

HYPERGRAPHS OF BOUNDED DISJOINTNESS

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Abstract. A k -uniform hypergraph is s -almost intersecting if every edge is disjoint from exactly s other edges. Gerbner, Lemons, Palmer, Patkós and Szécsi conjectured that for every k , and $s > s_0(k)$, every k -uniform s -almost intersecting hypergraph has at most $(s+1)\binom{2k-2}{k-1}$ edges. We prove a strengthened version of this conjecture and determine the extremal graphs. We also give some related results and conjectures.

1. Introduction. A k -uniform hypergraph \mathcal{F} is *intersecting* if $A \cap B$ is nonempty for all edges $A, B \in \mathcal{F}$. Erdős, Ko and Rado [6] showed that, for $n \geq 2k$, every k -uniform intersecting hypergraph $\mathcal{F} \subset \binom{[n]}{k}$ has size at most $\binom{n-1}{k-1}$; equality holds for the hypergraph of all k -sets containing a fixed element.

It is natural to vary the intersection condition and look at hypergraphs in which some pairs of edges are allowed to be disjoint. A number of authors have addressed the global problem of minimizing the number of disjoint pairs in a hypergraph of given size and order (see Frankl [8], Ahlswede [1], Ahlswede and Katona [2], Bollobás and Leader [4], Das, Gan and Sudakov [5]). This paper examines the local version of this question introduced by Gerbner, Lemons, Palmer, Patkós and Szécsi [11], where each edge is disjoint from a bounded number of other edges.

Following [11], we define a hypergraph \mathcal{F} to be $(\leq s)$ -almost intersecting if for all $A \in \mathcal{F}$, there are at most s sets $B \in \mathcal{F}$ satisfying $A \cap B = \emptyset$, and s -almost intersecting if for all $A \in \mathcal{F}$, there are exactly s sets $B \in \mathcal{F}$ satisfying $A \cap B = \emptyset$. More generally, let us also say that \mathcal{F} is $[a, b]$ -almost intersecting if for all $A \in \mathcal{F}$

$$a \leq |\{B \in \mathcal{F} : A \cap B = \emptyset\}| \leq b.$$

The maximum size of a k -uniform $(\leq s)$ -almost intersecting hypergraph was investigated in [11], where it was shown that the Erdős-Ko-Rado bound continues to hold provided $n > n_0(k, s)$. By contrast, it was also shown in [11] that the maximum size of a k -uniform s -almost intersecting hypergraph does not grow with the size of the ground set: every k -uniform s -almost intersecting hypergraph has at most $s\binom{2ks}{ks}$ edges. Gerbner, Lemons, Palmer, Pálvölgyi, Patkós and Szécsi [10] subsequently improved this bound to $(2s-1)\binom{2k}{k}$.

An example of a large k -uniform s -almost intersecting hypergraph is given by the family

$$\mathcal{F}(k, s) = \left\{ A \cup \{j\} : A \in \binom{[2k-2]}{k-1}, j \in \{2k-1, 2k, \dots, 2k+s-1\} \right\},$$

which has $(s+1)\binom{2k-2}{k-1}$ edges. In [11], Gerbner, Lemons, Palmer, Patkós and Szécsi conjecture that for every k and $s > s_0(k)$, this is the maximal size of any s -almost intersecting k -uniform hypergraph.

We prove this conjecture in Section 3. In fact, we prove a rather stronger result: we show that for every $k \geq 2$ there are $R = R(k)$ and $s_0(k)$ such that, for $s > s_0$,

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every k -uniform $[R, s]$ -almost intersecting hypergraph has at most $(s+1)\binom{2k-2}{k-1}$ edges. We also determine all the extremal hypergraphs. Among the extremal hypergraphs, the family $\mathcal{F}(k, s)$ minimizes the number of elements in the base set.

The bound on R that we obtain is rather large, as our argument depends on an application of the Sunflower Lemma of Erdős and Rado. It seems likely that something much smaller would suffice: in fact, we conjecture that for sufficiently large s , $R = 1$ is enough. Note that we cannot take $R = 0$, as there are intersecting k -uniform hypergraphs of unbounded size (and an intersecting hypergraph is automatically $[0, s]$ -almost intersecting). However, in Section 4, we consider the effect of weak disjointness assumptions. In particular, for the cases $k = 2$ and $k = 3$, we show that a single pair of disjoint edges suffices to recover the bound $(s+1)\binom{2k-2}{k-1}$ on the number of edges, and that we get the same family of extremal hypergraphs. (We remark that [11] fully characterizes extremal s -almost intersecting graphs for $k = 2$ and all s .)

We also prove sharp bounds for *multihypergraphs*, that is, uniform set systems in which repeated edges are allowed. As in the hypergraph case, there are $[0, s]$ -almost intersecting systems of unbounded size. In Section 2, we prove that the family $\mathcal{M}(k, s)$ consisting of $\binom{[2k]}{k}$, with each edge having multiplicity s , is the unique extremal examples over k -uniform multihypergraphs for the property of being $[1, s]$ -almost intersecting. Note that in the large k and large s limit, $s\binom{2k}{k}$ is about four times as large as $(s+1)\binom{2k-2}{k-1}$.

Finally, in Section 5, we discuss our results and raise some further questions.

We conclude this section with some definitions. We write $[n]$ for the set $\{1, \dots, n\}$ and $\binom{S}{j}$ for the set of all j -element subsets of a set S . Given a hypergraph \mathcal{F} , its *disjointness graph* $\text{DG}(\mathcal{F})$ has vertex set equal to \mathcal{F} , and $A \sim B$ in $\text{DG}(\mathcal{F})$ exactly when $A \cap B = \emptyset$. Note that \mathcal{F} is $[a, b]$ -almost intersecting exactly when the minimal and maximal vertex degrees in $\text{DG}(\mathcal{F})$ satisfy $a \leq \delta(\text{DG}(\mathcal{F})) \leq \Delta(\text{DG}(\mathcal{F})) \leq b$. The definitions of $[a, b]$ -almost intersecting and *disjointness graph* extend directly to *multihypergraphs*, that is, uniform set systems in which repeated edges are allowed. (In the disjointness graph, multiple copies of a single edge correspond to distinct vertices.)

2. Multihypergraphs. First, we fully characterize the extremal behavior in the multihypergraph case. Recall that $\mathcal{M}(k, s)$ is the multihypergraph consisting of $\binom{[2k]}{k}$, where each edge occurs with multiplicity s . Its disjointness graph consists of $\frac{1}{2}\binom{2k}{k}$ copies of the complete bipartite graph $K_{s,s}$.

THEOREM 2.1. *For $s \geq 1$, any k -uniform $[1, s]$ -almost intersecting multihypergraph has at most $s\binom{2k}{k}$ edges.*

The unique multihypergraph achieving this bound is $\mathcal{M}(k, s)$, which is s -almost intersecting.

We will use two classical theorems from extremal set theory. The first is the Bollobás theorem on intersections between pairs of sets.

THEOREM 2.2. [3] *Let $(A_1, B_1), \dots, (A_m, B_m)$ be a sequence of pairs of sets with $|A_i| = a$ and $|B_i| = b$ for every i . If*

1. $A_i \cap B_i = \emptyset$ for $1 \leq i \leq m$, and
2. $A_i \cap B_j \neq \emptyset$ for $1 \leq i, j \leq m$,

then $m \leq \binom{a+b}{a}$. Furthermore, if $m = \binom{a+b}{a}$ then there is some set S of cardinality $a+b$ such that the A_i are all subsets of S of size a , and $B_i = S \setminus A_i$ for each i .

We will also need the skew version of this theorem (see Frankl [9], Kalai [12], Lovász [13]).

THEOREM 2.3. [9, 12, 13] Let $(A_1, B_1), \dots, (A_m, B_m)$ be a sequence of pairs of sets with $|A_i| = a$ and $|B_i| = b$ for every i . If

1. $A_i \cap B_i = \emptyset$ for $1 \leq i \leq m$, and
2. $A_i \cap B_j \neq \emptyset$ for $1 \leq i < j \leq m$,

then $m \leq \binom{a+b}{b}$.

Note that the assumptions in Theorem 2.3 are weaker than in Theorem 2.2; however, there is not a unique extremal graph (for example, B_1 can be any b -element set disjoint from A_1).

We are now ready to proceed with the proof of Theorem 2.1.

Proof. [Proof of Theorem 2.1] Let \mathcal{F} be any such multihypergraph, and let $F = \text{DG}(\mathcal{F})$ be its disjointness graph (an edge of \mathcal{F} with multiplicity c is represented by c distinct vertices in F). We know the minimal and maximal degree satisfy $\delta(F) \geq 1$ and $\Delta(F) \leq s$, respectively. For $A \in \mathcal{F} = V(F)$, let $\Gamma(A) = \{B \in \mathcal{F} : A \cap B = \emptyset\}$ be its neighborhood in F .

We construct a sequence $(A_1, B_1), (A_2, B_2), \dots$ of pairs of vertices of F according to the following procedure, which we will call the *AB algorithm*: set $i = 1$ and $V_1 = \mathcal{F}$. Repeat the following steps until $V_i = \emptyset$:

1. Choose B_i arbitrarily from V_i .
2. Since $\delta \geq 1$, we know $\Gamma(B_i) \neq \emptyset$. Let A_i be an arbitrary element of $\Gamma(B_i)$. (Note that A_i is chosen from V_1 , the original vertex set, for every i .)
3. Set $V_{i+1} = V_i \setminus \Gamma(A_i)$ and increment i .

Let m be the length of the resulting sequence of pairs (A_i, B_i) . By the construction we immediately have $A_i \cap B_i = \emptyset$ for $i = 1, \dots, m$. Since at stage i we eliminate all sets disjoint from A_i as candidates for any future B_j , we know $A_i \cap B_j \neq \emptyset$ for $1 \leq i < j \leq m$. The hypotheses of Theorem 2.3 are satisfied, so $m \leq \binom{2k}{k}$.

Since at the i -th step in the AB algorithm we eliminate at most s vertices from V_{i+1} , we must have $|\mathcal{F}|/s \leq m$. Thus $|\mathcal{F}|/s \leq \binom{2k}{k}$, and we have proved the first claim in the theorem.

Now assume that \mathcal{F} is a $[1, s]$ -almost intersecting k -uniform multihypergraph with exactly $s \binom{2k}{k}$ edges, and apply the AB algorithm to $F = \text{DG}(\mathcal{F})$. The resulting sequences A_1, A_2, \dots, A_m and B_1, B_2, \dots, B_m have length at most $\binom{2k}{k}$, and so the algorithm must eliminate exactly s vertices from V_{i+1} at the i th step, for every i . Since this must hold for every possible sequence of choices in the algorithm, F must be s -regular, and so \mathcal{F} itself is $[s, s]$ -almost intersecting.

We claim that for $X, Y \in \mathcal{F}$, either $\Gamma(X) = \Gamma(Y)$ or $\Gamma(X) \cap \Gamma(Y) = \emptyset$. Why? Assume that the edges X, Y are a counterexample, so that there exists $Z \in \Gamma(X) \setminus \Gamma(Y)$. Note that there must then exist $W \in \Gamma(Y) \setminus \Gamma(X)$, since both $\Gamma(X)$ and $\Gamma(Y)$ contain s elements. Consider running the AB algorithm with $B_1 = Z$, $A_1 = X$, $B_2 = W$, and $A_2 = Y$. Then $|V_2 \setminus V_3| = s - |\Gamma(X) \cap \Gamma(Y)| < s$, which is impossible (see Figure 2.1).

Now let X and Z be two vertices adjacent in F . For any $Y \in \Gamma(Z)$, we know $\Gamma(Y) = \Gamma(X)$. That is, every neighbor of Z , including X , has the same neighborhood, which must be of size s and contains Z . Similarly, every neighbor of X must have the same neighborhood, which is of size s and contains X . We conclude that the connected component containing X and Z in F is isomorphic to the complete bipartite graph $K_{s,s}$, and that F itself consists of $\frac{1}{2} \binom{2k}{k}$ disjoint copies of $K_{s,s}$.

Let $K = \binom{2k}{k}$, and choose a sequence $F_1 G_1, \dots, F_{K/2} G_{K/2}$ of edges of F , one from each component. For each edge $F_i G_i$, we define two pairs (F_i, G_i) and (G_i, F_i) : altogether we get K pairs of k -sets, and these satisfy the conditions of Theorem 2.2.

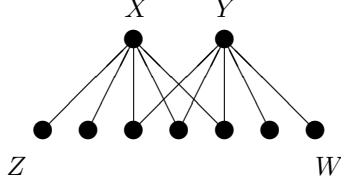


FIG. 2.1. Here $s = 5$, but the non-trivial intersection of $\Gamma(X)$ and $\Gamma(Y)$ forces $|V_2 - V_3| = 2$ when $B_1 = Z$, $A_1 = X$, $B_2 = W$, and $A_2 = Y$.

It follows that the pairs $\{F_i, G_i\}$ consist of all partitions of some fixed set S of size $2k$ into two sets of size k . Furthermore, replacing any F_i or G_i by a different vertex from (the same vertex class in) the same component must give the same graph. It follows that each part of each component of F must represent s copies of the same k -set. Thus \mathcal{F} is actually $\mathcal{M}(k, s)$. \square

3. Hypergraphs. For ordinary hypergraphs, we are able to significantly weaken the assumptions of the conjecture made in [11]. Although in Theorem 3.1 we assume only that our hypergraphs are $[R, s]$ -almost intersecting for some $R > R_0(k)$, we are able to show that the extremal systems are in fact all s -almost intersecting.

Let us describe the extremal families. Fix disjoint sets A, B with $|A| = 2k - 2$ and $|B| \geq s + 1$. Let $f : \binom{A}{k-1} \rightarrow \binom{B}{s+1}$ be any map such that $f(S) = f(A \setminus S)$ for every $S \in \binom{A}{k-1}$. We then define the k -uniform hypergraph \mathcal{M}_f by

$$\mathcal{M}_f = \left\{ S \cup \{x\} : S \in \binom{A}{k-1}, x \in f(S) \right\}. \quad (3.1)$$

Thus \mathcal{M}_f is the union of $\frac{1}{2} \binom{2k-2}{k-1}$ “double stars” of the form $\{S \cup \{x_1\}, \dots, S \cup \{x_{s+1}\}, (A \setminus S) \cup \{x_1\}, \dots, (A \setminus S) \cup \{x_{s+1}\}\}$.

Each edge of \mathcal{M}_f is disjoint from exactly those edges which have the complementary “core” in A and a different “petal” in B ; there are s such edges. Hence all \mathcal{M}_f are, in fact, s -almost intersecting, and all have the same disjointness graph: $\frac{1}{2} \binom{2k-2}{k-1}$ copies of $K_{s+1, s+1}$ minus a matching.

The hypergraphs $\mathcal{F}(k, s)$ defined in [11] correspond to $|B| = s + 1$ and $f(X) = B$ for all $X \in \binom{A}{k-1}$; they clearly minimize the size of the ground set over these families.

THEOREM 3.1. *Fix $k > 2$. Then there exist constants $R = R(k)$ and $s_0 = s_0(k)$ such that when $s > s_0$, any k -uniform $[R, s]$ -almost intersecting hypergraph has at most $(s + 1) \binom{2k-2}{k-1}$ edges.*

The only hypergraphs achieving this bound are those of the form \mathcal{M}_f for some f .

Remark. Note that the extremal $[R, s]$ -almost intersecting hypergraphs are, in fact, s -almost intersecting.

Remark. Of course [11] covers $k = 2$ completely for s -almost intersecting hypergraphs. In Section 4 below, we discuss $[1, s]$ -almost intersecting hypergraphs in the $k = 2, 3$ cases.

In general, a *sunflower* with r petals and core C is a collection of sets Y_1, \dots, Y_r such that $Y_i \cap Y_j = C$ for all $i \neq j$. The disjoint sets $Y_i - C$ are called the *petals*, and they are not allowed to be empty, although the center C can be. (Note that the “stars” in \mathcal{M}_f are in fact sunflowers with one-element petals and $(k - 1)$ -element cores.) The key fact about sunflowers, which we will use in the proof of Theorem 3.1, is the following classical result of Erdős and Rado [7]:

THEOREM 3.2 (Erdős-Rado Sunflower Lemma). *Fix $r, k \geq 1$. Any k -uniform hypergraph \mathcal{F} satisfying $|\mathcal{F}| > k!(r-1)^k$ contains a sunflower with r petals.*

The following lemma assures us that extremal examples for Theorem 3.1 avoid a particular kind of pathology. Note that it's simply not true for $k = 2$, as the complete bipartite graph $K_{2,s+1}$ contains disjoint edges.

LEMMA 3.3. *Fix $k > 2$. For $s > k^k$, no k -uniform $[0, s]$ -almost intersecting hypergraph with at least $(s+1)\binom{2k-2}{k-1}$ edges contains k mutually disjoint edges.*

Proof. [Proof of Lemma 3.3] Assume, to the contrary, that \mathcal{F} is a k -uniform $[0, s]$ -almost intersecting hypergraph containing mutually disjoint edges X_1, \dots, X_k . There are at most k^k edges $Y \in \mathcal{F}$ such that $Y \cap X_i \neq \emptyset$ for all $i \in [k]$. All other edges in \mathcal{F} are disjoint from at least one of the X_i . Look at degrees in $F = \text{DG}(\mathcal{F})$: we must have

$$d_F(X_1) + \dots + d_F(X_k) \geq (s+1)\binom{2k-2}{k-1} - k^k \geq (s+1)(k+1) - k^k,$$

since $\binom{2k-2}{k-1} > k+1$ for $k > 2$. But then $d_F(X_1) + \dots + d_F(X_k) > ks$, contradicting $\Delta(F) \leq s$. \square

Proof. [Proof of Theorem 3.1] It will be convenient to introduce a new parameter r and define $R = k^k r^k$. It is then enough to prove that there are functions $r_0(k)$ and $s_0(k, r)$ such that if $r > r_0(k)$ and $s > s_0(k, r)$, then every k -uniform (R, s) -almost intersecting hypergraph has at most $(s+1)\binom{2k-2}{k-1}$ edges. We will find such functions r_0 and s_0 in the course of the proof.

Let \mathcal{F} be a k -uniform $[R, s]$ -almost intersecting hypergraph with at least $(s+1)\binom{2k-2}{k-1}$ edges. By repeatedly applying Theorem 3.2 until too few edges are left to satisfy its hypotheses, we can decompose \mathcal{F} into a union of $\left\lceil \frac{|\mathcal{F}| - R}{r} \right\rceil$ sunflowers with r petals each, together with a collection of fewer than R leftover edges. Note that a single core might appear in many sunflowers; however, by Lemma 3.3, none of the sunflowers can have an empty core. Build the $(k-1)$ -uniform *core multihypergraph* \mathcal{G} by taking the cores of these sunflowers to be edges; if any core has fewer than $k-1$ elements, pad it with new dummy elements (distinct for each edge, so as to introduce no new intersections) to raise the cardinality to $k-1$.

CLAIM 1. \mathcal{G} is $[1, t]$ -almost intersecting, where $t = \frac{rs}{(r-k)^2} > \frac{s}{r}$.

Proof. [Proof of Claim 1] First consider the upper bound. Suppose a core $C \in \mathcal{G}$ is disjoint from T other cores, D_1, \dots, D_T . Consider a particular D_i :

- D_i can intersect at most $k-1$ petals around C , since $|D_i| = k-1$ and the petals at C are disjoint;
- similarly, each edge in the sunflower around C can meet at most k petals around D_i .

Thus, the number of disjoint pairs (X, Y) , where X is an \mathcal{F} -edge in the sunflower with core C , and Y is an \mathcal{F} -edge in the sunflower with core D_i for some i , is at least $(r - (k-1))(r - k)$. Summing the degrees in $F = \text{DG}(\mathcal{F})$ of the r edges in the sunflower with core C gives

$$T \cdot (r - k)(r - k + 1) \leq rs.$$

Now, for the lower bound: any edge X in \mathcal{F} is disjoint from at least R other edges of \mathcal{F} , and we omit fewer than R edges total as we construct sunflowers. Hence there must be at least one edge disjoint from X contributing to a core in \mathcal{G} , and disjointness is preserved by reducing to cores. \square

CLAIM 2. Fix $\epsilon > 0$. Suppose that $r > 6k(4^k)/\epsilon$ and $s > 2R/\epsilon$. Then

$$\left(\binom{2k-2}{k-1} - \epsilon \right) t \leq |\mathcal{G}| \leq \binom{2k-2}{k-1} t. \quad (3.2)$$

Proof. [Proof of Claim 2] The upper bound follows immediately from Claim 1 and Theorem 2.1.

For the lower bound, note first that

$$\frac{s}{r} \leq t \leq \frac{s}{r} \left(1 + \frac{3k}{r} \right), \quad (3.3)$$

with the second inequality true as long as $r > 5k$. Now, if $|\mathcal{G}| < \left(\binom{2k-2}{k-1} - \epsilon \right) t$, then $|\mathcal{F}| \leq r|\mathcal{G}| + R$ and (3.3) imply that

$$|\mathcal{F}| \leq \left(\left(\binom{2k-2}{k-1} - \epsilon \right) \left(1 + \frac{3k}{r} \right) s + R \right) < \binom{2k-2}{k-1} s. \quad (3.4)$$

Since $|\mathcal{F}| \geq \binom{2k-2}{k-1} (s+1)$, this is impossible. \square

Remark. At this point we have obtained an asymptotic version of the main conjecture. By construction we know $r|\mathcal{G}| \leq |\mathcal{F}| \leq r|\mathcal{G}| + R$, so for any ϵ, r, s satisfying the conditions of Claim 2 we have

$$|\mathcal{F}| \leq (1 + \epsilon)(s+1) \binom{2k-2}{k-1}. \quad (3.5)$$

We now look more closely at the structure of $G = \text{DG}(\mathcal{G})$, showing that it must be approximately regular (Claim 3) and has neighborhoods which are either identical or nearly disjoint (Claim 4).

CLAIM 3. If ϵ, r, s satisfy the conditions of Claim 2, then

$$(1 - \epsilon)t \leq \delta(G) \leq \Delta(G) \leq t.$$

Proof. [Proof of Claim 3] If $\delta(G) < (1 - \epsilon)t$, then it is possible to run the AB algorithm by taking B_1 to be a neighbor of a vertex of minimal degree, A_1 to be the vertex of minimal degree itself, and then continuing arbitrarily. We eliminate fewer than $(1 - \epsilon)t$ vertices after the first pair, and at most t at each of the following steps. Theorem 2.3 tells us that any run of the AB algorithm must terminate in at most $\binom{2k-2}{k-1}$ steps. Hence

$$|\mathcal{G}| < (1 - \epsilon)t + \left(\binom{2k-2}{k-1} - 1 \right) t = \left(\binom{2k-2}{k-1} - \epsilon \right) t, \quad (3.6)$$

contradicting Claim 2.

\square

CLAIM 4. If ϵ, r, s satisfy the conditions of Claim 2, then for all $X, Y \in \mathcal{F}$, either $\Gamma(X) = \Gamma(Y)$ or $|\Gamma(X) \cap \Gamma(Y)| \leq \epsilon t$.

Proof. [Proof of Claim 4] Assume, to the contrary, that there exist $X, Y \in \mathcal{G}$ with $\Gamma(X) \neq \Gamma(Y)$ and $|\Gamma(X) \cap \Gamma(Y)| > \epsilon t$. Exchanging X and Y if necessary, we may assume that $\Gamma(Y) \setminus \Gamma(X) \neq \emptyset$. Choose $W \in \Gamma(Y) \setminus \Gamma(X)$ and $Z \in \Gamma(X)$, and

run the AB algorithm with $B_1 = Z$, $A_1 = X$, $B_2 = W$ and $A_2 = Y$. After the first pair at most t vertices are eliminated. However, after the second pair, at most $(1 - \epsilon)t$ vertices are eliminated, because of the non-trivial intersection of the two neighborhoods. Again by Theorem 2.3 all vertices must be eliminated after at most $\binom{2k-2}{k-1}$ steps, so we conclude

$$|\mathcal{G}| \leq t + (1 - \epsilon)t + \left(\binom{2k-2}{k-1} - 2 \right) t. \quad (3.7)$$

Since the right-hand-sides of (3.7) and (3.6) are equal, Claim 2 again gives a contradiction. \square

CLAIM 5. *Let $r = 10k(64^k)$, and suppose that $s > 2(160^k)(k^{2k})(64^{k^2})$. Then G is a disjoint union of exactly $\frac{1}{2}\binom{2k-2}{k-1}$ complete bipartite graphs in which the size of each part of each component is between $(0.99)t$ and t .*

Proof. [Proof of Claim 5] First, set $\epsilon = \frac{1}{16^k}$ and note that ϵ, r, s then satisfy the conditions for Claims 2, 3 and 4. Recall that $R = k^k r^k$.

Let N_1, N_2, \dots be the distinct neighborhoods that occur in \mathcal{G} . We know that $(1 - \epsilon)t \leq |N_i| \leq t$ for each i , and $|N_i \cap N_j| < \epsilon t$ for distinct i, j . So for each i , $|N_i \setminus \bigcup_{j < i} \Gamma(X_j)| \geq |N_i| - \sum_{j < i} |N_i \cap N_j| \geq (1 - i\epsilon)t$. If there are at least $d = \binom{2k-2}{k-1} + 1$ distinct neighborhoods, then

$$|\mathcal{G}| \geq \left| \bigcup_{i=1}^d N_i \right| \geq \sum_{i=1}^d (1 - i\epsilon)t = \left(d - \binom{d+1}{2} \epsilon \right) t > \binom{2k-2}{k-1} t.$$

Since $\binom{d+1}{2} < 1/\epsilon$, this contradicts Claim 2.

We therefore have that there are at most $\binom{2k-2}{k-1}$ distinct neighborhoods. By Claim 3, all neighborhoods have size at most t . Since $\delta(\mathcal{G}) > 0$, the neighborhoods cover all the vertices. By the lower bound in Claim 2, there are exactly $\binom{2k-2}{k-1}$ neighborhoods.

Note that $\Gamma(x) = \Gamma(y)$, for x, y vertices of G , is an equivalence relation on the vertex set of G . No equivalence class can contain more than t vertices, since then any vertex in the corresponding neighborhood would have degree greater than t . If any equivalence class contains fewer than $\frac{2t}{3}$ vertices, then the total number of vertices in G is less than $\left(\binom{2k-2}{k-1} - \frac{1}{3} \right) t$, contradicting Claim 2. Hence every class contains at least $\frac{2t}{3}$ vertices.

Now suppose that $x \in \Gamma_1 \cap \Gamma_2$ witnesses the intersection of two distinct neighborhoods. Then x has degree at least $\frac{2t}{3} + \frac{2t}{3} = \frac{4t}{3}$, since x is adjacent to every vertex with neighborhood Γ_1 and every vertex with neighborhood Γ_2 . This is impossible, since $\Delta(G) \leq t$. We can conclude that distinct neighborhoods are in fact fully disjoint.

For any edge $\{x, y\}$ of G , we have $\Gamma(x) = \{z : \Gamma(z) = \Gamma(y)\}$. It follows that the component of $\{x, y\}$ is the complete bipartite graph with parts $\Gamma(x)$ and $\Gamma(y)$, and G therefore has the claimed structure. The lower bound on the size of the classes follows from Claim 3. \square

We are now ready to define the promised quantities $r_0(k) = 10k(64)^k$ and $s_0(k) = 2(160^k)(k^{2k})(64^{k^2})$, as in the conditions of Claim 5. Note that we have $R = k^k r^k = 10^k k^{2k} 64^{k^2}$.

The next step of the argument is identical to that at the end of the proof of Theorem 2.1. Extract a matching from G , taking one edge (X_i, Y_i) from each component

for $i = 1, \dots, \frac{1}{2} \binom{2k-2}{k-1}$. Recalling that vertices of G are actually edges in \mathcal{G} , consider the family of pairs of sets

$$(X_1, Y_1), (Y_1, X_1), (X_2, Y_2), (Y_2, X_2), \dots$$

By construction, this is a $(k-1)$ -uniform cross-intersecting family with $\binom{2k-2}{k-1}$ pairs. By Theorem 2.2, there exists a set S of size $2k-2$ such that every pair (X_i, Y_i) consists of complementary subsets of S , both of size $k-1$. As before, replacing any X_i or Y_i with another vertex from the same part of the same component does not change the graph. Hence we know that the core multihypergraph \mathcal{G} contains all $k-1$ element subsets of a fixed $(2k-2)$ -element set S , with all multiplicities between $(0.99)t$ and t . By the structure of \mathcal{G} , we know that no dummy elements were required in the construction of \mathcal{G} ; all the sunflowers had $(k-1)$ -element cores and single element petals.

We once again consider our original hypergraph, \mathcal{F} , which we know to be $[R, s]$ -almost intersecting and of size at least $(s+1)\binom{2k-2}{k-1}$.

CLAIM 6. *Every edge in \mathcal{F} intersects S in exactly $k-1$ elements, and every $(k-1)$ -element subset of S is contained in exactly $s+1$ edges of \mathcal{F} .*

Proof. [Proof of Claim 6] First, let X be a $(k-1)$ -element subset of S . Then there must be at least $(0.99)s$ edges in \mathcal{F} containing X , since X has multiplicity at least $(0.99)t$ in \mathcal{G} , each occurrence in \mathcal{G} corresponds to a sunflower with r petals in \mathcal{F} , and, by Claim 1, we must have $tr > s$.

Now, if any edge Z of \mathcal{F} intersects S in $q \leq k-2$ or fewer vertices, it would be disjoint from at least $((0.99)s - k)\binom{2k-2-q}{k-1}$ edges whose cores lie in $S \setminus Z$. This is far more than the allowed s disjointnesses. We conclude that $|Z \cap S| \geq k-1$.

What if $Z \subseteq S$? Then Z intersects every sunflower core; hence, by the preceding paragraph, Z intersects every edge in \mathcal{F} . This is also impossible.

Note also that if X is contained in *more* than $s+1$ edges of \mathcal{F} , then any edge W of \mathcal{F} containing $S \setminus X$ is disjoint from at least $s+1$ of the edges containing X , a contradiction.

Since \mathcal{F} has at least $(s+1)\binom{2k-2}{k-1}$ edges, every element of $\binom{S}{k-1}$ must be contained in exactly $(s+1)$ edges of \mathcal{F} . \square

We have shown that $|\mathcal{F}| \leq (s+1)\binom{2k-2}{k-1}$. To conclude the main proof, we now assume that \mathcal{F} has exactly $(s+1)\binom{2k-2}{k-1}$ edges. Let $X, Y \in \binom{S}{k-1}$ be a fixed pair of disjoint sets, and let x_1, \dots, x_{s+1} be the petal vertices over the core X . If there exists a vertex $y \notin \{x_1, \dots, x_{s+1}\}$ such that $Y' = \{y\} \cup Y \in \mathcal{F}$, then Y' is disjoint from every one of the $s+1$ edges containing X , which is impossible. Thus the $s+1$ edges of \mathcal{F} containing Y are $\{x_1\} \cup Y, \dots, \{x_{s+1}\} \cup Y$. This suffices to show that \mathcal{F} is indeed of the form \mathcal{M}_f for an appropriate function f . \square

4. Small values of k . In this section we specialize to the cases $k=2$ and $k=3$, and show that for both values we can in fact take $R=1$ in Theorem 3.1.

THEOREM 4.1. *For $s > 13$, any $[1, s]$ -almost intersecting graph has at most $2s+2$ edges. The only graph achieving this bound is $K_{2, s+1}$.*

Proof. Let \mathcal{F} be any $[1, s]$ -almost intersecting graph with $m \geq 2s+2$ edges. Let X be the vertex set of \mathcal{F} , and let $F = \text{DG}(\mathcal{F})$ be the disjointness graph of \mathcal{F} . For $e \in \mathcal{F}$ and $x \in X$, we shall write $d_F(e)$ for the degree in F of e (i.e. the number of edges disjoint from e) and $d(x)$ for the degree in \mathcal{F} of x (i.e. the number of edges that contain x). Note that $\delta(F) \geq 1$: we will carry out most of the proof under the weak assumption that no vertex meets every edge of \mathcal{F} , and only use the assumption that $\delta(F) \geq 1$ when we need it.

Next note that:

- Any edge $e = \{x, y\}$ of \mathcal{F} meets (in one vertex) $(m-1) - d_F(e) \geq (m-1) - s \geq s+1$ edges.
- However, $e = \{x, y\}$ also meets $(d(x) - 1) + (d(y) - 1)$ edges (in one vertex), so $d(x) + d(y) \geq s+3$.

Now choose a pair $e_1 = x_1y_1$ and $e_2 = x_2y_2$ of disjoint edges (which must exist as $m > 3$ and \mathcal{F} is not a star).

Suppose that $\min\{d(x_1), d(y_1)\} \geq 8$. Let f be any edge incident with x_1 that does not meet any of $\{y_1, x_2, y_2\}$. Then f is disjoint from at least 6 of the edges incident with y_1 and at least $(s+1) - 4 = s-3$ of the edges incident with e_2 ; since at most 2 edges are double-counted, f is disjoint from at least $6 + (s-3) - 2 > s$ edges, giving a contradiction. Arguing symmetrically for x_2 and y_2 (and relabelling if necessary) we may assume that $d(y_1), d(y_2) \leq 7$ and therefore $d(x_1), d(x_2) \geq (s+3) - 7 = s-4$.

Now if any edge is disjoint from $\{x_1, x_2\}$, it meets at most two edges incident with each, and so (as there may be an edge x_1x_2) misses at least $(d(x_1)-2) + (d(x_2)-2) - 1 = d(x_1) + d(x_2) - 5 \geq 2s - 13 > s$ edges. This is again a contradiction, so we see that all edges are incident with x_1 or x_2 .

The edge x_1x_2 is not present, or it would meet every other edge (this is the only place we use the condition $\delta(F) \geq 1$). Thus any edge incident with x_1 meets at most one edge incident with x_2 ; it follows that x_2 is incident with at most $s+1$ edges, and similarly x_1 is incident with at most $s+1$ edges. We deduce that $m = 2s+2$, and x_1 and x_2 are each incident with exactly $s+1$ edges. Furthermore, every edge incident with x_1 meets an edge incident with x_2 . It follows that \mathcal{F} is a copy of $K_{2,s+1}$. \square

We do not know the smallest possible value for s in Theorem 4.1; however there are only finitely many graphs with at most 30 edges and no isolated vertices, and so this could in principle be determined by a finite check. As noted in [11], there exist s -almost intersecting graphs with more than $2s+2$ edges for $s = 1, 3, 6$.

THEOREM 4.2. *For $s > 13$, any $[0, s]$ -almost intersecting graph that is not a star has at most $2s+3$ edges. The only graph achieving this bound is $K_{2,s+1}$ plus an edge connecting the vertices in the part of size 2.*

Proof. Suppose that \mathcal{F} satisfies the conditions of the theorem and has $m \geq 2s+3$ edges. We follow the proof of Theorem 4.1 through to the beginning of the final paragraph: at this point we have only used the conditions that $\Delta(F) \leq s$ and \mathcal{F} is not a star. We now delete the edge x_1x_2 if present, and complete the argument, finding that we are left with a copy of $K_{2,s+1}$. It follows that x_1x_2 must have been present, and adding it back gives the graph specified. \square

For $k = 3$, we can prove a similar strengthening of Theorem 3.1.

THEOREM 4.3. *For $s > 625$, any $[1, s]$ -almost intersecting 3-uniform hypergraph has at most $6s+6$ edges. The only hypergraphs achieving this bound are of the form \mathcal{M}_f for some function f .*

Proof. Let \mathcal{F} be any $[1, s]$ -almost intersecting 3-uniform hypergraph with $m \geq 6s+6$ edges. Let X be the vertex set of \mathcal{F} , and let $F = \text{DG}(\mathcal{F})$ be the disjointness graph of \mathcal{F} . As in the proof of Theorem 4.1, for $e \in \mathcal{F}$ and $x \in X$, we shall write $d_F(e)$ for the degree in F of e and $d(x)$ for the degree in \mathcal{F} of x . Note that $\delta(F) \geq 1$: we will carry out most of the proof under the weak assumption that for every edge e , and vertices $x, y \in e$, there is an edge disjoint from $\{x, y\}$, and only use the assumption that $\delta(F) \geq 1$ when we need it. We note that setting $s > 625$ is sufficiently large for all steps below.

We begin with a useful fact: suppose that some pair of vertices x and y have $t \geq 1$

common edges. There must be some edge e disjoint from $\{x, y\}$; since e is disjoint from all but at most 3 edges incident with x and y , we must have $t \leq s + 3$.

We now break into three cases, according to the structure of \mathcal{F} .

Case 1: \mathcal{F} contains three pairwise disjoint edges, say e_1, e_2, e_3 .

There are at most 27 edges meeting all three of these edges, and every other edge is disjoint from at least 1. It follows that $d_F(e_1) + d_F(e_2) + d_F(e_3) \geq m - 27 \geq 6s - 21 > 3s$, since s is sufficiently large. This contradicts $\Delta(F) \leq s$, so we conclude that \mathcal{F} does not contain three pairwise disjoint edges.

Case 2: There are edges e, f_1, f_2 such that e is disjoint from f_1 and f_2 , and $|f_1 \cap f_2| = 1$.

Suppose $f_1 \cap f_2 = \{y\}$. There are at most s edges disjoint from each of e, f_1, f_2 , and so at least $(6s + 6) - 3s = 3s + 6$ edges meet all of e, f_1, f_2 . At most 12 edges meet all of e, f_1, f_2 and miss y , and so at least $(3s + 6) - 12 = 3s - 6$ edges must meet e and y . At most 3 edges contain y and meet e in two vertices and so at least $3s - 9$ edges contain y and meet e in exactly one vertex. Let $e = \{x_1, x_2, x_3\}$, and, for $i = 1, 2, 3$, let

$$E_i = \{f \in \mathcal{F} : f \cap \{x_1, x_2, x_3, y\} = \{x_i, y\}\}.$$

Since, for each i , there are at most $s + 3$ edges incident with both y and x_i , it follows that $|E_i| \geq (3s - 9) - 2(s + 3) = s - 15$.

Now any edge that misses both y and x_i must be disjoint from at least $(s - 15) - 3 = s - 18$ edges from E_i . It follows that if f misses y then $|f \cap e| \geq 2$, since otherwise f would be disjoint from at least $2(s - 18) = 2s - 36 > s$ edges from $E_1 \cup E_2 \cup E_3$. Since there are at most s edges disjoint from e , and at most $3(s + 3) = 3s + 9$ edges incident with both e and y , it follows that there are at least $(6s + 6) - s - (3s + 9) = 2s - 3$ edges that meet e and miss y . Thus there are at least $2s - 3$ edges that meet e in exactly two vertices.

Finally, consider f_1 . There are at most 9 edges that meet e in two vertices and also meet f_1 . But then at least $(2s - 3) - 9 = 2s - 12$ edges that meet e in two vertices must miss f_1 , which gives a contradiction.

Case 3: For every edge e , and every pair of edges f_1, f_2 that are disjoint from e , $|f_1 \cap f_2| = 2$.

Set $K = 11$ and let G be the graph with vertex set X and $xy \in E(G)$ if there are at least K edges from \mathcal{F} that contain both x and y . For each edge xy of G , choose a set E_{xy} of K edges from \mathcal{F} that contain x and y .

It will be useful to note a relationship between edges of G and \mathcal{F} . If xy is an edge of G , then every edge that is disjoint from $\{x, y\}$ meets at most 3 edges from E_{xy} and so is disjoint from at least $K - 3$ edges from E_{xy} ; so if there are t edges of \mathcal{F} disjoint from $\{x, y\}$ then $t(K - 3) \leq \sum_{e \in E_{xy}} d_F(e) \leq sK$ and so $t \leq sK/(K - 3) = (11/8)s$.

We now consider the structure of G :

- Note first that G does not contain three independent edges, or else \mathcal{F} would contain 3 independent edges (we can pick these greedily).
- Next note that, for any edge x_1y_1 of G , there is at most one edge of G disjoint from x_1y_1 . For suppose x_2y_2 and x_2y_3 are disjoint from x_1y_1 . Since $K = 11$, we can greedily extend all three edges to $f_1, f_2, f_3 \in \mathcal{F}$ such that $f_2 \cap f_3 = \{x_2\}$ and f_1 is disjoint from $f_2 \cup f_3$. But we have already ruled this out in Case 2.

- We next show that $\Delta(G) \leq 3$. Suppose that xy_i , $i = 1, \dots, 4$ are 4 edges of G . Note that every edge of G must contain x , since any edge not containing x would be disjoint from at least two edges of form xy_i (which we have just shown does not happen). Let $f = \{z_1, z_2, z_3\}$ be any edge of \mathcal{F} that does not contain x . For each i , there are at most $s + 3$ edges of \mathcal{F} containing both x and z_i ; and there are at most s edges of \mathcal{F} disjoint from f , so there are at least $(6s + 6) - 3(s + 3) - s = 2s - 3$ edges of \mathcal{F} that do not contain x . There are at most $6(K - 1)$ edges of \mathcal{F} that contain at least two vertices from $\{y_1, y_2, y_3, y_4\}$ (since G does not have any edges among this set), and so there are at least $2s - 6K + 3$ edges of \mathcal{F} that miss x and meet $\{y_1, y_2, y_3, y_4\}$ in at most 1 vertex. It follows that some edge $\{x, y_i\}$ of G is disjoint from at least $(3/4)(2s - 6K + 3)$ edges of \mathcal{F} . But $(3/4)(2s - 6K + 3) > sK/(K - 3)$, giving a contradiction. We conclude that $\Delta(G) \leq 3$.

It follows from the facts above that all edges of G are contained in some set $S \subset X$ of size 4. We have not yet shown that G has any edges: if $e(G) = 0$, choose S to be any 4-set containing an edge of \mathcal{F} .

We next consider how the edges of \mathcal{F} intersect S .

- If S contains an edge of G there are at most $sK/(K - 3)$ edges of \mathcal{F} disjoint from S ; otherwise, S contains an edge of \mathcal{F} , so there are at most s edges disjoint from S .
- Suppose that at least $36K$ edges of \mathcal{F} meet S in exactly one vertex. Then at least $9K$ of these edges contain the same vertex $x \in S$: let E' be such a set of $9K$ edges. Now consider the graph H with vertex set $X - S$ and edges $\{e \setminus \{x\} : e \in E'\}$. H has at least $9K$ edges, and $\Delta(H) \leq K - 1$ (or we would have an edge of G from x to a vertex outside S). By choosing greedily, we can find a matching of size 5 in H ; let f_1, \dots, f_5 be the corresponding edges of \mathcal{F} . But now let e be any edge of \mathcal{F} that does not contain x : e meets at most 3 of the f_i , so there are two others that are disjoint from e and meet in one vertex, which is a configuration that we have already excluded. We conclude that at most $36K$ edges of \mathcal{F} meet S in exactly one vertex.
- At most 4 edges of \mathcal{F} are contained in S .

It follows that at least $(6s + 6) - sK/(K - 3) - 36K - 4 > s + 18$ (since s is large enough) edges of \mathcal{F} meet S in exactly 2 vertices. If any edge of \mathcal{F} is disjoint from S then it can meet at most 18 of these edges, giving a contradiction. We deduce that no edges of \mathcal{F} are disjoint from S , and so at least $(6s + 6) - 36K - 4 = 6s - 36K + 2$ edges of \mathcal{F} meet S in exactly two vertices. Since no pair of vertices in S belongs to more than $s + 3$ edges, it follows that every pair of vertices in S belongs to at least $(6s - 36K + 2) - 5(s + 3) = s - 36K - 13$ edges. Now if any edge of \mathcal{F} meets S in only one vertex, it is disjoint from at least $s - 36K - 16$ edges incident with any pair of vertices in S that it does not meet; there are 3 such pairs, giving a contradiction. We conclude that every edge meets S in at least 2 vertices.

Finally, let $S = \{s_1, s_2, s_3, s_4\}$. If \mathcal{F} is $[1, s]$ -almost intersecting, then no edge of \mathcal{F} is contained in S (or it would meet every other edge). No pair $\{s_i, s_j\}$ belongs to more than $s + 1$ edges, or any edge containing $S \setminus \{s_i, s_j\}$ would be disjoint from more than s of these edges. Thus if $|\mathcal{F}| \geq 6s + 6$ then every pair from S is incident with exactly $s + 1$ edges, and so every edge incident with $\{s_i, s_j\}$ must meet an edge incident with $S \setminus \{s_i, s_j\}$. It follows immediately that \mathcal{F} is an extremal system from the family described in the theorem. \square

THEOREM 4.4. *Let $s > 625$, and suppose that \mathcal{F} is an $[0, s]$ -almost intersecting*

3-uniform hypergraph. If for every edge $e \in \mathcal{F}$, and vertices $x, y \in e$, there is an edge of \mathcal{F} disjoint from $\{x, y\}$, then \mathcal{F} has at most $6s + 10$ edges. The only hypergraphs achieving this bound are given by adding up to 4 edges entirely within the 4-vertex “core” of the extremal hypergraphs of Theorem 4.3.

Proof. Follow the proof of Theorem 4.3 through the the final paragraph, and then delete all edges that lie entirely inside S . \square

5. Discussion. The results of Section 4 show that for $k = 2, 3$ we can take $R = 1$ in Theorem 3.1. We conjecture that a similar result holds for all k .

CONJECTURE 5.1. *Fix $k > 2$ and let $s > s_1(k)$ be sufficiently large. Then any k -uniform $[1, s]$ -almost intersecting hypergraph has at most $(s + 1)\binom{2k-2}{k-1}$ edges.*

It seems likely that the only hypergraphs achieving this bound are of the form \mathcal{M}_f for some function f , as described in (3.1).

Another way to weaken the conditions of Theorem 3.1 is to drop the condition that every edge needs to be disjoint from some other edge. Of course, the system can then have unbounded size, as we could take a large star. However, if we demand only that *some* pair of edges is disjoint then, for $k = 2, 3$, Theorems 4.2 and 4.4 again determine the hypergraphs of maximal size, which depends only on k .

One natural way of expressing the hypothesis of Theorem 4.2 is to say that no vertex meets all edges. We might hope for an extension to k -uniform hypergraphs for $k > 2$, by looking at hypergraphs in which no set of t vertices meets all edges. In this case, $t = k - 2$ is not enough as we can fix a set S of size $2k - 3$ and take all edges that meet S in exactly $k - 1$ vertices: the resulting hypergraph is intersecting and has unbounded size. On the other hand, $t = k$ is equivalent to the condition that every edge is disjoint from some other edge, which takes us back to Conjecture 5.1! So the only other interesting case is $t = k - 1$, for which we conjecture the following.

PROBLEM 5.2. *What is the maximum size of a k -uniform, $[0, s]$ -almost intersecting hypergraph \mathcal{F} in which no set of $k - 1$ vertices meets all the edges? What do the extremal hypergraphs look like?*

It is easy to see that the size of \mathcal{F} is bounded (as pointed out to us by Alexey Pokrovskiy [14]): \mathcal{F} cannot contain a sunflower with $k + s + 1$ edges (as the core of the sunflower would then have to meet every edge), so the number of edges is bounded by the Sunflower Lemma.

We conjecture that, for $k \geq 2$ and sufficiently large s , an extremal example can be obtained by filling in the $(2k - 2)$ -set at the centre of a hypergraph of form \mathcal{M}_f , giving hypergraphs of size

$$(s + 1)\binom{2k - 2}{k - 1} + \binom{2k - 2}{k}.$$

We conjecture that the same bound should hold under the weaker condition that no edge contains a set of $k - 1$ vertices meeting all the edges. The Sunflower Lemma again gives an upper bound, while Theorems 4.2 and 4.4 confirm the conjecture in the special cases $k = 2, 3$.

Another improvement to Theorem 3.1 would be to bring down the value of s_0 . We have not optimized the constants in the proof, but (due to the use of the Sunflower Lemma) the proof gives a bound of order $2^{O(k^2)}$. It is likely that this is far from the truth. Can s_0 be brought down to a polynomial in k ?

From below, a few examples show that we cannot hope to bring s all the way down to 1. For $k = 2$, [11] found several small graphs that are s -almost intersecting, but have more than $2s + 2$ edges. We note that the complete hypergraph $\binom{[7]}{3}$ is

4-almost intersecting and has $35 > (4 + 1)\binom{4}{2} = 30$ edges. Similarly, the complete hypergraph $\binom{[9]}{4}$ is 5-almost intersecting with $126 > (5 + 1)\binom{6}{3} = 120$ edges. For all $k \geq 2$, the complete hypergraph $\binom{[2k]}{k}$ is 1-almost intersecting and has more than $2\binom{2k-2}{k-1}$ edges, so we need $s_0 > 1$. But we cannot even rule out the possibility that we can take s_0 to be some constant (independent of k).

Finally, we mention that the problems above have all concerned almost intersecting hypergraphs and multihypergraphs. Let us define \mathcal{F} to be $[a, b]$ -almost t -intersecting if for all $A \in \mathcal{F}$

$$a \leq |\{B \in \mathcal{F} : |A \cap B| < t\}| \leq b.$$

It would be natural to try to extend the results (and questions) to almost t -intersecting hypergraphs for $t \geq 2$.

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