Educating awareness:
Using variation theory in task design

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ABSTRACT

Researchers claim that implementing highly demanding cognitive tasks helps students improve their mathematical thinking, which is critical for a deep understanding of mathematics (Stein et al., 1996; Charalambous, 2010). In this action research, I investigate how participation in a professional development program on variation theory increases teachers' awareness about the cognitive demands of the tasks they use in class. This study's data were collected from various sources: classroom observations, semi-structured interviews, field notes from meetings, and classroom artefacts. The main finding is that teachers source their tasks from the published curriculum materials and do not implement higher cognitive demand tasks than those in the textbooks or past exam questions. This research takes an original approach by using variation theory to increase teachers' awareness of cognitive demands and improve their subject matter and pedagogical knowledge.

1. Introduction

Students spend most of their study time engaged in tasks and activities designed by their teachers (Hiebert et al., 2003). This implies that tasks are a dominant factor in the learning outcomes as "the mathematical tasks implemented in classrooms influence students' opportunities to learn mathematics in important ways" (Stylianides and Stylianides, 2008). The opinion expressed by many researchers in mathematics education is that school mathematics is reduced to learning mathematical tools and does not promote the type of thinking necessary to do mathematics (Watson, 2008; Cuoco et al., 1996). But in what sort of activities should students be involved in class? Exploring, observing variance and invariance, conjecturing, noticing relationships, choosing and constructing representations are just a few examples that Watson (2008) considers as mathematics practices. In their article "Habits of mind", Cuoco et al. (1996) also suggest that students should be encouraged to be inventors, conjecturers, tinkerers, visualizers to help them develop mathematical skills. Lockhart (2009) points out that school mathematics was reduced to memorising and reciting formulae without understanding while teaching to test has become the main focus in lessons. These do not develop an interest in mathematics but rather deter the students from studying mathematics beyond the minimum required. One implication is that teachers need to select and devise mathematical tasks that are engaging and cognitively demanding.

Apart from introducing new content, the reformed A levels also emphasized the need to develop generalising skills, solve challenging problems, and make deductions and inferences (Department for Education, 2016). But as Drake (2006) noted, "mathematics education reform requires teachers to adopt new strategies not only for teaching mathematics but also for learning mathematics themselves." This suggests that teachers would benefit from a program where they

could reflect on their current practices and also would allow them to develop their mathematical and pedagogical knowledge.

Research studies show that teachers are heavily influenced by textbooks and other published curriculum materials when planning for classroom instruction (Sun, 2011). Doyle (1983) observed that these materials focus on procedures and fluency rather than conceptual understanding. More recent research confirms that teachers design their lesson using mainly the textbook tasks and follow the sequences suggested in them (Haggarty and Pepin, 2002; Eisenmann and Even, 2011). Third International Mathematics and Science Study (TIMMS) reports that 96% of teachers use the textbooks in lessons (Hiebert et al., 2003). One problem raised by this practice is that "textbooks offer mathematics which is lacking in variety, cognitive challenge, and linguistic complexity," and this will result in "an impoverished and restricted mathematical diet" (Haggarty and Pepin, 2002).

This research project is part of a program I run as a Teacher Educator at the school I am currently employed. The school I work at is located in China and caters for Chinese nationals looking to further their education at prestigious universities abroad. The vast majority of the students choose to continue studying mathematics courses in university and this requires a robust foundation to be built at high school level. From my own experience and also from the literature (Sun, 2011), I know that teachers in China rely on textbooks as the main resource to be used for teaching, revising, and homework. Teachers' culture of teaching to exam and students' habit to drill only past exam questions is a widespread practice in our school. Similar behaviour patterns have been documented by researchers in the US (Madaus, 1988) and Australia (Ayres et al., 2004). Other countries might fall into this category, as Madaus (1988) observes that "In every setting where a high-stakes test operates, a tradition of past tests develops, which eventually de

facto defines the curriculum". However, if we want students to get a better *relational understanding* instead of *instrumental understanding* (Skemp, 1978), these practices need changing.

In light of the A levels reform and considering students' needs, together with teachers' practices, the main issue raising for me as a Teacher Educator is how can we increase the teachers' awareness about the richness of the tasks they offer in the lesson?

The overall aim is to challenge the current "state of desirable school culture" (Jaworski, 2006) which is teaching to exam using mostly textbooks and past exam questions.

2. Literature review

The literature review presented below emerged from my attempt to find an answer to the question: what are the factors that might help or impede teachers offering mathematically rich tasks to students?

The research studies analysed below cover different topics, such as variation theory, teachers' beliefs or task design. Give that the topic of mathematical tasks is very large, I will particularly focus on cognitive demands of tasks.

2.1 Variation theory

In this section of the review, I will focus on variation as a theory developed by Marton and Booth (1997), a didactical practice in China (Bianshi 變式), and I will look at how variation

theory is applied in educational research. I will finish by showing what are the strengths and limitations of using this theory.

Variation theory is a learning theory rooted in phenomenography (Runesson, 2005). It tries to explain how learning occurs and why some people are better at learning than others (Marton and Booth, 1997). Marton and Tsui (2004) assert that learning is acquiring knowledge about something. In variation theory, that "something" is called the object of learning (Marton and Tsui, 2004). The learning actions such as memorising, defining, and interpreting refer to the general aspect of the object of learning, while the things they act on (for example: memorise a formula) define the specific aspect of it. The object of learning has a dynamical structure as it can change in time, depending on conditions. The intended object of learning is what's in the teacher's awareness before the lesson, while the enacted object of learning is what students encounter in class. The lived object of learning is what students actually learn in that lesson, and this might be different for each of them.

Marton and Booth (1997) argue that for learning to take place, learners must experience an object or a phenomenon in different ways, that is, to become aware of various critical features of that object or phenomenon. Critical features can be made visible by varying some them, while keeping the others invariant: "in other words, these features must be experienced as dimensions of variation" (ibid.). Possibilities for learning can be afforded or constrained by opening and varying dimensions of variation, thus creating a "space of learning" (Marton and Tsui, 2004). These critical features can be brought into focus by following specific patterns of variation: contrast, generalisation, separation, and fusion (Marton and Pang, 2006).

For example, in mathematics, teachers use contrast by giving counterexamples: the kite shows that not every quadrilateral with two pairs of congruent sides is a parallelogram. Generalisation

can be used to enforce the idea that a polygon is a closed shape with straight sides. By looking at triangles with the same base and same height, students can separate the isosceles triangles from the others, as the ones with the smallest perimeter. Using fusion, students can differentiate between a kite and a parallelogram by becoming aware of differences in size for both angles and congruent sides positioning.

Watson and Mason (2006) applied variation theory in a mathematical context and assert that the concept of 'dimension of variation' can be used to reveal the underlying mathematical structure of an exercise. They extend this concept and introduce the notions of 'dimensions of possible variation' and 'range of permissible change' as tools to be used in task design. They further claim that "tasks that carefully display constrained variation are generally likely to result in progress in ways that unstructured sets of tasks do not, as long as learners are working within mathematically supportive learning environments" (ibid.).

However, Rowland (2008) observes that careless selection of dimension of variation can impede students' learning by 'obscuring the variables'.

Research on teachers' choice of examples further highlights the importance of task sequencing in building the ability to distinguish between critical and non-critical features (Zodik and Zaslavsky, 2008). When planning is informed by variation theory, teachers should be aware of the dimensions of variation they choose to open and the range of values allocated to these dimensions (Kullberg et al., 2017; Rowland, 2008). This prompts the necessity of teachers to possess a *variation-pedagogy* (Watson, 2017), that is, the ability required in selecting and using the dimensions of variation and the range in which they change. In a study using variation theory as a tool to analyse teachers' use of patterns of variation, Al-Murani et al. (2019) also emphasize

the need for a special pedagogical knowledge as they conclude that "there is no single best way to use variation effectively across all mathematics."

In an article about how teaching with variation (Bianshi 變式) is applied in China, Gu et al. (2004) describe two types of variation that can be seen in a lesson: conceptual variation and procedural variation. The former allows students to experience the concept in multiple ways, including contrast with non-concept, while the latter refers to using different methods to solve a problem. Sun (2011) further acknowledges that these practices are deeply rooted in Chinese mathematics education and identifies them as *multiple solutions-one problem, one problem-multiple solutions, and one problem-multiple changes*. Sun (2011) also analyses how variation theory is used in Chinese mathematics textbooks and compares them with US textbooks. She claims that variation theory is an "indigenous" practice in task design in China, while in the US, this is not the case. However, her final conclusion is that more research is necessary on its effectiveness as a pedagogical tool.

The research literature above shows that variation theory has a diverse use in education. Sun (2011) presents the use of variation in textbooks, Al-Murani et al. (2019) use variation theory as an analytical tool, while Watson and Mason (2016) discuss how to use strategic variation for sequencing tasks to improve learning. Numerous studies have used of variation theory in a multitude of other areas. In professional development, Ekawati and Lin (2004) and Bragg (2017) used variation theory in action research projects aimed at developing teachers' ability to notice tasks' features. Variation theory is widely used in designing tasks for teachers' development or students learning, as mentioned by the studies presented next. Koichu et al. (2015) used variation theory to design tasks aimed at teachers' learning. Pang and Ling (2012) demonstrate the power of variation theory as a pedagogical tool in planning lessons and predicting the necessary

changes to foster learning. Kassa and Ding (2019) use variation theory to show how designing and implementing tasks using patterns of variation and invariance can improve students understanding of critical features of an object of learning.

Applications of variation theory are not limited to mathematics but it is also used in other fields, for example business (Cheng, 2016) and language teaching (Holmqvist et al., 2008).

According to Askew (2012, p. 74), one of the strengths of variation theory is that it maximises the chance of closing the gap between the intended object of learning and the lived object of learning. However, some limitations of variation theory have been acknowledged by researchers in education. Lam (2013) highlights a potential problem with the assumption that something has to vary in order to be learnt and gives examples where students learn the invariant aspect rather than the variant ones, as the theory suggests. He proposes that maybe a complementary theory of invariance might be a solution to this problem.

Bussey et al. (2013) claim that variation theory has certain limitations since it fails to account for two factors that might influence the intended and enacted object of learning: prior students' knowledge and instructional materials. Even more, these factors could have an impact on the studies informed by variation theory as a framework, as well on the type of research questions used in these studies. Following a constructivist approach, which posits that new understandings result from integrating new and prior knowledge, they argue that the features of the enacted object of learning and prior knowledge both contribute to the creation of the lived object of learning. They suggest that any research design informed by variation theory should include some methods of assessing students' knowledge before and after the learning event, creating a *pre-lived* and *post-lived object of learning*. A study carried previously by Guo and Pang (2011) confirms that students exposed to the same instruction perceive different critical features and

thus, attend to different things, depending on their prior knowledge. In the context of language learning, Holmqvist et al. (2008) also find that previous knowledge influences the outcomes of a lesson designed using variation theory.

Another issue raised by Bussey et al. (2013) is that by using instructional materials created by a third party, teachers might be unaware that these resources incorporate features that might have an unintentional influence on the object of learning.

These points are relevant for this study as I will look at how teachers use students' previous knowledge and whether they are aware of the cognitive demands of the tasks they select from published materials. As a Teacher Educator, it is also essential to be aware of the potential limitations caused by using variation theory and take the necessary steps in my research to counter these issues.

In summary, variation theory has proved to be a powerful tool that can be used in task design, teachers' development and to improve students' learning.

2.2 Awareness

Marton and Booth (1997, p. 108) try to define awareness in terms of experiences. They explain that experiencing something is being aware of certain aspects of that *something*. They further claim that at one given point, we are aware only of some aspects or things while others receded to the background. Since we have different things in our focus of attention at different times, our awareness structure is changing, so we have different experiences. Hence, they assert that "a person's awareness is the world as experienced by the person."

In the particular case of teachers, Smith and Kleine (1969) propose different categories:

> Teacher awareness is the broadest category and can be discriminated into academic and social categories; or into object, person, and social categories; or into symbolic, figural, and behavioural categories. (p. 253)

Using van Hiele's levels, Mason and Johnston-Wilder (2004, p. 60) build a hierarchy of awareness based on the ability to recognise properties and making connections between them. Differences in levels of awareness can be applied in education to explain the differences between beginner and experienced teachers: "what distinguishes expert from novice behaviour is the degree of complexity of awareness that can be sustained moment by moment and the richness of possible choices that come to mind" (Mason and Davis, 2013, p. 190). This implies that while *knowing more* by accumulating more facts is an essential component, combining it with *noticing more* would result in knowing in a richer way, giving the ability to respond in the moment by using mathematical, pedagogical or didactical actions. "This is implied by educating awareness" (ibid., p. 192).

Mason and Johnston-Wilder (2004, p. 161) assert that we can learn by becoming aware of an object's features and the relationships between them. Thus, we can learn not by memorising mechanically but by educating awareness.

Building on Gattengno's definition that awareness is "that which enables powers," Mason (1998) suggests three levels of awareness relevant to an educational context.

- First level is awareness-in-action, which refers to having the power to act.
- Awareness-in-discipline is the awareness of awareness-in-action. People at this level of awareness have the ability to act in a certain way and can explain why that is possible.

- Awareness-in-counsel refers to awareness of awareness-in-discipline and is achieved when people have the necessary level of self-awareness to notice what others require to build their awareness-in-action or shift to awareness-in-discipline.

This study focuses mainly on building participants' awareness-in-discipline with regards to the level of cognitive demands of tasks. Using Stein's et al. (1996) levels of cognitive demands of tasks as a framework, they will be able to identify which tasks are more difficult than others and explain *why* this is the case.

2.3 Tasks

2.3.1 Definitions and importance

In this section I will look at what is meant by a mathematical tasks, why they are important, and what principles can help teachers design more challenging lessons.

According to Doyle and Carter (1984) a task "is used to designate the situational structures that organise and direct thought and action" (p. 130). In educational settings, an *academic task* is characterized by three features: the goal that students are expected to achieve, the resources available, and the set of operations students need to perform to produce the desired result. They "form the basic unit of treatment in classrooms" (Doyle, 1983, p. 162). Stein et al. (1996) define a *mathematical task* as "a classroom activity, the purpose of which is to focus the students' attention on a particular mathematical idea" (p. 460). Thus, a task can be a single activity, a question or a set of problems (Boston and Smith, 2011). In this research, I extend the definition given by Stein et al. (1996) to include also the questions and problems assigned for outside the classroom time. The particular feature of tasks that is investigated in this project is the level of

cognitive demand, which is defined as “the kinds of thinking needed to solve tasks” (Stein et al., 2000, p. 3).

In a study across seven countries, Hiebert et al. (2003) found that students spend over 80% of the classroom time engaged in mathematical tasks. Consequently, students perceive what is meant by doing mathematics through the lens of tasks offered by their teachers (Stein et al., 1996; Boston and Smith, 2011; Jones and Pepin, 2016). The amount of time allocated to tasks reveals the vital role they play in mathematical instruction. A considerable body of research have been devoted to classifying them or studying their impact on learning.

One way tasks can be identified is by the purpose they are used for: learning new skills and knowledge, reviewing previously acquired knowledge, or assessing the understanding of this knowledge. “Tasks influence learners by directing their attention to particular aspects of content and by specifying ways to process the information” (Doyle, 1983, p. 161). Hence, by engaging with different tasks, students learn different things. When learners are required to memorise a text, they will be able to reproduce the words without necessarily understanding the text. To go beyond this surface aspect of it and understand the meaning of that text, a comprehension task is needed (ibid., p. 163). Stein and Lane (1996) claim that students develop a better understanding if they are engaged in tasks with a high level of cognitive demand. Furthermore, there is evidence that students exposed to challenging tasks perform better, even if the tasks’ difficulty is reduced during implementation (Boston and Smith, 2011). In the US, National Council of Teachers of Mathematics acknowledges that “effective, worthwhile tasks are used to introduce important mathematical ideas and to engage and challenge students intellectually” (NCTM 2000, p.18).

While appreciating the usefulness of challenging tasks, Boston and Wolf (2006) point out that tasks with a low-level of cognitive demand are not necessarily "inappropriate or bad instructional tasks" but should be judged in relation to the lesson's goal.

Doyle and Carter (1984) notice that tasks include an element of *ambiguity*, measured by the difficulty to implement well-established routines, and an element of *risk*, assessed by the likelihood that the tasks will be completed. It follows that tasks do not offer the same learning opportunities and students' learning is afforded or constraint by the information and operations necessary to complete the task (Doyle, 1983). In variation theory terms, learning is facilitated by the dimensions of variation opened to students, and the ranges of change permitted to them. (Kullberg et al., 2017; Watson and Mason, 2006).

As this project aims to increase awareness about the tasks' level of difficulty, I will present below two frameworks that describe the cognitive demands.

2.3.2 Academic tasks framework

A considerable amount of Walter Doyle's work in the 1980s involved examining the intellectual processes required by academic tasks and how these tasks are enacted in the classroom. Acknowledging that different tasks involve thinking and acting in different ways, Doyle (1983) identifies cognitive demand as one way to group tasks. He suggests that tasks can be divided into four categories: memory tasks, procedural or routine tasks, comprehension or understanding tasks, and opinion tasks. When students have to replicate content they learnt before, such as mathematical formulae, they perform a *memory task*. If they have to apply an algorithm to produce an answer, then they are engaged in a *procedural task*. In the case of mathematics, this

can be exemplified by doing long division. *Comprehension tasks* involve using prior knowledge and applying procedures in new contexts, choosing or providing a new, alternative method to solving a problem. The last type, *opinion tasks*, is defined as "tasks in which students are expected to state a preference for something (e.g., select a favourite short story)" (ibid., p. 163).

2.3.3 Mathematical tasks framework

Drawing on Doyle's work, Stein et al. (1996) conducted an extensive study on tasks capacity to build students' mathematical powers in a reform-oriented context. The focal points of the project were tasks' features and cognitive demands. In their view, task's features were understood as those aspects which engage students in reasoning and explaining. The team analysed 144 tasks in terms of the number of methods to solve it, how many representations can be used, and the type of explanations and justifications required. Inspired by Doyle's classification of tasks according to their cognitive demands, Stein et al. (1996) grouped tasks as follows:

> "memorization, the use of formulas, algorithms, or procedures without connection to concepts, understanding, or meaning; the use of formulas, algorithms, or procedures with connection to concepts, understanding, or meaning; and cognitive activity that can be characterized as "doing mathematics," including complex mathematical thinking and reasoning activities such as making and testing conjectures, framing problems, looking for patterns, and so on." (p. 466)

Tasks involving actions such as memorization or applying an algorithm (*procedure without connection*) are considered low cognitive demand tasks. At the other end of the spectrum, we have tasks with high level of cognitive demand, which require students to make connections between different areas of mathematics or select and justify their strategy.

2.3.4 Tasks in curriculum

Textbooks represent one of the primary resources that teachers turn to when planning for instruction. Researchers analysed tasks in the curriculum material from various perspectives. Some studies focused on sequencing or their cognitive demand (Baker et al., 2000) while others explored the alignment with the official standards (Porter, 2006).

Baker et al. (2000) did an extensive research about the changes in the content of elementary textbooks published in the US between 1900 and 2000. They found that while there was a drop in the tasks' level of difficulty towards the middle of the century, significant changes occurred during the second half, with more topics being added to the curriculum and more demanding strategies being required. However, they acknowledge the caveat that this trend represents the intended curriculum, not the enacted one. Examining the American middle school textbooks published during a period of 50 years, Jones and Tarr (2007) report a similar trend. Inspecting the tasks using the Task Analysis Guide (Stein et al., 1996), they find that most of the probability tasks published in the later years required high level of cognitive demand, as opposed to the ones published mid-century. Sears and Chavez (2014) looked at how geometry textbooks provide students with the opportunity to engage with proof tasks. They conclude that only a small proportion of tasks require rigorous proofs and textbooks offer excessive guidance, reducing tasks to "filling in the blanks". An extensive study was conducted by Stylianides (2009) to investigate the number of proof tasks in American textbooks. The results from analysing over 4000 tasks show that only about 40% of the tasks promoted reasoning and proving skills, considered cognitively demanding. The research also surveyed the distribution of proof tasks across different chapters. Tasks were grouped according to the mathematical content they address and then compared the proportion of *reasoning-and-proving* tasks in each of these

groups. The findings show that number theory chapters have more tasks involving proofs than geometry, while algebra offers little opportunities to engage with such tasks. Stylianides calls for a more balanced distribution of the *reasoning-and-proving* tasks, since these activities are present in all areas of mathematics. Mesa et al. (2012) coded 488 examples from algebra textbooks used in community colleges in the US and found that 91% of them were procedures without connection and only 2% involved doing mathematics.

While the studies above focused only on one geographical area, other comparative studies extended their investigation across multiple countries. Data collected by Zhu and Fan (2006) from the US and Chinese textbooks revealed that the tasks in both series were similar in many ways, for example when comparing proportion of open-ended to closed-ended questions, or when looking at their form of presentation. However, in terms of number of steps required to solve a problem, the US textbooks were less challenging than the Chinese ones.

Wang et al. (2017) explored how linear functions topic is treated in the UK and Shanghai curriculum. Testing students using mostly tasks from the textbooks, they conclude that while students from England made more calculation errors, their Chinese counterparts struggled with the visual representations.

These studies provide valuable information for me as, in my context, I work with Chinese students following a western curriculum. These findings help me to become more aware of the needs of our students and teachers.

2.3.5 Task unfolding

In their extensive research on tasks' features and cognitive demands, Stein et al. (1996) use the Mathematical Task Framework from the QUASAR project to describe the evolution of a task from one stage to another and various factors influencing them (Fig. 1). They identify as critical the moment in time when the task is announced to the students; this point marks the transition from a set-up phase to an implementation phase.

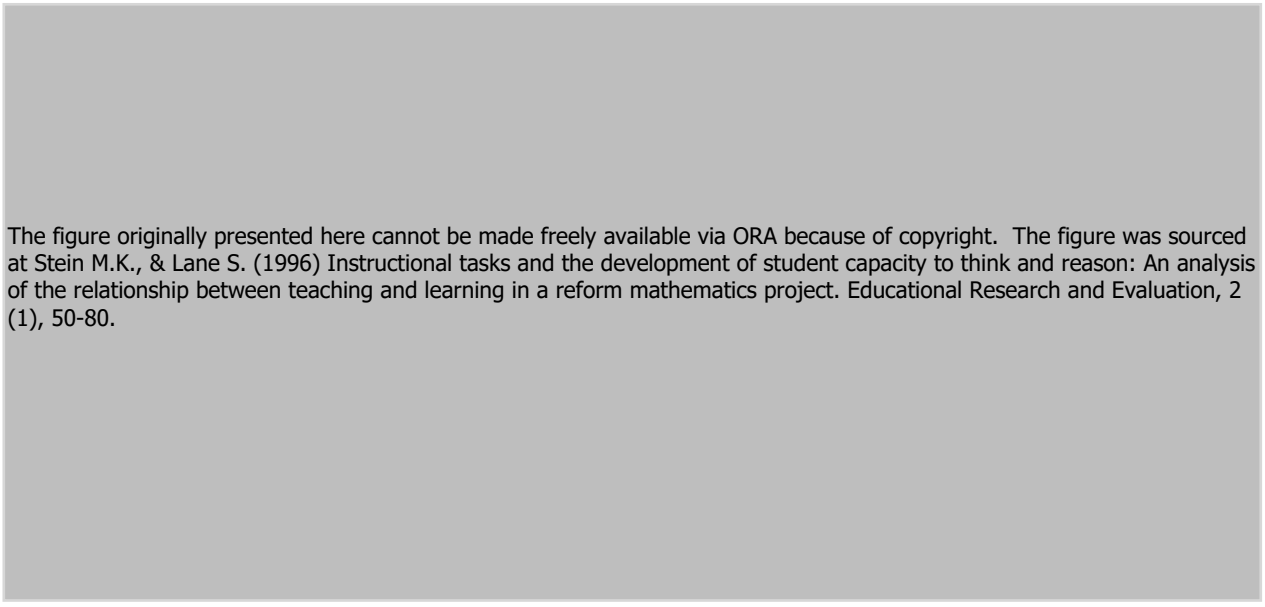


Fig. 1 Mathematical Tasks Framework (Stein et al., 1996)

As we can see from the diagram above, tasks have a dynamic nature as they can change their character under the action of different variables. The development of a task from selection to enactment is well documented by other researchers, as I will show below.

Tasks are usually selected from the curriculum material but often the task as intended by the publisher is not congruent to the task as intended by the teacher. Differences might be caused either unintentionally, for example teachers not attending to the same thing as the original author

(Jones and Tarr, 2007) or by teachers modifying it (Jones and Pepin, 2016). Reasons for altering the task could be related to many things, such as teachers' beliefs or the need to fit teachers' goals for students' learning (Stein et al., 2000). To match the task with the learning needs implies that teachers should be aware of the cognitive demand (Stein et al., 2000). Once it is announced to class, various conditions shape the unfolding of the task during the implementation phase. These conditions might include students' instructional habits or classroom norms (Doyle, 1983), students' responses or teachers' knowledge (Charalambous, 2010).

Of particular interest to this project is the development of cognitive demand during the task's life. Charalambous (2010) suggests that considering the high-low level classification of tasks, and three stages in their evolution, there are eight possible paths for tasks' development. An example of five paths he identified in his study can be seen in Appendix 3.

Analysing the evolution of 144 tasks, Stein et al. (1996) report that tasks starting as high-level of cognitive demand tend to decline to procedure without connection during implementation. Among the factors identified as contributing to the decline of tasks were: students' pressure to reduce the difficulty, teachers' intervention by solving the challenging part or students' disengagement due to lack of motivation. These findings confirm Brousseau's claim that teachers and students must respect a *didactical contract* for learning to take place (Brousseau, 1997).

In a case study involving two teachers, Charalambous (2010) finds similar results with Stein et al. (1996) and notice that teachers' knowledge can influence the way the task is implemented in class.

Henningsen and Stein (1997) analysed the classroom-based factors that contribute to students' engagement with high-level tasks and suggest various solutions to avoid the decrease of tasks'

difficulty during implementation. Their recommendation include strategies such as: allowing sufficient time for the unfolding of the tasks, designing tasks that build on students' prior knowledge, and providing appropriate scaffolding.

Stein and Kim (2009) identify two main reasons for teachers' difficulty to implement high-level cognitive demand tasks. First, teachers who were taught in a traditional manner might lack the knowledge of dealing with a more investigative approach of the lesson. Second, implementing unusual tasks might require students to work in groups and discuss ideas, leading to classroom management problems. While classroom management is not an issue in my context, the first reason mentioned above is something that I need to consider in this project.

Smith et al. (2008) offer a framework to support teachers in designing lessons that include tasks with increased difficulty. Their "Thinking Through a Lesson Protocol" TTLP (Appendix 4) provides a set of questions that teachers could ask at each of the three stages: selecting and setting up the task, supporting students while exploring the task, and sharing and discussing the task.

Jackson et al. (2012) examined the conditions that are conducive to effective implementation of tasks. They recommend that when launching the tasks, teachers should consider the following four critical elements: key contextual features, key mathematical ideas, develop an appropriate vocabulary that allows students to express their opinion about the key features, and finally, maintain the cognitive demand.

2.3.6 Task design

Designing tasks is a useful skill for educators as it frees them from the dependency on prescribed materials. Alan Bell was one of the first researchers to focus on principles for designing mathematical tasks. He suggests that students can experience "genuine and substantial mathematical activities" if teachers follow certain principles when preparing the learning activities (Bell, 1993). He further specified the following set of implementation principles:

> "First one chooses a situation which embodies, in some contexts, the concepts and relations of the conceptual field in which it is desired to work. Within this situation, tasks are proposed to the learners which bring into play the concepts and relations. It is necessary that the learner shall know when the task is correctly performed; hence some form of feedback is required. When errors occur, arising from some misconception, it is appropriate to expose the cognitive conflict and to help the learner to achieve a resolution. This is one type of intervention which a teacher may make to assist the learning process. "(ibid., p. 9)

Bell also recommends that tasks should be flexible, meaning that changes can be made to its components. This would enable teachers to fine-tune the intensity of learning by adjusting the degree of challenge. This resembles the idea of selecting the dimensions of possible variation and changing their range to create a learning experience. (Watson and Mason, 2006).

In an attempt to resolve the conflict between the students as consumers and producers of knowledge, Krainer (1993) introduces the concept of *powerful tasks*, which are units characterised by the following principles:

- Tasks are interconnected.
- They have a dynamic structure, meaning that they can facilitate further questioning.

- Initiate active processes of concept formation, accompanied by concept generation actions.
- Tasks foster reflection resulting in further questions from students.

Kieran et al. (2015) offer an overview of how design frameworks have developed over the years and how they can be classified using different criteria, such as the theories underpinning them or whether they focus on student learning or assessment.

Relevant for this particular study is the cognitive demand of tasks and variation theory. As seen above, researchers in the field propose some guidelines for planning lessons to include tasks with a high level of cognitive demand (Smith et al., 2008; Jackson et al., 2012).

Since paragraph 2.1 described variation theory in more detail, I will summarise here a few principles derived from this theory. The main idea is that variation must occur for learning to take place (Marton and Booth, 1997). Task designers must identify the dimension of variation and their associated range of change that allow learners to distinguish the intended object of learning. Following specific patterns of variation, teachers maximise the chance that learners discern the critical features of the intended object of learning.

2.4 Teachers' knowledge

It goes without saying that knowledge is a vital component in the process of teaching. Zodik and Zaslavsky (2008) assert that "what is taught and how it is taught" (p. 167) is dictated by the standard of teachers' knowledge.

As shown in the previous section, the way tasks are set up at the planning stage is influenced by teachers' knowledge (Charalambous, 2010). A deep understanding of mathematics also increases their ability to act-in-the-moment during the lesson (Mason and Spence, 1999).

Shulman (1986) asserts that mathematical content knowledge is not enough to be an effective teacher. He argues that special types of knowledge are necessary for teaching and suggests that knowledge should be divided into three categories: subject matter content knowledge (SMK), pedagogical content knowledge (PCK), and curricular knowledge (CK). Of most relevance for this project is PCK, which includes:

> "the most useful forms of representation of those ideas, the most powerful analogies, illustrations, examples, explanations, and demonstrations-in a word, the ways of representing and formulating the subject that make it comprehensible to others" (Shulman, 1986,p. 9).

In a later contribution to the field, Shulman (1987) introduces the concept of teacher's knowledge base and expands his list by adding four more knowledge categories related to classroom management, educational context, learners, and educational purposes and ends.

One aspect of *knowledge of learners* is described by Zodik and Zaslavsky (2008) as "teachers' awareness of the consequences of students' over-generalizing or under-generalizing from examples, and students' possible tendency to notice irrelevant features of an example instead of attending to its critical features" (p. 167). This resonates with this project's goal to increase teachers' awareness of critical features by using variation theory.

Building on Shulman's work, Ball, Thames, and Phelps developed a framework for mathematical knowledge for teaching (MKT) which they define as "the mathematical knowledge needed to carry out the work of teaching mathematics" (Ball et al., 2008). Similar with

Shulman's original categories, MKT is divided into subject matter knowledge (SMK) and pedagogical content knkowledge (PCK). Within subject matter knowledge, they differentiate between mathematics as used by people in everyday life and the mathematics needed for teaching only. The first subcategory is named common content knowledge (CCK) and the second one is specialized content knowledge (SCK). The third subcategory of SMK is called horizon content knowledge (HCK) and refers to how topics are connected. PCK is also further refined into three subdomains: knowledge of content and students (KCS), knowledge of content and curriculum (KCC), and knowledge of content and teaching (KCT). Being aware of students' common misconceptions in an example of KCS, while sequencing tasks could represent KCT. This classification is summarised in the diagram below.

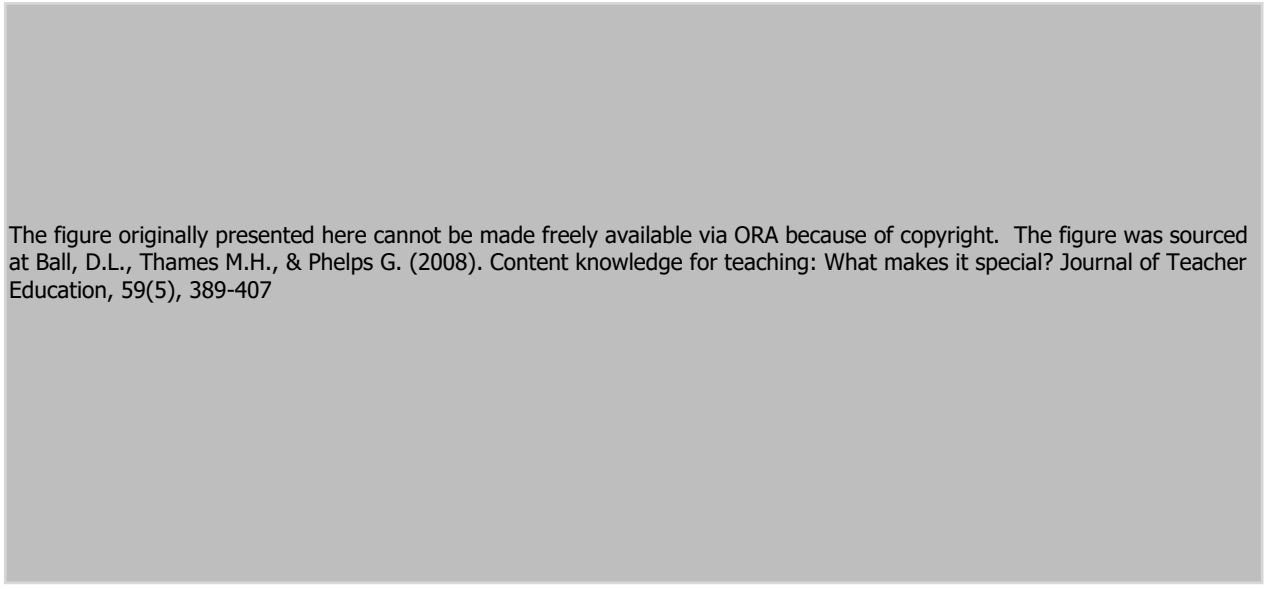


Fig. 2 (Ball et al., 2008)

However, since Ball et al. developed these constructs in a primary school context, some researchers query their validity when applied at a secondary or college level. This argument finds support in a study carried out by Speer et al. (2015), who claim that the definitions given by Ball

et al. are dependent on the context they are applied in. They provide the following example: "recognizing the mathematical accuracy of a definition, considered part of SCK for elementary teachers, is CCK for those with more mathematics education" (p.114).

Other researchers also sought to develop frameworks for mathematics teachers' knowledge. In a study involving preservice primary school teachers, Rowland et al. (2005) developed a list of 18 codes to analyse how SMK and PCK contribute to mathematics teaching development. These codes were then into four units: foundation, transformation, connection, and contingency, to create the Knowledge Quartet (KQ) framework. This new framework has proved to be an efficient tool in facilitating discussions between teacher educators, mentors, and mentees.

The foundation unit comprises teacher's beliefs and knowledge as acquired at university level. The transformation unit is defined as teacher's ability to transform their SMK by using examples and representations. Knowledge about connections between concepts and sequencing tasks are examples of codes that contribute to the connection dimension. Contingency unit refers to teacher's ability to answer to unexpected situations. This resonates with the concept of knowing-to act in the moment found in Mason and Spence (1999).

There are many overlaps between the categories proposed by Ball et al. (2008) and the four dimensions in the KQ. For example, CCK and SCK can be found in the foundation unit, while the transformation dimension can reveal KCT. The connection unit includes HCK, but the contingency unit can consist of all the categories due to its complexity. However, while the KQ is useful in getting information about the knowledge in use, it does not evaluate this knowledge. Weston (2013) and Liston (2015) proposed adaptations of this model so that data can be quantified.

2.5 Teachers' beliefs

It is generally agreed that teachers' knowledge plays a vital role in teachers' practices (Shulman, 1986; Ernest, 1989). However, there are reasons to believe that in the mathematics context, teachers' behaviour might be influenced by their beliefs (Thompson, 1994). While there is no agreement between researchers what is actually meant by the term beliefs (Furinghetti and Pehkonen, 2002), in this study, the notion of beliefs "refers to anything that an individual regards as true" (Beswick, 2005). This differentiates beliefs, as a subjective idea about something, from knowledge, which is something accepted as true by the wide community.

Ernest (1989) suggests that teachers' beliefs can be divided into beliefs about the nature of mathematics, beliefs about teaching mathematics and beliefs about learning mathematics.

The beliefs about the nature of mathematics are further divided into three categories, from a philosophical point of view. The first one is the Instrumentalist, who views mathematics as comprising of unrelated rules and facts. Platonists form the second category. They view mathematics as a unified body of knowledge that exists *apriori*; thus, it is discovered, not created. People in the last class have a problem-solving view of mathematics and consider it a dynamic construction, constantly under revision.

Ernest (1989) summarizes the connections between teachers' knowledge, beliefs, attitudes, and practices as follows:

- Instrumentalist: the teacher act as an instructor and is focused on content and performance; mathematics learning means passive reception of knowledge and skill mastery.
- Platonist: this teacher is an explainer. He is also focused on content but emphasises understanding, seeing learning as active construction of understanding.

- Problem-solving: being more student-focused, the teacher takes on a facilitator's role and allows them to explore their interests.

However, Philipp (2007) noticed that relations between beliefs, conceptions and attitudes are not generally agreed upon and researchers are not using them in a consistent way.

Many research studies show the influence of teachers' beliefs on their practice (Thompson, 1994; Stipek et al., 2001; Polly et al., 2013).

Studying the links between the students' learning outcomes and teachers' knowledge, beliefs and practices, Askew et al. (1997) distinguish three types of belief orientation:

- connection – teachers make connections between different areas of mathematics, they believe that students learn through challenging tasks that involve conjecturing, reasoning, and justifying. They also allows more autonomy to students to collaborate;
- transmission – teachers believe that students learn by following sets of rules and procedures to get the correct answer. They focus on one part of the curriculum at a time.
- discovery – teachers with this orientation believe that students learn better by discovering methods themselves.

The results of their study showed that students taught by teachers with a *connectionist* orientation were more likely to have higher learning gains than those in classes with a discovery or transmission orientation.

Smilar findings were reported by Stipek et al. (2001). In a study conducted with 21 teachers, they found that teachers who see mathematics as a set of rules to be learnt focused more on getting the correct answers and allowed little autonomy to students. On the other hand, those who

considered mathematics to be a tool for thought emphasised understanding more, took a more constructivist approach and gave students more freedom to explore.

An important observation made is that teachers in the first category rely heavily on textbooks and prescribed activities, which might mean that they have lower confidence in their mathematical knowledge. This observation has implications for me as a teacher educator, as it might be something I encounter in my context.

Polly et al. (2013) extend their study from looking at the connection between beliefs and practices to examining their relation to students' achievement. They report that students from classes with a teacher-centered approach achieved significantly lower grades in curriculum-based assessments than students from classes with a student-centered approach.

In her study about teachers' beliefs regarding mathematics as a school subject and as a discipline, Beswick (2012) shows that there could be incongruences between the espoused and enacted beliefs. Skott (2008) finds similar discrepancies and presents a case where the enacted beliefs are influenced by the context in which the teacher works.

More research can be found about how teachers' practice is influenced by beliefs regarding technology, gender, students' ability etc. but I considered them as not being so relevant for this study.

As we can see from the research above, beliefs might play an important role in the way teachers plan and enact their lesson, which is something that needs to be considered when analysing their activity.

2.6 Measuring growth

While professional development programs are supposed to improve teachers' knowledge and skills, Clarke and Hollingsworth (2002) claim that the traditional approach of one-shot workshops are inefficient. They suggest that along with Teacher Educators' intervention, many other factors contribute to facilitating or impeding changes in practice. Based on empirical data, Clarke and Hollingsworth designed the Interconnected Model of Professional Growth - ICMPG (figure 1) as a tool to be used in analysing how changes take place in time by looking at the interaction between four different domains: external domain, personal domain, domain of practice, and domain of consequence. A change in one domain may trigger a chain of reactions that might lead to changes in other domains. Interactions between different domains are mediated by reflection and enactment. Given the connections between different domains, the patterns of changes are not necessarily linear or circular.

This model can a very useful tool for Teacher Educators in examining the effect of their intervention and measuring teachers' growth.

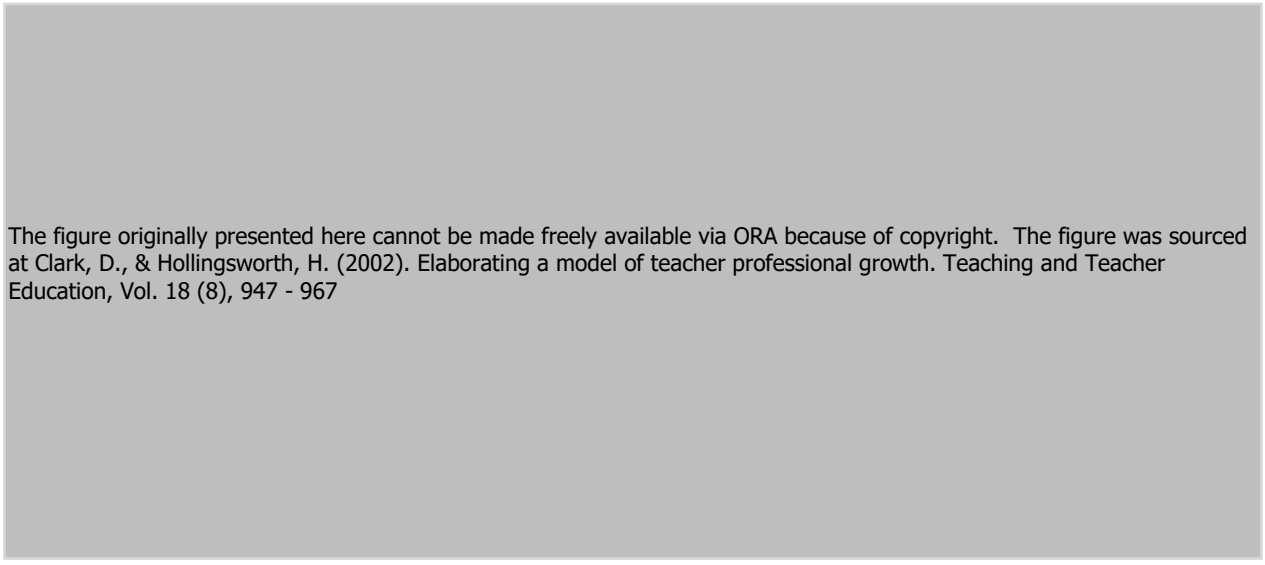


Figure 1. Interconnected Model of Professional Growth – Clarke and Hollingsworth (2002, p. 951)

3. Methodology

This study describes a professional development program for secondary mathematics teachers which is focused on task design. This investigation allows me to determine how the participation in such program can increase teachers' ability to assess the level of cognitive demands of mathematical tasks and how it can influence teachers' selection of tasks for lessons. The chapter begins with a description of the goals and context of the study. It then explains the procedures employed, including selection of participants, ethical considerations, and justification of action research choice. The last part of the chapter comprises the program phases and a discussion of how data was collected and analysed.

I would like to point out that since the school is in China, the unfolding of this research project was severely disrupted by COVID-19 pandemic. The school closed in December for the Christmas holiday, then opened for a two-week examination period in January and then stayed closed until August for on-campus lectures. That meant that the timeline of the project had to be adjusted accordingly and the amount of data that could be collected was seriously limited..

3.1 Research questions

The goal of this project is to increase teachers' awareness about the level of cognitive demand of mathematical tasks. In this study I will look at teachers' instructional practices with respect to selection of mathematical tasks and I will also analyse changes in teachers' ability to identify the level of cognitive demand. The aim of this research is to answer to the following questions:

1. What is the source and level of cognitive demands of tasks that teachers employ in their instruction?
2. What is the teachers' ability in identifying the level of cognitive demand of mathematical tasks and how does it change after participation in a professional development program focused on tasks' cognitive demands?
3. How does teachers' engagement in a professional development program using variation theory improve their ability to design tasks of different difficulty levels?

3.2 School context

This research project was undertaken at a mixed private school in China, where I work as a Maths teacher and Teacher Educator. The school offers A-level courses for Chinese nationals looking to further their studies at top universities abroad. Mathematics is the most popular subject in school, and all students take A-level Mathematics, with about half of the students choosing to continue with Further Mathematics every year. For the June 2019 session 255 students took the Mathematics A-Level exams and 65.5% of them attaining A*-A grades (85.7% A*-C). In Further Mathematics, 125 students sat the exams and their grades were 37.6% A*-A (83.2% A*-C). Since over 50 students apply for Oxbridge every year (yielding around 15 offers), A-levels are not the only tests this students take. They also prepare for interviews and exams specific to Oxbridge, which are much more difficult than the regular A level exams. This reflects the need to have highly qualified teachers who can develop students' mathematical thinking. The 17 teachers in the Mathematics department have a diverse educational background and their teaching experience ranges from beginners to 18 years.

3.3 Participants

The teachers participating in this project were recruited from the Mathematics Department of the secondary school. The project was presented during a departmental meeting, and teachers had the chance to ask questions and raise any issues they might have regarding the process. Initially, eight teachers agreed to take part in this professional development program. However, throughout the year, two teachers dropped out due to other commitments. Two teachers left the school unexpectedly, so data collected from these four teachers was discarded and not used in the current study. All teachers agreed to the pre and post-intervention interviews, lesson observations, and collection of artefacts. However, some teachers did not feel comfortable with too many class observations. Due to ethical issues, we agreed on having only one lesson observations per teacher, at the beginning of the project.

The participants' teaching experience ranges from 6 to 20 years, and their educational backgrounds include mathematics, finance, and computer science.

3.4 Ethical considerations

Special attention was given to the potential ethical issues raised by this research project. As published by the British Educational Research Association (BERA, 2018), the ethical guidelines for educational research were followed throughout the whole project. Before the start of the research, ethical approval was requested from the Central University Research Ethics Committee (CUREC) at the University of Oxford. The research ethics clearance procedure involved submitting an application form where details were given about the nature of the research, description of the methodology, recruitment of participants and use of data. The first step after

receiving the approval from CUREC was to inform the Head of School about the project and get written consent from him. The project's presentation, including why the teachers' presence is essential, and how data will be collected and used, were done during a departmental meeting. To ensure teachers take part in the project on a volunteer basis, a series of steps were taken. A special emphasis was placed on the fact that teachers have absolutely no obligation to take part in it, and there is no consequence for not doing so. Also, it was made clear that anyone can participate and withdraw their participation at any time without the need to explain it. To allow the potential participants some time to get more information about the program and also to avoid any peer pressure, the confirmation for participation was not asked on the spot but a window of one week was given. All teachers were provided with contact details of the Department of Education, where they could raise any concerns regarding this research project. To ensure the participants' privacy, pseudonyms replaced the real name. The chosen four names Alan, Bianca, Candy and Daniel were randomly assigned to participants. All the data was digitalised and stored on the Nexus One Drive, being available only to myself and my supervisor, and will be deleted at the completion of the MSc degree. Special considerations were taken to reduce the tensions and balance the power relationship between me as a researcher and the teachers. In order to minimise the distance between me as a teacher educator and the participants, assurances were given that information gathered from persons involved in this project will not be shared with the school management. All individual meetings took place in a neutral, welcoming, and safe space. Any extra work generated by participating in the project was mitigated by the professional development gained and was recognised as such by the school management. The project was not sponsored and there were no incentives for taking part in it.

3.5 Choice of design: action research

Finding the right methodology for this investigation was not a straightforward process. The initial plan was to take a learning study approach (Pang and Ling, 2012) as it involves a close collaboration between the participants, which was one of the program's aims. However, some teachers did not feel comfortable being under the scrutiny of their colleagues. Due to ethical considerations, this methodology had to be dropped and the program was redesigned in a way that would still keep collaboration at its heart but would eliminate the involvement of other participants, except for the researcher in lesson observations.

After reading the specialist literature to find the most suitable research methodology, it was decided that action research fits best the new design of the project:

> "Action research is critical (and self-critical) collaborative enquiry by reflective practitioners being accountable and making the results of their enquiry public, self-evaluating their practice and engaged in participatory problem-solving and continuing professional development." (Zuber-Skerritt, 2005, p. 86)

Some potential issues had to be clarified before starting the project. One of them was the extra workload involved by participating in this study. It was agreed with the school management that the time spent in this project would be recognised as professional development and would relieve teachers from the obligation to participate in other programs as the meetings would take place during the already allocated slots for professional development. Having already faced teachers' anxiety to be put in the spotlight, I had to make sure that participants are under no pressure and contribute only when they feel comfortable doing so.

This study has a practical nature (Denscombe, 2010) as it aims at improving the teaching methods and the use of resources in our school. This improvement comes as a result of a change, which is a critical feature of action research (ibid.). Participants in this study "take an active and thoughtful approach to changing themselves, their practices and the conditions under which they practice" (Kemmis et al., 2014). This project also seeks to contribute to the participants' personal growth by improving their pedagogical knowledge. Using their newly acquired pedagogical knowledge, teachers might change the way they select and implement tasks in lessons. A change in the conditions that enable or constrain their practice could be a shift in the resources they use and the redesign of tasks from the traditional resources they were using. All these changes further emphasize that action research is a "practice-changing practice" (ibid.). In order for changes to occur, action must also be complemented by reflection (McNiff, and Whitehead, 2002). Although this resembles the "reflective practitioner" (Schon, 1994), action research is distinguished by the fact that research, action and reflection is done in a systematic way, following a cyclic process of *plan-act-reflect* (Denscombe, 2010; Kemmis et al., 2014; McNiff, and Whitehead, 2002). In their "action research spiral" (Appendix 1), Kemmis et al. (2014) place reflection as the critical step to move on to the next stage. While many researchers view reflection as a solitary action, Kemmis et al. argue that self-reflection in a more collaborative way is desirable.

In this project, the *plan-act-reflect* cycle of action research is followed but a certain degree of flexibility is necessary to be able to respond to unpredicted situations (Burns, 1999). Participants collaboratively design tasks which then they implement in their lessons. By reflecting on the results both individually and as a group, teachers further refine these tasks and the way they are implemented, in order to improve their practice. By sharing ideas and thoughts, the participants

have the opportunity to grow and learn from other people's work (McNiff, and Whitehead, 2002). One of the advantages of using action research is that the researcher and the practitioners work alongside, as there is no distinction between them (ibid.). Being part of the team gives the researcher the opportunity to get a close view on how teachers think and how they develop in time and it also allows him to decide on what is the best course of action. Collaborative action research encourages teachers to openly discuss common problems in the organisation or their practice, therefore being potentially more efficient than individual action research (Burns, 1999). By engaging more closely with their practice, teachers become more aware of their strengths and weaknesses, which is one of the aims of this study. The cyclic nature of action research allows for building a record of change in practice, language, and discourse, enabling the researcher to document progress (Cohen et al., 2007). A personal benefit for the practitioners who take part in this action research is the professional development they get by learning about some aspects of education theory, such as variation theory and cognitive demand.

While action research has many advantages, some impediments must be acknowledged as well. Given the small scale of this project, its validity to generalise findings can be argued. However, its rigour makes it a suitable methodology for a situated problem (Cohen et al., 2007). Its internal validity is further strengthened by using the triangulation method in collecting data (ibid.), thus reducing participants' bias. As a researcher and participant in this project, I am also subject to bias. To minimise any potential bias, special consideration was given that all data collected was analysed without trying to get a predefined result.

In conclusion, action research is a suitable research design to address the research questions since it allows identifying the problems and provides a space and time for reflection, which can

lead to increased awareness. Through collaboration, it gives the participants the possibility to act as agents of change for others.

3.6 Stages of the study

3.6.1 Pre intervention

This project consisted of two initial phases, where data is collected and teachers are introduced to the theory, and a recurrent phase where teachers analyse their activity and act with the aim of improving their practice. .

Lesson observation – tasks collection - interview/source of tasks – sorting task activity – Workshop1 (Variation Theory) – Workshop 2 (Levels of Cognitive Demand) – chapter analysis – task collection – sorting activity – collaborative planning – act – observe - reflect

The last phase represent the action research spiral (Appendix 1).

Setting the scene

The idea to conduct this study was sparked by my interest in teacher education and also by the school management's desire to increase the number of students accepted in Oxbridge and have better prepared teachers. Before selecting the participants, a meeting with the school principal and the Head of Mathematics was arranged to discuss a potential focus for the project. Analysing the current situation and taking into consideration the school's interest, I suggested that a focus on cognitive demands of tasks would be beneficial for both teachers and students. This is in line

with researchers' recommendations that professional development programs should be topic specific (Lock et al., 2009).

From my previous experience as a teacher in this school, I knew there was a discrepancy between my practice and what students are used to. Acting according to my beliefs, I was not limiting myself to exam materials. Working with so many gifted students gave me the opportunity to work on more difficult tasks and discuss topics that go beyond the curriculum. However, initially, many students were mainly concerned with getting high scores in the final exams and considered unnecessary any work beyond what is required in the exams. Eventually, after I insisted on dedicating part of the lesson to solving unusual tasks, they started to enjoy the challenge and the new connections they were making between different areas of maths.

Regarding the culture in school, I was aware that it is common among the teachers to rely heavily on official textbooks and past exam papers when preparing for lessons. It has become a norm for all school departments to print booklets with past papers and offer them to students.

Data collection and analysis

The collection of data took place at all stages of this project.

The initial data was collected from lesson observations, submission of tasks, semi-structured interview, and sorting activities and was used as a baseline regarding teachers' practices and knowledge about tasks' cognitive demand and also to find the origin of these tasks. The use of multiple sources for data allowed for triangulation, which increases the reliability of data (Cohen et al., 2007).

Data collected during the intermediary period was used to decide any necessary intervention, while data from the final phase was compared against the initial one to determine the progress.

Lesson observation

The initial lesson observation was one of the sources used to confirm the necessity of having cognitive demand as the area of focus for the program. As suggested by Goodnough (2011), a view of the phenomenon from different perspectives helps establish a clear focus. The lesson observation would provide me with a window into what type of work students are engaged in (Doyle, 1983; Boston and Wolf, 2006) and how teachers respond to contingent situation (Rowland et al., 2005). This information could then later be used to decide on the program's priorities as it might indicate the necessity to develop content and pedagogical knowledge. Field notes were used to collect data about the source and nature of the task.

Tasks collection

Shortly after the lesson observation, teachers were asked to submit the tasks used over a period of five lessons. Participants had to provide all the tasks they used for the whole class, a group or an individual. Tasks included any exercise employed to convey the main idea of the lesson and also any starter, even if not be related to the objective of the lesson, as long as it represents a mathematical idea. All tasks were coded using the Task Analysis Guide (Stein and Lane, 1997) according to their level of cognitive demand.

Field notes

This tool was used throughout the whole project, from the first lesson observation until the last activity involving the participants. During the lesson observations, field notes were used to record any useful information regarding the tasks employed. Any discussion with teachers was

also documented using field notes. This was particular convenient as it is not always possible to audio record any spontaneous conversation with the participants.

Semi-structured interview

A semi-structured interview was planned just before the workshops (Appendix 13). This method was employed because it would allow me to draw certain conclusions about teachers' beliefs regarding the role of tasks utilised. Prior to the interview, the teachers supplied data about the main sources for the tasks they use in students' instruction (Appendix 5). Denscombe (2010) describe three types of interview: open, semi-structured and structured. I found the open interview as not suitable because I had certain things I wanted to address and discussions might diverge from my initial plan. It would also be very time consuming. The structured interview is very rigid and I felt that teachers might not express fully their thoughts on the issues I raise. The advantage of using a semi-structured interview is that it allows the researcher to have control over the topics discussed, while still letting the interviewees to develop their ideas more freely about the topics raised (Denscombe, 2010).

Sorting tasks

Before and after the workshops, sorting tasks activities were employed to assess teachers' ability to differentiate between tasks with various levels of cognitive demands. The tasks were selected mainly from textbooks familiar to the teachers but also from other sources (Appendices 8 and 9). Teachers were asked to sort the tasks into four groups according to their level of difficulty and

provide reasons for their choice. This activity was inspired by the work of Arbaugh and Brown (2006).

Self-reports

After the second workshop, teachers were asked to choose a chapter form a textbook they were using and analyse the questions in that chapter with regard to the level of cognitive demands and reflect on their suitability for the students. The aim was to raise awareness about the tasks they can find in the textbooks and the opportunities they offer to students to engage with a high level of mathematical thinking. Teachers were also asked to use TAG to analyse the tasks offered to students over five lessons and reflect on the students' interaction with the tasks. The results were then reported and discussed with the researcher.

Data analysis

Both quantitative (e.g. level of cognitive demand) and qualitative data (e.g. source of tasks) were collected in this project (Cohen et al., 2007). Following procedures similar to the ones used by Stein et al. (1996), all the tasks were coded according to their level of cognitive demand, using the Task Analysis Guide (Appendix 2). Levels 1 and 2 refer to tasks with a low cognitive demand, while 3 and 4 means the tasks involve a high level of thinking. Descriptive statistics such as mean and maximum value were used to analyse the level of cognitive demand of tasks from lesson observations, tasks collection and reports. The aim was to calculate the average difficulty of these tasks and to see if the whole range of difficulty is covered within a certain period of time. This information was then used to assess the difficulty of questions that teachers are employing in class. The tasks collected at the beginning of the project were coded by the

researcher, while the tasks used after the intervention were coded by the participants. This gave them the opportunity to get familiar with the TAG and also to reflect and become more aware of the level of cognitive demand they employ in lessons.

For the sorting tasks activity, teachers received one point for each tasks card that they placed in the right category. The pre and post intervention scores were then compared to measure teachers' ability to recognize the right level of cognitive demand. The criteria teachers used to discriminate between tasks were in the first round was also compared to the criteria provided by TAG but they were not scored.

The results from self-reflection were analysed and compared with the set of data from the task collection activity before intervention to see if there is any increase in the level of difficulty for the tasks used in class.

Qualitative data from field notes and semi-structured interview was analysed for indication of change of practice or beliefs. No particular framework was used for this, rather an open coding approach was taken.

3.6.2 Intervention

After the initial data collection, the intervention phase was deployed. As a teacher educator, my role was to prepare and deliver a session on the pedagogical tools intended to be used throughout this program. The main tools used in this professional development program are variation theory (Marton and Booth, 1997) and cognitive demands of tasks (Stein et al, 1996). The reason I chose to work on cognitive demands was because I felt this is something teachers often ignore in their lesson preparation or are not aware of a potential mismatch between what they offer to students and the potential level that the students can work at. Variation theory was chosen as it is very

flexible and allows teachers to design, discuss and change tasks to suit their needs, in other words, it helps teachers unpack mathematical knowledge.

The aim of the first session was to introduce teachers to the theoretical aspects of variation theory. Rather than introducing it directly, I decided that a practice-into-theory approach (Korthagen and Kessel, 1999) would be more useful as it would give the teachers an example of how they might use variation theory as a pedagogical tool in their classrooms. In my role as a TE, I designed and guided an activity and moderate the discussions generated during the session. The exercise I used for this session was on standard deviation and this topic was chosen because it can be used to exemplify the main features of variation theory. This activity allowed teachers not only to gain theoretical and pedagogical knowledge but also to experience a new method of teaching, which might contribute to a change in their beliefs. An important goal of this session was for teachers to become familiar with the specialist vocabulary such as dimensions of variation and range of permissible change (Marton and Booth, 1997; Watson and Mason, 2006) as this would provide a common language to be used by participants.

The second session involved learning about the levels of cognitive demands of tasks. Participants were distributed the Task Analysis Guide TAG (Appendix 2) and each strand was discussed to get an agreement on the suitability of using it. The topic used for this session was graph transformation. The goal was to use variation theory in order to create tasks with different levels of difficulty.

3.6.3 Post intervention

This phase consists mainly of the plan-act-reflect cycles of action research. Teachers design tasks collaboratively using variation theory and focusing on the level of cognitive demand. Data collected at this stage consists of teachers' self-reports and field notes from discussions with the teachers. An analysis of the data was carried out in search of evidence of change in teachers' knowledge, beliefs or practice. The interconnected growth model (Clarks and Hollingsworth, 2002) is used to measure any potential development.

4. Findings and discussion

In this section I will look at the data collected during the project and I will analyse how these results answer my research questions.

In the first part of this section I will discuss the findings regarding the source of tasks and then I will move on and look at teachers' knowledge about cognitive demands of mathematical tasks and finally, I will analyse teachers' participation in workshops and how this has influenced their ability to design tasks with various levels of difficulty.

4.1 Sources of levels of cognitive demands of tasks

In this sub-section I will analyse data from this study and I try to answer my first research question: What is the source and level of cognitive demands of tasks that teachers employ in their instruction?

The first data collection regarding the sources of tasks was done during my lesson observations. What I noticed during my classroom visits was that all four teachers relied exclusively on the

textbooks in their instruction. However, a general conclusion could not be drawn before collecting more data. Individual detailed results on sources of tasks as reported by teachers can be found in Appendix 5. The summary of those results (see Table 1) indicate that textbooks and similar materials are the main choice for teachers when preparing for instruction, with 91.65% of the tasks those sources. Even more, 81.6% of the tasks come from the official textbook and past papers only. This confirms my initial prediction and also the previous studies claiming that teachers use textbooks as the main source of tasks (Sun, 2011; Haggarty and Pepin, 2002; Eisenmann and Even, 2011; Hiebert et al., 2003).

	Percentage of tasks from textbook and similar materials	Percentage of tasks from textbook and past exam papers
Alan	91.30%	91.30%
Bianca	87.50%	57.50%
Candy	90.00%	80.00%
Daniel	97.50%	97.50%

Table 1: Proportion of tasks sourced from textbooks and similar materials.

Comparing individual teachers, we can see that the teacher with the highest percentage of textbook use is Daniel, who relies on the textbook and past exam questions 97.5% of the times. The teacher who relies least on the official printed materials is Bianca, who uses the textbook or similar materials for 87.5% of the times. Looking at the use of the official textbook and past exam papers only, there is a staggering difference between Bianca, with 57.5% and Daniel, with 97.5%. There is also a significant difference between Bianca and the next person in the group

who is using these materials for 80% of the time. When inquired about the sources of tasks, she made the following comments:

> "I don't use the textbooks very much because they have mistakes and also, they are difficult to follow. I think past papers are the best because they show students exactly the type of questions they will get and how to answer them. I also take questions from [gives the name of a website]. I think they are better than the textbook. For Further Maths we didn't have a textbook until last year, you know, I already have materials from other sources. I also like to take questions from the Chinese textbooks and Maths competitions because sometimes the textbook is too easy."

From these comments I can see that although she has a focus on exams, Bianca is aware of certain limitations of the textbook and I expect that she will be more open to trying to work on challenging tasks. More comments about her use of Chinese textbooks and Maths competitions will be made later, when discussing the levels of cognitive demands.

While relying on printed curriculum materials, Candy showed a preference for using activities from other sources when introducing new topics. She explained that because she finds working on questions form the book or past exam papers as being boring, when introducing a new topic is her chance to show videos or use activities that are more fun. Other than that, she prefers to use textbook because she knows the questions and that makes her feel very confident. This indicates that she might not have a very strong SMK to support her in a potential contingent situation (Rowland et al., 2005) emerging from using unfamiliar resources. Knowing to act in the moment (Mason and Davies, 1998) when unexpected situations arise is largely supported by SMK. As a Teacher Educator, this is one area I should focus on when working with Candy.

The other participant in this study, Alan, relies exclusively on the official textbook and past exam papers when using printed materials. However, he indicated that quite often he designs tasks himself when doing revision with students. He justified his action by the fact that certain topics don't come up very often in the exam hence the students don't have too many past exam questions to practice on. This means that if he wants the students to be well prepared, he has to design extra questions himself.

In conclusion, teachers rely heavily on prescribed curriculum materials when preparing tasks for instruction. Although some teachers may occasionally use other materials, they account for a very small percentage of the total. Apart from Bianca, no other teachers showed signs that they would employ tasks with difficulty exceeding the one found in the textbook or exam questions. It seems that teachers' main concern is getting students ready for the official exams, and they pitch the level of their instruction at that level.

In this sub-section I do an analysis of the level of cognitive demands that teachers employ in class. The first set of data regarding levels of cognitive demand was collected at the beginning of the project and included tasks from lesson observations and those submitted by the teachers. Tables with individual levels for each task and lesson can be found in Appendices 6 and 7. For reader's convenience, the summarised data from the pre and post intervention are presented in the two tables below (Table 2 and Table 3). They show the overall results for each participant, as well as the grand totals.

	Alan	%	**Bianca**	%	**Candy**	%	**Daniel**	%	**Total (per level)**	Percentage (per level)
Level 1	0	0%	0	0%	0	0%	0	0%	**0**	0%
Level 2	9	28%	13	38%	16	42%	26	48%	**64**	41%
Level 3	22	69%	18	53%	22	58%	28	52%	**90**	57%
Level 4	1	3%	3	9%	0	0%	0	0%	**4**	3%
Total (per person)	**32**	100%	**34**	100%	**38**	100%	**54**	100%	**158**	100%
Mean level	**2.75**		**2.71**		**2.58**		**2.52**		**2.62**	

Table 2: Round 1- number of tasks organised by level of cognitive demand

	Alan	%	**Bianca**	%	**Candy**	%	**Daniel**	%	**Total (per level)**	Percentage (per level)
Level 1	0	0%	0	0%	0	0%	0	0%	**0**	0%
Level 2	16	42%	17	43%	16	40%	24	47%	**73**	43%
Level 3	22	58%	22	55%	20	50%	25	49%	**89**	53%
Level 4	0	0%	1	3%	4	10%	2	4%	**7**	4%
Total (per person)	**38**	100%	**40**	100%	**40**	100%	**51**	100%	**169**	100%
Mean level	**2.58**		**2.60**		**2.70**		**2.57**		**2.61**	

Table 3: Round 2 - number of tasks organised by level of cognitive demand

Comparing the overall results, we can see that there were 11 more tasks analysed in the second round than in the first one. This result might seem unnatural considering that the first round included tasks from six lessons (including lesson observation) instead of five. However, a plausible explanation for this result might be the length of lesson when the tasks was collected.

Some lessons are only 45 minutes long while others are 90 minutes long, hence more tasks can be done during the longer lessons.

A striking similarity is the overall average level of cognitive demands, 2.62 in the first round compare to 2.61 in the second round. While some people might argue that this consistency means that teachers do not implement tasks with higher degree of difficulty I dispute the validity of such conclusion without doing a more detailed analysis of the data. For example, tasks in the second round might be sourced from lessons involving mainly procedures, such as divisions of polynomials.

Looking at levels of cognitive demand, one can see that there are no Level 1 tasks present in the two tables. The reason is that memorisation tasks are not given explicitly to students. However, in the second round of tasks collection, some teachers did report using Level 1 tasks but after discussing with them, they changed them to Level 2. This confusion usually comes from teachers not distinguishing between learning a formula and applying it. Below is an example of a dialog with a teacher who initially identified a question as Level 1:

In how many ways can the six letters A, B, C, D, E and F be arranged in a row?
Question from Chalmers (2018, p. 127)

Researcher: Why do you think this is a Level 1 task?

Teacher: All you have to do is apply the formula.

R: Yes, but does it involve a procedure? Any steps to follow?

T: No, just apply the formula.

R: OK, so what formula?

T: n factorial.

R: so is the answer n factorial?

T: Yes, I mean no, it is 6 factorial in this case.

R: Right! So is there any procedure to follow?

T: well, you have to count the letters but that is easy.

R: Yes but it means you have to identify *n* first, and then apply the formula, so you follow a procedure.

These discussions were very useful in helping teachers identify the correct level but also for me as a Teacher Educator as it challenged my knowledge about this topic.

An important result is that Level 3 questions represent the more than half of the tasks in both tables. That means the students are exposed to a significant amount of questions that require connections, which is essential in building a solid mathematical knowledge base for working on Level 4. Comparing Level 2 and Level 3, we can see that there is an increase in number of the former one. While there is no easy answer to why this is the case, one possibility is that tasks were collected from lessons that involve a lot of practice on procedures, such as numerical methods, solving second order differential equations, or integration by substitution.

An encouraging sign is the increase in the number of Level 4 tasks, although it is too early to link it to the participation in this project. Although all teachers reported working on Level 4 questions at one point, only Bianca reported using challenging questions in both rounds. This is consistent with her comments she made during the interview when she mentioned that sometimes she uses questions from Chinese books and Maths competitions, as they are more difficult. This shows a certain awareness regarding the degree of difficulty present in various curriculum materials.

Comparing individual teachers, it is interesting to see that the ranking in terms of number of tasks used in class is the same in both sets of data. Also, there is a big gap between the teachers with the highest and the lowest number of implemented tasks. Overall, Daniel reported 50 Level 2 tasks and 53 Level 3 tasks, while Alan only 25 and 44, respectively. My concern as a Teacher Educator is that although he uses a lot more Level 3 tasks, Daniel might reduce the level of

cognitive demand during the implementation, thus reducing the efficiency of the tasks Stein et al., 1996).

I will now conclude and try to answer my first research question.

My research question 1 was:

> What is the source and level of cognitive demand of tasks that teachers implement in class?

Form the section above, I can say that teachers use nearly exclusively the textbook and past exam papers as the main source for the tasks for instructions. The bulk of these tasks are Level 2 and 3, with Level 3 being predominant. Level 4 tasks are rarely employed while Level 1 tasks are never employed explicitly.

4.2 Sorting tasks activities

In this sub-section I will try to use data collected before and after the intervention to answer my second research question:

> What is the teachers' ability in identifying the level of cognitive demand of mathematical tasks and how does it change after participation in a professional development program focused on tasks' cognitive demands?

Teachers' ability to identify the level of cognitive demands was measured before and after the intervention using two sorting tasks activities. The results were analysed to detect any increase of teachers' knowledge about cognitive demands of mathematical tasks. Copies of the tasks used can be found in Appendix 8 and 9, while tables with detailed answers given by teachers can be found in Appendix 10. The summary of the data is presented below in Tables 4 and 5. Overall, the data shows that teachers have improved their skills in identifying the levels of cognitive

demands. The percentage of correctly identified levels increased significantly, from 36% to 61 %. While there is not a big change in identifying the extremes (Level 1 and Level 4), there is a huge change in the number of Level 2 and Level 3 tasks correctly identified. In the initial sorting task activity, only 13% of Level 2 tasks and 34% of Level 3 tasks were labelled correctly. In the second round, there was a threefold increase in the number of tasks correctly identified at Level 2, while for Level 3, the numbers nearly doubled. An interesting result we get if we reduce the number of levels and we classify the tasks using only two criteria: low demand and high demand. This means that if the teacher identified a task with Level 3 as being a Level 4, we still count it as correct since the task has been identified as high level. If a task with Level 2 was identified as Level 3, then it was counted as wrong, since it was labelled as high level instead of low level. Summarised data using these new criteria is presented in tables 6 and 7. Looking at the results from these two tables, we can see that the percentage of correctly identified is much higher than the ones form the tables with individual levels.

	Level 1 (2 tasks)	%	**Level 2 (6 tasks)**	%	**Level 3 (8 tasks)**	%	**Level 4 (2 tasks)**	%	**Total (18 tasks)**	%
Alan	2	100%	0	0%	3	38%	2	100%	**7**	**39%**
Bianca	2	100%	2	33%	5	63%	0	0%	**9**	**50%**
Candy	1	50%	0	0%	1	13%	1	50%	**3**	**17%**
Daniel	2	100%	1	17%	2	25%	2	100%	**7**	**39%**
Mean	**1.75**	88%	0.75	13%	2.75	34%	1.25	63%	**6.5**	**36%**

Table 4: Sorting activity 1 - Number of tasks correctly identified according to their level of cognitive demands

	Level 1 (2 tasks)	%	**Level 2 (6 tasks)**	%	**Level 3 (7 tasks)**	%	**Level 4 (3 tasks)**	%	**Total (18 tasks)**	%
Alan	1	50%	2	33%	3	43%	1	33%	**7**	**39%**
Bianca	2	100%	4	67%	5	71%	3	100%	**14**	**78%**
Candy	2	100%	3	50%	4	57%	1	33%	**10**	**56%**
Daniel	2	100%	3	50%	5	71%	3	100%	**13**	**72%**
Mean	1.75	88%	3	50%	4.25	61%	2	67%	**11**	**61%**

Table 5: Sorting activity 2 - Number of tasks correctly identified according to their level of cognitive demands

	Low level (8 tasks)	%	High level (10 tasks)	%	Total (18 tasks)	%
Alan	7	88%	5	50%	**12**	**67%**
Bianca	8	100%	7	70%	**15**	**83%**
Candy	6	75%	4	40%	**10**	**56%**
Daniel	7	88%	6	60%	**13**	**72%**
Mean	7	88%	5.50	55%	**12.50**	**69%**

Table 6: Sorting activity 1 - Number of correctly identified low and high levels

	Low level (8 tasks)	%	High level (10 tasks)	%	Total (18 tasks)	%
Alan	7	88%	5	50%	**12**	**67%**
Bianca	6	75%	8	80%	**14**	**78%**
Candy	8	100%	8	80%	**16**	**89%**
Daniel	6	75%	9	90%	**15**	**83%**
Mean	6.75	84%	7.50	75%	**14.25**	**79%**

Table 7: Sorting activity 2 - Number of correctly identified low and high levels

This means that teachers can differentiate between easy and difficult tasks but they make mistakes when separating between individual levels. It also indicates that teachers might have a *phronesis* type of knowledge regarding levels of difficult (Korthagen and Kessels, 1999). Taking the analysis further, we can see that the low demand tasks were identified in over 80% of the cases in both session, while the high demand tasks were labelled correctly only 55% of the time in the first session and 75% of the time after the intervention. This indicates that the intervention had a positive effect on teachers' ability to identify high level tasks but more work is necessary to differentiate between Level 3 and Level 4 tasks. An analysis of individual performance reveals that Candy had the highest improvement in identifying the level of cognitive demand. Tables 4 and 5 show that Candy had a success rate of only 17% in the first round, but then it increased to 56% after participating in the workshop. She is also the teacher with highest increase in performance when using data from tables 6 and 7. Candy was able to

match correctly 89% of the tasks using the low-high level criteria. However, she fails to distinguish very well between tasks when finer filters are applied. This shows the importance of using both sets of tables and analysing teachers' performance. For example, tables 6 and 7 show a decline in Bianca's success rate. However, looking at table 5, we can see that she is the teacher with the highest accuracy in identifying the levels of cognitive demand. This discrepancy might be caused by the small sample of tasks used in the study, which is something I should consider in future investigations.

As a Teacher Educator, I was very interested in how the participants discriminate between tasks so after the first sorting activity asked the teachers to say what criteria they use to group them. Out of many rules they mentioned, one was common to all of them when labelling a task as difficult: familiarity with the task. Bianca mentioned that she considers a tasks difficult if similar type was not encountered before, while Candy said that a task is difficult if the method is not like in the textbooks. In a similar fashion, tasks were considered easy if they involve procedures that were taught before, even if they require connections between topics. For example, when inquired why he labelled some Level 3 tasks as Level 2, Alan replied: "they look standard, it is just practice". Bianca also downgraded some tasks based on the same principle: "it is in the syllabus, we do it all the time". When challenged that the questions do require some connections, hence they should be Level 3, they claimed that with enough practice, those connections are obvious. These comments further justify the need for teachers to employ Level 4 tasks if developing strong mathematical thinking skills is desired. Those types of questions cannot be done by repeating procedures only, they require students "to make and justify choices, integrate ideas, to plan strategies, and to explain their thinking" (Sullivan et al., 2013).

I will now conclude and answer my second research question:

What is the teachers' ability in identifying the level of cognitive demand of mathematical tasks and how does it change after participation in a professional development program focused on tasks' cognitive demands?

Data shows that without any training, teachers have a phronesis type of knowledge (Korthagen and Kessels, 1999) that allows them to discriminate between tasks with low and high level of difficulty in the majority of cases. However, the success rate is much lower when finer grading is necessary. After participating in the workshop, teachers' ability in identifying the levels of cognitive demands is significantly higher. Not only they increased their scores in the low-high level divide, but also their accuracy in distinguishing between all four levels has improved. I believe that this is an important step in reaching my goal of educating awareness: teachers can act (separate the tasks) and also know why that action is right, hence they have an awareness-in-discipline (Mason, 1998).

4.2 Workshops

In this section I will present a short summary of the workshops I ran with the teachers. Through my interventions, my intention was to engage teachers in mathematical discovery activities and provide them with the opportunity to learn mathematics. Oliveira et al., 2008 suggest that hese are efficient ways to change teachers' beliefs, which would bring me one step closer to changing teachers' practices.

The first workshop was on variation theory. The exercise used was about standard deviation (Appendix 12). All diagrams were designed symmetrical on purpose, in a hope that teachers will notice it explore the implications of non-symmetrical data. At the beginning, I was introducing

the diagrams one by one, asking the teachers to identify what is the same and what is different. Very soon teachers figured it out that it is about standard deviation. However, when I asked them to think about what students would answer, they started to analyse the exercise in more detail. I will reproduce below a dialog that took place between participants:

> "Alan: this is actually quite good. It makes it so obvious that it is about the distance from the mean. I can see this working with the students.
>
> Candy: yes, but what if the data was not symmetrical? Would they still figure it out?
>
> Alan: It depends. But the idea is to make it obvious that it is about distance from the mean, so symmetry helps.
>
> Researcher: Daniel, what do you think?
>
> Daniel: I think it's good. It gives students a feeling of what standard deviation is. The textbook doesn't explain much.
>
> Researcher: So what if, as Candy said, we make the data not symmetrical?"

After the last question, the teachers had a long conversation about how this activity can be modified for students with different abilities. They even brought up the idea of doing something similar for other topics in Statistics. At one point, during the discussion, Alan started to draw some sketches on the board to show how the mean can move without changing the median. Candy made the following comment: "I always knew the rules with mean and median for skewed data, but I could never explain exactly why."

This is an example of how discussions can facilitate development of mathematical understanding, which might result in teachers being more open to students' ideas and more able to deal with challenging questions (Leikin and Zaskis, 2007).

This first session was successful at engaging teachers in discussions, sharing ideas and improving their SMK and PCK. In fact, at the end of the workshop, one of the teachers asked for a copy of the diagrams so that he can use it with students. Almost immediately, another teacher said: “Yes, can you please send us all, a copy? I might use it as well.”

Research confirms that being part of a community, helps teachers develop their knowledge and also act on them, determining their practices. In this case, Alan improved his PCK by learning an interactive method of working on skewed data and Candy improved her SMK by getting a deeper understanding of the concept of skewed data. The extent to which this session has an impact in practice, is something to be determined in the future.

The second session with teachers involved learning about levels of cognitive demand and using variation theory to design tasks.

Using the function $f(x) = \sin(x)$ and graph transformations as a theme, teachers had to identify as many dimension of variations as possible and create examples for each instance. They then had to match their example with one of the levels from TAG. A significant part of this session was dedicated to identifying multiple problem structures that could be used in this context, such as: drawing the graph of a transformation, match multiple expressions with a graph, study the behaviour of the inverse function under the transformation, use of multiple choice etc. This approach is similar to what Sun (2011) describes as “one problem multiple solutions” or “one problem multiple changes”.

Teachers were very quick in identifying dimensions of variation that are present in textbooks, such as $f(x) = a\sin(bx + c) + d$. However, nobody thought of changing the powers of *x* to get

functions like $f(x) = a\sin(x^2)$ or $f(x) = x\sin(x^{-1})$. Nevertheless, teachers were very receptive and showed interest in solving this questions.

At the end of the workshop I offered a couple of challenging questions to teachers and I asked them to use them in class. From my own experience, those questions always get students very excited. My hope was that students' reaction will be seen by teachers as a salient outcome and will trigger a change in beliefs and practices (Clarke and Hollingsworth, 1999).

After this session, I asked the teachers to choose a chapter from the textbook, analyse the levels of cognitive demands and reflect on their suitability for all of their students. This action is also meant to contribute to a change in teachers beliefs (Oliveira et al., 2008).

Overall, I believe this was a good start in building a collaborative team and hopefully this will develop into a "community of inquiry involving teachers, students and teacher educators, in which all participants are learners" (Jaworski, 2006).

4.4 Teachers' growth

The school's closure for a long period of time has seriously affected my opportunities to work with teachers and collect data about changes in their beliefs, knowledge and practices. However, I would still like to analyse their short participation in the study and I will try to discern any effect this involvement might have had on them so far.

I believe it is too early to apply the ICMPG to highlight any changes in knowledge, behaviour or practice but still, some useful conclusions can be drawn.

4.4.1 The case of Alan

Alan is a Mathematics graduate and has been in school for many years. Based on the data I collected, I would say that he has a connectionist orientation (Askew et al., 1997). During the lesson observation I could see that he spends most of the time on level 3 tasks. He insists on students understanding the concepts and is not relying only students applying algorithms. This could be the reason why he reported a lower number of tasks used. During the interview, he made the following comment: "I don't care that much about the exam results, like others. I do my best to make students understand stuff". His concern for understanding explains why he coded a Level 1 task as Level 4 in the second round of sorting tasks. The task was: "Define sec(x)". We had the following dialog

> "R: why do you think it is a Level 4?
>
> A: I don't look at it as the definition only. Students know the definition, but how many can explain why we use it or what does it mean?
>
> R: Can you show me what secant means?
>
> A: To be honest, I don't know either."

This was an opportunity to show him how to use the trigonometric circle to represent secant and tangent functions and thus contributing to his SMK and PCK (Ball et al., 2008).

Alan also seemed to be willing to implement difficult tasks in class. When I asked him about using the challenging questions I provided, he said that he finds them very good but he keeps them for the period before Christmas. This prompts me to consider if availability of ready-to-use tasks is one reason why they are missing in lessons.

As this project was considered as part of the professional development, when reporting on it to school, Alan showed me his comment:

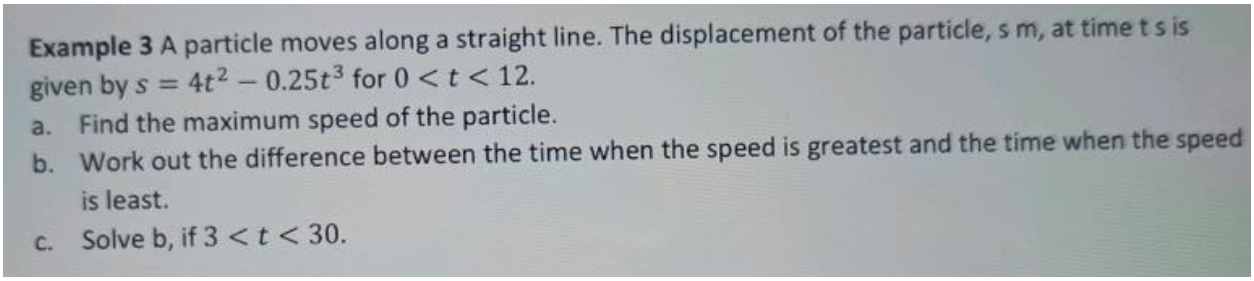
Overall Comments

One of the best PD sessions i have experienced in 10 years at [redacted]. The teachers in the room were engaged throughout, and the staff left the room asking further questions and talking about further collaboration

Although there is not enough data to apply ICMPG and detect any changes, I believe that Alan has gained so far by participating in this project.

4.4.2 The case of Bianca

Bianca has a degree in Finance and is the youngest participant in this project. She joined the school not long ago and this might be the reason why she had little contribution during the workshops. However, important information for me as a Teacher Educator were revealed during the interview and casual talks we had. The lesson I observed was with one of the new groups and it proved to be a bit challenging for me as it was conducted mainly in Chinese. While I was not able to understand all the talk going on, I could still follow tasks they were working on as they were from textbooks in English. Her actions during the lessons showed a connectionist orientation. From the diagrams she was drawing on the board I could infer that she was consistently linking the current task to previous learnt knowledge. She even spontaneously modified a task from the textbook to approach a more general case. After the lesson, when I inquired about it, she said that she often does that, as sometimes the questions are too easy. She showed me an example from a previous lesson where point b) was modified from the original question to make it more general. Below is a picture of the modified task.



Example 3 A particle moves along a straight line. The displacement of the particle, s m, at time t s is given by $s = 4t^2 - 0.25t^3$ for $0 < t < 12$.

a. Find the maximum speed of the particle.
b. Work out the difference between the time when the speed is greatest and the time when the speed is least.
c. Solve b, if $3 < t < 30$.

This practice of teachers modifying tasks confirm previous studies mentioned in the literature review (Stein et al., 2000; Jones and Pepin, 2016).

Her previous work on changing tasks difficulty probably helped in the work we did on identifying levels of cognitive demand, Bianca having the highest accuracy in both rounds. She also said that she uses questions from the Chinese textbooks as they are more difficult.

This shows a certain level of awareness about the cognitive demands in tasks and judging by her progress in tasks sorting, I am confident she will prove a success in this project.

4.4.3 The case of Candy

Candy has a degree in mathematics and has been working in this school for two years. I observed her teaching a lesson on probabilities and her behaviour during the lesson indicates a transmission orientation. When working with the students, she constantly insisted on following procedures and applying formulae. For example, when a student asked something about a question on conditional probability, she replied: “Just apply the formula, you will get the answer”, without acknowledging that the question could be solved using Venn diagrams. Throughout the lesson, she showed a constant concern about sticking to her methods and did not use multiple representations when the questions allowed it. Her lack of confidence in exploring other ways might show that I need to work with her on improving her SMK first. Also, when I asked to try and use task I from sorting activity 1, she found excuses to not do it. This confirms research results that teachers might not implement high-level tasks even when they are available (Charalambous, 2006).

4.4.4 The case of Daniel

Daniel is the most experienced teacher in the department.

In the initial interview, when talking about the use of curricular materials, we had the following dialog:

> "*Interviewer:* Do you think we should use only the official textbooks?
>
> *Daniel:* Well, all students have the textbooks and it is handy to use it. I looked at some other resources but these one follow the syllabus closely and the others have other things in them sometimes…these are easy to follow. I use mainly the textbook in lesson because they only give the answers at the back, so if students have questions, I can explain them in class. The past papers are very good to for homework…students have the mark scheme and can check their work, they don't need my help for that, usually.
>
> *Researcher:* How about the self-designed tasks you say you use?
>
> *Daniel:* Those are just some examples I use to introduce new topics, nothing too complicated. I just like my examples better than the ones in the book, but only for some chapters, not all".

During the lesson observation I could see that Daniel designed the lesson exclusively around the examples and questions given in the textbook. He repeatedly asked the students to follow the examples on the board, which were taken from the textbook and check the answers with the back of the book. These point to a possible transmission orientation he might have (Askew et al., 1997). Another reason is the fact that during the interview, he did not show much interest in doing anything else from the syllabus because as he said: "I am judged by the exam results, not by the extra work I do."

5. Conclusions and implications

The aim of this study was to increase teachers' awareness about the levels of cognitive demand they employ in class and also to provide teachers with a tool to help them design tasks with different levels of difficulty.

To answer my first research question, I collected and analysed data regarding the source and levels of cognitive demand of tasks that teachers employ in class. My results confirm previous studies that teachers use textbooks for most of the time (Sun, 2011; Haggarty and Pepin, 2002; Eisenmann and Even, 2011; Hiebert et al., 2003). Subsequently, the tasks offered to students do not possess a high degree of difficulty (Mesa et al., 2012; Stylianides, 2009). My data also show that a significant proportion of tasks that students work on, are procedures without connection.

To find an answer to the second research question, I collected data pre and post intervention. Overall, the results show evidence that teachers have improved their accuracy in assessing these levels. An important merit to this success can be attributed to the quality of the framework developed by Stein and her collaborators.

Finally, to address the last research question, I designed and delivered a professional development program where teachers had the chance to improve their subject and pedagogical knowledge. This also offered them the opportunity to discuss the availability and use of resources.

Although the pandemic had a serious impact on the unfolding of the project and results are not up to my expectation, I still consider that it was a good start of an action research which I hope will become a long term activity in the maths department.

However, some important caveats need to be acknowledged. The research was done in a school with a specific profile, which means that the findings cannot be generalised and applied to other

schools. Even more, only four teachers out of the whole school are analysed in this study, which makes the results even more limited.

As shown in the study, the size of samples might affect the accuracy of the conclusion. Also, teachers' and researchers' lack of previous experience in coding tasks might mean that the results are not totally reliable.

Implication for future practice

Being the first larger scale research that I initiated and ran, this study had a significant impact on my developing as a Teacher Educator. Although some things went smoothly, I also had to overcome many challenges. Throughout the project, I have sharpened my skills in reading specialist literature, and applying and modifying methodologies. Also, working with other teachers has proved a unique experience, as they challenged me in a different way than students do. While I learnt many things about different ways to approach and tackle a research, I still have a long way to go. Looking back I can see certain areas that need to be addressed.

From the previous section, it becomes obvious that in future studies I need to ensure an increased reliability of the results. Taking a larger size of samples is part of the solution. Also, working with more experienced researchers on task coding would help increasing the validity of our data. Coding tasks is one area of my foundation knowledge (Rowland et al., 2005) as a Teacher Educator that needs improving.

As the time allowed only for one cycle of the action research to be performed, this study reports only on the tasks selection phase. In future, I will extend my focus to include the implementation of tasks with high level of cognitive demand. If this project will prove successful, it might be extended to other departments in school and even in other branches across the network.

I believe that so far, this study contributed to research by confirming the utility of the work carried out by Stein and her collaborators.

I consider that this study could prompt the wider research community to take into account the adequacy of using the Task Analysis Guide in their teacher training programs and the benefits it might bring to novice teachers.

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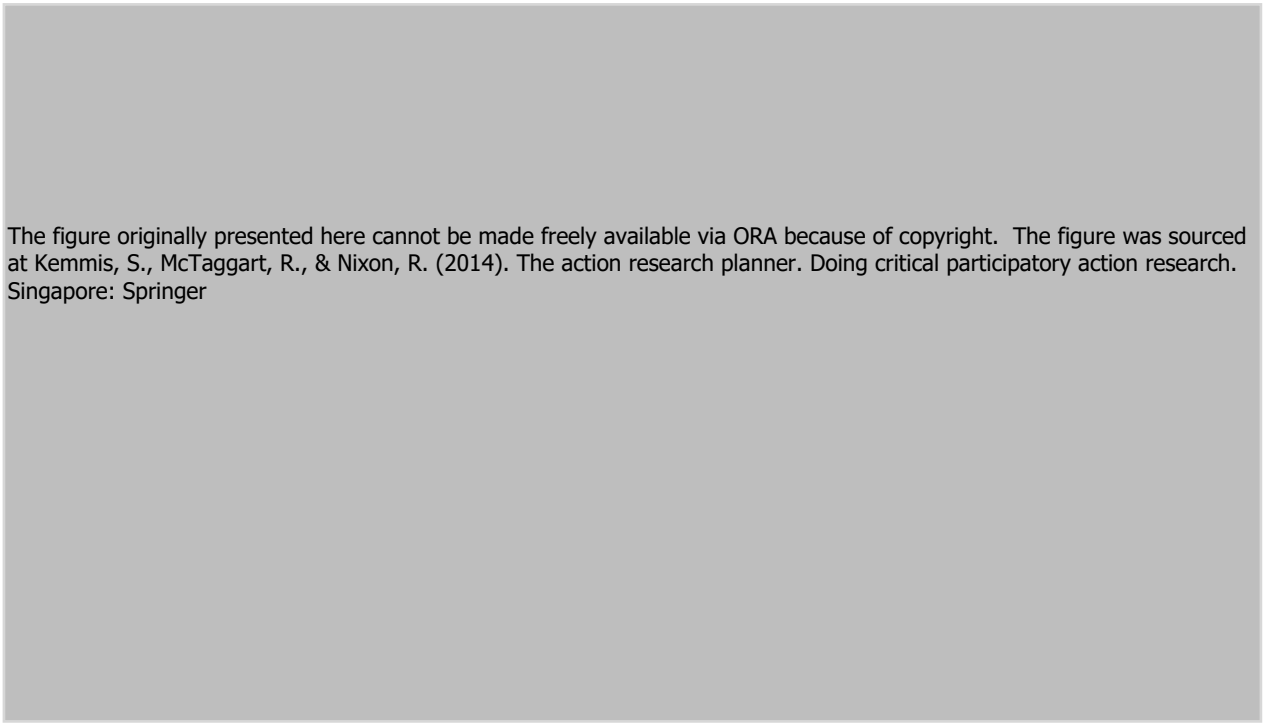
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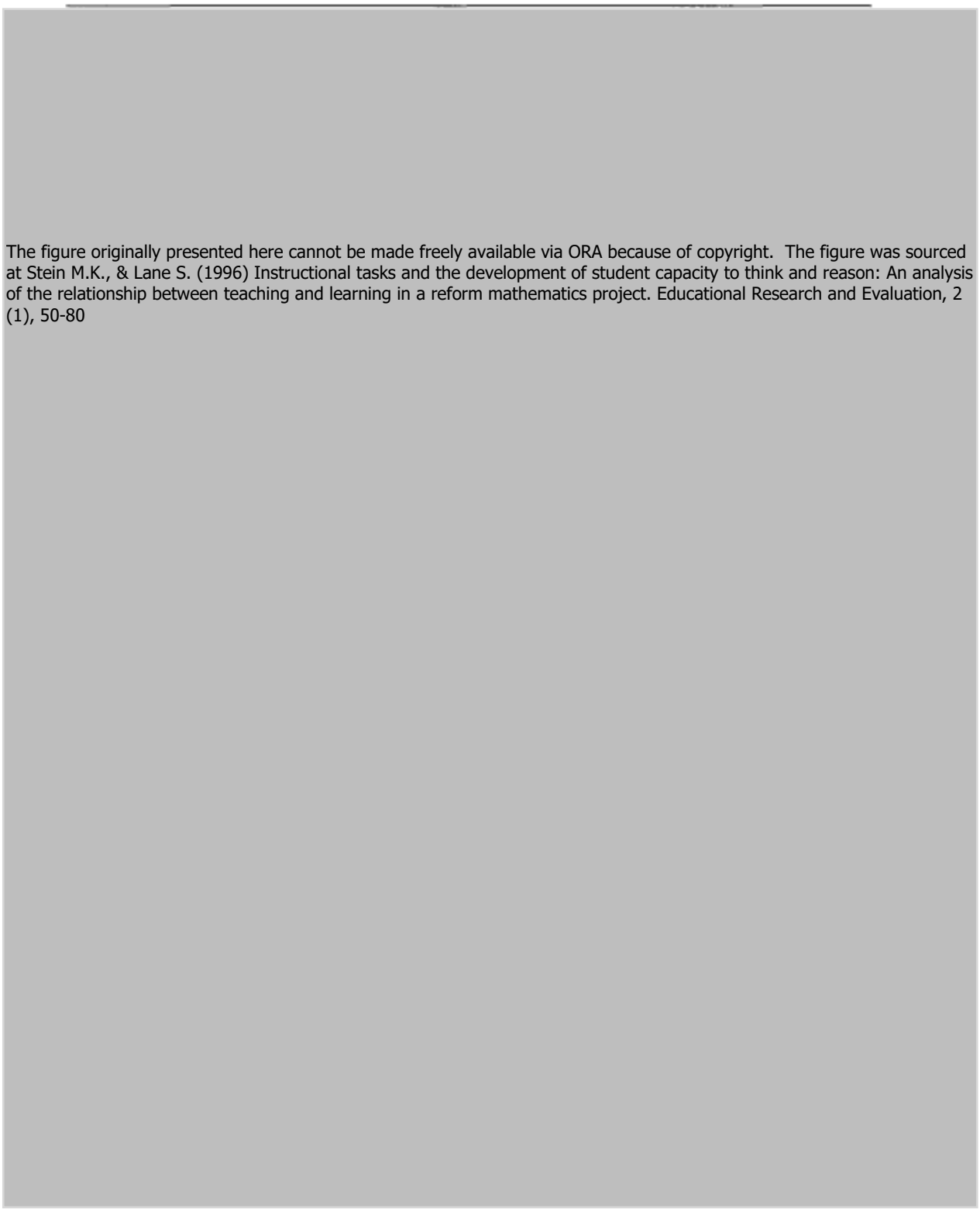
Appendix 1

Cyclical nature of action research (Kemmis et al., 2014, p. 28)



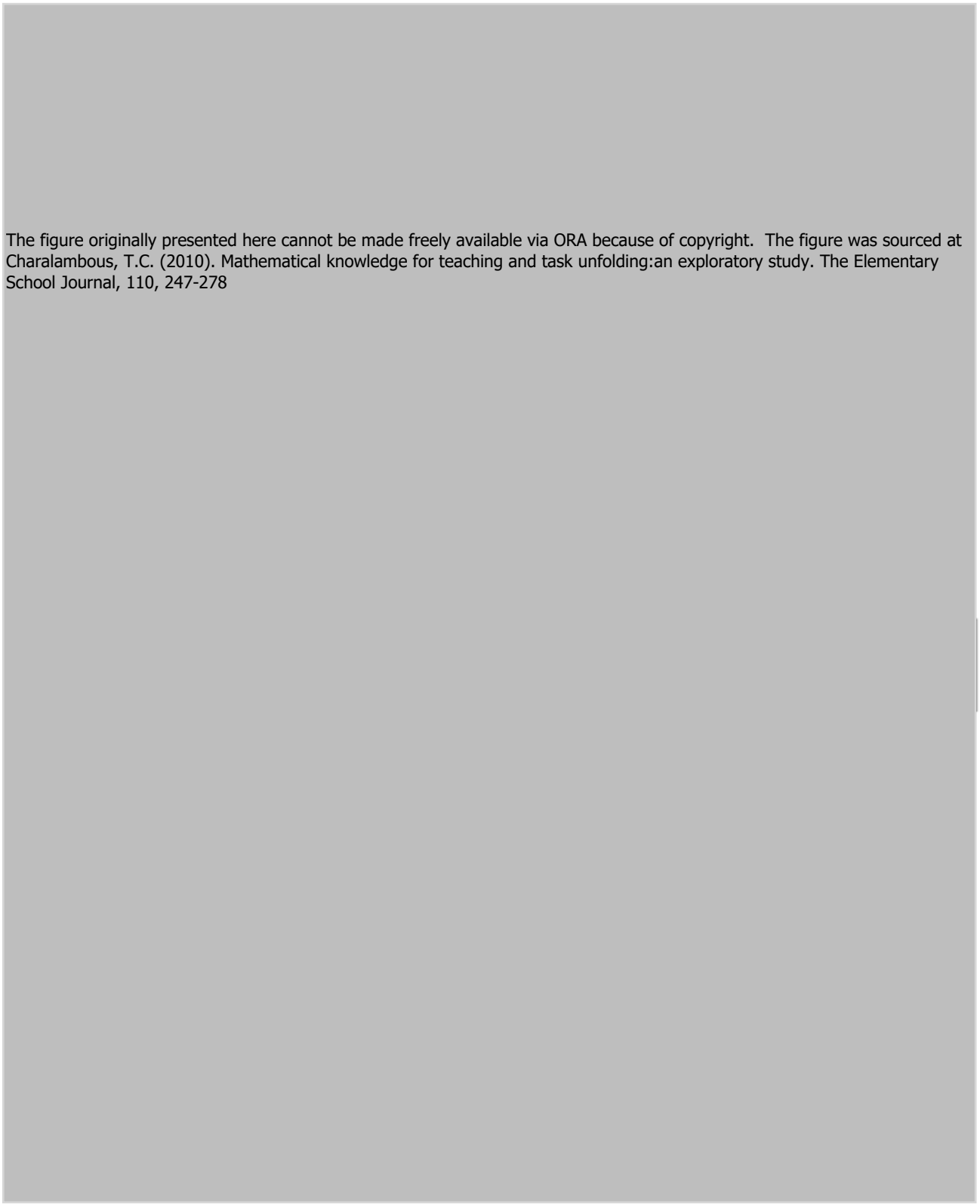
Appendix 2 – Task Analysis Guide TAG

Levels of cognitive demand (Stein and Lane, 1996)

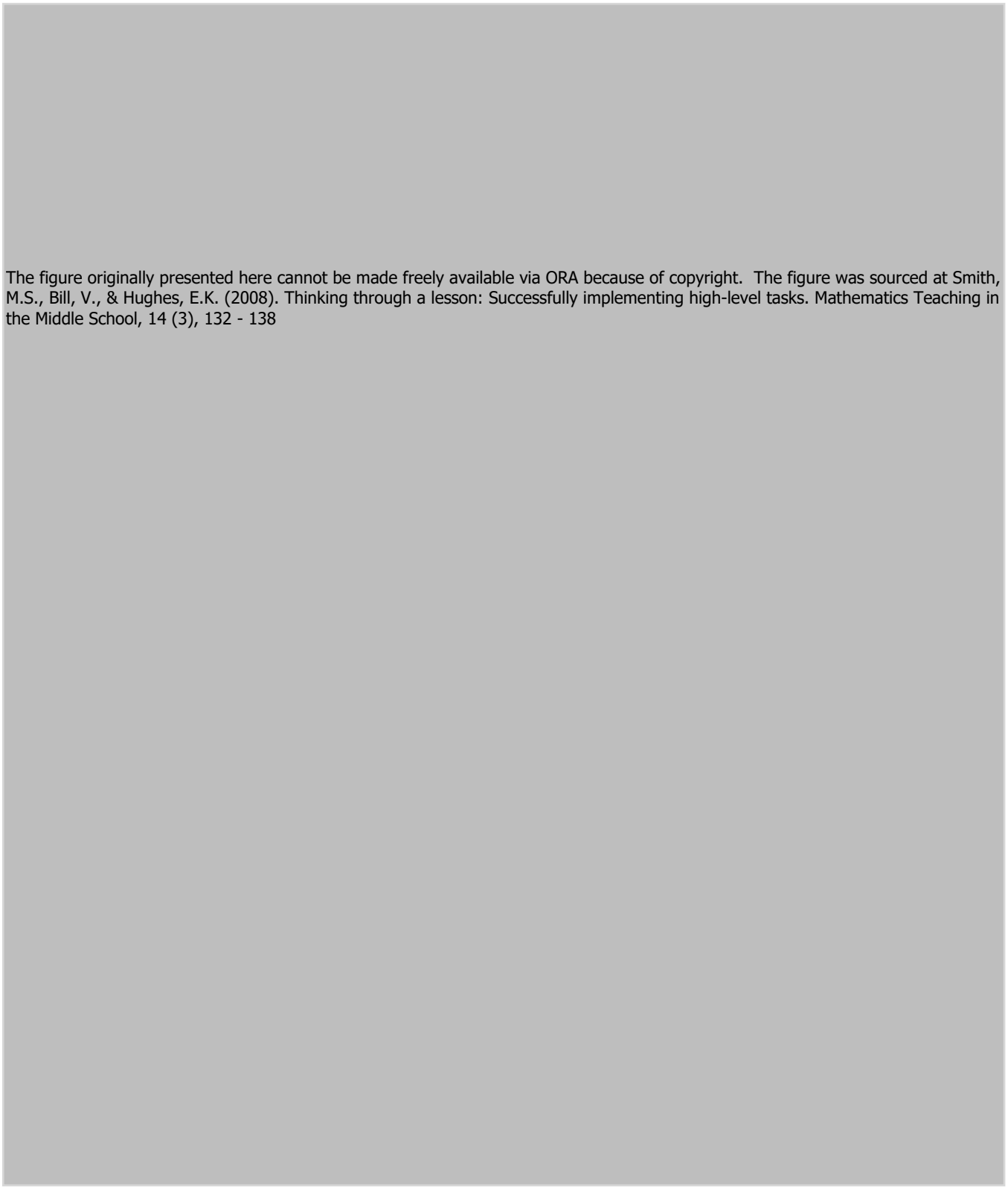


The figure originally presented here cannot be made freely available via ORA because of copyright. The figure was sourced at Stein M.K., & Lane S. (1996) Instructional tasks and the development of student capacity to think and reason: An analysis of the relationship between teaching and learning in a reform mathematics project. Educational Research and Evaluation, 2 (1), 50-80

Appendix 3 – Possible paths for task unfolding (Charalambous, 2010)



Appendix 4 –Teaching Through a Lesson Protocol TTLP (Smith et al., 2008)



Appendix 5 – Sources of tasks

ALAN	Textbook	Past papers	Hodder or Collins (alternative textbooks)	Others (similar to txtb)	Self-designed	Others (different)	TOTAL
Introducing new topic	60	30	0	0	10	0	**100**
Practice in class	70	25	0	0	5	0	**100**
Homework	60	40	0	0	0	0	**100**
Revision	10	70	0	0	20	0	**100**
Average	**50**	**41.25**	**0**	**0**	**8.75**	**0**	**100**
Cumulative average	**50**	**91.25**	**91.25**	**91.25**	**100**	**100**	

BIANCA	Textbook	Past papers	Hodder or Collins (alternative textbooks)	Others (similar to txtb)	Self-designed	Others (different)	**TOTAL**
Introducing new topic	50	10	30	0	10	0	**100**
Practice in class	20	30	10	20	10	10	**100**
Homework	30	20	10	20	0	20	**100**
Revision	20	50	20	10	0	0	**100**
Average	**30**	**27.5**	**17.5**	**12.5**	**5**	**7.5**	**100**
Cumulative average	**30**	**57.5**	**75**	**87.5**	**92.5**	**100**	

CANDY	Textbook	Past papers	Hodder or Collins (alternative textbooks)	Others (similar to txtb)	Self-designed	Others (different)	**Total**
Introducing new topic	50	0	0	10	0	40	**100**
Practice in class	40	50	10	0	0	0	**100**
Homework	40	50	10	0	0	0	**100**
Revision	0	90	10	0	0	0	**100**
Average	**32.5**	**47.5**	**7.5**	**2.5**	**0**	**10**	**100**
Cumulative average	**32.5**	**80**	**87.5**	**90**	**90**	**100**	

DANIEL	Textbook	Past papers	Hodder or Collins (alternative textbooks)	Others (similar to txtb)	Self-designed	Others (different)	**Total**
Introducing new topic	90	0	0	0	10	0	**100**
Practice in class	80	20	0	0	0	0	**100**
Homework	60	40	0	0	0	0	**100**
Revision	20	80	0	0	0	0	**100**
Average	**62.5**	**35**	**0**	**0**	**2.5**	**0**	**100**
Cumulative average	**62.5**	**97.5**	**97.5**	**97.5**	**100**	**100**	

Appendix 6 – Round 1 of tasks collection: levels of cognitive demands of tasks

Highlighted cells are homework tasks

Alan								Mean	Max
Lesson observation	2	3	3	3	2	3		2.67	3
Lesson 1	3	2	2	3				2.50	3
Lesson 2	2	3	3	4	2	3	3	2.86	4
Lesson 3	2	3	3	2	3			2.60	3
Lesson 4	3	3	3	3				3.00	3
Lesson 5	3	3	3	3	3	2		2.83	3

Bianca										Mean	Max
Lesson observation	2	2	3	2	3	3	3	3	3	2.67	3
Lesson 1	2	3	3	3	3	2	2			2.57	3
Lesson 2	2	4	3	3	3					3.00	4
Lesson 3	4	3	3	4						3.50	4
Lesson 4	2	2	2	3	2	3	2			2.29	3
Lesson 5	3	3	2	3						2.75	3

Highlighted cells are homework tasks

Candy									Mean	Max
Lesson observation	2	3	3	2	3	2	3		2.57	3
Lesson 1	2	2	2	3	3				2.40	3
Lesson 2	3	3	3	3					3.00	3
Lesson 3	2	2	3	3	2	3	2		2.43	3
Lesson 4	2	2	2	2	2	3	3	3	2.38	3
Lesson 5	3	2	3	3	3	3	3		2.86	3

Daniel														Mean	Max
Lesson observation	3	3	3	2	2	3	3	3							
Lesson 1	2	2	2	2	3	3	2	2	3	2	2	2	3	2.31	3
Lesson 2	2	2	3	3	2	2	3	3	2	2	3			2.45	3
Lesson 3	2	3	3	3	2	2								2.50	3
Lesson 4	2	2	2	3	3	3								2.50	3
Lesson 5	2	3	3	3	3	3	2	3	3	2				2.70	3

Appendix 7 – Round 2 of tasks collection: levels of cognitive demands of tasks

Highlighted cells are homework tasks

Alan										Mean	Max
Lesson 1	3	2	3	3	3					2.80	3
Lesson 2	3	2	2	2	2	2	3	3		2.38	3
Lesson 3	2	3	2	2	3	3	3	3	3	2.67	3
Lesson 4	3	3	2	2	3	2	2	3		2.50	3
Lesson 5	2	3	3	3	2	2	3	3		2.63	3

Bianca													Mean	Max
Lesson 1	2	2	2	3	3	2	2	3	3	3			2.50	3
Lesson 2	2	2	3	3	2	3	3						2.57	3
Lesson 3	2	2	2	2	3	3	3	3	3	2	2	3	2.50	3
Lesson 4	3	3	3	4									3.25	4
Lesson 5	2	3	2	2	3	3	3						2.57	3

Candy											Mean	Max
Lesson 1	3	4	3	3	2	2	1	2	2	2	2.40	4
Lesson 2	3	3	3	3	3	3					3.00	3
Lesson 3	2	2	2	3	3	2	3	3	2		2.44	3
Lesson 4	3	3	2	2	3	3	3				2.71	3
Lesson 5	2	2	3	2	3	3					2.50	3

Daniel												Mean	Max
Lesson 1	2	2	2	3	3	3	3	2	2	3	3	2.55	3
Lesson 2	2	2	2	3	3	3	3	3	3	3		2.70	3
Lesson 3	2	2	2	2	2	4	3	3	3	2	2	2.45	4
Lesson 4	2	2	2	2	2	3	3	3	3			2.44	3
Lesson 5	2	2	2	3	2	3	3	3	3	4		2.70	4

Appendix 8 - Tasks sorting 1

Sources: Pemberton and Hughes (2018); Chalmers (2018); Stewart (2016);

https://www.cut-the-knot.org/Curriculum/Probability/TriProbability.shtml

M

In how many ways can the six letters A, B, C, D, E and F be arranged in a row?

K

Seven cars and x vans can be parked in a line in 39 916 800 ways. Find the number of ways in which five cars and $x+2$ vans can be parked in a line.

J What is the definition of cosec(x)?

R

Find the number of distinct five-letter arrangements that can be made from:

a two As and three Bs **b** two identical vowels and three Bs

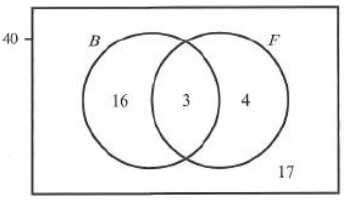
c two identical vowels and any three identical consonants.

N

A group of 40 students took a test in Economics. The following Venn diagram shows that 19 boys (B) took the test and that seven students failed the test (F).

a Describe the 21 students who are members of the set B'.

b Find the probability that a randomly selected student is a boy or someone who failed the test.



E Given that $P(A)=0.4$, $P(B)=0.7$ and that $P(A\cup B)=0.8$, find:

a $P(A\cup B')$ **b** $P(A'\cap B)$

H

Each of 27 tourists was asked which of the countries Angola (A), Burundi (B) and Cameroon (C) they had visited. Of the group, 15 had visited Angola; 8 had visited Burundi; 12 had visited Cameroon; 2 had visited all three countries; and 21 had visited only one. Of those who had visited Angola, 4 had visited only one other country. Of those who had not visited Angola, 5 had visited Burundi only. All of the tourists had visited at least one of these countries.

a Draw a fully labelled Venn diagram to illustrate this information.

b Find the number of tourists in set B' and describe them.

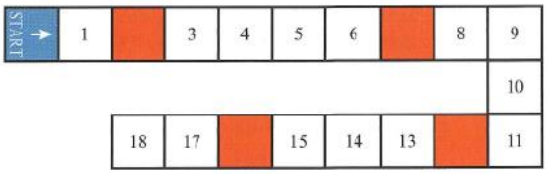
c Describe the tourists in set $(A\cup B)\cap C'$ and state how many there are.

d Find the probability that a randomly selected tourist from this group had visited at least two of these three countries.

G Y and Z are independent events. $P(Y)=0.7$ and $P(Z)=0.9$. Find $P(Y\cap Z)$.

D

A game board is shown in the diagram.



Players take turns to roll an ordinary fair die, then move their counters forward from 'start' a number of squares equal to the number rolled with the die. If a player's counter ends its move on a coloured square, then it is moved back to the start.

a Find the probability that a player's counter is on 'start' after rolling the die:

i once **ii** twice.

b Find the probability that after rolling the die three times, a player's counter is on:

i 18 **ii** 17

C

A survey of a random sample of 23 people recorded the number of unwanted emails they received in a particular week. The results are given below.

9 18 13 18 21 17 22 27 8 11 26 26 32 17 31 20 36 15 13 25 35 29 14

a Represent the data in a stem-and-leaf diagram. [3]

b Draw, on graph paper, a box-and-whisker diagram to represent the data. [4]

F

In a particular discrete probability distribution the random variable X takes the value $\frac{120}{r}$ with probability $\frac{r}{45}$, where r takes all integer values from 1 to 9 inclusive.

i Show that $P(X = 40) = \frac{1}{15}$. [2]

ii Construct the probability distribution table for X. [3]

iii Which is the modal value of X? [1]

iv Find the probability that X lies between 18 and 100. [2]

O

The equation of a curve is $y = 3 + 4x - x^2$.

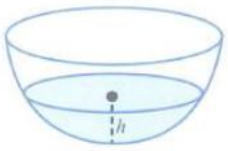
i Show that the equation of the normal to the curve at the point (3, 6) is $2y = x + 9$. [4]

ii Given that the normal meets the coordinate axes at points A and B, find the coordinates of the mid-point of AB. [2]

iii Find the coordinates of the point at which the normal meets the curve again. [4]

B

Water is poured into the hemispherical bowl of radius 5 cm at a rate of $3\pi\,\text{cm}^3\text{s}^{-1}$.

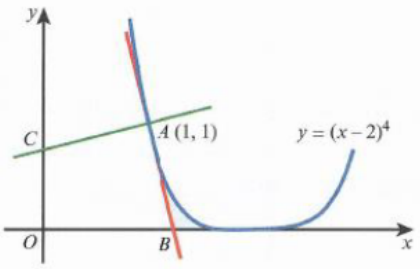


After t seconds, the volume of water in the bowl, $V\,\text{cm}^3$, is given by $V = 5\pi h^2 - \frac{1}{3}\pi h^3$, where h cm is the height of the water in the bowl.

a Find the rate of change of h when $h = 1$.

b Find the rate of change of h when $h = 3$.

A



The diagram shows part of the curve $y = (x-2)^4$ and the point $A(1, 1)$ on the curve.
The tangent at A cuts the x-axis at B and the normal at A cuts the y-axis at C.

i Find the coordinates of B and C. [6]

ii Find the distance AC, giving your answer in the form $\frac{\sqrt{a}}{b}$, where a and b are integers. [2]

iii Find the area of the shaded region. [4]

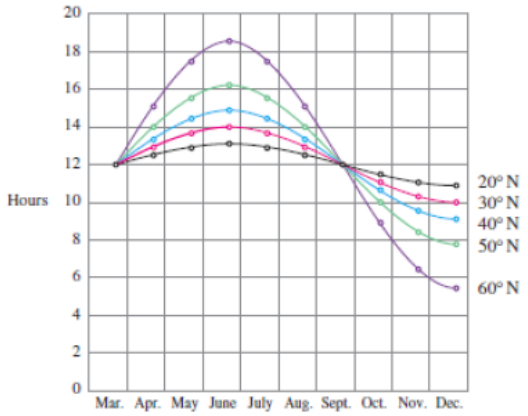
I

A stick is broken at two points at random. What is the probability that the resulting three parts will form a triangle?

Q What is Newton's law of restitution?

L

EXAMPLE 4 Figure 9 shows graphs of the number of hours of daylight as functions of the time of the year at several latitudes. Given that Philadelphia is located at approximately 40°N latitude, find a function that models the length of daylight at Philadelphia.



P

The table shows the number *N* of species of reptiles and amphibians inhabiting Caribbean islands and the area A of the island in square miles.

Model *N* as a function of *A*.

Island	*A*	*N*
Saba	4	5
Monserrat	40	9
Puerto Rico	3,459	40
Jamaica	4,411	39
Hispaniola	29,418	84
Cuba	44,218	76

Appendix 9 – Task sorting 2

Sources: Chalmers (2018); Pemberton (2018); Povey (2015).

A

Solve $\frac{5}{x+2} + \frac{3}{x-4} = 2$. Leave your answers in surd form.

B What is the law of conservation of momentum?

C

a Split 10 into two parts so that the difference between the squares of the parts is 60.

b Split N into two parts so that the difference between the squares of the parts is D.

D Prove that $x^2 + kx + 2 = 0$ has real roots if $k \geqslant 2\sqrt{2}$.

For which other values of k does the equation have real roots?

E P is the point $(a, a-2)$ and Q is the point $(4-3a, -a)$.

a Find the gradient of the line PQ.

b Find the gradient of a line perpendicular to PQ.

c Given that the distance PQ is $10\sqrt{5}$, find the two possible values of a.

F What is the area of an n-sided regular polygon inscribed within a circle of radius r?

G Three ordinary fair dice are rolled. Find the number of ways in which the number rolled with the first die can exceed the sum of the numbers rolled with the second and third dice. Hence, find the probability that this event does not occur in two successive rolls of the three dice.

H The probability distribution for the random variable X is given in the following table.

x	2	3	4	5
$P(X = x)$	p	$2p$	$\frac{1}{2}p$	$3p$

Find the value of p and work out $P(2 < X < 5)$.

I The probability distribution for the random variable X is given in the following table.

x	0	1	2	3
$P(X = x)$	0.10	0.12	0.36	0.42

Calculate $E(X)$ and $Var(X)$.

J Define sec*(x)*

K A THIN HOOP of diameter d is thrown on to an infinitely large chessboard with squares of side L. What is the chance of the hoop enclosing two colours?

L Solve the quadratic equation $ax^2 - bx + c = 0$, giving your answers in terms of a, b and c.

M Given that $X \sim B(12, 0.3)$, find the mean, the variance and the standard deviation of X.

N The first four terms, in ascending powers of x, in the expansion of $(1 + ax + bx^2)^7$ are $1 - 14x + 91x^2 + px^3$. Find the values of a, b and p.

O Without using a calculator, find the exact values of:

i $\sin 120°$

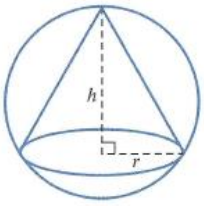
P A curve has equation $y = \left(4 - \sqrt{x}\right)^3$.

The normal at the point $P(4, 8)$ and the normal at the point $Q(9, 1)$ intersect at the point R.

a Find the coordinates of R.

b Find the area of triangle PQR.

Q The diagram shows a right circular cone of base radius r cm and height h cm cut from a solid sphere of radius 10 cm. The volume of the cone is V cm^3.



a Express r in terms of h.

b Show that $V = \frac{1}{3}\pi h^2(20 - h)$.

c Find the value for h for which there is a stationary value of V.

d Determine the magnitude and nature of this stationary value.

R Given that $Z \sim N(0, 1)$, find the following probabilities correct to 3 significant figures.

a $P(Z < 0.567)$ **b** $P(Z \leq 2.468)$ **c** $P(Z > -1.53)$ **d** $P(Z \geq -0.077)$

Appendix 10 – Results for sorting tasks activities

Task sort 1

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
Researcher	3	3	2	2	2	3	2	3	4	1	3	3	2	2	3	4	1	3
Alan	2	2	1	3	1	2	1	3	4	1	3	3	1	1	1	4	1	1
Bianca	2	3	1	2	2	3	1	3	3	1	3	2	1	1	3	1	1	3
Candy	3	1	1	3	3	4	1	2	4	2	2	4	1	1	2	2	1	2
Daniel	1	1	1	4	2	2	1	3	4	1	4	3	1	1	1	4	1	4

Number of tasks in round 1

Level 1	Level 2	Level 3	Level 4	TOTAL
2	6	8	2	18

Task sort 2

	A	B	C	D	E	F	G	H	I	J	K	L	M	N	O	P	Q	R
Researcher	2	1	3	3	3	4	4	3	2	1	4	2	2	3	2	3	3	2
Alan	2	1	3	3	2	3	2	2	1	4	4	1	1	2	1	2	3	2
Bianca	2	1	3	2	3	4	4	3	3	1	4	2	2	2	2	3	3	3
Candy	1	1	4	3	2	4	3	3	1	1	3	2	1	2	2	3	3	2
Daniel	3	1	4	3	3	4	4	2	2	1	4	4	2	3	1	3	3	2

Number of tasks in round 2

Level 1	Level 2	Level 3	Level 4	TOTAL
2	6	7	3	18

Appendix 11

Correct number of answers for sorting activity 1

	Level 1	%	**Level 2**	%	**Level 3**	%	**Level 4**	%	**Total (out of 18)**	%
Alan	2	100%	0	0%	3	38%	2	100%	**7**	**39%**
Bianca	2	100%	2	33%	5	63%	0	0%	**9**	**50%**
Candy	1	50%	0	0%	1	13%	1	50%	**3**	**17%**
Daniel	2	100%	1	17%	2	25%	2	100%	**7**	**39%**
Mean	1.75	88%	0.75	13%	2.75	34%	1.25	63%	**6.5**	**36%**

Correct identification of low/high level in sorting activity 1

	Low level	%	**High level**	%	**Total**	%
Alan	7	88%	5	50%	**12**	**67%**
Bianca	8	100%	7	70%	**15**	**83%**
Candy	6	75%	4	40%	**10**	**56%**
Daniel	7	88%	6	60%	**13**	**72%**
Mean	7	88%	5.50	55%	**12.50**	**69%**

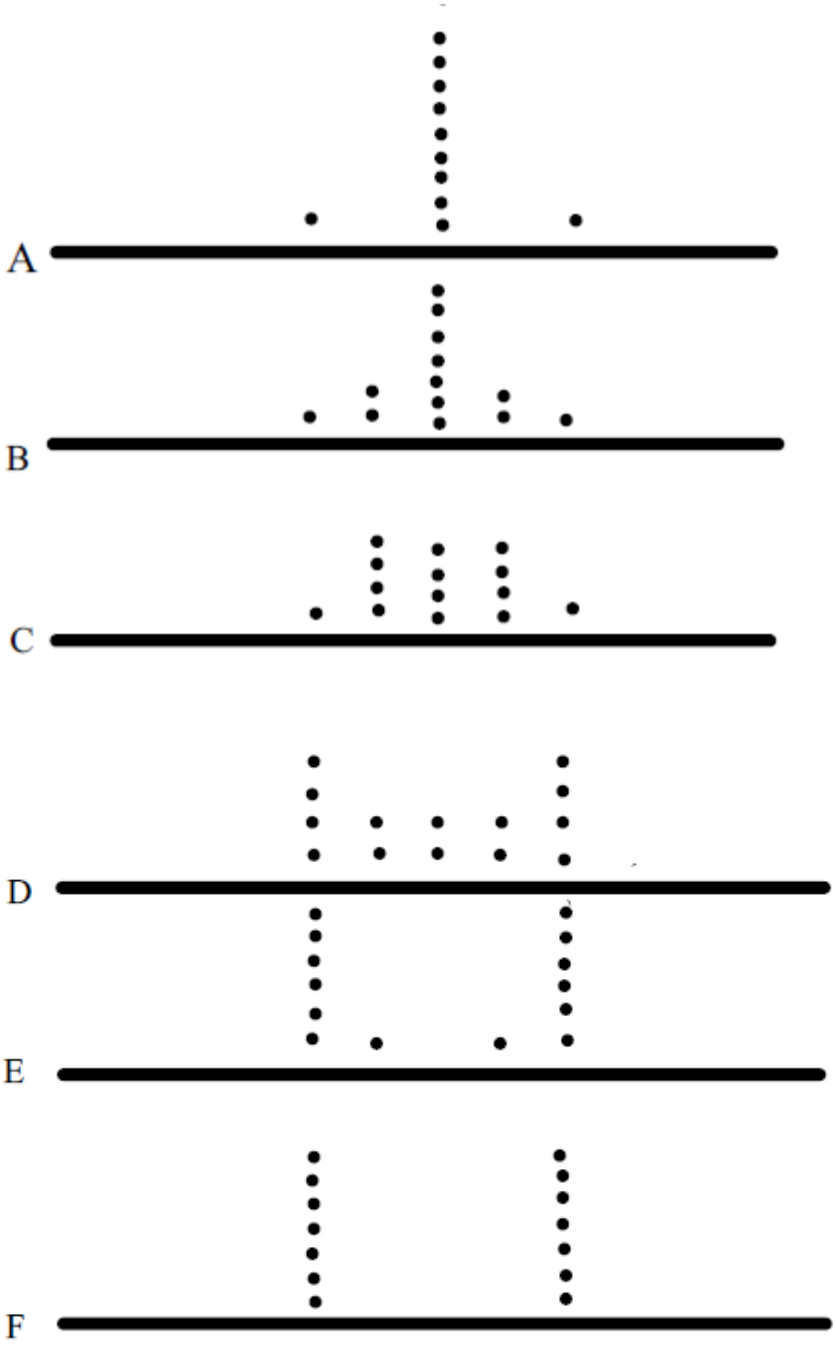
Correct number of answers for sorting activity 2

	Level 1	%	**Level 2**	%	**Level 3**	%	**Level 4**	%	**Total**	%
Alan	1	50%	2	33%	3	43%	1	33%	**7**	**39%**
Bianca	2	100%	4	67%	5	71%	3	100%	**14**	**78%**
Candy	2	100%	3	50%	4	57%	1	33%	**10**	**56%**
Daniel	2	100%	3	50%	5	71%	3	100%	**13**	**72%**
Mean	1.75	88%	3	50%	4.25	61%	2	67%	**11**	**61%**

Correct identification of low/high level in sorting activity 1

	Low level	%	**High level**	%	**Total**	%
Alan	7	88%	5	50%	**12**	**67%**
Bianca	6	75%	8	80%	**14**	**78%**
Candy	8	100%	8	80%	**16**	**89%**
Daniel	6	75%	9	90%	**15**	**83%**
Mean	6.75	84%	7.50	75%	**14.25**	**79%**

Appendix 12 – Diagrams used for Wirkshop 1 (variation theory)



Appendix 13 - List of questions for the semi-structured interview

1. What other materials, apart from textbook, do you use in class?
2. Are you aware of any other resource we have in school?
3. What criteria do you use when choosing tasks for instruction?
4. How do students respond to the tasks?
5. What do you think about the questions in the textbook?
6. Do you think students can learn mathematics by exploring and working on difficult questions?
7. Is it important for students to know where formulae come from or it is enough to just apply them?
8. How confident are you in working on more difficult tasks?
9. What do you think students' reaction would be if exposed to challenging questions?