and it always reflects all the available information in the market. Returns in the future only rely on future signals and thus are uncorrelated with historical returns. The market is informational efficient without over- or under- reaction to news. On the other hand, the equilibrium price in the mixed market is a linear combination of all state variables and one

[11]In Brown and Jennings (1989), Singleton (1986), the myopic optimal holding is also $X_t^R = \dfrac{E_R\left(P_{R,t+1}|\mathcal{F}_t\right) - P_{R,t}}{\lambda Var_R(P_{R,t+1}|\mathcal{F}_t)}$. See Brunnermeier (2003) for a review.

can guess that the return sequence will exhibit some serial correlations over time. Lemma 7 below presents further extension to Proposition 10, where all the traders in the market are gambler's fallacy type.

Lemma 7 *(Equilibrium Price in the Single Type Market)*
The equilibrium price in a market with only type i investors takes the following format:

$$\forall\, i \in \mathcal{I}, \quad P_{i,t} = V^i_{t+1|t}$$

Lemma 7 tells us that in a single-type market, the equilibrium price always only loads on the expected value from that type but not the "lucky streak" term $\epsilon_{t+1|t}$. This is the case even in a market with only gambler's fallacy investors. Therefore, the gambler's fallacy market is "rational" in the sense that the price only reveals the fundamental value. This result is similar to a classic CREE model. The "lucky streak" does not enter into the equilibrium price in a pure fallacy market due to the failure of fallacy type in recognising their mistakes. It becomes important in pricing the mixed market because of the effort of the rational type in rationalising the market.

The "lucky streak" indicates how lucky in the history by its definition, but it can also be viewed as a market "sentiment" evaluation. From the perspective of the gambler's fallacy traders, a negative $\epsilon_{t+1|t}$ indicates a recovering regime while a positive $\epsilon_{t+1|t}$ indicates just the opposite. In the next session, the implications of the "lucky streak" will be fully discussed.

3.5 Equilibrium Analysis

In this section, we consider the mixed market with both types of traders. The study first look into the properties of optimal portfolios followed by the study of market reactions to information flows. It also examines interactions of two types of traders and their roles in pricing. At the end of this section, we explore the asset pricing implications.

3.5.1 Optimal Portfolio

We first give the properties of the optimal portfolio holdings by each type in the following Proposition.

Proposition 11 *(Optimal Portfolio)*

1. *The optimal portfolio of i'th type is:*

$$X_t^i = \frac{1}{\lambda}\Gamma_t^i E_i\left(Q_{t+1}^i|\mathcal{F}_t\right) - \frac{1}{\lambda}\Gamma_t^i g_t^i \Psi_t$$

2. *The optimal demand is a linear function of the difference in valuations of two types and the "lucky streak":*

$$X_t^i = \frac{1}{\lambda}F_{d,t}^i\left(V_{t+1|t}^R - V_{t+1|t}^G\right) + \frac{1}{\lambda}F_{\epsilon,t}^i \epsilon_{t+1|t}$$

where $i \in \mathcal{I}$; Q_{t+1}^i is the dollar return at $t+1$; Γ_t^i, g_t^i, $F_{d,t}^i$ and $F_{\epsilon,t}^i$ are variables uniquely defined by the equilibrium and are shown in appendix.

The first part of Proposition 11 shows that the optimal portfolio is composed of two parts. The first component is a mean-variance efficient portfolio similar to a static CREE model, reflecting the trade-off between expected return and risk[12], where $E_i(Q_{t+1}^i|\mathcal{F}_t)$ is the expected return in the next period and Γ_t^i is the inverse of the normalised estimation of the variance. Like other intertemporal models, where investors are non-myopic (Merton, 1973; Wang, 1994; He and Wang, 1995), the second component $-\frac{1}{\lambda}\Gamma_t^i g_t^i \Psi_t$ is a hedging portfolio reflecting the additional position today due to the predictions of future price changes. The existence of heterogeneous beliefs creates non-zero correlations in returns over time. Since investors in our model are non-myopic, they are willing to take additional positions today to maximise their lifetime wealth. The conclusions at this step hold without the market clearing constraint.

By adding in the market clearing condition, the study further presents the conclusions in the second part of Proposition 11. It shows that the optimal portfolio is also a linear function of system state variables. It further decomposes the linear optimal portfolio into two parts– the difference in two types's asset valuations and the "lucky streak"with $F_{d,t}^i$ the weight on the valuation difference and $F_{\epsilon,t}^i$ the weight on the "lucky streak". Figure 3.3 in appendix shows that $F_{d,t}^R$ and $F_{\epsilon,t}^R$ are positive, while $F_{d,t}^G$ and $F_{\epsilon,t}^G$ are negative[13]. The intuition for these results is the following. Gambler's fallacy type tends to long the risky asset $-F_{d,t}^G > 0$ when their estimated value is higher than that of rational investors. Due

[12]In a standard Grossman model, the mean-variance efficient portfolio takes this format: $X = \frac{E(V-P)}{\lambda Var(V)}$

[13]Simulation results are similar for a wide range of parameters.

to the gambler's fallacy, they are willing to short the asset ($F^G_{\epsilon,t} < 0$) when the asset has a long lucky history ($\epsilon_{t+1|t} > 0$). The rational type, instead, takes the opposite strategy of gambler's fallacy type.

Figure 3.3 also shows that the absolute values of the weights on the valuation difference and "lucky streak" $|F^i_{d,t}|$ and $|F^i_{\epsilon,t}|$ increase over time, indicating that given the equilibrium price, both types are willing to trade more as they become better informed since the perceived risk of the asset decreases as more information comes in. In the later section 3.5.3, we further analyse the role of two types in determining the equilibrium price, which is affected by the relative riskiness of the asset perceived by two types due to their different beliefs.

3.5.2 Over-reaction and Under-reaction

In this subsection, we study how expectations of price, return, optimal holding and trading volume evolve in face of information shock.

Proposition 12 *(Expectations in the Mixed Market)*

$$E_R\left[P_{t+k}|\mathcal{F}_t\right] = \alpha_{t+k}\prod_{l=0}^{t-1} A^R_{\Psi,t+k-l}\Psi_t, \tag{3.9}$$

$$E_R\left[Q_{t+k}|\mathcal{F}_t\right] = A_{Q,t+k}\prod_{l=0}^{t-2} A^R_{\Psi,t+k-1-l}\Psi_t, \tag{3.10}$$

Compared to the rational benchmark, the expectation of price is no longer constant but changes over time. The expected price now loads on the fallacy type's expectations of the value and the "lucky streak". To understand the results in Proposition 12, let's consider the one period version of Eq. (3.9) and its gambler's fallacy counterpart below.
The expectation of the rational type is:

$$\begin{aligned} E_R\left[P_{t+1}|\mathcal{F}_t\right] = {} & \alpha_{1,t+1}V^R_{t+1|t} + \alpha_{2,t+1}V^G_{t+1|t} + \alpha_{3,t+1}(\delta-\beta)\epsilon_{t+1|t} \\ & + (\alpha_{2.t+1}K^G_{t+1} + \alpha_{3,t+1}k)V^R_{t+1|t} - (\alpha_{2,t+1}K^G_{t+1} + \alpha_{3,t+1}k)V^G_{t+1|t} + \\ & (\alpha_{2,t+1}\frac{\beta}{\delta-\beta}K^G_{t+1} + \alpha_{3,t+1}\frac{\beta}{\delta-\beta}k)\epsilon_{t+1|t}. \end{aligned}$$

The expectation of the gambler's fallacy type is:

$$\begin{aligned} E_G\left[P_{t+1}|\mathcal{F}_t\right] = {} & \alpha_{1,t+1}V^R_{t+1|t} + \alpha_{2,t+1}V^G_{t+1|t} + \alpha_{3,t+1}(\delta-\beta)\epsilon_{t+1|t} \\ & - \alpha_{1.t+1}K^R_{t+1}V^R_{t+1|t} + \alpha_{1,t+1}K^R_{t+1}V^G_{t+1|t} - \alpha_{1,t+1}\frac{\beta}{\delta-\beta}K^R_{t+1}\epsilon_{t+1|t}. \end{aligned}$$

We leave the full analysis of the parameter α_t to the next section, but claim that $\alpha_{1,t}$, $\alpha_{2,t}$ and $\alpha_{3,t}$ are all positive. The first three items in these two expressions are the same, representing the price of the last period. Other terms represent the adjustments to the previous price based on the new information. Consistent with the intuition, it is shown that both types over-weigh their own value forecasting and under-weigh the other type's value forecasting. Due to the gambler's fallacy, the "lucky streak" $\epsilon_{t+1|t}$ has a negative weight $-\alpha_{1,t+1}\frac{\beta}{\delta-\beta}K^R_{t+1}$ in the fallacy expectation but has a positive weight $(\alpha_{2,t+1}\frac{\beta}{\delta-\beta}K^G_{t+1} + \alpha_{3,t+1}\frac{\beta}{\delta-\beta}k)$ in the rational expectations. These results are consistent with each type's optimal portfolio in equilibrium described by Proposition 11.

With the full decomposition of the expectations, we are able to look into the effect of information on the market. Since the expected price and returns are no longer constant, a curious question is how will the equilibrium price and return react to some unexpected news? The simulation results with a single signal shock are presented below. [14]

Figure 3.4 plots the simulation results of an information shock on the rational price expectations in three types of markets.

The solid line represents the price expectation in the market with only rational investors. The price expectation is constant and set at the expectation of the liquidity value conditional on information at t. Correspondingly, the expected return is just 0 in the rational market. The dash line represents the rational expectations on the price in the pure fallacy market. Expectation of the price converges to the price expectation in the rational market. Expectations of the return turns from positive to negative and also converges to 0 as shown in Figure 3.5 in the appendix. This is because more information resolves all the uncertainties about the value for both types. The market with gambler's fallacy makers under-react to the information initially after the shock but over-react afterwards.

Intuitively, the gambler's fallacy type reacts slowly to recent news compared to the rational traders ($K^G_t < K^R_t$) due to the reversing prediction. Their prediction of the value is more conservative at the beginning after observing some good news. This initial insufficient updates makes following up news seems to be very optimistic, even though the following up news are merely the confirmation of the initial good information. The gambler's fallacy makers begin to boost their value estimations during these periods. Because they always predict a reversal in signals, the accumulation of "lucky" information makes them even more pessimistic about future luck. The lack of observation in strong reversal forces them

[14]We assume that both types share the same prior on the fundamental with no information observed before a public signal arrives. The new signal is different from the prior expectation and therefore creates a signal shock (information shock) for both types as indicated by v^i_t in the inference process described by Corollary 8. To better understand the effect of the signal, we simply assume that all future signal realisations take the same value as the first one. Intuitively, this assumption implies that the signal is a full revelation of the fundamental value.

to explain the good news by incorrectly believing the real value is very high, leading to overreactions afterwards. However, observed signals can not beat their expectations forever. Since the power of past signals diminishes overtime, after several periods, the predictions of the signals fall below the real observations because of the overestimation in the underlying value. This process lasts until the fallacy agents' value estimation converges back to the rational level. Finally, the rational expectations in the mix market are plotted by the dotted line. The price pattern in mixed market is similar to the pure fallacy market, but the fluctuation is less intensive due to the existence of the rational traders. Our results show that the under-reaction and over-reaction in the market may share the same root–the gambler's fallacy. The existence of gambler's fallacy traders creates arbitrage opportunities and is beneficial to the rational traders.

We further analyse the changes in the optimal holdings and the trading volume following an information shock.

Proposition 13 *(Optimal Holdings and Trading Volume)*
The expected optimal holdings for both types from the perspective of the rational agents are

$$E_R\left[X^R_{t+k}|\mathcal{F}_t\right] = \frac{1}{\lambda}F^R_{t+k}\prod_{l=1}^{k}A^R_{\Psi,t+l}\Psi_t; \quad E_R\left[X^G_{t+k}|\mathcal{F}_t\right] = \frac{1}{\lambda}F^G_{t+k}\prod_{l=1}^{k}A^R_{\Psi,t+l}\Psi_t$$

The expected trading volume Ξ is

$$E_R\left[\Xi_{t+k}|\mathcal{F}_t\right] = \frac{m}{\lambda}\left|F^R_{t+k}\prod_{l=1}^{k}A^R_{\Psi,t+l}\Psi_t - F^R_{t+k-1}\prod_{l=1}^{k-1}A^R_{\Psi,t+l}\Psi_t\right|$$

where m is the proportion of rational traders in the market.

Based on Proposition 13, Figure 3.6 reports the simulation results of both rational and fallacy traders' expected positions over time.

Following a positive information shock, rational traders initially long the asset because of the under-reaction of the gambler's fallacy trades, but gradually enters into the short position as gambler's fallacy traders become over-reacting. Together with the pattern of price over time, we can see that rational traders tend to chase the price trends. By introducing in the gambler's fallacy traders, the rational traders in the market act as if they have short term hot-hand fallacy on prices. As shown before, the equilibrium price contains no more information other than the signals observed. Our assumption of gambler's fallacy on information, therefore, can be interpreted as beliefs on price. To be more specific, assuming gambler's fallacy on the information flow is similar to assuming that part of the investors in the market have gambler's fallacy, while others have short-term hot-hand fallacy on prices. Our model is capable to accommodate the survey evidence in Greenwood and Shleifer (2013) which pointed out that extrapolators in the market hold optimistic predictions on returns when recent prices are high (hot-hand fallacy). Therefore, as long as there exists gambler's fallacy traders who believe an increase in price should be quickly reversed, or say, believers of mean-reversal, the momentum trading strategy is beneficial. Our assumption of gambler's fallacy actually rationalises the existence of hot-hand fallacy and as well as the momentum trading strategy.

Besides, as more information comes in, beliefs in the market converge. However, the speed of convergence in positions is very slow because both types become more confident in their estimations and are willing to take advantage of any difference in valuations. Simulation results show that this is the case for a very long period until the time approaches the last few periods when both types must unwind their positions. The simulation results of expected trading volume are shown in Figure 3.7 in the appendix. Trading volume decreases overtime after the announcement shock and will bounce back as time approaches the realisation date if there is no further information. The intuition is similar to He and Wang (1995). Due to fewer trading opportunities in the future, investors will unwind their risky positions as the liquidation date gets close. Since trading occurs due to the difference in valuations, the trading volume maximises when the market contains equal proportion of two types, that is $m = 0.5$ and decreases as the market approaches one type from either direction.

Our results on the equilibrium price in this section are similar to Rabin and Vayanos (2010) in which a representative agent who has gambler's fallacy on the dividends series creates under- and over-reaction in market. They show that people's prediction on the signal initially exhibits gambler's fallacy, but turns into hot-hand fallacy later on as the change in state dominates the total effect. Different from their work, we further introduce in the rational type and give the full equilibrium. Our analysis focuses on the equilibrium price (the estimation of the unobserved states) instead of on the signals. We show that following a signal shock, the price will stay away from the fundamental value for a long time. The rational traders are not powerful enough to rationalising the price to the right level quickly. We will further analyse these facts in the next section. By presenting the equilibrium, we also derive results on each type's trading strategies and the market trading volume following an information shock.

A more recent study by Barberis et al. (2013) developed a consumption-based asset pricing model in which rational investors trading against investors form beliefs about future prices changes in the stock market by extrapolating past price changes—similar to hot-hand fallacy. Simulation results showed that their model successfully derived the momentum reversal but not momentum due to the existence of rational investors and the assumption that the extrapolators' demand for the risky asset depends on the most recent price change. Compare to their work, by simply assuming gambler's fallacy on information series, we successfully derive both momentum and reversal. The rational traders in our model act as having short term hot-hand fallacy in price changes, close to their assumption of the irrational type.

So far, we complete our analysis of the effect of an information shock on the equilibrium price and optimal holdings. In the next subsection, we further look into the properties of the equilibrium in order to understand the roles of each type in determining the price.

3.5.3 Market Dominance

We have shown that over- and under-reaction are still significant even though there are rational traders living in the market. To better understand the interaction of rational and fallacy traders, we further explore the linear competitive equilibrium in this subsection.

Equilibrium price in the mixed market is different from the single type market as it is determined by a combination of $V^R_{t+1|t}$, $V^G_{t+1|t}$ and $\epsilon_{t+1|t}$ as stated in Proposition 9. The weights $\alpha_{t,1}$ and $\alpha_{t,2}$ on $V^R_{t+1|t}$, $V^G_{t+1|t}$ represent the pricing power of each type, since they tell us how much the equilibrium price is determined by the estimation of the value from that type. The following proposition formally presents the properties of α_t, which we define as "Market Dominance".

Proposition 14 *(Market Dominance)*
In a linear equilibrium of the mixed market, where $P_t = \alpha_t \Psi_t$, $\alpha_t = [\alpha_{0,t},\ \alpha_{1,t},\ \alpha_{2,t},\ \alpha_{3,t}]$, $\Psi_t = [1,\ V^R_{t+1|t},\ V^G_{t+1|t},\ \varepsilon^\delta_{t+1|t}]'$, we have
i) $\alpha_{0,t} = 0$.
ii) $\alpha_{1,t} + \alpha_{2,t} = 1$.
iii) $\alpha_{1,t} > 0$ decreases in t and $\alpha_{2,t} > 0$ increases in t.
iv) $\alpha_{3,t} > 0$ decreases in t.

The first part of Proposition 14 points out that the equilibrium price contains no constant term. It is a pure linear combination of all the state variables of the economy. This result is consistent with the previous studies as it is assumed that the net supply Θ is 0[15]. In the case of $\Theta > 0$, $\alpha_{0,t}$ becomes non-zero. The value of Θ does not affect our major results. Therefore, the full analysis of $\Theta > 0$ is presented in the appendix.

The second part states that the equilibrium price contains a weighted average of two types' value expectations. It indicates that a change in the fundamental value captured by the investors also changes the price by the same amount; a result similar to a single type market. Besides, it is consistent with previous CRRE models in which the price is usually a weighted average of all investors' expectations.

The third statement indicates that rational investors lose market influence over time since the equilibrium price over-weighs the fallacy trader's valuation as the liquidation date gets closer. Figure 3.8 provides the simulation results of $\alpha_{1,t}$, $\alpha_{2,t}$ and $\alpha_{3,t}$. It is shown that even with a large proportion of rational traders, for example $m = 0.9$ in Figure 3.8, the market still ends up in a gambler's fallacy regime, even though initially, the equilibrium price is largely determined by the rational trader's valuation.

To understand these results, the study first considers the traditional CREE in Grossman (1976) where the market power of an investor is determined by the risk attitudes[16]. Less risk-averse investors are more powerful in price determination because they are fearless and trade more aggressively. Similar intuition can be applied here. However, the difference in risk attitudes no longer comes from different Arrow-Debreu risk-aversion parameters, but from different estimations of the riskiness of the asset—the conditional variance of the liquidation value. The conditional variance reflects the confidence level of an investor in the value estimations. A lower variance implies a safer asset, making investors act as if they are less risk-averse. Even though in general, both types become more confident as more information comes in, the fallacy traders are comparably less "risk-averse" than rational ones. Intuitively, gambler's fallacy investors fail to update the value estimation sufficiently with small Kalman Gains K_t^G. They believe that their predictions are very accurate because

[15]An example is Grossman (1976). The equilibrium price is

$$P = \left(\frac{1}{1+r}\right) \frac{\sum_{i=1}^{n} \dfrac{\mu + \frac{\sigma^2}{\sigma^2+\sigma_i^2}(s_i - \mu)}{\alpha_i \sigma^2 \left(1 - \dfrac{\sigma^2}{\sigma^2+\sigma_i^2}\right)} - Z}{\sum_{i=1}^{n} \dfrac{1}{\alpha_i \sigma^2 \left(1 - \dfrac{\sigma^2}{\sigma^2+\sigma_i^2}\right)}}$$

, where $\mu + \frac{\sigma^2}{\sigma^2+\sigma_i^2}(s_i - \mu)$ is agent i's expectation. Z is the net supply of the stock. r is the risk free return. By setting the risk-free rate r to 0 and the supply of stock to 0, the equilibrium price is a weighted average of the expectations from all agents in the market.

[16]Grossman (1976) showed that risk-tolerant agents with lower Arrow-Debreu risk-aversion parameters have a strong influence on the equilibrium price. This result can be seen from their equilibrium price, where α_i is agent i's risk-aversion parameter.

they put those unpredicted shock into the "lucky streak" and believe it will be reversed in the next period. This failure leads to smaller estimated variance of the asset ($\Sigma_{V,t}^R > \Sigma_{V,t}^G$)[17] and over-confidence in their predictions of fundamentals[18]. Under-estimation in the asset risk enables the fallacy traders to bear extra risks and trade more aggressively. They, therefore, become powerful in pricing as the realisation date approaches.

It is also shown that the rational traders tend to be more powerful in determining the price initially than around the liquidation date. This is due to the other factor that governs the risk attitudes in a non-myopic model as explained below—the intertemporal diversification effect.

As pointed out before, heterogeneous beliefs create correlations in returns over time, and therefore, non-myopic investors always take additional hedging positions today to maximise their lifetime utility. To be more specific, the assets tomorrow and in the future, are treated as another risky asset for determining the optimal position today. The new portfolio can provide same returns with lower risks due to the diversification as long as these two assets are not perfectly positive correlated. This effect is similar to the diversification effect in a static portfolio building problem with different assets. For this reason, the type of investors who face more negative correlated periods in the future are more willing to trade aggressively. A short intuition is provided here but full analysis of correlations is given in the next section. It is shown in the last section that from the perspective of rational traders, the slow reactions of fallacy traders create short-term under-reaction and long-term over-reaction. Therefore, under-reaction dominates the total effect when the liquidation date is close, while over-reaction is more powerful when the liquidation is far. As shown in the next section, over-reaction generates negative correlation between current and future returns, making rational traders more willing to take heavier positions and trade aggressively in the early trading periods. As the liquidation date approaches, returns across periods become positively correlated due to the short-term under-reaction, making hedging less desirable for rational traders. On the other hand, the fallacy traders take just the opposite view of the rational ones on the reactions of the market and become more willing to hedge as time goes on. The changes in the incentive of hedging is responsible for the decrease of $\alpha_{1,t}$ and the increase of $\alpha_{2,t}$. In general, over-reaction (under-reaction) encourages rational investors to trade aggressively (conservatively) but discourages mean-reverting believers to trade conservatively (aggressively).

[17]Proof of this result is given in the proof of Proposition 15.

[18]In Grossman (1976), the well-informed agent with lower signal noise σ_i^2 have stronger influence on the equilibrium price. A comparable explanation in our model is fallacy agents view signals as having more noise in estimating the value because they need to explain the lucky streak term from the signals as well.

Proposition 15 also states that α_3 is positive and decreases overtime as shown in Figure 3.8. The "lucky streak" term becomes important in determining the price in the mixed market due to the differences in expectations. As stated before, the "lucky streak" term can be viewed as a market sentiment indicator. When $\epsilon_{t+1|t}$ is positive, the fallacy makers are over-pessimistic and falsely believe the market is reverting, while the real market is not. The equilibrium price, therefore, loads positively on $\epsilon_{t+1|t}$ due to the effort of the rational agents in rationalising the price[19].

α_3 decreases for the same reason as α_1. As shown, the "lucky streak"represents the price rationalising process due to the trading behaviour of the rational investors. Since the rational type loses the market dominance overtime, α_3 also decreases. Simulation results (Appendix Figure 3.9) also show that the larger the proportion of fallacy investors in the population, the smaller the α_3, which is consistent with pricing power of two types.

Proposition 15 below presents a sufficient conditional for the gambler's fallacy investors to dominate the market.

Proposition 15 *(Switch in Market Dominance)*
i) Gambler's fallacy investors dominate the market for $t \to +\infty$ if

$$\frac{m}{n} < \lim_{t \to +\infty} \frac{\Sigma^R_{V,t|t-1}}{\Sigma^G_{V,t|t-1}}$$

where $m+n=1$ with m and n representing the proportion of rational and gambler's fallacy type in the population. ii) $\lim_{t \to +\infty} \frac{\Sigma^R_{V,t|t-1}}{\Sigma^G_{V,t|t-1}}$ increases in gambler's fallacy parameters β and δ.

Part i) of Proposition 15 points out a sufficient condition for the gambler's fallacy investors to dominate the market. The left hand side is the ratio of two types in the population, while the right hand side reflects the relative riskiness of the asset captured by different types. Proposition 15 stataes that the switch is more likely to happen when the proportion of rational traders is small and when gambler's fallacy traders feel the asset is safer as compared to the rational traders. Furthermore, it is noticed that the right hand side of the inequality $\frac{\Sigma^R_{V,t|t-1}}{\Sigma^G_{V,t|t-1}}$ is always greater than 1. This is because gambler's fallacy investors attribute signal shocks partially to previous realisation, leading to stable value estimations. Therefore, when there are equal proportions of two types in the market, that is $m = n = 0.5$, the fallacy investors will finally dominate the market when the trading period is long enough.

[19]This results can also be seen from the decomposition of optimal demands in Proposition 11 in which the rational portfolio load positively on $\epsilon_{t+1|t}$.

Part ii) of Proposition 15 indicates that the right hand side limit in Part i) increases in gambler's fallacy (β and δ). The limit is larger when $\Sigma^G_{V,t|t-1}$ decreases faster to 0 than $\Sigma^R_{V,t|t-1}$, which means gambler's fallacy agents become over-confident in their prediction quicker than rational agents. Indeed, gambler's fallacy (law of small number) is beneficial by making people learn the value faster even though their predictions contain large errors. In other words, they believe their estimations are accurate while they are not. Since strong gambler's fallacy makes people over-confident in their forecasting, the switch is more likely to happen in this situation. In the extreme case when $\delta = 1$, that is, past information does not decay in power for forecasting, $\underset{t\to+\infty}{lim} \frac{\Sigma^R_{V,t}}{\Sigma^G_{V,t}}$ explodes and a switch in dominance will always happen.

3.5.4 Asset Pricing Implications

In this section, we present the detailed analysis of asset pricing implications of our model including 1. Momentum and Reversal, and 2. Market Volatility.

3.5.4.1 Momentum and Reversal

Empirically, returns on the stock market are positively autocorrelated at short lags, and negatively autocorrelated at long lags. It is shown that following an information shock, the market with gambler's fallacy traders exhibits short-term under-reaction and long-term over-reaction. The general properties of this result are further studied by examining the autocorrelations in returns.

Proposition 16 *(Autocorrelation)*
For any two periods t_1, t_2 with $t_1 > t_2$, autocovariance between the returns Q_{t_1} and Q_{t_2} is

$$Cov_R\left(Q_{t_1},\, Q_{t_2}\right) = A^R_{Q,t_1} \prod_{k=0}^{t_1-t_2-1} A^R_{\Psi,t_1-k} \left[Var\left(\Psi_{t_2}\right) A^{R\mathbf{T}}_{Q,t_2} + B^R_{\Psi,t_2} Var\left(v_{t_2}\right) B^{R\mathbf{T}}_{Q,t_2}\right] \tag{3.11}$$

where $Var_R\left(\Psi_t\right) = \left(\sum\limits_{n=1}^{t} D_n\right) \sigma^2_V \left(\sum\limits_{n=1}^{t} D_n\right)^{\mathbf{T}} + \sum\limits_{n=1}^{t} \left(D_n D_n^{\mathbf{T}}\right) \sigma^2_\xi$,
and $D_n = \left(\prod\limits_{l=n+1}^{t} G_l\right) \bar{K}_n$.

Table 3.1 gives the autocovariance of one-period ahead returns at lags of 1, 3, 5, 7 and 9 based on the same information set at $t = 10$. The proportion of rational investors is 0.5.

Compared to the rational benchmark where any two returns are uncorrelated, the autocovariance between one-period returns in the mixed market are no longer 0. Simulation

Table 3.1: COVARIANCE MATRIX OF RETURNS

$\mathbf{Cov^R(Q_{t_1},\ Q_{t_2}\|\mathcal{F}_{10})}$				
$Cov(Q_{11},\ Q_{12})$	$Cov(Q_{11},\ Q_{14})$	$Cov(Q_{11},\ Q_{16})$	$Cov(Q_{11},\ Q_{18})$	$Cov(Q_{11},\ Q_{20})$
0.02677	-0.00398	-0.00436	-0.00313	-0.00219
$Cov(Q_{13},\ Q_{14})$	$Cov(Q_{13},\ Q_{16})$	$Cov(Q_{13},\ Q_{18})$	$Cov(Q_{13},\ Q_{20})$	$Cov(Q_{13},\ Q_{22})$
0.00631	0.00008	-0.00075	-0.00074	-0.00061
$Cov(Q_{15},\ Q_{16})$	$Cov(Q_{15},\ Q_{18})$	$Cov(Q_{15},\ Q_{20})$	$Cov(Q_{15},\ Q_{22})$	$Cov(Q_{15},\ Q_{24})$
0.00378	0.00085	0.00009	-0.00012	-0.00017
$Cov(Q_{17},\ Q_{18})$	$Cov(Q_{17},\ Q_{20})$	$Cov(Q_{17},\ Q_{22})$	$Cov(Q_{17},\ Q_{24})$	$Cov(Q_{17},\ Q_{26})$
0.00230	0.00075	0.00022	0.00003	-0.00004
$Cov(Q_{19},\ Q_{20})$	$Cov(Q_{19},\ Q_{22})$	$Cov(Q_{19},\ Q_{24})$	$Cov(Q_{19},\ Q_{26})$	$Cov(Q_{19},\ Q_{28})$
0.00148	0.00057	0.00021	0.00006	0.00000
Parameters: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$				

results in Table 3.1 confirm that returns at short lags are positively correlated and are negatively correlated at longer lags. The autocovariance converges to 0 as the prediction date becomes more distant since the uncertainty resolves as more information is achieved. Our model conclusions support the empirical evidence and provide explanations to the short term momentum and long term momentum reversal.

Some previous work, for example, Long et al. (1990), Barberis, Shleifer and Vishny (1998), Hong and Stein (1999), Barberis and Shleifer (2003), Barberis et al. (2013) also generated short-term momentum, long-term reversals or both. All these papers assumed momentum traders directly. Long et al. (1990) and Hong and Stein (1999), for example, assumed that the demand of the risky asset today depends positively on the previous price changes. Our model assumption is fundamentally different from these papers in that we assume gambler's fallacy in information flow and the hot-hand trading behaviour is rational optimal choices in a market with gambler's fallacy.

Moreover, it is also shown that a better informed (less volatile) market has a longer period of momentum as compared to the length of reversal. This is because people are more reluctant to adjust to any new signals when they are relatively confident in their forecasting. Therefore, it takes longer time for the initial under-reaction to change into over-reaction. Together with dynamic diversification effect as shown in the last section, this result tells that a market sensitive to news is more likely to be rational, while a less sensitive market tends to be dominated by fallacy valuation.

3.5.4.2 Volatility

We further compare the volatility of price in a mixed market with the rational benchmark. Empirically, observed stock market returns are thought to exhibit excess volatility, which could not be fully explained by predicted fluctuations in cash flows. Besides, the market volatility is highest in crisis but lowest in the stable economy growth. It is now examined whether this is the case in our model by deriving the price volatility in a mixed market.

Proposition 17 *(Volatility)*
The variance of price in the mixed market is:

$$
\begin{aligned}
Var_R\left(P_t\right) &= \alpha_t Var_R\left(\Psi_t\right)\alpha_t^{\mathbf{T}} \\
&= \sigma_V^2\left[\alpha_{1,t}\sum_{n=1}^{t}\left(\prod_{l=n+1}^{t}(1-K_l^R)K_n^R\right)+\alpha_{2,t}C_{t,1}+\alpha_{3,t}C_{t,2}\right]^2 \\
&\quad + \sigma_\xi^2\left[\sum_{n=1}^{t}\left(\alpha_{1,t}\prod_{l=n+1}^{t}(1-K_l^R)K_n^R+\alpha_{2,t}C_{t,3}+\alpha_{3,t}C_{t,4}\right)^2\right]
\end{aligned}
$$

where $Var_R\left(\Psi_t\right)=\left(\sum_{n=1}^{t}D_n\right)\sigma_V^2\left(\sum_{n=1}^{t}D_n\right)^{\mathbf{T}}+\sum_{n=1}^{t}\left(D_nD_n^{\mathbf{T}}\right)\sigma_\xi^2$,
and $D_n=\left(\prod_{l=n+1}^{t}G_l\right)\bar{K}_n$. $C_{n,t}$ *for* $n=1,2,3,4$ *are determined by the products of the transition matrices.*

In the rational benchmark, the variance of price is:

$$
Var_R\left(P_t\right)=\sigma_V^2\left[\sum_{n=1}^{t}\left(\prod_{l=n+1}^{t}(1-K_l^R)K_n^R\right)\right]^2+\sigma_\xi^2\left[\sum_{n=1}^{t}\left(\prod_{l=n+1}^{t}(1-K_l^R)K_n^R\right)^2\right].
$$

In the mixed market, the variance is still a linear combination of σ_V^2 and σ_ξ^2 but with different weights. Based on Proposition 17, Table 3.2 reports the variance of price for several $(m,\ t)$ pairs. Simulation results show that the price becomes less volatile as more fallacy investors enter into the market. The following lemma gives a more straightforward intuition about this result by deriving the variance of price one period ahead.

Lemma 8 *(One-period Ahead Volatility)*
In the special case, the volatility of the price one period ahead is:

$$
\sigma_{P,t+1|t}^R=\left(\alpha_{1,t+1}K_{t+1}^R+\alpha_{2,t+1}K_{t+1}^G+\alpha_{3,t+1}k_{t+1}\right)\sigma_{v,t+1|t}^R
$$

Table 3.2: VARIANCE OF PRICES

$Var_R(P_t)$						
	$t = 5$	$t = 10$	$t = 15$	$t = 20$	$t = 30$	$t = 40$
$m = 0$	0.0543	0.0057	0.0016	0.0007	0.0002	0.0001
$m = 0.4$	0.0674	0.0085	0.0026	0.0011	0.0004	0.0002
$m = 0.7$	0.0778	0.0121	0.0040	0.0019	0.0007	0.0003
$m = 1$	0.0800	0.0200	0.0089	0.0050	0.0022	0.0013
Parameters: $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$						

The volatility of price in the mixed market is a linear combination of the Kalman Gains weighted by the market power parameters times the volatility of the shock. Instead, the volatility of price in the rational market is simply K_t^R times the volatility of the shock. Simulation results in Figure 3.10 presents the one period ahead price volatility in a mixed market and a rational market. It is shown that with the fallacy investors, the price becomes less volatile than it is in the rational market. This result comes from $K_t^R > K_t^G$. Intuitively, the slow reaction to new information of the fallacy investors reduces the immediate effect of an information shock on the market.

This result is just opposite of the result in X-CAPM (Barberis et al., 2013) where investors have hot-hand fallacy. Given that we also have hot-hand like investors in our model, the question is why our model does not show similar results? In Barberis et al. (2013), the excess volatility comes from the self-confirmation beliefs of extrapolators, who amplify the volatility by further pushing the trend up by following an initial price increase. However, the "hot-hand fallacy" traders in our model are more rational and conservative, depending on their judgements of whether the market is in the over- or under-reaction regime. The "hot-hand fallacy" in our model is only short-term and the direction of beliefs always switches after a period of time[20]. To make our work comparable to the previous work, the sign of β is changed into positive. The gambler's fallacy assumption becomes real hot-hand fallacy. The simulation results in Figure 3.11 show that the price now exhibits excess volatility in a mixed market, similar to what Barberis and Shleifer (2003) derived in their paper.

Since the estimation of the fundamental value is equal to the price in a pure one type market, the existence of the fallacy investors actually reduces the risk faced by rational investors. Empirically, the stock market prices are thought to exhibit excess volatility than the predictions under the rational assumption. However, in the case that the signals in the

[20]This can be seen from the simulation results with a signal shock in Figure 3.4.

market are in fact mean-reversal, the market will request higher premium to compensate for the additional risks the "mean-reversal" investors now take. Besides, our results may be able to explain the low volatility when the market is in stable growth. More people turn into gambler's fallacy due to cautiousness after observing a long trend, leading to lower volatility in the market.

3.6 Conclusion and Further Discussions

This paper develops a multi-period competitive rational expectations model of stock trading in which rational traders trade against investors with gambler's fallacy. The model gives an insight into the widely existed market anomalies–momentum and reversal. In contrast to previous efforts in addressing these anomalies, which typically assume a representative investor in the market, our model studies an aggregate market with heterogeneous belief holders.

Moreover, the internalisation of hot-hand fallacy in our model by assuming its opposite error, gambler's fallacy makes our model successfully capture both the short-term under-reaction and long term over-reaction in the market. Following some good information, the market initially under-reacts due to the reversal predictions from gambler's fallacy. The strong beliefs in reversal and the absence of such observations make the market gradually enter into the over-reaction mode since valuations on the fundamental are irrationally high.

Our analysis on the market dominance further rationalises the existence of momentum and reversal since it is identified that the fallacy trader's valuation plays a significant role in pricing the asset and this power does not diminish even with the existence of rational arbitragers. To be specific, the market tends to switch from "rational" regime to "fallacy" regime as more information is achieved. When the liquidation date is far away, the over-reaction and the corresponding negative correlated returns over time reduce the overall risks faced by the rational traders, making them trade aggressively. But as the liquidation date gets closer, under-reaction dominates the market from the perspective of the rational traders, making them trade conservatively. Since the fallacy traders hold just the opposite view of the market, they become very powerful in pricing as the time goes on. Moreover, the nature of gamber's fallacy is conservatism towards new information, making the valuations relatively stable for the fallacy traders. The fallacy traders therefore, view the asset as less riskier compared to a rational trader and are more willing to take excess risks in general.

The existence of a linear competitive equilibrium in our model sheds lights on a new asset pricing model with momentum and reversal. We show that the market in general requires higher compensation for risks, when investors are randomness believers rather than mean-reversal believers. Our results also rationalise the momentum trading strategy and it shows that aggressive trading behaviour can be rooted in conservatism due to gambler's fallacy.

One important critique on biased belief-based literature is where should the cognitive bias be implemented. Our model setting partially survived the critique by assuming gambler's fallacy in beliefs on signals; price predictions from the fallacy agents also exhibit gambler's fallacy. In spite of the consistent error for fallacy makers, rational belief holders are only rational in dealing with the signal process but exhibit hot-hand fallacy in the stochastic process of price. In other words, the rational assumption on signal sequence is equivalent to hot-hand assumption on price sequence, leaving this part of the model open to critique. Moreover, the hot-hand results seem to be consistent with the survey evidence in Greenwood and Shleifer (2013) in which subjects extrapolate on prices directly.

Another issue which our model fails to address is the equity premium puzzle since the gambler's fallacy makers in fact reduce the risk of an asset by taking opposite position to its previous trend. In the paper from Barberis et al. (2013) where investors perform similar to hot-hand fallacy makers, the market price are more volatile than the rational level. Even though the rational traders take a similar role as hot-hand traders in their paper, however, the existence of gambler's fallacy traders cancels this effect in our model. A potentially interesting extension to our current work is to explore the model by replacing gambler's fallacy with hot-hand assumption and it is shown in the main body that hot-hand fallacy assumption can generate additional volatility in the market.

Finally, future work may include formal modelling of asset pricing with momentum and reversal. In order to study the impact of gambler's fallacy closely, the model in this paper assumes a stable fundamental value over time. Calibrations of the pricing model with a stochastic fundamental value and immediate consumptions are possible subjects for future studies.

3.7 Mathematical Appendix A

Proposition 8: (Kalman Filtering)

We first reformulate agents' beliefs into state space models. The beliefs evolution therefore is described by Kalman Filtering[21]. To be specific, the beliefs of the rational type described by (3.2) is already a standard state space model and is presented as below in (3.12)(3.13):

$$y_t^R = V_t + \xi_t^R \tag{3.12}$$

$$V_t = V_{t-1} + \omega_t^R. \tag{3.13}$$

where $V_1 \sim \mathcal{N}(\mu, \sigma_V^2)$, $\xi_t^R \sim N(0, \sigma_\xi^2)$, $\omega_t^R \sim N(0, \sigma_\omega^2)$ and $\sigma_\omega^2 = 0$.

Instead, the beliefs of type G in (3.3)(3.4) can be reformulated into a state space model as in (3.14)(3.15) by letting $\epsilon_t \equiv \sum_{k=0}^{t-1} \delta^k \kappa_{t-k}$:

$$y_t^G = ZS_{t-1} + \eta_t \tag{3.14}$$

$$S_t = \Upsilon S_{t-1} + \tau_t \tag{3.15}$$

where

$$S_t = [V_t \,, \epsilon_t]',$$

$$Z = [1, -\beta],$$

$$\Upsilon = \begin{bmatrix} 1 & 0 \\ 0 & \delta - \beta \end{bmatrix},$$

$$\eta_t = \omega_t^G + \xi_t^G,$$

$$\tau_t = \left[\omega_t^G \,, \xi_t^G\right]'.$$

Similar to the rational case, we assume $V_1 \sim \mathcal{N}(\mu, \sigma_V^2)$, $\xi_t^G \sim N(0, \sigma_\xi^2)$, $\omega_t^G \sim N(0, \sigma_\omega^2)$, and $\sigma_\omega^2 = 0$. We further assume $\epsilon_0 = \xi_0^G$ and is randomly drawn from $\mathcal{N}(0, \sigma_\xi^2)$ implying both types of investors have the same priors on V so that our assumption $\mathcal{F}_0^i = \mathcal{F}_0$ above is satisfied. Compare to the state space model for the rational agents with V_t as the state variable, the fallacy agents' problem have two state variables, V_t and ϵ_t.

The rational investor's recursive-filtering problem is standard. The results are directly from Tsay (2005), Chapter 11.1. We only prove the problem for the fallacy investors.

[21] The problem of the rational type is very simple. The reason for applying Kalman Filter to this problem is to make the rational case comparable to the fallacy one.

The problem described in (3.14)(3.15) can be reformulated as below:

$$y_t = Z\Upsilon^{-1}S_t - Z\Upsilon^{-1}\tau_t + \eta_t$$

$$S_t = \Upsilon S_{t-1} + \tau_t$$

And we have

$$S_{t+1|t} = E_G\left(\Upsilon S_t + \tau_{t+1}|\mathcal{F}_t\right) = \Upsilon S_{t|t}$$

$$v_t^G = y_t - y_{t|t-1}^G = Z\Upsilon^{-1}\left(S_t - S_{t|t-1}\right) - Z\Upsilon^{-1}\tau_t + \eta_t$$

Use the formula for the conditional expectation of the bivariate normal distribution, we have

$$S_{t|t} = E_G\left(S_t|\mathcal{F}_t,\ v_t\right) = S_{t|t-1} + Cov_G(S_t,\ v_t|\mathcal{F}_{t-1}) \cdot Var_G(v_t|\mathcal{F}_{t-1})^{-1}v_t$$

where

$$\begin{aligned}
Cov_G(S_t,\ v_t|\mathcal{F}_t) &= Cov_G\left(S_t,\ Z\Upsilon^{-1}(S_t - S_{t|t-1} - Z\Upsilon^{-1}\tau_t + \eta_t)|\mathcal{F}_{t-1}\right)\\
&= \Sigma_{S,t|t-1}^G\left(Z\Upsilon^{-1}\right)^{\mathbf{T}} + Cov_G\left(S_t, -Z\Upsilon^{-1}\tau_t + \eta_t|\mathcal{F}_{t-1}\right)\\
&= \Sigma_{S,t|t-1}^G\left(Z\Upsilon^{-1}\right)^{\mathbf{T}} + Cov_G\left(\tau_t, -Z\Upsilon^{-1}\tau_t + \eta_t|\mathcal{F}_{t-1}\right)\\
&= \Sigma_{S,t|t-1}^G\left(Z\Upsilon^{-1}\right)^{\mathbf{T}} - L_t\left(Z\Upsilon^{-1}\right)^{\mathbf{T}} + \Omega_t
\end{aligned}$$

$$\begin{aligned}
Var_G\left(v_t|\mathcal{F}_{t-1}\right) &= Var_G\left(Z\Upsilon^{-1}(S_t - S_{t|t-1}) - Z\Upsilon^{-1}\tau_t + \eta_t|\mathcal{F}_{t-1}\right)\\
&= Z\Upsilon^{-1}\Sigma_{S,t|t-1}^G\left(Z\Upsilon^{-1}\right)^{\mathbf{T}} + Var_G\left(-Z\Upsilon^{-1}\tau_t + \eta_t|\mathcal{F}_{t-1}\right)\\
&+2Cov_G\left(Z\Upsilon^{-1}S_t,\ -Z\Upsilon^{-1}\tau_t + \eta_t|\mathcal{F}_{t-1}\right).
\end{aligned}$$

Furthermore, it is easily to show that

$$Var_G\left(-Z\Upsilon^{-1}\tau_t + \eta_t|\mathcal{F}_{t-1}\right) = H_t + Z\Upsilon^{-1}L_t\left(Z\Upsilon^{-1}\right)^{\mathbf{T}} - 2Z\Upsilon^{-1}\Omega_t$$

$$Cov_G\left(Z\Upsilon^{-1}S_t,\ -Z\Upsilon^{-1}\tau_t + \eta_t|\mathcal{F}_{t-1}\right) = -Z\Upsilon^{-1}L_t\left(Z\Upsilon^{-1}\right)^{\mathbf{T}} + Z\Upsilon^{-1}\Omega_t.$$

Let $K_t = \Upsilon Cov_G\left(S_t,\ v_t^G\right)Var_G\left(v_t\right)^{-1}$, we proved that

$$S_{t+1|t} = \Upsilon S_{t|t-1} + K_t v_t^G$$

Next, we prove the recursive equation for the conditional variance.

$$\Sigma^G_{S,t+1|t} = \Upsilon\Sigma^G_{S,t|t}\Upsilon^{\mathbf{T}} + L_{t+1}.$$

It is easy to show that

$$\Sigma^G_{S,t|t} = \Sigma^G_{S,t|t-1} - Var_G\left(v^G_t\right)\Upsilon^{-1}K_tK_t^{\mathbf{T}}\left(\Upsilon^{\mathbf{T}}\right)^{-1}.$$

Bring it back to the original formula, we prove that

$$\Sigma^G_{S,t+1|t} = \Upsilon\Sigma^G_{t|t-1}\Upsilon^{\mathbf{T}} - [Y\Sigma^G_{t|t-1}Y^{\mathbf{T}} + H - YLY^{\mathbf{T}}]K_tK_t^{\mathbf{T}} + L,$$

The formula of K_t can be easily derived from here. *Q.E.D.*

Lemma6: (Steady State)

Similar to Rabin and Vayanos (2010), the steady state of the Ricatti equations exists. Σ^R, Σ^G, K^R, K^G and k are determine by the solutions to

$$\Sigma^R = \Sigma^R(1 - K^R)$$

$$\Sigma^G = \Upsilon\Sigma^G\Upsilon^{\mathbf{T}} - [Z\Upsilon^{-1}\Sigma^G(Z\Upsilon^{-1})^{\mathbf{T}} + H - Z\Upsilon^{-1}L(Z\Upsilon^{-1})^{\mathbf{T}}]KK^{\mathbf{T}} + L$$

$$K^R = \frac{\Sigma^R}{\Sigma^R + \sigma^2_\xi}$$

$$\begin{aligned} K &= [K^G,\ k]' \\ &= \Upsilon\left(\Sigma^G(Z\Upsilon^{-1})^{\mathbf{T}} - L(Z\Upsilon^{-1})^{\mathbf{T}} + \Omega\right)\left(Z\Upsilon^{-1}\Sigma^G(Z\Upsilon^{-1})^{\mathbf{T}} + H - Z\Upsilon^{-1}L(Z\Upsilon^{-1})^{\mathbf{T}}\right)^{-1}\end{aligned}$$

Given λ and δ, it is easily to show that $\Sigma^G = \begin{bmatrix} 0 & 0 \\ 0 & \sigma^2_\xi \end{bmatrix}$, $\Sigma^R = 0$, $K^R = K^G = 0$ while $k = \dfrac{(\delta-\beta)^2}{-\beta(\delta-\beta) + (\delta-\beta)(\beta+1)\sigma^2_\xi}$. *Q.E.D.*

Proposition 9: (Linear Equilibrium)

3.7.0.3 Step 1: Conjectured Price

Assumption 4 *Given the economy defined in Section 3.3.1 and two type of investors with beliefs defined in Eq.*(3.2)-(3.4)*, there exists a competitive equilibrium in which the price of the risky asset is a linear function of the state variables of the economy:*

$$P_t = \alpha_t \Psi_t,$$

where $\alpha_t = [\alpha_{0,t},\ \alpha_{1,t},\ \alpha_{2,t},\ \alpha_{3,t}]$ *is a* 4×1 *vector.*

We use P_t and Q_t to denote the stochastic process of real price and return. Due to different beliefs, we further use P_t^i and Q_t^i to denote the beliefs of type i on the stochastic process of P_t and Q_t, and Ψ_t^i to denote the beliefs on the stochastic process Ψ_t[22]. Given the assumption in Step 1, we have the following Lemma 9.

Lemma 9 *(Price and Return Process)*
In a linear equilibrium where $P_t = \alpha_t \Psi_t$*,* Q_t *and* Ψ_t *are both measurable with respect to* $\mathcal{F}_t$ *and take the following Gaussian processes under information* $\{\mathcal{F}_t : 1 \leq t \leq T\}$ *for type* i*,* $i \in R, G$*:*

$$Q_{t+1}^i = A_{Q,t+1}^i \Psi_t^i + B_{Q,t+1}^i v_{t+1}^i \tag{3.16}$$

$$\Psi_{t+1}^i = A_{\Psi,t+1}^i \Psi_t^i + B_{\Psi,t+1}^i v_{t+1}^i \tag{3.17}$$

where v_{t+1}^i *are signal shocks defined as before and are normally distributed conditional on* $\mathcal{F}_t$*;* $A_{\Psi,t}^i$ *are* 4×4 *matrices,* $A_{Q,t}^i$ *are* 1×4 *vectors,* $B_{\Psi,t}^i$ *are* 4×1 *vectors and* $B_{Q,t}^i$ *are constants given in Appendix.*

Assume $P_t = \alpha_t \Psi_t$ holds, we first prove the Lemma 1. The way to derive $A_{\Psi,t}^i$ and $B_{\Psi,t}^i$ is from the recursive process presented in Corollary 8. We only give the proof for $A_{\Psi,t}^G$ and $B_{\Psi,t}^G$ and use similar method, we can derive $A_{\Psi,t}^R$ and $B_{\Psi,t}^G$.

[22]We do not differentiate between the realisations and the stochastic process here in the notations. But we point out that conditional on each information set $\mathcal{F}_t$, P_t, Q_t, and Ψ_t are no longer random variables and are public information for both types. However, P_{t+k}, Q_{t+k}, and Ψ_{t+k} are still random based on information set $\mathcal{F}_t$. Therefore, conditional on the $\mathcal{F}_t$, we have $P_t^i = P_t$, $Q_t^i = Q_t$, and $\Psi_t^i = \Psi_t$, all take their realisations. Here we implicitly assume the information set $\mathcal{F}_j$ has $j < t$. Furthermore, P_t^i, Q_t^i, and Ψ_t^i are different processes for different information sets $\mathcal{F}_j$. But, since we only care about the one step ahead expression, the information set j is eliminated from the expression.

From G's perspective,

$$\begin{aligned}
V^R_{t+1|t} &= V^R_{t|t-1} + K^R_t v^R_t \\
&= V^R_{t|t-1} + K^R_t \left(V^G_{t|t-1} - \frac{\beta}{\delta - \beta} \epsilon^G_{t|t-1} + v^G_t - V^R_{t|t-1} \right) \\
&= \left(1 - K^R_t\right) V^R_{t|t-1} + K^R_t V^G_{t|t-1} - \frac{\beta}{\delta - \beta} K^R_t \epsilon^G_{t|t-1} + K^R_t v^G_t;
\end{aligned}$$

similarly,

$$V^G_{t+1|t} = V^G_{t|t-1} + K^G_t v^G_t$$

$$\epsilon^G_{t+1|t} = (\delta - \beta)\, \epsilon^G_{t|t-1} + k^G_t v^G_t.$$

These equations above give us the expression that

$$A^G_{\Psi,t} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - K^R_t & K^R_t & -\frac{\beta}{\delta-\beta} K^R_t \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta - \beta \end{pmatrix}, \qquad B^G_{\Psi,t} = \begin{pmatrix} 0 \\ K^R_t \\ K^G_t \\ k_t \end{pmatrix}.$$

Similarly, we have

$$A^R_{\Psi,t} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & K^G_t & 1 - K^G_t & \frac{\beta}{\delta-\beta} K^G_t \\ 0 & k_t & -k_t & \delta - \beta + \frac{\beta}{\delta-\beta} k_t \end{pmatrix}, \qquad B^R_{\Psi,t} = \begin{pmatrix} 0 \\ K^R_t \\ K^G_t \\ k_t \end{pmatrix}.$$

Furthermore, given the assumption that $P_t = \alpha_t \Psi_t$, we can show that from G's perspective,

$$\begin{aligned}
Q^i_{t+1} &= a_{0,t+1} - a_{0,t} \\
&+ a_{1,t+1} \left[V^R_{t+1|t} + K^R_{t+1} \left(V^G_{t+1|t} - \frac{\beta}{\delta - \beta} \epsilon_{t+1|t} + v^G_{t+1} - V^R_{t+1|t} \right) \right] \\
&a_{2,t+1} \left[V^G_{t+1|t} + K^G_{t+1} v^G_{t+1} \right] \\
&a_{3,t+1} \left[(\delta - \beta)\, \epsilon_{t+1|t} + k^G_{t+1} v^G_{t+1} \right] \\
&- a_{1,t} V^R_{t+1|t} - a_{2,t} V^G_{t+1|t} - a_{3,t} \epsilon_{t+1|t}
\end{aligned}$$

Reorganise the right hand according to $V^R_{t+1|t}$, $V^G_{t+1|t}$, $\epsilon_{t+1|t}$ and v^G_t will give give us the matrix $A^G_{Q,t+1}$ and $B^G_{Q,t+1}$ in the Proposition, that is: $A^G_{Q,t} = \alpha_t A^G_{\Psi,t} - \alpha_{t-1}$, $B^G_{Q,t} = \alpha_t B^G_{\Psi,t}$. Similarly, we have $A^R_{Q,t} = \alpha_t A^R_{\Psi,t} - \alpha_{t-1}$, $B^R_{Q,t} = \alpha_t B^R_{\Psi,t}$. $\quad Q.E.D.$

Notice that based on Definition 3 and given our Proposition 9 holds, the real stochastic process of the state vector of the economy and excess return must coincide with the beliefs of the rational investors because their beliefs y^i_t are correct. Therefore, the real process of Ψ_t and Q_t can be written as

$$\Psi_{t+1} = A^R_{\Psi,t+1}\Psi_t + B^R_{\Psi,t+1}v^R_{t+1}$$

$$Q_{t+1} = A^R_{Q,t+1}Q_t + B^R_{Q,t+1}v^R_{t+1}.$$

We further move on to the optimisation problem which determines the optimal portfolios of two types of investors.

3.7.0.4 Step 2: Optimisation

With Gaussian assumptions on the price process in Step 1 and Lemma 9 and under the CARA utility function assumption, the optimisation problem defined by equations (3.7), (3.8) can be solved using dynamic programming.

Let $J(W^i_t;\ \Psi^i_t,\ t)$ be the value function of type i, we have the Bellman equation below:

$$J(W^i_t;\ \Psi^i_t,\ t) = \underset{X^i_t}{Max} E_i[J(W^i_{t+1};\ \Psi^i_{t+1},\ t+1)|\mathcal{F}_t].$$

The optimisation problem (3.7), (3.8) is equivalent to the following problem:

$$0 = \underset{X^i_t}{Max}\left\{E_i[J(W^i_{t+1};\ \Psi^i_{t+1},\ t+1)|\mathcal{F}_t] - J(W^i_t;\ \Psi^i_t,\ t)\right\}$$

$$s.t.\ \ W^i_{t+1} = W^i_t + X^i_t Q^i_{t+1}$$

$$J(W^i_T;\ \Psi^i_T,\ T) = -e^{-\lambda W^i_T},$$

Given the Lemma 9, we can derive Proposition 11.

Proposition 11: (Optimal Holdings)

We prove the Proposition 11 part (i) by solving the optimisation problem.

Assume the value function take the following format:

$$J(W_t^i;\ \Psi_t,\ t) = -exp\left\{-\lambda W_t^i - \frac{1}{2}\Psi_t^{i\mathbf{T}} U_t^i \Psi_t^i\right\}$$

The first term represents the utility from the current investment, and the second term represents a risk adjusted utility from future investment.

We plug the Gaussian process of Q_{t+1}^i and Ψ_t^i into the proposed value function and we have:

$$\begin{aligned} E_i[J(W_{t+1}^i;\ \Psi_{t+1}^i,\ t+1)|\mathcal{F}_t] \quad &= -\rho_{t+1} exp\left\{-\lambda W_t^i - \tfrac{1}{2}\Psi_t^{i\mathbf{T}} A_{\Psi,t+1}^{i\mathbf{T}} U_{t+1}^i A_{\Psi,t+1}^i \Psi_t^i\right\} \\ &\times exp\left\{-\lambda X_t^i A_{Q,t+1}^i \Psi_t^i + \tfrac{1}{2} C_{t+1}^{i\mathbf{T}} \theta_{t+1}^i C_{t+1}^i\right\} \end{aligned} \tag{3.18}$$

where

$$C_{t+1}^i = \lambda B_{Q,t+1}^{i\mathbf{T}} X_t^i + B_{\Psi,t+1}^{\mathbf{T}} U_{t+1}^i A_{\Psi,t+1}^i \Psi_t^i,$$

$$\theta_{t+1}^i = \left(Var(v_{t+1}^i)^{-1} + B_{\Psi,t+1}^{i\mathbf{T}} U_{t+1}^i B_{\Psi,t+1}^i\right)^{-1},$$

$$\rho_{t+1}^i = (1 + Var(v_{t+1}^i) B_{\Psi,t+1}^{i\mathbf{T}} U_{t+1}^i B_{\Psi,t+1}^i)^{-\frac{1}{2}}.$$

Differentiate the value function with respect to X_t^i, and from the F.O.C., we have

$$X_t^i = \frac{1}{\lambda} F_t^i \Psi_t,$$

where

$$F_t^i \equiv \left(B_{Q,t+1}^i \theta_{t+1}^i B_{Q,t+1}^{i\mathbf{T}}\right)^{-1} \left(A_{Q,t+1}^i - B_{Q,t+1}^i \theta_{t+1}^i B_{\Psi,t+1}^{i\mathbf{T}} U_{t+1}^i A_{\Psi,t+1}^i\right),$$

The second order condition for optimality holds since $B_{Q,t+1}^i \theta_{t+1}^i B_{Q,t+1}^{i\mathbf{T}} > 0$. Since $A_{Q,t+1}^i$ is defined by Eq. (3.16) and (3.17), we have $A_{Q,t+1}^i \Psi_t = E_i(Q_{t+1}^i|\mathcal{F}_t)$. Define $\Gamma_t^i \equiv \left(B_{Q,t+1}^i \theta_{t+1}^i B_{Q,t+1}^{i\mathbf{T}}\right)^{-1}$ and $g_t^i \equiv B_{Q,t+1}^i \theta_{t+1}^i B_{\Psi,t+1}^{i\mathbf{T}} U_{t+1}^i A_{\Psi,t+1}^i$, we then have the conclusion in Proposition 11 part (i).

Furthermore, we need to show that

$$J(W_t^i;\ \Psi_t,\ t) = -exp\left\{-\lambda W_t^i - \frac{1}{2}\Psi_t^{i\mathbf{T}} U_t^i \Psi_t^i\right\} \tag{3.19}$$

is the right format of the value function. This is the correct value function if we can find the right format of U_t^i. To do this, we substitute $X_t^i = \dfrac{1}{\lambda} F_t^i \Psi_t$ into the Bellman equation, that is

$$J(W_t^i;\ \Psi_t,\ t) = \underset{X_t^i}{Max} E_i[J(W_{t+1}^i;\ \Psi_{t+1}^i,\ t+1)|\mathcal{F}_t]^{23}.$$

Define

$$\begin{aligned} M_{t+1}^i \equiv\ & F_{t+1}^{i\mathbf{T}} \left(B_{Q,t+2}^i \theta_{t+2}^i B_{Q,t+2}^{i\mathbf{T}} \right) F_{t+1}^i + A_{\Psi,t+2}^{i\mathbf{T}} U_{t+2}^i A_{\Psi,t+2}^i \\ & - \left(B_{\Psi,t+2}^{i\mathbf{T}} U_{t+2}^i A_{\Psi,t+2}^i \right)^{\mathbf{T}} \theta_{t+2}^i \left(B_{\Psi,t+2}^{i\mathbf{T}} U_{t+2}^i A_{\Psi,t+2}^i \right), \end{aligned}$$

we have

$$-\rho_{t+1}^i exp\left\{ -\lambda W_t^i - \tfrac{1}{2} \Psi_t^{i\mathbf{T}} M_t^i \Psi_t^i \right\} = \ -exp\left\{ -\lambda W_t^i - \tfrac{1}{2} \Psi_t^{i\mathbf{T}} U_t^i \Psi_t^i \right\}\ .$$

From here, we can derive the right form of U_t^i:

$$U_t^i = -2ln\rho_{t+1}^i I_{11} + M_t^i,$$

and we complete our proof for Proposition 11. Our conclusion at this step holds without the market clearing constraint.

Using the result derived in part (i), Result in part (ii) can be easily derived by substituting $A_{Q,t+1}^i$, $A_{\Psi,t+1}^i$ with E.q. 3.16 and 3.17 and α_t with the result in Proposition 14. *Q.E.D.*

3.7.0.5 Step 3. Market Clearing and Equilibrium Price

The last step to prove our Proposition 9 is to show that the equilibrium price derived from the market clearing condition is consistent with the proposed equilibrium price function in Step 1.

Since investors in our model are assumed to be non-myopic, given the optimal demands of both types solved in Step 2, the market clearing conditions can be applied backwards recursively to derive the equilibrium price in each period.

To be more specific, the market clearing condition is $\sum_{i\in\mathcal{I}} X_t^i = 0$, for $t = 1, ..., T-1$. We first find the equilibrium price function at $T-1$. The dollar return at T is $Q_T = V - P_{T-1}$ and the equilibrium price can be solved explicitly. There is a continuum of investors on $[0,\ 1]$, with m proportion of rational investors and $n = 1-m$ proportion of fallacy investors

[23] In Proposition 11, the optimal holdings X_t^i is now a function of Ψ_t rather than Ψ_t^i. At time t, both Ψ_t and P_t are non-random variables and are public information contained in the information set $\mathcal{F}_t$. Their realisations are common knowledge to both types of traders so that $\Psi_t^i = \Psi_t$, $P_t^i = P_t$ and $Q_t^i = Q_t$.

in the market, investor i's optimal demand at $T-1$ has the linear form:

$$X^i_{T-1} = \frac{1}{\lambda}\frac{E^i[Q_T|\mathcal{F}_{T-1}]}{\Sigma^i_{V,T|T-1}}.$$

From market clearing condition at $T-1$:

$$mX^R_{T-1} + nX^G_{T-1} = 0,$$

we derive the explicit format for $P_{T-1} = \alpha_{T-1}\Psi_t$,
where $\alpha_{T-1} = [0, \dfrac{m\Sigma^G_{V,T|T-1}}{\left(m\Sigma^G_{V,T|T-1} + n\Sigma^R_{V,T|T-1}\right)}, \dfrac{n\Sigma^R_{V,T|T-1}}{\left(m\Sigma^G_{V,T|T-1} + n\Sigma^R_{V,T|T-1}\right)}, 0]$. Proposition 18 below shows that from α_{T-1}, the parameter vector α_t for any period t exist and is uniquely determined recursively.

Proposition 18 *(Recursive Process)*
The equilibrium price P_t is linear in Ψ_t with the parameter vector α_t uniquely determined by the following recursive process:

$$\alpha_t = \alpha_{t+1}\overline{E}_{t+1}, \tag{3.20}$$

where $\overline{E}_{t+1}$ is a 4×4 matrix fully determined by m, λ, $A^i_{\Psi,t+1}$, B_{Ψ,t_1}, $A^i_{Q,t+1}$, B_{Q,t_1}, $\Sigma^i_{V,t|t-1}$, $Var(v^i_{t+1})$.

Proposition 18 completes the proof for the Proposition 9 which states that there exists a linear competitive equilibrium in the economy. By Definition 3, our proof for Proposition 9 can also be extended to derive a competitive equilibrium where the real signals have serial correlation. The equilibrium price process is still linear in the state variables of the economy but the real stochastic process will evolve according to the beliefs of the fallacy agents.
Proof of Proposition 18 RECURSIVE PROCESS

The final step to complete the proof of linear equilibrium in Proposition 9 is to show that we can recursively solve the problem as shown in Proposition 18. We shall show that there exist a unique solution under the current setting.

We first present our Proposition 18 completely and show the detailed format of related matrices involved.

Recursive Process—Revisited

Parameters in the equilibrium are determined by the following recursive process:

$$\alpha_t = \alpha_{t+1}\overline{E}_{t+1} \tag{3.21}$$

$$\overline{E}_{t+1} = \left(\frac{m}{\theta^R_{t+1}} + \frac{n}{\theta^G_{t+1}}\right)^{-1}\left(\frac{mE^R_{t+1}}{\theta^R_{t+1}} + \frac{nE^G_{t+1}}{\theta^G_{t+1}}\right) \tag{3.22}$$

$$E^i_{t+1} = A^i_{\Psi,t+1} - B^i_{\Psi,t+1}B^{i\mathbf{T}}_{\Psi,t+1}\theta^i_{t+1}U^i_{t+1}A^i_{\Psi,t+1}, \quad for\ i = R,\ G \tag{3.23}$$

$$U^i_{t+1} = M^i_{t+1} - 2log(\rho^i_{t+1})II, \tag{3.24}$$

with II a 4×4 zero matrix except the up-left corner set to 1 and

$$U^i_{T-1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & u^i_{T-1} & -u^i_{T-1} & 0 \\ 0 & -u^i_{T-1} & u^i_{T-1} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$u^R_{T-1} = \frac{n^2\Sigma^R_{V,T|T-1}}{(m\Sigma^G_{V,T|T-1} + n\Sigma^R_{V,T|T-1})^2}, \quad u^G_{T-1} = \frac{m^2\Sigma^G_{V,T|T-1}}{(m\Sigma^G_{V,T|T-1} + n\Sigma^R_{V,T|T-1})^2}$$

$$\theta^i_{t+1} = \left(Var(v^i_{t+1})^{-1} + B^{i\mathbf{T}}_{\Psi,t+1}U^i_{t+1}B^i_{\Psi,t+1}\right)^{-1}, \quad for\ i = R,\ G \tag{3.25}$$

$$\rho_{t+1} = (1 + Var(v^i_{t+1})B^{i\mathbf{T}}_{\Psi,t+1}U^i_{t+1}B^i_{\Psi,t+1})^{-\frac{1}{2}} \tag{3.26}$$

$$\begin{aligned} M^i_{t+1} &= F^{\mathbf{T}}_{t+1}\left(B^i_{Q,t+2}\theta^i_{t+2}B^{i\mathbf{T}}_{Q,t+2}\right)F_{t+1} \\ &- \left(B^{i\mathbf{T}}_{\Psi,t+2}U^i_{t+2}A^i_{\Psi,t+2}\right)^{\mathbf{T}}\theta_{t+2}\left(B^{i\mathbf{T}}_{\Psi,t+2}U^i_{t+2}A^i_{\Psi,t+2}\right) + A^{i\mathbf{T}}_{\Psi,t+2}U^i_{t+2}A^i_{\Psi,t+2} \end{aligned} \tag{3.27}$$ [24]

$$F^i_{t+1} = \left(B^i_{Q,t+2}\theta^i_{t+2}B^{i\mathbf{T}}_{Q,t+2}\right)^{-1}\left(A_{Q,t+2} - B^i_{Q,t+2}\theta^i_{t+2}B^{i\mathbf{T}}_{\Psi,t+2}U^i_{t+2}A_{\Psi,t+2}\right) \tag{3.29}$$ [25]

[24]

$$M^i_t = F^{\mathbf{T}}_t\left(B^i_{Q,t+1}\theta^i_{t+1}B^{i\mathbf{T}}_{Q,t+1}\right)F_t - \left(B^{i\mathbf{T}}_{\Psi,t+1}U^i_{t+1}A^i_{\Psi,t+1}\right)^{\mathbf{T}}\theta_{t+1}\left(B^{i\mathbf{T}}_{\Psi,t+1}U^i_{t+1}A^i_{\Psi,t+1}\right) + A^{i\mathbf{T}}_{\Psi,t+1}U^i_{t+1}A^i_{\Psi,t+1}$$

[25]

$$F^i_t = \left(B^i_{Q,t+1}\theta^i_{t+1}B^{i\mathbf{T}}_{Q,t+1}\right)^{-1}\left(A_{Q,t+1} - B^i_{Q,t+1}\theta^i_{t+1}B^{i\mathbf{T}}_{\Psi,t+1}U^i_{t+1}A_{\Psi,t+1}\right). \tag{3.29}$$

We shall prove the Proposition presented here.

From Lemma 9, we have $A^i_{Q,t} = \alpha_t A^i_{\Psi,t} - \alpha_{t-1}$ and $B^i_{Q,t} = \alpha_t K_t$. Substitute these two formulas into F^i_t:

$$F^i_t = \frac{1}{(\alpha_{t+1}K_{t+1})^2 \theta^i_{t+1}} \alpha_{t+1} E^i_{t+1} - \frac{1}{(\alpha_{t+1}K_{t+1})^2 \theta^i_{t+1}} \alpha_t \tag{3.30}$$

where $E^i_{t+1} = A^i_{\Psi,t+1} - K_{t+1} B^{i\mathbf{T}}_{\Psi,t+1} \theta^i_{t+1} U^i_{t+1} A^i_{\Psi,t+1}$. Since $\Psi^R_t = \Psi^G_t$, substitute $X^i_t = \frac{1}{\lambda} F^i_t \Psi_t$ into the market clearing condition $mX^R_t + nX^G_t = 0$, we have

$$mF^R_t + nF^G_t = \mathbf{0}. \tag{3.31}$$

Furthermore, substitute (3.30) into (3.31):

$$m\left(\frac{\alpha_{t+1}E^R_{t+1}}{\theta^R_{t+1}} - \frac{\alpha_t}{\theta^R_{t+1}}\right) + n\left(\frac{\alpha_{t+1}E^G_{t+1}}{\theta^G_{t+1}} - \frac{\alpha_t}{\theta^G_{t+1}}\right) = 0 \tag{3.32}$$

Define $\overline{E}_{t+1} \equiv \left(\frac{m}{\theta^R_{t+1}} + \frac{n}{\theta^G_{t+1}}\right)^{-1} \left(\frac{mE^R_{t+1}}{\theta^R_{t+1}} + \frac{nE^G_{t+1}}{\theta^G_{t+1}}\right)$, we have $\alpha_t = \alpha_{t+1}\overline{E}_{t+1}$ and $\overline{E}_{t+1}$ is a 4×4 matrix. Therefore, as long as we know the α_{T-1}, we can solve the linear equilibrium price and the parameter matrices $A^i_{Q,t}$ and $B^i_{Q,t}$ at any date t recursively.

The last period problem is fairly easy to be solved. It is equivalent to the last period solution as in a 3 period model presented in Brunnermeier (2003) text book. The last period U^i_{T-1} is solved directly from the last period Bellman equation. *Q.E.D.*

Our proof of this proposition is completed by the proof of Lemma 9, Proposition 11 and Corollary 18 above.

A 3-period example of this proof is given in Appendix B.

Proposition 10: (Rational Benchmark)

Since the rational benchmark is a special case of the mixed market with $m = 1$, and we have $\alpha_{t,1} + \alpha_{t,2} = 1$, it is easily to derive from here that $\alpha_{t,1} = 1$ for all t. Therefore, the price in the rational market is $P_t = V^R_{t+1|t}$.

Furthermore, the result that the equilibrium price $P_{R,t}$ is a martingale sequence comes directly from the nature of Kalman filtering since the $P_{R,t}$ takes the expectation of the liquidation value. We also prove that $E_R\left[Q_{R,t+k}|\mathcal{F}_t\right] = 0$ because the returns $Q_{R,t}$ is a martingale difference sequence.

Finally, we point out that $Corr_R\left(Q_{t_1}^R,\ Q_{t_2}^R\right) = 0$ comes directly from the property of a martingale difference sequence. $Q.E.D.$

Proposition 12 and Proposition 13: (Expectations in the Mixed Market)

We only give the proof for Proposition 12. Results in Proposition 13 can be easily derived by expressing the state variables of the economy recursively as in our proof for Proposition 12.

$$\begin{aligned} E_R\left[P_t|\mathcal{F}_0\right] &= E_R\left[\alpha_t\Psi_t^R|\mathcal{F}_0\right] \\ &= E_R\left[\alpha_t\left(A_{\Psi,t}^R\Psi_{t-1}^R + B_{\Psi,t}^R v_t^R\right)|\mathcal{F}_0\right] \\ &= E_R\left[\alpha_t A_{\Psi,t}^R\Psi_{t-1}^R|\mathcal{F}_0\right] \\ &= \alpha_t\prod_{l=0}^{t-1}A_{\Psi,t-l}^R\Psi_0. \end{aligned}$$

$$\begin{aligned} E_R\left[Q_t|\mathcal{F}_0\right] &= E_R\left[A_{Q,t}^R\Psi_t^R + B_{Q,t}^R v_t^R|\mathcal{F}_0\right] \\ &= E_R\left[A_{Q,t}^R\Psi_t^R|\mathcal{F}_0\right] \\ &= A_{Q,t}\prod_{l=0}^{t-2}A_{\Psi,t-1-l}^R\Psi_0. \end{aligned}$$

$Q.E.D.$

Proposition 14: (Market Dominance)

The proof is easy but tedious and we only present the proof for the last two periods. It is easy to derive that at $t = T-1$, we have

$$\alpha_{T-1} = \left(0,\ \frac{m\Sigma_{V,T|T-1}^G}{m\Sigma_{V,T|T-1}^G + n\Sigma_{V,T|T-1}^R},\ \frac{n\Sigma_{V,T|T-1}^R}{m\Sigma_{V,T|T-1}^G + n\Sigma_{V,T|T-1}^R},\ 0\right)$$

$$U_{T-1}^R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & u_{T-1}^R & -u_{T-1}^R & 0 \\ 0 & -u_{T-1}^R & u_{T-1}^R & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},\ U_{T-1}^G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & u_{T-1}^G & -u_{T-1}^G & 0 \\ 0 & -u_{T-1}^G & u_{T-1}^G & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$u_{T-1}^R = \frac{1}{\lambda}\frac{n^2\Sigma_{V,T|T-1}^R}{\left(m\Sigma_{V,T|T-1}^G + n\Sigma_{V,T|T-1}^R\right)^2},\ u_{T-1}^G = \frac{1}{\lambda}\frac{m^2\Sigma_{V,T|T-1}^G}{\left(m\Sigma_{V,T|T-1}^G + n\Sigma_{V,T|T-1}^R\right)^2}.$$

Define $b_d \equiv \dfrac{\beta}{\delta - \beta}$, we have

$$A^R_{\Psi,T-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & K^G_{T-1} & 1 - K^G_{T-1} & b_d K^G_{T-1} \\ 0 & k_{T-1} & -k_{T-1} & \delta - \beta + b_d k_{T-1} \end{pmatrix},$$

$$A^G_{\Psi,T-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - K^R_{T-1} & K^R_{T-1} & -b_d K^R_{T-1} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \delta - \beta \end{pmatrix},$$

$$B^R_{\Psi,T-1} = B^G_{\Psi,T-1} = \begin{pmatrix} 0 \\ K^R_{T-1} \\ K^G_{T-1} \\ k_{T-1} \end{pmatrix}$$

Furthermore, we have

$$\begin{aligned} \theta^i_{t+1} &= \left(\left(\Sigma^i_{v,T-1|T-2} \right)^{-1} + B^{i\mathbf{T}}_{\Psi,t+1} U^i_{t+1} B^i_{\Psi,t+1} \right)^{-1} \\ &= \left(\left(\Sigma^i_{v,T-1|T-2} \right)^{-1} + \left(K^R_{T-1} - K^G_{T-1} \right)^2 u^i_{T-1} \right)^{-1}; \end{aligned}$$

$$\begin{aligned} E^i_{T-1} &= A^i_{\Psi,T-1} - B^i_{\Psi,T-1} B^{i\mathbf{T}}_{\Psi,T-1} \theta^i_{T-1} U^i_{T-1} A^i_{\Psi,T-1} \\ &= \left(I_4 - B^i_{\Psi,T-1} B^{i\mathbf{T}}_{\Psi,T-1} \theta^i_{T-1} U^i_{T-1} \right) A^i_{\Psi,T-1} \end{aligned}$$

For simplicity, we omit the subscription $T-1$ and we have

$$E^R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - L^R K^R + L^R K^R K^G & L^R K^R - L^R K^R K^G & b_d L^R K^R K^G \\ 0 & -L^R K^G + K^G + L^R \left(K^G\right)^2 & 1 + L^R K^G - K^G - L^R \left(K^G\right)^2 & b_d K^G + b_d L^R \left(K^G\right)^2 \\ 0 & -L^R k + L^R K^G k + k & L^R k - L^R K^G k - k & b_d L^R K^G k + \delta - \beta + b_d k \end{pmatrix}$$

$$E^G = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - K^R - L^G K^R + L^G \left(K^R\right)^2 & K^R - L^G \left(K^R\right)^2 + L^G K^R & -b_d K^R + b_d L^G \left(K^R\right)^2 \\ 0 & -L^G K^G + L^G K^R K^G & 1 - L^G K^R K^G + L^G K^G & -b_d L^G K^R K^G \\ 0 & -L^G k + L^G K^R k & -L^G K^R k + L^G k & \delta - \beta + b_d L^G K^R k \end{pmatrix}$$

where $L^i = (K^R - K^G)u^i\theta^i$. Since $E = \dfrac{\frac{m}{\theta^R}E^R + \frac{n}{\theta^G}E^G}{\frac{m}{\theta^R} + \frac{n}{\theta^G}}$, define $\bar{E} \equiv \frac{m}{\theta^R}E^R + \frac{n}{\theta^G}E^G$, $D^R \equiv \dfrac{L^R}{\theta^R} = \left(K^R - K^G\right)u^R$, $D^G \equiv \dfrac{L^G}{\theta^G} = \left(K^R - K^G\right)u^G$, we have the first, second, third and fourth columns of $\bar{E}$ are

$$\bar{E}_1 = \begin{pmatrix} \frac{m}{\theta^R} + \frac{n}{\theta^G} \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\bar{E}_2 = \begin{pmatrix} 0 \\ \dfrac{m}{\theta^R} - mD^R K^R + mD^R K^R K^G + \dfrac{n}{\theta^G} - \dfrac{n}{\theta^G}K^R - nD^G K^R + nD^G\left(K^R\right)^2 \\ -mD^R K^G + \dfrac{m}{\theta^R}K^G + mD^R\left(K^G\right)^2 - nD^G K^G + nD^G K^R K^G \\ -mD^R k + mD^R K^G k + \dfrac{m}{\theta^R}k - nD^G k + nD^G K^R k \end{pmatrix},$$

$$\bar{E}_3 = \begin{pmatrix} 0 \\ mD^R K^R - mD^R K^R K^G + \dfrac{n}{\theta^G}K^R - nD^G\left(K^R\right)^2 + nD^G K^R \\ \dfrac{m}{\theta^R} + mD^R K^G - \dfrac{m}{\theta^R}K^G - mD^R\left(K^G\right)^2 - mD^G K^R K^G + \dfrac{m}{\theta^G} + mD^G K^G \\ mD^R k - mD^R K^G k - \dfrac{m}{\theta^R}k - nD^G K^R k + nD^G k \end{pmatrix},$$

$$\bar{E}_4 = \begin{pmatrix} 0 \\ b_d m D^R K^R K^G - \dfrac{n b_d K^R}{\theta^G} + n b_d D^G\left(K^R\right)^2 \\ b_d\dfrac{m}{\theta^R}K^G + b_d m D^R\left(K^G\right)^2 - n b_d D^G K^R K^G \\ b_d m D^R K^G k + \dfrac{m(\delta-\beta)}{\theta^R} + \dfrac{m b_d}{\theta^R}k + \dfrac{n(\delta-\beta)}{\theta^G} + n b_d D^G K^R k \end{pmatrix}.$$

Since $\alpha_{T-2} = \alpha_{T-1} \cdot E_{T-1}$, we have

$$\alpha_{T-2,0} = 0;$$

$$\alpha_{T-2,1}\left(\frac{m}{\theta^R} + \frac{n}{\theta^G}\right) = \alpha_{T-1}\bar{E}_2 = \alpha_{T-1,1}\left(\frac{m}{\theta^R} + \frac{n}{\theta^G}\right);$$

$$\alpha_{T-2,2}\left(\frac{m}{\theta^R} + \frac{n}{\theta^G}\right) = \alpha_{T-1}\bar{E}_3 = (1 - \alpha_{T-1,1})\left(\frac{m}{\theta^R} + \frac{n}{\theta^G}\right).$$

It is easily seen that $\alpha_{T-2,1} + \alpha_{T-2,2} = 1$

Next, we give our proof to the third and fourth parts of Proposition 14. To show that $\alpha_{t,1}$ decreases overtime, we shall prove $\alpha_{T-2,1} - \alpha_{T-2,2} > \alpha_{T-1,1} - \alpha_{T-1,2}$. It is easy to show that

$$\alpha T-2,1-\alpha_{T-2,2} \tag{3.33}$$
$$=\left(\frac{m}{\theta^R}+\frac{n}{\theta^G}\right)(\alpha_{T-1,1}-\alpha_{T-1,2}) \tag{3.34}$$
$$+\left(\alpha_{T-1,1}K^R+\alpha_{T-1,2}K^G\right)\left(-2mD^R+2nD^G\right) \tag{3.35}$$
$$+\alpha_{T-1,1}\left(2mD^RK^RK^G\right)+\alpha_{T-1,2}\left(2nD^GK^RK^G\right) \tag{3.36}$$
$$+\alpha_{T-1,1}\left(2nD^GK^{R^2}\right)+\alpha_{T-1,2}\left(2mD^RK^{G^2}\right) \tag{3.37}$$
$$+\alpha_{T-1,1}\left(-\frac{2n}{\theta^G}K^R\right)+\alpha_{T-1,2}\left(\frac{2m}{\theta^R}K^G\right) \tag{3.38}$$

Since $D^R=(K^R-K^G)u^R$, $D^G=(K^R-K^G)u^G$, and $0<K^R-K^G<1$, $u^R<1$, $u^G<1$, we have (3.35)$=o^4$, (3.36)$=o^5$ and (3.37)$=o^5$.

Since

$$\theta^i_{t+1}=\left(\left(\Sigma^i_{v,T-1|T-2}\right)^{-1}+\left(K^R_{T-1}-K^G_{T-1}\right)^2u^i_{T-1}\right)^{-1},$$

where $\left(K^R_{T-1}-K^G_{T-1}\right)^2u^i_{T-1}=o^3$, The sign of the summation from (3.35)-(3.38) is determined by (3.38), to be more specifically, by $\left(\Sigma^R_{v,T-1|T-2}\right)^{-1}$ and $\left(\Sigma^G_{v,T-1|T-2}\right)^{-1}$.

The original question can be simplified to determine the sign of

$$\alpha_{T-1,1}\left(-2n\left(\Sigma^G_{v,T-1|T-2}\right)^{-1}K^R\right)+\alpha_{T-1,2}\left(2m\left(\Sigma^G_{v,T-1|T-2}\right)^{-1}K^G\right) \tag{3.39}$$

Because $\left(\Sigma^R_{v,T-1|T-2}\right)^{-1}\gg\left(\Sigma^G_{v,T-1|T-2}\right)^{-1}$, we have (3.39)$>0$. Therefore,

$$\alpha_{T-2,1}-\alpha_{T-2,2}=\frac{\left(\frac{m}{\theta^R}+\frac{n}{\theta^G}\right)(\alpha_{T-1,1}-\alpha_{T-1,2})+\oplus}{\left(\frac{m}{\theta^R}+\frac{n}{\theta^G}\right)}>\alpha_{T-1,1}-\alpha_{T-1,2}.$$

Similarly, it is easy to show that

$$\alpha_{T-2,3}=-\alpha_{1,T-1}nK^R\left(\Sigma^G_{v,T-1|T-2}\right)^{-1}+\alpha_{2,T-1}mK^G\left(\Sigma^R_{v,T-1|T-2}\right)^{-1}$$

and is positive. It is easy to show that the following periods are similar to the last two periods and therefore, $\alpha_{3,t}$ decreases overtime. *Q.E.D.*

Proposition 15: (Switch in Market Dominance)

In the final period T, the proportion $\frac{\alpha_{t,1}}{\alpha_{t,2}}$ is $\frac{m\Sigma^G_{V,T|T-1}}{n\Sigma^R_{V,T|T-1}}$. If $\frac{m\Sigma^G_{V,T|T-1}}{n\Sigma^R_{V,T|T-1}}>1$, obviously, the switch will happen. Further more, we have shown that as $t\to T$, the rational investors are exposed to positive correlated returns while fallacy investors are exposed to negative correlated returns and $\alpha_{t,1}$ decreases while $\alpha_{t,2}$ increases. Since there is no correlations over future returns after T, for t large and close enough to T, $\alpha_{T,1} > \alpha_{t,1}$ and $\alpha_{T,2} < \alpha_{t,2}$. Therefore, a sufficient condition for a switch to happen is if $\lim_{t\to+\infty} \frac{\Sigma^G_{V,T|T-1}}{\Sigma^R_{V,T|T-1}} > \frac{m}{n}$. We further prove the properties of $\lim_{t\to+\infty} \frac{\Sigma^G_{V,T|T-1}}{\Sigma^R_{V,T|T-1}}$.

Write out the matrices in the inference problem, we have the following difference equations

$$\Sigma^R_{V,t+1|t} = \Sigma^R_{V,t|t-1} - \frac{(\Sigma^R_{V,t|t-1})^2}{\Sigma^R_{V,t|t-1} + \sigma^2_\xi}$$

$$\Sigma^G_{V,t+1|t} = \Sigma^G_{V,t|t-1} - \frac{(\Sigma^G_{V,t|t-1} + \frac{\beta}{\delta-\beta}c_{t|t-1})^2}{\Sigma^G_{V,t|t-1} + \frac{2\beta}{\beta-\delta}c_{t|t-1} + \left(\frac{\beta}{\delta-\beta}\right)^2 \Sigma^G_{\epsilon,t|t-1} + \left[1 - \left(\frac{\beta}{\delta-\beta}\right)^2\right]\sigma^2_\xi}$$

$$c_{t+1|t} = (\delta-\beta)c_{t|t-1} - \frac{(\Sigma^G_{V,t|t-1} + \frac{\beta}{\delta-\beta}c_{t|t-1})\left[(\delta-\beta)c_{t|t-1} - \beta\Sigma^G_{\epsilon,t|t-1} + \delta\sigma^2_\xi\right]}{\Sigma^G_{V,t|t-1} + \frac{2\beta}{\beta-\delta}c_{t|t-1} + \left(\frac{\beta}{\delta-\beta}\right)^2 \Sigma^G_{\epsilon,t|t-1} + \left[1 - \left(\frac{\beta}{\delta-\beta}\right)^2\right]\sigma^2_\xi}$$

$$\Sigma^G_{\epsilon,t+1|t} = (\delta-\beta)^2\Sigma^G_{\epsilon,t+1|t} - \frac{\left[(\delta-\beta)c_{t|t-1} - \beta\Sigma^G_{\epsilon,t|t-1} + \delta\sigma^2_\xi\right]^2}{\Sigma^G_{V,t|t-1} + \frac{2\beta}{\beta-\delta}c_{t|t-1} + \left(\frac{\beta}{\delta-\beta}\right)^2 \Sigma^G_{\epsilon,t|t-1} + \left[1 - \left(\frac{\beta}{\delta-\beta}\right)^2\right]\sigma^2_\xi} + \sigma^2_\xi$$

where $c_{t+1|t}$ is the conditional covariance for the fallacy investors. Linearise all four equations at around the stable state, that is $\Sigma^R_{V,t+1|t} = \Sigma^G_{V,t+1|t} = c_{t+1|t} = 0$ and $\Sigma^G_{\epsilon,t+1|t} = \sigma^2_\xi$ we have

$$\Sigma^R_{V,t+1|t} \approx \Sigma^R_{V,t|t-1} \tag{3.40}$$

$$\begin{bmatrix} \Sigma^G_{V,t+1|t} \\ c_{t+1|t} \\ \Sigma^G_{\epsilon,t+1|t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \beta-\delta & \delta & 0 \\ (\beta-\delta)^2 & 2\delta(\beta-\delta) & \delta^2 \end{bmatrix} \begin{bmatrix} \Sigma^G_{V,t|t-1} \\ c_{t|t-1} \\ \Sigma^G_{\epsilon,t|t-1} - \sigma^2_\xi \end{bmatrix} \tag{3.41}$$

From the linearised difference equation, we can easily see that $\lim_{T\to+\infty} \frac{\Sigma^G_{V,T|T-1}}{\Sigma^R_{V,T|T-1}}$ exists and approximately equals $\frac{\Sigma^G_{V,0}}{\Sigma^R_{V,0}}$ where $\Sigma^G_{V,0}$ and $\Sigma^R_{V,0}$ are two constants.

Reorganised these equations, we have

$$\frac{\Sigma^R_{V,t+1|t}}{\Sigma^R_{V,t|t-1}} = 1 - \frac{\Sigma^R_{V,t|t-1}}{\Sigma^R_{V,t|t-1} + \sigma^2_\xi}$$

$$\frac{\Sigma^G_{V,t+1|t}}{\Sigma^G_{V,t|t-1}} = 1 - \frac{\Sigma^G_{V,t|t-1} + \dfrac{2\beta}{\delta-\beta}c_{t|t-1} + \left(\dfrac{\beta}{\delta-\beta}\right)^2 \dfrac{c_{t|t-1}}{\Sigma^G_{V,t|t-1}}}{\Sigma^G_{V,t|t-1} + \dfrac{2\beta}{\beta-\delta}c_{t|t-1} + \left(\dfrac{\beta}{\delta-\beta}\right)^2 \Sigma^G_{\epsilon,t|t-1} + \left[1 - \left(\dfrac{\beta}{\delta-\beta}\right)^2\right]\sigma^2_\xi}$$

If $\dfrac{\Sigma^G_{V,t|t-1}}{\Sigma^R_{V,t|t-1}}$ increases in t, we must have $\dfrac{\Sigma^R_{V,t+1|t}}{\Sigma^R_{V,t|t-1}} - \dfrac{\Sigma^G_{V,t+1|t}}{\Sigma^G_{V,t|t-1}} > 0$

$$\frac{\Sigma^R_{V,t+1|t}}{\Sigma^R_{V,t|t-1}} - \frac{\Sigma^G_{V,t+1|t}}{\Sigma^G_{V,t|t-1}} = \frac{\left(\Sigma^G_{V,t|t-1} + \dfrac{2\beta}{\beta-\delta}c_{t|t-1}\right)\sigma^2_\xi - \Sigma^R_{V,t|t-1}S^2_\xi}{\left(\Sigma^G_{V,t|t-1} + \dfrac{2\beta}{\beta-\delta}c_{t|t-1} + S^2_\xi\right)\left(\Sigma^R_{V,t|t-1} + \sigma^2_\xi\right)} + \frac{(\dfrac{\beta}{\delta-\beta})^2 \dfrac{c^2_{t|t-1}}{\Sigma^G_{V,t|t-1}}}{\Sigma^G_{V,t|t-1} + \dfrac{2\beta}{\beta-\delta}c_{t|t-1} + S^2_\xi} \tag{3.42}$$

where $S^2_\xi = \left(\dfrac{\beta}{\delta-\beta}\right)^2 \Sigma^G_{\epsilon,t|t-1} + \left[1 - \left(\dfrac{\beta}{\delta-\beta}\right)^2\right]\sigma^2_\xi \approx \sigma^2_\xi$ and therefore, the second term in the equation is positive. Next, we show that the first term is also positive.

The denominator of the first term is positive since it approaches σ^2_ξ. By plugging the general solutions derived from equations (3.40)(3.41) into the first term of equation (3.42) and assuming $\Sigma^R_{V,1|0} = \Sigma^G_{V,1|0} = \Sigma_{V,0}$, the numerator of the first term becomes

$$\left[\Sigma_{V,0}\left(\frac{2\beta}{1-\delta}(1-\delta^t) + 1\right) + \frac{2\beta\delta^t}{1-\delta^2}c_0\right]\sigma^2_\xi - \Sigma_{V,0}\cdot\left[\frac{A_t(1-\delta^{2t})}{1-\delta^2} + \delta^{2t}\Sigma_{\epsilon,0}\right]$$

where $A_t = (\delta-\beta)\Sigma_{V,0} + 2\delta(\beta-\delta)\left(\dfrac{(\beta-\delta)\Sigma_{V,0}(1-\delta^t)}{1-\delta} + \delta^t c_0\right) + (1-\delta^2)\sigma^2_\xi$. when $t \to +\infty$, and for $\delta < 1$, $\beta < 1$, the numerator becomes

$$\frac{2\beta}{1-\delta} - \frac{(\delta-\beta) + \dfrac{2\delta(\beta-\delta)^2}{1-\delta}}{1-\delta^2}\Sigma_{V,0} \tag{3.43}$$

It is greater than 0 when $\Sigma_{V,0}$ is close enough to 0. Therefore, we proved that $\dfrac{\Sigma^R_{V,t+1|t}}{\Sigma^R_{V,t|t-1}} - \dfrac{\Sigma^G_{V,t+1|t}}{\Sigma^G_{V,t|t-1}} > 0$. Since the second term approaches 0 while the denominator in the first term approaches a constant for any given σ_ξ^2, $\frac{\Sigma^R_{V,t+1|t}}{\Sigma^R_{V,t|t-1}} - \frac{\Sigma^G_{V,t+1|t}}{\Sigma^G_{V,t|t-1}}$ is mainly determined by the numerator we derived and is increasing in β and δ.

Since when $\beta = 0$ and $\delta = 0$, both types are rational and $\frac{\Sigma^G_{V,T|T-1}}{\Sigma^R_{V,T|T-1}} = 1$, we have $\underset{T\to+\infty}{lim} \frac{\Sigma^G_{V,T|T-1}}{\Sigma^R_{V,T|T-1}} \geq 1$ and increases in δ and β. When $\delta = 1$, (3.43) explodes, therefore, $\underset{T\to+\infty}{lim} \frac{\Sigma^G_{V,T|T-1}}{\Sigma^R_{V,T|T-1}}$ also explodes at $\delta = 1$. *Q.E.D.*

Proposition 17 and Lemma 8: (Volatility)

$$\begin{aligned} \Psi_t^R &= A^R_{\Psi,t}\Psi^R_{t-1} + B^R_{\Psi,t}\left(y_t - V^R_{t|t-1}\right) \\ &= G^R_t \Psi^R_{t-1} + B^R_{\Psi,t} y_t \\ &= \prod_{j=1}^{t} G^R_j \Psi_0 + \sum_{n=1}^{t-1}\left(\prod_{l=n+1}^{t} G_n\right) B^R_{\Psi,n} y_n + B^R_{\Psi,t} y_t, \end{aligned} \tag{3.44}$$

Further more, we have

$$y_n = V + \xi_n \tag{3.45}$$

Substitute equation (3.45) into equation (3.44) and reformat the equation we have

$$\Psi_t^R = \prod_{j=1}^{t} G^R_j \Psi_0 + \sum_{n=1}^{t}\left(\prod_{l=n+1}^{t} G_n\right) B^R_{\Psi,n} V + \sum_{n=1}^{t}\left[\left(\prod_{l=n+1}^{t} G_n\right) B^R_{\Psi,n}\xi_n\right].$$

Therefore, we have

$$Var\left(\Psi_t^R\right) = \left(\sum_{n=1}^{t} D_n\right)\sigma_V^2 \left(\sum_{n=1}^{t} D_n\right)^{\mathbf{T}} + \sum_{n=1}^{t}\left(D_n D_n^{\mathbf{T}}\right)\sigma_\xi^2$$

where $D_n = \left(\prod_{l=n+1}^{t} G_n\right) B^R_{\Psi,n}$. Expand the last equation, we can easily derive our conclusion in the Proposition.

To derive the result in Lemma 8, write

$$\begin{aligned} Var(P_{t+1}|\mathcal{F}_t) &= \alpha_{t+1} Var(\Psi^R_{t+1}|\mathcal{F}_t)\alpha^{\mathbf{T}}_{t+1} \\ &= \alpha_{t+1} Var(A^R_{\Psi,t+1}\Psi^R_t + B^R_{\Psi,t+1} v^R_{t+1}|\mathcal{F}_t)\alpha^{\mathbf{T}}_{t+1} \\ &= \left(\alpha_{1,t+1}K^R_{t+1} + \alpha_{2,t+1}K^G_{t+1} + \alpha_{3,t+1}k_{t+1}\right)^2 Var_R(v_{t+1}|\mathcal{F}_t). \end{aligned}$$

We complete our proof for Lemma 8. *Q.E.D.*

Proposition 16: (Autocorrelation)

Expand Q_i recursively and use the results in Proposition 10 on the $Cov^R(v_i,\ v_j)$ we have

$$\begin{aligned} Cov^R\left(Q_i,\ Q_j\right) &= A^R_{Q,i} Cov\left(\Psi_i,\ \Psi_j\right) A^{R\mathbf{T}}_{Q,j} + A_{Q,i} E_R\left(\Psi_i \cdot v_j^{\mathbf{T}}\right) B^{\mathbf{T}}_{Q,j} \\ &= A^R_{Q,i} \prod_{k=0}^{i-j-1} A^R_{\Psi,i-k} \left[Var\left(\Psi_j\right) A^{R\mathbf{T}}_{Q,j} + B^R_{\Psi,j} Var\left(v_j\right) B^{R\mathbf{T}}_{Q,j}\right] \end{aligned}$$

where the format of the $Var^R\left(\Psi_t\right)$ is shown in the proof of Proposition 17. $Q.E.D.$

3.8 Mathematical Appendix B

A 3-period Proof of Proposition 11, 18 and 9

Consider a simple 3-period problem where the traders' expected utility functions are given by:

$$E[-exp(-\lambda W_3^i|\mathcal{F}_t)].$$

Using backward induction, at period $t = 2$,the optimisation problem is

$$\underset{X_2^i}{Max} E_{i,2}\left[-exp\left\{-\lambda W_2^i - X_2^i(V - P_2)\right\}|\mathcal{F}_2\right]$$
$$= \underset{X_2^i}{Max} - exp\left\{-\lambda W_2^i - \lambda X_2^i\left(V_{3|2}^i - P_2\right) + \frac{1}{2}\lambda^2 X_2^{i2}\Sigma_{V,3|2}^i\right\}$$

From the first order condition, we have $X_2^i = \dfrac{V_{3|2}^i - P_2}{\lambda\Sigma_{V,3|2}^i}$. Together with $P_2 = \alpha_2\Psi_2$, we have $X_2^i = \dfrac{(I_1 - \alpha_2)\Psi_2}{\lambda\Sigma_{V,3|2}^i}$, where $I_1 = [0, 1, 0, 0]$, and $F_2^i = \dfrac{I_1 - \alpha_2}{\Sigma_{V,3|2}^i}$.

Plug $X_2^i = \dfrac{(I_1 - \alpha_2)\Psi_2}{\lambda\Sigma_{V,3|2}^i}$ into the original utility function, we can derive the value function at $t = 2$:

$$\begin{aligned} J(W_2; X_2; 2) &= -exp\left\{-\lambda W_2^i - \frac{1}{2}\frac{\left(V_{3|2}^i - P_2\right)^2}{\Sigma_{V,3|2}^i}\right\} \\ &= -exp\left\{-\lambda W_2^i - \frac{1}{2}\frac{\Psi_2^{\mathbf{T}}(I_1 - \alpha_2)^{\mathbf{T}}(I_1 - \alpha_2)\Psi_2}{\Sigma_{V,3|2}^i}\right\} \\ &= -exp\left\{-\lambda W_2^i - \frac{1}{2}\Psi_2^{\mathbf{T}}U_2^i\Psi_2\right\} \end{aligned}$$

where $U_2^i = \dfrac{(I_1 - \alpha_2)^{\mathbf{T}}(I_1 - \alpha_2)}{\Sigma_{V,3|2}^i}$.

Use the market clearing condition: $mX_2^R + nX_2^G = 0$, we have

$$m\frac{V_{3|2}^R - P_2}{\lambda\Sigma_{V,3|2}^R} + n\frac{V_{3|2}^G - P_2}{\lambda\Sigma_{V,3|2}^G} = 0,$$

thus

$$P_2 = \frac{mV_{3|2}^R\Sigma_{V,3|2}^G + nV_{3|2}^G\Sigma_{V,3|2}^R}{m\Sigma_{V,3|2}^G + n\Sigma_{V,3|2}^R}$$

Therefore, $\alpha_2 = [0, \frac{m\Sigma^G_{V,3|2}}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}}, \frac{n\Sigma^R_{V,3|2}}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}}, 0]$. Plug it into U^i_2, we have

$$
\begin{aligned}
U^R_2 &= \begin{bmatrix} 0 \\ \frac{n\Sigma^R_{V,3|2}}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}} \\ \frac{-n\Sigma^R_{V,3|2}}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}} \\ 0 \end{bmatrix} \cdot [0, \frac{n\Sigma^R_{V,3|2}}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}}, \frac{-n\Sigma^R_{V,3|2}}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}}, 0]/\Sigma^R_{V,3|2} \\
&= \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \left(\frac{n}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}}\right)^2 \Sigma^R_{V,3|2} & -\left(\frac{n}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}}\right)^2 \Sigma^R_{V,3|2} & 0 \\ 0 & -\left(\frac{n}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}}\right)^2 \Sigma^R_{V,3|2} & \left(\frac{n}{m\Sigma^G_{V,3|2}+n\Sigma^R_{V,3|2}}\right)^2 \Sigma^R_{V,3|2} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{aligned}
$$

This is exactly what we have in the general problem. Similar for U^G_2.

Using backward induction, the problem we want to solve at $t = 1$ is

$$
\begin{aligned}
\underset{X^i_1}{Max} E_i \left[J(W_2; X_2; 2)|\mathcal{F}_1\right] &= \underset{X^i_1}{Max} E_i \left[-exp\left\{-\lambda W^i_2 - \frac{1}{2}\Psi^{\mathbf{T}}_2 U^i_2 \Psi_2\right\}|\mathcal{F}_1\right] \\
&= \underset{X^i_1}{Max} E_i \left[-exp\left\{-\lambda X^i_1 Q_2 - \frac{1}{2}\Psi^{\mathbf{T}}_2 U^i_2 \Psi_2\right\}|\mathcal{F}_1\right]
\end{aligned}
$$

Since in a linear equilibrium, we have

$Q^i_2 = A^i_{Q,2}\Psi_1 + B^i_{Q,2}v^i_2$, and $\Psi^i_2 = A^i_{\Psi,2}\Psi_1 + B^i_{\Psi,2}v^i_2$. Plug these two processes into the optimisation problem, the expectation can be removed and replaced by a linear function of conditional expecation and variance. That is equation (3.18).

$$
\begin{aligned}
&\underset{X^i_1}{Max} E_i \left[J(W_2; X_2; 2)|\mathcal{F}_1\right] \\
&= \underset{X^i_1}{Max} - \rho_2 exp\left\{-\lambda W^i_1 - \frac{1}{2}\Psi^{i\mathbf{T}}_1 A^{i\mathbf{T}}_{\Psi,2} U^i_2 A^i_{\Psi,2}\Psi^i_1\right\} \times exp\left\{-\lambda X^i_1 A^i_{Q,2}\Psi^i_1 + \frac{1}{2} C^{i\mathbf{T}}_2 \theta^i_2 C^i_2\right\}
\end{aligned}
$$

where C^i_2, θ^i_2 and ρ_2 are simply functions in U_2 as expressed in the equation (3.18).

The first order condition of the maximisation problem gives that
$X^i_1 = \frac{1}{\lambda} F^i_1 \Psi_1$, where $F^i_1 \equiv \left(B^i_{Q,2}\theta^i_2 B^{i\mathbf{T}}_{Q,2}\right)^{-1}\left(A^i_{Q,2} - B^i_{Q,2}\theta^i_2 B^{i\mathbf{T}}_{\Psi,2} U^i_2 A^i_{\Psi,2}\right)$.

Substitute $X_1^i = \frac{1}{\lambda} F_1^i \Psi_1$ into

$$-\rho_2 exp \left\{ -\lambda W_1^i - \frac{1}{2} \Psi_1^{i\mathbf{T}} A_{\Psi,2}^{i\mathbf{T}} U_2^i A_{\Psi,2}^i \Psi_1^i \right\} \times exp \left\{ -\lambda X_1^i A_{Q,2}^i \Psi_1^i + \frac{1}{2} C_2^{i\mathbf{T}} \theta_2^i C_2^i \right\}$$

, we have

$$J(W_1; X_1; 1) = -\rho_2 exp \left\{ -\lambda W_1^i - \frac{1}{2} \Psi_1^{i\mathbf{T}} A_{\Psi,2}^{i\mathbf{T}} U_2^i A_{\Psi,2}^i \Psi_1^i - F_1^i \Psi_1^i A_{Q,2}^i \Psi_1^i + \frac{1}{2} C_2^{i\mathbf{T}} \theta_2^i C_2^i \right\}.$$

By our assumption that $J(W_1; X_1; 1) = -exp\{-\lambda W_1^i - \frac{1}{2} \Psi_1^{\mathbf{T}} U_1^i \Psi_1\}$, we let

$$\begin{aligned} &- exp\{-\lambda W_1^i - \frac{1}{2} \Psi_1^{\mathbf{T}} U_1^i \Psi_1\} \\ &\qquad = -\rho_2 exp \left\{ -\lambda W_1^i - \frac{1}{2} \Psi_1^{i\mathbf{T}} A_{\Psi,2}^{i\mathbf{T}} U_2^i A_{\Psi,2}^i \Psi_1^i - F_1^i \Psi_1^i A_{Q,2}^i \Psi_1^i + \frac{1}{2} C_2^{i\mathbf{T}} \theta_2^i C_2^i \right\} \end{aligned}$$

which give us that

$U_1^i = -2ln\rho_2^i I_{11} + M_1^i$, where M_1^i is a function of U_2^i and is known. The expression of M_1^i is given in the general proof.

Similar as in deriving α_2, the only undetermined parameter at this stage is $A_{Q,2}^i$, which contains α_1 and our U_1^i is still a function of α_1. To derive our α_1, we further apply the market clearing condition: $mX_1^R + nX_1^G = 0$. Substitute $X_1^i = \frac{1}{\lambda} F_1^i \Psi_1$ into the market clearing condition, we have

$$\left(mF_1^R + nF_1^G \right) \Psi_1 = 0.$$

Since it holds for any value of Ψ_1, we have

$$mF_1^R + nF_1^G = 0_{1\times 4}.$$

Substitute $A_{Q,2}^i = \alpha_2 A_{\Psi,2}^i - \alpha_1$ and $B_{Q,2}^i = \alpha_2 K_2$ into

$$F_1^i \equiv \left(B_{Q,2}^i \theta_2^i B_{Q,2}^{i\mathbf{T}} \right)^{-1} \left(A_{Q,2}^i - B_{Q,2}^i \theta_2^i B_{\Psi,2}^{i\mathbf{T}} U_2^i A_{\Psi,2}^i \right),$$

we have $F_1^i = \frac{1}{(\alpha_2 K_2)^2 \theta_2^i} \alpha_2 E_2^i - \frac{1}{(\alpha_2 K_2)^2 \theta_2^i} \alpha_1$, where $E_2^i = A_{\Psi,2}^i - K_2 B_{\Psi,2}^{i\mathbf{T}} \theta_2^i U_2^i A_{\Psi,2}^i$ and is known. Plug the formula into the market clearing condition, we have

$$m\left(\frac{\alpha_2 E_2^R}{\theta_2^R}-\frac{\alpha_1}{\theta_2^R}\right)+n\left(\frac{\alpha_2 E_2^G}{\theta_2^G}-\frac{\alpha_1}{\theta_2^G}\right)=0,$$

which gives us the expression of α_1 as a function of α_2, that is

$$\alpha_1=\alpha_2\overline{E}_2,$$

where $\overline{E}_2 \equiv \left(\frac{m}{\theta_2^R}+\frac{n}{\theta_2^G}\right)^{-1}\left(\frac{mE_2^R}{\theta_2^R}+\frac{nE_2^G}{\theta_2^G}\right)$ and is known. *Q.E.D.*

3.9 Appendix C: Non-zero Net Supply

We assume $\Theta = 0$ for simplicity since a positive Θ together with the learning process generates non-zero correlations in prices, making the analysis of a signal shock on the prices and returns less clear.

By changing the net supply of the stock Θ into positive, the equilibrium price now contains a constant term, that is $\alpha_{0,t} \neq 0$.

It is easy to show that $\alpha_{1,t}$, $\alpha_{2,t}$ and $\alpha_{3,t}$ still evolves as shown in the main paragraph while $\alpha_{0,t}$ can be derived as below:

$$\alpha_{0,t}=\alpha_{0,t+1}\overline{E}_1-\frac{\Theta}{\frac{m}{\theta_{t+1}^R}+\frac{n}{\theta_{t+1}^G}}$$

where $\overline{E}_1$ is the first column of matrix $\overline{E}$. Consistent with previous studies, $\alpha_{0,t}$ is negative, reflecting the risk-aversion attitude. It is also shown that $\alpha_{0,t}$ increases over time since more information reduces the risk in variance, thus strengthening demands and boosting the asset price.

Another change in our conclusion is the optimal demand. F_t^R and F_t^G now also contain a constant term. That is

$$X_t^i=\frac{1}{\lambda}F_{1,t}^i+\frac{1}{\lambda}F_{d,t}^i\left(V_{t+1|t}^R-V_{t+1|t}^G\right)+\frac{1}{\lambda}F_{\epsilon,t}^i\epsilon_{t+1|t}$$

The effect of $F_{1,t}^i$ is independent of $F_{d,t}$ and $F_{\epsilon,t}$. Figure 3.12 gives the simulation result for $F_{1,t}^i$.

Signs of constant terms $F_{1,t}^i$ reflect that the rational investors turns from long to short positive. The constant term reflect the risk attitudes of the investors, indicating that the rational investors are comparatively more risk-averse in the market when the liquidation date is close. This result is consistent with our analysis in Propostion 14.

With a positive net supply, even the price in the pure rational market increases as the risk in variance decreases over time. Here we provide the simulation results for the prices and returns following a signal shock. Results are similar to what we have before except the price has a general increasing trend.

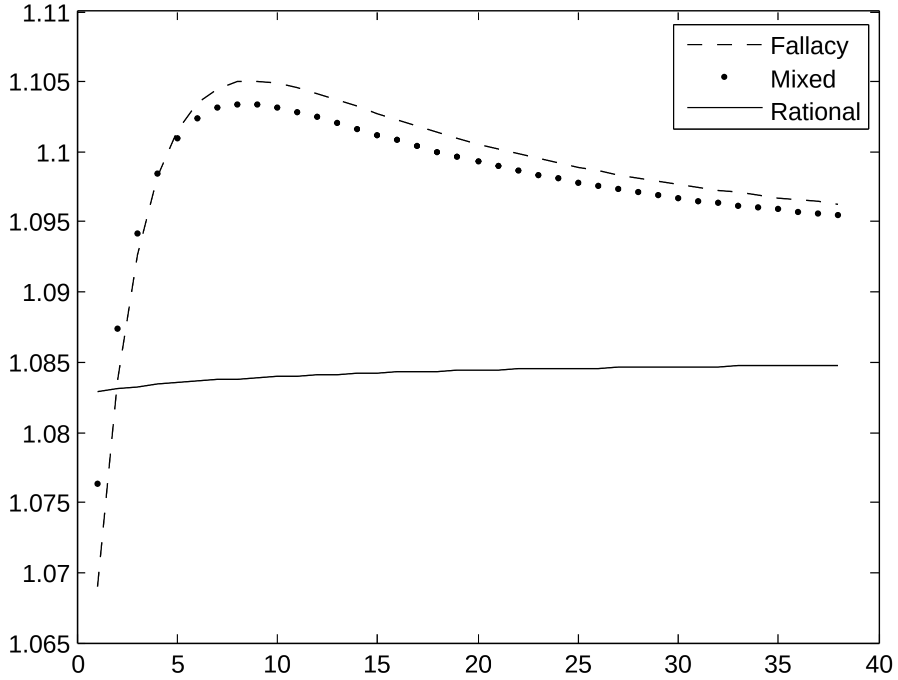


Figure 3.1: INFORMATION SHOCK ON EXPECTED PRICE
This figure plots the rational expectation in price after a signal shock at $t = 10$. Parameters are set at the following values: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$.

3.10 Appendix D: Figures

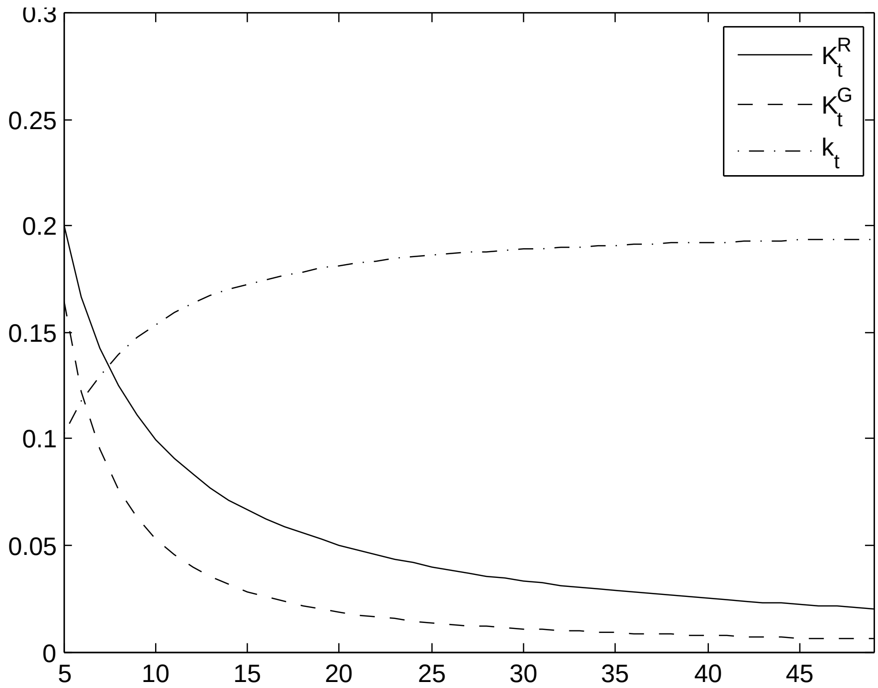


Figure 3.2: KALMAN GAINS
This figure plots the Kalman Gains K_t^R for the rational agents and the Kalman Gains $K_t = [K_t^G,\ k_t]'$ for the fallacy agents over time. Parameters are set at the following values: $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$. First 5 periods are not shown in the figure to remove the initial parameter setting effects.

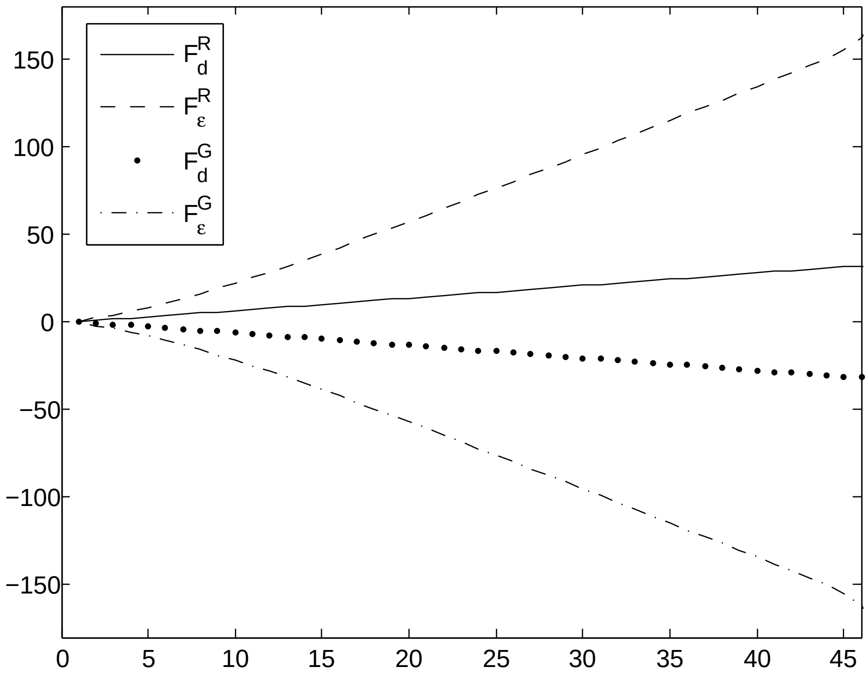


Figure 3.3: $F^R_{d,t}$, $F^R_{\epsilon,t}$, $F^G_{d,t}$, $F^G_{\epsilon,t}$

$F^R_{d,t}$, $F^R_{\epsilon,t}$, $F^G_{d,t}$, $F^G_{\epsilon,t}$, the weight on $V^R_{t+1|t} - V^G_{t+1_t}$ and $\epsilon_{t+1|t}$ in the optimal portfolio. Parameters are: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma^2_\xi = 1$, $\sigma^2_V = 1$. The last 4 periods are eliminated to remove the liquidation effect.

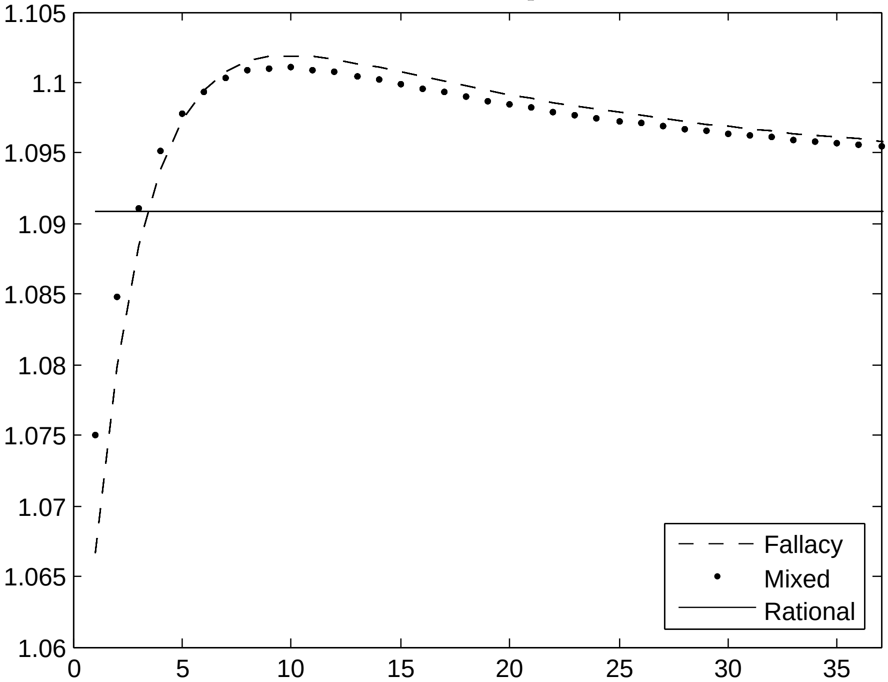


Figure 3.4: INFORMATION SHOCK ON EXPECTED PRICE

This figure plots the rational expectations in price after a signal shock at $t = 0$. Parameters are set at the following values: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma^2_\xi = 1$, $\sigma^2_V = 1$.

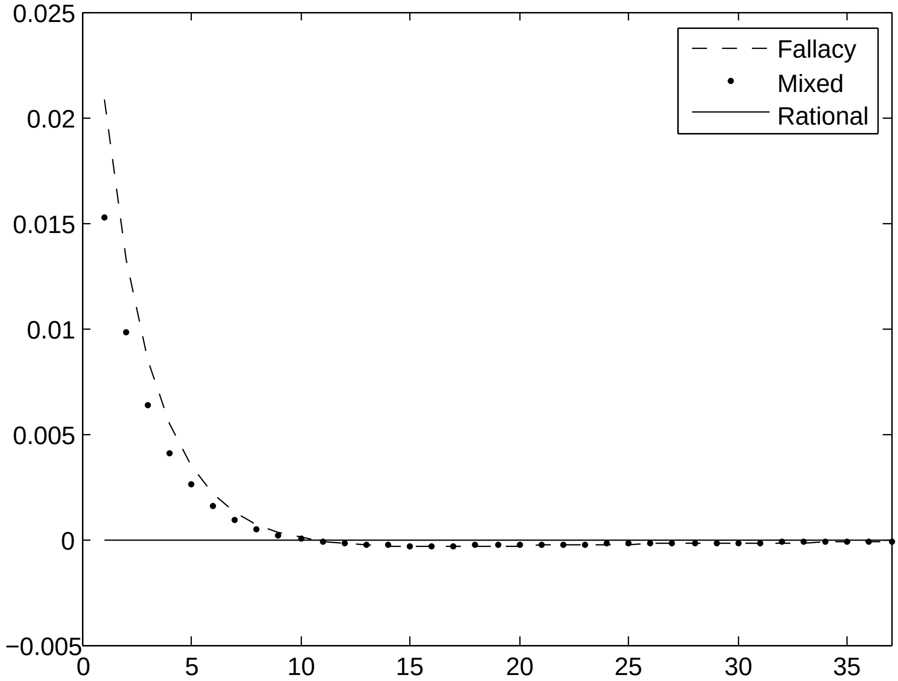


Figure 3.5: INFORMATION SHOCK ON EXPECTED RETURN
Figure 3.5 plots the rational expectations in the dollar returns after a single signal shock at $t = 0$. Parameters are set at the following values: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$.

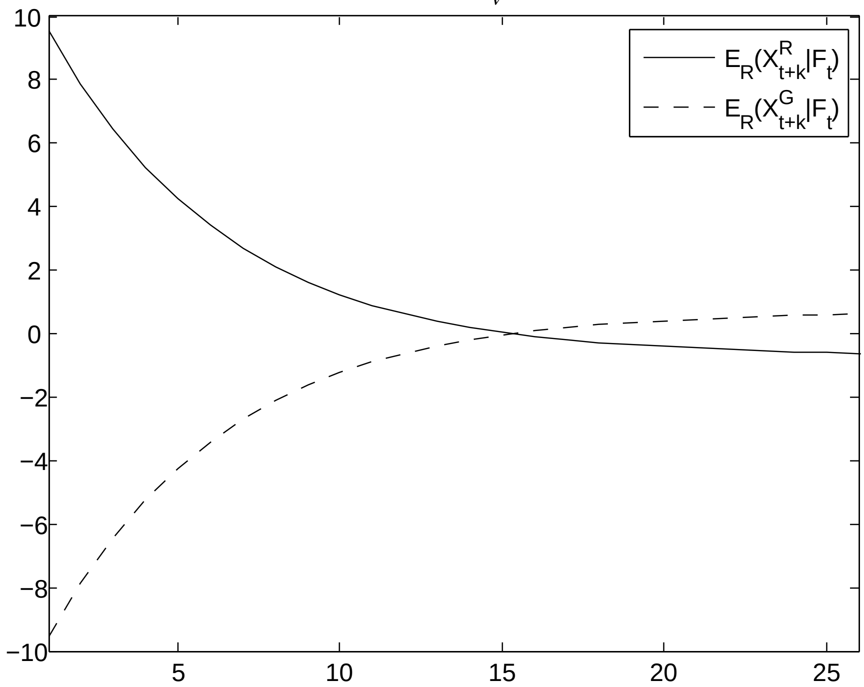


Figure 3.6: INFORMATION SHOCK ON EXPECTED OPTIMAL HOLDINGS
Figure 3.6 plots the optimal holding positions following a signal shock at $t = 0$ Parameters are set at the following values: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$. Figure only shows $t = 0$ to $t = 28$. The last several periods are removed to make the results easier to be read.

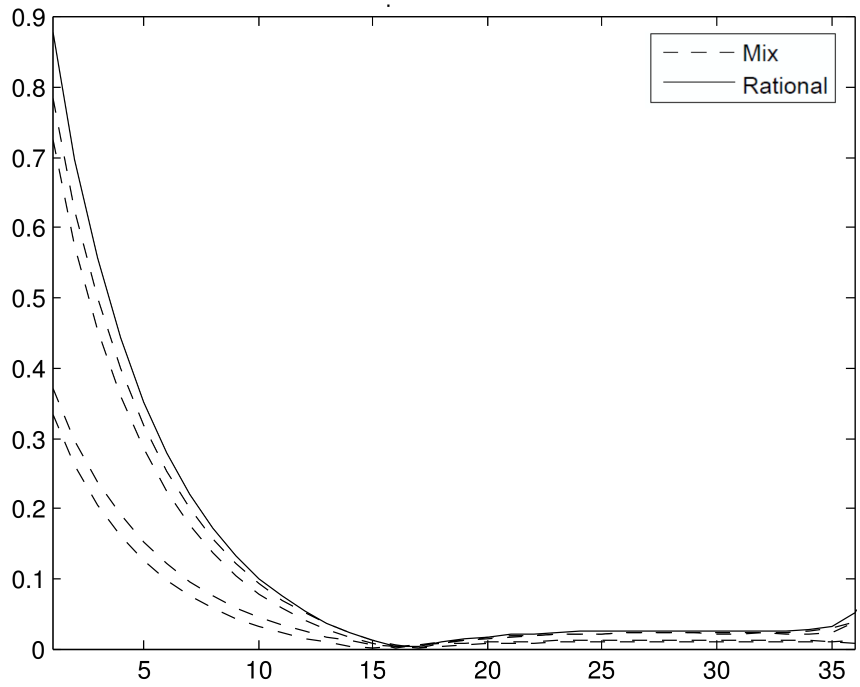


Figure 3.7: INFORMATION SHOCK ON EXPECTED VOLUME
Figure 3.7 plots the trading volume following a signal shock at $t = 0$. Parameters are set at the following values: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$. Figure only shows $t = 0$ to $t = 28$. The solid line represents $m = 0.5$ while the dash lines from top to down are representing $m = 0.1, 0.3, 0.7, 0.9$. We remove the last several periods to make results easier to be read.

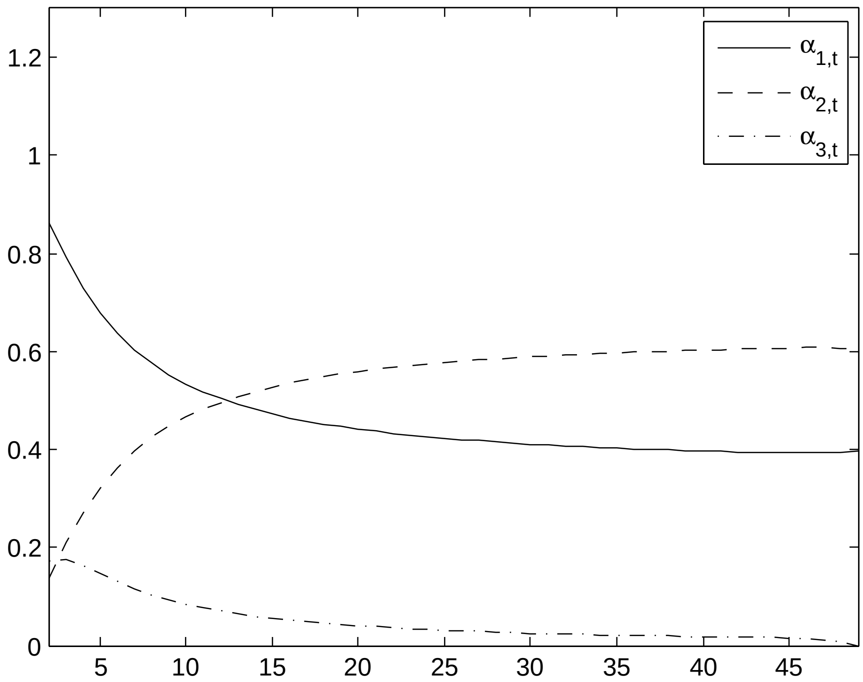


Figure 3.8: $\alpha_{1,t},\ \alpha_{2,t},\ \alpha_{3,t}$
This figure plots $\alpha_{1,t},\ \alpha_{2,t},\ \alpha_{3,t}$, the weight on $V_{t+1|t}^R$, $V_{t+1_t}^G$ and $\epsilon_{t+1|t}$ in the equilibrium price P_t overtime. Parameters are set at the following values: $m = 0.9$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$.

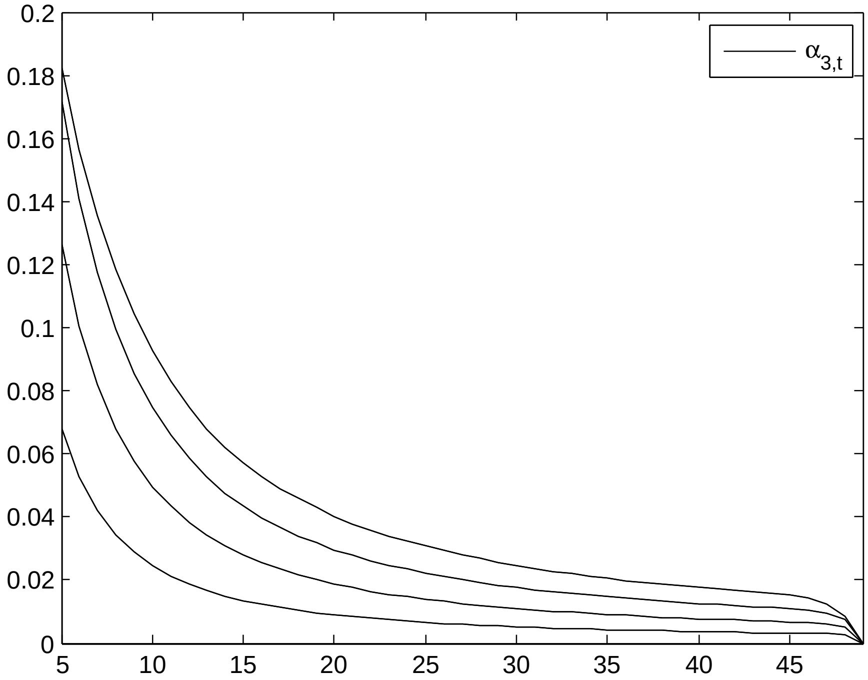


Figure 3.9: $\alpha_{3,t}$

This figure plots $\alpha_{3,t}$ for different proportion of rational agents in the market. Lines from bottom to top represents $m = 0.2,\ 0.4,\ 0.6,\ 0.8$. Parameters are set at the following values: $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$.

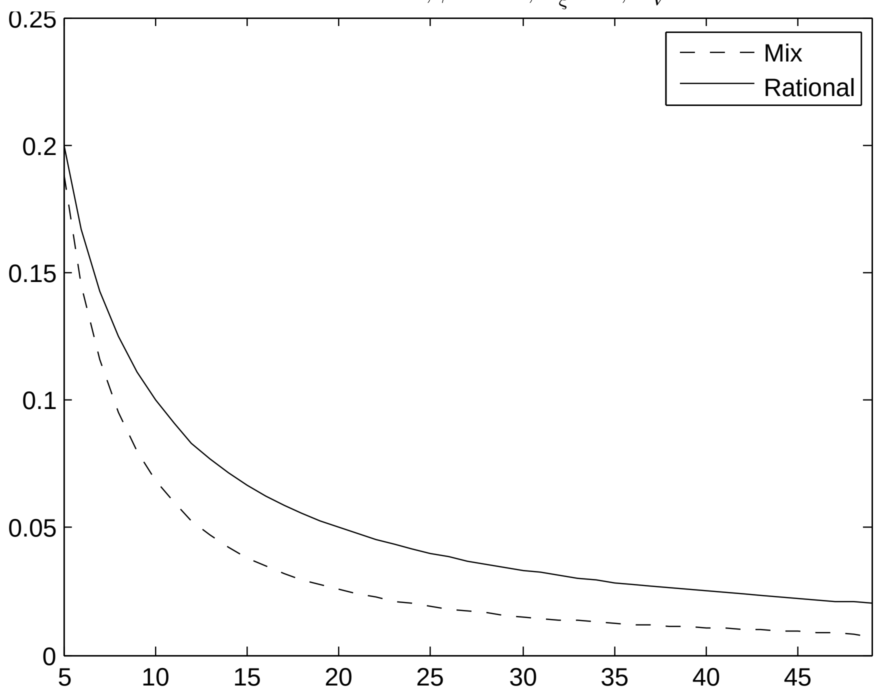


Figure 3.10: PRICE VOLATILITY

Figure 3.10 plots $\frac{\sqrt{Var^R(P_{t+1}|\mathcal{F}_t)}}{\sigma^R_{v,t+1|t}}$, the one period ahead conditional price volatility normalised by the value volatilities in the rational and a mixed market. Parameters are set at the following values: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma_\xi^2 = 1$, $\sigma_V^2 = 1$.

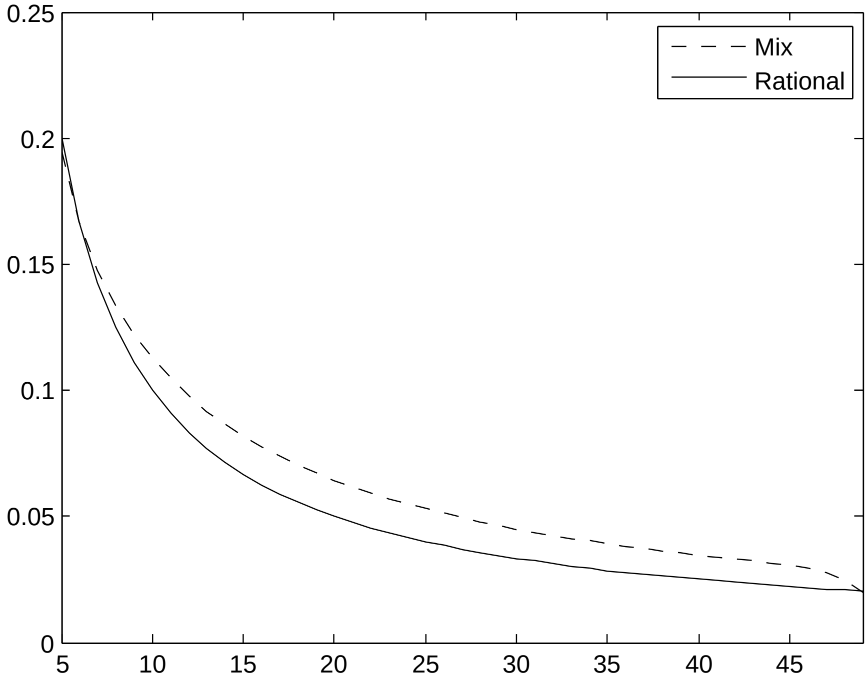


Figure 3.11: PRICE VOLATILITY WITH HOTHAND FALLACY

This figure plots $\frac{\sqrt{Var^R(P_{t+1}|\mathcal{F}_t)}}{\sigma^R_{V,t+1|t}}$, the price volatilities normalised by the value volatilities in a mixed market and a rational market. Parameters are set at the following values: $T = 50$, $\lambda = 1$, $\delta = 0.5$, $\beta = -0.5$, $\sigma^2_\xi = 1$, $\sigma^2_V = 1$.

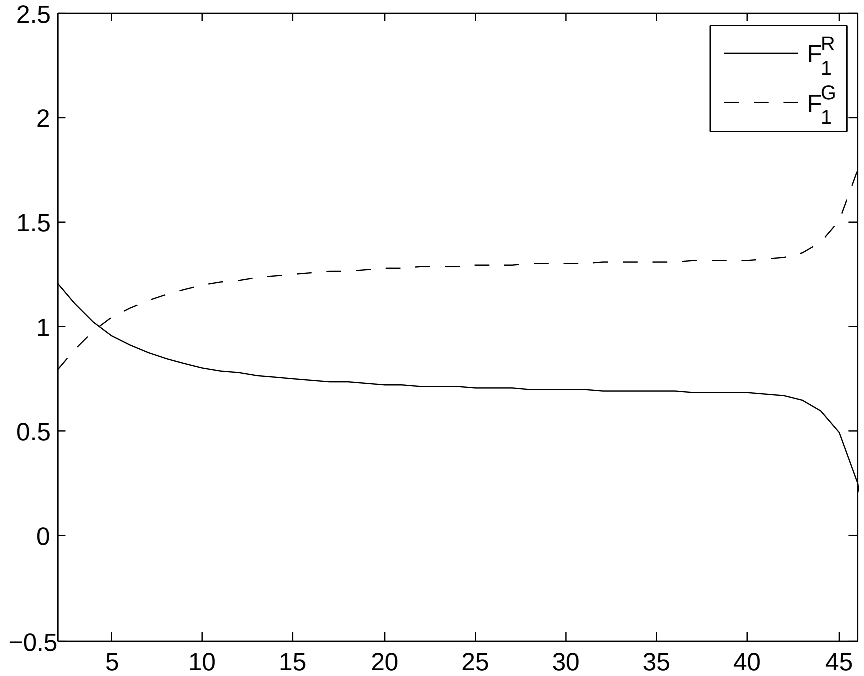


Figure 3.12: $F^R_{1,t}$, $F^G_{1,t}$

This figure plots $F^R_{1,t}$, $F^G_{1,t}$ overtime Parameters are set at the following values: $m = 0.5$, $T = 50$, $\lambda = 1$, $\delta = 0.8$, $\beta = 0.6$, $\sigma^2_\xi = 1$, $\sigma^2_V = 1$. The last 4 periods are eliminated to remove the liquidation effect.

Chapter 4

Regime Switching in Financial Market and Portfolio Choice

Abstract

This paper considers a variety of regime switching models with time varying transition probabilities for the joint distribution of stocks and bonds returns. The paper results support a two-regime univariate model for stocks with ISM (Manufacturing Index) and P/E ratio, as leading predictors for the transition probabilities and support the fixed transition probability model for univariate distribution of bond returns. Under joint distribution assumption, our model selects a three regime model with ISM, unemployment rate and P/E ratio as predictors for the time varying transition probabilities. Even though both fixed and time varying transition probability models identify three regimes in the financial market, however, the time varying transition probability model provides better out of sample predictions, based on the regime-dependent portfolio performance.

4.1 Introduction

The boom-bust cycles in the financial market have been as old as capitalism itself. The cycles have repeated and will also repeat in the future. In reality, people can hardly ignore these cycles due to their threat to the financial markets and to the entire economy. Furthermore, investment opportunities also vary over cycles since different asset classes have different performance over time. One strand of econometric literature treats the cycles as a Markov chain, since the financial market time series sometimes exhibit fundamental changes in their behaviour, associated with events like financial crisis (Jeanne and Masson, 1998; Mouratidis, 2008; Cerra and Saxena, 2005; Hamilton, 2005), or discrete policy, economic and political events (Hamilton, 1988; Sims and Zha, 2006; Davig, 2004).

This paper explores a variety of econometric models for the joint distribution of stocks and bonds returns in the US market under the regime switching framework. While the development of Markov switching extensions to time series modelling has provided a useful way of characterising financial market cycles, these models are not without their weakness. One problem of the classic regime switching model is that the transition probabilities are constant and can hardly reflect the changes in the economy environment. This paper attempts to build a transition probabilities matrix that is time varying, predicted by some leading indicators and also has the classic feature of regime switching models for identifying the regimes in both single and joint markets.

The paper proceeds as follows. Section II presents the univariate Markov regime switching model for individual time series of stock and equity, starting with the univariate Fixed Transition Probability (FTP) model. The model results suggests a 2- regime (bullish and bearish market) AR(0) process for stocks and 3-regime AR(0) process for bonds. Moving to the Univariate Time Varying Transition Probability (UTVTP) Model, it is shown that 2-regime model with P/E ratio and ISM as the predictors for the transition probability matrix for the stock market is preferred to the FTP model. While for the bond market, even though the FTP model achieves lower AIC, the regimes identified by the TVTP model with AR(1) and ISM and PE ratio as leading indicators are more meaningful. Section III extends the univariate model to bivariate model settings by first exploring the results of a bivariate FTP model using the joint distribution of stocks and bonds. BFTP model supports the 3-regime assumption (stable, transition, crisis) with AR(0) for stocks and AR(1) for bonds. Section IV formally presents the Bivariate TVTP model. The BTVTP models still supports the 3-regime AR(0)-AR(1) assumption, but with improved identifications on all 3 regimes. The BTVTP also achieves better AIC as compared to the BFTP model. Furthermore, all three leading predictors (ISM, P/E ratio, unemployment rate) are significant. Finally, section V examines the out-of-sample regime-dependent portfolio performance using the BFTP and BTVTP models in section III and IV. We build an expected-mean-expected-variance portfolio using filtered probabilities. Both BFTP and TVTP models outperform the SP500 benchmark. Besides, the dynamically optimised TVTP portfolio provides significantly better returns compared its fixed allocation benchmark, while the FTP strategy does not.

4.2 Relevant Literature

Previous work on regime switching models mainly focuses on univariate models with fixed transition probabilities. The regime switching framework has been applied to study the economic variables such as exchange rates (Engel and Hamilton, 1990), output growth (Hamilton, 1989), interests (Gray, 1996; Ang and Bekaert, 1998), commodity indices (Fong and K.H., 2001), and stock returns (Rydn, Tersvirta and sbrink, 1998; Turner and C., 1989; Whitelaw, 2000)[1].

Extensions to multivariate fixed transition probability models include Timmermann and Guidolin (2006), Ang and Bekaert (2002), Perez-Quiros and Timmermann (2000), Hamilton and Lin (1996) and Chen (2009). Two previous papers studied the joint distribution of stock and bond returns with fixed transition probabilities. Timmermann and Guidolin (2006) explored the joint distribution of large capital stock, small capital stock and bond returns with fixed transition probabilities. Their paper identified four regimes as crash, slow growth, bull and recovery states. Chen (2009) studied the correlation and volatilities of the bond and stock markets in a regime switching bivariate GARCH model in which they found the "low-to-high"switching in stock volatility is more likely to be associated with the "high-to-low"switching in correlation, while the "low-to-high"switching in bond volatility is more likely to be associated with that the "low-to-high"switching in correlation.

A significant extension to the Markov Switching model was developed by Diebold, Lee and Weinbach (1994), who created a framework that allows the transition probabilities to depend on the economy predictors and thereby vary over time. There are two alternative methods in the study of Markov Switching models, the maximum likelihood method, which is initially developed by Diebold, Lee and Weinbach (1994) and the Bayesian inference method (Gibbs-Sampling), which is developed by Filardo and Gordon (1998). Further applications include Peria (2002), Calvet and Fisher (2004) and Sims and Zha (2006).

This paper is the first for examining the regime switching model with time varying transition probabilities in the context of joint distribution of bond and equity. The expected

[1]See Timmermann and Guidolin (2006) and Hamilton (2008) for a review

maximum likelihood method is used in parameter estimations[2]. Our model identifies three regimes in the financial market with the joint distribution of US stocks and bonds, the crisis, recovery and stable state. The results choose ISM (ISM Manufacturing Index)[3], unemployment rate and P/E ratio as predictors for the transition probabilities. Our model also assumes regime-dependent volatilities and correlations of stocks and bonds.

4.3 Univariate Model

In this section, we first consider the univariate Markov regime switching model of stock and bond returns, starting with the univariate FTP model. It is important to understand the univariate dynamics of each asset class before moving on to a thorough search of a bivariate TVTP model. The FTP model is the bench mark for the TVTP model while univariate model is the starting point to justify the bivariate model. The study in this section also helps in reducing the number of parameters to be estimated in the complicated bivariate TVTP model later on.

4.3.1 Data

Monthly returns from both the stock and bond markets indices are used to estimate the model. Stock market returns are computed as monthly returns on S&P500 index, as adjusted for any splits and dividends. The bond market returns are calculated using the 10-year T-bonds. The raw data for S&P500 and treasury bond are obtained from the Federal Reserve database in St. Louis (FRED). The treasury bond returns include coupon and price appreciation. The unemployment and ISM are all available on FRED. P/E ratio are available from CRSP database. The sample period is from February 1, 1962 to May 31, 2015, and include a total of 629 observations.

4.3.2 Methodology

We use EM (expectation-maximization) algorithm which was first proposed by Dempster, Laird and Rubin (1977) and extended by Hamilton (1988) for the estimation of parameters in the framework of Markov regime switching model. The EM algorithm is an iterative

[2]Alternative popular method in the estimation of regime switching model is Gibbs Sampling.

[3]The ISM Manufacturing Index is based on surveys of more than 300 manufacturing firms by the Institute of Supply Management. The ISM Manufacturing Index monitors employment, production, inventories, new orders and supplier deliveries. A composite diffusion index monitors conditions in national manufacturing and is based on the data from these surveys.

method for finding maximum likelihood or maximum a posteriori estimates of parameters in statistical models, where the model depends on unobserved latent variables. The EM iteration alternates between performing an expectation step, which creates a function for the expectation of the log-likelihood, evaluated using the current estimate for the parameters, and a maximization step, which computes parameters maximizing the expected log-likelihood found on the expectation step. These parameter-estimates are then used to determine the distribution of the latent variables in the next expectation step. To better understand the algorithm process, the following part gives an example of the algorithm with 4 states. This process can easily be extended to n states.

Let $\{s_t\}_{t=1}^{T}$ be the sample path of a first-order, four-state Markov process with transition probability matrix illustrated in the following table. As is apparent in the figure, the transition probabilities are time-varying, evolving as functions of $x'_{t-1}\beta_{ij},\ i,\ j = 1, ..., 4$, where the $(k \times 1)$ conditioning vector x_{t-1} contains economic variables that affect the state transition probabilities. We define $\beta = (\beta'_{11}, ..., \beta'_{ij}, ..., \beta'_{43})'$. In the case of fixed transition probabilities, x is a vector of 1s. The condition that $\sum_j P_t^{ij} = 1$ must also be satisfied.

	Regime 1	Regime 2	Regime 3	Regime 4
Regime 1	$P_t^{11} = f(x'_{t-1}\beta_{11})$	$P_t^{12} = f(x'_{t-1}\beta_{12})$	$P_t^{13} = f(x'_{t-1}\beta_{13})$	$P_t^{14} = f(x'_{t-1}\beta_{14})$
Regime 2	$P_t^{21} = f(x'_{t-1}\beta_{21})$	$P_t^{22} = f(x'_{t-1}\beta_{22})$	$P_t^{23} = f(x'_{t-1}\beta_{23})$	$P_t^{24} = f(x'_{t-1}\beta_{24})$
Regime 3	$P_t^{31} = f(x'_{t-1}\beta_{31})$	$P_t^{32} = f(x'_{t-1}\beta_{32})$	$P_t^{33} = f(x'_{t-1}\beta_{33})$	$P_t^{34} = f(x'_{t-1}\beta_{34})$
Regime 4	$P_t^{41} = f(x'_{t-1}\beta_{41})$	$P_t^{42} = f(x'_{t-1}\beta_{42})$	$P_t^{43} = f(x'_{t-1}\beta_{43})$	$P_t^{44} = f(x'_{t-1}\beta_{44})$

Let $\{y_t\}_{t=1}^{T}$ be the sample path of 2 time series $\{y_t\}_{t=1}^{T} = (y_{1t},\ y_{2t})'$, and $\{z_t\}_{t=1}^{T}$ be the sample path of m time series with the first dimension setted at constant 1. It is assumed that $y_t = A_i z_t + \epsilon_{it}$ and ϵ_{it} depends on $\{s_t\}_{t=1}^{T}$ as follows:

$$(\epsilon_t | s_t = i;\ \alpha_i) \overset{i.i.d.}{\sim} N(0,\ \Sigma_i)$$

and therefore,

$$(y_t | s_t = i;\ \alpha_i) \overset{i.i.d.}{\sim} N(A_i z_t,\ \Sigma_i)$$

where A_i is a $2 \times m$ matrix, $\Sigma_i = \begin{pmatrix} \sigma_{1i}^2 & c_i \\ c_i & \sigma_{2i}^2 \end{pmatrix}$, $\alpha_i = (A_{i11},\ ...,\ A_{i23},\ \sigma_{1i}^2,\ \sigma_{2i}^2,\ c_i)\ (9 \times 1)$, and define $\alpha = (\alpha_1,\ \alpha_2,\ \alpha_3\ \alpha_4)$.

The density of ϵ_t conditional upon s_t is:

$$f(y_t | s_t = i;\ \alpha_i) = (2\pi)^{-\frac{1}{2}} |\Sigma_i|^{-\frac{1}{2}} exp\left\{ -\frac{1}{2}(y_t - A_i z_t)' \Sigma_i^{-1} (y_t - A_i z_t) \right\} \tag{4.1}$$

for $i = 1, 2, 3, 4$.

Define $P(S_1 = 1) = \rho_1$, $P(S_1 = 2) = \rho_2$, $P(S_1 = 3) = \rho_3$ and $\rho = (\rho_1,\ \rho_2,\ \rho_3)$.

Define $\beta_{ij} = (\beta^1_{ij},\ \beta^2_{ij},\ \beta^3_{ij})'$, $i = 1, ..., 4$, $j = 1, ..., 3$ and $\beta = (\beta^1_{11}, ..., \beta^3_{43})$.

Define $\theta == \alpha', \beta', \rho'$ to be the vector of all model parameters.

The EM algorithm follows the steps below. Detailed process is given in Appendix A.

- 1. Pick $\theta^{(0)}$
- 2. Get $P(s_t = i|\underline{y_T},\ \underline{x_T};\ \theta^{(0)})$, for all t and $i = 1, 2, 3, 4$.
  Get $P(S_t = i,\ S_{t-1} = j|\underline{y_T},\ \underline{x_T};\ \theta^{(0)})$ for all t, $i, j = 1, ..., 4$.
  Construct $Elogf(\underline{y_T},\ \underline{s_T}|\underline{x_T};\ \theta^{(0)})$
- 3. Set $\theta^{(1)} = \underset{\theta}{argmax} E\left\{logf(\underline{y_T},\ \underline{s_T}|\underline{x_T};\ \theta^{(0)})\right\}$
- 4. Iterate to converge $||\theta^{(k)} - \theta^{(k-1)}||$.

4.3.3 Univariate FTP Model

Before the estimation of univariate TVTP model, we first consider the regimes in each asset class under fixed transition probability assumption. The aim is to examine the basic properties and the coherence in regimes of these two return series. Higher degree of coherence can reduce the number of regimes to be estimated in the bivariate model.

Let $\{s_{it}\}_{t=1}^T$ be the sample path of a first-order, k_i regimes Markov process S_{it} for the asset class $i = 1, 2$ with the transition probability $p_{i,s_{it}s_{it-1}} = P(S_{it} = s_{it}|S_{it-1} = s_{it-1})$. Each return series r_{it} is modeled as Markov switching process with the parameters determined by the asset-specific state variable $s_{i,t}$:

$$r_{it} = \mu_{is_{it}} + \sum_{j=1}^{p} \beta_{is_{it}j} r_{it-j} + \sigma_{is_{it}} \xi_{it}, \quad i = 1, 2, \quad \xi_{it} \sim IIN(0, 1), \tag{4.2}$$

For $k_i = 1$, the model is simply a standard linear model, while as k_i increases, it becomes easier for the model to fit the data dynamics but at the cost of over-fitting, since the number of parameters to be estimated increases fast. The model over-fitting problem becomes even more severe if the transition probabilities are time varying, as parameters determining each of the two states transition probabilities also need to be estimated. Besides, larger k_i also reduces the out-of-sample forecasting accuracy. Therefore, it is very important to start with the simple univariate FTP model in order to understand the basic features of each return series.

Before the estimation of regimes, we first filter each return series by an $AR(p)$ model, with p chosen as the best fit according to the Schwarz's Bayesian Information Criterion

(BIC). Results suggest the first-order autoregressive term in the bond returns[4]. With this result, we set $p = 1$ for bond returns and search extensively over k_i up to 3 to determine the optimal k_i for each asset.

To assist the economic interpretation, Figure 4.1, 4.2 and 4.3, 4.4 plots the smoothed probabilities, fitted to each single return series under 2 and 3 regime assumptions.

Our results suggest 2 regimes for stocks based on information criteria and it supports 3 states for bond return based on AIC. The intuition for the stock market is clear; the first regime of the stock comes with a low volatility and high return, representing bullish market. Under the second regime, the market delivers a low return but is accompanied with the highest volatility, representing a bear market. The 3-regime model identifies a similar bullish market to the 2-regime model and divides the second regime into another two regimes–one with relatively higher return at 0.6761 and lower volatility at 4.0702 and the other one with return and volatility at -0.2847 and 6.5538, respectively. From the figure, the third regime seems to capture the financial market crisis peak, however, as compared to the 3-regime model's smoothed probabilities with historical crisis period, the assumption is not supported by the results. Hence, this thesis sticks to the 2-regime model for the stock market.

The result selects a 3-regime model in the bond market without autoregressive terms. The entire data period are in general divided into 2 regions, before and after 1980s. The first regime mainly covers 1962 to 1979. During this period, the treasury bonds provided an average monthly return of -0.1521, mainly due to the interest hike before 80s. Bonds also had lower volatility during that period–the monthly volatility was 1.7288. On the contrary, the second regime mainly covers late 80s to now, with a relatively higher return at 0.3633 and a moderate volatility at 3.0942. The return during this period was positive due to the overall interests cuts. The third regime appears in all of the data history. It reflects the most volatile periods in bond market. As compared to the two-regime model, it is easily seen that the two regimes before and after 80s are classified into one. Therefore, the model identified by the 3-regime assumption is similar to the 2-regime model in economic sense.

4.3.4 Univariate Time Varying Transition Probability Model

This subsection further moves on to the univariate TVTP model. In the FTP model, the transition probability is fixed over time. However, in the real world, economic considerations suggest the desirability of allowing the transition probabilities in the financial market to

[4]Krolzig (1998) suggests the autoregressive order p in the regime switching model can be selected based on the single state model.

vary. Rather than assuming the transition probability is fixed overtime, in this section, the transition probabilities are modelled as functions of economic variables, as presented before:

$$P_{it}^{n,m} = f(x'_{i,t-1}\beta_{n,m})$$

where m and n are the corresponding states at time $t-1$ and t and $P_{it}^{n,m}$ is the transition probability from a state m at $t-1$ to a state n at t. As mentioned before, $\sum_n P_{it}^{n,m} = 1$ must be satisfied for every m. The transition matrix P_{it} is constructed in the following way:

$$P_{it} = Q_{it} \circ R_{it}$$

where $\circ$ represents the Hadamard elementwise matrix product and we define

$$Q_{it} = \begin{bmatrix} q_{it,11} & q_{it,12} & \cdots & q_{it,1k_i} \\ q_{it,21} & q_{it,22} & \cdots & q_{it,2k_i} \\ \vdots & \vdots & \ddots & \vdots \\ q_{it,k_i1} & q_{it,k_i1} & \cdots & q_{it,k_ik_i} \\ 1 & 1 & \cdots & 1 \end{bmatrix}, \tag{4.3}$$

$$R_{it} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1-q_{it,11} & 1-q_{it,12} & \cdots & 1-q_{it,1k_i} \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{l=1}^{k_i-2}(1-q_{it,l1}) & \prod_{l=1}^{k_i-2}(1-q_{it,l2}) & \cdots & \prod_{l=1}^{k_i-2}(1-q_{it,lk_i}) \\ \prod_{l=1}^{k_i-1}(1-q_{it,l1}) & \prod_{l=1}^{k_i-1}(1-q_{it,l1}) & \cdots & \prod_{l=1}^{k_i-1}(1-q_{it,l1}) \end{bmatrix}. \tag{4.4}$$

We assume the probit model for the conditional transition probability $q_{it,nm}$: $q_{it,nm} = \Phi(x_{it}\beta_{i,nm})$ with $\Phi(\cdot)$ as the cumulative normal density function, X_{it} as the vector of economic predictors and $b_{i,nm}$ as the parameters to be estimated for the predictors. The $q_{it,nm}$ with $s_{it} = n$ and $s_{it-1} = n$ is the probability of transferring from state m to n at t conditional on the economy at state m is not transferring to a state l where $l < n$. With the above definition of Q_{it} and R_{it}, the condition that $\sum_n P_{it}^{n,m}$ is automatically satisfied.

Similar to the FTP model, each return series r_{it} is still modelled as Markov switching process with the parameters determined by the asset-specific state variable $s_{i,t}$:

$$r_{it} = \mu_{is_{it}} + \sum_{j=1}^{p} \beta_{is_{it}j} r_{it-j} + \sigma_{is_{it}} \xi_{it}, \quad i = 1, 2, \quad \xi_{it} \sim IIN(0, 1), \tag{4.5}$$

One difficulty in modelling time varying transition probabilities is to choose the leading predictors. Unfortunately, few theoretical studies have been done on this topic. Therefore, both macroeconomics indicators as well as market indicators are tested in our model. We use ISM and Unemployment data as macroeconomics predictors and Price/Earnings ratio as the market predictors. This paper exhausts all the combinations of these three indicators in order to find which one contributes the most to the variations of transition probabilities in the financial market. Results are presented in table 4.2 and table 4.3.

The TVTP results support 2 regimes assumption for stocks with P/E ratio and ISM together as the leading indicators. For the bond market, none of the TVTP model is preferred to the FTP model based on AIC. However, we still present the results of two alternative models with the lowest AIC. The Univariate TVTP model results are given in Table 4.4, 4.5 and 4.6 with their smoothed probabilities in Figure 4.5, 4.6 and 4.7.

Starting with the stock market, the information criteria AIC prefers the TVTP model to the FTP model. The regimes identified by TVTP and FTP models are basically the same, except that under TVTP assumption, the properties of each regime are less diversified. The first regime is the good regime with higher returns and lower volatility, while the second regime is the bad one with lower returns and higher volatility[5]. Model indicates that the transition probability to stay in the good regime is insignificant (decreasing) in ISM and increasing in PE, while the transition probability to stay in a bad regime is decreasing in ISM and insignificant (increasing) in PE.

Comparing the state probabilities with historical bear market periods, it can be seen that the first regime in generally covers the most stable market. On the other hand, crisis period and the unstable short periods, before and after the crisis are included in the regime 2. Therefore, the results simply states that a strong increase in P/E predicts further bullish trends. Irrational boosting in P/E does not necessarily lead to the crisis in this model. On the contrary, within the bad regime, the ISM is a strong predictor for recovery. With an decreasing production, the economy can hardly pick up and return to the good regime.

All the TVTP models have higher AIC as compared to the FTP model. Among all the TVTP models for the bond market, the information criteria selects the two regime AR(1)

[5]TVTP Returns: 0.8006, 0.3984, with std 5.8617 and 27.2464. FTP returns: 0.9769 , 0.3679, 5.6136 and 24.7851.

model with ISM and P/E ratio together as predictors. We also present the results of three-regime AR(1) model with all the three indicators since the model AIC is only slightly higher than the two-regime models while the results have very different and strong economic sense.

Under the 2-regime assumption, the states captured by the TVTP model are similar to the FTP model. The parameters and smoothed probabilities are given in Table 4.5 and Figure 4.5. The first regime has a long term average return at 0.1712 and volatility at 1.6187 while the second regime has a return-volatility pair at 0.5684 and 3.6712[6]. The probability to stay in regime 1 is slightly increasing in both ISM and PE, while the transition probability from regime 2 to 1 is increasing in ISM and slightly increasing in PE.

Under the three regime assumption, the AR(1) model with all three indicators as predictors achieves the best AIC. Even though its AIC is slightly worse than the FTP model and the 2 regimes TVTP model, however, the regimes captured by the 3 regimes TVTP model provide more clear economic intuition. The first regime has an average monthly return of -1.9579%, with a low volatility 1.2791. The second regime has the highest level of return (5.3517%) and moderate volatility (2.0608). The third regime has a moderate monthly return (5.1615%) but has the highest volatility 3.4464. The first regime covers most periods in history during which the interest is stable and slowly increasing. The second period instead, covers most of the post recession recovery periods during which the interests rates are relatively low and decreasing, leading to higher returns. The bond volatility is relatively low due to the stable monetary policy in recovery. The third period covers the most volatile bond market periods. The returns are lower as compared to the recovery periods but higher compared to the stable returns. The entire 1980s with high interests rate and the subprime mortgage crisis fall into this period.

The model indicates that under the stable regime, the probability to stay in regime 1 is increasing in ISM, decreasing in UEM and increasing in PE. Conditional on not staying in regime 1, the transition probability from regime 1 to 2 is decreasing in ISM, increasing in UEM and increasing in PE. Compare to ISM and PE, UEM is the most important factor in predicting the switch between the stable and recovery state. The economy will return to the recovery state if the employment get worse.

On the other hand, the transition probability from regime 2 to 1 increases in ISM, decreases in UEM and decreases in PE. Furthermore, conditional on non-transfer to the stable state, the transition probability to stay in regime 2 (recovery) is increasing in ISM, increasing in UEM and decreasing in PE. Therefore, UEM is still important for determining whether the economy will leave the recovery state.

[6]Longrunmean=Shortrunmean/(1-β), Longrunvariance=Shortrunvariance/(1-β^2)

Finally, the transition probability from regime 3 to regime 1 is increasing in ISM, decreasing in UEM and decreasing in PE. Conditional on not moving to regime 1, the transition probability to to regime 2 is deceasing in ISM, increasing in UEM and increasing in PE. For the volatile period to end with stable state, the stock market need to be less valued, while both the production and unemployment are getting improved. Otherwise, the economy will move to a recovery state.

4.4 Bivariate FTP Model

In this section, we further consider the joint distribution of stock and bond returns. We start our analysis with a joint fixed transition probability model and focus on the changes in estimations from the univariate model. The model to be analysed in this section is

$$\mathbf{r}_t = \boldsymbol{\mu}_{s_t} + \sum_{j=1}^{p} \boldsymbol{\beta}_{s,j} \mathbf{r}_{t-j} + \boldsymbol{\varepsilon_t}, \quad \boldsymbol{\varepsilon_t} \sim N(\mathbf{0}, \Omega_{s_t}), \tag{4.6}$$

where $\mathbf{r}_t = (r_{1t},\ r_{2t})'$, $\boldsymbol{\mu}_{s_t} = (\mu_{1,s_t},\ \mu_{2,s_t})'$ are the corresponding vectors of stock and bond returns and mean returns in state s_t. $\boldsymbol{\beta}_{s,j}$ is the 2×2 autoregressive coefficients matrix of lag j in state s_t. Ω_{s_t} is the state-dependent covariance matrix. The transition probability matrix is the same as before with $p_{i,s_{it}s_{it-1}} = P(S_{it} = s_{it} | S_{it-1} = s_{it-1})$

Model results suggest a 3-regime model with AR(0) and AR(1) processes for the stocks and bonds [7]. Parameter estimations and smoothed probabilities are given in Table 4.7, 4.8 and Figure 4.8.

The first regime is the stable state, where the stock market provides the highest returns among all the three regimes and the bond market has the lowest returns. Both the markets are stable with the lowest volatilities. The correlation between the equity and bond is also the smallest.

The second regime comes with higher market volatilities and highest correlations in equity and bond. The stock market has a relatively high return and the bond market has the lowest return. By looking at the smoothed probabilities, the second regime covers short early stage of crisis and the period after crisis. The high correlation between two markets reflects that both markets are simultaneously moved by the change in the monetary policy.

The third regime captured by the model is basically a crisis period. Under the third regime, both markets are very volatile but the correlation between two markets is very low. The bond market has its highest returns while the stock market returns are at the lowest

[7] Previous results such as Timmermann and Guidolin (2006) and **?**) also support AR(0) process for large cap stocks.

level, reflecting the flight to safety effect. Figure X shaded the most serious financial crisis period in the history. The smoothed probabilities of the third regime indicates that the model captures all the crisis periods. It also captured the 1987 Black Monday and the European debt crisis of 2010. Under the first and third regimes, previous bond returns have positive effect on the current return, while under the second regime, the effect is negative.

Furthermore, the transition probability matrix indicates that the stable state is the most stable one, while the second regime, a transition regime is the least stable regime.

4.5 Bivariate TVTP Model

This section explores a bivariate TVTP model. The transition probability structure is the same as in section 4.2.4, while the Markov switching process is replaced by section 4.3.

Information criteria supports 3 regimes AR0-AR1 model with all the three predictors. Three regimes all have very clear economic identities. The first regime is still the stable regime with the lowest volatilities with positive returns in both the markets. Two markets are positively correlated under this regime. The second regime is the crisis regime. Returns in the stock market are at its lowest level while the bond market achieves the highest returns. Under the crisis regime, the stock market has the highest volatility while the bond volatility is higher than the stable regime. The correlations between two markets becomes clearly negative due to the flight to safety effect. This result is different from the FTP model, where the correlation is a small and positive number. Finally, the third regime is a recovery regime under which stocks achieve the best returns, even though the volatility is still very high. Bond markets become most volatile during the recovery regime and it moves in the same direction as the stock market.

The regimes captured by the TVTP model are similar to that under FTP model but there are still several differences between the two. The major difference comes in identifying the 80s. The FTP models take the 80s into the crisis regime, while the TVTP model does not. Other crisis periods identified by the TVTP model are more accurate than the FTP model. Besides, under FTP model, the crisis peak tends to be lagged of the real crisis period before 1980s due to the weakness in differentiating the recovery and crisis.

From the average transition probability matrix, it can be seen that the first regime is still the most stable state, while the third regime–recovery regime is the least stable one. Furthermore, the chance of fall back to the crisis state once the economy is in recovery is almost 0 (0.0000). To better understand the effects of each indicators in predicting the changes in transition probabilities, the standard deviation analysis is presented, in which, we change the value of the predictors by one standard deviation under each state to see their effects on the transition matrix. The results are provided in Table 4.12.

Higher ISM increases the chance to move into a crisis from the stable state and the chance to stay in the crisis state. But when the economy is in the recovery state, the chance of turning into the stable state increases as ISM gets better.

On the other hand, as unemployment increases, the financial market is more likely to move out of the crisis or the stable state. The second result is intuitive and straightforward, while the first result is because the real economy cycle is lagged of the financial market. When the unemployment gets worse, the financial market is already out of the bottom. Furthermore, the recovery state tends to be longer when the unemployment is worse.

Finally, the P/E ratio effects are similar to the ISM. A higher P/E makes the crisis periods more likely when the market is in the stable or crisis regime, but makes the stable state more likely when the market is in recovery regime.

4.6 Out of Sample Portfolio Performance

In this section, the out-of sample regime-dependent portfolio performance is examined using filtered probabilities derived from the FTP model and the TVTP model. The forecasting period is from Jan, 2007 to May, 2014, with 87 monthly data points. Only 7 year period is choosen since the number of parameters to be estimated is large. Even though the out-of-sample period is short, it covers an entire business cycle.

The portfolio strategy in this section is an expected mean-variance efficient portfolio. It provides the maximum expected returns given the choice of risk in variance (in expectation terms). It is assumed that the portfolio is rebalanced monthly.

The expected variance of the portfolio O_{t+1} conditional on information set $\mathcal{F}_t$ available at t is

$$E[Var(O_{t+1})|\mathcal{F}_t] = \sum_{k=1}^{3} P(S_{t+1} = k|\mathcal{F}_t) \cdot Var(O_{t+1}|S_{t+1} = k,\ \mathcal{F}_t)$$

where $Var(O_{t+1}|S_{t+1} = k,\ \mathcal{F}_t)$ is the variance of the portfolio return conditional on the regime being k, that is

$$\begin{aligned} &Var(O_{t+1}|S_{t+1} = k,\ \mathcal{F}_t) \\ &\quad = w_{E,t}^2 Var(R_{Equity,t+1}|S_{t+1} = k,\ \mathcal{F}_t) + w_{B,t}^2 Var(R_{Bond,t+1}|S_{t+1} = k,\ \mathcal{F}_t) \\ &\qquad + 2w_{E,t}w_{B,t}Cov(R_{Equity,t+1}, R_{Bond,t+1}|S_{t+1} = k,\ \mathcal{F}_t). \end{aligned}$$

$w_{E,t}$ and $w_{B,t} = 1 - w_{E,t}$ are the corresponding weight allocated to equity and bond at period t. Furthermore, the variance and covariance of stocks and bonds are presented as below:

$$Var(R_{Stock,t+1}|S_{t+1} = k,\ \mathcal{F}_t) = Var(R_{Stock,t}|S_t = k,\ \mathcal{F}_t),$$

$$Var(R_{Bond,t+1}|S_{t+1} = k,\ \mathcal{F}_t) = Var(R_{Bond,t}|S_t = k,\ \mathcal{F}_t),$$

$$Cov(R_{Stock,t+1}, R_{Bond,t+1}|S_{t+1} = k,\ \mathcal{F}_t) = Cov(R_{Stock,t}, R_{Bond,t}|S_t = k,\ \mathcal{F}_t).$$

By choosing the expected variance level (in the experiments, we assume the target variance is $Var = 10, 20$), one can solve the quadratic equation below and derive two sets of candidate weights: $(w^1_{E,t},\ w^1_{B,t})$ and $(w^2_{E,t},\ w^2_{B,t})$,

$$E[Var(O_{t+1})|\mathcal{F}_t] = Var.$$

Using the paired weights above, we calculate the expected return of the portfolio by

$$\begin{aligned} E[O_{t+1}|\mathcal{F}_t] &= \sum_{k=1}^{3} P(S_{t+1} = k|\mathcal{F}_t) \cdot E(O_{t+1}|S_{t+1} = k,\ \mathcal{F}_t) \\ &= \sum_{k=1}^{3} P(S_{t+1} = k|\mathcal{F}_t) \cdot \{w_{E,t} \cdot E(R_{Equity,t+1}|S_{t+1} = k,\ \mathcal{F}_t) \\ &\qquad + w_{B,t} \cdot E(R_{Bond,t+1}|S_{t+1} = k,\ \mathcal{F}_t)\} \end{aligned}$$

where $E(R_{Equity,t+1}|S_{t+1} = k,\ \mathcal{F}_t) = E(R_{Equity,t}|S_t = k,\ \mathcal{F}_t)$, and $E(R_{Bond,t+1}|S_{t+1} = k,\ \mathcal{F}_t) = E(R_{Bond,t}|S_t = k,\ \mathcal{F}_t)$. The optimised allocations $w^o_{E,t}$ and $w^o_{B,t} = 1 - w^o_{E,t}$ simply takes the set of weights that generates higher expected portfolio return.

Notice that the probabilities used in the portfolio constructions are out-of-sample filtered probabilities, instead of smoothed probabilities (as derived in the previous sections). The application enables us to properly compare the forecasting ability of the FTP model and TVTP model. Figure 4.10 presents the the filtered probabilities under FTP model (3-regime AR0AR1 model) and the TVTP model (3-regime AR0AR1 model with ISM, UEM, PE as leading predictors). The comparison between smoothed and filtered probabilities show that the filtered probabilities are similar but contain more uncertainties when the underlying state changes.

Given the optimal allocation $w^o_{E,t}$, $w^o_{B,t}$ and the realised returns of equity and bonds at $t + 1$, we calculate the realised out-of-sample returns of the regime-dependent portfolio strategy and compare its performance with our predetermined benchmarks. We select

3 portfolios as the corresponding benchmarks. The benchmark for the TVTP model is a portfolio with a constant weight taking the value of the average weights of the corresponding TVTP strategy. Similar benchmark is constructed for the FTP model. We also compare both dynamic portfolio strategies to the performance of SPY. Figure 4.11 and 4.12 presents the performance of two dynamic strategies overtime as compared to SPY under different risk tolerance and with leverage or not. Figure 4.13 gives the performance comparison of two dynamic strategies with respect to their fixed allocation benchmarks.

Results show 1. both the FTP and the TVTP regime-dependent strategies outperforms the SPY benchmark. 2. TVTP strategy has better performance as compared to the FTP strategy. 3. TVTP strategy provides significantly better returns compared to its fixed allocation benchmark, while FTP strategy does not. Summary of the portfolio performance is presented in table 4.13.

Overall, the TVTP model outperforms the FTP model, as well as similar fixed allocation benchmarks based on the out-of-sample results.

4.7 Conclusion and Further Discussions

This paper considers a variety of regime switching models with time varying transition probabilities for the joint distribution of stocks and bonds returns. The paper results support a two-regime univariate model for stocks with ISM and P/E ratio as the leading predictors for the transition probabilities and support the fixed transition probability model for univariate distribution of bond returns.

Under joint distribution assumption, the result selects a three regime model with ISM, unemployment rate and P/E ratio, together as predictors for the time varying transition probabilities. Even though both fixed and time varying transition probability models identify three regimes in the financial market, however, the time varying transition probability model provides better out of sample predictions on the basis of the regime-dependent portfolio performance.

One critique on portfolio construction based on the regime switching model is the timing issue. A classic Markov model usually fails to pick up the changes in the regime ahead of time and thus can hardly provide early reallocation signals. The time varying transition probability frame-work instead, by using leading indicators, are better at predicting a regime earlier by using leading indicators, leads to better out-of-sample performance in the portfolio.

Another popular critique on Markov regime switching model is the model usually fails to handle the regime breakdowns. One example is the 1980s, where the interest rates' behaviour were non-repeatable in the history observed. Both classic and our models fails to classify this period correctly. Based on my knowledge, this difficulty can hardly be addressed by the current regime switching frame work and needs innovative econometric models, which are beyond the scope of this paper.

4.8 Mathematical Appendix A

EM algorithm in general

- 1. Pick $\theta^{(0)}$
- 2. Get $P(s_t = i|\underline{y_T},\ \underline{x_T};\ \theta^{(0)})$, for all t and $i = 1, 2, 3, 4$. $(T \times 4) numbers.$
  Get $P(S_t = i,\ S_{t-1} = j|\underline{y_T},\ \underline{x_T};\ \theta^{(0)})$ for all t, $i, j = 1, ..., 4$, $(T) numbers.$
  Construct $Elogf(\underline{y_T},\ \underline{s_T}|\underline{x_T};\ \theta^{(0)})$
- 3. Set $\theta^{(1)} = \underset{\theta}{argmax} E\left\{logf(\underline{y_T},\ \underline{s_T}|\underline{x_T};\ \theta^{(0)})\right\}$
- 4. Iterate to converge $||\theta^{(k)} - \theta^{(k-1)}||$.

EM algorithm in details

Step 2: The Expectation Steps Need to get the expected complete-data log likelihood below:

$$
\begin{aligned}
&Elogf(\underline{y_T},\ \underline{s_T}|\underline{x_T};\ \theta^{(k-1)}) \\
&\quad = \sum_{j=1}^{4} P(S_1 = j|\underline{y_T},\ \underline{x_T};\ \theta_j^{(k-1)}) \cdot \left[logf(y_1|S_1 = j;\ \alpha_i^{(k-1)}) + log\rho_j^{(k-1)}\right] \\
&\qquad + \sum_{t=2}^{T} \left\{ \sum_{i=1}^{4} P(S_t = j|\underline{y_T},\ \underline{x_T};\ \theta_j^{(k-1)}) \cdot logf(y_t|S_t = j;\ \alpha_j^{(k-1)}) \right\} \\
&\qquad\qquad + \sum_{t=2}^{T} \left\{ \sum_{j=1}^{4}\sum_{i=1}^{4} P(S_t = j,\ S_{t-1} = i|\underline{y_T},\ \underline{x_T};\ \theta^{(k-1)}) \cdot logP_t^{ij} \right\}
\end{aligned} \tag{4.7}
$$

The following steps describe how to calculated the probabilities in this equation.

- $f(y_t|s_t = i;\ \alpha_i)$

  Given $\theta^{(k-1)}$, then $\mu_i^{(k-1)}$, $\Sigma_i^{(k-1)}$ are known

  $$f(y_t|s_t = i;\ \alpha_i^{(k-1)}) = (2\pi)^{-\frac{1}{2}}|\Sigma_i^{(k-1)}|^{-\frac{1}{2}} exp\left\{-\frac{1}{2}(y_t - A_i^{(k-1)} \cdot z_t)' \left(\Sigma_i^{(k-1)}\right)^{-1} (y_t - A_i^{(k-1)} \cdot z_t)\right\} \tag{4.8}$$

  for $t = 1, ..., T$, $i = 1, ..., 4$. $(T \times 4)$ numbers.

- $P(S_t = j,\ S_{t-1} = i|\underline{y_T},\ \underline{x_T};\ \theta^{(k-1)})$

Step 1: Calculate the transition probabilities given by Figure 1. $(T-1\times 16)$ matrix.

$$P(S_t=j|S_{t-1}=i,\ x_{t-1};\beta_{ij}^{(k-1)}) = \Phi\left(x_{t-1}\beta_{ij}^{(k-1)\mathbf{T}}\right) \tag{4.9}$$

Step 2: Iterations on $t=2,...,T$ to get $P(S_t=j,\ S_{t-1}=i|\underline{y_t},\ \underline{x_t};\ \theta^{(k-1)})$.

- 2a. $f(y_t,\ S_t,\ S_{t-1}|\underline{y_{t-1}},\ \underline{x_{t-1}};\ \theta^{(k-1)}),\ t=2,...,T$.
  For $t=2$,

$$f(y_2,\ S_2,\ S_1|\underline{y}_1,\ \underline{x}_1;\ \theta^{(k-1)}) = f(y_2|S_2=j;\ \alpha_j^{(k-1)})\cdot P(S_2=j|S_1=i,\ x_{t-1};\beta_{ij}^{(k-1)})\cdot\rho_i^{(k-1)} \tag{4.10}$$

  For $t=3,...,T$,

$$f(y_t,\ S_t,\ S_{t-1}|\underline{y_{t-1}},\ \underline{x_{t-1}};\ \theta^{(k-1)}) = \sum_{l=1}^{4} f(y_t|S_t=j;\ \alpha_j^{(k-1)}) \cdot P(S_t=j|S_{t-1}=i,\ x_{t-1};\beta_{ij}^{(k-1)})\cdot P(S_{t-1}=i,\ S_{t-2}=l|\underline{y_{t-1}},\ \underline{x_{t-1}};\ \theta^{(k-1)}) \tag{4.11}$$

- 2b. $f(y_t|\underline{y_{t-1}},\ \underline{x_{t-1}};\ \theta^{(k-1)}) = \sum_{j=1}^{4}\sum_{i=1}^{4} f(y_t,\ S_t=i,\ S_{t-1}=j|\underline{y_{t-1}},\ \underline{x_{t-1}};\ \theta^{(k-1)})$
- 2c. $P(S_t=i,\ S_{t-1}=j|\underline{y_t},\ \underline{x_t};\ \theta^{(k-1)}) = \dfrac{f(y_t,\ S_t,\ S_{t-1}|\underline{y_{t-1}},\ \underline{x_{t-1}};\ \theta^{(k-1)})}{f(y_t|\underline{y_{t-1}},\ \underline{x_{t-1}};\ \theta^{(k-1)})}$

Step 3: Smoothed Probabilities: $P(S_t=i,\ S_{t-1}=j|\underline{y_T},\ \underline{x_T};\ \theta^{(k-1)})$.

- 3a. Iterations on $t=2,...,T$ and $\tau=t+2,...,T$ to calculate
  $P(S_\tau,\ S_{\tau-1},\ S_t,\ S_{t-1}|\underline{y_\tau},\ \underline{x_\tau};\ \theta^{(k-1)})$.

$$P(S_\tau=m,\ S_{\tau-1}=n,\ S_t=i,\ S_{t-1}=j|\underline{y_\tau},\ \underline{x_\tau};\ \theta^{(k-1)}) = \frac{\sum_{S_{\tau-2}=1}^{4} f(y_\tau|S_\tau;\ \alpha_m^{(k-1)})\cdot P(S_\tau|S_{\tau-1},\ x_{\tau-1};\beta_{ij}^{(k-1)})}{f(y_\tau|\underline{y_{\tau-1}},\ \underline{x_{\tau-1}};\ \theta^{(k-1)})} \cdot P(S_{\tau-1},\ S_{\tau-2},\ S_t,\ S_{t-1}|\underline{y_{\tau-1}},\ \underline{x_{\tau-1}};\ \theta^{(k-1)}) \tag{4.12}$$

when $\tau = t+2$, the last term in the numerator is initialised by

$$P(S_{t+1} = n,\ S_t = i,\ S_{t-1} = j|\underline{y_{t+1}},\ \underline{x_{t+1}};\ \theta^{(k-1)}) = \frac{f(y_{t+1}|S_{t+1} = n;\ \alpha_n^{(k-1)}) \cdot P(S_{t+1} = n|S_t = i,\ x_t; \beta_{ij}^{(k-1)})}{f(y_{t+1}|\underline{y_t},\ \underline{x_t};\ \theta^{(k-1)})} \cdot P(S_t = i,\ S_{t-1} = j|\underline{y_t},\ \underline{x_t};\ \theta^{(k-1)}) \quad (4.13)$$

For each τ, we produce a (16×1) vector of probabilities corresponding to the four possible cases of $(S_\tau,\ S_{\tau-1})$, Upon reaching $\tau = T$, a $(T-3 \times 16)$ matrix is produced.

- 3b. Upon reaching $\tau = T$,
  $P(S_t,\ S_{t-1}|\underline{y_T},\ \underline{x_T};\ \theta^{(k-1)}) = \sum_{S_T=1}^{4} \sum_{S_{T-1}=1}^{4} P(S_T,\ S_{T-1},\ S_t,\ S_{t-1}|\underline{y_T},\ \underline{x_T};\ \theta^{(k-1)})$
- 3c. Steps 3a and 3b are repeated for all possible time t valuations of $(S_t,\ S_{t-1})$, until a smoothed probability has been calculated for each of the four possible valuations. At this point, we have a (1×16) vector of smoothed joint state probabilities for $(S_t,\ S_{t-1})$
- 3d. Steps 3a-3c are repeated for $t = 3, ..., T$, yielding a total of $(T-1 \times 16)$ smoothed joint state probabilities.

- Smoothed Marginal State Probabilities $P(S_t|\underline{y_T},\ \underline{x_T};\ \theta^{(k-1)})$

$$P(S_t = i|\underline{y_T},\ \underline{x_T};\ \theta^{(k-1)}) = \sum_{j=1}^{4} P(S_t = i,\ S_{t-1} = j|\underline{y_T},\ \underline{x_T};\ \theta^{(k-1)}), \qquad i = 1, ..., 4 \quad (4.14)$$

- Solutions of model parameters are given by maximising the log likelihood functions.

Univariate Fixed Transition Probability Model–Two Regimes

The figures plot the smoothed probabilities of regime 1 estimated from the univariate Markov switching model

$$r_{it} = \mu_{is_{it}} + \sum_{j=1}^{p} \beta_{is_{it}j} r_{it-j} + \sigma_{is_{it}} \xi_{it}, \quad i = 1, 2, \quad \xi_{it} \sim IIN(0, 1),$$

with fixed transition probabilities $p_{i,s_{it}s_{it-1}} = P(S_{it} = s_{it} | S_{it-1} = s_{it-1})$.

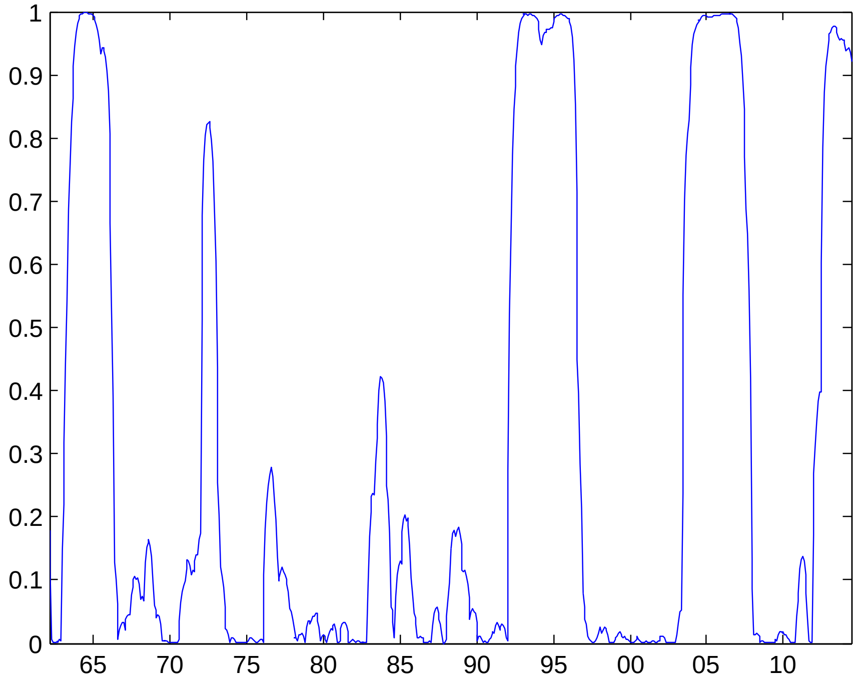


Figure 4.1: UFTP: SMOOTHED PROBABILITIES OF STOCKS

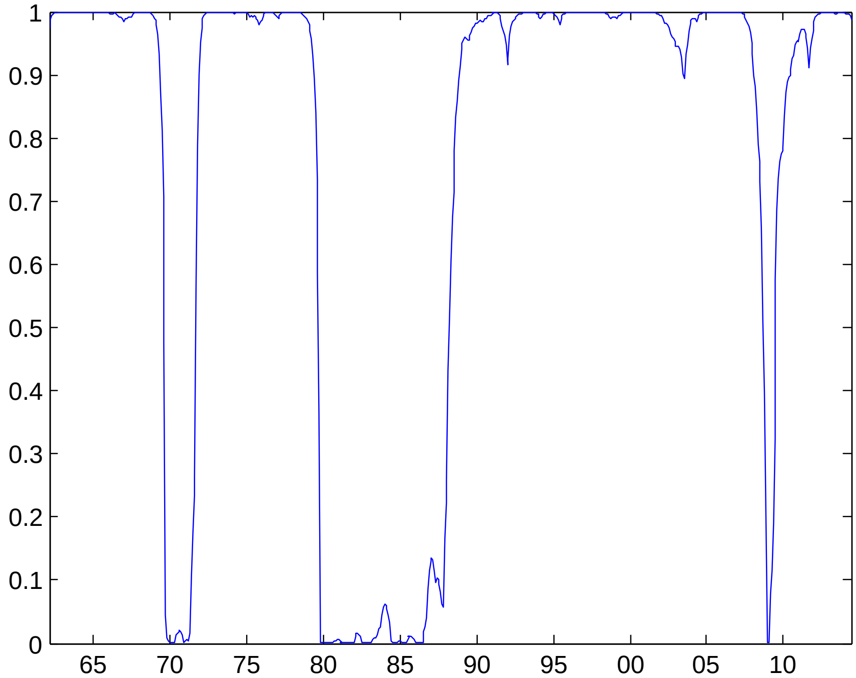


Figure 4.2: UFTP: SMOOTHED PROBABILITIES OF BONDS

Univariate Fixed Transition Probability Model–Three Regimes

The figures plot the smoothed probabilities of regime 1 estimated from the univariate Markov switching model

$$r_{it} = \mu_{is_{it}} + \sum_{j=1}^{p} \beta_{is_{it}j} r_{it-j} + \sigma_{is_{it}} \xi_{it}, \quad i = 1, 2, \quad \xi_{it} \sim IIN(0, 1),$$

with fixed transition probabilities $p_{i,s_{it}s_{it-1}} = P(S_{it} = s_{it} | S_{it-1} = s_{it-1})$.

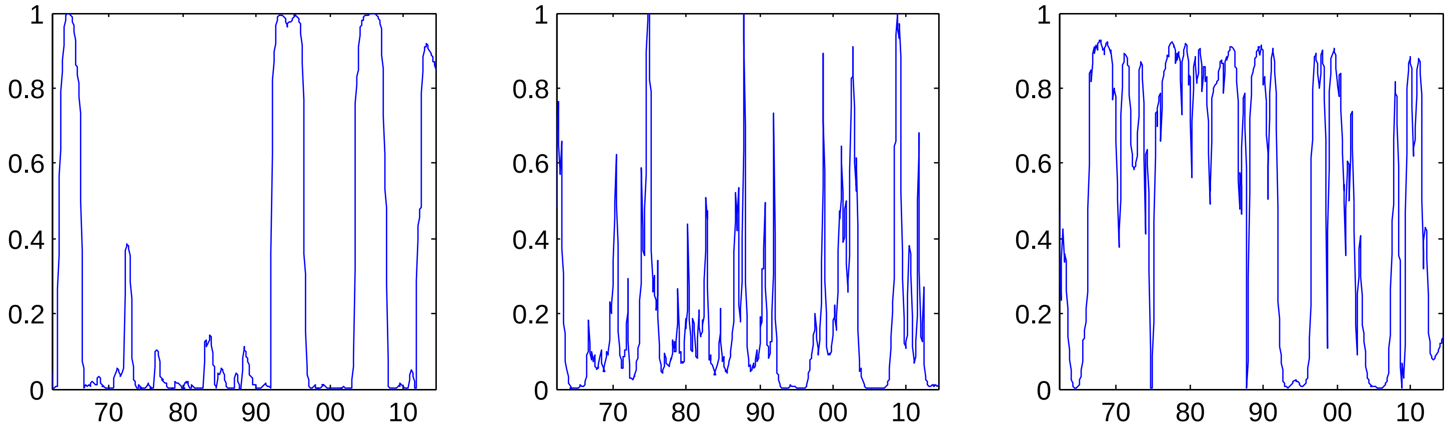


Figure 4.3: UFTP: SMOOTHED PROBABILITIES OF STOCKS

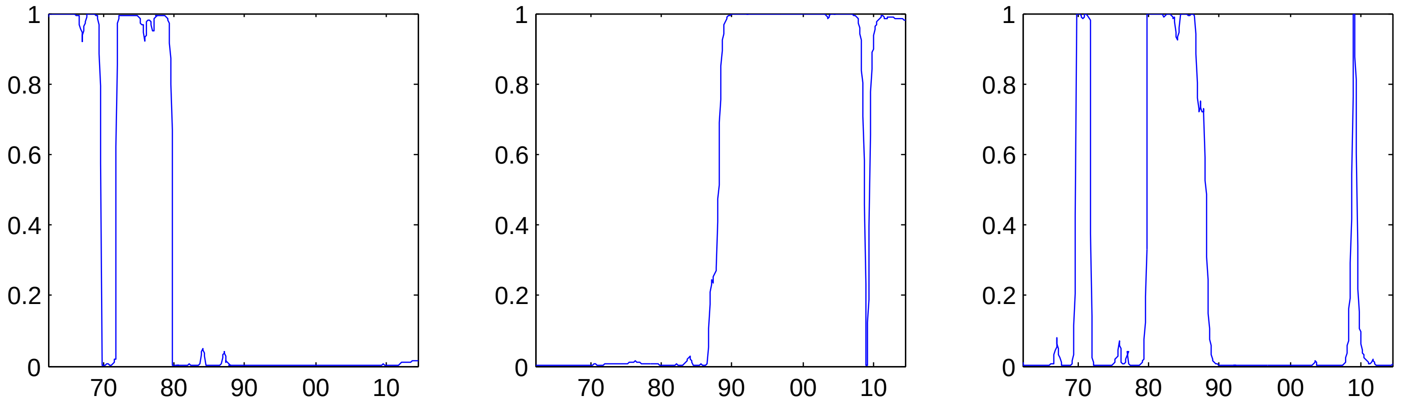


Figure 4.4: UFTP: SMOOTHED PROBABILITIES OF BONDS

Univariate Time Varying Transition Probability Model of Stocks

The figures plot the smoothed probabilities estimated from the univariate Markov switching model

$$r_{it} = \mu_{is_{it}} + \sum_{j=1}^{p} \beta_{is_{it}j} r_{it-j} + \sigma_{is_{it}} \xi_{it}, \quad i = 1, 2, \quad \xi_{it} \sim IIN(0, 1),$$

with fixed transition probabilities $p_{i,s_{it}s_{it-1}} = P(S_{it} = s_{it} | S_{it-1} = s_{it-1})$.

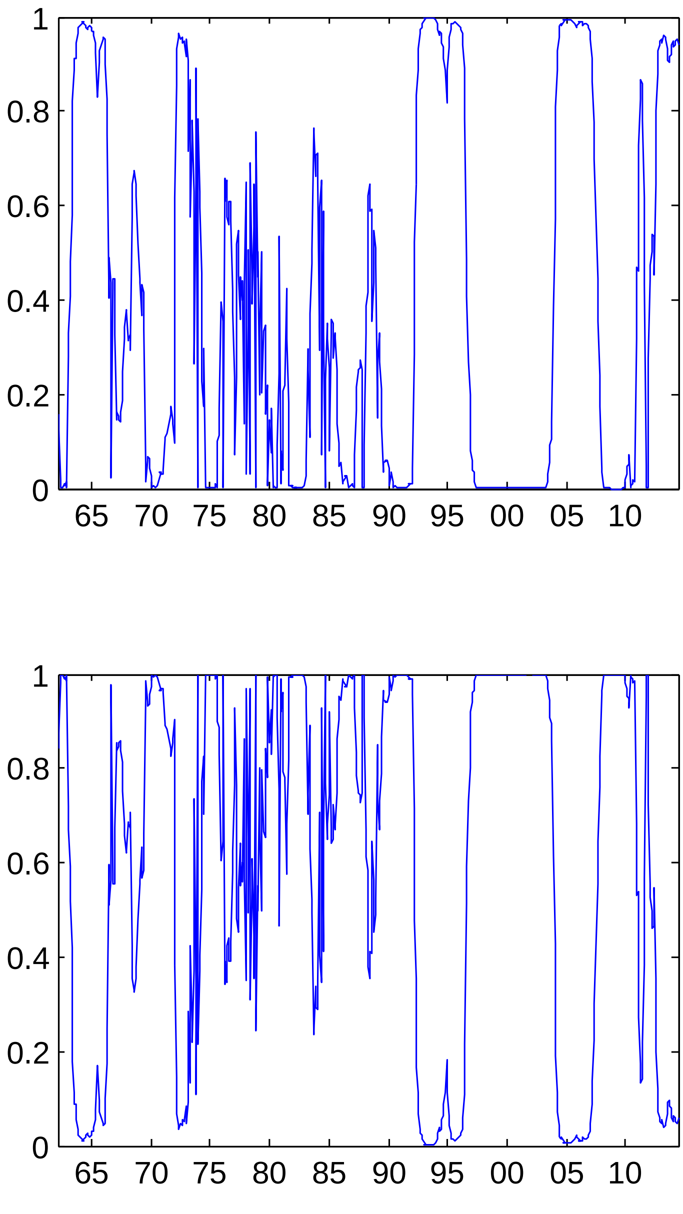


Figure 4.5: UTVTP: SMOOTHED PROBABILITIES OF STOCKS

Univariate Time Varying Transition Probability Model of Bonds

The figures plot the smoothed probabilities estimated from the univariate Markov switching model

$$r_{it} = \mu_{is_{it}} + \sum_{j=1}^{p} \beta_{is_{it}j} r_{it-j} + \sigma_{is_{it}} \xi_{it}, \quad i = 1, 2, \quad \xi_{it} \sim IIN(0, 1),$$

with fixed transition probabilities $p_{i,s_{it}s_{it-1}} = P(S_{it} = s_{it} | S_{it-1} = s_{it-1})$.

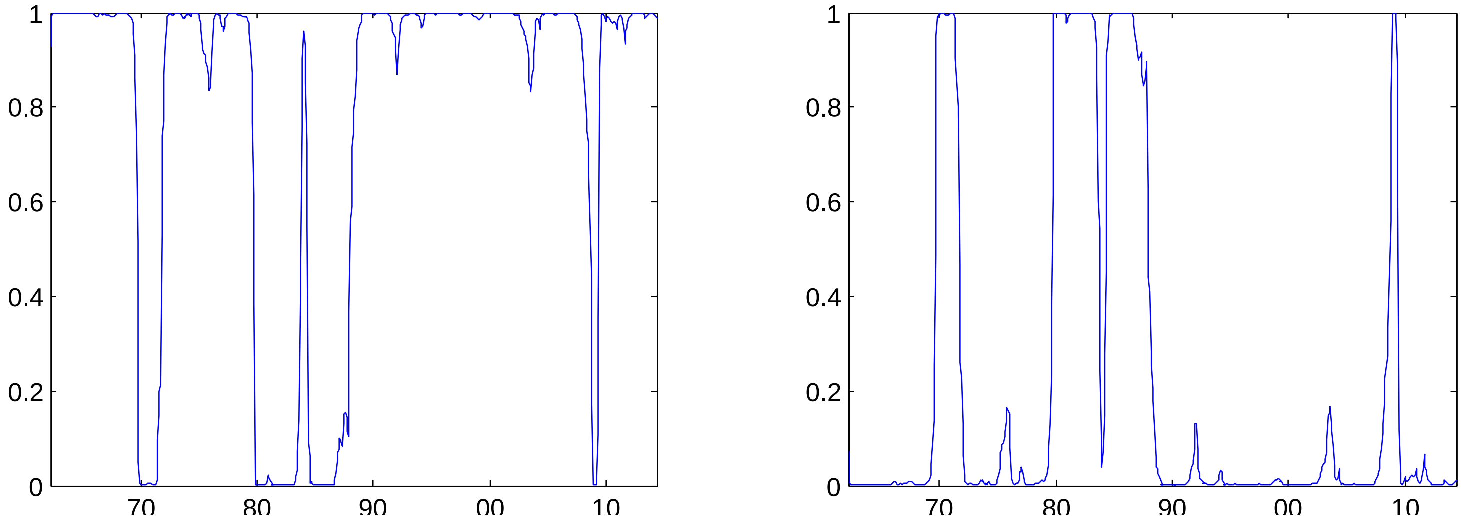


Figure 4.6: UTVTP: SMOOTHED PROBABILITIES OF BONDS

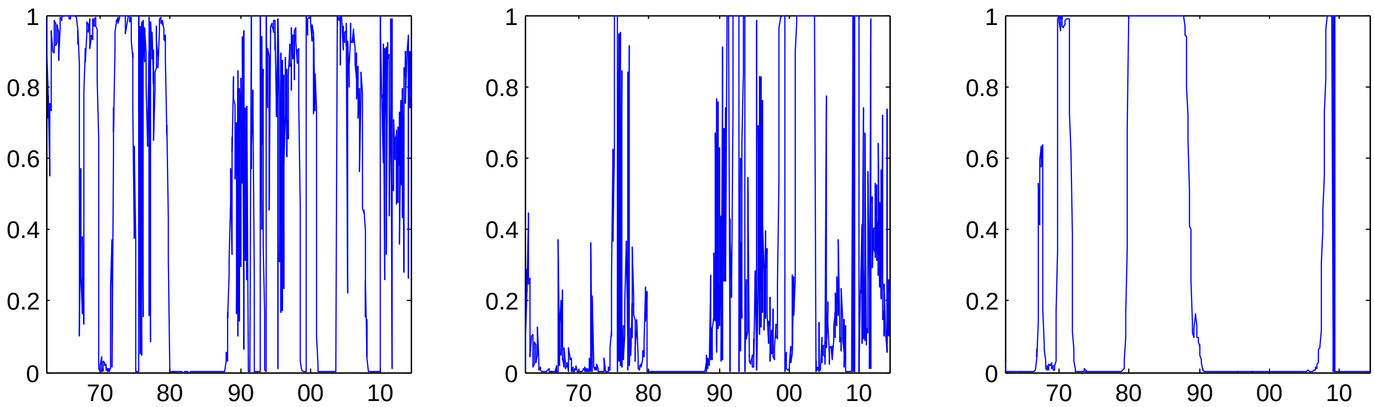


Figure 4.7: UTVTP: SMOOTHED PROBABILITIES OF BONDS

Bivariate Fixed transition Probability Model

The figures plot the smoothed probabilities estimated from the univariate Markov switching model

$$r_{it} = \mu_{is_{it}} + \sum_{j=1}^{p} \beta_{is_{it}j} r_{it-j} + \sigma_{is_{it}} \xi_{it}, \quad i = 1, 2, \quad \xi_{it} \sim IIN(0, 1),$$

with fixed transition probabilities $p_{i,s_{it}s_{it-1}} = P(S_{it} = s_{it} | S_{it-1} = s_{it-1})$.

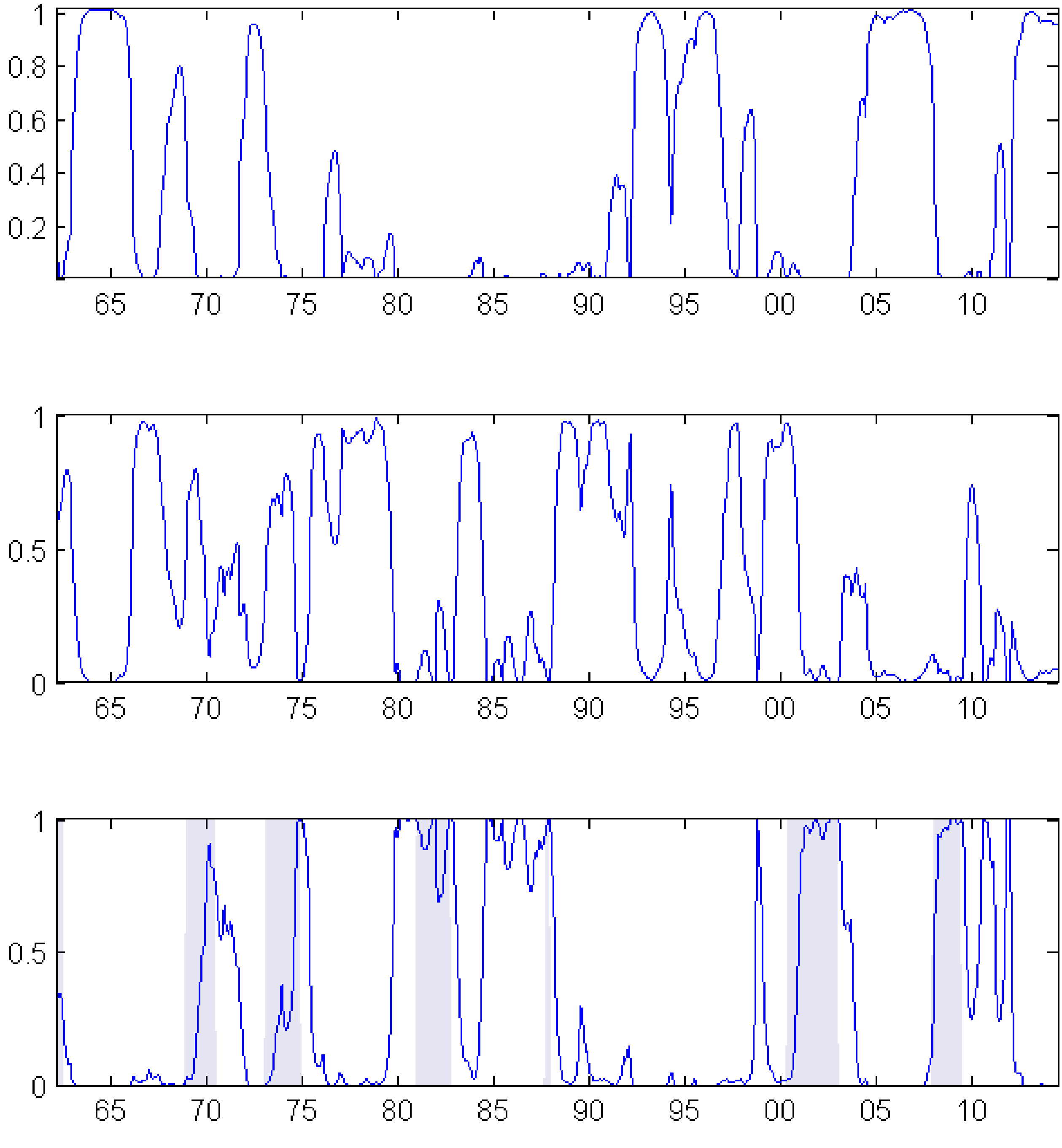


Figure 4.8: BFTP: SMOOTHED PROBABILITIES

Bivariate Time Varying Transition Probability Model

The figures plot the smoothed probabilities estimated from the univariate Markov switching model

$$r_{it} = \mu_{is_{it}} + \sum_{j=1}^{p} \beta_{is_{it}j} r_{it-j} + \sigma_{is_{it}} \xi_{it}, \quad i = 1, 2, \quad \xi_{it} \sim IIN(0, 1),$$

with fixed transition probabilities $p_{i,s_{it}s_{it-1}} = P(S_{it} = s_{it} | S_{it-1} = s_{it-1})$.

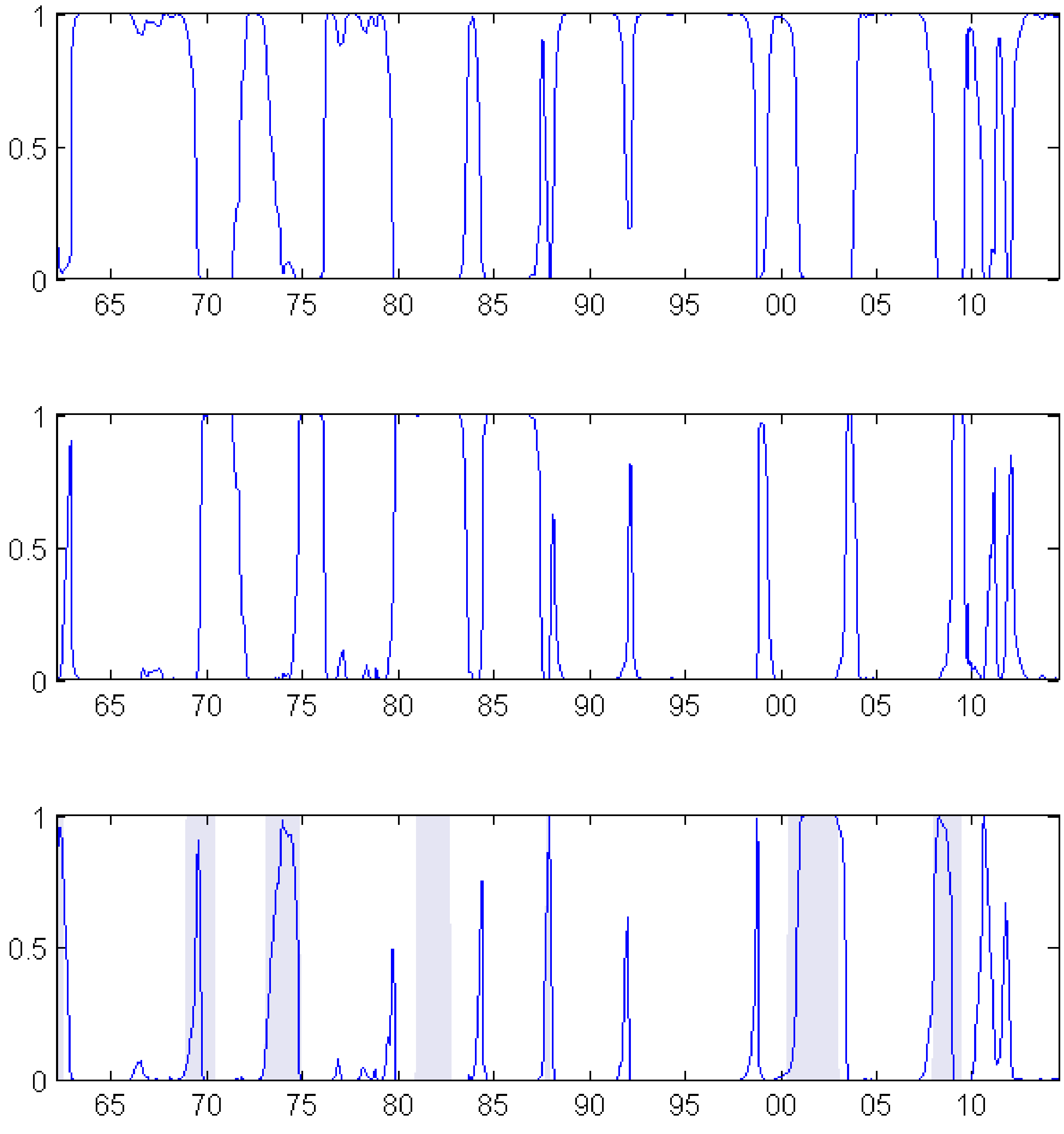


Figure 4.9: BTVTP: SMOOTHED PROBABILITIES

Filtered probabilities versus Smoothed Probabilities

The figures plot the filtered and smoothed probabilities estimated from the FTP model(3-regime AR0AR1) and TVTP model(3-regime AR0AR1 with ISM, UEM, PE as leading indicators)

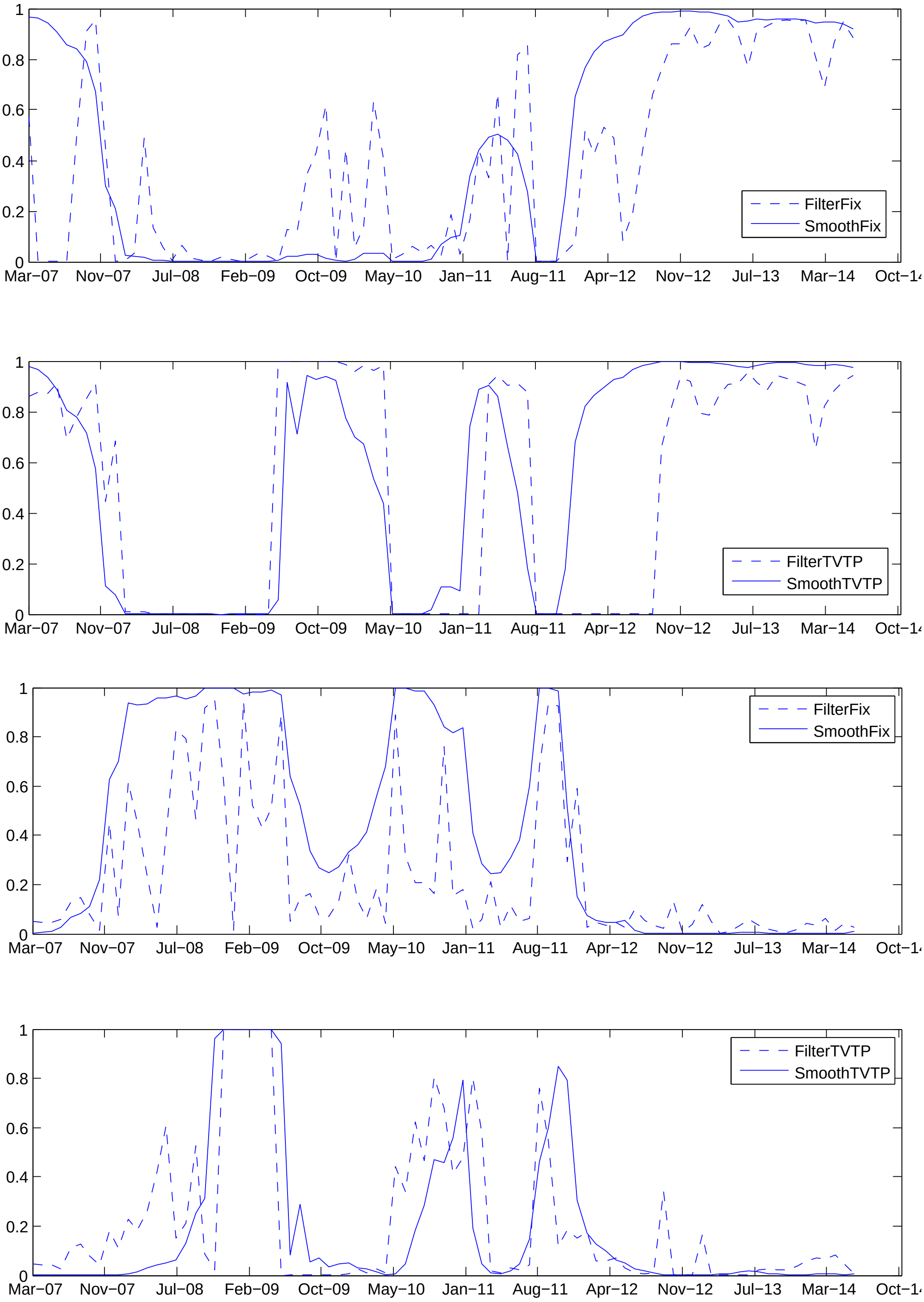
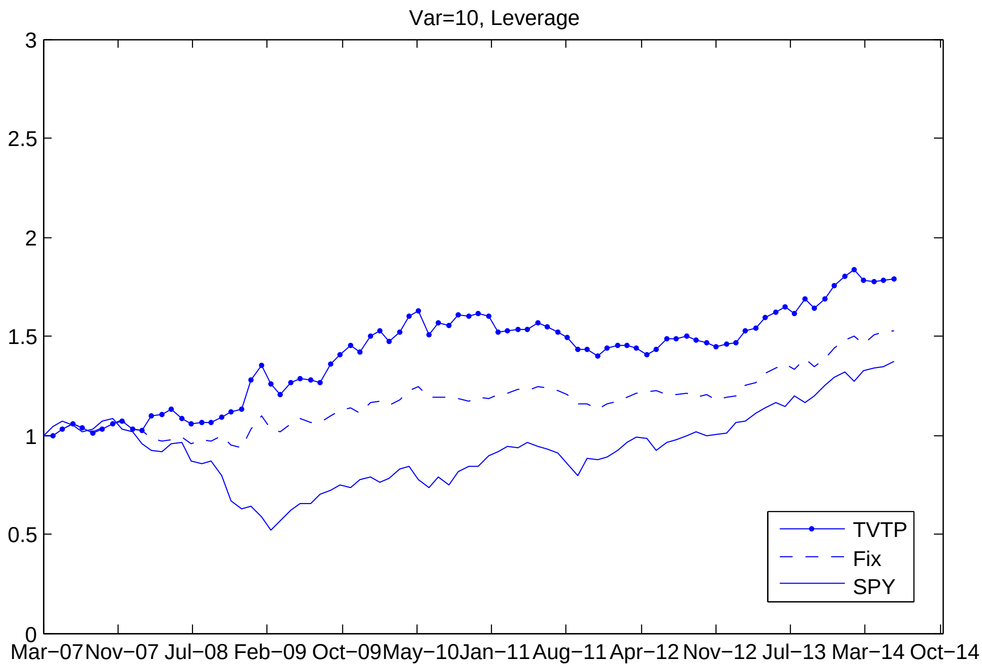


Figure 4.10: Filtered Probs versus Smoothed Probs

Leveraged Portfolio Performance with Variance Tolerance at 10, 20

The figures plot the portfolio returns based on out of sample filtered probability.



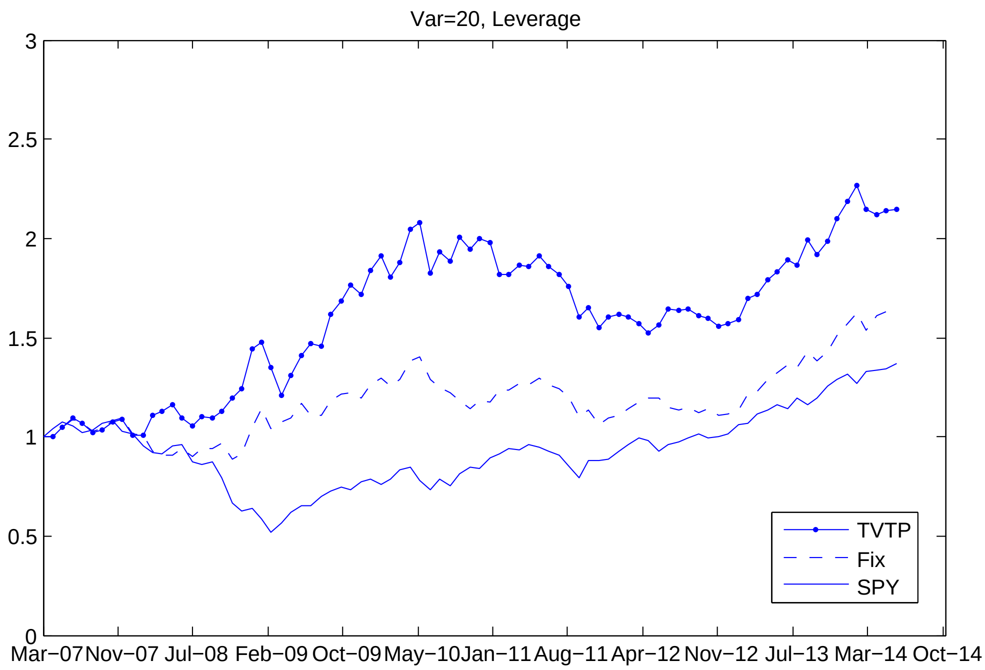
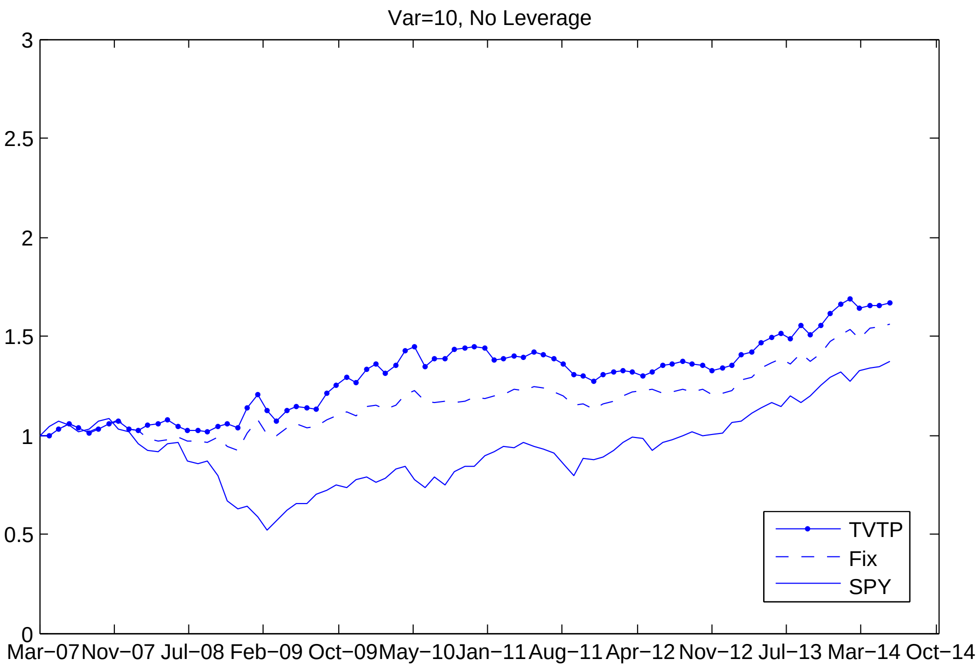


Figure 4.11: Leveraged Portfolio Performance versus SPY

Non-leveraged Portfolio Performance with Variance Tolerance at Var=10, 20
The figures plot the portfolio returns based on out of sample filtered probability.



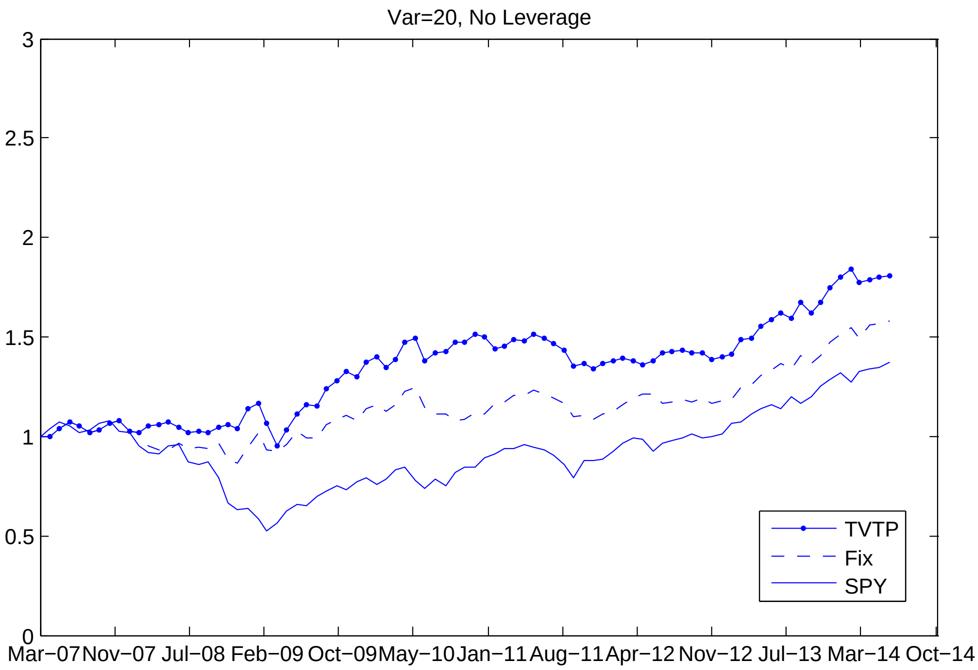
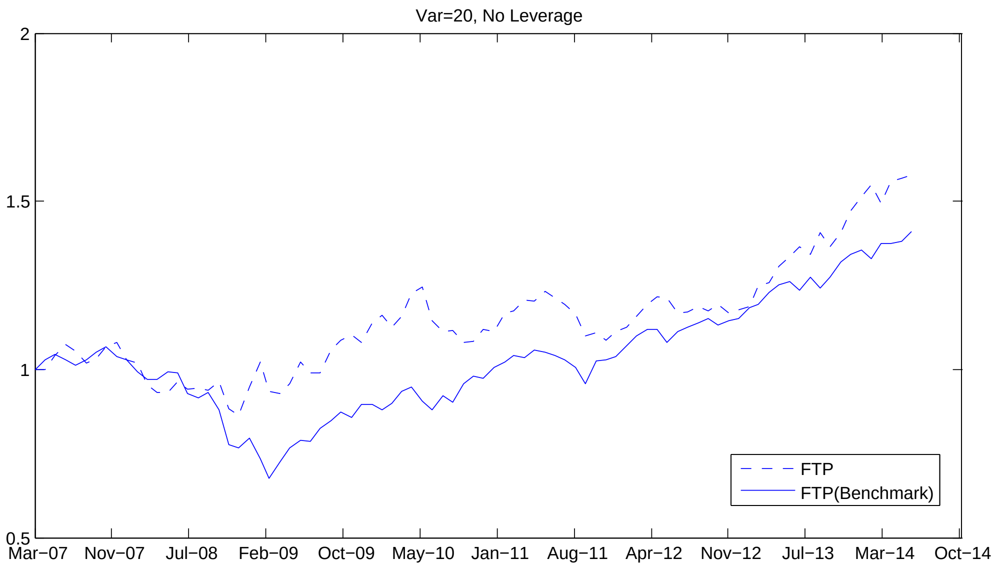


Figure 4.12: Non-leveraged Portfolio Performance versus SPY

Non-leveraged Portfolio Performance with Variance Tolerance at Var=20 compared to the fixed allocation benchmark

The figures plot the portfolio returns based on out of sample filtered probability.



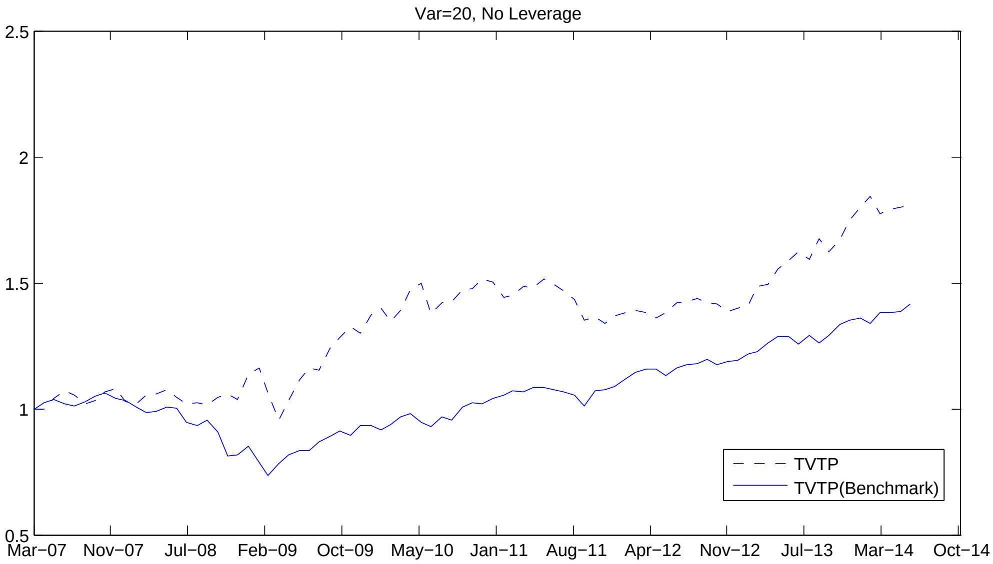


Figure 4.13: Regime-Dependent Strategy versus Fixed Allocation Benchmark

Table 4.1: Univariate Fixed Transition Probability Models

Univariate Fixed Transition Probability Models				
	Two-State AR(0)		Three-State AR(0)	
Parameter	Stocks	Bonds	Stocks	Bonds
μ_1	0.9769	0.1726	1.0538	-0.1521
μ_2	0.3679	0.5405	-0.2847	0.3633
μ_3	NA	NA	0.6761	0.5616
σ_1	2.3693	1.6168	2.3152	1.3149
σ_2	4.9785	3.5953	6.5538	1.7590
σ_3	NA	NA	4.0702	3.6383
p_{11}	0.9620	0.9920	0.9712	0.9871
p_{22}	0.9817	0.9722	0.8018	0.9961
p_{33}	NA	NA	0.9329	0.9735
p_{12}	NA	NA	0.0440	0.0000
p_{13}	NA	NA	0.0027	0.0104
p_{21}	NA	NA	0.0015	0.0000
Log-Likelihood	-1772.6911	-1323.8989	-1780.4815	-1312.0230
AIC	3569.3823	2659.7978	3572.9631	2648.0461
	Two State AR(1)		Three State AR(1)	
Parameter	Stocks	Bonds	Stocks	Bonds
μ_1	1.1187	0.1552	1.0384	0.1330
μ_2	0.3550	0.4629	0.8450	1.5007
μ_3	NA	NA	-1.2049	0.5608
β_1	-0.1308	0.1043	-0.1181	0.1178
β_2	0.0640	0.1346	0.0408	-0.9517
β_3	NA	NA	-0.0460	0.1933
σ_1	2.2990	0.3668	2.2360	1.6263
σ_2	4.9463	3.5481	4.1453	0.4997
σ_3	NA	NA	6.8579	3.6314
p_{11}	0.9617	0.9919	0.9637	0.9837
p_{22}	0.9825	0.9726	0.9114	0.0000
p_{33}	NA	NA	0.6951	0.9618
p_{12}	NA	NA	0.0177	1.0000
p_{13}	NA	NA	0.0000	0.0000
p_{21}	NA	NA	0.0363	0.0071
Log-Likelihood	-1778.4229	-1319.9995	-1770.4450	-1316.4234
AIC	3572.8458	2655.9989	3570.8900	2662.8468

Table 4.2: Univariate TVTP Model Selection of Stocks

Univariate TVTP Model Selection of Stocks								
Model(2,0)								
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	6	8	8	8	10	10	10	12
Loglikelihood	-1772.6911	-1779.0341	-1779.6375	-1775.7101	-1777.1003	-1772.7985	-1773.9111	-1774.7488
AIC	**3569.3823**	3574.0681	3575.2750	3567.4202	3574.2007	**3565.5971**	3567.8222	3573.4976
Model(2,1)								
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	8	10	10	10	12	12	12	14
Loglikelihood	-1778.42291	-1776.73661	-1776.573478	-1786.93286	-1775.039755	-1770.859048	-1785.582245	-1769.349112
AIC	3572.845821	3573.47322	3573.146957	3593.865719	3574.07951	3565.718096	3595.164491	3566.698223
Model(3,0)								
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	12	18	18	18	24	24	24	30
Loglikelihood	-1780.4815	-1772.7556	-1770.1094	-1768.4417	-1766.4676	-1760.7548	-1763.7934	-1756.4765
AIC	3572.9631	3581.5112	3576.2188	3572.8834	3580.9352	3569.5095	3575.5868	3572.9530
Model(3,1)								
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	15	21	21	21	27	27	27	33
Loglikelihood	-1770.445022	-1770.218068	-1772.118363	-1773.706129	-1766.998427	-1763.879408	-1758.495876	-1767.96877
AIC	3570.890045	3582.436136	3586.236726	3589.412258	3587.996854	3581.758816	3570.991753	3601.937541

Table 4.3: Univariate TVTP Model Selection of Bonds

Univariate TVTP Model Selection of Bonds								
Model(2,0)								
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	6	8	8	8	10	10	10	12
Loglikelihood	-1323.8989	-1322.4847	-1323.6947	-1322.2133	-1322.2291	-1319.6107	-1321.0539	-1318.1741
AIC	2659.7978	2660.9694	2663.3895	2660.4266	2664.4582	2659.2214	2662.1078	2660.3483
Model(2,1)								
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	8	10	10	10	12	12	12	14
Loglikelihood	-1319.9995	-1318.4395	-1319.7721	-1318.4007	-1318.1924	-1315.8610	-1317.1934	-1314.2975
AIC	2655.9989	2656.8789	2659.5442	2656.8014	2660.3847	**2655.7220**	2658.3868	2656.5951
Model(3,0)								
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	12	18	18	18	24	24	24	30
Loglikelihood	-1312.0230	-1322.3439	-1316.0657	-1322.0504	-1310.8451	-1314.6084	-1320.9037	-1308.0253
AIC	**2648.0461**	2680.6877	2668.1313	2680.1008	2669.6903	2677.2168	2689.8074	2676.0506
Model(3,1)								
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	15	21	21	21	27	27	27	33
Loglikelihood	-1316.4234	-1310.1667	-1316.1253	-1312.0576	-1301.6963	-1301.0831	-1311.6132	-1295.0536
AIC	2662.8468	2662.3335	2674.2505	2666.1153	2657.3926	2656.1662	2677.2265	2656.1072

Table 4.4: Univariate TVTP Model Parameters-Stocks

Univariate Time Varying Transition Probability Models-Stocks				
	FTP Two-Regime AR(0)		TVTP Two-Regime AR(0) ISM PE	
Parameter	Regime 1	Regime 2	Regime 1	Regime 2
μ	0.9769	0.3679	0.8006	0.3984
σ	2.3693	4.9785	5.8617	27.2464
α_C	NA	NA	-1.1443	-7.0445
α_I	NA	NA	-0.0224	0.1326
α_P	NA	NA	0.2254	-0.0912
Average Transition	0.9620	0.0183	0.8259	0.1279
Probability Matrix	0.0380	0.9817	0.1741	0.8721
No. Parameters	6.00		10	
Loglikelihood	-1772.6911		-1772.7985	
AIC	3569.3823		3565.5971	

Table 4.5: Univariate TVTP Model Parameters-Bonds-2 Regimes

Univariate Time Varying Transition Probability Models-Bonds				
	FTP Two-Regime AR(1)		Two-Regime AR(1) ISM PE	
Parameter	Regime 1	Regime 2	Regime 1	Regime 2
μ	0.1552	0.4629	0.1537	0.4909
β	0.1043	0.1346	0.1017	0.1363
σ	0.3668	3.5481	2.5930	13.2276
α_C	NA	NA	0.6188	-11.3043
α_I	NA	NA	0.0225	0.1654
α_P	NA	NA	0.0276	0.0324
Average Transition	0.9919	0.0274	0.9873	0.0844
Probability Matrix	0.0081	0.9726	0.0127	0.9156
No. Parameters	6		12	
Loglikelihood	-1319.9995		-1315.8610	
AIC	2655.9989		2655.7220	

Table 4.6: Univariate TVTP Model Parameters-Bonds 3 Regimes

Univariate Time Varying Transition Probability Models-Bonds						
	FTP Three-Regimes AR(0)			Three-Regime AR(1) ISM UEM PE		
Parameter	Regime 1	Regime 2	Regime 3	Regime 1	Regime 2	Regime 3
μ	-0.1521	0.3633	0.5616	-0.0161	0.5550	0.4471
β	NA	NA	NA	0.1761	-0.0371	0.1338
σ	1.3149	1.7590	3.6383	1.2791	2.0608	3.4464
$\alpha_{1,C}$	NA	NA	NA	-9.8770	-13.2575	-2.7133
$\alpha_{1,I}$	NA	NA	NA	0.2234	13.7478	0.0728
$\alpha_{1,U}$	NA	NA	NA	-0.3054	-42.6736	-0.5095
$\alpha_{1,P}$	NA	NA	NA	0.0179	-17.5716	-0.0070
$\alpha_{2,C}$	NA	NA	NA	-5.9105	2.5673	-2.2377
$\alpha_{2,I}$	NA	NA	NA	-0.0163	18.4027	-7.3887
$\alpha_{2,U}$	NA	NA	NA	1.1317	4.9965	6.9699
$\alpha_{2,P}$	NA	NA	NA	0.1465	-9.3011	2.7355
Average Transition	0.9871	0.0000	0.0104	0.6175	0.8203	0.0637
Probability	0.0000	0.9961	0.0161	0.3680	0.1718	0.0080
Matrix	0.0129	0.0039	0.9735	0.0145	0.0080	0.9283
No. Parameters	6			33		
Loglikelihood	-1312.0230			-1295.0536		
AIC	2648.0461			2656.1072		

Table 4.7: Bivariate FTP Model Selection

Bivariate FTP Model Selection			
Model(k, p_1, p_2)	No.Parameters	Loglikelihood	AIC
(2,0,0)	12	-3103.0346	6230.0693
(3,0,0)	21	-3091.1962	6224.3924
(4,0,0)	32	-3083.2112	6230.4224
(5,0,0)	45	-3073.3795	6236.7590
(6,0,0)	60	-3058.7249	6237.4497
(2,0,1)	14	-3096.2321	6220.4641
(3,0,1)	24	-3084.8246	6217.6493
(4,0,1)	36	-3075.6353	6223.2706
(5,0,1)	50	-3066.0179	6232.0357
(6,0,1)	66	-3069.7563	6271.5126

Table 4.8: Bivariate FTP Model

Bivariate Fixed Transition Probability Model			
Three-Regimes AR0AR1			
Parameter	Regime 1	Regime 2	Regime 3
μ_1	1.0365	0.4887	0.4187
μ_2	0.2288	0.0811	0.4255
β_2	0.1957	-0.1610	0.1996
σ_{11}^2	6.6013	17.5041	33.9139
σ_{22}^2	1.6763	3.5417	9.7288
σ_{12}	0.0832	3.6381	0.6410
Long-run Mean	Regime 1	Regime 2	Regime 3
Stock Long-run Mean	1.0365	0.4887	0.4187
Bond Long-run Mean	0.2844	0.0698	0.5316
	Regime 1	Regime 2	Regime 3
Stocks Volatility	2.5693	4.1838	5.8236
Bonds Volatility	0.2941	1.3118	1.9206
Correlation	0.1101	0.6629	0.0573
Transition Prob. Matrix	Regime 1	Regime 2	Regime 3
Regime 1	0.9574	0.0406	0.0114
Regime 2	0.0272	0.9178	0.0632
Regime 3	0.0154	0.0416	0.9254

Table 4.9: Bivariate TVTP Model Selection

Bivariate TVTP Model Selection								
Model(k,p)	k=2,p1=0,p2=0							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	12.0000	14.0000	14.0000	14.0000	16.0000	16.0000	16.0000	18.0000
Loglikelihood	-3103.0346	-3097.6223	-3103.4551	-3102.9030	-3096.8901	-3095.3842	-3103.1202	-3094.6514
AIC	6230.0693	6223.2445	6234.9102	6233.8060	6225.7802	6222.7685	6238.2404	6225.3029
Model(k,p)	k=3,p1=0,p2=0							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	21.0000	27.0000	27.0000	27.0000	33.0000	33.0000	33.0000	39.0000
Loglikelihood	-3091.1962	-3098.8375	-3094.3021	-3092.5723	-3073.4071	-3100.0613	-3071.0974	-3088.8058
AIC	6224.3924	6251.6751	6242.6043	6239.1447	6212.8141	6266.1226	6208.1947	6255.6116
Model(k,p)	k=4,p1=0,p2=0							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	32.0000	44.0000	44.0000	44.0000	56.0000	56.0000	56.0000	68.0000
Loglikelihood	-3083.2112	-3074.3200	-3073.7061	-3082.3167	-3059.2650	-3055.0399	-3069.8649	-3054.5420
AIC	6230.4224	6236.6400	6235.4122	6252.6335	6230.5300	6222.0798	6251.7299	6245.0839
Model(k,p)	k=5,p1=0,p2=0							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	45.0000	65.0000	65.0000	65.0000	85.0000	85.0000	85.0000	105.0000
Loglikelihood	-3073.3795	-3082.9872	-3078.0484	-3098.3329	-3086.2322	-3095.4372	-3101.4567	-3092.7267
AIC	6236.7590	6295.9743	6286.0969	6326.6658	6342.4643	6360.8745	6372.9134	6395.4533
Model(k,p)	k=6,p1=0,p2=0							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	60.0000	90.0000	90.0000	90.0000	120.0000	120.0000	120.0000	150.0000
Loglikelihood	-3056.0652	-3093.2939	-3072.6346	-3099.3801	-3099.7207	-3091.6244	-3097.0955	-3089.7952
AIC	6232.1304	6366.5879	6325.2693	6378.7601	6439.4414	6423.2488	6434.1909	6479.5905
Model(k,p)	k=2,p1=0,p2=1							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	14.0000	16.0000	16.0000	16.0000	18.0000	18.0000	18.0000	20.0000
Loglikelihood	-3096.2299	-3090.5882	-3096.0280	-3096.1789	-3089.5905	-3088.6391	-3095.5743	-3089.6318
AIC	6220.4598	6213.1765	6224.0559	6224.3578	6215.1811	6213.2782	6227.1486	6219.2636
Model(k,p)	k=3,p1=0,p2=1							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	24.0000	30.0000	30.0000	30.0000	36.0000	36.0000	36.0000	42.0000
Loglikelihood	-3084.8246	-3092.3105	-3092.1736	-3096.7183	-3066.0320	-3095.0512	-3070.0587	-3058.8431
AIC	6217.6493	6244.6210	6244.3473	6253.4367	6204.0641	6262.1024	6212.1175	6201.6861
Model(k,p)	k=4,p1=0,p2=1							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	36.0000	48.0000	48.0000	48.0000	60.0000	60.0000	60.0000	72.0000
Loglikelihood	-3075.6353	-3065.5668	-3057.9940	-3064.7127	-3059.4978	-3061.7199	-3047.8382	-3062.6750
AIC	6223.2706	6227.1335	6211.9881	6225.4255	6238.9955	6243.4398	6215.6764	6269.3500
Model(k,p)	k=5,p1=0,p2=1							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	50.0000	70.0000	70.0000	70.0000	90.0000	90.0000	90.0000	110.0000
Loglikelihood	-3066.0179	-3083.4523	-3067.5505	-3093.9693	-3080.2866	-3093.6430	-3093.8217	-3090.1642
AIC	6232.0357	6306.9045	6275.1010	6327.9385	6340.5732	6367.2860	6367.6435	6400.3284
Model(k,p)	k=6,p1=0,p2=1							
Predictors	FTP	TVTPI	TVTPU	TVTPE	TVTPIU	TVTPIE	TVTPUE	TVTPIUE
No. Parameters	66.0000	96.0000	96.0000	96.0000	126.0000	126.0000	126.0000	156.0000
Loglikelihood	-3069.7563	-3095.5818	-3063.0163	-3091.3735	-3092.5348	-3091.3104	-3091.5115	-3087.6262
AIC	6271.5126	6383.1636	6318.0325	6374.7469	6437.0696	6434.6208	6435.0229	6487.2524

Table 4.10: Bivariate Time Varying Transition Probability Models

Bivariate Time Varying Transition Probability Models			
Three-Regimes			
Parameter	Regime 1	Regime 2	Regime 3
μ_1	0.8949	-2.2081	1.2862
	(0.0000)	(0.0006)	(0.0034)
μ_2	0.1102	0.2790	0.4765
	(0.1577)	(0.1737)	(0.0688)
β_2	0.0698	0.2209	0.0766
	(0.1690)	(0.0483)	(0.3017)
σ_{11}	10.1377	29.2325	29.0820
	(0.0000)	(0.0000)	(0.0000)
σ_{22}	2.2678	3.4664	11.7006
	(0.0000)	(0.0000)	(0.0000)
σ_{12}	1.4188	-4.7897	5.1897
	(0.0000)	(0.0000)	(0.0001)
	Regime k		
	k=1	k=2	k=3
$\alpha_{1k,C}$	2.8298	-2.9354	-17.3978
	(0.0000)	(0.0000)	(0.0000)
$\alpha_{1k,I}$	-0.0074	-0.0682	0.2990
	(0.0027)	(0.8978)	(0.0000)
$\alpha_{1k,U}$	-0.0700	-0.2227	-0.1144
	(0.0021)	(0.8802)	(0.0023)
$\alpha_{1k,P}$	-0.0087	0.0464	0.0444
	(0.2326)	(0.9241)	(0.0000)
$\alpha_{2k,C}$	-1.9843	-4.0090	-2.1156
	(0.0111)	(0.0000)	(0.0002)
$\alpha_{2k,I}$	2.6603	0.0831	-0.0064
	(0.9463)	(0.0000)	(0.9891)
$\alpha_{2k,U}$	0.2133	-0.1382	-1.2120
	(0.8195)	(0.0002)	(0.0000)
$\alpha_{2k,P}$	-2.5765	0.0768	-0.0014
	(0.0000)	(0.0000)	(0.9966)

Table 4.11: Bivariate Time Varying Transition Probability Models Summary

Bivariate Time Varying Transition Probability Models Summary			
Long-run Mean	Regime 1	Regime 2	Regime 3
Stock Long-run Mean	0.8949	-2.2081	1.2862
Bond Long-run Mean	0.1185	0.3581	0.5160
	Regime 1	Regime 2	Regime 3
Stocks Volatility	3.1840	5.4067	5.3928
Bonds Volatility	1.5096	1.9090	3.4307
Correlation	0.2952	-0.4641	0.2805
Avg. Transition Matrix	Regime 1	Regime 2	Regime 3
Regime 1	0.9656	0.0001	0.2474
Regime 2	0.0319	0.7705	0.0000
Regime 3	0.0025	0.2294	0.7526

Table 4.12: Transition Probability Analysis

Transition Probability Analysis									
Change in ISM									
	Regime 1			Regime 2			Regime 3		
ISM	49.9490	55.0894	60.2298	46.7251	53.1660	59.6069	41.2520	48.1488	55.0456
UEM	5.8116	5.8116	5.8116	5.9068	5.9068	5.9068	7.0888	7.0888	7.0888
PE	18.5839	18.5839	18.5839	22.0803	22.0803	22.0803	17.2344	17.2344	17.2344
Regime 1	0.9706	0.9680	0.9651	0.0000	0.0000	0.0000	0.0000	0.0012	0.1623
Regime 2	0.0294	0.0320	0.0349	0.7738	0.9008	0.9657	0.0000	0.0000	0.0000
Regime 3	0.0000	0.0000	0.0000	0.2262	0.0992	0.0343	1.0000	0.9988	0.8377
Change in UEM									
	Regime 1			Regime 2			Regime 3		
ISM	55.0894	55.0894	55.0894	53.1660	53.1660	53.1660	48.1488	48.1488	48.1488
UEM	4.3078	5.8116	7.3154	4.2236	5.9068	7.5899	5.4453	7.0888	8.7324
PE	18.5839	18.5839	18.5839	22.0803	22.0803	22.0803	17.2344	17.2344	17.2344
Regime 1	0.9748	0.9680	0.9596	0.0000	0.0000	0.0000	0.0021	0.0012	0.0006
Regime 2	0.0252	0.0320	0.0404	0.9356	0.9008	0.8540	0.0000	0.0000	0.0000
Regime 3	0.0000	0.0000	0.0000	0.0644	0.0992	0.1460	0.9979	0.9988	0.9994
Change in PE									
	Regime 1			Regime 2			Regime 3		
ISM	55.0894	55.0894	55.0894	53.1660	53.1660	53.1660	48.1488	48.1488	48.1488
UEM	5.8116	5.8116	5.8116	5.9068	5.9068	5.9068	7.0888	7.0888	7.0888
PE	8.7821	18.5839	28.3857	12.2843	22.0803	31.8763	0.3939	17.2344	34.0748
Regime 1	0.9736	0.9680	0.9614	0.0000	0.0000	0.0000	0.0001	0.0012	0.0107
Regime 2	0.0264	0.0320	0.0386	0.7035	0.9008	0.9792	0.0000	0.0000	0.0000
Regime 3	0.0000	0.0000	0.0000	0.2965	0.0992	0.0208	0.9999	0.9988	0.9893

Table 4.13: Portfolio Performance

Porfolio Performance				
Var=20, No Leverage	TVTP	TVTP_Benchmark	FTP	FTP_Benchmark
Annulised Return	8.73%	5.02%	6.64%	4.97%
Standard Deviation	11.99%	9.83%	12.11%	11.36%
Sharpe Ratio	0.73	0.51	0.55	0.44
Average Equity Allocation	59.82%	59.82%	69.59%	69.59%

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