# Factors Contributing to Children's Understanding of Fractions: Quantitative Reasoning and Arithmetic




Rossana Barros Baertl

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Supervisors: Terezinha Nunes and Lars-Erik Malmberg

Department of Education

Harris Manchester College

## Abstract

The aim of this study was to investigate the origin of children's understanding of fractions by testing a theoretical model in which quantitative reasoning and arithmetic knowledge are proposed as independent and significant factors for learning fractions. The model also includes non-verbal ability as a moderator of how much children learn when they participate in fractions lessons.

Quantitative reasoning is the ability to use relations between quantities to come to new conclusions; for example, knowing that a cake shared fairly among three children and two cakes shared fairly among six children produces equivalent amounts of cake. Arithmetic knowledge is the ability to understand number relations in the operations of addition, subtraction, multiplication and division.

This study used a combination of longitudinal and intervention methods involving three steps: (1) the development of measures and (2) an analysis of their predictive strength in the context of a longitudinal study; and (3) an intervention study.

Participants in the longitudinal study were 9-year-olds (N=124; Year 4) who had received limited formal instruction on fractions. They completed two tests, the Quantitative Reasoning and the Arithmetic Knowledge test, which were subjected to a confirmatory factor analysis to investigate the statistical independence of the measures. They also completed a non-verbal ability test, which was used as a control in the longitudinal predictions. Children were assessed a second time six months later using the same measures to investigate whether the factor structure of the measures was invariant over time. At the final assessment point, 12 months later, at the end of Year 5 and after formal instruction about fractions in school, participants answered a sample of items from a standardised Fractions test (CSMS Fractions) (Hart, Brown, Kerslake, Kuchemann, and Ruddock, 1985).

The confirmatory factor analyses showed a better fit of a two-factor model than of a one-factor model, supporting the hypothesis that the two theoretically different abilities, quantitative reasoning and arithmetic, can be differentiated statistically. Hierarchical regressions (N=105) investigated the independence and relative strength of quantitative reasoning and arithmetic in the prediction of performance in the CSMS Fractions items after school instruction. The regressions showed that each measure contributed independently to the prediction of performance in the CSMS Fractions items, after controlling for non-verbal ability; quantitative reasoning was the stronger predictor of the two.

The intervention study aimed to assess the impact of two fraction intervention programmes on the CSMS Fractions test: one intervention promoted quantitative reasoning and the other promoted arithmetic skills. Both groups were compared to an unseen control group, who participated in the assessments but did not receive any intervention from the researcher. A total of 103 Year 4 children in four schools participated. All children were pre-tested before random allocation to one of three groups. Each of the intervention groups participated in 8 weekly half-hour sessions with a researcher in small groups outside the classroom. Analyses of covariance, controlling for pre-test performance and non-verbal ability, showed a significant effect of the quantitative reasoning intervention: Cohen's *d* effect size=0.44 at the immediate post-test and *d*=0.45 at the delayed post-test. The arithmetic intervention had a significant impact only at the delayed post-test (*d*=0.28). Pre-test performance did not moderate children's learning in the intervention, whereas non-verbal ability acted as a significant moderator at the delayed post-test.

Results confirm that quantitative reasoning and arithmetic are different mathematical abilities. Both make independent contributions to learning fractions; quantitative reasoning is a stronger predictor. This result is consistent with training results, as quantitative reasoning training yields the best performance in fractions learning. This learning remained stable after several weeks and non-verbal reasoning moderates their learning in the long term.

## Publications and Presentations

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# Chapter 1 Introduction

The aim of this research programme is to investigate the origin of children's understanding of fractions by testing a theoretical model in which two different mathematical abilities, quantitative reasoning and arithmetic knowledge, are proposed as crucial for children's learning of fractions.

The first section of this introduction considers the definitions of fractions and the factors to be studied in the proposed theoretical model of children's learning of fractions. This includes a theoretical perspective about how children learn mathematics, in order to highlight two mathematical abilities that could influence their learning. The second section describes the theoretical model proposed for the study. The third section describes the research methods employed to test the theoretical model proposed. Finally, the organisation of this thesis is outlined.

## Definitions

Fractions are one type of rational numbers. The root in the word rational number is "ratio":

> Formally a rational number is a number represented as the ratio of one integer to another. Examples of rational numbers are 2/3 and -5/7. In general, a rational number is a number of the form $n/m$, where n and m are integers and … we must restrict m so that $m \neq 0$. $n/m$ is commonly called a *fraction*. The number n is the *numerator* of the fraction and m is the fraction's *denominator*. (King, 2010, p. 109)

According to a second definition, fractions, "a ratio or division of two whole numbers, numerator and denominator, are considerably more complex than a single whole number" (Torbeyns, Schneider, Xin, and Siegler, 2015, p. 6). The term "fraction" is most often used for a situation in which a whole is divided into parts, the sum of which adds up to the whole; thus the term fraction often denotes extensive quantities (Tolman, 1917).

Rational numbers are used also to represent intensive quantities, in which n and m represent different extensive quantities, and thus they do not add to form a whole. The present research does not cover intensive quantities.

Theoretical perspectives about the origins of understanding fractions have been proposed and differentiated from the origins of whole numbers. Whole numbers come from additive relations, by putting things together and taking them apart (Nunes and Bryant, 1996); fractions come from quotients (Kieren, 1993), from dividing. If a cake were to be divided into two equal parts, each piece would be numerically represented as ½; and if one cake were divided into four equal parts, each piece would be numerically represented as ¼. Although fractions come from division, the relations between the terms in a fraction involve the operation of division and of multiplication as well, because the inverse of ½ is 2. The terms multiplication and division thus refer to arithmetic operations, and the term multiplicative relations refers to the relations between the quantities, such as proportional relations (Vergnaud, 1988). This study tests this perspective of understanding fractions based on understanding multiplicative relations between quantities, as well as representing them symbolically with numbers and being able to carry out operations with such representations.

Many researchers have claimed that it is important to distinguish two mathematical abilities, one related to thinking about quantities and the other related to thinking about numbers, which are a symbolic representation for quantities and for relations between quantities (Nunes and Bryant, 2015; Nunes, Bryant, Barros, and Sylva, 2012a; Piaget, 1952; Thompson, 1993; Vergnaud, 1979).

Importantly, Thompson (1993) notes that:

> Quantities, when measured, have numerical value, but we need not measure them or know their measures to reason about them. You can think of your height, another person's height, and the amount by which one of you is taller than the other without having to know the actual values. Quantities are more concrete than numbers. (p. 3)

Quantitative reasoning is "the ability to represent quantitative information and act on the representations to come to conclusions not previously known about the quantities represented or about the relationships between them" (Nunes, Bryant, Evans, and Barros, 2015, p. 178). Tasks that assess quantitative reasoning demand thinking about the quantities and the relation between them, and children can reason about quantities smaller than a unit without using a fraction as a numerical representation. For example, a child can understand that if a pizza is shared equally between four friends, each one would get a smaller piece than if the same pizza was equally shared between two friends. This requires understanding of multiplicative relations between quantities, as the bigger the number of sharers the smaller each share will be. Another example: a child can recognise that if a pizza is cut in half and one of the halves is cut in half again, although you can see three pieces, the piece resulting from cutting one half in half is a quarter of the whole, even without knowing the word "quarter". The child demonstrates this by explaining that the whole pizza would be four equal pieces, which are the same size as the one resulting from cutting one half in half. That understanding would demonstrate reasoning about multiplicative relations between quantities, by using quantitative information to come to a conclusion about the quantities represented.

In contrast, arithmetic is the ability to understand relations between numbers and to operate on them. Guedj (1997) defined arithmetic as "the science of numbers, which analyses the behaviors of various numbers in four operations: addition, subtraction, multiplication and division" (Guedj, 1997, p. 63). Numbers have meaning on their own, defined by the numerical system – for example, 5 means 4 + 1 (Nunes and Bryant, 1997, 2015). Learning about fractions also involves learning about the relations between fractions (i.e. equivalence, order) and how to operate with fractions. Calculating with fractions draws on children's previous knowledge of arithmetic with whole numbers to the extent that they need to use addition, subtraction, multiplication and division facts when implementing calculations with fractions, even though they need to follow new procedural rules. For example, to solve $\frac{2}{3} + \frac{3}{7}$ they need to know how to multiply 3 x 7 to find a common denominator, how to divide 21 by 3 and multiply 7 by 2, and they also need to know how to add 14 + 9 and follow the procedural steps to solve this sum. Rules for calculating with fractions can be taught to children in school without making an explicit connection between the calculation and understanding fractions.

There is ample evidence that children find fractions challenging. For example, although children realise that if three people share a pizza they will get more than if five people share an identical pizza, they think that 1/3 is smaller than 1/5 (Mack, 1995; Nunes and Bryant, 2008), because 3 is smaller than 5. There is also evidence that understanding fractions is important for further mathematical learning and more advanced mathematical topics (Booth and Newton, 2012; Hecht, Close, and Santisi, 2003; Nunes and Bryant, 2009a, 2009b; Thompson and Saldanha, 2003; Torbeyns et al., 2015); for example, in algebra they might need to know the inverse of ½. This prompts the question as to whether there are any identifiable factors in children's mathematical abilities before they are taught

about fractions that could facilitate the learning process. In other words, do quantitative reasoning and arithmetic with whole numbers contribute to children's learning of fractions independently? If so, what is their relative importance for learning fractions?

**Theoretical model**

Although extensive research about children's learning of fractions has been conducted, the origins of fractions knowledge and the relative contribution of previous mathematical abilities to learning about fractions remains uncertain. It is unclear whether it is better to focus on children's ability to reason quantitatively or to focus on their arithmetical skills in order to prepare them for learning about fractions.

Previous research has established the importance of each of these two cognitive abilities to children's mathematical success later (De Smedt et al., 2009; Geary, 2011; Jordan et al., 2013; Nunes et al., 2012a; Nunes et al., 2007), but has not tested their relative contribution to the success specifically for learning fractions when the children are formally taught in school.

Nunes and her colleagues (Nunes et al., 2012a; Nunes, Bryant, Pretzlik, and Hurry, 2006) investigated the relation between quantitative reasoning, arithmetic and mathematical achievement using longitudinal and intervention studies. One of their longitudinal studies included measures of both quantitative reasoning and arithmetic and analysed their relative predictive value for mathematics achievement at school (Nunes et al., 2012a). This study investigated the contribution of quantitative reasoning to predict achievement in the domain of whole numbers. Nunes and her colleagues (2012a) found that both quantitative reasoning and arithmetic made independent contributions to the prediction of mathematical achievement after five years; quantitative reasoning was the

strongest predictor even after controlling for general intelligence, working memory and arithmetic, assessed by WISC-III (Wechsler, 1991).

Intervention studies have also found that teaching children about quantitative reasoning related to fractions can lead to better outcomes on a fractions test. In their randomised controlled study, Nunes, Bryant, Pretzlik, and Hurry (2006) used a training programme on quantitative reasoning about fractions, which ran over five to eight sessions. They found significant differences between the intervention group and the control group at the immediate post-test and at the delayed post-test taken eight weeks later. They stated that their effect size was small but remained constant after two months. This study did not include an intervention on arithmetic knowledge; to the best of my knowledge no study has compared two interventions, one based on quantitative reasoning and one on arithmetic, to compare their impact on children's learning of fractions.

Longitudinal (Jordan et al., 2013) and intervention studies that aim at improving children's arithmetic knowledge of fractions have also found promising results at improving children's knowledge of fractions (Cramer et al., 2019; Jordan et al., 2013; Saxe, Diakow, and Gearhart, 2013; Saxe et al., 2007). Number line interventions are based on the relations between numbers without a reference to quantities. Saxe and his colleagues (2013) reported an intervention study in which 4th- and 5th-grade classrooms were matched and assigned either to receive the number line intervention or be the comparison group. The intervention was implemented for 19 days, of which 10 days were focused on integers and 9 on fractions. The intervention included professional development for the teachers, 3 days and 4 evenings in total. The aim of the professional development was to build teachers' knowledge of mathematics content; teachers solved and discussed number line problems along with other activities such as role-played key

lessons. Two evenings of the professional development were focused on fractions. Their results showed that overall children in the experimental group outperformed children from the comparison group. The fractions subtest showed a significant difference between the groups, confirming that working on a number line can have a positive impact on the results of a fractions test. Similarly, Cramer and her colleagues (2019) reported an intervention study using a number line task which leads to children's improvement in fractions understanding. These studies show that teaching children about arithmetic, specifically about relations between numbers on the number line, can lead to improvement of outcomes in a fractions test.

These studies provide evidence that both mathematical abilities, quantitative reasoning and arithmetic, are important and can improve children's learning of fractions, but no study has compared their relative importance in learning fractions, which this study aims to test. Although studies have shown that individual differences in quantitative reasoning and general mathematical abilities are relatively stable (Nunes et al., 2012a; Nunes et al., 2015; Strand, 2004), they can be changed by instruction (Cramer et al., 2019; Nunes et al., 2007; Saxe et al., 2013). In this study, two intervention programmes will be designed aiming to change children's knowledge of fractions and the difference in their effectiveness will be assessed.

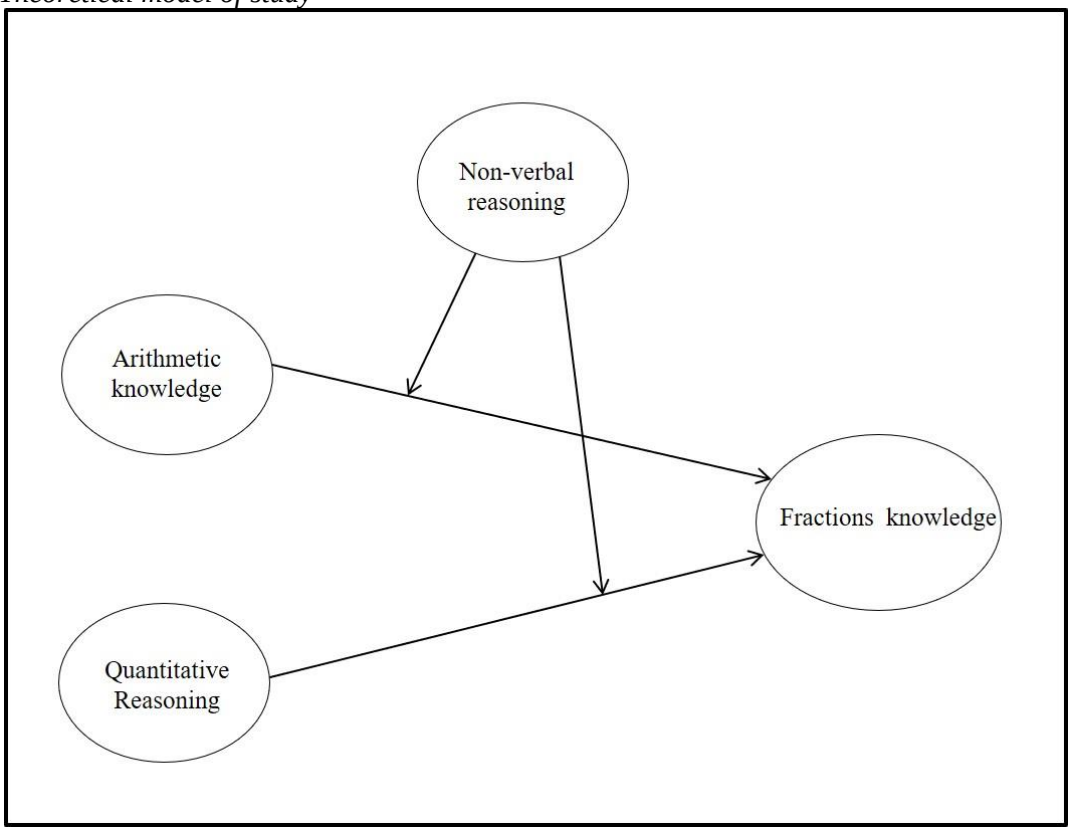
Children certainly are expected to differ before participating in an intervention to improve their knowledge of fractions: they can differ in what they already know and also in general cognitive abilities. There undoubtedly are individual differences in children's knowledge of fractions and in how they learn about fractions (Barros, 2007; Hallett, Nunes, and Bryant, 2010; Hallett, Nunes, Bryant, and Thorpe, 2012; Hecht, 1998; Hecht et al., 2003; Siegler and Pyke, 2013; Vukovic et al., 2014) and many researchers have

attempted to predict these individual differences (Barros, 2007; Hallett et al., 2010; Hallett et al., 2012; Jordan et al., 2013) as well as to control for them statistically when children participate in an intervention. Previous studies have also controlled for cognitive differences between children participating in intervention studies. In this study, the aim is not to control for cognitive differences but to investigate whether these differences moderate the outcomes of an intervention. To the best of my knowledge, no study so far has tested whether non-verbal ability is a moderator of how well children learn about fractions in an intervention study. It is quite plausible that in any intervention study there are cognitive moderators that affect the effectiveness of the intervention, but so far there is little information about this effect in intervention research.

The theoretical model presented in Figure 1 is proposed for this study, which aims to test the importance of these two mathematical abilities, quantitative reasoning and arithmetic, for learning fractions from instruction and the role of children's non-verbal ability as a moderator of their learning of fractions. In view of previous longitudinal and intervention studies (Hallett et al., 2010; Hecht et al., 2003; Nunes et al., 2012a; Nunes et al., 2006), it was hypothesised that quantitative reasoning would be a stronger predictor of children's ability to learn fractions when these are formally introduced in the school curriculum the following year. It was also hypothesised that, given the specific connection between arithmetic with whole numbers and arithmetic with fractions, children's knowledge of arithmetic will also predict how well they learn the new rules for calculating with fractions, but the connection between knowledge of arithmetic with whole numbers and learning of fractions is predicted to be weaker than the connection between quantitative reasoning and learning of fractions.

The proposed theoretical model (as shown in Figure 1) also considers the role of children's non-verbal ability in their learning. In line with the model, predictions for the intervention study are that an intervention that improves children's quantitative reasoning about situations that involve fractional quantities as well as an intervention that improves their knowledge of arithmetic will have an impact on their performance on an outcome measure that assesses knowledge of fractions. This impact will be moderated by their cognitive ability.

**Figure 1**
*Theoretical model of study*




A child with higher cognitive abilities may be able to transfer their arithmetic knowledge to quantitative reasoning knowledge or their quantitative reasoning knowledge to arithmetic knowledge, but a child with lower cognitive abilities may not be able to transfer between these two different types of knowledge.

**Research strategy**

The research project was divided into four main steps:

1. the development of the measures of quantitative reasoning and arithmetic knowledge;
2. a longitudinal study to assess their relative importance in the prediction of fractions learning measured at a later time point;
3. an intervention study, which evaluates whether teaching children about quantitative reasoning or about arithmetic have similar or different impact on children's understanding of fractions;
4. an analysis of cognitive ability as a moderator of the impact of the interventions on the outcome measure.

**The development of measures**: It was reasoned that the development of the measures to assess their statistical difference was an important first step for this thesis. Two theoretically different mathematical abilities are proposed as precursors for learning fractions; however, one may argue that these two abilities, although theoretically different, may be closely related empirically. The selection of measures that assess children's understanding of quantitative reasoning and arithmetic is of great importance in a study that investigates their relative importance for their future learning of fractions. If the measures are closely related, they would not be able to be statistically independent and therefore it would be impossible to compare them as predictors for learning fractions.

**The need for a longitudinal and an intervention study:** Bradley and Bryant (1983) stated that to make a foundation claim, such as what is the origin of fractions knowledge, it is important to combine longitudinal and intervention designs because each design has a

weakness that the other one can compensate in the support of a foundation model. They argued that:

> Properly controlled training studies demonstrate cause-effect relationships, but these could be arbitrary; one cannot be sure that such relationships exist in real life. On the other hand longitudinal studies which control for other variables such as intelligence demonstrate genuine relationships; but it is not certain that these are causal… would not exclude the possibility that both are determined by some unknown *tertium quid*. (Bradley and Bryant, 1983, p. 419)

Thus, their weaknesses and strengths together would allow for making a foundation claim. A combination of longitudinal and intervention data in the same study is rare, but it has been implemented to investigate foundation connections both with literacy and mathematics learning (Bradley and Bryant, 1983; Nunes et al., 2007).

It was reasoned that in order to investigate the origin of children's understanding of fractions, the theoretical model should be tested using a combination of longitudinal and intervention methods. Together the combination of methods would provide robust evidence to investigate the origins for learning fractions. Thus, the second and third step are a longitudinal and intervention study respectively.

On the one hand, the longitudinal study aims to test the relation between quantitative reasoning and arithmetic with learning fractions over time. This study tests how children's knowledge of quantitative reasoning and arithmetic may impact their learning of fractions when they are introduced in school the following year. The longitudinal study will test the relationship between children's knowledge of quantitative reasoning and arithmetic with their learning of fractions over time, controlling for non-

verbal ability. This is important as non-verbal ability may have common variance with these abilities, and to claim a genuine relationship one must control for possible underlying abilities.

On the other hand, the intervention study investigates the cause and effect of two training programmes, one on quantitative reasoning and one on arithmetic, by comparing each to a third group that did not participate in any training. Any difference in an outcome measure of children's fractions knowledge, after they received (or not) one of the trainings, could be attributed to the training.

**Cognitive ability as a moderator of the effects of an intervention:** It has been long stated that cognitive abilities play a crucial role in learning, including mathematics (Inhelder and Piaget, 1958; Kyttälä and Lehto, 2008). There is vast evidence that longitudinal and intervention studies should include meaningful control variables, such as cognitive abilities, as these act as explanatory variables in the social sciences. For example, non-verbal ability may act as the underlying driving variable for children's learning.

It was reasoned that if cognitive abilities play a role in learning, they could moderate the effect of an intervention aiming at children's learning of fractions. There is limited research on the moderator effect of general cognitive processes on fractions learning. Recently, Fuchs and his colleagues (2014) found that working memory could moderate the effect of a fraction's intervention for students with mathematics difficulties.

Non-verbal ability is the capacity to analyse information to find patterns, solve problems and find relations using visual, or hands-on reasoning. This thesis considered non-verbal ability as the cognitive ability which could moderate children's learning of fractions.

**Context of this research**

This research study seems to be timely to investigate the differentiation of these abilities because in 2016 separate reasoning and arithmetic tests were introduced in England for primary school children. These national tests are designed to assess the new curriculum. The Standards and Testing Agency of England claim that these changes reflect areas of the curriculum that have changed (Standards and Testing Agency of England, 2016).

Rational number was included in the English National Curriculum (2014) and it was expected that children would be taught more systematically about fractions during Year 4 and Year 5 of primary school. The statutory requirements of the National Curriculum in England for Year 4 children in fractions are: (1) to recognise equivalent fractions, (2) to add and subtract fractions with the same denominator, (3) to write decimal equivalents to some fractions and to solve problems involving fractions to calculate quantities, (4) to use fractions to divide quantities including non-unit fractions where the answer is a whole number. For Year 5 children the statutory requirements are: (1) to compare and order fractions with the same and different denominators, (2) to identify, name and write equivalent fractions, (3) to recognise numbers and improper fractions and to convert to one another, (4) to write decimals as fractions, (5) to add, subtract and multiply fractions. The multiplication and division section for both Years 4 and 5 refers mainly to arithmetic learning, such as recalling multiplication tables; there is no mention of reasoning with quantities.

Although the English National Curriculum clearly states that children should learn about fractions, the small number of questions related to fractions in the Key Stage Tests is surprising; other standardised mathematical achievement tests, such as Progress in Maths

(NFER), which are designed aiming to assess expected learning from the National Curriculum, also include very few questions on fractions.

More specifically, the national standardised tests of mathematical achievement, designed by the UK government and administered by teachers at the end of Key Stage 2, present very few items related to fractions. These tests are taken when children are aged 11 approximately, undertaken in Year 6. Similarly, the NFER test Progress in Maths 10 only presents 4 items out of 31 to assess fractions knowledge; while Progress in Maths 11 (Clausem-May, Vappula, and Ruddock, 2004) presents 6 items out of 32 to assess fractions knowledge. Thus, despite fractions being in the curriculum, they do not receive the importance they deserve as a new number and a basis for further mathematical knowledge.

This study is of both practical and theoretical importance. The practical importance of this study is that it could potentially influence educational approaches to teaching fractions; to regulating their assessment; and to understanding children's general cognitive development. It might provide evidence that specific experiences which a child could have before they receive formal teaching of fractions could act as precursors for this learning. The theoretical importance of this study is that it could also contribute to the understanding of individual differences in cognitive ability and the impact this has on learning about fractions. The results from this study would provide evidence for establishing that quantitative reasoning and arithmetic act as the foundation for fractions knowledge.

**Outline of thesis**

The thesis is divided into six chapters. In the following paragraphs an outline of each chapter is provided.

Chapter 2 presents the literature review. There has been a great deal of research concerning children's learning of fractions; the literature review here focuses on evidence directly relevant to the relations between quantitative reasoning, arithmetic and learning fractions and to the educational debates about teaching and learning fractions. The chapter has six sections. The first section includes differences between whole numbers and rational numbers. The second section presents a cognitive approach to mathematical learning which includes the theoretical aspects of cognitive development of mathematics. The third sections offer a general debate about teaching mathematics, the conceptual and procedural knowledge debate, their differences and its relation to empirical evidence on children's learning of fractions; it also reviews specific educational debates about fractions, which include previous research on quantitative reasoning and arithmetic and its relation to children's success to learn fractions. The fourth section examines evidence of research programmes about how to teach fractions. The fifth section revises the role of general cognitive abilities in learning fractions. Finally, the last section presents the rationale and hypotheses of the present study.

Chapter 3 presents the first step of the research project, the development of the two measures. One measure aims to assess children's quantitative reasoning and the second measure aims to assess children's arithmetical knowledge. Although these are two theoretically different abilities, it is important to establish whether these two constructs are distinguishable at the empirical level. The data was analysed using confirmatory factor analyses, which allows a researcher to test an a priori construct. It was expected that the two tests would derive two separable, but correlated, constructs. To test if these were two separable constructs, two models were compared in terms of their model fit. The models were (a) a two-factor model, in which the items of each of the constructs are loaded to its

correspondent factor, compared to (b) a one-factor model, in which the items from both tests loaded together into one factor. If a two-factor model presents a better model fit, the results would justify the distinction of quantitative reasoning and arithmetic as two empirically different mathematical abilities. The results from this chapter will provide the basis for a longitudinal study using the measures to test the relative importance of these two abilities towards the learning of fractions.

Chapter 4 describes a longitudinal study that examined if quantitative reasoning and arithmetical knowledge are independent predictors of children's attainment in a measure of fractions knowledge twelve months later, after they were taught about fractions in school. A description of the participants, measures and methods is presented, as well as the analyses and results. Hierarchical fixed-order multiple regression analyses were used to test whether quantitative reasoning and arithmetic made independent contributions to the prediction of performance in a test of knowledge of fractions. This kind of analysis allows one to compare the contributions made by each ability, quantitative reasoning and arithmetic, even after controlling non-verbal ability. These results provide the basis for an intervention study.

Chapter 5 presents the intervention study. The intervention study aims to compare the effect of two different intervention programmes, one related to quantitative reasoning and one related to arithmetic, with a third group that acts as a control, and to test the role of cognitive ability as moderator of the impact of the intervention. The study was carried out with children in Year 4. After all children were pre-tested, children were randomised within schools and assigned to one of three groups: (1) a Quantitative reasoning intervention group; (2) an Arithmetic Knowledge intervention group, and (3) an unseen Control group, which only participated in pre-, post- and delayed post-test assessments and

did not receive any training. The interventions were carried out by the student researcher in small groups of six children at the school, during eight weekly sessions delivered in approximately 30 minutes per session. It was expected that the children who received the quantitative reasoning training would make the most improvements in their learning of fractions, followed by the arithmetic knowledge group and then the control group. Analyses of covariance were carried out to compare the results between the three groups, taking into account their pre-test scores, non-verbal ability and the effect of a moderator term to consider whether non-verbal ability differentiates children's performance across intervention programmes.

The empirical and theoretical contributions that can be drawn from these three studies in relation to children's learning of fractions are discussed in Chapter 6, as well as their strengths and limitations and suggestions for future research. The most important contribution of this thesis is the test of a theoretical model about the origins of learning fractions based in the independent contribution of each of these points.

# Chapter 2 Literature Review

The aim of the current chapter is to present the background to the theoretical model for this study and relevant past research in the domain of fractions. The theoretical model proposed is based on the idea that two main abilities underlie mathematics learning: children's understanding of relations between quantities and children's understanding of arithmetic. Both of these specific abilities are related to general cognitive skills, which predict and mediate the learning of these specific skills when children are taught mathematics in school. The research will investigate the relative importance of these two abilities for learning about fractions and the moderating role of cognitive skills. The relations between quantities relevant to understanding fractions are multiplicative relations between quantities, which encompass the operations of multiplication and division. Children's numerical abilities with fractions include understanding fractions as numbers and arithmetic rules. In this chapter evidence supporting that these two factors are important for children's fractions learning is reviewed.

This review is divided into six sections. The first section compares whole numbers with rational numbers and evidence regarding difficulties with this new type of number when children are introduced to it are presented. In the second section a cognitive approach to children's mathematics learning that forms the basis of this thesis is described. In the third section, the debate in mathematics education about differences between conceptual and procedural knowledge and the relations between them is presented. It is argued that this debate could be reformulated by adopting the distinction between quantitative reasoning and arithmetic knowledge as aspects of mathematics learning, including fractions learning. The third section also reviews evidence about the measurement of conceptual and procedural knowledge as well as the measurement of

quantitative reasoning and arithmetic. The fourth section presents specific debates related to fractions teaching with empirical evidence supporting quantitative reasoning and arithmetic. The fifth section briefly discusses the role of general cognitive abilities in mathematics learning and also specifically in fraction learning. Finally, the sixth section presents the rationale and the hypotheses of this study.

## 2.1 Differences between whole numbers and fractions

Most of the mathematics curriculum up to Year 4 in England is about whole numbers: what they represent, how they are written, and how to calculate with them. When fractions are introduced, fractions are a new type of number for children, and it is expected that, when children are exposed to something new, they interpret it using their previous knowledge.

Children learn that the properties of whole numbers don't apply to all numbers when they encounter fraction arithmetic (Vamvakoussi and Vosniadou, 2010). For example, until then they expect that all the numbers go in order of magnitude or that a multiplication always leads to a larger number, but that is not true for fractions. Fractions are a new kind of number. Many have studied this bias to applying whole number properties to fractions as the whole-number bias (Behr, Lesh, Post, and Silver, 1983; Mack, 1995; Obersteiner, Van Dooren, Van Hoof, and Verschaffel, 2013; Siegler, Thompson, and Schneider, 2011; Van Hoof, Verschaffel, Ghesquière, and Van Dooren, 2017).

Nunes and Bryant (2009b) described some differences between rational and whole numbers:

(a) When children learn to count, they learn numbers in an *order* which also represents their order of magnitude. With rational numbers this is different

because the order of fractions depends on the relation between the numerator and the denominator. If the denominators are the same, then the larger the numerator the larger the fraction. However, it is the opposite when the numerators are the same: the larger the denominator, the smaller the fraction.

(b) Another difference between rational numbers and whole numbers is that two fractions may be different numbers but represent *equivalent quantities*, for example ½ is equivalent to 2/4. But it is also possible that two fractions may use the same number label and represent different quantities: for example, ½ of 10 and ½ of 20.

(c) Differently from whole numbers, fractions are numbers represented by two digits with an inverse multiplicative relation between them, such that the larger the denominator, the smaller the quantity represented, when the numerator is held constant.

(d) The *density* of numbers is different in rational and whole numbers; there are infinitive fractions between two whole numbers. This in contrast to whole numbers where there are no natural numbers between, for example, 2 and 3.

Children must recognise fractions as a new number and that properties and rules that apply to whole numbers don't apply to all numbers including fractions. One property that applies to all real numbers is that they possess magnitudes and that numbers can be ordered on number lines according to their magnitude (Siegler, Fazio, Bailey, and Zhou, 2013). Studies have aimed to assess if children with maths difficulties (Van Hoof et al., 2017), normally developing children (Vamvakoussi and Vosniadou, 2010), and adults with different levels of expertise in mathematics (Obersteiner et al., 2013; Van Hoof, Lijnen,

Verschaffel, and Van Dooren, 2013) present a tendency to apply whole number properties to fractions.

Van Hoof et al.'s (2017) study aimed to investigate rational number understanding of learners with dyscalculia compared to learners without dyscalculia. They compared three groups of children: 6th graders with dyscalculia (n=16), a chronological age match group (6th graders without dyscalculia, n=56), and an ability match group (4th graders, n=51). Students with dyscalculia also presented other difficulties, but they showed the same level of intelligence and had the same home language (Dutch). All participants completed a shortened version of the Rational Number Sense Test (RNST; Van Hoof, Verschaffel et al., 2015) as a measure of their rational number understanding. This test instrument was constructed based on a broad literature with the aim to measure learners' natural number bias in rational number tasks. They were also assessed on non-verbal ability (Raven's progressive matrices), verbal intelligence (SIBO test), reading achievement, mathematics achievement, word recognition ability and a second test of reading ability. Analyses of covariance showed that, even after controlling for learners' age, IQ and reading achievement, the 6th-grade control group significantly outperformed learners with dyscalculia on consistent rational number tasks (natural number knowledge leads to correct answers) $F(1.67) = 4.35$, $p = 0.04$, $\eta 2 = 0.06$, and on inconsistent rational number tasks (natural number knowledge leads to incorrect answers), $F(1.67) = 102.16$, $p < 0.001$, $\eta 2 = 0.60$. Their results showed that 60% of the variance in learners' accuracy on inconsistent items could be explained by whether a learner was in the group of learners with dyscalculia, while this was only 6% of the variance for the consistent rational number items. The same analyses were carried out to compare children with dyscalculia and children in Year 4, who were the mathematical ability comparison group; no significant

difference was found between both groups' accuracy on consistent and on inconsistent rational number tasks. Their results showed that the rational number understanding of learners with dyscalculia is significantly lower than that of typically developing peers, but not significantly different from younger learners who acted as the mathematical age comparison group, after statistically controlling for mathematics achievement. Learners with dyscalculia are more affected by the natural number bias compared to their peers, which leads to even more difficulties in understanding the rational number system.

Vamvakoussi and Vosniadou (2010) aimed to investigate the development of understanding density of rational numbers, and to show that this understanding is demonstrated at different levels depending on the type of number. They hypothesised that age and level of instruction would influence the students' understanding of number density. The participants were 181 7th graders, 166 9th graders and 202 11th graders (a total of 549). All students completed one of two multiple-choice questionnaires, each consisting of 14 questions. The questionnaires asked how many finite or infinite number of numbers and what type of numbers lie between two rational numbers. They showed students' mean performance in the four item groups (natural numbers, natural and decimal numbers, decimals and fractions). The results showed main effects for item group type [$F(3, 546) = 52.5$, $p < .001$], and for grade [$F(2, 546) = 25.4$, $p < .001$]. Their results exposed that students tended to believe that numbers must be of the same type as the interval numbers (only decimals between decimals) and that the type of number shown as the interval endpoints (natural numbers, decimals or fractions) influenced students' judgments of the number of intermediate numbers in those intervals. Based on their results, they concluded that is difficult for students to see a set of rational numbers as a unified system of numbers and that students presented whole number bias.

Obersteiner et al. (2013) aimed to examine if expert mathematicians with high levels of mathematics knowledge overcome the whole number bias on fraction comparison problems, and if they use componential or holistic mental representations to solve these problems. They hypothesised that although the experts would be able to solve all the fraction problems presented, they would show some whole number bias in terms of longer response times in some items, which are incongruent with whole number properties. A total of 44 staff members of the Department of Mathematics or Computer Science in a Belgian university, who were studying mathematics or working as mathematicians, took part in the study. The participants were invited to individual sessions in which they were presented with the items on a laptop using a software which recorded their response time and accuracy rate. Items were designed to represent different types of items related to whole number bias, in two blocks that were presented in random order. At the beginning of each block, participants were informed about the types of items that would be presented in the following block (numerators or denominators would be equal/would not be equal). The items presented two fractions at a time, and participants had to choose the larger fraction. There was no time limit, but the tester asked the participants to answer the 90 items as quickly and as accurately as possible. They used a general estimation of equation approach to compare response times between the different types of items within each of the two blocks. Items were selected for the analyses to avoid the items where participants could use a benchmark strategy, for example close to 1. Within block items, there was a significant effect of congruency [$\chi^2(1, N=44) = 61.09, p < .001$], with larger response times for incongruent than congruent items, confirming a natural number bias. Their results show that even experts tend to be biased by natural numbers when fraction pairs have common numerators or denominators.

These studies show some of the difficulties that fractions present, such as density or fraction magnitude, mainly because of a whole number bias, thus recognising that a fraction is a different kind of number, and the importance of research on children's understanding of fractions.

In conclusion, research on whole number bias shows that children and even adults have not become fully aware of the fact that rational numbers are a new type of number, which differs from the type of number about which they have been taught previously and which they have used to represent quantities to calculate. Since children learn whole numbers first, if rational numbers are not introduced as a new concept then these differences could become obstacles to learning fractions. Despite these differences between whole numbers and rational numbers, there may be some concepts that could provide continuity between whole numbers and fractions, which could serve as a precursor for learning fractions. This study will investigate if multiplicative reasoning and arithmetical knowledge can provide that continuity.

## 2.2 A cognitive approach to mathematics learning

Psychologists and educators have studied cognitive processes and how children develop concepts and ideas that allow them to acquire mathematical skills. These perspectives describe how children reach certain levels of mathematical understanding that allow them to learn further. Children reach the understanding of concepts that may develop in steps or simultaneously, but children's development of mathematical skills is based on their understanding of these concepts.

To understand a particular aspect of mathematical development, such as how children understand fractions, it is necessary to revise the approach taken in this research

programme as to what counts as mathematical knowledge, and more specifically as fraction understanding. Based on Piaget (1965), Vergnaud (1979, 1988) and Nunes and Bryant's (1996) views of mathematical knowledge, this thesis recognises mathematical knowledge and learning as a combination of concepts, schemas of action, invariants, situations and symbolic representations. The first section of this chapter presents these general aspects of cognitive development.

**Schemas of action and invariants**

A schema of action refers to "a cohesive, repeatable action sequence possessing component actions that are tightly interconnected and governed by a core meaning" (Piaget, 1952, p. 7). Piaget's theory proposes that experiencing a situation several times in slightly different ways helps the child to acquire meaning and to transfer this meaning to similar situations. A mathematical experience is experience in which the information comes from the child's own actions and from the coordination of his/her actions. When the child finds equilibrium between external information and his/her internal structure for these activities, this knowledge becomes a schema, a unit of knowledge related to objects, actions and abstract concepts (Piaget, 1977).

Piaget emphasised that, after acquiring a schema of action, children must reach reversible thinking, as the coordination that permits a mental construction back and forth; the child realises that a change may be cancelled out by mentally reversing the steps and returning to how it was to begin with (Piaget, 1977, p. 12): for example, putting things together and taking them apart, as pieces of a whole. Piaget believed that children must understand the reversibility of transformations in order to be credited with full understanding of a concept and of the operations involved; for example, they must

understand that subtraction is the inverse of addition and that multiplication is the inverse of division.

Piaget, Inhelder, and Szeminska's (1960) study of children's understanding of fractions was part of their study of the development of length and area concepts. They observed children from 3 to 8 years in situations in which they had to divide a length or area into an equal number of parts. Piaget proposed a task that required the child to be able to reverse from whole to parts, back and forth, and to understand the part–whole and part–part relationships. Reversible thinking should make it possible for children to understand proportions and fractions.

Two schemas of action that have been proposed to support children's understanding of multiplicative structures, thereby relevant to learning fractions, are: one-to-many correspondence and sharing (Nunes and Bryant, 1996). One-to-many correspondence refers to the relation between two quantities, such that one item of the first quantity corresponds to several of the second set, mainly related to multiplication. Children first use correspondences to solve multiplicative reasoning problems. Sharing refers to the action of dividing a set of items between a different set of items, which is mainly related to dividing. Nunes and Bryant's (2015) research evidenced that multiplicative reasoning has its origins in children's ability to use correspondences and maintain fixed ratios when solving multiplicative problems. They proposed that one-to-many correspondence is the core of all multiplicative reasoning, showing that young children can use one-to-many correspondences as a schema of action to solve multiplicative reasoning problems. Nunes and Bryant recognised that very young children may not be aware of the ratio between the quantities, but the situation and manipulatives allow them to apply the schema in action and solve problems. Nunes and Bryant proposed

that children face sharing situations efficiently, understanding the relation between the dividend, the divisor and the quotient. These relations may seem simple but imply the understanding of different types of relation: for example, the more people sharing something, the less each person will receive, which is an inverse relation; the more items to be shared, the more items each person will receive, which is a direct relation. Considering multiplicative reasoning, they recognise that, in order to understand fully multiplicative problems, one must understand that there is an inverse relation between multiplication and division, however they propose that "these two operations originate from different schemas of action, and this means that children need to learn to coordinate the two concepts later on, in order to attain a more integrated understanding of multiplicative situations" (Nunes and Bryant, 2015, p. 734).

Schemas provide the basis for children to understand the invariants of concepts: the notion of operational invariant is associated with mathematical objects, their properties and their relations (Vergnaud, 2009). In fractions, an invariant is, for example, equivalence (Mamede and Nunes, 2008); the ratio that keeps fractions equivalent remains invariant.

Piaget's ideas of learning by action and reversibility of thinking provided a basis for many future theories (e.g. Steffe, 1994, 2001; Thompson and Saldanha, 2003) that influenced the theoretical model proposed in this thesis. Nunes and Bryant's proposal is that two action schemas, one-to-many correspondences and sharing all allow children to establish fixed ratios between the two variables in multiplicative situations. Their theory proposes that multiplicative reasoning schemas support children's understanding of rational numbers. These ideas are central to this thesis as it applies to fractions, through sharing and ratio; the assessment and intervention designed for this research project will include these aspects.

**Situations**

Vergnaud (1988, 1997a) proposed his Theory of Conceptual Fields to describe how mathematical concepts develop under different categories of situations and the procedures used to deal with these situations. He proposed that children learn different concepts and how to apply them in different situations over a period of time. He believed it is important to provide many different situations, not only to make the learning meaningful but also to lead to full mastery of the concept. In the educational context, the teacher is the one who intervenes by choosing the situations in any conceptual field (Vergnaud, 2013). He proposed that these conceptual fields consist of situations related to additive structures or multiplicative structures, for example, making learning a concept meaningful. In that sense a conceptual field is:

> at the same time a set of situations and a set of concepts tied together. By this, I mean that a concept's meaning does not come from one situation only but from a variety of situations and that, reciprocally, a situation cannot be analysed with one concept alone, but rather with several concepts, forming systems… the development of a conceptual field requires children's meeting and being faced with contrasting situations (Vergnaud, 2009, p. 86).

Vergnaud (1997a) differentiated additive structures from multiplicative structures as the two main conceptual fields of ordinary arithmetic. He claimed that the variety of problems which required multiplication or division, or a combination of both, usually comprise two different kinds of quantity (two variables); and concepts such as proportion, scalar ratio, rate, linear and multilinear function, linear combination, dimensional analysis fall under the conceptual field of multiplicative structures. The concepts of fraction and rational number are also involved in such relationships (Vergnaud, 1997b). Most

multiplicative situations involve two quantities, and these situations demand fraction concepts; for example, two pizzas (first quantity) being shared by four friends (second quantity).

There are different classifications of situations in the fractions domain, which they mainly refer to as subconstructs (Behr et al., 1983; Kieren, 1976; Nunes et al., 2007; Post, Wachsmuth, Lesh, and Behr, 1985). A subconstruct includes procedures and invariants used to understand a situation.

Kieren (1976) proposed that the concept of fractions consists of several subconstructs and that a solid understanding of the general concept depends on gaining an understanding of each of these different meanings of fractions as well as of their coming together. Kieren proposed that the concept of a fraction can be viewed as the composition of five related but distinct subconstructs: part–whole, quotient, operator, ratio and measure. Merlyn J. Behr, Wachsmuth, Post, and Lesh (1984) extended Kieren's classification in their Rational Number project, stating that rational numbers can be interpreted in at least six ways (referred to as *subconstructs*): a part-to-whole comparison, a decimal (which is a notation and not a situation), a ratio, an indicated division (quotient), an operator, and a measure of continuous or discrete quantities. Nunes, Bryant and their colleagues (2007) proposed four types of situations in which fractions are used by considering what the numbers in the fractions represent: part–whole, quotient, operator and intensive quantities. These situations are described in the following paragraph and research evidence for each one is presented in the next section.

This thesis will focus on Nunes and Bryant's classification of situations in which fractions are needed. This classification was chosen because it has in common with Kieren's situations part–whole, quotient, operator and ratio, which Nunes and Bryant refer

to as intensive quantities. Behr and colleagues presented more subconstructs, but some of them are notations such as decimals, which although they are rational numbers are not fractions. Behr and colleagues and Kieren have one subconstruct in common which is not part of the classification used by Nunes and Bryant, and this is measurement. The measurement subconstruct involves determining the fractional size of one magnitude, such as an area which could be seen as the part–whole subconstruct, as it is a part of a whole. Thus, it seemed that the classification proposed by Nunes and Bryant was the most appropriate for a classification of situations in which fractions are used to contextualise fractions in mathematics primary education. The next section will describe each situation.

*Nunes and Bryant's classification of situations in which fractions are used:*

Part–whole

Part–whole situations present the partitioning of a whole into equal shares. The denominator defines the number of parts into which a whole has been divided, and the numerator defines the number of parts taken: e.g. cutting a pizza into 4 parts, so that one part is 1/4.

Part–whole situations are typically used with children, this situation involves cutting a whole into equal pieces. To understand fractions, children must be aware that the more parts a whole is cut into, the smaller each part gets (Nunes and Bryant, 2015). Part–whole is the situation most commonly used to introduce fractions in primary schools (Campos and Silva, 2009; Nunes and Bryant, 2009b) and it has been extensively studied (Behr, Wachsmuth, Post, and Lesh, 1984; Kamii and Clark, 1995; Kerslake, 1986).

Kerslake (1986) carried out a research programme that aimed at investigating in depth some of the difficulties commonly experienced by secondary school pupils in different areas of mathematics, with the aim of designing teaching modules that would

alleviate these difficulties. Fractions was one of these areas, and within the fractions study the aims were: (1) to see if children could think of fractions as numbers, (2) to discover the models of fractions that children use and (3) to determine children's understanding of equivalence. They worked with adolescents aged 12-14 years old, in studies that took five phases.

The first phase was a survey study, which involved designing a fractions test (246 1st-year, 309 2nd-year, 308 3rd-year and 215 4th-year pupils); the second phase involved interviews (23 pupils aged 12-14); a third phase involved further interviews. The results of these interviews led to the design of teaching experiments (phase four) which involved both small groups and whole classes (phase five) with a pre-test, immediate post-test and delayed post-test.

In the survey study, pupils were tested on four tests on fractions. The first two tests contained items that were presented in problem or diagrammatic form. The third test contained items on recognition of fractions, equivalence, addition and subtraction. The fourth test contained items on multiplication and division. The items on these tests were also matched by a parallel set that was presented with no words or diagrams but in a straight computational form. The survey study reported that although pupils can reason about parts of a unit when using a part–whole situation, pupils find it difficult to relate that knowledge to fractions. Pupils were better at answering the problems than answering the computation questions. Pupil's responses to some questions in the test showed interesting insights into their understanding of fractions, and she concluded that students do not seem to make connections between a question presented as a situation and one presented as a computational question.

In the interview study, pupils were asked several questions of which four were about the equivalence of fractions using part–whole situations. Pupils were shown models of fractions, for example two exact figures that were divided differently but the shaded parts were equivalent. Kerslake (1986) reported that when shown a symbolic form (a circle with parts shaded), 76.5% of the pupils were able to say that 1/4 and 5/20 are the same, and 94% were also able to say that ¾ and 12/16 are the same or equivalent. However, no child was able to explain the use of the idea of equivalence in adding fractions with different denominators. She believed that children were repeating a routine when answering a part–whole question, and for that reason children failed to use that knowledge to add fractions.

In a study about fraction equivalence, Kamii and Clark's (1995) study aimed to investigate children's understanding of equivalent fractions and their conservation of the whole and the parts. Following the work of Piaget, Inhelder, and Szeminska (1960), they interviewed 120 5th and 6th graders individually and asked them to compare and form equivalent fractions. For example, two identical rectangles were folded and cut differently in front of the child: one rectangle was folded horizontally into 4 parts, and one strip out of the four was cut in half. The other rectangle was folded vertically into 8 parts and cut into 8 strips. Children were asked how many of the 8 strips would make ¾. They found that only 13% of the 5th graders and 32% of the 6th graders answered correctly to ¾=6/8. They reported that the children approached the task perceptually instead of making use of equivalent fractions. They concluded that teaching children fractions this way was not appropriate.

Other studies support the idea that part–whole should be the main way of teaching fractions, as they consider it the underlying factor for understanding fractions.

Charalambous and Pitta-Pantazi's (2007) study aimed at testing Kieren's theoretical model. They tested 646 children in Year 5 and 6 on their knowledge of fractions; the test presented items of all situations proposed by Kieren's model. They examined the associations between the subconstructs and their impact on children's overall performance on fractions by structural equation models (SEM). They designed a test comprising a number of tasks for each situation as described by Kieren (1993). Their results showed that 43 of the 50 items fell under the factor that it was designed for: part–whole, ratio (as intensive quantities), operator, quotient and measure (size of one relative to another by determining the number of times one fits into the other). The proportion of correct responses was highest in the part–whole subconstruct (proportion correct =0.75, SD= 0.20), followed by ratio (proportion correct= 0.64, SD=0.25), quotient (proportion correct= 0.55, SD=0.29), operator (proportion correct= 0.45, SD=0.35) and measure (proportion correct= 0.25, SD=0.33). They further carried out a SEM to analyse the relation between the subconstructs according to their theoretical model in which part–whole/partitioning acts as the latent variable that loads onto latent variables ratio, operator, quotient and measure. Their results showed that the factor loading is higher from part–whole to ratio and operator, but lower to quotient and measure. They claimed that their findings justify the popular use of part–whole situations to teach fractions. Although they also specified that using part–whole is not enough to fully understand fractions for more complex mathematical topics, as learning fractions mainly through a part–whole situation would provide limited support to understanding fractions in other kind of situations as suggested by Vergnaud (1997), children must experience different kinds of situations to master a concept. Their results suggest that quotient and measure are different situations than part–whole, ratio and operator, confirming that if children were already taught about fractions

they may have been taught these subconstructs, as others stated that method is the most commonly used (Campos and Silva, 2009; Nunes and Bryant, 2009b) and that reflects on their performance, but not necessarily that fractions should be introduced that way or that they predict better understanding of fractions.

Thompson and Saldanha (2003) stated that part–whole situations cause problems for children when they encounter improper fractions, as many students understand a/b as "a part of b", for example 4/5 would mean "4 parts of 5", which seems reasonable until they reach improper fractions, for example 5/4, because children reason that you can't have 5 parts out of 4.

Nunes and Bryant (2015) considered another reason why children may experience difficulties with the part–whole situation, as in this kind of situation understanding the inverse relation between the number of parts and the size of the parts is crucial. Thus, the situation of part–whole relations is conceptually demanding, and children demonstrate great difficulty in solving problems in which this situation is required.

Quotient

In the quotient situation, two quantities are involved: one represented by the dividend and another represented by the divisor. This contrasts with part–whole situations, where there is one quantity, a whole which is divided in equal parts (Nunes and Bryant, 2009b). In this situation, the denominator defines the number of recipients and the numerator defines the number of quantities being shared: e.g. one pizza shared by four children means 1 divided by 4, which is ¼; and two pizzas divided by four children will be 2/4.

Kornilaki and Nunes (2005) compared inferences of 5-, 6- and 7-year-olds about equivalence and order in two division situations: (A) the quantities were discrete, and the

dividends were larger than the divisors (e.g. 4 fishes were fairly shared among 2 cats); (B) the quantities were continuous and dividends smaller than the divisors (e.g. 2 fishcakes were shared among 4 cats). Children were presented with materials, cats and fish toys or cats and fishcake toys; the researcher shared fairly the fishes or fishcakes among two groups of cats: a group of white cats and a group of brown cats. After the researcher shared the fishes or fishcakes, the children were asked about equivalence and whether each cat in one group would receive the same amount of fish as each cat in the other group. In some trials, the number of fish (dividend) and cats (divisor) was the same; in other trials, the dividend was the same, but the divisor was different (e.g. 2 white cats shared 4 fishes and 4 brown cats shared 8 fishes). Most of the children succeeded in all the items where the dividend (fish) and the divisor (cats) were the same: 62% of the 5-year-olds, 84% of the 6-year-olds and all the 7-year-olds answered all the questions correctly. However, when the dividend (fish) was the same and the divisors (cats) were different, only 31% of the 5-year-olds answered correctly, 50% of the 6-year-olds and 81% of the 7-year-olds. There was no difference in the level of success attained by the children with discrete versus continuous quantities. They concluded that children were capable of reasoning about the relations and arrived at the answer by comparing the quantities rather than counting.

Operator

In the operator situation, a discrete quantity is divided, for example ¾ of 12 eggs. The denominator refers to the number of shares that the discrete quantity was divided into, and the numerator the eventual number of groups, being a discrete quantity as there is only a range of possible values. In this example, dividing 12 by 4 and the result in each share multiplied by 3.

Hart et al. (1985) designed a set of diagnostic tests, two of them related to fractions, to be taken in England by 246 pupils in 1st year aged 11-12 and 309 pupils in 2nd year aged 12-13. They reported the percentage correct for an operator item, which provided information on how difficult this kind of situation may be for children. Pupils were shown a drawing of 12 marbles and asked to draw a ring around 1/3 of the marbles and to write how many were 2/3 of the marbles. They found that 77% of 1st-year pupils answered the first question correctly and 71% the second question; 78% of 2nd-year pupils answered the first question correctly and 73% the second question. Although these percentages may seem high, pupils in 1st and 2nd year should be able to solve 1/3 of 12 even by counting. Thus, it is surprising and shows the level of difficulty of this kind of situation. Hart reported that a problem that may arise with this item is that some pupils state that a third of 12 is 3 without consideration, as they concentrate on the number 3 in the denominator of 1/3. In fact, 18% of pupils in Year 1 and 18% of pupils in Year 2 answered 3 to the question 1/3 of 12.

Mamede (2015) reported a study that aimed to investigate the effects of an intervention in children's reasoning on fractions in three different situations: quotient, part–whole and operator. A cohort of 37 Portuguese children aged 6-7 were pre-tested and post-tested individually, the test comprising items assessing fraction knowledge in three different situations: quotient, part–whole and operator. Children were randomly assigned to one of four groups: a group that received a quotient intervention, a group that received a part–whole intervention, a group that received an operator intervention, and a comparison group. Each small group (of five children) participated in two teaching sessions of about 35 minutes; during teaching, each intervention group learned about fractions in only one of the three situations. Post-hoc (Bonferroni) tests showed that the intervention on quotient

situations significantly increased children's performance compared to both the part–whole intervention group, $t(32)=-3.15$, $p< .05$), and the control group ($t(32)=-319$, $p<.05$), but not with the operator intervention group ($t(32)=-2.07$, n.s). In the post-test there were almost no correct responses to reasoning problems presented in part–whole or operator situations. In the naming fraction items, there were improvements in children's responses, but Mamede (2015) states that these improvements depended on the intervention, as children who received teaching on quotient situations improved their performance in quotient situations only, whereas children in the part–whole and operator intervention groups improved their accuracy in naming fractions in both types of situation. However, children who were taught in the quotient situation made significant progress in reasoning about fractions, but did not transfer this learning to the other situations. Children taught in the part–whole or in the operator situations only learned how to label fractions, showing no progress in reasoning items. Thus, the kind of situation in which fractions are presented to children has an effect on their understanding about fractions.

Intensive quantities

Nunes and Bryant (2009a) describe an intensive quantity as a quantity that is measured by the relation between two variables expressed as a ratio, such as price per kilo, ratio of orange juice concentrate and water, kilometres per hour, etc.

A characteristic of intensive quantities is that the quantities may vary but the relation may remain the same. Nunes and Bryant (2009a) explain it with a probability example: "The probability of drawing a blue marble out of a bag with 12 marbles, where 3 are white and 9 are blue, is the same as the probability of drawing a blue marble out of a bag with 24 marbles, where 6 are white and 18 are blue" (Nunes et al., 2009, p. 511). Another example of intensive quantities used in research has been a paint mixture, where

the relation or ratio remains: for example, the paint mixture is the same if one mixes 1 white paint with 2 blue paints, or 2 white paints with 4 blue paints.

Howe, Nunes, and Bryant's (2011) study on intensive quantities reports a teaching programme with 535 children aged 9-11. They designed a four-session programme to encourage learning about intensive quantities such as speed, density and temperature. They developed two versions of the teaching programme, one using ratio representation and the other using fractions. Results show a significant difference between the three groups, ratio, fractions and control: $F(2,514) = 4.64$, $p < 0.01$, $d = 0.30$ (Ratio vs. Fraction), 0.27 (Ratio vs. Control), the ratio group outperforming the other groups. These results remained similar in the delayed post-test. Howe et al. (2011) showed that both versions of the teaching programme promoted learning of fractions. However, the ratio version also supported proportional reasoning.

In summary, different studies have shown what children learn and the challenges they face depending on the fraction situation that they are faced with. Situations provide children with experiences that give meaning to a concept while experiencing the world. The situations proposed by Nunes and Bryant in which fractions are needed show the importance of understanding and dealing with fractions, as children and adults will be faced with these situations in their lives. These situations may involve operations on symbolized quantities, or may not.

Research evidence was revised in order to select situations to be included in the intervention study; it was reasoned that it would be best to choose the situations that children found easier in previous research studies. Studies (Correa et al., 1998; Frydman and Bryant, 1988; Kamii and Clark, 1995) have reported children's learning and difficulties in each situation – part–whole, quotient, operator and intensive quantities – but

no research has compared all four fractions situations. Mamede's study (2015) provided evidence that supports performance advantage for those children who were given fractions in quotient situations, compared to those in the other two situations. The children who were given quotient problems performed better than children given other types of problems. Mamede has compared three types of situation, and research regarding intensive quantities has demonstrated that, in general, children have more difficulty understanding intensive quantity situations than extensive quantity situations (Nunes, Desli, and Bell, 2003).

Another comparative study carried out by Nunes and Bryant (2009a) aimed to investigate the impact of situations on the development of mathematical concepts by comparing children's understanding of equivalence of fractions depending on the type of situation: (A) part–whole and (B) quotient. They present two types of evidence on how situations may affect children's thinking about rational numbers. Children were in Year 4 and 5 in England. Adopting the method used by Kornilaki and Nunes (2005), children were asked to compare the shares that one group would receive with the shares received by another group, in two types of situation. In (A) part–whole, children were asked to shade 2/3 of shapes divided in 3, 6 and 9 equal parts. In (B) quotient, children were presented with two groups and were asked to share fairly a continuous quantity. For example, children were asked to share one pie among four girls in the same group, and compared it to another group of eight boys that were sharing fairly two pies. Children were asked if each child from both groups, girls and boys, would receive the same amount of pie. The dividend and divisors were different, but their ratio was the same. The proportion of correct responses to part–whole situations was .31 (SD= .39), in contrast with a higher rate in quotient situations which was .73 (SD= .37). These results suggest that children from

age 7 can make inferences about equivalence of rational numbers, and that they find it easier using a quotient situation. Equivalence is an important aspect of understanding fractions; in the correspondence schema of action, the reasoning is based on direct proportion (twice as many chocolate bars and twice as many children means that everyone gets the same) (Nunes and Bryant, 2015).

These studies (Nunes et al, 2009; Mamede, 2015) compared which approach would give more advantage to children learning fractions, favouring the quotient situation. It is expected that both types of situations, part–whole and quotient, may contribute to the learning of fractions; but empirical evidence suggests that children experience less difficulty when reasoning about a quotient situation, which may lead into a smoother transition from whole numbers to fractions. This evidence will serve as a basis to select the quotient situation to be used in the intervention programme of the current study.

The results from these studies indicate that (a) children find it challenging to work with quantities smaller than a unit, but are able to do so depending on the situation they are presented with; (b) children are able to reason about quantities smaller than a unit, and they do so before they are able to label or operate with fractions; (c) depending on the type of situation, children may struggle to use the knowledge they have from equivalence to move on to carry out operations with fractions.

In conclusion, these data suggest that teaching fractions from a quotient perspective may have some advantages. This is consistent with Streefland's (1993) proposal that quotient situations are easier to understand and follow, as they are based on sharing fairly, which is a common experience to which most children are exposed from a very early age: for example, they share sweets among their friends or books among their classmates.

In the current study the quotient situation will be used as the starting point for teaching in the intervention study, because it is a precursor that can lead to insights about rational numbers.

**Symbols**

Symbols can be culturally developed (such as the number system), but they can also be self-generated graphic and gestural symbols that children use as support for their mathematical thinking (Nunes and Bryant, 2015; Vygotsky, 1978); numbers are symbols culturally developed. As Thompson (1993) stated, numbers represent quantities when measured, and quantitative relations, but numbers, quantities and quantitative relations are not the same. Nunes and Bryant (2015) made an important distinction when referring to the meaning of numbers: they distinguished between the referential and analytical meaning of numbers. They stated that "numbers … can be used to represent quantities: This is the representational meaning of numbers. But numbers have meaning in and of themselves, even when not being used to represent quantities. This meaning of numbers is termed *analytic* because it is given by definitions within a number system." (Nunes and Bryant, 2015, p. 718) Children need to learn numbers and to understand the representational and analytical meaning of numbers.

The number line is a graphical representation of the analytical meaning of numbers. Different studies claim that children's understanding of the number line provides a foundation for students' subsequent work with the representation of fractions on a number line (Cramer, Behr, and Lesh, 2009; Saxe et al., 2013). The number line can be used to demonstrate fraction equivalence and order of magnitude, in this sense understanding fractions as numbers.

Siegler et al. (2011) reported the importance of estimation of the position of fractions on a number line as an indication of understanding fraction magnitudes, which is an important aspect of understanding fractions. Their study aimed to test an integrated theory of acquisition of knowledge of numbers and fractions, connecting them by their knowledge of numerical magnitude. They tested 6th and 8th graders (11- and 12-year-olds and 13- and 14-year-olds) with three assessments of fraction magnitude knowledge: 0 to 1 number line estimation, 0 to 5 number line estimation, and 0 to 1 magnitude comparison. They also included an assessment on fraction arithmetic problems, verbal reports of strategy use after each task on number line estimation and arithmetic problem. They also collected student's mathematics achievement test scores. They reported correlations of .54 to .86 between fraction number line estimation and mathematics achievement at 6th and 8th grade. They concluded that arithmetic fraction knowledge as overall mathematics achievement is closely related to accuracy of magnitude representations beyond what is explained by fraction knowledge.

In summary, these studies provide evidence of the importance of understanding numbers independently to quantities, especially understanding their analytical meaning materialised in numerical magnitude.

## 2.3 Conceptual and procedural knowledge, or quantitative reasoning and arithmetic

In the first section, a traditional distinction between two types of mathematics knowledge is discussed; in the second section, a different distinction is proposed, that of quantitative reasoning and arithmetic, which replaces the previous one and stems from the distinction between the two types of number meaning.

## Conceptual and procedural knowledge

Two types of mathematical ability are traditionally recognised in psychology and education, albeit under different terms: namely, conceptual and procedural knowledge. This debate has been important in the domain of fractions, because it has been claimed that procedures can be learned without an understanding of the underlying concepts and of how the procedures relate to situations in which quantities are divided. For example, Ball (1990) noted that just under 90% of the prospective teachers in her study were able to implement a division of an improper fraction or mixed number by a fraction (1¾ ÷ ½), but only 26% were able to come up with a situation that could be used to illustrate this computation (see also Nillas, 2003; Simon, 1993).

It is worth noting that researchers differ in the terms used for conceptual and procedural knowledge. A reason for these differences has its origins in differences among disciplines. Interpretations and research designs preferences vary across disciplines. Star and Stylianides (2013) argue that psychology research refers to types of knowledge, while education research refers to quality of knowledge. However, interdisciplinary work is central to progress in many areas, including mathematics education. It was considered important to acknowledge this difference in interpreting terminology and describe which definitions will be considered in this study, which according to Star and Stylianides (2013) is described as the psychological perspective, based on quantitative research and aiming to assess conceptual and procedural knowledge.

Different definitions have been proposed for conceptual knowledge within this terminology framework. Rittle-Johnson, Siegler, and Alibali (2001) defined conceptual knowledge as an implicit or explicit understanding of principles and their interrelations with different situations. Children faced with a new situation would rely on their

conceptual knowledge to find a solution. Hecht (1998) defined conceptual knowledge as a mental model which contains the meaning of mathematical symbols. Byrnes and Wasik (1991) argued that conceptual knowledge consists of concepts and relational representations. Similarly, Hallett, Nunes, and Bryant (2010) defined conceptual knowledge as the understanding of the relations between mental representations. Each of these definitions brings one or more aspects of what Vergnaud (1997a) proposed as a concept: a construct composed by situations, operational invariants and symbolic representations. Although these definitions use different terms, they make essentially the same claim, and thus in this thesis conceptual knowledge about fractions will be considered as understanding the principles related to the situations in which fractions are used. For example, knowing that if a cake is shared in three equal parts between three children, then each one gets 1/3 of the cake (Byrnes and Wasik, 1991; Hallett et al., 2010; Hecht, 1998). It is also related to the understanding of the order of fractions and to the ability to make rough estimates of fractions (Byrnes and Wasik, 1991; Hecht, 1998; Nunes and Bryant, 1996).

Procedural knowledge in fractions refers to following specific steps required to solve fraction computations (Hecht, 1998). For example, there are specific procedures for multiplying, adding and dividing fractions. Hallett et al. (2010) defined procedures as self-contained and separate from meaning. Procedures do not have connections with anything outside the procedure. They state that procedural knowledge cannot confirm the output that it produces. For example, procedural knowledge would allow a child to solve an addition of a fraction problem that requires the application of common denominator procedure. Although conceptual and procedural knowledge about fractions corresponds to two different approaches to solve problems, there is no reason why a child cannot use both

(Hallett et al., 2010); in fact, real understanding of fractions would rely on applying both kinds of knowledge.

Research into conceptual and procedural knowledge aims to distinguish these two kinds of knowledge by relying on different kinds of evidence, mainly in differences in performance; however, differences in performance don't necessarily provide evidence to claim that these two are, in fact, different abilities. Lenz and his colleagues (Lenz, Dreher, Holzäpfel, and Wittmann, 2019) studied whether conceptual and procedural knowledge were empirically different. They aimed to measure conceptual and procedural knowledge in the fractions domain and test whether these two theoretically different abilities could be empirically separated. They tested 235 German students in 8th and 9th grade on conceptual and procedural knowledge of fractions. They carried out confirmatory factor analyses and compared the fit indices of a one-factor and a two-factor model for explaining differences in pupils' performance solving fractions items. They found that the two-factor model based on the two types of knowledge showed better fit than a one-factor model, which supports the idea that conceptual and procedural knowledge should be treated as two different kinds of knowledge. They also found that conceptual knowledge correlated more strongly with general cognitive ability than procedural knowledge. Their results confirmed that these two abilities should be treated differently.

Hecht, Close, and Santisi (2003) also evaluated whether conceptual and procedural knowledge are distinct abilities. They carried out a Confirmatory Factor Analysis to determine whether a latent conceptual knowledge factor and a latent procedural knowledge factor, which they called fraction outcomes, represented two distinct domains. They considered two kinds of items as conceptual knowledge: items assessing part–whole and cardinality of fractions. Their fraction outcomes measure comprises items related to

fraction computation (procedural knowledge) and word problems that involved fractional quantities presented in a standard symbolic form and fraction estimation. They reported a good fit to the data with a comparative fit index, which suggests that it is feasible to assess quantitatively the independence of these two types of knowledge: conceptual knowledge and procedural knowledge in the fraction domain. Their results supported the hypothesis that a conceptual knowledge factor and a procedural knowledge factor represent two distinct abilities.

Acknowledging that conceptual and procedural knowledge are different kinds of knowledge highlights the importance of ensuring that, when claiming that a child understands fractions, the assessment to measure this should contain both kinds of knowledge: conceptual and procedural knowledge about fractions.

It is also worth noting that although traditionally the fractions domain is divided into conceptual and procedural knowledge, there is a grey area in which both types of knowledge may be required or relied upon. For example, if a child is asked to paint two-thirds of a figure, they may do it just by counting as a procedure, or they may use the concept and rely on their understanding of what 2/3 mean. Kerslake (1986) showed that children answered questions related to equivalence of fractions successfully by following a procedure they had learned, not necessarily understanding the concept of equivalence. She came to that conclusion because the children who successfully answered some equivalence questions by counting, following a procedure, failed to use equivalence to add fractions.

*Does one of these types of knowledge precede the other?*

This distinction between conceptual and procedural knowledge about fractions also prompts the question about which type of knowledge children are more likely to adopt at the start when learning about fractions. It is worth noting that researchers who aim to study

the quality of conceptual and procedural knowledge claim that if one conceives of conceptual and procedural knowledge as qualities, then it does not matter whether conceptual knowledge or procedural knowledge is taught first (Star and Stylianides, 2013). However, research into types of knowledge in the fractions domain has focused on three different approaches related to how children learn fractions and how knowledge develops: the procedures-first view, the concepts-first view and the iterative view. The way children start learning about fractions may have implications as to which precursors are stronger predictors of learning fractions, as quantitative reasoning relies more on conceptual knowledge (Hecht and Vagi, 2012) and arithmetical skills can contribute to procedural knowledge (Jordan et al., 2013). If such an ordered acquisition is found, it would provide evidence relevant to the theoretical model proposed in this thesis.

The procedures-first view (Kerslake, 1986; Pantziara and Philippou, 2012) states that procedural understanding precedes conceptual. Pantziara and Philippou (2012) tested 302 6th graders to measure their knowledge of fractions. The aim of their study was to analyse the difficulty levels of the items to classify how children's abilities in fractions may develop. They found that the addition of fractions with the same denominators, which required children's use of a procedure, was found the easiest of all items by children. They stated that this finding shows that students initially develop procedural understanding.

Kerslake's (1986) research programme aimed to locate students' difficulties in mathematics and find ways to help them overcome these difficulties. In the fractions section, after carrying out a big survey, they interviewed 23 students in the UK. She found that addition of fractions was the easiest item for pupils aged 12-14, maybe because students are taught first about procedures, following steps through memorisation and practice, with little understanding of fractions. It is important to note that Kerslake (1986)

doesn't argue that procedures knowledge develops naturally in children, but that it is taught first, without a connection to children's conceptual knowledge of fractions.

The concepts-first view (Byrnes and Wasik, 1991) suggests that understanding of concepts is necessary for children to discover and use procedures. Piaget believed that learning develops by understanding concepts, and suggested (Inhelder and Piaget, 1958) that understanding concepts can increase procedural knowledge, when children attempt to understand their procedural outcome. Byrnes and Wasik (1991) aimed to explore the relation between conceptual and procedural knowledge. They carried out two experiments designed to explore specifically the role of conceptual knowledge in fractions procedural learning. Their first experiment aimed to determine 4th- and 6th-grade children's conceptual and procedural knowledge. They compared younger children who had received formal instruction about concepts of fractions, but not computations, to older children who had received computation and concept instructions. All the children were given the Fractions Knowledge Task, in which nine questions assessed conceptual knowledge and six items assessed procedural knowledge. The conceptual items were divided into three types: picture-symbol, simple-morphism and order items. The procedural questions presented two kinds of problems: multiplication and addition. Both groups of children demonstrated conceptual knowledge of fractions. However, the younger children in 4th grade scored badly with the procedural questions. This can be attributed to lack of instruction on computations received by 4th-grade children. These results suggest that procedural knowledge may be more dependent on instruction than conceptual knowledge. Interestingly, even the 6th-grade children made many procedural errors, for example adding both numerators and then both denominators as 1/3 +1/5 = 2/8. In Experiment 2, they designed an intervention to teach children how to add fractions, and to avoid adding

numerators and denominators. They told the children that 'fractions are not ordinary numbers' and used concrete manipulatives to illustrate why adding numerators and denominators was wrong. They concluded that their positive results suggested that conceptual knowledge fosters relearning of the lower common denominator method. However, it could also be considered that it was the use of manipulatives that supported the learning. Nonetheless, they stated that in some cases conceptual understanding is a necessary but not sufficient condition for the acquisition of procedures.

A similar study, carried out by Barros (2007), aimed to compare children with different levels of mathematical achievement on their conceptual and procedural understanding of fractions. In their study, 608 children from Grades 5 and 6 (374 from Grade 5 and 234 from Grade 6) were assessed for their procedural and conceptual knowledge in a fractions test, for their mathematical achievement and non-verbal ability. Their conceptual and procedural performance was compared between Grade 5 and Grade 6 children, and between children with three different levels of mathematical achievement. They found that mathematical achievement relies more on conceptual knowledge, while school grade influences procedural knowledge. The authors concluded that children's performance in procedural knowledge was related to formal instruction about fractions, as there was a clear improvement of procedural scores from Grade 5 to Grade 6 after receiving formal instruction, while the conceptual scores did not differ across years. The improvement in procedural knowledge across years is due to instruction, and therefore it would not necessarily develop if children were not receiving that instruction. This result is consistent with Byrnes' conclusions (Byrnes, 1992; Byrnes and Wasik, 1991), that procedural knowledge may be related to instruction effects, while conceptual knowledge may develop through different life experiences. It also is in line with evidence that initial

development in conceptual knowledge may improve procedural development (Inhelder and Piaget, 1958; Rittle-Johnson and Alibali, 1999; Rittle-Johnson et al., 2001). Barros (2007) stated that their results provided evidence to establish that in both academic years, the contribution of conceptual knowledge to mathematics achievement is independent of procedural knowledge, and significant even after taking into account age, non-verbal ability and procedural knowledge.

The third approach, the iterative-view (Rittle-Johnson et al., 2001), implies that both forms of knowledge develop simultaneously. They argue that conceptual knowledge can lead to progress in procedures, which allows children to confirm the output that procedural knowledge produces. Rittle-Johnson, Siegler, and Alibali (2001) argued that it may not be the case that conceptual knowledge development in the fractions domain leads to procedural knowledge development, but that they develop in an iterative process, which means that each kind of knowledge affects the other. They carried out two micro-genetic studies with 5th- and 6th-grade children who were learning decimal fractions, to explore their conceptual and procedural knowledge development in fractions. In their first study, they assessed children's conceptual and procedural knowledge to evaluate if these scores were correlated and could predict learning. Procedural knowledge was assessed by the ability to place a decimal fraction on a number line, while conceptual knowledge was assessed by understanding decimal fraction concepts, such as relative magnitude, relations to fixed values, continuous quantities, equivalent values and plausible addition solutions. In conceptual tasks, children were asked to circle the larger of two decimal fractions, to choose the decimal fraction that it is nearest to, greater than, or less than, to write a number that comes between decimal fractions, to circle all the numbers that are equivalent to a given decimal fraction, and to evaluate the correctness of possible answers to decimal

fraction addition problems. Results from the first study showed that learning correct representations of the value of decimal fractions improved procedural performance later on, and improvements in procedural tasks were followed by an increase in conceptual understanding. They concluded that children's initial conceptual knowledge predicted gains in procedural knowledge, and gains in procedural knowledge predicted improvements in conceptual knowledge; thus, these results supported the iterative model. In their second study, Rittle-Johnson and colleagues took a more active part by providing prompts to children to guide them to correct responses: the first prompt was to ensure the child noticed the tenth digit in the target number, while the second prompt was to show a number line divided in ten equal parts. Both prompts aimed to provide a composite representation of a tenth. They hypothesised that these guided manipulations would improve children's problem representation and thus improve procedural knowledge. They stated that their results provide evidence for an iterative process: when children improved their conceptual knowledge of fractions, they usually improved their procedural knowledge as well, and vice versa, in an iterative process.

Bailey et al. (2015) studied children's development of fraction concepts and procedures by comparing US and Chinese students. Their aim was to compare two populations that differ in procedural knowledge of fractions in order to test whether knowledge of fraction concepts is mediated by differences in knowledge of fraction procedures. They assessed 131 Grade 6 and Grade 8 children on their procedural knowledge, using fraction arithmetic problems; and their conceptual knowledge of fractions, using number line estimation and fraction magnitude comparison. They found that Chinese middle-school children had higher knowledge of fraction concepts and procedures than US children in the same grades; however, the difference in procedural

knowledge was larger. They claim that the differences between the conceptual knowledge of fractions was mediated by their procedural knowledge, since Chinese students showed a higher performance in procedural knowledge of fractions. This mediated their conceptual knowledge, so the Chinese children outperformed their US counterparts in conceptual knowledge. They also found that the differences were larger when comparing low achievers. They concluded that if procedural knowledge could mediate their performance in conceptual knowledge, this was because conceptual and procedural knowledge of fractions develop iteratively.

These studies relating to how children learn fractions based on three different approaches provide contradictory findings, because all of them have found positive results: the procedures-first view, the concepts-first view and the iterative view. As an answer to that, recent research suggests an alternative approach, which may explain these conflicting results by considering individual differences (Hallett et al., 2010; Hallett, Nunes, Bryant, and Thorpe, 2012; Hecht and Vagi, 2012). The authors explain that children could rely on both skills in different ways, and studying only one would not give us answers. Hallett et al. (2010) report cluster analyses of 318 Year 4 and 5 students in the United Kingdom (mean age 9.0 years), when the participants were assessed on their understanding of fractions, which included subscales of conceptual and procedural knowledge. They found five distinct clusters, which suggests that there may be more than one way in which children draw on conceptual and procedural knowledge. Hecht and Vagi (2012) replicated the study done by Hallett et al. (2010) in a longitudinal study, in order to assess the stability of the clusters. They aimed to explore clusters of different learning profiles for procedural and conceptual knowledge by investigating children's strength and weakness patterns about their knowledge of fractions. They assessed 181 4th graders on procedural

knowledge as fraction calculation; conceptual knowledge using picture symbols (shaded parts of a figure to give a fraction) and a symbol picture (given a fraction to shade parts of figure); fraction comparison and computation from pictures, by adding parts of shaded pictures and shading a blank picture provided; fraction knowledge and intelligence assessed by WASI. As Hallett and colleagues (2010), they decided to use residuals which show the variability in each kind of knowledge that is not shared: residuals for conceptual knowledge were obtained by regressing conceptual knowledge on procedural knowledge; residuals for procedural knowledge were obtained by regressing procedural knowledge on conceptual knowledge. Residuals provide a purer estimate of students' conceptual knowledge by controlling for redundancy with procedural knowledge and vice versa. They found similar results to those found by Hallett et al. (2010).

The debate between conceptual and procedural knowledge is an important one in psychology and mathematics education, and it influenced this thesis in three aspects. First, it is possible to differentiate conceptual and procedural types of knowledge empirically, which provides support for the possibility of distinguishing quantitative reasoning and arithmetic. Secondly, contradictory evidence regarding classification of items and which kind of knowledge precedes learning suggests that conceptual and procedural knowledge may not be a clear distinction when referring to two different kinds of mathematical ability. And thirdly, fraction knowledge assessments should be analysed in terms of classifying the items in different kinds of knowledge. It is also worth noting that some tasks may be classified as assessing procedural knowledge by some researchers and assessing conceptual knowledge by others: for example, items related to addition of fractions were classified as procedural by Kerslake (1986) and conceptual by Byrnes and Wasik (1991).

These different perspectives on how to define and assess conceptual knowledge and procedural knowledge, not just in the domain of fractions but in general mathematical learning, may be evidencing the fact that the distinction between these two abilities is not necessarily clear-cut and theoretically-based, but may depend rather on how children learned something (Nunes, 2018).

The reason why it is not clear-cut may lie in Nunes and Bryant's proposal that numbers have two meanings: analytical and referential. Nunes (2019) argued that the aim of the distinction between conceptual and procedural knowledge is to differentiate two types of mathematical knowledge and suggested that the distinction between quantitative reasoning and arithmetic works better.

**Reformulating the distinction as quantitative reasoning and arithmetic**

In this thesis, a different approach to the distinction between two types of mathematical knowledge is proposed: two theoretically different mathematical abilities are hypothesised as the basis for learning fractions, which are quantitative reasoning and arithmetic. This different distinction replaces the previously discussed conceptual and procedural knowledge. This different distinction stems from the distinction between the two types of number meaning. It should be acknowledged that conceptual knowledge relies more on quantitative reasoning, and procedural knowledge relies more on arithmetic skills. This is because, on the one hand, conceptual knowledge involves children's understanding concerning what quantities represent (Hecht et al., 2003), for example 1/3 of a pizza; while on the other hand, procedural knowledge involves operating with numbers.

Nunes (2019) stated:

> This suggestion is not simply about changing the names of the two forms of knowledge but a different approach to characterising them. Both arithmetic and quantitative reasoning have conceptual and operational aspects. Both are learned in the context of situations and, like procedural and conceptual knowledge, they are correlated. (p. 9)

It is important to note that quantitative reasoning includes using numbers on many occasions, but it is the understanding of the relations between quantities that matters in solving a problem, not the numbers per se. Arithmetic includes reasoning, and children can use their reasoning skills to solve an arithmetic problem, such as 34 + 25, where they could think 30+20 = 50+4+5=59. However, it should be noted that when reasoning to solve arithmetic sums, this reasoning will always be related to the analytical meaning of numbers.

Based on the general cognitive aspects of mathematics theories, quantitative reasoning is based on understanding the relations between quantities, experienced in different situations. It has been stated that in primary school, children should understand two kinds of relations between quantities: one that refers to additive relations, and the other that refers to multiplicative relations. Additive relations include relations such as additive composition, inverse relation between addition and subtraction and additive reasoning (Gilmore and Bryant, 2008; Nunes and Bryant, 1996, 2015; Piaget, 1952; Thompson, 1993; Vergnaud, 1979). All these relations are closely related to the idea of part–whole relations or the action of putting things together and separating them (Nunes and Bryant, 1996). The other kind of relation that comprises quantitative reasoning is multiplicative relations. In multiplicative reasoning, relations between quantities are based

on one-to-many correspondences and proportional relations (Nunes and Bryant, 2015; Nunes et al., 2007; Vergnaud, 1988). Situations in which quantities present an additive relation will require children to add or subtract to find an answer, while multiplicative relations will demand understanding a ratio between two quantities. In both kind of situations, children may be able to reason about the quantities without using numbers or an arithmetic procedure to come to a conclusion; it is the nature of the relation which varies and differentiates multiplicative and additive relations.

Arithmetic is the branch of mathematics dealing with the properties and manipulation of numbers (Guedj, 1997). Numbers may represent quantities, but numbers also have their own meaning within the number system: analytical meaning (Nunes and Bryant, 2015). Learning arithmetic is more than memorisation of addition and multiplication tables: it comprises rules and procedures related to numbers.

**Measuring arithmetic and quantitative reasoning**

The theoretical question proposed in this study is about the relative importance of multiplicative reasoning and arithmetic knowledge for children's fractions learning. Before one can discuss their relative importance, it is crucial to answer how these abilities differ empirically and how independent they are from each other. In order to discuss if these abilities are empirically different, first one must be able to measure them. It is important to note that different studies claim to assess quantitative reasoning or arithmetic, but in this review only research that uses definitions similar to those adopted in this study for quantitative reasoning and arithmetic will be considered.

*Measuring arithmetic*

Arithmetic skills involve understanding relations between numbers and procedures to operate with them. These have been measured for a very long time in school and in IQ

tests or ability tests. The British Ability Scale (Elliott and Smith, 2011), the Weschler Intelligence Scale for Children (WISC) (Wechsler, 1949) and the Wide Range Achievement Test (WRAT) (Jastak and Wilkinson, 1984; Robertson, 2010) are just three of many measures that include a subscale related to the ability to operate with numbers. In the WISC, the arithmetic scale involves the presentation of word problems that do not require much thinking about the relations between quantities. In the WRAT, the arithmetic scale includes items to assess counting, reading number symbols, solving oral problems and performing written computations. These tests commonly present either oral or written sums or simple problems. Tests that measure arithmetic skills have shown good validity and reliability.

In order to assess children's knowledge of arithmetic, the assessment must include questions in different aspects of numbers, operations and rules.

As studies suggest that learning about rules of whole numbers support but also bias arithmetic with fractions, it is important to assess children's knowledge of whole number arithmetic: they must be assessed on addition, subtraction, multiplication and division of whole numbers as well as fractions, if one wants to assess a good predictor of learning fractions.

One aspect of understanding number refers to magnitude and number comparisons; children present great difficulty when comparing fractions (Mack, 1995). Kerslake (1986) carried out a survey and interview study that aimed to investigate in depth some of the difficulties commonly experienced by pupils learning fractions. Over 500 secondary-age students participated in the survey and 23 in the interviews. Their results show that students experienced difficulties with fractions arithmetic. For example, she found that students presented difficulties with addition of fractions, reporting as a very common error

the addition of numerators and denominators. Kerslake (1986) reported that "about one-quarter of the 13-, 14- and 15-year-olds added numerators and denominators for 1/10 + 3/5". These results could also extend to subtraction sums. Kerslake (1986) also stated that children found challenging the idea that there are an infinite number of fractions between two fractions; she claimed that "when asked 'How many fractions lie between 1/4 and 1/2?', 29.5 per cent of the14-year-olds and 30.2 per cent of the 15-year-olds gave the answer 'one': this 'one' could refer to the fraction 1/3" (Kerslake, 1986, p. 12). This evidence relates to the challenges of understanding fraction as a new number.

Many studies have found that children and even adults find fractions challenging (Siegler et al., 2013), implying that an assessment of fraction knowledge should be chosen carefully, assuring that all aspects of number understanding are included: equivalence, addition, subtraction, density, order and comparison. In order to assess children's knowledge of arithmetic before they are taught about fractions, children must be assessed on their ability to operate with whole numbers: adding, subtracting, multiplying and dividing.

Studies have shown that whole number knowledge may interfere with fraction arithmetic performance (Obersteiner et al., 2013; Van Hoof et al., 2013); however, it should be noted that learning about fraction arithmetic relies on some arithmetic knowledge with whole numbers. Some studies have reported on the contribution of arithmetic to fraction knowledge, specifically to fraction arithmetic procedures (Hecht and Vagi, 2012; Jordan et al., 2013).

Jordan et al. (2013) provided evidence that calculation fluency can predict fraction understanding. Jordan and her colleagues (2013) carried out a longitudinal study that aimed to explore the developmental predictors of children's fraction concepts and

procedures. In their study they included a predictor that can provide evidence towards the hypothesis that arithmetic knowledge of whole numbers can contribute to the arithmetic of fractions. Participants were 357 children in Year 4 who had started the study in Year 3. Children were assessed on eight predictors: language, non-verbal ability, attentive behaviour, working memory, approximate number system, number line estimation, calculation fluency and reading fluency. Most of these predictors are general characteristics; only two predictors were related to mathematical skills, number line estimation and calculation fluency; and their measures do not include reasoning about quantities. The measure used to test calculation fluency was the Addition Fluency subtest of the Wechsler Individual Achievement Test, in which students are given one minute to solve 48 addition problems with addends between 0 and 10. The outcome measure of fraction knowledge was divided in two types of items, conceptual and procedural knowledge items. Three types of items measured conceptual knowledge of fractions: (1) part–whole understanding (shade 1/3 of the rectangle or shade the figure to represent the fraction symbol), (2) fraction comparison and equivalence (e.g. "Which picture shows that 3/4 is the same as 6/8?"), (3) and number line accuracy (e.g. "On the portion of the number line, a dot shows where 1/2 is"). The procedural knowledge items included fraction computation (addition and subtraction with the same denominators and mixed numbers) and their corresponding fraction word problems (involving fractions with the same denominators). For example, the word problem was, "Sara painted 2/5 of a picture on Monday and 1/5 of the picture on Tuesday. How much of the picture did she paint during both days?" The corresponding computation item was, "2/5 + 1/5 = ___." (Jordan et al., 2013, p. 50). Their results suggest that each of the predictors contributed to later conceptual understanding of fractions. Nevertheless, four of the predictors, number line

estimation, attentive behaviour, calculation fluency and working memory, made unique contributions to learning procedural knowledge of fractions. Their results suggest that arithmetical knowledge of whole numbers can predict conceptual knowledge of fractions; but the items they used to assess conceptual knowledge were mainly related to part–whole understanding via shading; also, equivalence and comparison of fractions was evaluated with part–whole items. Children were not assessed on order of fractions, density of fractions and equivalence of fractions in a different format other than part–whole. Therefore, further items would be necessary to conclude that calculation fluency can predict conceptual knowledge overall. This study provides evidence that arithmetical knowledge will contribute to arithmetic knowledge of fractions.

*Measuring quantitative reasoning*

In order to assess children's understanding of multiplicative reasoning, the assessment must include questions in different situations that involve proportions, ratios and inverse relations between multiplication and division. It is possible to reason about quantities without numerical representations, so questions involving multiplicative relations may or may not include numbers. Assessment of children's knowledge of multiplicative structures will be based on the schemas proposed by Nunes and Bryant as important for understanding multiplicative structures, which was hypothesised as the basis for learning fractions. Thus, an assessment should be included of children's knowledge of multiplicative reasoning, including quantities that can be represented by whole number questions as well as quantities smaller than a unit.

Nunes and Bryant proposed mainly two schemas of action that act as the basis for understanding multiplicative reasoning: these are one-to-many correspondences and sharing. The *one-to-many correspondence* schema of action relates to a child creating sets

of objects by keeping a ratio. For example: each bus can take 6 children on an excursion, and there are 18 children, so how many buses do we need so that all the children can go on the excursion? Another example: there are 3 houses and 2 dogs live in each house, so how many dogs are there in total? Both problems are solved by establishing one-to-many correspondence between two quantities. The relation is a ratio; children are not just separating items, but they must keep a ratio by replicating each set with the same quantity (Nunes and Bryant, 1996). This situation can be related to fractions when children are dividing.

One-to-many correspondences can represent two different kinds of questions: one is partitive and another is quotative. A partitive question presents the child with the number of recipients and the number of items, and the child must find the number of items each recipient will get. This kind of question may be solved by counting, but it should not be underestimated that the child must do two things at the same time: count on while grouping. For example: there are 3 houses and we need to put 15 guinea pigs in the houses, so how many guinea pigs will go to each house? A quotative question is more challenging for a child, as in this kind of question the child has a known quantity of items and a known ratio but needs to find out how many recipients there could be. For example: We are going to have a party. Each child who comes will get 2 balloons. We have 18 balloons. How many children can we invite? This kind of question assesses the child's ability to reason about two quantities connected by a fixed ratio; the children are told the amount of each quantity, but they need to find the ratio. Another example: There are 8 sweets in 2 rolls. Each roll has the same number of sweets. How many sweets are there in 5 rolls?

Some questions require one to find the set of all possible outcomes; they are called product of measures (Vergnaud, 1988) or Cartesian problems (Brown, 1981). The ability

to find out the number of possibilities or outcomes by combining two or more categories also relies on multiplicative relations. For example: If a boy has 2 pairs of shorts and 3 T-shirts, how many outfits could he combine?

Multiplicative reasoning problems are more demanding when the unit ratio is not presented; the information in the problem is about the relation between two quantities and the child must either find the ratio or use a scalar solution to reach the answer. These questions require children to understand proportions and may include direct or inverse proportional relations. For example: If 1 cat eats a certain amount of food in 8 days, how many days will the same amount of food last for 2 cats?

Reasoning about quantities smaller than a unit also relies on understanding multiplicative relations, based on children's understanding that when a quantity is divided by a number of parts, this would result in smaller parts. For example, if one pizza is shared by two children or two pizzas are shared by four children, will all children get the same amount of pizza?

Nunes and her colleagues (Nunes et al., 2012a; Nunes et al., 2015; Nunes et al., 2007) have shown that is possible to assess multiplicative quantitative reasoning in young children and adolescents, and how quantitative reasoning relates to further mathematical performance in school (Nunes et al., 2012a; Nunes et al., 2007).

Nunes et al. (2015) present the subscales of a test used with young children that aims to assess their quantitative reasoning knowledge. This test included quantitative relations of additive and multiplicative nature. The aim of their study was to present an assessment that aimed to assess young children's quantitative reasoning; they also address the distinction between concepts and relations between quantities and number and procedures. They assessed 112 children on three occasions: starting in their first year of

primary school and the last assessment carried out in their second year of primary school. They describe each of the subscales and items used to assess the children's quantitative reasoning knowledge. Their results suggest that is possible to assess quantitative reasoning with a reliable measure, showing also that it is a stable ability even after controlling for working memory and general cognitive ability. Quantitative reasoning and arithmetic skills made independent contributions to their mathematical achievement even after six years. These results provide strong support to justify that quantitative reasoning can be measured and their statistical distinction to arithmetic skills.

Although the distinction between quantitative reasoning and arithmetic skill could provide valuable insights to both theory and practice, not many studies have compared their relative importance. To the best of my knowledge, only two longitudinal studies have included measures of both arithmetic and quantitative reasoning and analysed their predictive value for mathematics achievement at school (Nunes et al., 2007). These studies investigated the contribution of reasoning to predict achievement in the domain of whole numbers, providing evidence for their relative importance in children's mathematical development. Both studies used as outcome measures standardized tests of mathematical achievement, designed by the UK government and administered by teachers, referred to as Key Stage Assessments. One study used Key Stage 1 Assessments (Nunes et al., 2007), the second study used Key Stage 2 and 3 as outcome measures (Nunes et al., 2012a). These tests don't present rational number items; Key Stage 2 Paper A 2014 presented three items or marks outs of 40 on fractions, and Key Stage 2 Paper B 2014 presented only one item on fractions. Although both studies make an important contribution to evaluating the relative contribution of these two mathematical abilities, they do not clarify whether the finding can be generalised to rational numbers. It seems the case that each ability can

provide independent contributions, but no longitudinal studies have investigated whether quantitative reasoning and arithmetic can predict children's learning of rational numbers, which this study will investigate.

In the current study, a measure to assess multiplicative reasoning and a measure to assess arithmetic skills will be evaluated to test whether they differ statistically when children are in 4th grade, between 9 and 10 years old.

The distinction between conceptual and procedural knowledge is not the terminology used in this thesis; it is similar but not identical to the distinction between quantitative reasoning and arithmetic. Good knowledge of arithmetic is not just procedural; it draws on the understanding of numbers and operations. When a child solves an addition of fractions, for example, the child can draw on knowledge of fraction magnitude: adding 2/3 and 4/5 is likely to be a number that is closer to 2 than to 1. Such items are used to test conceptual knowledge but are arithmetic items; they are not about relations between quantities but about relations between numbers.

In this thesis a fraction knowledge assessment developed independently to this research is needed; a fraction test was reviewed and its items were classified into three kinds of knowledge: conceptual knowledge about fractions, procedural knowledge about fractions, and the third was not necessarily conceptual or procedural because children could solve these items either by following a routine or by following an understanding of quantities, as stated by Kerslake (1986). For example, given a whole divided into three parts, children are asked to shade one third. These items will be referred to as naming fractions. This measure will be described in greater detail when referring to the measures used within each study of this thesis.

In summary, quantitative reasoning, arithmetic and fractions knowledge are measurable concepts. In order to assess them properly, one must select items carefully to include different schemas in action, invariants, situations and numbers, different operations and arithmetical procedures.

Based on the evidence analysed from the studies described above, it is hypothesised that arithmetical knowledge developed in the context of whole numbers can contribute to arithmetic knowledge in the domain of fractions, as some arithmetical knowledge with whole numbers is required, such as adding, subtracting, multiplying and dividing. It is also the case that both whole numbers and fractions can be ordered on a number line.

Based on previous research, it is expected that quantitative reasoning may also influence arithmetic knowledge, whereas arithmetic is only expected to predict fraction arithmetic knowledge and not fraction reasoning (Hecht et al., 2003). This evidence serves as a basis for the theoretical model proposed: that quantitative reasoning will be a stronger predictor of fractions learning, although arithmetic should also predict how well children learn new arithmetical rules.

## 2.4 Research programmes about how to teach fractions

There are many debates about fractions in the educational community: some are related to an a priori classification of situations in which fractions may be used, while other debates are related to how to start teaching fractions, (a) from a situation, be it from a quotient situation or from a part–whole situation, or (b) from symbols and arithmetic. Each of these perspectives provides insights into the possible roots of children's understanding about fractions.

Different researchers have started from different bases: some start from a quotient situation, for example by sharing; while others may start by arithmetic or number line use. In this section a review of some of these approaches is discussed.

**Streefland: fractions in quotient situations**

"Too often, we ignore the child's experience with ratio and proportions outside of formal mathematics lessons and teach children algorithms, which utilize techniques that are alien to them." (Singh, 2000, p. 291)

Streefland (1997) proposed a realistic approach to teaching fractions, in which children were presented with everyday situations that required them to reason about division problems that made use of dividends, divisors, quotients and fractions. He proposed that teaching fractions with meaning should be done by illustrating two activities: fair sharing and splitting up the group of sharers into subgroups (Streefland, 1993). Children start by sharing, which means that they use two quantities set into correspondence by a fixed ratio; this situation focuses on equivalence (following a ratio) and the number of parts does not matter. He based his guides to teaching fractions to primary school children on the following situations: (1) Part–whole, or 'bird's-eye view', a division situation which generates fractions by dividing a whole into equal parts; he argues that this approach does not appear successful. (2) Quotient, or 'some chocolate bars for some children', a situation in which he believes the estimation becomes more meaningful and it allows division to be done in various ways. He believes in the importance of fair sharing as an access to fractions. (3) How do we proceed? He believes that fractions are about relations, and that children should learn to describe and mathematise these relations as models to describe the world. (4) Table arrangements (Streefland, 1997), in which he

describes a situation to help children to construct symbols and to schematise sharing situations, which Nunes and Bryant (1996) described as N-splits.

**Rational number project: fractions in part–whole situations**

The rational number project, led by Behr, Post, Lesh, Cramer and others, was a multi-university research effort funded by the National Science Foundation from 1979 through 2018. The project developed instructional and assessment materials concerned with the learning of rational number. This project was focused on a multiple representation approach and extensive student participation in the learning process (Cramer, Behr, Post and Lesh, 2009):

> Traditionally we have taught children the arithmetic of numbers with the ultimate objective of their being able to use this arithmetic to model and solve real-world problems. The order of this approach has been first to teach the arithmetic of numbers, essentially divorced from a social context, and then to teach problem solving making attachments between numbers and operations on numbers with measurements of and operations on quantities to model relationships between quantities. (Behr, Harel, Post, and Lesh, 1994, pp. 121-122)

Behr, Lesh, Post, and Silver (1983) believed that children's understanding of rational number goes through stages and progressively integrates concepts from different situations. They based their materials on Kieren's classification of situations and argued that the part–whole situation is the best point to start as its understanding depends directly on the ability to partition either a continuous quantity or a set of discrete objects into equal-sized subparts or sets. They also assured that understanding part–whole situations is the basis for learning other situations in which fractions are involved.

Behr et al. (1984) designed an intervention study to assess students' understanding of the order and equivalence of rational numbers using manipulatives in a part–whole situation. Their study aimed to investigate the impact of manipulative materials on the learning of concepts related to fractions. Their hypothesis was that the ability to make translations among and within modes of representation (verbal, pictorial, symbolic) is what makes ideas meaningful to learners. The participants were 12 4th graders from two schools. The study lasted 18 weeks, the intervention consisting of 13 lessons, from 3 to 8 days each. The children worked on different kinds of activities: some carried out individually and others in a group. The children were introduced to the part–whole interpretation of rational number by means of circular and rectangular pieces of laminated coloured construction paper. Each unit fraction 1/n, for n from 1 to 10, 12, and 15, was represented by a different coloured paper. The task consisted of deciding which of two fractions was smaller and then explaining the reasons for the decision. Equivalence items required the child to decide whether two fractions were equivalent, or to generate a fraction equivalent to a given one by supplying a missing numerator or denominator. The intervention covered five themes: naming fractions, equivalent fractions, comparing fractions, adding fractions with the same denominators, and multiplying fractions. Each child was interviewed individually on 11 separate occasions. The interviews were conducted approximately every 8 days during the 18-week research study. Children used different strategies and gave different explanations to their solutions, which were classified by the authors. Results suggest that most children were successful after extensive instruction, but some continued to demonstrate inadequate understanding. These results suggest that although some children can learn about equivalence of quantities smaller than a unit with instruction on part–whole situations, it seems that for some children a different

approach could offer a firmer foundation for learning fractions. The authors suggested that the difficulty for some children could have been that their previous knowledge relating to whole numbers sometimes interfered with their learning of equivalence on rational numbers; that being the case, then a different approach could offer an easier transition from whole numbers to fractions.

**Fractions as numbers on a number line**

Some researchers have focused on fractions on a number line. They claim that introducing fractions using a number line provides continuity with whole numbers and improves children's understanding of fraction magnitude.

Hamdan and Gunderson (2016) studied children's ability to place fractions on a number line. Their aim was to assess whether this ability was contributing to children's fractions learning or whether children presented fraction knowledge. In their study they questioned whether the number line could support fraction learning as it provided a mental representation. The cohort was 114 children in Grade 2 and Grade 3 participating in their intervention study. Children were allocated to one of three groups: the first group received training on the number line, the second group received training on area models and the third group acted as a control. Their results showed that children from the training groups improved their fraction knowledge on the subject taught: the number line group improved their knowledge of fractions in number lines, and the group of area models improved their knowledge of fractions related to area models. However, it was the children who received training on the number line who were able to transfer this knowledge to fraction magnitude knowledge. They stated that the number line acts as the basis for children's understanding of fraction magnitude.

In a second study, Gunderson, Hamdan, Hildebrand, and Bartek (2019) studied children's ability to estimate fractions on a number line and hypothesised that fraction understanding on a number line can support their understanding of fraction magnitude, which they claim is strongly related to algebra and overall high school maths achievement. The aim of their study was to test if an intervention using number lines would promote understanding of fraction magnitude. Continuing with their previous result, they added more characteristics to their number line intervention such as "unidimensionality", always teaching left to right orientation on the number line and labelled endpoints. They assessed 148 children in 2nd and 3rd Grades and randomly allocated them to one of four intervention groups: three of them received number line training and the fourth received area model training. All the children who were allocated to one of the three groups that received number line training were taught with a number line, but in each intervention group the number line had different conditions. The first group had a pure unidimensional number line (17.5 cm horizontal line), the second group had a hybrid unidimensional number line (17.5 x 0.6 cm rectangle), and the third were presented with a square number line (17.5 x 17.5 cm). In the fourth group which received area model training, children were taught to segment and shade a square. Their results showed significant differences among the intervention groups in a fraction magnitude task at post-test after controlling for their previous knowledge of fraction magnitude (pre-test), reading achievement and age. The intervention group that received training with a unidimensional number line showed the best results. The authors concluded that "unidimensionality" is an important feature of number lines to promote fraction magnitude concepts.

Saxe et al. (2007) report an interesting three-step study that guided them in the design of a classroom intervention using number lines to learn about fractions. Their aim

was to understand how number lines can help students develop understanding of fraction concepts. They started with the premise that 5th- and 6th-grade students can use the number line as a vehicle for understanding number concepts such as relations between whole numbers and fractions, the density of rational numbers, and fraction equivalence in the sense that a number can be named with many labels (1/2=2/4). They took a three-step study: in the first step they interviewed 3rd-grade and 5th-grade students to find out their intuitions and actual knowledge about number lines and fractions on number lines. They asked them to identify points on the number line depicting whole numbers and fractions. They used children's answers to design individual tutorials for 21 3rd-grade and 23 5th-grade students that aimed to develop strategies for two aspects of fractions on a number line: (1) conceptualising and labelling a point as a fraction, and (2) naming equivalent fractions. Students were assessed before and after receiving the tutorials. The 3rd-graders showed no improvement with equivalent fractions, while 5th graders showed improvement in fraction indication and strategies for naming and equivalence, but they reported that their progress was limited. The third step was to design a class intervention using the tutorial material but adapted for classroom context: students worked independently and then were encouraged to discuss in small groups. They reported students' improvement on worksheets and believed that the number line supported fraction learning.

## 2.5 The role of general cognitive abilities

Cognitive abilities play a crucial role in learning mathematics in general and arguably also in learning about fractions. There is limited research on the moderator effect of general cognitive processes on fractions learning, but there is more on how they contribute variability to performance on measures of knowledge of fractions (e.g. attentiveness in the classroom (Fuchs et al., 2013; Hecht et al., 2003); language

comprehension (Fuchs et al., 2013; Namkung and Fuchs, 2016); working memory (Bailey, Siegler, and Geary, 2014; Fuchs et al., 2016; Fuchs et al., 2014; Hecht et al., 2003; Siegler and Pyke, 2013); and reasoning ability (Krowka and Fuchs, 2017).

Such predictive studies suggest that these abilities might moderate how much children learn when they are taught about fractions. Fuchs et al. (2014) pursued this hypothesis in a study that investigated whether individual differences in working memory moderated the impact of two different interventions designed to improve fraction knowledge. They selected 243 4th-grade at risk students, who were randomly assigned to one of three groups. Two groups received a 12-week intervention, one which aimed to improve conceptual knowledge while the other aimed to improve calculation fluency; the third group acted as a control. Both interventions led to better outcome performance on a fraction test. They found that working memory moderated the effect of the interventions, as students with very low working memory benefited from the conceptual intervention whereas the fluency intervention favoured students with better working memory.

There is evidence that non-verbal cognitive abilities play a part in mathematical learning (Inhelder and Piaget, 1958; Kyttälä and Lehto, 2008). The Kyttälä and Lehto (2008) study aimed to investigate whether and how active and passive visuo-spatial working memory (VSWM) and non-verbal ability, as assessed by Raven's Progressive Matrices, are related to mathematics performance in a normal school population. They assessed 128 students, aged 15-16, in VSWM using a modified Corsi Blocks task and a matrix pattern task; and in non-verbal ability using Raven's Progressive Matrices. They stated that non-verbal ability and passive VSWM accounted for variance in overall maths performance. Active VSWM exhibited significant correlations with maths measures, but in a series of regression analyses most of its effect was observed to be mediated by non-

verbal ability. They argued that non-verbal ability measured by Raven's Progressive Matrices contributed to different subscores of mathematical skills. This evidence suggests that non-verbal ability could have an effect on scores of fractions knowledge and could potentially moderate learning.

In the present study, it is hypothesised that children's general cognitive ability as measured by a non-verbal test, the Raven's Matrices, acts as a moderator of the impact of interventions to teach children about fractions. Children can sometimes solve mathematical problems without being able to explain verbally how they did it; and they can use manipulatives or imagination to solve a fraction item without using words or language. Based on research analysed previously, children use schemas of action to solve mathematical problems even before they can use words to explain, but they may use other symbols and actions that help them solve the problem. It was reasoned that non-verbal ability, as the capacity to analyse information and identify patterns without using words, could act as a moderator of fraction learning. To the best of my knowledge, this moderation has not been studied.

## 2.6 Rationale for the study

Studies have shown that children have difficulties with fractions, even after being taught about fractions (Hecht et al., 2003; Hecht and Vagi, 2012; Kerslake, 1986; Nunes and Bryant, 2015; Thompson and Saldanha, 2003). The difficulty in achieving fraction understanding makes learning about fractions an important issue educationally, as well as theoretically. Although extensive studies have been conducted on children's understanding of fractions, some questions to support the teaching and learning of fractions remain; we

are still unsure about precursors to learning fractions, or if there is another unknown variable which determines the learning of fractions (Nunes and Bryant, 2009b).

Theoretical and empirical evidence from the literature pointed out that there may be precursors, quantitative reasoning and arithmetic knowledge, to learning fractions. Nunes and Bryant (2009a) and Howe et al. (2011) have shown that teaching children about multiplicative reasoning, especially quotient situations and intensive quantities, can lead to better outcomes on a fractions test, but they were not able to verify these as longitudinal predictors. Thus, they pointed out the need to verify this in order to create a predictive model of learning fractions. Similarly, studies (Saxe et al., 2007) have shown that teaching children about fraction arithmetic can lead to a better outcome in a fractions test. However, the relative importance of these abilities to learning fractions remains unknown. This study aims to fill such gaps in the literature.

This study is of both practical and theoretical importance. Its practical significance is that it could potentially influence educational approaches to teaching fractions and contribute to further understanding of children's general cognitive development. The theoretical contribution is that the combination of longitudinal and intervention studies can clarify the foundational role played by quantitative reasoning and arithmetic in providing a basis for children's learning of fractions.

The child's cognitive ability is hypothesised to moderate their learning of fractions: a child with higher cognitive abilities may achieve transfer from quantitative reasoning to arithmetic or vice versa, but a child with lower cognitive abilities may not achieve transfer between these two different types of knowledge. The intervention study will test the hypothesis that non-verbal ability acts as a moderator of learning quantitative reasoning and arithmetic knowledge of fractions.

## 2.7 Hypotheses and research strategy

The findings of the studies described in the literature review reveal that children understand the relations between quantities in sharing situations before they know the arithmetic involved in division; they understand quotient situations better than part–whole, but they must connect fractions to all situations ultimately, in order to understand fractions. The choice of the situations to be used in this study was based on research (Correa et al., 1998; Frydman and Bryant, 1988; Kornilaki and Nunes, 2005; Streefland, 1997) revealing consistently that children find starting from a quotient situation easier, and their performance in tasks measuring quotient situations was higher than their performance in tasks measuring other situations. Based on these studies, the interventions in this thesis will start from quotient situations. Streefland (1997) proposes ways of teaching with quotient situation and part–whole situations; his guidelines along with others from empirical research such as Howe et al. (2011) and Nunes et al. (2009a) will serve as a basis for the design of the quantitative reasoning intervention study. Evidence shows that children can learn about fractions when exposed to a number line; this idea will be used for the design of the arithmetic intervention study.

These results lead to the following hypotheses:

(1) Quantitative reasoning and arithmetic knowledge are distinct mathematical abilities.

(2) Measures of quantitative reasoning and of arithmetic are longitudinal predictors of children's knowledge of fractions after the children are taught about fractions in school; quantitative reasoning is the stronger predictor of the two.

(3) Two intervention programmes, one based on quantitative reasoning and the other based on arithmetic, designed to teach children about fractions will

improve their fractions knowledge; the intervention based on quantitative reasoning will lead to better outcomes on a fractions knowledge test than the intervention based on arithmetic.

(4) Non-verbal cognitive ability is a moderator of children's learning during the intervention; children with higher measured non-verbal ability will benefit more from the intervention than those with lower measured non-verbal ability.

In order to test these hypotheses, this thesis will follow a research strategy to carry out three studies:

(a) The first study will aim to test the empirical distinction between quantitative reasoning and arithmetic by means of a confirmatory factor analysis.

(b) The second study will be longitudinal with the aim of identifying which of these two abilities, quantitative reasoning or arithmetic, is the better predictor of learning fractions.

(c) The third study will be an intervention randomised controlled trial with the aim of testing whether quantitative ability and arithmetic act as a foundation for fractions learning; this study will also analyse the moderator effect of cognitive ability.

# Chapter 3 Development of Measures

The aim of this study is to assess the importance of two predictors of children's fractions learning after they are taught about fractions in school. The predictors are quantitative reasoning and arithmetic knowledge. The first step of the longitudinal study was to design two tests, one to assess children's quantitative reasoning and the second to assess children's arithmetic knowledge. The purpose of this chapter is to analyse whether the measures designed to evaluate these two precursors of fractions learning can be treated as two different measures by examining (1) the factor structure of the measures, (2) whether the factor structure is invariant over time, and (3) if it is independent from non-verbal ability. The following sections discuss each of the aims of this chapter.

*The factor structure of the measures*

The items in each of the measures were designed on the basis of the distinction between quantitative reasoning and arithmetic presented in the previous chapter. Although these two mathematical abilities can be distinguished theoretically, one must ascertain if they are too closely related to be distinguished empirically. Children who are good at quantitative reasoning may also be good at arithmetic knowledge, and children who find quantitative reasoning challenging may also find arithmetic challenging. This would make it difficult to separate the measures empirically. If the measures are not too closely related statistically, they would be treated as different precursors and contribute independently as predictors of fractions learning. If the measures are statistically distinguishable, the results will provide evidence that these are two different mathematical abilities and thus form the basis for the longitudinal study to assess their relative importance for learning fractions over time.

Quantitative reasoning and arithmetic knowledge are abstract constructs – one cannot measure them directly. Children can solve problems and computation exercises and it is possible to make inferences about their quantitative reasoning and about their arithmetic knowledge from their performance on these tasks. In psychometrics, latent variables are variables that represent a concept that cannot be measured directly but are usually measured through many indirect observable variables (Skrondal and Rabe-Hesketh, 2007). Quantitative reasoning should be one latent variable and arithmetic knowledge should be another latent variable. There are different statistical analyses that use latent variable modelling, such as factor analysis, item response theory and structural equation modelling among others, to test if a group of variables form a latent variable. In this chapter, factor analysis will be used to investigate the factor structure of the measures and to examine whether the items designed for each of the measures can be connected to their correspondent latent variable, quantitative reasoning or to arithmetic knowledge.

There are two types of factor analysis: one is exploratory and the other is confirmatory. Both are used to analyse statistically how items in a measure can be grouped into latent variables (Mueller, 1978). In exploratory factor analysis (EFA), the researcher makes no a priori assumptions about relationships among the items and how these will group under different latent variables. In contrast, confirmatory factor analysis (CFA) is theoretically driven and is used to test whether items of a construct are consistent with a researcher's hypothesis about the nature of that construct. In CFA the researcher determines the latent variables a priori, to test if the theoretical structure of a measure exists by analysing how the empirical data fits with the hypothesised model. If the observed data present a good fit for the factor analysis model, it is reasonable to assume that the theory is backed by the empirical data and that the measure presents a strong factor

structure, hence structural validity. This study will use CFA to test the hypothesis that each of the constructs corresponds to an independent latent variable, and thus to two different mathematical abilities.

If one wants to determine if the items of both measures are measuring one or two latent variables, one would need to compare the fit of two models. One model would group all the items into one factor, which represents the underlying theory that all the questions fall under one latent variable. A second model would split the items theoretically based on a hypothesis that each group of items measures two different latent variables. The fit of the model is a measure of the discrepancy between the observed data and the values expected using the proposed model, for example the comparison between an architect's model of a building and the actual building. If the observed data fit better with the one-factor model, one would assume that all items are measuring one latent variable or ability. If the observed data show a better fit with the two-factor model, one would assume there are two underlying abilities being measured.

In this chapter, the goodness-of-fit indices of two CFAs are compared. In the first CFA, all the items were specified to load on one construct, representing a general mathematical ability. In the second CFA, the items were specified to load on their two a priori constructs: the first construct being children's skill to operate with numbers and the second construct their ability to reason about quantities. It was hypothesised that if the indices of fit are statistically better in a CFA with one factor, then the results would indicate that the items of both measures are assessing one ability. In contrast, if the indices of fit are statistically better in a CFA with two factors, the results would confirm that each test is assessing a different ability (Brown, 2014). If the factor analyses determine that there are in fact two different mathematical abilities, these results would confirm the factor

structure of two separate constructs. Thus, these would be used to test whether they are independent predictors of children's learning of fractions.

*The factor structure invariance over time*

The second aim of this chapter is to assess the factor structure invariance over time. This is the property of constructs to measure the same latent variable at different time points, regardless of changes in the conditions of measurement (Horn and McArdle, 1992; Meredith, 1993); it also allows for comparisons over different groups (Floyd and Widaman, 1995). Horn and McArdle (1992, p. 117) stated that "The general question of invariance of measurement is one of whether or not, under different conditions of observing and studying phenomena, measurement operations yield measures of the same attribute."

Rather than assuming that the constructs are measuring quantitative reasoning and arithmetic knowledge, and that this would be the case not only at different times of assessment but with different groups, one needs to test formally the factor structure invariance of the measures. This would ensure that quantitative reasoning or arithmetic knowledge is being measured over time, even if it is tested at different times and across different groups.

In order to test whether the factor structure is invariant over time, one can measure the factor structure of the same construct at different time points and compare them. Some authors have argued that to test the structure invariant one needs to use CFA (Cunningham, 1982; Horn and McArdle, 1992) and establish different types of measurement invariance. For example, whether factor loadings, intercepts or residual variances are equivalent across groups or time. To do this, CFA models with different constraints should be duplicated with data from different time points and tested for differences between these models. For

example, the same CFA model should be carried out twice, first with the data from the first time point of measurement and then with the data from the second time point of measurement, using the same constructs. Some authors also recommend comparing the difference of the goodness-of-fit indices of the models at each time point (Van De Schoot, Schmidt, De Beuckelaer, Lek, and Zondervan-Zwijnenburg, 2015).

In this chapter, the factor loadings and goodness-of-fit indices of the CFA model, carried out to test the factor structure that presents better goodness of fit, will be duplicated with data collected six months later, using the same constructs. Thus, if the models at two different time points present similar indices of fit and factor loadings, this would show that the factor structure is invariant over time.

*Independence from non-verbal ability*

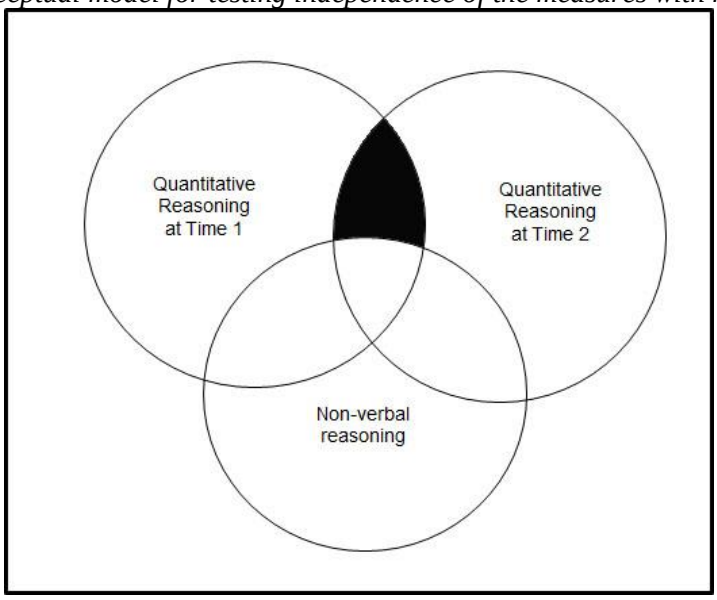
If the factor structure invariance of the measures over time is shown to be highly consistent, one may still argue that this is because some abilities are dependent on, or driven by, intelligence (Kyttälä and Lehto, 2008; Raven, 2000). The items designed for each measure should relate to each other not because of intelligence but because of their underlying latent variable. For example, quantitative reasoning measured at time 1 should relate to quantitative reasoning measured at time 2, independently of intelligence.

The model presented in Figure 2 shows the conceptual model for independence of the constructs and non-verbal ability. In this conceptual model, independent variance from quantitative reasoning, measured at time 1 and time 2, and non-verbal ability are represented in white; the black section represents the common variance between quantitative reasoning at time 1 and at time 2, which is not explained by non-verbal ability. This variance is the common variance shared only by quantitative reasoning measured at

time 1 and time 2. If this shared variance is high and significant, one can conclude that the construct can measure quantitative reasoning independently from non-verbal ability.

In order to test this conceptual model, partial correlations can be used. Partial correlations allow one to test the correlation coefficient between quantitative reasoning measured at time 1 and quantitative reasoning measured at time 2, excluding the variance explained by non-verbal ability. The same conceptual model could be tested for arithmetic knowledge.

**Figure 2**
*Conceptual model for testing independence of the measures with non-verbal ability*




To summarise, it is of importance that measures of children's quantitative reasoning knowledge and arithmetic knowledge are analysed to understand to what extent these two abilities are related but are still different mathematical abilities. Results from this study would provide further understanding of children's learning processes, but would also provide evidence with empirical implications for education and further studies. There is little evidence to show to what extent these mathematical abilities are independent, separate abilities, or whether they are independent of children's non-verbal ability. The

results could provide suggestions for how children are taught mathematics and lend support to separating these two skills in assessments, as is currently done in the Key Stage 1 and 2 tests designed by the UK government.

In the following sections, both measures will be analysed. First their hypothesised factor structure will be tested using CFA, then their factor structure invariance over time and their independence from non-verbal ability will also be measured. It is predicted that CFA models will show that quantitative reasoning and arithmetic knowledge are two different abilities, although highly correlated. If so, these results will provide the basis for a longitudinal study to test the independent contribution of each mathematical ability to children's learning of fractions.

### 3.1 Method

This longitudinal study draws from a single group of participants, measuring their knowledge of quantitative reasoning and arithmetic knowledge at two testing points, six months apart.

This study follows the British Psychological Society Code of Ethics and Conduct and the British Educational Research Association Ethical Guidelines for Educational Research. The whole research project was approved by the Central University of Oxford Research Ethics Committee (Appendix 1 and Appendix 2).

The following sections will describe the participant characteristics, the sampling procedures, the sample for the study, the measures and the research strategy followed in this study.

### 3.1.1 Participant characteristics

Year 4 children were chosen for this study, after consideration of what they had been taught and what they had not yet been taught:

(1) Year 4 children have received limited instruction about fractions in school; a greater focus on teaching fractions is included in the curriculum for Year 5 children.

(2) The preparation of children for this topic before Year 5 consists of introducing fraction labels using words (half, quarter, third, tenth) and the corresponding numerical representations, without much emphasis on equivalence or ordering of fractions in general. Equivalence refers to the understanding that $\frac{2}{3} = \frac{4}{6}$ and ordering of fractions is the understanding that $\frac{1}{1000} < \frac{1}{100} < \frac{1}{10} < \frac{1}{5} < \frac{1}{2}$

(3) Year 4 children are also introduced to addition and subtraction of fractions with the same denominator and are expected to learn the decimal representation for a half, a quarter and three-quarters.

(4) At the end of Year 4, children are expected to memorise times tables to 12, which should give them a basis for working with fractions with different denominators that have a common multiple. Children have also learned division and about decimals.

### 3.1.2 Sampling procedures

A database of primary schools in Oxfordshire was used to send invitations: 24 schools from outside central Oxford were selected; central Oxford schools were avoided as they are saturated with requests to participate in research studies. Schools were selected based on being accessible by public transport. The recruitment process started by sending an initial letter inviting schools to take part in the study; 7 schools agreed to participate, but 1 school did not respond to further communication and consequently only 6 schools

remained in the study. Letters to guardians or parents of the children in Year 4 seeking their agreement to take part in the study were then sent via their teachers. Opt-in consent was used, as this method is considered to be of the highest ethical standard when carrying out research with young participants. Using opt-in methods presents difficulty in recruitment. This could be because parents may forget to send their signed permission, or the child may have lost the letter on the way home. To avoid this issue some studies may decide to send an invitation where parents are asked to send the letter back if they do not wish their child to participate; these studies tend to have more participation. In this study it was decided that opt-in was the best option, even after considering the possible difficulty of recruiting participants. It was thought that children were going to be assessed on different occasions and were going to miss some classes; also the study was carried over a spread of time and parents may forget that they had agreed to their children's participation. Therefore, letters asking for parents' signed permission to participate were sent through teachers; without knowing the specific number of children in each class, it was estimated a maximum of 30 children per class.

A total of 300 parent letters were sent to schools (estimating there would be a maximum of 30 children per class, 10 classes participating), then these letters were sent to parents by staff at the schools. The number of letters sent to parents is unknown as teachers were asked to send them. The parents of 124 children agreed for them to take part. It is important to mention that in three schools all children in the class participated, but in the other three schools a smaller group of children agreed to participate. This response rate is a consequence of using opt-in consent forms. Children were given a further opportunity to agree to participate before every test started. It was explained to the children that the activities were being carried out to support research, that they were not obliged to

participate, that the research would not have any impact on their school scores and that no individual data would be shared or disclosed at any point.

*Sample*

A total of 124 children (62 girls and 62 boys) in Year 4 agreed to participate in the study. Their mean age was 9 years and 2 months (range 8 years 6 months to 9 years 6 months). The six schools represented varied socio-economic status, thus a more representative sample of the population in Oxfordshire. Although this study did not collect data on socio-economic status, data available from the National Statistics databases were used, which provide data at school level on deprivation indices such as entitlement to free school meals. Figures for England show that in 2015 approximately 15.86% of school children were entitled to free school meals, although only around 13.1% took them up. In this study, the percentage of pupils from the participating schools known to be eligible for free school meals ranged from 3.2% to 15.4%, with an average of 8.7%. This suggests that, on average, the participating schools present a lower deprivation index than the national average. Two children in the study were identified as having special educational needs, and so their data were discarded. All children who had parental permission to take part and did not report special educational needs were included in the study.

### 3.1.3 Procedure

The participants took part in three testing points, each six months apart. During the first testing point, the participants were tested as a whole class group on one day in two consecutive 40-minute sessions. The tests were introduced as an activity to find out how much children at that age know about mathematics. At the first testing point, children were assessed on the Quantitative Reasoning test, the Arithmetic Knowledge test and Raven's

Progressive Matrices (Raven, Court, and Raven, 1996). The order in which the tests were presented to the children was the same in all schools: (1) Quantitative Reasoning test, (2) Arithmetic Knowledge test and (3) Raven's Progressive Matrices. This order was carried out in all schools and testing points. It was important to maintain the order due to practical reasons and for consistency. The procedure started with the Quantitative Reasoning test, because the researcher must read each question aloud twice following a script and wait for pupils to answer before moving on to the next question. Once this test was finished, it was followed by the Arithmetic Knowledge test, in which children were asked to answer all the arithmetic questions independently. Children take different time to answer, thus it was reasoned that it was better to give this test after the Quantitative Reasoning test which is answered by all children at the same time. The children were given positive encouragement without specific feedback throughout. Once all children had finished the Arithmetic Knowledge test, all the children received instructions as a group on how to answer the Raven's Progressive Matrices test. They were all given a Raven's booklet, and after they had worked individually for 35 minutes, all the answer sheets and booklets were collected. Children tend to like this test, so the testing finishes on a positive note.

At the second testing point, children were also tested as an entire class, in a session of approximately 40 minutes. Children were assessed on three tests: Quantitative Reasoning, Arithmetic Knowledge and Fractions. Only the Quantitative Reasoning and Arithmetic Knowledge tests will be discussed in this chapter.

The third testing point will be described in the next chapter, which presents the second step of this longitudinal study. This chapter presents the development of the measures used as predictive measures in the longitudinal study.

### 3.1.4 Measures

Three measures will be described and their psychometric properties for internal consistency presented. First the Quantitative Reasoning test and the Arithmetic Knowledge test will be described. Then their psychometric properties will be presented. Finally, the description of a non-verbal ability test will be provided followed by the psychometric properties of internal consistency of the measure estimated with the sample from the current study.

***Description of the Quantitative Reasoning and Arithmetic test***

This study addresses the distinction between quantitative reasoning and arithmetic; there is a clear theoretical distinction, but it may not be reflected empirically when these abilities are measured. The choice of items to measure children's quantitative reasoning and the development of items to measure their arithmetic knowledge had to be theoretically driven and systematic, in order to have a balance of number of items between scales and to provide grounds for identifying two latent variables, quantitative reasoning and arithmetic.

In order to have a balance of number of items, it was reasoned that both scales should present two domains: in quantitative reasoning, whole quantities and quantities smaller than a unit; in arithmetic, whole numbers and fractions. Each domain should present the same number of items in each scale, as shown in Table 1. For example, the section assessing children's knowledge of relations between whole units should have the same number of items as the whole numbers section. It was hypothesised that quantitative reasoning would be one latent variable comprising whole units and quantities smaller than a unit; arithmetic was hypothesised to be a different latent variable comprising items measuring knowledge of how to operate with whole numbers and fractions.

**Table 1**
*Item classification*

| | Whole units/ Whole numbers | Quantities smaller than a unit/ Fractions | Total number of questions by test |
|---|---|---|---|
| Arithmetic | 9 questions | 6 questions | 15 |
| Quantitative Reasoning | 9 questions | 6 questions | 15 |
| Total number of items in both tests | | 30 questions | |

In the following section, items for each of the scales are described.

## Quantitative Reasoning test

The Quantitative Reasoning test was designed by Nunes and Bryant (Nunes et al., 2012a); a selection of 15 items was used, as these were relevant to this project. Each question sets a problem with visual support and aims to assess children's ability to reason about quantities and multiplicative relations between quantities, with minimal demand on knowledge of arithmetic. No reading is required by the children; the researcher reads each question, which is supported by a visual image. The 15 items were recorded into 17 answers, because two items comprised two answers. The 17 questions are categorised into two domains: (a) *Whole numbers*: multiplicative relations between whole quantities that require simple calculations (11 items), and (b) *Fractions*: reasoning about quantities smaller than a unit with no computation required (6 items).

### *(a) Quantitative Reasoning – multiplicative relation between whole units*

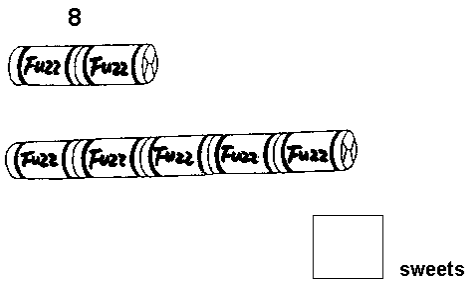
The section which assessed multiplicative relations between whole units consisted of 11 questions that required simple calculations. The 11 questions in the quantitative reasoning measure assessed children's ability to think about ratios or solve a product of measures problem (Vergnaud, 1988).

In the quantitative multiplicative reasoning questions that assess the child's ability to reason about two quantities connected by a fixed ratio, the children are told the amount of each quantity, but they need to find the ratio. These questions required children to

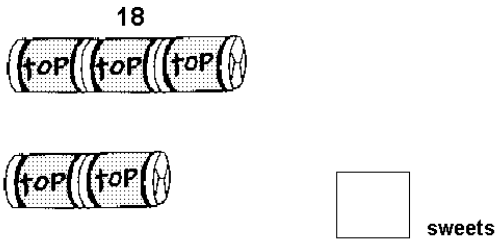
choose carefully when to divide and when to multiply, to find the ratio between these two quantities and to keep the ratio in order to find a new value.

**Figure 3**

*Examples of Quantitative Reasoning questions (ratio)*




*Instructions read to the children:*
The roll on top has 8 sweets.
How many sweets do you think there are in the big roll below?
Write your answer in the empty box.




*Instructions read to the children:*
The roll of sweets on top has 18 sweets.
How many sweets do you think there are in the small roll below?
Write your answer in the empty box.

For example, in the first question shown on the left in Figure 3, children were presented with two rolls of sweets, composed of smaller rolls. The roll on top was composed of two smaller rolls and it was stated that it had 8 sweets. The correspondence between rolls and sweets is 1 to 4. The roll bellow was composed of 5 smaller rolls. Children were asked how many sweets there were in the rolls below. Children can easily share the 8 sweets contained in the overall roll in two parts, one corresponding to each section of the roll. This allows them to form a composite unit: one section, 4 sweets. Steffe (1992) suggests that situations like this, in which one measure is like a container for the other, can be treated like a box (the section in the roll) with a certain number of items inside (the sweets); he found that children's ability to think in terms of composite units is a predictor of their understanding of proportionality. The composite unit can then be replicated as many times as necessary so that the problem is solved.

The item on the right presents one roll of sweets on top with a total of 18 sweets, which are organised in three sections. When the children find the correspondence between

each section in the roll and the sweets, they form a composite unit: every time they add one section, they add 6 sweets to a roll. This reasoning by composite units allows them to find how many sweets there are in the small roll below.

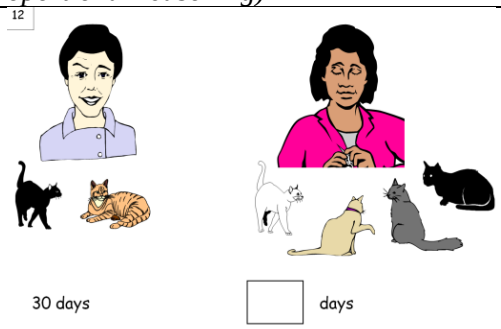
In some questions there is a ratio between two quantities that are different; the composite unit is thus a ratio between two different quantities. Two examples are shown in Figure 4. In the question on the left, the taste of the medicine will be the same if the ratio 1:4 is maintained. In the question on the right, the composite unit is 2 cats:30 days. The children will need to use inverse proportional reasoning: if the number of cats is doubled, the number of days has to be halved.

**Figure 4**

*Examples of Quantitative Reasoning questions (proportional reasoning)*




*Instructions read to the children:*
There is a medicine that is very bitter, and the chemist mixes it with syrup for the children to make it taste better.
Yesterday she mixed 1 spoon of medicine with 4 spoons of syrup.
Today she had to make more mixture and she will have to use 2 spoons of medicine.
How many spoons of syrup will she need for the mixture to taste the same as yesterday?
Draw the spoons or write the number under the syrup bottle.



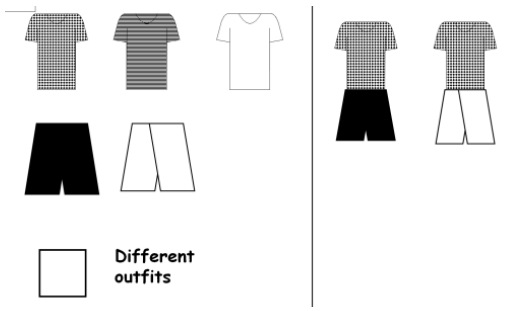

*Instructions read to the children:*
Mrs Green has two cats. Mrs Patel has four cats. They went out together to buy cat food. They bought the same amount of cat food.
Mrs Green said that the food she bought is enough for her cats to eat for 30 days. How long do you think the food Mrs Patel bought will last?

In other items, as shown in Figure 5, on the left an item shows two pairs of shorts and three T-shirts. Children are asked to provide the total number of possible outfits. The children need to reason that for each pair of shorts, they can have 3 different outfits because there are 3 different tops. The reasoning is termed product of measures by

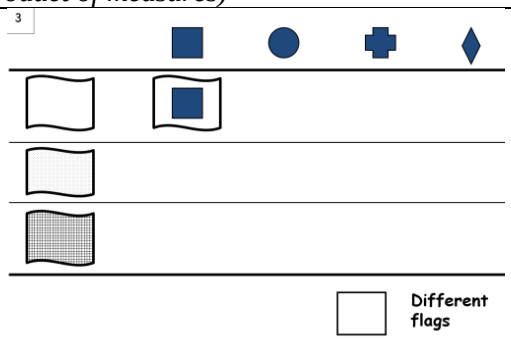
Vergnaud (1983). The same applies to the item shown in Figure 5 on the right, which shows some flags. Children are asked to provide the total number of possible flags. The children need to reason that for each different background, you can make 4 different flags by changing the figures.

**Figure 5**

*Examples of Quantitative multiplicative questions (product of measures)*




*Instructions read to the children:*
Rebecca has three different shirts and two different pairs of shorts. You can see them on the left side of the page. She can combine the shirts with shorts and wear different outfits. You can see on the right side of the page how she can wear the same shirt with the two shorts and have different looking outfits.
If she changes her shirts and shorts around to make different outfits, how many different outfits can she make? Write your answer in the box.



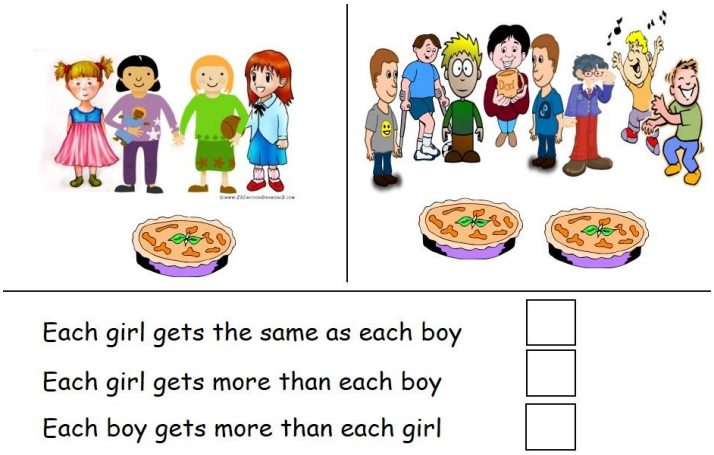

*Instructions read to the children:*
At Pear Tree School they have three different types of material that the students can use to make flags. One is plain, one has some dots, and the third has squares.
The teacher bought boxes with 4 different figures for the children to paste on the centre of the flags. The teacher wants the children to make as many different flags as they can.
Imran has made one flag using plain material and a square figure.
How many different flags can the children make? Write your answer in the box.

### *(b) Quantitative Reasoning in the context of quantities smaller than the unit*

Six questions were designed to assess quantitative reasoning in the domain of quantities smaller than a unit. In Figure 6, two questions are presented. Notice that children were not asked to read or write a numerical representation of a fraction, but to select an answer from three options read by the researcher. These questions also did not require any computation but relied on the children's understanding that when a quantity is divided by a number of parts, this would result in smaller parts. In Figure 6, two groups of children and pies are presented: on the left, four girls and one pie; on the right, eight boys and two pies,

four boys for each pie. Children had to reason how many children in each group shared one pie so that they could conclude if they were all receiving the same amount of pie or not. Children were asked to select one option from: (a) each girl received the same as each boy; (b) each girl received more than each boy; or (c) each boy received more than each girl.

**Figure 6**
*Example of Quantitative Reasoning fractions questions*




*Instructions read to the children:*
There are two groups of children. In the first group, 4 girls are going to equally share one pie. In the second group, 8 boys are going to equally share two pies. Which of these sentences is correct? Tick the box for the right one.

It is expected that the type of situation and the specific ratio will together influence the level of difficulty of the items; ratios 1:2, 1:3 and 1:4 are much more easily handled than a ratio of 1:1.5.

**The Arithmetic Knowledge test**

This test seeks to assess children's skills in solving calculations, some of which might be known from memory, whereas others require the use of procedures following arithmetical rules. The test was given to children as a written calculation in an answer booklet; the children were tested as a whole class. The arithmetic knowledge assessment

has 15 questions that are categorized in two groups: (a) operating with whole numbers (9 items), and (b) operating with symbolic representation of numerical fractions (6 items).

Items require different types of arithmetical knowledge:

***(a) Operating with whole numbers***

(1) Two multiplication items with carrying (e.g. 45 × 5) and one item without carrying (e.g. 13 × 3)

(2) Four division items without a remainder (e.g. 84 divided by 4; 96 divided by 6) and one item with a remainder (e.g. 28 divided by 13)

(3) One item of subtraction with borrowing (e.g. 72 – 43).

***(b) Operating with symbolic numerical representations of fractions***

(1) Three items of addition and subtraction of proper fractions with the same denominator (e.g. 3/8 + 2/8)

(2) Two items of addition of proper fractions with common multiple denominators (e.g. ½ + ¼)

(3) One item where one term is a proper fraction and the other term a whole number (e.g. 1 – 2/5)

All the items from both tests, Quantitative Reasoning and Arithmetic Knowledge, are presented in Appendix 4.

***Psychometric properties of the Quantitative Reasoning and Arithmetic Knowledge test***

Internal consistency is a measure based on the correlations between items on the same scale. It assesses the degree to which the items are interrelated, assuming each test to be one latent construct. Cronbach's alpha is a measure of internal consistency that uses pairwise correlations. It has been suggested that above .90 shows excellent internal consistency, although one must be careful if that is the case as some items may be

redundant. An alpha of .80 to .90 shows good internal consistency, and .70 to .80 suggests acceptable internal consistency (Bland and Altman, 1997; Cronbach, 1951; Tavakol and Dennick, 2011). Cronbach's alpha was calculated for each of the tests, Quantitative Reasoning and Arithmetic Knowledge, separately.

The test of Quantitative Reasoning consisting of 17 items showed a good level of internal consistency at time 1 and time 2 of data collection ($\alpha = .81$ and .82, respectively).

The test of Arithmetic Knowledge consisting of 15 items showed a good level of internal consistency at time 1 and time 2 of data collection ($\alpha = .86$ and .87, respectively). This also shows that the scores of items are correlated.

### *Description of the non-verbal ability test*

#### Raven's Progressive Matrices

Raven's Progressive Matrices aim to assess general ability from a non-verbal perspective as described by Spearman's G. The test aims to measure eductive ability, which is the ability to create new ideas, find patterns, discern meaning and identify relationships (Raven, 2000).

The items are presented in a printed booklet and children answer on an answer sheet. It lasts approximately 35 minutes. This test contains sets A to E, with 12 items per set and 60 items in total. In each item, the child is asked to identify the missing element that completes a pattern and to choose the answer from multiple-choice options. It is designed for the general population from 6 to 17+ years old and should be administered as a group using pen and paper. The Raven's Standard Progressive Matrices produces a single raw score as well as percentile rank to indicate the child's ability, compared to a norm group. In this study, each item score was added as a linear composite.

*Psychometric properties of the non-verbal ability test*

Internal consistency using Cronbach's alpha was calculated for the Progressive Raven's Matrices using the data of the current study. The scale of 52 items showed a good level of internal consistency ($\alpha = .91$).

The test presented a mean of 32.56 (SD 8.60) with a minimum of 9 and a maximum of 46 for the sample of this study.

## 3.2 Results

In this section the analysis to examine the measures will be presented in the following order: (1) first, all the items from both tests, Quantitative Reasoning and Arithmetic Knowledge, were analysed to assess their level of difficulty using descriptive statistics; (2) then the factor structure of the measures was tested using CFA, two different models were presented and their model fit compared; (3) subsequently, there was an analysis to measure whether the factor structure was invariant over time, which was assessed by repeating the models of CFA over time and comparing them; (4) finally, the independence of the constructs from non-verbal ability was assessed by using partial correlations.

### 3.2.1 Level of difficulty of the items

Descriptive analysis aims to provide information about the measures and the level of difficulty of each item. The descriptive analyses were used to calculate the proportion correct of each of the questions. Each item has a score of 0 for incorrect or 1 for correct responses. A proportion of correct responses close to 1 indicate that the item was easier than one where the mean is close to 0. The proportion of correct responses of all the items is presented in Table 2. The first column of the table presents the visual item as printed on

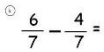
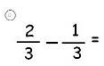
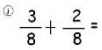
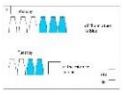
the children's tests. The second column indicates to which test and subtest the item belongs, as well as a description of the item. The third column presents the proportion of correct responses to each item from 0 to 1. All items from both scales, Quantitative Reasoning and Arithmetic Knowledge, were included together in the descriptive analysis and then ordered by level of difficulty as shown in the third column.

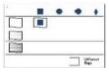
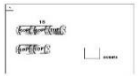
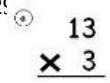
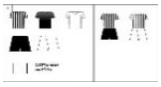
The results show that the Arithmetic Knowledge items were consistently more challenging for children than the Quantitative Reasoning items. The most difficult item required children to calculate the addition of a proper fraction with common multiple denominators; proportion correct = .06. Division of whole numbers also presented higher levels of difficulty. The seven most difficult items belonged to the arithmetic scale.

**Table 2**

*Items from Quantitative Reasoning and Arithmetic Knowledge tests by level of difficulty*

| | **Item description** | **Descriptive** | | | |
|---|---|---|---|---|---|
| | Scale | **Time 1** | | **Time 2** | |
| **Visual Item** | *Description* | Proportion correct | SD | Proportion correct | SD |
| 1 | Arithmetic, Fractions<br>*Addition of proper fractions with common multiple denominators* | 0.06 | 0.23 | 0.09 | 0.28 |
| 2 | Arithmetic, Whole numbers<br>*Division without a remainder* | 0.06 | 0.25 | 0.12 | 0.32 |
| 3 | Arithmetic, Fractions<br>*Addition of proper fractions with common multiple denominators* | 0.10 | 0.30 | 0.12 | 0.32 |
| 4 | Arithmetic, Whole numbers<br>*Multiplication with carrying* | 0.11 | 0.32 | 0.20 | 0.40 |
| 5 | Arithmetic, Whole numbers<br>*Division with remainder* | 0.13 | 0.34 | 0.26 | 0.44 |
| 6 | Arithmetic, Fractions<br>*One term a proper fraction and the other term a whole number* | 0.13 | 0.34 | 0.21 | 0.41 |
| 7 | Arithmetic, Whole numbers<br>*Division without a remainder* | 0.23 | 0.43 | 0.26 | 0.44 |
| 8 | Quantitative Reasoning, Whole numbers<br>*Proportion required division* | 0.26 | 0.44 | 0.42 | 0.50 |

| | Item description | Descriptive | | | |
|---|---|---|---|---|---|
| | Scale | Time 1 | | Time 2 | |
| Visual Item | *Description* | Proportion correct | SD | Proportion correct | SD |
| 9  | Arithmetic, Whole numbers<br>*Division without a remainder* | 0.28 | 0.45 | 0.41 | 0.49 |
| 10  | Quantitative Reasoning, Fractions<br>*Compare parts: 2 out 3 and 4 out of 6* | 0.29 | 0.46 | 0.33 | 0.47 |
| 11-12  | Quantitative Reasoning, Whole numbers<br>*Proportion* | 0.32 | 0.47 | 0.44 | 0.50 |
| | Quantitative Reasoning, Whole numbers<br>*Proportion* | 0.32 | 0.47 | 0.40 | 0.49 |
| 13  | Arithmetic, Fractions<br>*Subtraction of proper fractions with the same denominator* | 0.33 | 0.47 | 0.39 | 0.49 |
| 14  | Arithmetic, Whole numbers<br>*Division without a remainder* | 0.35 | 0.48 | 0.47 | 0.50 |
| 15  | Quantitative Reasoning, Fractions<br>*Compare parts: 4 out of 8 and 2 out of 4* | 0.35 | 0.48 | 0.44 | 0.50 |
| 16  | Arithmetic, Fractions<br>*Subtraction of proper fractions with the same denominator* | 0.36 | 0.48 | 0.47 | 0.50 |
| 17  | Arithmetic, Whole numbers<br>*Multiplication with carrying* | 0.40 | 0.49 | 0.54 | 0.50 |
| 18  | Quantitative Reasoning, Fractions<br>*Compare shares 2:3 and 3:6* | 0.44 | 0.50 | 0.46 | 0.50 |
| 19  | Quantitative Reasoning, Whole numbers<br>*Proportion with equal numbers* | 0.44 | 0.50 | 0.59 | 0.49 |
| 20  | Arithmetic, Fractions<br>*Addition of proper fractions with the same denominator* | 0.45 | 0.50 | 0.63 | 0.48 |
| 21  | Quantitative Reasoning, Proportional | 0.48 | 0.50 | 0.52 | 0.50 |
| 22  | Arithmetic, Whole numbers<br>*Subtraction with borrowing* | 0.48 | 0.50 | 0.58 | 0.50 |

| Visual Item | Item description<br>Scale<br>*Description* | Descriptive<br>Time 1<br>Proportion correct | SD | Time 2<br>Proportion correct | SD |
|---|---|---|---|---|---|
| 23  | Quantitative Reasoning, Whole numbers<br>*Inverse relation between the divisor and the quotient* | 0.49 | 0.50 | 0.58 | 0.50 |
| 24  | Quantitative Reasoning, Whole numbers<br>*Sample space* | 0.57 | 0.50 | 0.61 | 0.49 |
| 25  | Quantitative Reasoning, Fractions<br>*Compare shares 2:3 and 3:4* | 0.60 | 0.49 | 0.61 | 0.49 |
| 26  | Quantitative Reasoning, Fractions<br>*Compare a third and two thirds* | 0.60 | 0.49 | 0.72 | 0.45 |
| 27  | Quantitative Reasoning, Whole numbers<br>*Proportion, double* | 0.60 | 0.49 | 0.72 | 0.45 |
| 28  | Quantitative Reasoning, Whole numbers<br>*Division (third), then subtract* | 0.62 | 0.49 | 0.71 | 0.46 |
| 29  | Arithmetic, Whole numbers<br>*Multiplication without carrying* | 0.64 | 0.48 | 0.71 | 0.46 |
| 30  | Quantitative Reasoning, Whole numbers<br>*Division (half), then add* | 0.65 | 0.48 | 0.70 | 0.46 |
| 31  | Quantitative Reasoning, Whole numbers<br>*Multiplication* | 0.67 | 0.47 | 0.72 | 0.45 |
| 32  | Quantitative Reasoning, Fractions<br>*Compare shares 1:4 and 2:8* | 0.70 | 0.46 | 0.74 | 0.44 |

Three items from the Quantitative Reasoning test were ranked as the easiest. The question with the highest correct proportion was from the subscale Quantitative Reasoning, fractions. This question was presented in Figure 6, showing two groups of children, one group of four with one pie and one group of eight with two pies, under which

pupils are asked to tick the right sentence: Each girl gets the same as each boy. This question had a proportion correct = .70, which shows that most children (70%) answered it correctly.

### 3.2.2 Factor structure of the measures

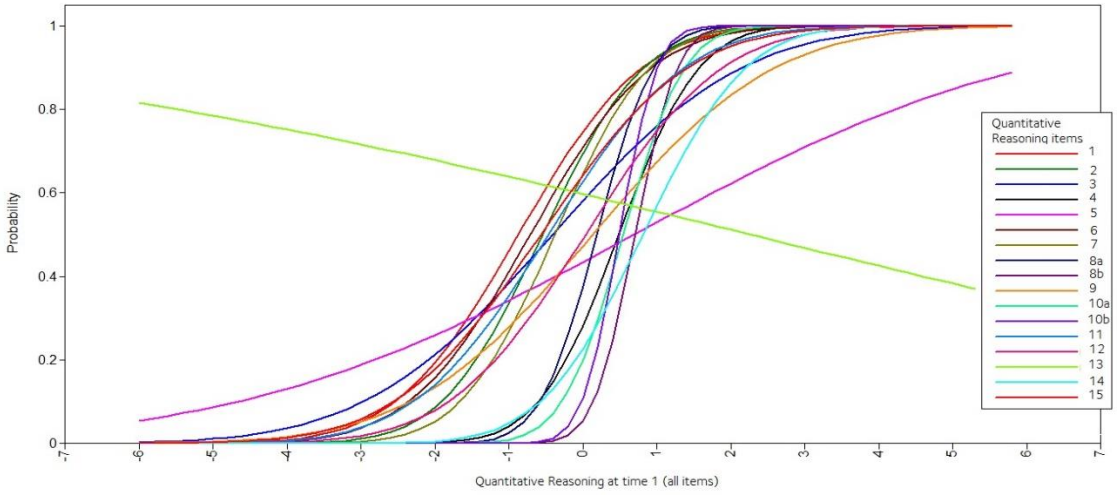
In order to assess whether the Quantitative Reasoning and the Arithmetic Knowledge tests could be seen as measures of distinct though related competencies ($r(124) = .63$, $p < .01$), a series of CFA models were carried out. As the items were dichotomous, the Weighted Least Square estimator was utilised using MPlus software; the model is equivalent to a two-parameter logistic (2PL) item response model, where the β observed as factor loadings are equivalent to logistic regression parameters.

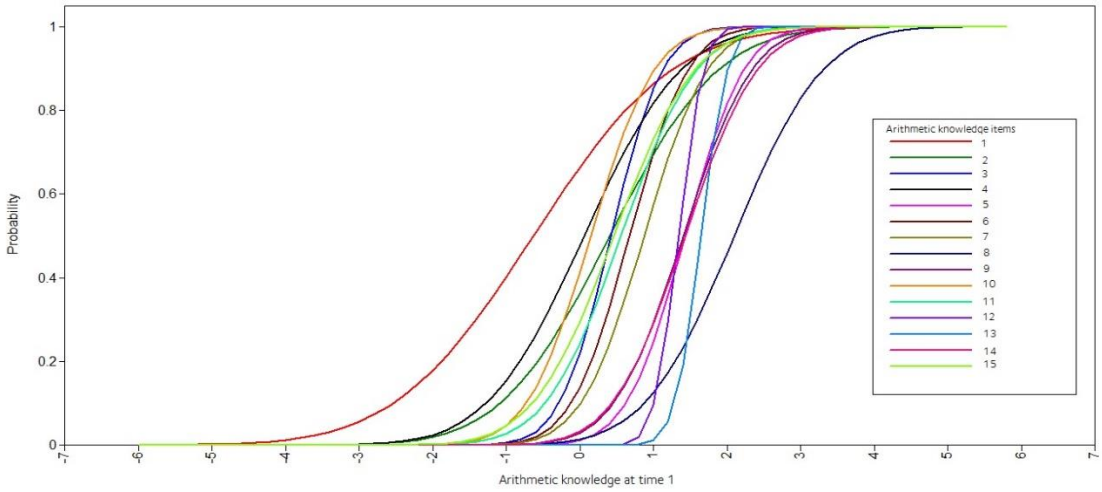
It was argued that the model with the best goodness of fit indices would be the model to best represent the empirical data. The indices used in this study are the confirmatory fit index (CFI), the Tucker-Lewis index (TLI) and the root mean square error of approximation (RMSEA). For these indices, fit is considered adequate if the CFI and TLI values are > 0.90 and the RMSEA < 0.08 (Chen, 2007; Cheung and Rensvold, 2002). Goodness of fit indices are descriptive, and it is difficult to compare them statistically; one can compare and say the highest CFI is the best fit, or the lowest RMSEA is the better fit. It is possible to estimate the ΔRMSEA and ΔCFI; for example, by calculating the difference in RMSEA between the model with two factors and the model with one factor (One-factor model RMSEA; Two-factor model RMSEA).

Two models were proposed: the first model included the 32 items from both tests, Arithmetic Knowledge and Quantitative Reasoning. In the second model, 15 items from Arithmetic Knowledge were specified to load into one factor and 17 items from the

Quantitative Reasoning test were specified to load into the second factor. Two different models were analysed. Figure 7 shows a line per item; most items behave similarly, and this shows coherence between them. However, the line representing item 13 is crossing in a different direction to all other items, which shows that this item is negatively correlated to the rest. Although the goodness of fit indices were acceptable, the factor loading and item curve characteristic showed that item 13 (green line) from the Quantitative Reasoning measure presents a negative correlation with the other items, presenting a curve that goes in the opposite direction to the curves of the other items, as shown in Figure 7. It was concluded that item 13 should be removed from this measure, so this item is not included in any of the following analyses. Thus, the Quantitative Reasoning measure will contain 16 items after removing item 13. All items from the Arithmetic Knowledge measure were included. The Arithmetic Knowledge measure consists of 15 items.

**Figure 7**
*Item curve characteristics from CFA*






After removing item 13 the models were run again, the two different CFA models were carried out, analysed and their indices of fit compared. Table 3 shows the indices of fit for each of the models. Although the fit indices of CFI and RMSEA both relate to the chi-square distribution, they relate differently. The CFI compares the proposed model to a null or baseline model (uncorrelated covariance matrix), whereas RMSEA used the chi-square distribution of the target model (it compares the error by difference between the observed and the expected). The first model comprised just one factor, in which all 31 items from both scales (Quantitative Reasoning and Arithmetic Knowledge) were included together in the one factor. Results were analysed, the values of the goodness of fit indices of the CFI and TLI were .90, which is acceptable, and the value of the RMSEA was below .08, which is adequate (Chen, 2007; Cheung and Rensvold, 2002). To test if the goodness of fit could be improved, a second model was designed. The second model had two a priori factors: one was composed of all the Quantitative Reasoning items and the second factor was composed of all the Arithmetic Knowledge items. The model with two a priori factors showed better goodness of fit indices, with a comparative fit index value (CFI= .945) that well exceeds the generally accepted cut-off criterion of .90 (Byrne, 1994, 2001) and an overall better fit than the analysis with one factor (see Table 3). The improvement of the

goodness of fit indices was calculated and found that the ΔRMSEA was slightly bigger than .015 (ΔRMSEA = .016) and the ΔCFI was bigger than .01 (ΔCFI= .042). These results show that the two-factor model fits the data better (Chen, 2007; Cheung and Rensvold, 2002).

**Table 3**
*Confirmatory Factor Analysis of the measures at Time 1*

| Model | $\chi^2$ | df | CFI | ΔCFI (CFI a- CFI b) | TLI | RMSEA | ΔRMSEA (RMSEA a- RMSEA b) |
|---|---|---|---|---|---|---|---|
| One factor | 664.7* | 434 | .903 | | .896 | .065 | |
| Two factors | 563.7* | 433 | .945 | -.042 | .941 | .049 | .016 |

Note. $\chi 2$ = chi square goodness of fit statistic; df = degrees of freedom; CFI = Comparative Fit Index; TLI = Tucker Lewis Index; RMSEA = Root Mean Square Error of Approximation; WRMR = Weighted Root Mean Square Residual).
* Indicates $\chi 2$ are statistically significant ($p < .001$); N=124

These results give support to the hypothesis that these are indeed two different abilities and that understanding Quantitative Reasoning differs from Arithmetic Knowledge.

### 3.2.3 Factor structure invariance

Six months later, children were tested using the same measures of Quantitative Reasoning and Arithmetic Knowledge. In this section, analyses were divided into two steps:

(1) First, it was important to test whether the factor structure results from time 1 could be duplicated with time 2 data. In order to test if the results could be duplicated, the same CFA models were carried out.

(2) Then, to test the factor structure invariance, the two-factor CFA model from time 1 was compared to the same two-factor CFA model from time 2; their indices of fit and their factor loadings were compared.

*Test if results from time 1 could be replicated at time 2*

As described in the section on factor structure, two models were tested and compared to each other to assess which one presented a better model fit. The first model was composed of one factor, in which all items from both scales (Quantitative Reasoning at time 2 and Arithmetic Knowledge at time 2) were included. Results from time 2 were very similar to those of time 1; this model presented acceptable CFI and TLI values (.95 and .94 respectively) and RMSEA (.06) as shown in Table 4.

**Table 4**

*Confirmatory Factor Analysis of the measures at Time 2*

| Model | $\chi^2$ | df | CFI | ΔCFI (CFI a-CFI b) | TLI | RMSEA | ΔRMSEA (RMSEA a-RMSEA b) |
|---|---|---|---|---|---|---|---|
| One Factor | 594.6* | 434 | .945 | | .941 | .060 | |
| Two Factors | 535.4* | 433 | .965 | -.02 | .962 | .048 | .012 |

Note: $\chi^2$ = chi square goodness of fit statistic; df = degrees of freedom; CFI = Comparative Fit Index; TLI = Tucker Lewis Index; RMSEA = Root Mean Square Error of Approximation; WRMR = Weighted Root Mean Square Residual
*Indicates $\chi^2$ are statistically significant ($p < .001$); N=104

However, in order to test if the model fit could be improved, a second CFA model with the two a priori factors was carried out. In this model, one factor comprised all the Quantitative Reasoning items at time 2, and the second factor comprised all the Arithmetic Knowledge items at time 2. Results from time 1 were also duplicated and it was found that the model with two a priori factors showed a better fit than the analysis with one factor (see Table 4). The goodness of fit indices were compared by calculating the ΔRMSEA and ΔCFI between the model with two factors and the model with one factor. Results confirmed the previous results found at time 1, showing that a two-factor model fits better than a one-factor model according to ΔRMSEA < .016 and ΔCFI < .015 (Chen, 2007; Cheung and Rensvold, 2002).

Results from time 1 data collection were replicated at time 2 data collection, thus confirming that Quantitative Reasoning and Arithmetic Knowledge are two different mathematical abilities.

*Factor structure invariance*

In order to assess the factor structure invariance, the two-factor models, one from time 1 and one from time 2, were compared. Results shown in Table 5 confirm that the goodness of fit indices are similar at both time points. The ΔRMSEA was smaller than .016 and ΔCFI was -.02, which is slightly higher than the suggested .015 (Chen, 2007; Cheung and Rensvold, 2002).

**Table 5**
*Comparison of goodness of fit indices of CFA two-factor models at Time 1 and at Time 2*

| Model | $\chi^2$ | df | CFI | ΔCFI (CFI a- CFI b) | TLI | RMSEA | ΔRMSEA (RMSEA a- RMSEA b) |
|---|---|---|---|---|---|---|---|
| Two Factors at Time 1 | 563.7* | 433 | .945 | | .941 | .049 | |
| Two Factors at Time 2 | 535.4* | 433 | .965 | -.020 | .962 | .048 | 0.001 |

Note: $\chi^2$ = chi square goodness of fit statistic; df = degrees of freedom; CFI = Comparative Fit Index; TLI = Tucker Lewis Index; RMSEA = Root Mean Square Error of Approximation; WRMR = Weighted Root Mean Square Residual
* Indicates $\chi^2$ are statistically significant ($p < .001$); N = 104

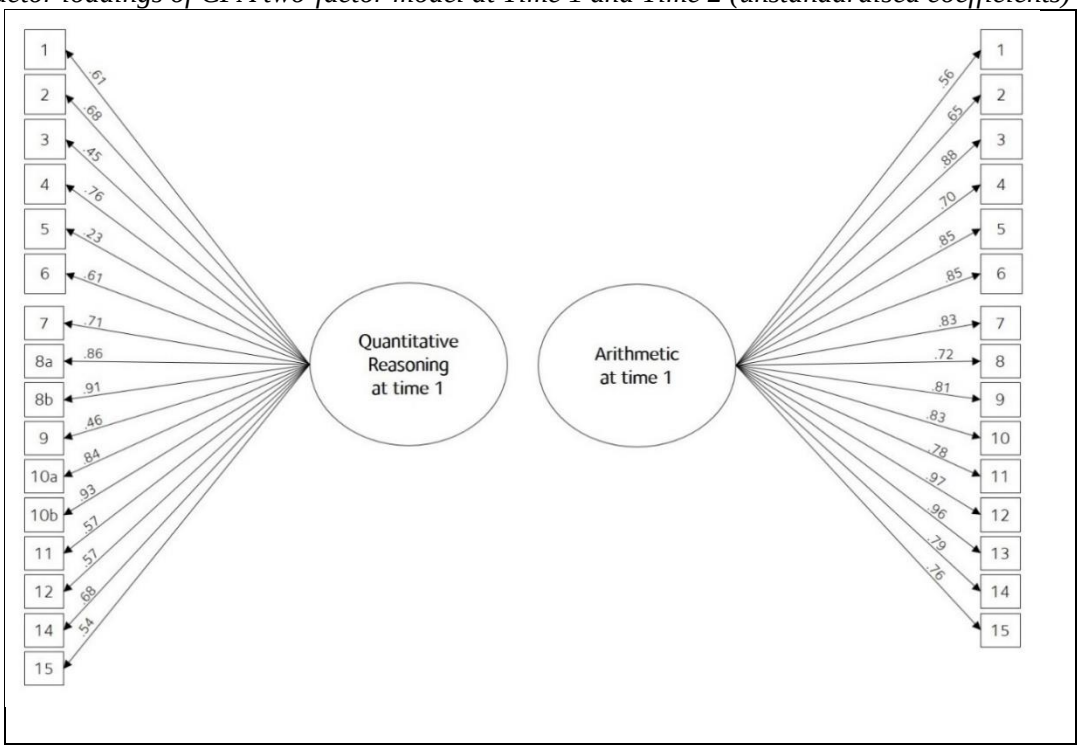
These results demonstrate that the factor structure remains the same at two different time points.

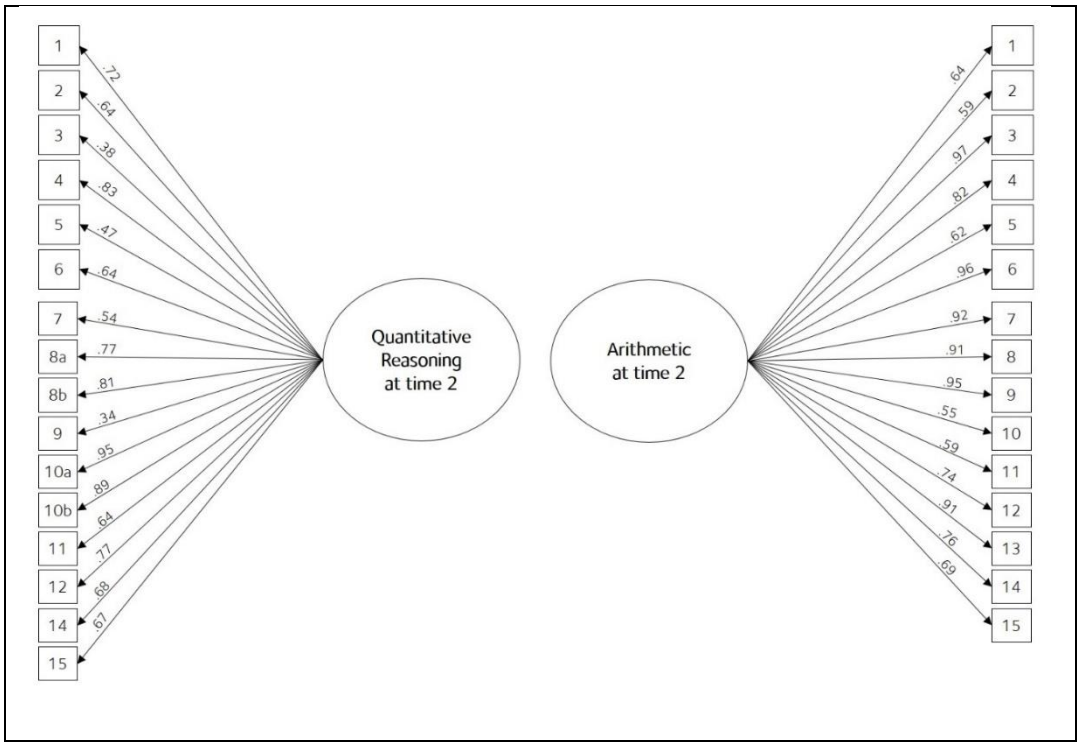
In CFA, the relationship of each variable loaded on to a latent variable is expressed by the factor loading. It can be interpreted as standard regression coefficients or correlations. Thus, the higher the factor loading, the stronger the relation with the latent variable. The factor loadings of the CFA two-factor models, from time 1 and time 2, were compared. Factor Quantitative Reasoning at time 1 comprised 16 items reported as right or wrong, with factor loadings from .23 to .93. Factor Arithmetic Knowledge at time 1 comprised 15 items reported as right or wrong, with factor loadings from .56 to .97. Quantitative Reasoning items show slightly lower factor loadings to the latent variable

than their counterpart items in Arithmetic Knowledge. At time 2, the variables also showed an association to the latent variables. Quantitative Reasoning variables presented a range of factor loadings from 0.34 to 0.95. Arithmetic Knowledge variables at time 2 presented a range of factor loadings from 0.55 to 0.97. These results suggest that the variables are correlated to the latent variable and are appropriate to measure that latent variable. These are presented in Figure 8.

**Figure 8**

*Factor loadings of CFA two-factor model at Time 1 and Time 2 (unstandardised coefficients)*






### 3.2.4 Independence of the measures from non-verbal ability

*Partial correlations*

Although the factor structure shows that there are two distinct abilities and this result was replicated using data from children's scores when assessed six months after using the same measures, Quantitative Reasoning and Arithmetic Knowledge, there may still be an underlying ability that explains the results. This could be non-verbal ability. To test if Quantitative Reasoning and Arithmetic Knowledge are stable measures independent of children's non-verbal ability, they should correlate highly even after controlling for non-verbal ability.

First correlations of all the measures, including non-verbal ability, were computed. Then, partial correlations of Quantitative Reasoning and Arithmetic Knowledge at time 1 and time 2, controlling for non-verbal ability, were analysed.

Table 6 shows the result of the correlations between the measures. Quantitative Reasoning at time 1 and Quantitative Reasoning at time 2 correlate more highly (r=.80, p <.001, N=104) than with the Arithmetic Knowledge measure obtained at the same time (r=.61, p<.001, N=124). This difference is statistically significant (z= 2.89, p<0.005). Similarly, Arithmetic Knowledge at time 1 and Arithmetic Knowledge at time 2 correlate more highly with each other than with Quantitative Reasoning obtained at the same time. This difference is statistically significant (z=3.44, p <.001, N=124 and 104). Both measures show a lower correlation with non-verbal ability.

**Table 6**
*Correlations of all measures*

| | Quantitative Reasoning at Time 1 | Quantitative Reasoning at Time 2 | Arithmetic Knowledge at Time 1 | Arithmetic Knowledge at Time 2 | Non-verbal ability |
|---|---|---|---|---|---|
| **Quantitative Reasoning at Time 1** | 1 | | | | |
| **Quantitative Reasoning at Time 2** | .80** | 1 | | | |
| **Arithmetic Knowledge at Time 1** | .61** | .64** | 1 | | |
| **Arithmetic Knowledge at Time 2** | .68** | .69** | .86** | 1 | |
| **Non-verbal ability** | .57** | .54** | .44** | .50** | 1 |

** p<0.01

Table 7 shows the partial correlations between the measures at two assessment points. Both the Quantitative Reasoning and the Arithmetic Knowledge measures showed high correlations. The Quantitative Reasoning items showed a partial correlation of r= .71, p <.001, N=101. The Arithmetic Knowledge items showed a partial correlation of r= .83, p <.001, N=101.

Quantitative Reasoning at time 1 and Quantitative Reasoning at time 2 correlate more highly to each other, even after controlling for non-verbal ability (r= .72, p <.001,

N=101), than with the Arithmetic Knowledge measure obtained at the same time (r= .55, p<.001, N=101). This difference is statistically significant (z= 2.03, p= 0.021, N=101).

Quantitative Reasoning and Arithmetic Knowledge remain highly correlated even after partialling out the variance that they share with non-verbal ability.

**Table 7**
*Partial correlations of Time 1 and 2 scores for Quantitative Reasoning and Arithmetic Knowledge measures*

| Control variable | Measures at two testing points | Quantitative Reasoning at Time 1 | Quantitative Reasoning at Time 2 | Arithmetic Knowledge at Time 1 | Arithmetic Knowledge at Time 2 |
|---|---|---|---|---|---|
| Non-verbal ability | Quantitative Reasoning at Time 1 | 1.00 | | | |
| | Quantitative Reasoning at Time 2 | .72 | 1.00 | | |
| | Arithmetic Knowledge at Time 1 | .55 | .55 | 1.00 | |
| | Arithmetic Knowledge at Time 2 | .56 | .58 | .83 | 1.00 |

All correlations were significant at $p<0.001$

These results show that these two abilities continue to share variance with each other after partialling out what they have in common with non-verbal ability. As a further assessment of the consistency of each measure independent of its relation with other abilities, new partial correlations were computed. The partial correlation between scores in Quantitative Reasoning at time 1 and scores in Quantitative Reasoning at time 2, controlling for non-verbal ability and Arithmetic Knowledge at time 1, was equal to .6, which is still strong and statistically significant ($p<.01$). The partial correlation between scores in Arithmetic Knowledge at time 1 and scores in Arithmetic Knowledge at time 2, controlling for non-verbal ability and Quantitative Reasoning at time 1, was equal to .75, which is high and statistically significant ($p<.01$).

## 3.3 Conclusions

In this chapter the Quantitative Reasoning and Arithmetic Knowledge measures were analysed to determine the quality of the measures by investigating their construct validity, factor structure invariance over time and independence from non-verbal ability. It

was hypothesised that the factor structure, tested using CFA, would show that there are two theoretically and empirically different mathematical abilities. The analyses of the measures produced results confirming this prediction. First, it is possible to measure them separately by using distinct sets of items, even though Quantitative Reasoning and Arithmetic Knowledge are related. These results confirm that there is clear justification for making a distinction between Quantitative Reasoning and Arithmetic skills.

The constructs' factor structure is invariant over time and is independent from non-verbal ability. These results show that the measures could be used to carry out assessments under different conditions, for example, at different time points and measuring the same attributes; also, that they could explain children's learning of fractions independently from non-verbal ability.

The justification allows for using the measures in a longitudinal study that investigates whether children's understanding of Quantitative Reasoning or their Arithmetic Knowledge is a better predictor of their learning of fractions. If this is the case, it can provide information for mathematics curricula, such as more focus on promoting multiplicative reasoning than hitherto, in order to prepare children for learning about rational numbers.

Further analyses in the longitudinal project will provide valuable additional details regarding the interactions between children's knowledge of multiplicative reasoning and their knowledge of arithmetic as predictors of their subsequent learning of fractions.

# Chapter 4 Precursors of Fraction Learning: A Longitudinal Study

In the previous chapter, it was established that Quantitative Reasoning and Arithmetic Knowledge are distinct abilities. These results raise an interesting possibility about how children learn fractions, which is whether each of these two distinct abilities makes separate and independent contributions to children's future learning and understanding of fractions. If this is the case, it would be important to determine their specific contribution to children's learning of fractions.

This chapter presents the second step of the longitudinal study. The aim of this step is to investigate how children's mathematical abilities measured before they are taught systematically about fractions in school predicts their learning outcomes. This chapter presents the last testing point of the longitudinal study, which aims to assess whether Quantitative Reasoning and Arithmetic Knowledge can make independent contributions to children's attainment in a fractions test after 12 months; and if so, to evaluate their comparative importance, controlling for non-verbal ability. This will be carried out by:

(1) Assessing two abilities, Quantitative Reasoning and Arithmetic Knowledge, previous to the teaching of fractions (as analysed in the last chapter)

(2) Assessing children's fractions knowledge 12 months later, after they have been taught about fractions

(3) Considering the contribution of non-verbal cognitive ability.

In the following sections, a description of the precursors to fractions knowledge is presented, then the rationale for a control variable in longitudinal studies and a description of the study.

*Assessing two precursors: Quantitative Reasoning and Arithmetic Knowledge*

In order to understand fractions, children must learn at least two things: Quantitative Reasoning and Arithmetic Knowledge. If one wants to test these two abilities as precursors to learning fractions, one should take into account that fractions are numbers in the domain of quotients (Kieren, 1993) and therefore require understanding of multiplication and division. Both mathematical abilities should assess children's understanding of multiplication and division: for Quantitative Reasoning, their knowledge of quantitative multiplicative relations; and for Arithmetic Knowledge, their ability to carry out arithmetical procedures that allow them to operate with fractions, such as multiplication and division. Longitudinal studies allow one to test more than just a correlation between the measures; they provide the proportion of variance that each factor contributes to the outcome measure.

In the following paragraphs, first assessing Quantitative Reasoning is discussed, followed by assessing Arithmetic Knowledge.

*Assessing Quantitative Reasoning*

Children's Quantitative Reasoning about multiplicative relations has been assessed in previous studies (Kornilaki, 1999; Nunes et al., 2015; Nunes et al., 2007; Nunes, Bryant, Barros and Sylva, 2012a). In the previous chapter, a group of selected items that aimed to assess children's understanding of multiplicative relations was examined, and it was concluded that it would be appropriate to use this selection of items as the Quantitative Reasoning test for the current longitudinal study. Some of the items require easy computations, and others no computation at all. For example, the children are asked to consider whether if a child cuts a chocolate into eight equal pieces and eats four pieces, and another child cuts an identical chocolate into six equal pieces and eats three pieces, is

the amount of chocolate each child eats the same or not (Kornilaki and Nunes, 2005). The children are not asked to represent the amount of chocolate by using fractions. They are asked to judge if the amounts eaten are equivalent, and if not, which will be larger. This is an example of multiplicative reasoning about quantities without the need to calculate.

A previous study (Nunes et al., 2012a) has found that Quantitative Reasoning was a stronger predictor for children's further mathematical learning than Arithmetic Knowledge. Although the outcome measure did not include rational numbers, the study is suggestive that Quantitative Reasoning is likely to be a stronger predictor for learning fractions.

Studies have also investigated children's reasoning about multiplicative relations between quantities before they are taught about fractions in school (Nunes et al., 2012a; Nunes et al., 2007). However, these studies have not analysed longitudinally whether children's multiplicative reasoning predicts their learning of fractions subsequently when the topic is taught systematically as part of the curriculum in school. Therefore, this study would provide new information about how children learn about fractions.

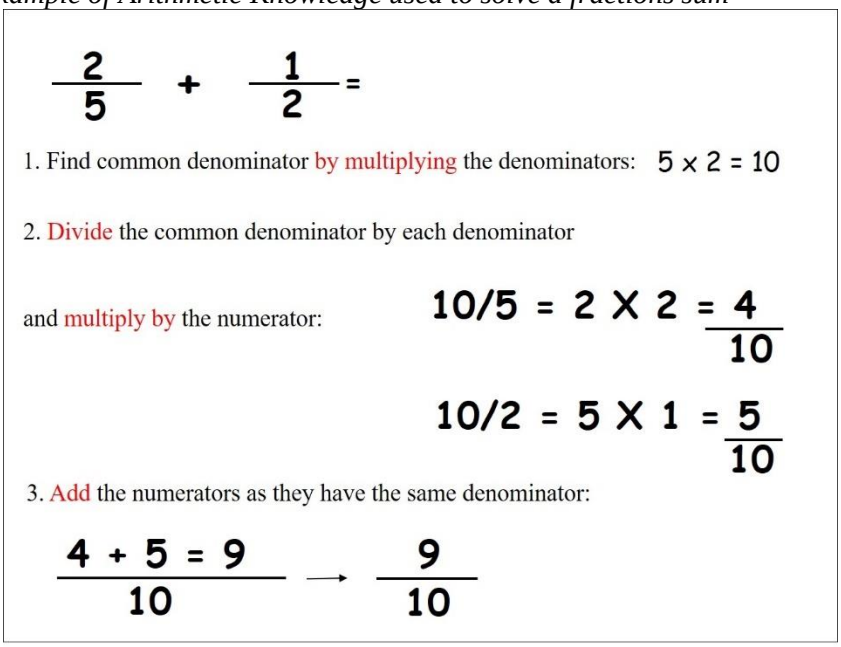
*Assessing Arithmetic Knowledge*

Arithmetic Knowledge should also play an important role in children's learning about fractions and provide an independent contribution from what Quantitative Reasoning may provide. Because children must have some Arithmetic Knowledge of whole numbers in order to calculate with fractions, as they need to use addition, subtraction, multiplication and division when implementing calculations with fractions, it is important to assess the contribution of Arithmetic Knowledge to learning fractions. They also need to learn the procedural rules of how to calculate with fractions, but this can be taught in school without explicitly explaining the connection between the calculation and the meaning of fractions.

Therefore, children may learn the arithmetic to operate with fractions without establishing a connection with their understanding about Quantitative Reasoning.

To calculate with fractions, children must use their previous knowledge of arithmetic as they implement a calculation and follow arithmetical rules. For example, to solve a simple addition of fractions, children must find the common denominator, which may require the multiplication between both denominators; then they must divide the common denominator by each of the denominators, multiply it by each of the numerators and add them together. These steps may be learned and followed without understanding the meaning of a fraction, or why the denominators need to be the same to add the numerators. These steps may be followed, and the arithmetic procedures carried out without understanding. However, children with or without understanding of fractions would need to know how to carry out arithmetic sums in order to add fractions. Please see this example illustrated in Figure 9.

**Figure 9**
*Example of Arithmetic Knowledge used to solve a fractions sum*



In the previous chapter, a group of items that aimed to assess children's knowledge of calculating with numbers and to follow procedural rules was examined, and it was concluded that it would be appropriate to use this selection of items as the Arithmetic Knowledge test for the current longitudinal study.

These are two distinct mathematical abilities and children need both to operate and understand fractions, and to learn more complex mathematical topics. However, the question remains as to which mathematical ability has a more important role for children's success in learning fractions.

*Assessing children's fractions knowledge 12 months later*

Children's knowledge of fractions can be assessed, as some tests have been developed previously. A widely used and recognised set of diagnostic tests on specific mathematical knowledge is the Chelsea Diagnostic Mathematics Tests (CMSM) (Hart et al., 1985); for this study the fractions test of this set was considered. It was reasoned that the outcome measure for the longitudinal and the intervention study should be the same, and the design should be independent from the current study. The Chelsea test was used in the UK with an older population (Hart et al., 1985) but then adapted and used in research with a younger group in the UK (Hallett et al., 2012), so it was considered appropriate for this study. This test was reviewed by two researchers who agreed that it was an appropriate test in relation to the curriculum that the children were taught in Year 5 in the UK. Although much has been investigated about fractions since the CSMS tests were developed, a large portion of the subsequent research focuses on children's misconceptions and on the natural number bias (Vamvakoussi, Van Dooren, and Verschaffel, 2012; Van Hoof et al., 2013; Van Hoof et al., 2017). These studies are extremely interesting, but they would not be appropriate measures of what children in the UK have learned after being taught about fractions. The studies suggest that some of the misconceptions can persist

well into secondary school (Van Hoof et al., 2013); this would mean that a floor effect was likely should such tasks be used with children in Year 5, who still have to complete one more year in primary school before moving to secondary school.

In this test, fractions knowledge is evaluated in three areas: children's reasoning about fractions, children's ability to carry out arithmetical procedures using fractions and children's knowledge of fractions labels. The first two areas have been described before. The third area, Naming fractions, connects a quantity with a number, but it may not require quantitative reasoning because children are taught procedures to apply fraction labels to parts of a whole. Such items assess the ability to write a fraction numerically when presented with a whole cut into a number of parts: for example, the child is presented with an image of a square that has been cut into four pieces and one square has been shaded; the child is expected to label the shaded piece as ¼. This type of item is considered under knowledge of naming fractions, because a child may learn how to do it without making a connection to the division of the quantity, but rather by counting the total number of parts and the number of shaded parts.

Based on previous research, it is expected that Quantitative Reasoning may also influence Arithmetical Knowledge, whereas arithmetic is only expected to support further arithmetical knowledge of fractions and not reasoning (Hecht et al., 2003). Naming fractions could be related to both.

***Considering other cognitive abilities***

This study aims to test the relative importance of two precursors to learning fractions, and it may be argued that general intelligence affects how well children do in mathematics. Thus, the child's general intelligence could explain why Arithmetic Knowledge and Quantitative Reasoning contribute to the results in a general attainment fractions test. A carefully designed longitudinal study should control for general

intelligence in order to test whether predictors, such as Quantitative Reasoning and Arithmetic Knowledge, can have a specific contribution to children's learning of fractions (Bradley and Bryant, 1983). With the purpose of testing if the contributions are specific, one should control statistically for other aspects of learning, such as general intelligence, and establish their importance and contribution to the learning of fractions.

It was reasoned that to assess general intelligence it was best to use a non-verbal ability test, because this kind of test does not rely on linguistic instructions or mathematical language. The Raven's Progressive Matrices test is a well-researched measure of non-verbal ability and is appropriate for the age range of primary school children (Raven et al., 1996). This study uses the Raven's Standard Progressive Matrices to assess children's non-verbal ability.

*This study*

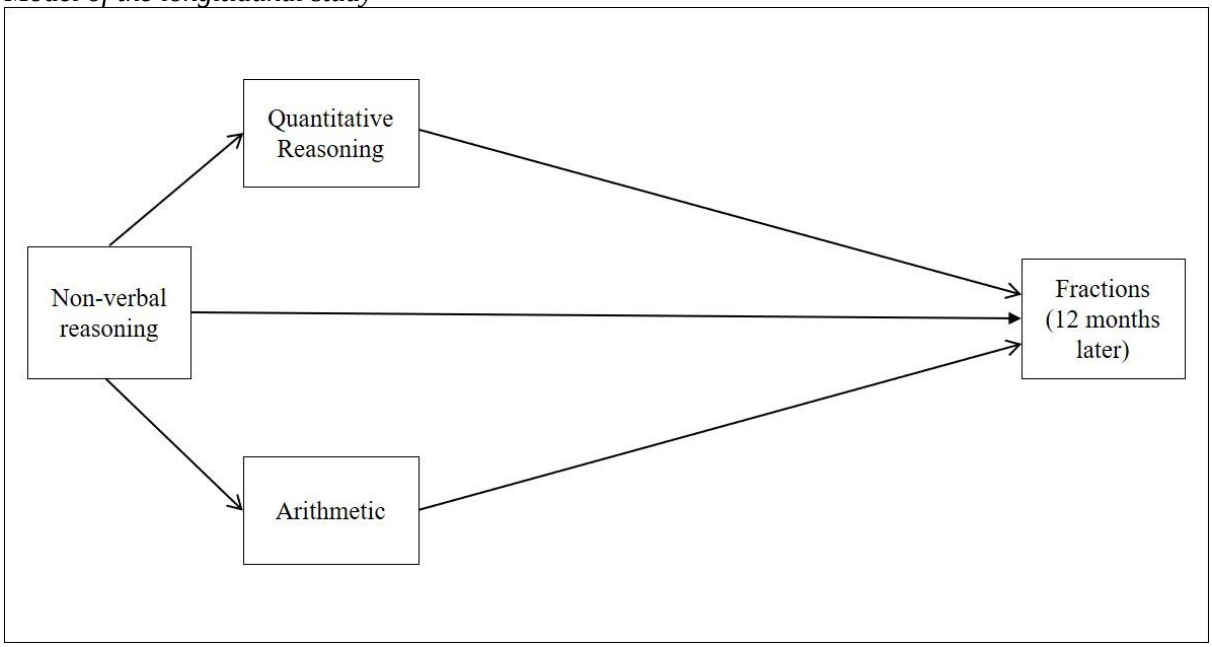
A longitudinal study with children in Year 4 was carried out to test if Quantitative Reasoning and Arithmetic Knowledge can make independent contributions to children's learning of fractions, after considering non-verbal ability.

Year 4 was chosen as the first data collection time, because testing children in Year 4 about their Quantitative Reasoning and Arithmetic Knowledge would provide information on their knowledge of these two mathematical abilities at a stage in their education before they received more systematic instruction on fractions in school according to the National Curriculum of England (Department for Education, 2014). This is important in a longitudinal study that investigates the origins of learning fractions, because extensive previous knowledge of fractions would prevent an analysis of precursors of this knowledge.

Results from the previous chapter gave support to the choice to use the Quantitative Reasoning test and the Arithmetic Knowledge test for this study. Children were also assessed on their non-verbal ability in order to take this into account and be able to attain a clear conclusion as to whether the contributions of these two mathematical abilities are independent of children's intelligence.

As shown in Figure 10, the model of this longitudinal study aims to test the contribution of Quantitative Reasoning and Arithmetic Knowledge, controlling for non-verbal ability, to predict children's learning of fractions as assessed by a selection of items from the Chelsea Diagnostics Mathematics Tests – Fractions (CSMS) (Hart, Brown, Kerslake, Kuchemann, and Ruddock, 1985) 12 months after the initial assessment of the predictors took place. Testing the children's knowledge of fractions 12 months later would allow time for children to learn about fractions in school, as expected by the National Curriculum of England.

**Figure 10**
*Model of the longitidunal study*



The quantitative analyses in this study will test the contributions of Quantitative Reasoning and Arithmetic Knowledge towards children's learning of fractions 12 months later, when they have been systematically taught about fractions in school.

Multiple regression analysis is commonly used by psychologists and educators to test predictions (Ferguson and Takane, 1989; Field, 2013). There are different kinds of multiple regressions, such as linear or logistic regressions. Linear regression is used when the outcome measure is continuous, whereas logistic regression is appropriate when the outcome measure has a restricted number of possible values, for example when the outcome is categorical (Field, 2013). Linear multiple regressions can be carried out through different methods, such as hierarchical or stepwise. The hierarchical method allows the inclusion of each predictor (variable) as a new step in the regression as a fixed-order and an explanation of the amount of variance that the predictor contributes to the dependent variable, while the previous steps act as control variables. This kind of method allows researchers to measure the variance explained by each step, thus each predictor. The first predictor included in the equation will explain all the variance it has in common with the subsequent predictors. As a result, it usually takes most of the variance, whereas the last predictor will only explain variance beyond what was already explained. As the fractions test is a continuous variable, it was reasoned that hierarchical fixed-order multiple regression analyses would allow testing of the specific and independent contribution of each predictor.

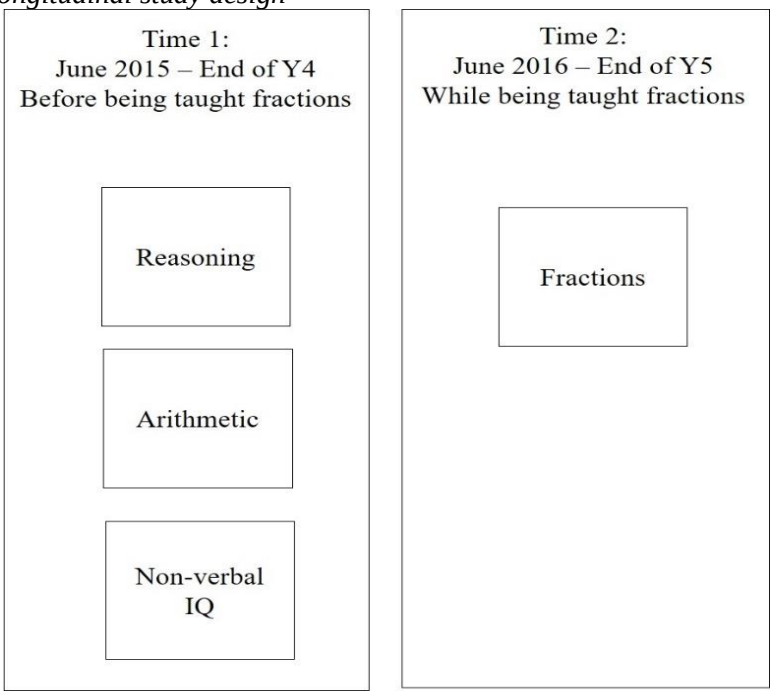
The results of this study will have theoretical and practical implications. It will provide a better understanding of the precursors needed for learning fractions. As time is a precious resource in schools, these results could bring insight to help schools invest in the most crucial factor that will lead to learning fractions in primary school: they will show

new paths that schools can use to prepare children for learning fractions before the topic is introduced.

## 4.1 Method

This step of the longitudinal study seeks to explain the predictive relationship between children's Quantitative Reasoning and Arithmetic Knowledge and their fractions knowledge 12 months later. The study starts with a test in the school year before they are taught fractions and concludes with their score in a fractions test taken one year later; it started when children were in Year 4 and finished when children were in Year 5, as shown in Figure 11.

**Figure 11**
*Longitudinal study design*

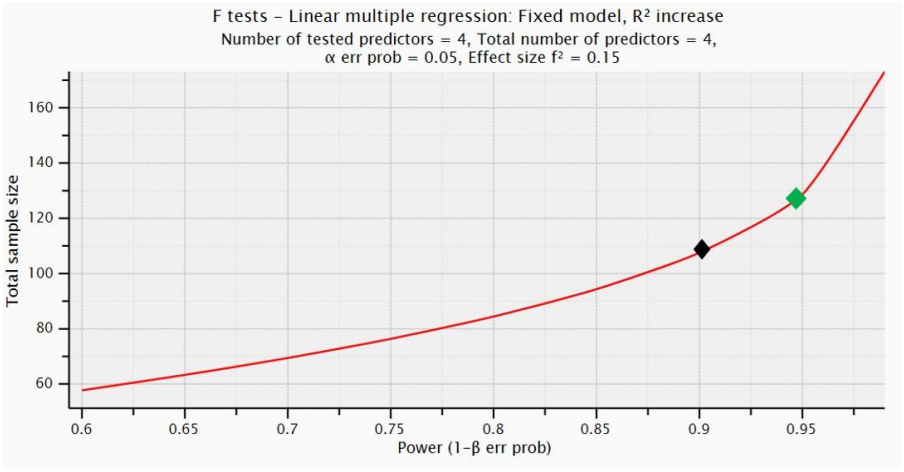


*Sample size*

Power calculations were carried out to estimate the sample size needed to be able to carry out the analyses proposed. The software G-Power 3 was used to calculate the sample size (Faul, Erdfelder, Buchner, and Lang, 2009) with a Fixed Model $R^2$ increase; Input Parameters: $f^2$= 0.15 (medium) (Cohen, 1988), Power of 0.90, 4 predictors (Age, Non-verbal ability, Quantitative Reasoning and Arithmetic), Output Parameters: Critical F 2.46, 108 participants. The aim was to recruit 120 children from five state primary schools in Oxfordshire. The target number of participants anticipated a possible loss of approximately 10% of the children by the end of the study, due to reasons such as the child leaving the school or absence on the day of testing.

Figure 12 shows the total sample required depending on the power aimed for the analysis. The black rhombus shown in the red curve indicates the sample size required for a power of .90; the green rhombus indicates the actual sample size in the study. This shows that the analyses will have enough power to be carried out.

**Figure 12**
*Power calculation curve of the longitudinal study*



### 4.1.1 Sample

A total of 124 children were recruited from six state primary schools in Oxfordshire. The participants were the same children who answered the Quantitative Reasoning and Arithmetic Knowledge tests described in Chapter 3, who were assessed on their fractions knowledge using the CSMS Fractions test 12 months later, at the end of Year 5. The following sections will describe the measures and the procedures used in the current study. All the schools that agreed to take part remained throughout the project. During the study 19 children were lost (15.3% of the initial number), leaving a total number of 105 participants. The children's attrition was observed in different schools, for example some children moved school and it was not possible to include them in the final testing point.

### 4.1.2 Measures

#### 4.1.2.1 Measures of the predictors

The two measures analysed in the previous chapter, Quantitative Reasoning and Arithmetic Knowledge tests, showed that both measures are independent of each other, present a sustainable factor structure and are independent from non-verbal ability. It was concluded that these measures were appropriate to use in the current study as predictors of fractions learning.

In the following sections a brief description of each measure is provided, and their internal consistency reported.

**The Quantitative Reasoning test** was designed by Nunes and Bryant (Nunes, Bryant, Barros and Sylva, 2012a). A selection of 16 questions divided into two sub-domains was used: (a) *Whole numbers*: Multiplicative relations between whole quantities

that require simple calculations (10 items) and (b) *Fractions*: reasoning about quantities smaller than a unit with no computations required (6 items).

The Quantitative Reasoning test showed good internal consistency, Cronbach's alpha = .83.

**The Arithmetic Knowledge test** is composed of 15 questions, divided into two sub-domains: (a) operating with whole numbers (9 items) and (b) operating with symbolic representation of numerical fractions (6 items). Items require different types of arithmetic knowledge and procedures: (1) multiplication items with carrying (e.g. 45x5) and without carrying (e.g. 13x3); (2) division items without a remainder (e.g. 84 divided by 4) and with a remainder (e.g. 28 divided by 13); (3) subtraction with borrowing (e.g. 72 – 43); (4) addition and subtraction of proper fractions with the same denominator (e.g. 3/8 + 2/8); (5) addition of proper fractions with common multiple denominators (e.g. ½ + ¼ ); and (6) one term a proper fraction and the other term a whole number (e.g. 1 – 2/5).

The measure showed a good level of Cronbach's alpha = .86.

#### 4.1.2.2 Control measure

***Non-verbal ability: Raven's Progressive Matrices*** aims to assess children's non-verbal ability. The child is asked to identify the missing element that completes a pattern from multiple-choice options. It is designed for the general population from 6 to 17+ years old and to be administered as a group using pen and paper (Raven et al., 1996).

#### 4.1.2.3 Outcome measure

***Fractions test*** A selection of items from the Chelsea Diagnostic Mathematics Tests (CMSM) Fractions test (Hart et al., 1985) was used as the basis of the fractions knowledge measure. The CMSM is a set of tests designed as diagnostic instruments to be used for

assessing a child's level of understanding of a mathematical topic. There are different tests available, such as Algebra, Place-Value and Fractions among others.

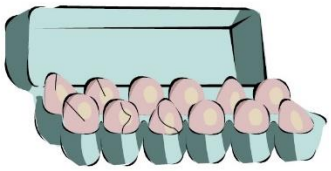
This study uses an adaptation of 37 items from the Fractions test for 11-16 year-olds. The tests Fractions 1 and Fractions 2 were adapted by Nunes and Bryant (Hallett et al., 2012) for presentation to children younger than 11 years old by adding drawings and excluding the items that were too difficult for the children in this study according to the level of difficulty described for 11-year-olds (Hart et al., 1985). For example, Figure 13 shows one original item and the adaptation. On the left the original item has a written question, whereas the adaptation on the right includes a drawing and the question is read to the children.

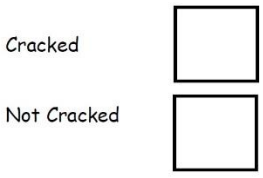
**Figure 13**
*Original items of CMSM test and Nunes and Bryant adaptation for younger children*

**Original**

6. 5 eggs in a box of 12 are found to be cracked.
(a) What fraction of the box of eggs is cracked? .....
(b) What fraction of the box of eggs is not cracked? .....

**Adaptation**



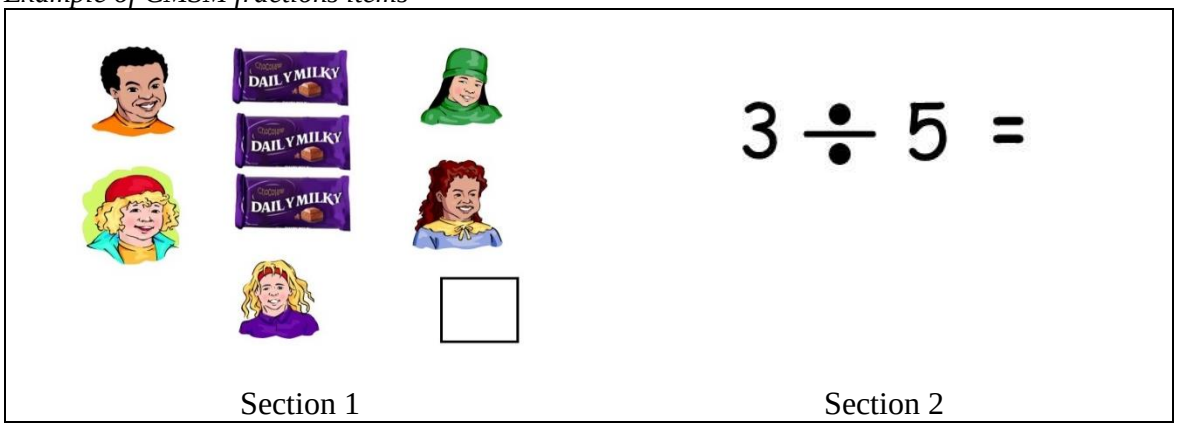



The researcher read the question: "5 eggs in a box of 12 are found to be cracked.
(a) What fraction of the box of eggs is cracked? Write the answer in the box.
(b) What fraction of the box of eggs is not cracked? Write the answer in the box."

Both papers, Fractions 1 and Fractions 2, are composed of two sections: the first section includes pictorial/word problems, followed by the second section that presents a series of fractional computations. The questions in section 1 require children to reason

about quantities in ways that are similar to the arithmetic questions in section 2. For example, in section 1 the question says, "Three bars of chocolate are to be shared equally between 5 children. What fraction of a chocolate bar does each child get? Please write your answer in the box." The parallel arithmetic item in section 2 is "3 ÷ 5=" and assesses whether the child will write 3/5. Figure 14 shows how the item in section 1 was adapted by adding a pictorial representation, which was used in the present study.

**Figure 14**
*Example of CMSM fractions items*




A total of 37 items were selected, of which 14 items were classified as arithmetic items, 9 items were classified as reasoning about fractions items and 14 were classified as naming fractions.

The complete Fractions test is included in Appendix 5.

## 4.2 Results

The aim of this study is to test the independent contribution of each predictor to children's knowledge of fractions after being taught about fractions during Year 5. In order to test this, it was decided to carry out a series of hierarchical fixed-order multiple regressions. This kind of analysis provides the variance of the last factor entered in the

equation, taking into account the contribution of all the predictors included already. If this model is carried out twice by swapping the last factor entered, Quantitative Reasoning in one model and Arithmetic Knowledge in a second model, this would show the independent contribution of each factor.

The results section is organised into two sub-sections. In the first section, the outcomes of the preliminary analyses, which tested whether the assumptions for the use of multiple regression techniques were met in this data set, are reported. In the second section, two models of hierarchical multiple regressions are used to test whether the measures of arithmetic and quantitative reasoning make independent contributions to the prediction of children's attainment in a measure of knowledge of fractions, after controlling for non-verbal ability.

### 4.2.1 Preliminary analyses

The aim of the preliminary analyses was to describe the data measured by the predictors and the outcome measure. These analyses are also important to verify the compliance of the data with assumptions to carry out the proposed statistical analysis, for example multiple regression techniques. This section is divided into two sub-sections: first the descriptive statistics are presented, and then the assumptions for multiple regression are tested.

*Reliability of the outcome measure*

The fractions measure showed a good level of internal consistency, Cronbach's alpha with 37 items = .92.

*Descriptive statistics*

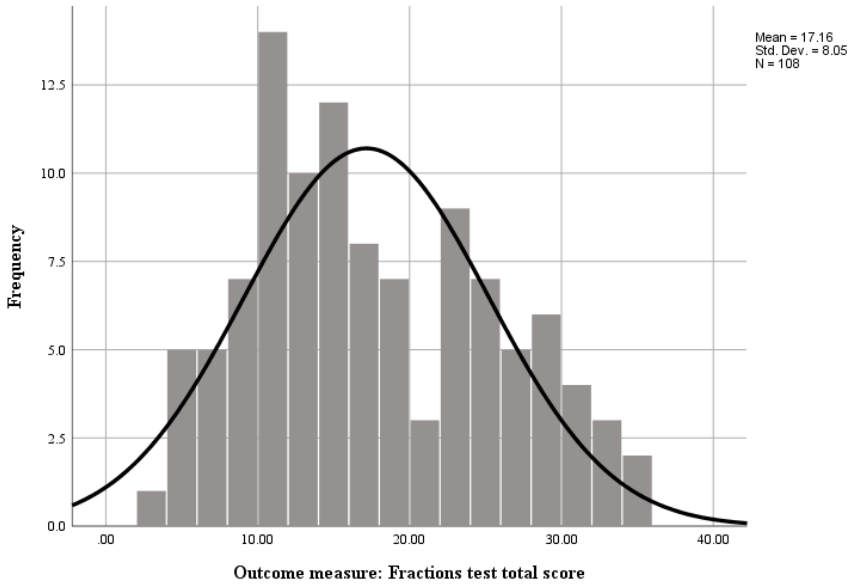
In this section, data were summarised to observe some patterns, and the distribution of the outcome measure is presented.

All items of the outcome measure were scored 0 for incorrect and 1 for correct. The scores of each of the items in each measure were added as a linear composite to provide the total score. For example, if a child had 7 items correct of the 37 items, the child would have been assigned a total score of 7. The possible range of scores in the outcome measure was 0 to 37. Children in this study scored minimum 3 and maximum 34.

The mean score of the children in this study in the outcome measure was 17.16 (SD=8.05).

To assess the distribution of the data, Skewness assesses the symmetry of the data distribution and Kurtosis measures the tails of the data (Field, 2013). The z-score of Skewness and Kurtosis should fall within a range of ±2 for normal distributed data (Field, 2013). These two measures provide an assessment of the spread of the data. The outcome measure showed a Z of Skewness of 1.63 and Z of Kurtosis of -1.80.

**Figure 15**
*Distribution of outcome measures*



The level of difficulty of the items was revised. A proportion of correct responses close to 1 indicates that the item was easier than one where the mean is close to 0. The proportion of correct responses of all the items is presented in Table 8. All items from the outcome measure were included together.

The most difficult item required children to divide a whole number by an improper fraction; proportion correct = .009. This result is aligned with that of the predictor measure in which division of whole numbers also presented a high level of difficulty. The easiest item was to select the bigger fraction between two fractions, both with the same denominator; proportion correct = .972. This result could potentially indicate a misconception of fractions, as the children may have used their knowledge of whole numbers and applied it to this item.

**Table 8**
*Level of difficulty of each item of the outcome measure*

| Item | Proportion correct | Standard deviation |
|---|---|---|
| 40 divided by 10 ½ | 0.009 | 0.096 |
| I am putting tiles on the floor; they are shown shaded. What fraction of the floor has been tiled? (9/24) | 0.074 | 0.263 |
| 3 divided by 5 | 0.102 | 0.304 |
| Pieces of wood of 1 ¼ cm from a 8 3/4 cm piece | 0.167 | 0.374 |
| Anna also spends 1/10 on bus (2/5+1/10) | 0.167 | 0.374 |
| 3 chocolates shared by 5 friends | 0.194 | 0.398 |
| What fraction of whole disc was shaded | 0.213 | 0.411 |
| 2 x 1/8 | 0.222 | 0.418 |
| Mary spent 1/4 and Paul ½ but we don't know how much they had. Is it possible that Paul spent more money than Mary? | 0.231 | 0.424 |
| 3 x 2 ½ | 0.250 | 0.435 |
| 17cm ribbon divided by 4 (select answer from multiple choice) | 0.269 | 0.445 |
| 2/3 of 12 | 0.287 | 0.454 |
| 1- 2/5 | 0.306 | 0.463 |
| Shade 2/3 of shape (Shape divided in 6 pieces) | 0.333 | 0.474 |
| A relay race is run in stages of 1/8 km each. Each runner runs one stage. How many runners would be required to run a total distance of ¾ km? | 0.352 | 0.480 |
| 1/10 + 2/5 | 0.361 | 0.483 |
| Shade 2/3 of shape (Shape divided in 6 pieces) | 0.380 | 0.488 |
| 2/3 of 12 marbles | 0.389 | 0.490 |
| Shade in 1/6 of the dotted part | 0.407 | 0.494 |

| Item | Proportion correct | Standard deviation |
|---|---|---|
| Select most shaded (3/12 or 3/10 or 3/9) | 0.435 | 0.498 |
| Order (1/4, 1/2,1/100, 1/3) | 0.491 | 0.502 |
| Anna spends 2/5 of her pocket money (1-2/5). How much has she left? | 0.500 | 0.502 |
| Select bigger fraction (3/5 and 3/4) | 0.509 | 0.502 |
| This pizza has to be shared equally between 5 girls. What fraction of the pizza does each girl get? | 0.556 | 0.499 |
| Fill in the missing number (6/8=3/?) | 0.583 | 0.495 |
| Fill in the missing number (5/10=?/30) | 0.611 | 0.490 |
| Bakers used 3/8 of their flour on bread and 2/8 of their flour making cake. How much flour did they use in total? | 0.657 | 0.477 |
| Fill in the missing number (1/3 = 2/?) | 0.676 | 0.470 |
| What fraction is shaded (7/16) | 0.694 | 0.463 |
| What fraction is shaded (3/8) | 0.750 | 0.435 |
| Write fractions of not cracked eggs (7/12) | 0.769 | 0.424 |
| Select bigger fraction (1/5 and 3/10) | 0.787 | 0.411 |
| Write fraction of cracked eggs (5/12) | 0.787 | 0.411 |
| Shade 2/3 of shape (Shape divided in 3 pieces) | 0.843 | 0.366 |
| What fraction is shaded (4/8) | 0.880 | 0.327 |
| Select shape with half shaded | 0.944 | 0.230 |
| Select bigger fraction (1/7 and 5/7) | 0.972 | 0.165 |

N=108

*Testing the assumptions for multiple regression*

In order to carry out multiple regression analysis, the data must comply with some assumptions (Field, 2013). For example, that the residuals are normally distributed and uncorrelated to each other (homoscedasticity), that the independent variables are not highly correlated with each other (absence of multicollinearity) and that the variance of error terms are similar across the values of the independent variables.

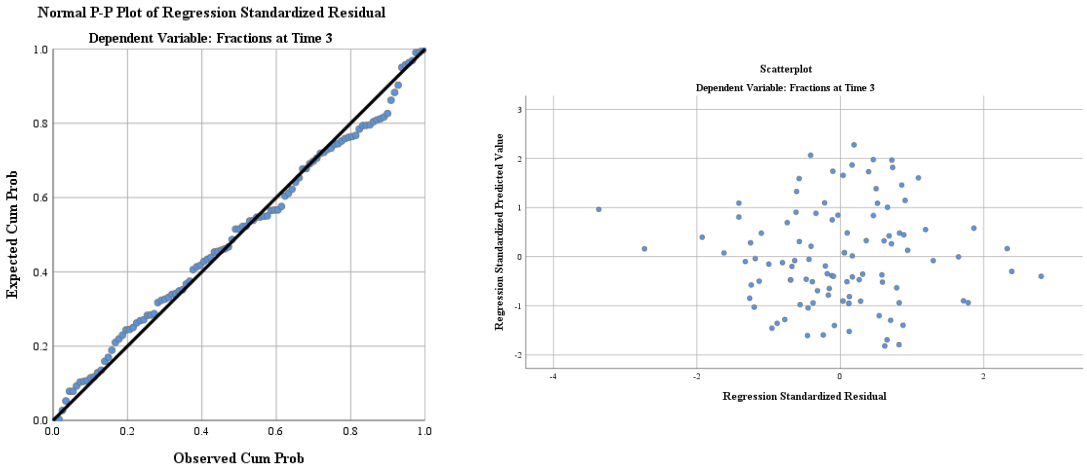
Normality and homoscedasticity assumptions

In order to carry out multiple regressions, one must test if the normality and homoscedasticity assumptions are met. This means that the residuals of the regression should present a normal distribution and that the residuals are uncorrelated to each other; these residuals are the differences between the observed value of the dependent variable and the predicted value. To test the assumption of normality of the data, one can use

diagnostic plots, such as the predicted probability plot and a scatterplot of the residuals. If the predicted probability plot shows a diagonal line, it indicates normality. If the scatterplot of the residuals show that these are spread across without showing a pattern or a tendency to group together, this indicates that the residuals are not correlated. The plots and scatterplots will be carried out for the multiple regression: Outcome measure Fractions, Predictors quantitative reasoning and arithmetic knowledge at time 1.

Figure 16 shows two columns: the first column presents the predicted probability plot of the observed and predicted values of the outcome measure of the model, and the second column presents a scatterplot showing the residuals and the fitted values. All the predicted probability plots showed a diagonal line indicating normality, and all the scatterplots demonstrated that the residuals and the fitted values are uncorrelated presenting no patterns or groups. Thus, the data comply with the normality and homoscedasticity assumptions to use the outcome measures in the statistical methods proposed.

**Figure 16**
*Diagnostic plots for normality of the data*



Assumption of absence of multicollinearity

Multicollinearity refers to the relation between the predictors, and if the predictors are highly correlated to each other it would mean that the predictors are multicollinear. In order to test the absence of multicollinearity between the predictors, one can use correlations analyses. With the predictor measures as revised in Chapter 3, although they are correlated, all correlations are below .80, which would indicate there is no multicollinearity.

The results from the analyses carried out to test the assumptions to use multiple regression techniques show that it is appropriate to use the statistical analyses proposed in this study with the data collected.

### 4.2.2 Multiple regressions

Hierarchical fixed-order multiple regression analyses will test whether each of the measures has a specific relation to the outcome measure, after controlling for the other predictors. The strength of the relations will be assessed comparing the standardised coefficient (β) of each factor and the $R^2$ change. Each of these provides different information about the relation between the predictors and the outcome measure. The standardised coefficient provides the strength of the effect of the predictor on the outcome measure, and allows comparison of the relative importance of each coefficient in a regression model (Field, 2013). The $R^2$ change provides information about how much variability in the outcome measure is accounted for by the predictors, and assesses the additional variance that each predictor accounts for in the outcome measure (Field, 2013). Together the standardised coefficient and the $R^2$ change compare the strength of the effect

of each predictor to the outcome measure and the variance contributed by the predictor to the outcome measure.

The following section presents the results of two multiple regressions to test the specific contribution of Quantitative Reasoning and Arithmetic Knowledge to the outcome measure. Based on a previous study (Nunes et al., 2012a), it was expected that Quantitative Reasoning scores would be the strongest predictor of fractions competence (after controlling for age, non-verbal ability and Arithmetic Knowledge). The longitudinal regression analyses will test relations over time, providing much stronger evidence that Quantitative Reasoning can predict fractions competence.

### *Model: Outcome measure Fractions (at end of Year 5), predictors Quantitative Reasoning and Arithmetic Knowledge (at end of Year 4)*

The quantitative analyses test the relative importance of each of the two predictors, Quantitative Reasoning and Arithmetic Knowledge, to predict fractions competence after 12 months. The predictors were measured when children had not yet been taught formally about fractions, and thus these results will show which predictor is more important to prepare children for fractions learning.

Two multiple regressions were carried out and are presented in the following sections: the first section presents the specific contribution of Arithmetic Knowledge, while the second section presents the specific contribution of Quantitative Reasoning.

*The specific contribution of Arithmetic Knowledge*

The results presented in Table 9 show that Arithmetic Knowledge accounted for a significant amount of additional variance: after controlling for age and non-verbal ability in the analysis ($R^2$=.74, F (1,100) =24.66, p<.01), it accounted for 6%. The standard

coefficient of Arithmetic Knowledge was significant and higher than that of non-verbal ability (β=.33, p<.01).

**Table 9**
*Multiple regression – Last step Arithmetic at Time 1 Outcome measure: Fractions*

| Regression | Steps | Variables | $R^2$ | $R^2$ change | β coefficient | B | Standard error of B |
|---|---|---|---|---|---|---|---|
| Controls | 1 | Age | .00 | .00 | -.09 | -.19 | .11 |
| | 2 | Non-verbal ability | .39 | .39* | .23* | .22 | .06 |
| Predictors | 3 | Quantitative Reasoning T1 | .69 | .29* | .46* | 3.80 | .59 |
| | 4 | Arithmetic Knowledge T1 | .74 | .06* | .33* | 2.79 | .56 |

* *p<0.001,* N=105

*The specific contribution of Quantitative Reasoning*

The second multiple regression carried out in the model aims to test the specific contribution of Quantitative Reasoning before children have received formal instruction about fractions in school, towards their learning of fractions one year later.

Table 10 shows the order in which the variables were entered into the model, the $R^2$ change of each variable and the standardised coefficient. These two will be examined for each predictor and compared to assess the relative importance and contribution of each predictor.

**Table 10**
*Multiple regression - Last step Reasoning at Time 1 Outcome measure: Fractions*

| Regression | Steps | Variables | $R^2$ | $R^2$ change | β coefficient | B | Standard error of B |
|---|---|---|---|---|---|---|---|
| Controls | 1 | Age | .00 | .00 | -.09 | -.19 | .11 |
| | 2 | Non-verbal ability | .39 | .39* | .23* | .22 | .06 |
| Predictors | 3 | Arithmetic Knowledge T1 | .65 | .25* | .33* | 2.79 | .56 |
| | 4 | Quantitative Reasoning T1 | .75 | .10* | .46* | 3.80 | .59 |

* *p<0.001,* N=105

Quantitative Reasoning explained a significant amount of variance of the outcome measure, even after controlling for age, non-verbal ability and Arithmetic Knowledge

($R^2$=.75, F (1,100) =41.46, p<.01). The standard coefficient also showed that the effect of Quantitative Reasoning on the outcome measure fractions at time 3 was the strongest (β=.46, p<.01).

Results presented in the tables given in this section (Table 9 andTable 10) show that each of the two predictors accounted for a significant amount of additional variance, after controlling for age and non-verbal ability in the analysis. Thus, each predictor made an independent contribution to predicting children's knowledge of fractions over the 12 months. It is noteworthy that the standardised coefficient value of quantitative reasoning was higher than all the other predictors, including the measure for non-verbal ability, in the prediction of children's knowledge of fractions (β coefficient=.46). Thus, these results provide empirical evidence that children's understanding of Quantitative Reasoning before they are taught about fractions in school is the most important predictor amongst those included in the analysis of their fractions learning. It should also be noted that the three predictors, non-verbal ability, Arithmetic Knowledge and Quantitative Reasoning, accounted for 75% of the variance in the CSMS Fractions test given 12 months later, which makes this a very important result from an educational perspective.

## 4.3 Conclusions

The aim of the longitudinal study was to determine whether two different mathematical abilities, Quantitative Reasoning and Arithmetic Knowledge, make significant and independent contributions to children's later knowledge of fractions after they are taught in school. The study showed that both Quantitative Reasoning and Arithmetic Knowledge play a crucial role in preparing children for learning about fractions in school. Each mathematical ability made an independent and significant contribution to the prediction of scores in fractions knowledge. Nevertheless, the importance of each

ability to predict their learning of fractions was not the same, and Quantitative Reasoning was a stronger predictor.

# Chapter 5 Teaching and Learning Fractions: An Intervention Study

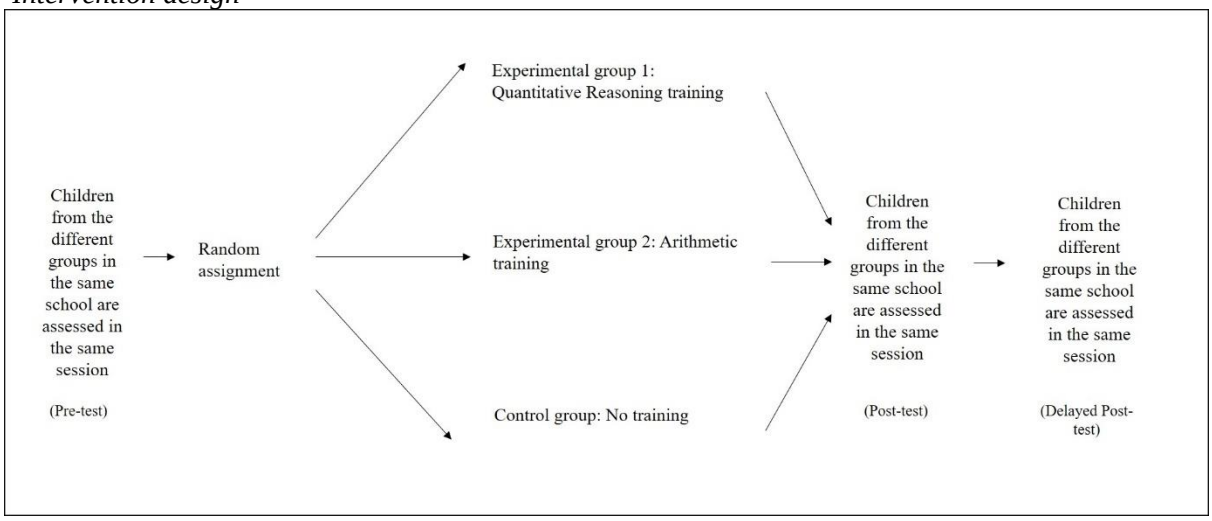
In the previous chapters it was established that Quantitative Reasoning and Arithmetic Knowledge are distinct abilities and that they make independent contributions to predicting children's later performance in a standardised test of knowledge of fractions. Quantitative Reasoning was found to be a stronger predictor than Arithmetic Knowledge when both measures were given before the children were taught formally about arithmetical procedures to solve computations with fractions. These findings confirm the importance of the research questions: can these mathematical abilities have an effect on children's learning of fractions, or was there another underlying factor not measured in the longitudinal study influencing children's knowledge of fractions besides the two abilities used in the longitudinal study as predictors? If there is a connection and these abilities could act as a foundation for learning fractions, it would be important to determine whether we can teach these mathematical abilities to support fractions learning; and if so, which would have a stronger impact on children's fractions knowledge.

This chapter presents the third step of the overall research programme. The aim of this study was to investigate the possible impact of Quantitative Reasoning and Arithmetic Knowledge on children's learning about fractions by means of an intervention study, and to test the possible role of non-verbal ability as a moderator of the intervention effect. In the study reported in this chapter, it was tested whether training in Quantitative Reasoning or training in Arithmetic Knowledge would have a greater impact on children's attainment in a measure of fractions knowledge when compared to a baseline group that received no training at all.

*Testing if these mathematical abilities can act as foundations for children's learning of fractions*

In order to investigate the origins of learning fractions, one must provide rigorous evidence that something specifically is causing the effect. Intervention studies provide such a rigorous tool for testing causal hypotheses. Intervention studies that include a group that receives the intervention and a group that does not receive the intervention are also called Randomised Control Trials (RCTs). The schema for the current study is depicted in Figure 17.

**Figure 17**
*Intervention design*




RCTs are designed to establish causal conclusions; the design itself ensures that a positive result in the experiment confers a high confidence level on the causal conclusion (Cartwright, 2009). The design represented in Figure 17 includes a pre-test before randomisation in order to avoid any possible bias in testing that could happen if this had taken place after randomization and two post-tests, an immediate and a delayed post-test.

*Controlling for previous knowledge*

One may argue that the intervention only worked because children knew or didn't know about fractions before they received the intervention. In order to make the comparison considering how much children knew before they started receiving the intervention, one must test all children on their fractions knowledge, even if they are not familiar with fractions. If all children are tested again after receiving the intervention, then it can be assumed that the change in their knowledge from before they started to after they finished the intervention must be due to the intervention, which is the only thing that the groups have done differently.

*Considering confounding effects*

Conventional RCTs suggest that the outcome measure in the study should be independent from the person developing and conducting the training programme. This caution allows for avoiding a main bias due to the test and the intervention being designed by the same person or group, which is known as "teaching to the test". This study uses items from the CSMS Fractions Test, developed by Hart et al. (1985), and thus were designed independently from what the intervention designer used.

Intervention studies provide strong support for a foundation link, but one could still argue that children may be learning about fractions at the time in school, and that would have an impact on what they learn in the intervention. One could attribute to the intervention an effect which is actually due to the school or teacher. However, if each group of children taught by the same teacher is randomly divided into three groups, one that receives the Quantitative Reasoning training, one that receives the Arithmetic Knowledge training, and one that only participates in the tests as a control group, this design would justify the claim that the only difference between these groups would be the

input from the intervention, as they are all receiving the same teaching during a school day. Thus, it is possible to conclude that if there is any difference in the children's knowledge of fractions between the three groups at post-test, it must be due to the intervention.

*Children's cognitive abilities*

One possible source of confounding is the children's cognitive abilities, i.e. what they bring to the learning sessions to begin with. One could argue that learning might be due to the children's abilities, which could act as a moderator of their learning. A moderator has the potential to alter the strength of the relationship (Baron and Kenny, 1986).

> There is, scattered in the literature, a remarkable amount of evidence of significant, predictable differences in the way people learn…. In general, unless one treatment is clearly best for everyone, treatments should be differentiated in such a way as to maximize their interactions with aptitude variables. (Cronbach, 1957, p. 681)

This study will test whether there is a moderation effect of non-verbal ability on treatment outcomes by analysing whether children with high non-verbal ability progress better than those with low non-verbal ability within a specific intervention group.

The choice of a measure to be used as a control and moderator needs careful consideration. Measures of verbal intelligence have been considered biased by psychologists for almost a century now (see Philpott, 1936, for example), and this criticism led to the development of non-verbal measures, which focus more on the person's ability to identify patterns in relations between figures. The best known and most extensively used and validated measure (Jastrzębski, Ciechanowska, and Chuderski, 2018) of non-

verbal cognitive ability is the Raven's Matrices test, which was chosen for this study. Non-verbal ability is the capacity to analyse information and identify patterns without the use of words in problem presentation and in the answers. In mathematics this is important as a child may be able to solve and reason about mathematical problems irrespective of what they could explain using words.

*Controlling for bias at post-test*

Conventional and rigorous RCTs require the tester to be blinded to the group allocation of the children (i.e. not to know the allocation) in order to avoid any risk of helping children during the test depending on their group allocation. A study in which the researcher who tested the children cannot be blinded to the children's allocation at post-test would suffer a risk of biased testing implementation. The alternative approach used in this study to avoid any potential bias was to test all children from the three groups in the same school at the same time. In this way all children would receive the same instructions, and any clarifications in response to questions would be given to all children.

Thus, in this study children were tested in groups according to their school class, and randomisation was implemented at the child level within each class. This circumvented two potential biases: (1) If the tester presented any bias during the testing, it would be delivered equally to all three groups. (2) Confounding effects of school and class teacher would be controlled as randomisation was within each class, with the allocation of the same number of children in each class into the three different groups.

*How can we teach these mathematical abilities to support fractions learning?*

Learning and teaching fractions in primary schools has been a challenge for children and teachers. Many research studies about fractions learning have been carried out, but despite the interest in fractions there is a dearth of intervention studies that test the

origins of what is crucial for children's success. Confrey (Confrey, 2008; Confrey and Maloney, 2010), for example, suggested that equipartitioning of a single whole is a crucial concept for children to learn fractions; whereas Streefland (1997) proposed that the basic concept is ratio between two different quantities. Confrey's argument is based on arithmetic: children must realise that the sum of the parts is equal to the whole (i.e. ½ + ½ = 1 just as ¼ + ¼ + ¼ + ¼ = 1 and so on). Streefland's argument is based on quantitative reasoning: if there are more people sharing the same total quantity, each one will receive less; if the ratio between the quantity being shared and number of sharers is the same in two groups, then recipients in both groups receive the same amount, even if the number of recipients and the number of items being shared is different. This study will test whether Quantitative Reasoning and Arithmetic Knowledge could act as a foundation for fractions learning. It is possible that both Arithmetic Knowledge and Quantitative Reasoning contribute to achievement in learning about fractions, but that the levels of their contributions differ.

It was predicted that both teaching programmes would have an impact on children's learning of fractions. However, as previous results showed that Quantitative Reasoning is a stronger predictor of children's learning of fractions, it was expected that the Quantitative Reasoning programme would have a stronger impact on children's learning of fractions.

## 5.1 Method

A three-group intervention study was carried out; the two intervention groups were compared with the control group, which served as a baseline. Two intervention groups received weekly sessions, either on Quantitative Reasoning or on Arithmetic Knowledge,

whereas the children in the control group only participated in the assessments. All children from the three groups were assessed together at three points: before the study started, immediately after the intervention sessions finished and several weeks after the intervention finished. At the first testing point children were also assessed on non-verbal ability.

The following sections describe the participants' characteristics, the sampling procedures, sample size calculations, randomisation and the measures used in this study.

### 5.1.1 Participants' characteristics

The selection of academic year is crucial in the intervention study. In order for the teachers to agree to take part in an intervention study that involves eight sessions, it is necessary that they think of fractions as an important topic in mathematics and that they think that children at that age level can learn about fractions. It was also important that the children had not yet received much teaching about fractions in school, otherwise the teachers would expect them to know what children can learn about fractions without the input of an intervention. Year 4 children were chosen for the intervention study after considering what they had been taught and what they had not yet been taught. According to the English National Curriculum (Department for Education, 2014), Year 4 children are expected to receive limited instruction about fractions in school. They are only expected to be taught about fractions labels (e.g. that half is represented by ½) and a restricted number of fraction equivalences (namely that ½ is equal to two ¼). In Year 4 children are expected to learn to calculate multiplication and division with whole numbers, which are the operations relevant for understanding fractions.

### 5.1.2 Sampling procedures

The same schools who participated in the longitudinal study were invited to participate in the intervention study. When the children who participated in the longitudinal study were in Year 5, children in Year 4 of the same school were invited to participate in the intervention study. So, the children who participated in the longitudinal study did not participate in the intervention study, but the schools were the same and thus the recruitment areas for the sample were the same.

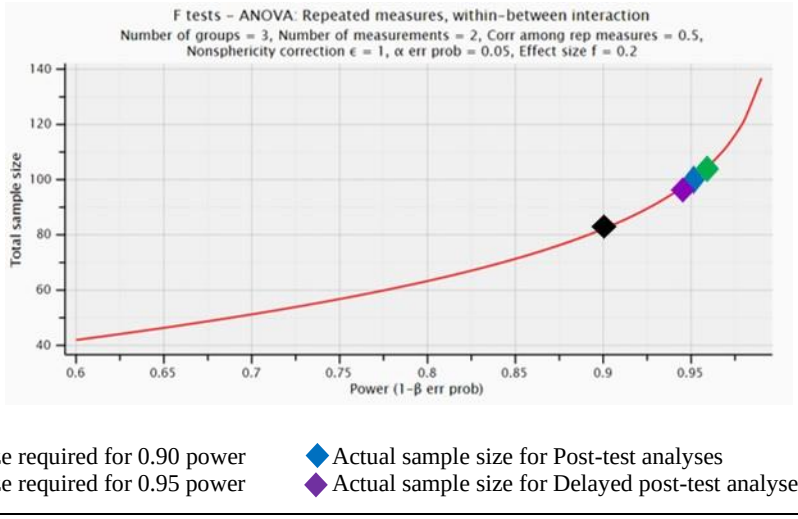
The recruitment process started by sending an initial letter inviting schools to take part in the study: six schools were invited to participate; four agreed to participate, but one school could not take part due to staff changes. One extra school which was not in the longitudinal study was invited and agreed to participate. Thus, four schools remained in the study. Letters to guardians or parents of the children in Year 4 seeking their agreement for the children to take part in the study were sent via the teachers. Opt-in consent was used as this method is of the highest ethical standard for carrying out research with young participants. A total of 190 parent letters were sent to schools, and school staff sent the letters to the children's parents or carers, but the number of letters sent to homes is unknown; ultimately 108 parents and children agreed to take part. Children were asked for their assent when the study started, and were asked at the start of each session if they wished to continue in the study. The recruitment and consent procedures were approved by the University of Oxford Research Ethics Committee (see Appendix 2).

### 5.1.3 Sample size

Power calculations were carried out to estimate the sample size needed. G-Power 3 was used to calculate the sample size (Faul, Erdfelder, Buchner, and Lang, 2009) with

ANOVA repeated measures within-between interaction; Input Parameters: $f^2$= 0.2 (Cohen, 1988), α error probability 0.05, Power of 0.95, Measures 2, Groups 3, Output Parameters: Critical F 3.09, 102 participants. The number of children recruited and pre-tested (108) and present at post-tests (103 children were present at the immediate post-test) was higher than the number required for power of 0.90, but at the delayed post-test only 98 children were present, and thus the number was short by 4 participants. Each analysis will include all the children who were present at the times of testing included in the analysis. Figure 18 shows the power calculation curve, on which the actual and required sample sizes for different levels of power are marked.

**Figure 18**
*Power calculation curve of the intervention study*




### 5.1.4 Sample

A total of 108 children (and their parents) agreed to participation in the study. The children were from six Year 4 classes in four state-supported primary schools. Only 103 children participated in the post-test (59 girls and 44 boys) and 98 children took part at the delayed post-test (58 girls and 40 boys). At the time of pre-test, their mean age was 9 years

(range 8 years 1 month to 9 years 9 months). The mean ages of the 103 children present at post-test and of the 98 children present at the delayed post-test did not differ (N=103, mean=9.35, N=98, mean=9.35) from the ages at pre-test.

### 5.1.5 Randomisation

Prior to randomisation, children were assessed on their fractions knowledge and their non-verbal ability. After all children were assessed, randomisation was implemented using SPSS. Children within each class in each participating school were randomly allocated to one of the three groups, of which the teachers were not informed. The children in the two intervention groups were called every session for the intervention. Thus, the teacher did know which children fell in the control group, who did not receive any kind of intervention.

Random numbers were generated for all children using SPSS. Children were ordered by these random numbers in ascending order within their class. The children allocated the highest random numbers in each class were allocated to the Quantitative Reasoning group until one-third of the children had an allocation; those in the next third were allocated to the Arithmetic Knowledge group; and the last third with the lowest random numbers were allocated to the control group. This procedure resulted in three equal groups of 36 children.

*The SPSS syntax used for this randomisation process was:*
***Syntax**:*
*COMPUTE random=RV.UNIFORM(1,2).*
*EXECUTE.*
*SORT CASES BY class(A) random(A).*

After randomisation, the data had to be cleaned; it was noticed that 4 children had not been able to finish the non-verbal ability test. This test should not be given twice in a

short period of time, so it was reasoned that these children should not be included in the final data set. Table 11 presents the number of children by group allocation included in the analyses. The three groups were indeed equal at pre-test (see Table 12).

**Table 11**
*Number of children present at each testing point*

| | Pre-test | Post-test | Delayed Post-test |
|---|---|---|---|
| Reasoning | 36 | 36 | 35 |
| Arithmetic | 32 | 32 | 31 |
| Control | 35 | 35 | 32 |
| Total | 103 | 103 | 98 |

**Table 12**
*Means, standard deviation and one-way analysis of variance at pre-test*

| Measure | Control | | Arithmetic Knowledge | | Quantitative Reasoning | | *F* | *p* |
|---|---|---|---|---|---|---|---|---|
| | *M* | *SD* | *M* | *SD* | *M* | *SD* | (2,101) | |
| Age at pre-test | 9.01 | 0.40 | 9.04 | 0.34 | 8.96 | 0.33 | 0.39 | 0.67 |
| Non-verbal ability | 30.83 | 7.93 | 28.82 | 8.25 | 29.42 | 6.60 | 0.63 | 0.53 |
| Pre-test | 17.78 | 7.77 | 17.91 | 7.50 | 17.83 | 6.59 | 0.02 | 0.98 |

### 5.1.5 Measures

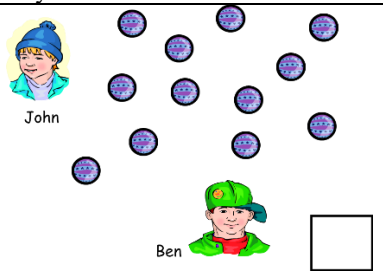
***Non-verbal ability: Raven's Progressive Matrices.*** This measure aims to assess children's non-verbal ability. The child is asked to identify, amongst four to six options, the missing element that completes a visually presented pattern. The measure was designed for the general population from 6 to 17+ years old to be administered as a group using pen and paper (Raven, Court, and Raven, 1996).

***Fractions knowledge: The Chelsea Diagnostic Mathematics Tests (CSMS) Fractions test*** (Hart et al., 1985). This measure was adapted for younger children by Nunes et al. (2007). It was selected as the fractions knowledge measure for this study and was administered at pre-test and at both the immediate and the delayed post-test. Some items designed by Nunes and Bryant were added to ensure that items related to all the situations as classified by Nunes and Bryant were included.

The CSMS is a set of tests designed as diagnostic instruments to be used for assessing pupils' level of attainment in fractions. There are different tests available: for example, Algebra, Place-value, Fractions among others. This study uses an adaptation by Nunes and Bryant of 49 items from the Fractions test 1 and 2 (Nunes, 2008; Nunes et al., 2007). The items were adapted to be used with younger children by including drawings to illustrate all the items that made references to a situation. For example, as shown in Figure 19, the drawing of a pizza and children were included when a question referred to sharing a pizza between five children. In another question a drawing of children was included, but the original CSMS item only included the drawing of marbles.

**Figure 19**

*CSMS questions and example of adaptation of questions by Nunes and Bryant*

| *CSMS question* | *Adaptation by Nunes and Bryant* |
|---|---|
| 1. This is a stick of rock. It is to be shared equally between 5 children. What fraction of the rock does each child get? .....  |  Question read by researcher: "This pizza has to be shared equally between 5 girls. What fraction of the pizza does each girl get? Write your answer in the box." |
| 2. (a) John wins $\frac{1}{3}$ of these marbles. Draw a ring round his marbles. (b) Jane wins $\frac{2}{3}$ of the marbles. How many marbles does she win? .....  |   Question read by researcher: "John wins 1/3 of these marbles. Draw a ring around the marbles. Ben wins 2/3 of the marbles. How many marbles does Ben get? Write your answer in the box." |

In the original CSMS test, there is no separation of items as measuring different aspects of the children's knowledge of fractions. It was reasoned that a balance of items that assessed reasoning and arithmetic was desirable, so that the measure could not be biased towards either aspect of children's knowledge of fractions. In order to obtain a balance between items that assessed the children's reasoning about relations between quantities and those that assessed their arithmetic knowledge, two judges independently classified the CSMS items into categories. It was originally expected that this classification would be into two types of item, reasoning or arithmetic; but a third category, naming fractions, was added because it was agreed by the judges that items that describe a quantity and require children to produce a numerical label are not clearly either about reasoning nor about arithmetic. Thus, three item types were used in the categorisation: (1) reasoning about quantities that are represented by fractions; (2) arithmetic knowledge of fractions; and (3) naming fractions.

**Reasoning about quantities represented by fractions**: Nunes and Bryant (2009b). This measure suggests that in order to understand how mathematical concepts develop, one needs to consider the concept as related to a set of situations that are functional and meaningful for children. It was noted that most of the items in the fractions test focused on part–whole situations; there was only one item related to quotient situations, and there were no items that referred to intensive quantities, which are measured by the ratio between two other quantities (for example, the concentration of a juice is measured by the ratio between the amounts of concentrate and water used to make the juice). In order to fill this gap, 4 items related to a quotient situation were added and 2 items related to intensive quantities. This section of the test had 16 items; 10 were from the original test.

**Arithmetic with fractions**: The arithmetic items from the original test included 7 items about comparing the magnitude of fractions and 8 items involving computations; one extra item was included to assess children's knowledge of subtraction of fractions, because there were items about addition but not about subtraction with fractions with the same denominator. This section of the test had 16 items; 15 were from the original test.

**Fraction labels**: These items present a quantity and require children to represent it with a fraction label; consequently their classification is neither as a reasoning nor as an arithmetic item. It was noted that there were no items related to intensive quantities. In order to include items with intensive quantities, 2 items were added. In total 15 items were all from the original test and 2 items added; all 17 items were treated as a separate group of items. The complete test is included in Appendix 5.

At pre-test, the children were given the 49 items described in the previous paragraphs and a non-verbal ability measure, the Raven's Progressive Matrices (Raven et al., 1996). The pre-test assessments were administered during the spring term (February 2016) before the intervention, which was carried out in the spring and summer terms. The post-test measure was administered in the summer term immediately after the intervention was completed (mid-May 2016), and the delayed post-test was administered at the end of the summer term (end June 2016).

The assessments were delivered to the whole class; the children answered first the fractions knowledge test and then the non-verbal ability test. The fractions knowledge test was presented in two parts: the first part was read by the researcher at the pace of the class; the second part was carried out independently by the children at the pace of each child. After all children had finished, the Raven's Matrices test was administered. The post-test and delayed post-test were delivered as a class and then scored by a marker unaware of the

children's group membership. The scores were entered to SPSS by the same unaware marker to avoid any bias.

### 5.1.6 The intervention

The aim of the overall design and method of delivery were common to both intervention groups. Children were taught in small groups of 6 children in 8 half-hour weekly intervention sessions, which consisted of either quantitative reasoning or arithmetic fractions exercises. Each session contained several exercises which were introduced as games to the children; the exercises differed according to the group's allocation. These sessions were in addition to their regular maths lessons. In order to avoid causing the children to miss the same lesson every week, the researcher visited the school on different days of the week and at different times, previously agreed with the class teacher.

The aim of the intervention was to promote explicit understanding either of quantitative reasoning about quantities that are represented by fractions or of arithmetical rules related to calculations with fractions. The intervention sessions were designed to help children improve their understanding of fractions. The quantitative reasoning training aimed to maximise their understanding of relations between quantities when they are divided and shared. In contrast, the arithmetic training aimed to maximise their understanding of arithmetical rules and fractional representations.

The procedures and instructions were provided by the researcher orally with the support of PowerPoint presentations and answer sheets designed especially for children in each of the training groups. The training programme materials were designed for 8 weekly sessions, with each session lasting for approximately 30 minutes. All groups covered all the material designed, but because different sets of children progressed at varying speeds,

the amount of time spent varied; four groups of children (two from the quantitative reasoning and two from the arithmetic group) finished covering the total amount of material the material in seven sessions.

For each session, the researcher took a small group out of their class and sat with the children around a table outside their classroom. During the sessions all children were always asked to write their answer on their answer sheet, and then the researcher asked individual children to answer orally, taking turns around the group. After an answer was provided, there was a discussion between the children and the researcher, following guidelines including probes to maintain standard procedures across different groups. Feedback was always provided, both by the PowerPoint presentation and by the researcher reinforcing it orally.

Children were given stickers for participating at the end of each session and were told it was a reward for their hard work. Children seemed motivated throughout the study and showed a good rapport with the researcher.

The teaching to both groups targeted the same main ideas, but one from a quantitative reasoning perspective and the other from an arithmetic perspective. The five main aims were:

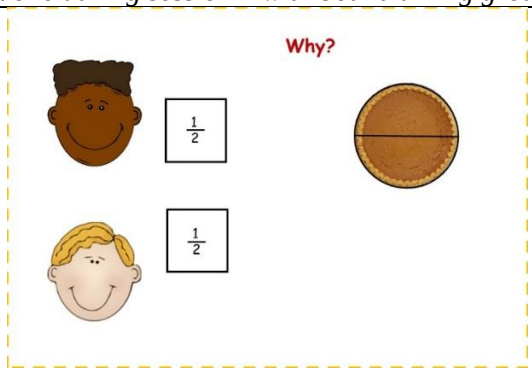
1. Fractions are numbers that represent quantities
2. A fraction can represent different things
3. Counting fractions as a precursor to adding
4. Ordering and the inverse relation between the divisors and the quotient
5. Equivalence, adding and subtracting fractions

The order of the sessions sought to represent a progression. It was reasoned that the children needed to learn how to represent different quantities using fractions in order to

give meaning to the numbers. Thus, in some sense both groups had to think initially about how to interpret the two numbers that are used jointly to produce a fractional representation. This means that in the first session, which involved understanding that fractions are numbers that represent quantities, the children in both training groups were shown the same slides to start with and discussed how to interpret a fraction (see Figure 20). Subsequently, the target concept for each session was the same for both groups, but one group was taught by referring to relations between quantities and the other group was taught the relevant arithmetical knowledge.

**Figure 20**

*An example of an exercise done during session 1 with both training groups*




Two boys are going to share one tart fairly between them. How much do you think each one gets? Quickly write your answer in the box.
Now can you explain why this is the right number to your friend?
Usual answer from children: "because you cut it in half", "because you chop it", "because each one gets half".
Researcher: "But why a half?"
After a bit of discussion, the researcher clarifies and reinforces by saying:
"It is 1 divided by 2. We cut the tart in two equal pieces and each one gets one piece, which is half of the tart."

The following sections describe material designed for each group.

## The two training programmes designed for the study

### *Quantitative Reasoning*

Materials were designed for a set of aims. This section describes the materials for each aim.

**Aim 1: Fractions are numbers that represent quantities; fractions are needed because quantities may be smaller than a unit.**

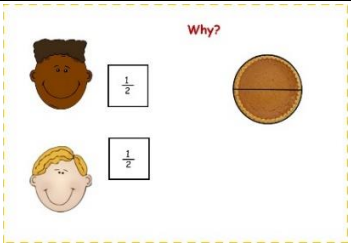
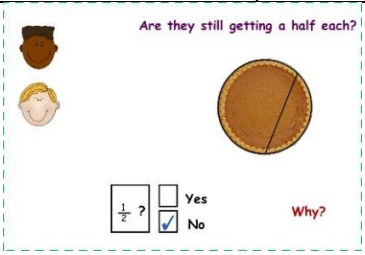
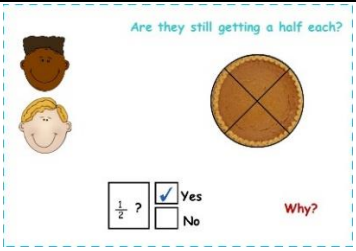
Materials were designed for children to understand that:

1. A fraction is a number that represents an amount of a quantity that cannot be represented by a whole number.
2. The amount must be divided into equal parts to be a fraction; a division into unequal parts is not represented by a fraction.
3. There is a relation between the two numbers, they are not interpreted separately: 1 divided by 2, 2 divided by 3 and so on. This division must be into equal parts.
4. This relation between division and quantity means that the more recipients there are, the smaller the share; or the more parts there are, the smaller each part is.

Figure 21 shows examples of some of the questions given to the children to discuss as a group and promote understanding of fractions.

**Figure 21**

*Example of exercises presented to children in session 1 of both training groups*

<table>
<tr>
<td>
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<tr>
<td></td>
<td>How about if they cut the tart like this? Are they still getting a half?<br>Quickly tick yes or no! (Most of them say No.)<br>Now can you explain why you think so to your friend?<br>(They usually say: because there is a big size and there is a little one…)<br>So, you ticked yes or no? Who can tell me why?<br>(They usually say: because there is a big size and there is a little one…)<br>So, if there is a big piece and small piece, would each one be a half? (They usually say no, and one of them says because the two parts have to be the same.)</td>
<td>How about if they cut the tart like this? Are they still getting a half?<br>Quickly tick yes or no! (Most of them say Yes.)<br>Now can you explain why that is to your friend?<br>Can someone tell me yes or no? Why?<br>(Most of them say: because they are equal and if this one gets this one – one boy to one quarter – and this one gets this one, they would be getting half.)<br>Reinforce, saying: That's right, because the parts are equal, and if one tart is being shared by two, each one gets half. In this case each</td>
</tr>
</table>

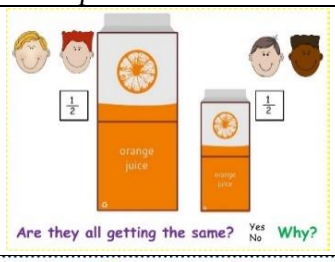
| | | |
|---|---|---|
| | Reinforce, saying: That's right, because the two pieces are not equal; to be half the amount, it has to be cut/divided into 2 equal parts. | one gets two pieces, so they would be getting half the tart each. |

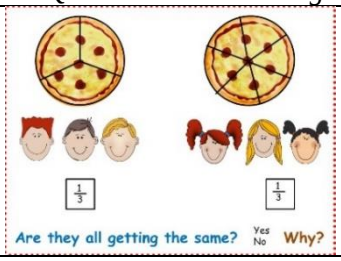
## Aim 2: A fraction can represent different things

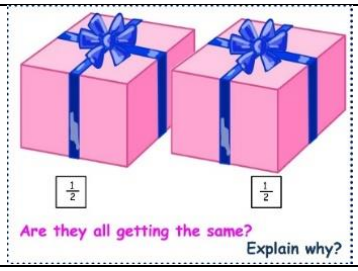
Materials were designed aiming to help children understand that the same fraction could represent different amounts, because it involves a ratio between two quantities. For example, half of 4 is not the same as half of 20. Figure 22 shows some examples of the questions that the children were asked with the aim of promoting their understanding of fractions. The examples use extensive quantities and unknown quantities. The questions with an unknown amount should prompt the children to think about the fact that, even if they receive half of something, that doesn't mean that they are receiving the same amount (half of a big cake or half of a very small cake are not the same amount).

**Figure 22**

*Examples of exercises presented to children in session 2 Quantitative Reasoning training*






After the discussion, the boxes were removed showing the children what was under each of the boxes.

## Aim 3: Inverse relation between the divisor and the quotient

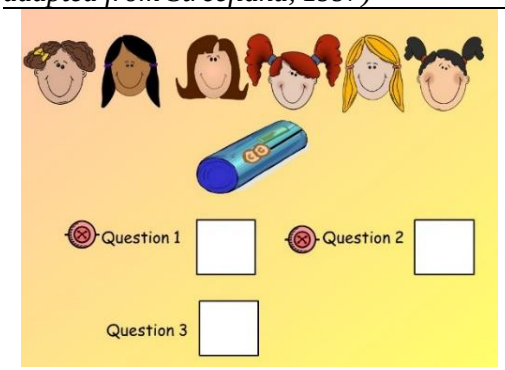
Material was designed to support children's learning about counting fractions and the inverse relation between the divisor and the quotient. The material encouraged children's discussion and practice of adding fractions without the use of an algorithm.

They already knew that 3 chocolates divided by 4 is ¾; if they get a quarter from each chocolate, ¼+¼+¼ is ¾, so they start to put together concepts and ideas they already knew.

Figure 23 shows an example of the kind of questions that children were asked to discuss. These questions were adapted from Streefland (1997) by Nunes et al. (2007); the aim of these questions was to encourage discussion and promote understanding about relations between quantities. The questions involve thinking about equivalence and addition of fractions.

**Figure 23**

*Examples of exercises presented to children in session 3 Quantitative Reasoning training (an item adapted from Streefland, 1997)*

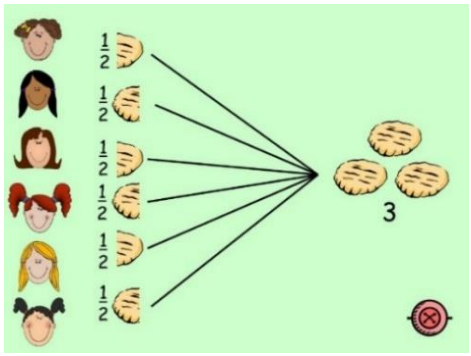



Six girls are going to share a packet of biscuits. The packet is closed, so we don't know how many biscuits are in the packet.

1. If each girl received one biscuit and there were no biscuits left, how many biscuits were in the packet? _____

2. If each girl received a half biscuit and there were no biscuits left, how many biscuits were in the packet? _____

3. If some more girls join the group, what will happen when the biscuits are shared? Do the girls now receive more or less each than the six girls did? _____

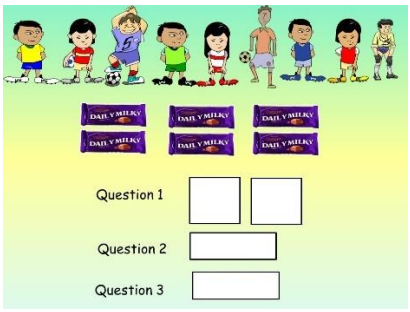



Example of feedback to Q2: Two halves should go to one biscuit, the next two halves to the other biscuit, and the last two to the third biscuit. This was presented to the children using animation.

**Aim 4: To reason about equivalences and addition of fractions and to relate fractions to the whole**

The material was designed to encourage children's reasoning about equivalences and addition of fractions and to relate fractions to the whole by encouraging children's discussions about the different results that could be obtained if the sharing were to be done in different ways, using numbers to represent something they already understood. Material was also designed to explore the idea of improper fractions, as one might have more than one whole shared among a certain number of recipients (for example: how many halves in 3 whole biscuits?). Figure 24 shows an item to exemplify the kinds of question that were presented to children for discussion during session 4. These questions were adapted from Streefland (1997) by Nunes et al. (2007). The aim was to try addition of fractions and to discuss different ways to do it, based on the children's understanding of relations between quantities.

**Figure 24**
*Example of exercises presented to children in session 4 Quantitative Reasoning training (an item adapted from Streefland, 1997)*




Nine children have six chocolate bars to share.
Q1 Find two different ways in which they could share. How much will each one get?
_____ or _____
Q2 Are both of these fair ways of sharing? Yes or No?
Q3 Do the children get the same amount if they share the chocolates in one way or the other? Yes or No?
Why? _____
Prompt:
If the child is uncertain, start with one chocolate bar.
If you chopped each bar into ½ would it work? (No) Why not?
If you chopped each bar into 3 parts would it work? (Yes) 2/3

If you chopped each bar into 9 parts would it work? (Yes) 6/9. If one bar is chopped into 9 pieces each child can have one bit (ninth). So how many of those bits (ninths) will each child get from six chocolate bars? 6/9
Others 4/6, 8/12, 12/18
What do these different ways tell us?

**Aim 5: Practise adding fractions by reasoning about quantities**

The children were also exposed to the idea of improper fractions by exemplifying that you can have fractions from different wholes. Figure 25 shows an example of a question that encourages their reasoning with improper fractions. In this question the children must add ½ + ½ + ½ and then subtract in order to find out how much chocolate was left, which would leave Gail with 2½ chocolate bars.

**Figure 25**
*Example of exercises presented to children in session 5 Quantitative Reasoning training*




Yesterday Gail had four bars of chocolate. She ate half a bar, gave half a bar to her brother and half a bar to her sister.
Q1 How many chocolate bars did the three of them eat altogether?
Q2 How much does she have left for today?

Further material supported reasoning of equivalence: after the children had already reasoned about the equivalence of one third and two sixths, material prompted them to think about how many thirds and how many sixths in a chocolate, and transform one fraction into the equivalent in order to add them.

**Figure 26**
*Example of exercises presented to children in session 6 Quantitative Reasoning training*



Eliza had some chocolate. She ate 2/6 during break and 1/3 at lunch. How much does she have left?

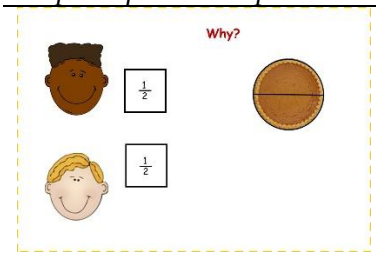
## Arithmetic

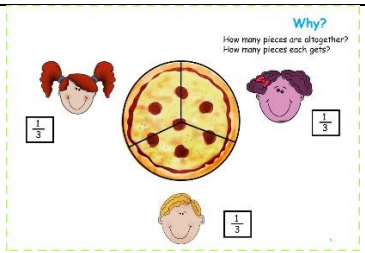
The arithmetic group was taught about fractions as numbers. They explored how to count fractions, equivalence, addition and subtraction of fractions.

### Aim 1: A fraction is a number that represents a division

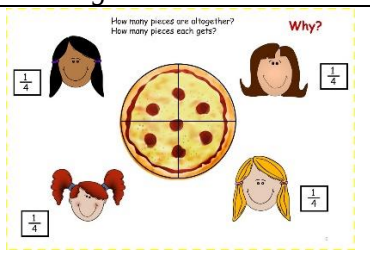
The material aimed to support the understanding that a fraction is a number that represents a division (i.e. a quantity divided by 2 or by 3 and so on).

**Figure 27**
*Examples of exercises presented to children in session 1 Arithmetic training*




Three girls are going to share one pizza fairly. How much do you think each one gets?
Write your answer in the box.
Now can you explain why this is the answer to your friend?
Can you tell me what you wrote? Why is that?
It is 1 divided by 3. We cut one pizza in three pieces and each one gets one piece.



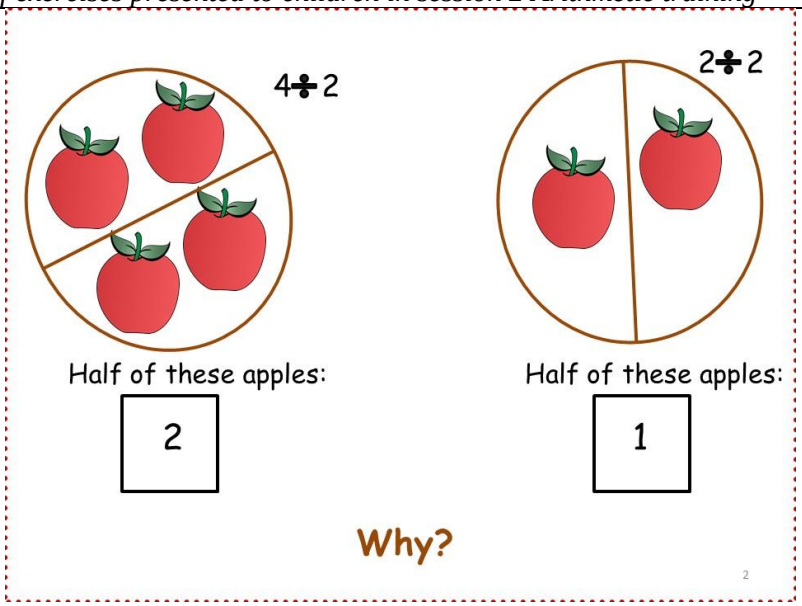

Four girls are going to share one pizza equally. How much do you think each one gets?
Write your answer in the box.
Now can you explain why this is the answer to your friend?
Can you tell me what you wrote? Why is that?
Reinforce: It is 1 divided by 4. We cut one pizza in four pieces and each one gets one piece.

## Aim 2: A fraction can represent different things

The material was designed to help children understand that a fraction can represent different things and to help them connect fractions with division (i.e. the line between the numbers represents a division).

**Figure 28**

*Examples of exercises presented to children in session 2 Arithmetic training*

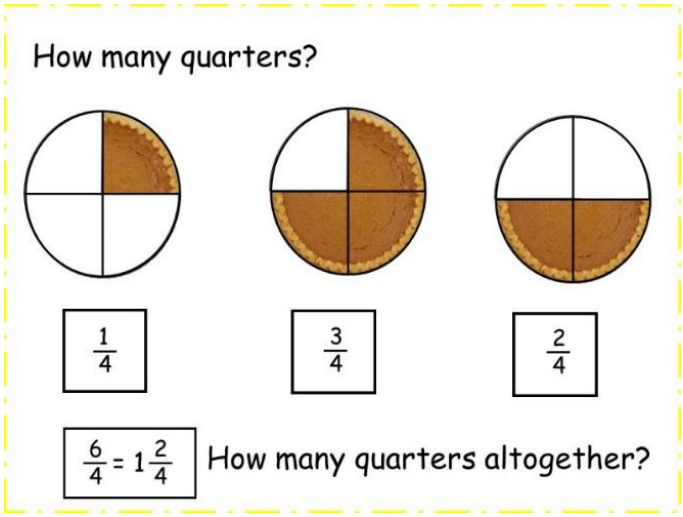



How many apples are half of these apples? (point to the first group) Write in the box.
Why? (They usually say because two 2 and 2 are 4, or because they drew a line between them.)
That means 1 (1 group of apples) divided by 2. So we have (click) 4/2=2
Now how many apples are half of these apples? (point to the second group) Write in the box. Now in the other group is 2/2= 1
*If you realise some of them can't divide, give counters and faces, so they can do it by correspondences: 4 counters divided into 2 groups; share them with 2.

## Aim 3: Count and add fractions

Materials were designed to encourage children to count and add fractions. The children were exposed to the idea that you can have fractions from different wholes, and therefore can have improper fractions.

**Figure 29**
*Example of exercise presented to children in session 3 Arithmetic training*

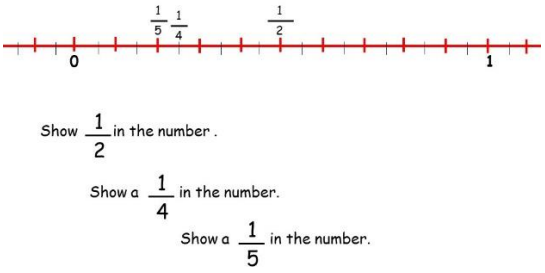



**Aim 4: Compare and add fractions using a number line**

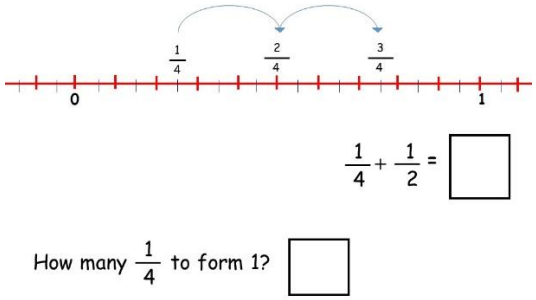
The number line was used only with the arithmetic group, as it aims to help children to focus on relations between numbers rather than relations between quantities. The size of the intervals marked on the number line by dividing it into a different number of parts can be used to compare unitary fractions (e.g. Divide into 2: half and half make 1. Divide into 4, for which children usually divide each half into 2: you need four quarters to make 1. The more pieces, the smaller they are).

**Figure 30**

*Example of exercises presented to children in session 4 Arithmetic training*




Show the number line and count the sections.
Show a half, show a quarter, and show a fifth.
Show a half plus ¼
Put marks on the number line.

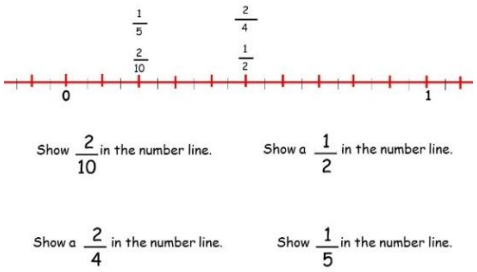



Show the number line, and count the sections.
Find a quarter and find a half.
Show a half plus ¼ and write in the box.
Now count all the quarters.

The equivalence of fractions is demonstrated on the number line so that the children realise that they can treat the same position as having different fraction labels and use the same labels when adding the fractions.

**Figure 31**

*Example of exercises presented to children in session 5 Arithmetic training*

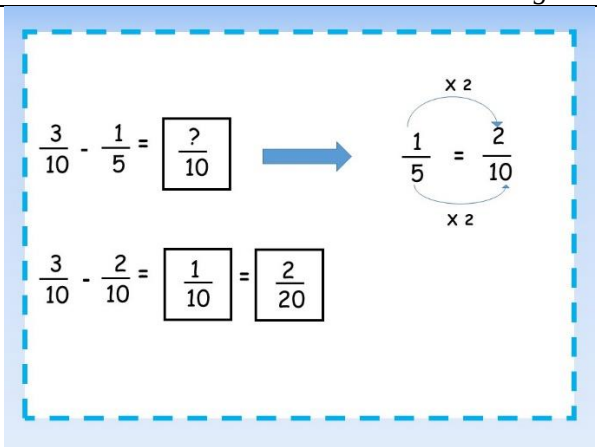



Show on the number line $\frac{2}{10}$
*If they can't do it, ask them roughly where it should go, i.e. closer to 0 or to 1?
If they want to do it exactly, ask them to count the lines and say that if there are 20 lines, how many lines would be $\frac{1}{10}$? And $\frac{2}{10}$?
If they still can't do it, give them 20 counters and ask them to give you half of 20 counters, then ¼, then $^1/_5$, then $^1/_{10}$
Show ½ on the number line.
*If they can't do it, give them 20 counters and ask them to give you a half. If they are just counting/adding to find a half, ask into how many groups you divide to have a half?
Show ¼ on the number line.
*If they can't do it, give them 20 counters and ask them to give you a quarter. If they try to do half and half again, ask into how many groups you divide to have a quarter?

## Aim 5: Adding fractions with different denominators

Material was designed to discuss rules about adding fractions, for example how one cannot just add the numerators of fractions that have different denominators. One rule to change the denominator is that one must change the numerator as well. Once fractions have the same denominator, we can add or subtract the numerators.

**Figure 32**
*Example of exercises presented to children in session 6 Arithmetic training*

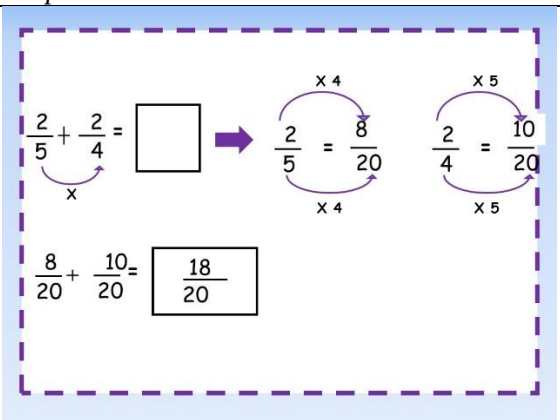



We want to try to subtract 1/5 from 3/10.
So first, are the bottoms the same?
If not, what do we need to do?
We need to make them the same, but how can we do that? Does someone know?
I'll show you first: how can we make 5 into 10?
Then take them through the animation….

Children practised the rule of multiplying the denominators to be able to add or subtract fractions with different denominators. The procedure for finding the smallest common denominator was not used, due to difficulties with multiplication among some children.

**Figure 33**

*Example of exercises presented to children in session 7 Arithmetic training*




Let's see if you remember…
Can we add two fractions that have different numbers at the bottom?
So here, can we add 1/5 and 1/4? Why not?
But can we make 5 and 4 into the same number? How about if we multiply 5 times 4 and then 4 times 5?
What is 5 x 4? 20
What is 4 x 5? 20
So the bottom number can be the same!
Let them try the rest of the sum; if they have difficulty, take them through the animation.

## 5.2 Results

The first step in the statistical analyses was to describe performance of the three intervention groups at pre-test and the reliability of the measures used in the study. This is a necessary preliminary analysis as it establishes whether the measures are reliable and whether the randomisation procedure worked as expected. In any RCT although randomisation is employed, there is still the possibility that the groups are not comparable at pre-test. The next step in the analysis of results was to run all the preliminary analyses required in order to use parametric statistical analyses. Subsequently, the statistical analyses were used to test the prediction that the intervention groups would differ significantly from the control group; if so, we'd compare the intervention groups with each other. Analyses of covariance and multiple regressions analyses were carried out to test if the mathematical abilities could act as a foundation for learning fractions.

### 5.2.1 Preliminary analyses

The aim of the preliminary analyses was to compare the groups at pre-test to establish whether the randomisation procedure worked as expected; to describe the reliability of the measures; and to test whether the dataset met the assumptions of analyses of covariance.

*Performance at pre-test: Randomisation procedure check*

Table 13 shows the groups' means in the fraction test at pre-test and the Raven's Matrices. Two one-way analyses of variance were carried out to compare the three groups' performance in the fractions measure at the time of pre-test and their non-verbal ability. As shown in Table 13, at pre-test all groups show similar means. The reasoning intervention group presented a mean of 17.83 (n=36) on the fractions test and of 29.42 (n=36) on the non-verbal ability test; the arithmetic group achieved a mean of 18.18 (n=32) on the fractions test and of 28.82 (n=32) on the non-verbal ability test; and the control group had a mean of 17.97 (n=35) on the fractions test and of 30.83 (n=35) on the non-verbal ability test. There were no statistically significant differences between group means on fraction knowledge at the time of pre-test, according to a one-way ANOVA ($F(2,103)=0.03$, $p=1.00$). Levene's test was used to test homogeneity of variance across groups at pre-test, and it was not significant ($F(2,103)=0.82$, $p=.44$). The group means on non-verbal ability assessed by the Raven's Progressive Matrices test were found to be statistically not significantly different, as ascertained by a one-way ANOVA ($F(2,101)=0.63$, $p=0.53$); the test of homogeneity of variance across groups was not significant ($F(2,101)=1.18$, $p=.31$). No significant differences were found, which confirms that the randomisation procedure produced comparable groups; and there was no association between group membership and performance in the measures to be used as covariates in the subsequent analyses.

These analyses confirm that all groups are at the same starting point, and therefore the difference between them at the end of the study would be due to the intervention they received or not, as this would be the only difference between the groups.

**Table 13**
*Descriptive statistics for pre-test total score and Raven's score by group allocation*

| Measure | Control | | Arithmetic Knowledge | | Quantitative Reasoning | |
|---|---|---|---|---|---|---|
| | *M* | *SD* | *M* | *SD* | *M* | *SD* |
| Age at pre-test | 9.01 | 0.40 | 9.04 | 0.34 | 8.96 | 0.33 |
| Non-verbal ability | 30.83 | 7.93 | 28.82 | 8.25 | 29.42 | 6.60 |
| Pre-test | 17.78 | 7.77 | 17.91 | 7.50 | 17.83 | 6.59 |

*Reliability of the measures*

During the analysis of the pre-test data, one item was removed from the analysis because no child had answered the item correctly, and thus the covariance matrix could not be computed.

The same analysis was carried out to assess the reliability of the measure at the time of post-test and delayed post-test. The same item that had been dropped from the analysis at pre-test was dropped at post-test, because only one child answered it correctly; and at the delayed post-test, because no child had answered the item correctly; and thus the covariance matrix cannot be computed. This was an arithmetic item (40 ÷ 10 ½ from the original test); no child answered correctly this item at pre-test nor at delayed post-test, and only one child answered it correctly at the immediate post-test.

The measure showed a good level of internal consistency at pre-test, Cronbach's alpha = .87 (N=104, n of items 48); and on both occasions of the post-test: Cronbach's alpha = .89 (N=103, n of items 48) for the immediate post-test, and Cronbach's alpha= .91 at the delayed post-test (N=98, n of items 48).

The correlations between the pre-test and the post-tests were high and significant at the .001 level (with the post-test r(101)=.84, $p < .001$ with the delayed post-test r(96)=.83, $p < .001$); as was the correlation between the scores in the two post-tests: r(95)=.92, $p < .001$. Thus, the measure had a good level of internal consistency on all testing occasions and showed high stability over time.

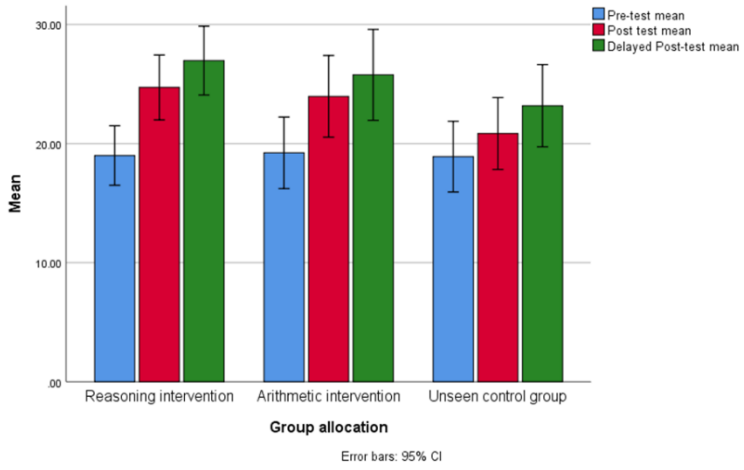
*Descriptive statistics for the three testing occasions*

In this section, the data are summarised to allow for the observation of patterns, if there are any. There was a loss of participants at the time of post-test: the numbers of children assessed were (a) 36 children from the Reasoning intervention group, (b) 32 children from the Arithmetic intervention group and (c) 35 children from the group that did not receive any teaching and is referred to as the control group. Only 98 children were assessed at delayed post-test: (a) 35 children in the Reasoning intervention group, (b) 31 children in the Arithmetic intervention group and (c) 32 children in the comparison group. The cumulative loss of participants was 3.7% at the pre-test, 4.6% at post-test and 9.3% at the delayed post-test. Cumulative calculations of loss of participants are used because it was not possible to analyse the impact of the intervention for children who did not participate in the pre-test. This loss of participants is considered sufficiently low as to make it unnecessary to implement imputations; Schulz and Grimes (2002) suggest as a rule of thumb that imputation should be seen as required if more than 20% of the data are missing at post-test.

Inspection of the figures (see Figure 34) shows that there was little difference between the groups at pre-test and that there was improvement in all three groups from pre- to the immediate and the delayed post-test. This means that the children were learning something relevant to the measure during this time. It is noteworthy that the control

group's performance was still slightly lower at the delayed post-test than that of the intervention groups at the immediate post-test, which suggests that the intervention groups learned more than the control group across testing occasions. The performance in the fractions test at pre-test varied slightly across the groups, but the difference between the groups was not statistically significant, as indicated earlier on. The fractions test mean scores at post-test and delayed post-test varied by a larger amount; the Quantitative Reasoning group showed the highest mean on both post-testing occasions, and the unseen control group showed the lowest mean on both post-testing occasions.

**Figure 34**
*Graph of sub-test means by group allocation*




Thus, the differences between groups at the immediate and the delayed post-tests were in the expected direction. It remains to be seen if these differences were statistically significant, which is tested with the analyses of covariance.

*Testing whether the assumptions for analyses of covariance are met*

In order to carry out an ANCOVA, there are several key assumptions that underlie its use and affect interpretation of the results. Four assumptions will be assessed in the next

section: (a) homogeneity of error variances, (b) normality of error terms and homoscedasticity, (c) multicollinearity and (d) homogeneity of regression slopes.

(a) Homogeneity of error variance

Homogeneity refers to the equality of error variances; the variance of the dependent variable must be equal for all groups. This assumption can be tested using the Levene's Test of Equality of Error Variances. Test of homogeneity of variance at post-test was not significant ($F(2,100)=0.76$, $p=.47$), and at the delayed post-test was also not significant either ($F(2,100)=1.03$, $p= .36$); the assumption of homogeneity was not violated on either occasion of the measurement of the dependent variable.

(b) Normality and homoscedasticity

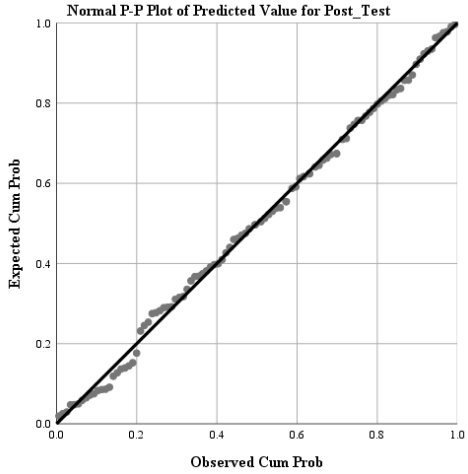
One must test if the normality and homoscedasticity assumptions are met by testing that the residuals present a normal distribution and are uncorrelated with each other: the fraction test outcome at post and delayed post-test with fractions knowledge at pre-test score. To test the assumption of normality of the data, diagnostic plots can be used. If the predicted probability plot shows a diagonal line, this indicates normality. If the scatterplot of the residuals shows that these are spread across without showing a pattern, this indicates that the residuals are not correlated.

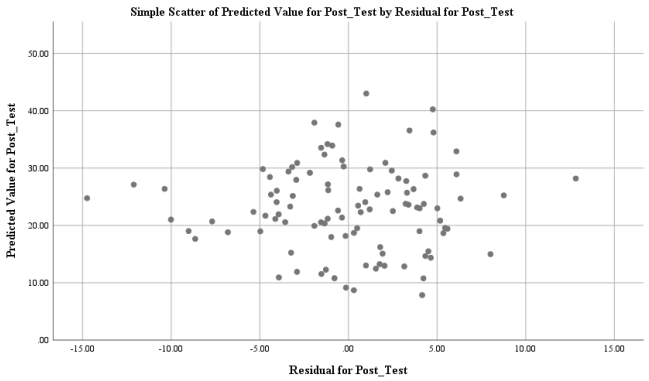
The plots and scatterplots were carried out for two ANCOVAs: (1) Dependent Fractions Post-test, Fixed Factor Group allocation and Covariates non-verbal ability and pre-test score and (2) Dependent Fractions Delayed Post-test, Fixed Factor Group allocation and Covariates non-verbal ability and pre-test score. As shown in Figure 35, both data sets adhere to the assumptions by showing a diagonal line in the predicted probability plot and no pattern in the scatterplot of the residuals.

**Figure 35**

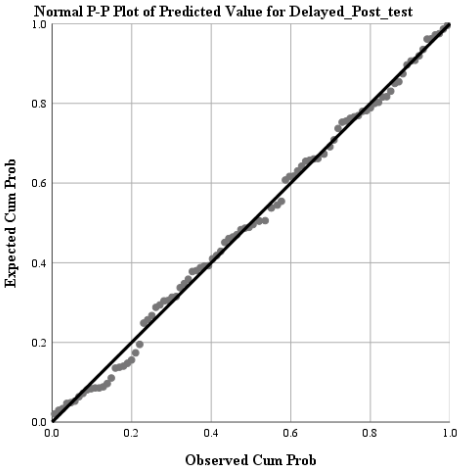
*Testing assumptions of normality and homoscedasticity using diagnostic plots*

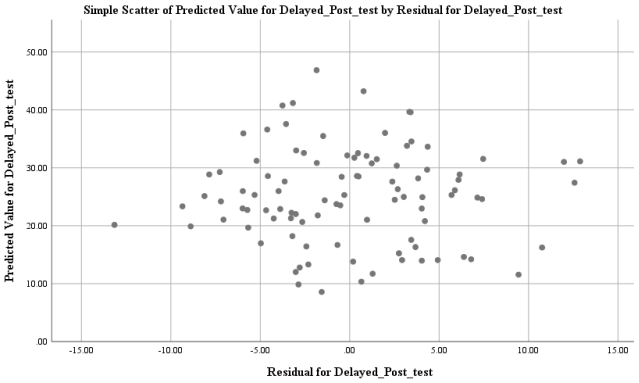
*Dependent Fractions Post-test*




*Dependent Fractions Delayed Post-test*




These results were confirmed by the Shapiro-Wilk test of normality shown in Table 14.

**Table 14**

*Shapiro-Wilk test of normality of dependent variables by group*

| Measure | Control | | | Arithmetic Knowledge | | | Quantitative Reasoning | | |
|---|---|---|---|---|---|---|---|---|---|
| | *W* | *df* | *p* | *W* | *df* | *p* | *W* | *df* | *p* |
| Post-test | .97 | 35 | .54 | .95 | 32 | .18 | .98 | 36 | .70 |
| Delayed post-test | .96 | 32 | .20 | .96 | 31 | .25 | .96 | 35 | .24 |

(a) Multicollinearity

A covariate measures a characteristic of the participant, excluding the intervention. By adding a covariate to the model, this characteristic is considered in the analyses. This reduces the error in the model and controls for a confounding effect. Non-verbal ability differs from child to child, and it could make a difference to how much children learn. Therefore, by adding it to the model, it controls for pre-existing differences; similarly, the measure of fractions knowledge at pre-test controls for what children knew already before the intervention started. Because it is possible that the Raven's scores are correlated with the pre-test performance on the fractions measure, it is necessary to inspect whether one of the measures used at pre-test is statistically redundant, due to multicollinearity. The correlation between the scores in the Raven's Matrices and the scores in the fractions measure at pre-test was significant at the .01 level but low: $r(102)= .33$, $p=.001$. This indicates that it is worth including the Raven's Matrices scores in the ANCOVA and pre-test as a control.

It has also been claimed that the covariates should be an independent variable but correlated with the dependent variable (Bump, 1992). The correlations between the pre-test and the post-test were reported earlier on: they were all significant at $p<.001$ level, $r(105)= .84$, $p<.001$. The correlations between non-verbal ability and the fractions score at post-test and at the delayed post-test were statistically significant: with the immediate post-test, $r(101)= .35$, $p<.001$; with the delayed post-test, $r(96)= .37$, $p<.001$. These are comparable to the correlation between the Raven's Matrices scores and the fractions pre-test, and considerably lower than the correlations between the fractions pre-test and the post-tests. Thus, the level of correlations observed among the pre-tests and post-tests indicates that

ANCOVA with the pre-test measures used in the study is an appropriate statistical analysis.

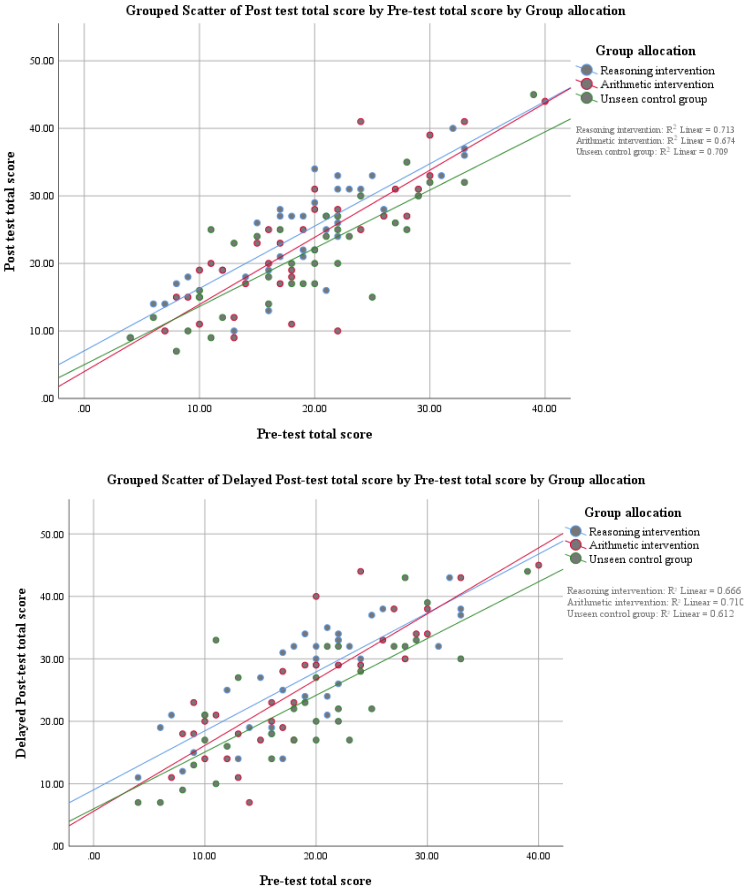
(b) Homogeneity of regression slopes

The homogeneity of regression slopes refers to the linear relationship between covariate and the dependent variable (Department for Education, 2014). The lines representing these linear relations should be parallel for each independent variable: in this case, the outcome in the fractions measure at the immediate and the delayed post-test. In this study, there are two covariates, the fractions measure at pre-test and non-verbal ability, and one moderator term, group by the measure of non-verbal ability. In the theoretical model presented in the introduction, it was hypothesised that the outcome of the interventions would vary with the children's level of cognitive ability, as measured by the Raven's test of non-verbal ability. Thus, it is expected that the assumption of homogeneity of regression slope will be met for the fractions measure before the intervention, which will be tested. The measure of non-verbal ability, which is hypothesised to moderate the effect of the intervention, will be discussed in the moderation section of the results.

In this study it means that each group should have parallel lines in a scatterplot showing the fractions test score at post-test by pre-test fractions score. Figure 38 shows the slopes for the covariate pre-test with fraction score at post-test and delayed post-test. Visual inspection of the slopes shown in Figure 38 suggest that the assumption is met for the covariate performance in the fractions measure at pre-test. Parallel analyses were carried out to test if the assumption was met when the dependent variable was the delayed post-test. Test of between-subject effects showed that the interaction group by pre-test score was not significant, and the assumption was met at post-test $F(2,97)=.58$ $p=0.56$, $\eta^2=.01$; and delayed post-test $F(2,92)=.68$ $p=0.51$, $\eta^2=.02$.

**Figure 36**

*Testing assumption of homogeneity of post-test and delayed post-test regression slopes*




In summary, all assumptions were met and the two dependent variables, fractions knowledge at post-test and fractions knowledge at delayed post-test, with covariates of fractions knowledge at pre-test and non-verbal ability, are suitable for an ANCOVA analysis to assess the effect of the interventions on children's fractions learning.

### 5.2.2 Analyses of covariance: impact of the intervention

The effectiveness of the training programme was analysed by a comparison between the intervention groups and the control group. Two analyses of covariance were carried out: in both, the z-scores of children's non-verbal ability and the pre-test fractions test scores were used as covariates; an interaction term of group allocation by non-verbal

ability was included to test the significance of non-verbal ability as a moderator of the effect of the intervention. In the first analysis, the dependent variable was the immediate post-test fractions test score; in the second analysis, the dependent variable was the delayed post-test fractions test score.

*Immediate post-test ANCOVA results*

Table 15,Table 16 and Table 17 show the outcomes of this analysis. Table 15 shows the raw means and standard deviations. The group of children who received the quantitative reasoning intervention showed the highest score on fractions knowledge at the post-test, followed by the arithmetic group.

**Table 15**
*Raw means and standard deviations by intervention group at the post-test*

| Measure | Control (n=35) | | Arithmetic Knowledge (n=32) | | Quantitative Reasoning (n=36) | |
|---|---|---|---|---|---|---|
| | *M* | *SD* | *M* | *SD* | *M* | *SD* |
| Non-verbal ability | 30.83 | 7.93 | 28.82 | 8.25 | 29.42 | 6.60 |
| Fraction knowledge | | | | | | |
| Pre-test | 17.97 | 7.79 | 18.18 | 7.50 | 17.83 | 6.59 |
| Post-test | 20.83 | 8.13 | 23.13 | 9.54 | 24.33 | 7.61 |

Table 16 shows the statistical information for the analysis of between-participants effects, presented in four sequential models of analyses of variance with the immediate post-test scores as the dependent variable. The first model is an ANCOVA with one covariate, pre-test by group allocation. The second model adds one more covariate, including pre-test and non-verbal ability; this model will be described in the next section referring to the interventions effect on fractions knowledge at post-test. The last model includes a moderator term, pre-test, non-verbal ability, group and group by non-verbal ability; this model will be described in the following section, related to the hypothesised moderation of non-verbal ability on the intervention effects.

**Table 16**
*Analyses of variance-dependent variable fractions knowledge at immediate post-test*

| Model | *F* ratio | df | p | $\eta^2$ |
|---|---|---|---|---|
| **Model 1: One covariate** | | | | |
| Pre-test | 272.27 | 1,99 | <.001 | .73 |
| Group | 6.16 | 2,99 | .003 | .11 |
| **Model 2: Two covariates** | | | | |
| Pre-test | 231.72 | 1,98 | <.001 | .70 |
| Non-verbal ability | 3.76 | 1,98 | .055 | .04 |
| Group | 6.86 | 2,98 | .002 | .12 |
| **Model 3: Two covariates and moderator** | | | | |
| Pre-test | 230.45 | 1,96 | <.001 | .71 |
| Non-verbal ability | 4.46 | 1,96 | .037 | .04 |
| Group | 6.45 | 2,96 | .002 | .12 |
| Group * Non-verbal ability | 1.73 | 2,96 | .182 | .04 |

Interventions effect on fractions knowledge test at post-test

The intervention effect is assessed in the second model, which includes two covariates: the pre-test score and non-verbal ability. In the second model shown in Table 16, the pre-test score is a significant covariate ($F(1,98)=231.72$, $p<0.001$). The group allocation also is statistically significant, which indicates that the groups differed at the immediate post-test, even after controlling for the covariates, but it remains to be tested whether both intervention groups differed significantly from the control group.

The overall group effect was significant, $F(2, 98) = 6.86$, $p < .05$; non-verbal ability was not a significant covariate, even though it approached significance ($p=.055$), but the fractions pre-test measure was a significant covariate. A priori planned comparison tests, which compared each of the intervention groups with the control group, are displayed in Table 17, which shows that after controlling for pre-test and non-verbal ability, the estimated marginal means continue to vary in the expected direction. The pairwise comparisons of the control group with the quantitative reasoning group was significant. The mean fraction knowledge at post-test of the control group ($M=20.95$, $SE=.73$) was significantly lower than that in the quantitative reasoning group ($M=24.65$, $SE=.72$). The control group and the arithmetic intervention group were not statistically different,

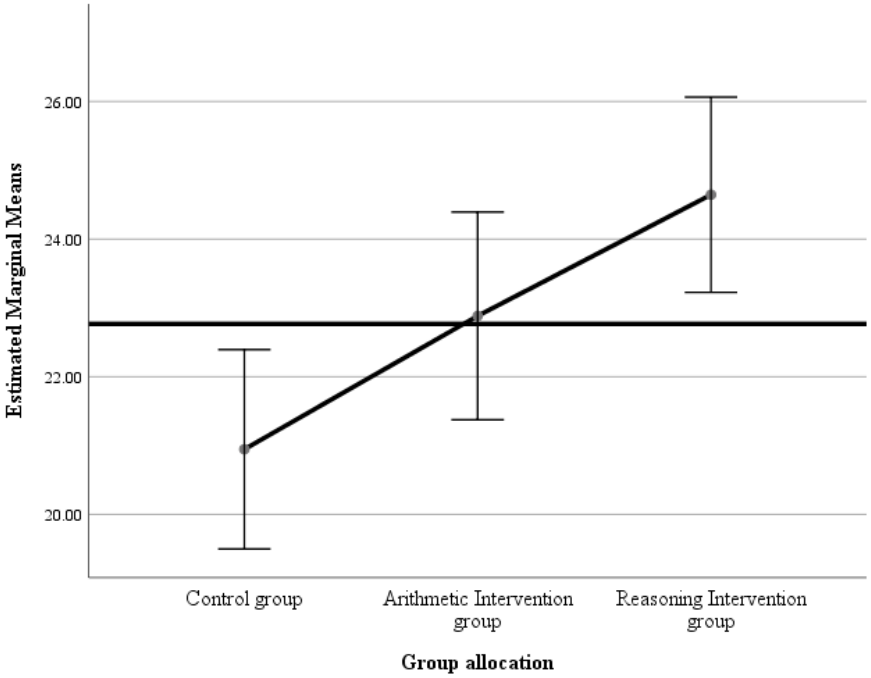
although the arithmetic intervention group presented a higher score (M=22.88, SE=.76). Also, the quantitative reasoning intervention group and the arithmetic intervention group were not statistically different. Figure 37 shows that the means increase in the hypothesised direction.

**Table 17**
*Estimated marginal means and pairwise comparisons at post-test*

| Measure | Control (n=35) | | Arithmetic (n=32) | | Quantitative Reasoning (n=36) | | Control vs. Arithmetic | | Control vs. Reasoning | | Arithmetic vs. Reasoning | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *M* | *SE* | *M* | *SE* | *M* | *SE* | *Mean diff.* | *SE* | *Mean diff.* | *SE* | *Mean diff.* | *SE* |
| Post-test | 20.95 | .73 | 22.88 | .76 | 24.65 | .72 | -1.94 | 1.05 | -3.70* | 1.02 | -1.76 | 1.04 |

*$p$<.05

**Figure 37**
*Estimated marginal means of post-test score by group*




The horizontal line is the observed grand mean.
Covariates: Pre-test score=18.08, Non-verbal ability=.018
Error bars: 95% confidence interval

The difference in performance between the reasoning group and the control group was statistically significant, after controlling for the effect of non-verbal ability, pre-test

score and moderator group allocation by non-verbal ability; Cohen's *d* effect size for the reasoning group in comparison to control: 0.44. This effect size can be considered moderate for an 8-week intervention in education. Hattie (2012) argued in meta-meta-analysis study that an effect size of 0.2 should be considered small, 0.4 moderate, and a *d* above 0.6 and greater as strong, even though 0.44 is considered a small effect size by Cohen (1988). The arithmetic group did not differ significantly from the control group at the immediate post-test, but the difference between the means in the outcome measure was in the predicted direction. Pairwise comparisons show that the two intervention groups did not differ from each other.

Non-verbal ability moderation of the intervention effect at immediate post-test

Figure 38 shows the slopes for the outcome measures at immediate post-test by non-verbal ability; the lines for the different groups are not parallel, which suggests that non-verbal ability could moderate the effect of the interventions.

**Figure 38**

*Non-verbal ability by immediate post-test regression slopes by group allocation*

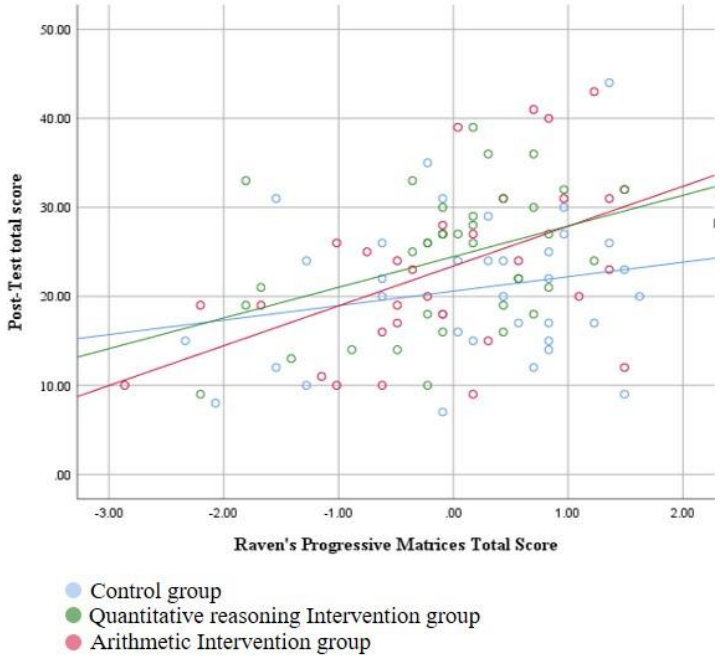


Table 16 shows the last ANCOVA model, which includes two covariates and a moderator term, group by non-verbal ability. The overall group effect was significant, F (2, 96) = 6.45, $p$ <.05; non-verbal ability was a significant covariate, and the fractions pre-test measure was a significant covariate. Although visual inspection of Figure 38 suggests an interaction between the covariate by group, the interaction term of group allocation by non-verbal ability was not significant at the time of the immediate post-test, which indicates that non-verbal ability was not a significant moderator of the intervention effects; thus, non-verbal ability has an effect on children's knowledge at the immediate post-test, but the effect does not vary across groups. This result leads to the rejection of our hypothesis that levels of non-verbal ability would moderate the effect of the interventions at the time of immediate post-test.

*Delayed post-test ANCOVA results*

Table 18 shows the outcomes of this analysis. It shows the raw means and standard deviations of children's performance on the fractions test at the delayed post-test, six weeks after they finished the intervention. The group of children who received the quantitative reasoning intervention showed the highest score on fractions knowledge at the post-test, followed by the arithmetic group.

**Table 18**
*Raw means and standard deviations by intervention group at the delayed post-test*

| Measure | Control (n=32) | | Arithmetic Knowledge (n=31) | | Quantitative Reasoning (n=35) | |
|---|---|---|---|---|---|---|
| | *M* | *SD* | *M* | *SD* | *M* | *SD* |
| Non-verbal ability | 30.83 | 7.93 | 28.82 | 8.25 | 29.42 | 6.60 |
| Fraction knowledge | | | | | | |
| Pre-test | 17.97 | 7.79 | 18.18 | 7.50 | 17.83 | 6.59 |
| Delayed post-test | 21.44 | 9.14 | 24.10 | 10.07 | 25.31 | 8.13 |

Table 19 shows the statistical information for the analysis of between-participants effects, presented in four sequential models of analyses of variance with the immediate delayed post-test scores as the dependent variable. The same procedure of analyses used with the post-test were used with the delayed post-test. The first model is a one-way ANOVA, delayed post-test by intervention group. The following ANCOVA models include covariates and a moderator term. The second model adds one covariate to the first model, pre-test. The third model adds one more covariate, including pre-test and non-verbal ability; this model will be described in the next section, referring to the interventions effect on fractions knowledge at delayed post-test. The last model includes a moderator term, pre-test, non-verbal ability, group and group by non-verbal ability; this model will be described in the subsequent section, related to the hypothesised moderation of non-verbal ability on the intervention effects.

**Table 19**

*Analyses of variance-dependent variable fractions knowledge at delayed post-test*

| Model | *F* ratio | df | p | $\eta^2$ |
|---|---|---|---|---|
| **Model 1: One covariate** | | | | |
| Pre-test | 241.26 | 1,94 | <.001 | .72 |
| Group | 6.30 | 2,94 | .003 | .12 |
| **Model 2: Two covariates** | | | | |
| Pre-test | 197.79 | 1,93 | <.001 | .68 |
| Non-verbal ability | 3.32 | 1,93 | .072 | .03 |
| Group | 6.82 | 2,93 | .002 | .13 |
| **Model 3: Two covariates and moderator** | | | | |
| Pre-test | 206.43 | 1,91 | .<.001 | .69 |
| Non-verbal ability | 4.23 | 1,91 | .042 | .04 |
| Group | 7.02 | 2,91 | .001 | .13 |
| Group * Non-verbal ability | 4.44 | 2,91 | .014 | .09 |

Interventions effect on fractions knowledge test at delayed post-test

The intervention effect on the delayed post-test score is assessed in the second model, which includes two covariates: the pre-test score and non-verbal ability. In the second model, shown in Table 19, the pre-test score is a significant covariate ($F(1,93)=197.79$, $p<0.001$). The group allocation also is statistically significant, which

indicates that the groups differed at the delayed post-test, even after controlling for the covariates, which suggests that the knowledge gained from the interventions is stable after six weeks, but it remains to be tested whether both intervention groups differed significantly from the control group.

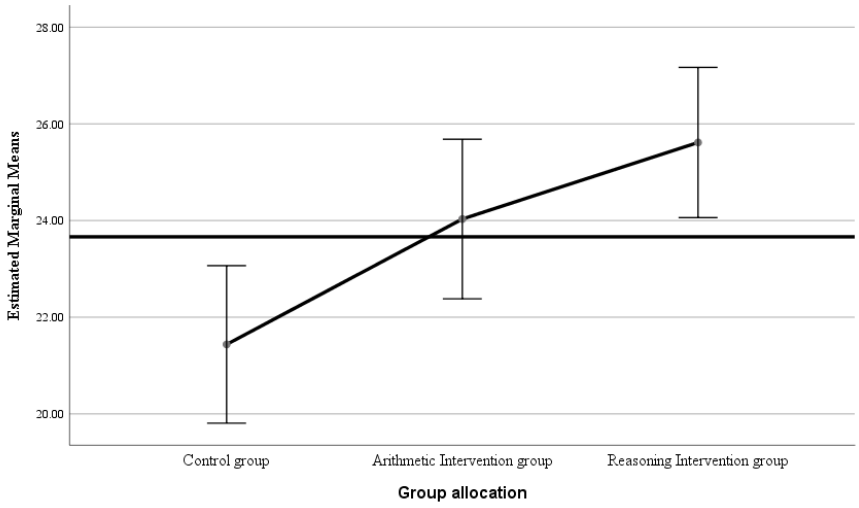
The overall group effect was significant, $F(2, 93) = 6.82$, $p < .05$; non-verbal ability was a significant covariate, and the fractions pre-test measure was also a significant covariate. A priori planned comparison tests, which compared each of the intervention groups with the control group, are displayed in Table 20; after controlling for the covariates, the estimated marginal means continue to vary in the expected direction. Although all three groups' scores improved from the immediate to the delayed post-test, the difference between both intervention groups and the control group increased. The pairwise comparisons of the control group with the quantitative reasoning group was significant. The difference in scores between the reasoning group and the control group, after controlling for the effect of non-verbal ability and pre-test score, was statistically significant: Cohen's *d* effect size for the reasoning group in comparison to control: 0.45. This effect size did not decrease from the post-test, which suggests that the learning advantage resulting from the intervention was to some extent durable. At the delayed post-test, the difference between the arithmetic and the control group was not statistically significant, although the arithmetic group had a higher mean than the control group; Cohen's *d* effect size for the arithmetic group in comparison to control was: 0.28, which is smaller than the effect observed for the reasoning group. Pairwise comparisons show that the two intervention groups did not differ from each other.

**Table 20**
*Estimated marginal means and pairwise comparisons at delayed post-test*

| Measure | Control (n=35) | | Arithmetic (n=32) | | Quantitative Reasoning (n=36) | | Control vs. Arithmetic | | Control vs. Reasoning | | Arithmetic vs. Reasoning | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | *M* | *SE* | *M* | *SE* | *M* | *SE* | *Mean diff.* | *SE* | *Mean diff.* | *SE* | *Mean diff.* | *SE* |
| Delayed post-test | 21.27 | .85 | 23.98 | .86 | 25.57 | .81 | -2.72 | 1.21 | -4.30* | 1.17 | -1.59 | 1.18 |

*$p$<.05

**Figure 39**
*Estimated marginal means on delayed post-test by group*




The horizontal line is the observed grand mean.
Covariates: Pre-test score=18.04, Non-verbal ability= - .015
Error bars: 95% confidence interval

Non-verbal ability moderation of the intervention effect at immediate post-test

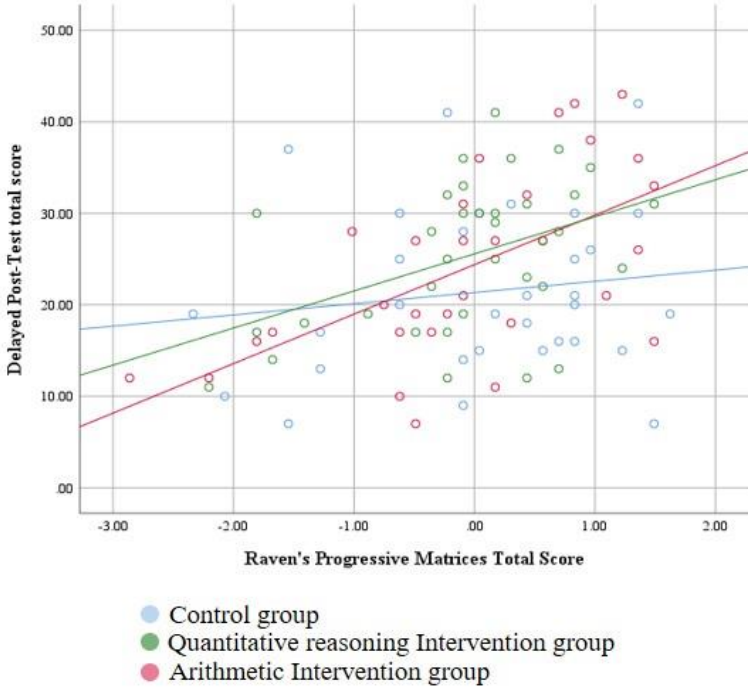
Figure 40 shows the slopes for the outcome measure delayed post-test by non-verbal ability; the lines for the different groups are not parallel, which suggests that non-verbal ability could moderate the effect of the interventions.

Table 19 shows the last ANCOVA model, which includes two covariates and a moderator term, group by non-verbal ability. The overall group effect was significant, $F(2,91)=7.021$, $p<.01$; non-verbal ability and the pre-test were significant covariates. The

interaction term was also significant, showing that the intervention had a different effect depending on the non-verbal ability of the child.

**Figure 40**

*Non-verbal ability by delayed post-test regression slopes by group allocation*




*Secondary analyses by group of items*

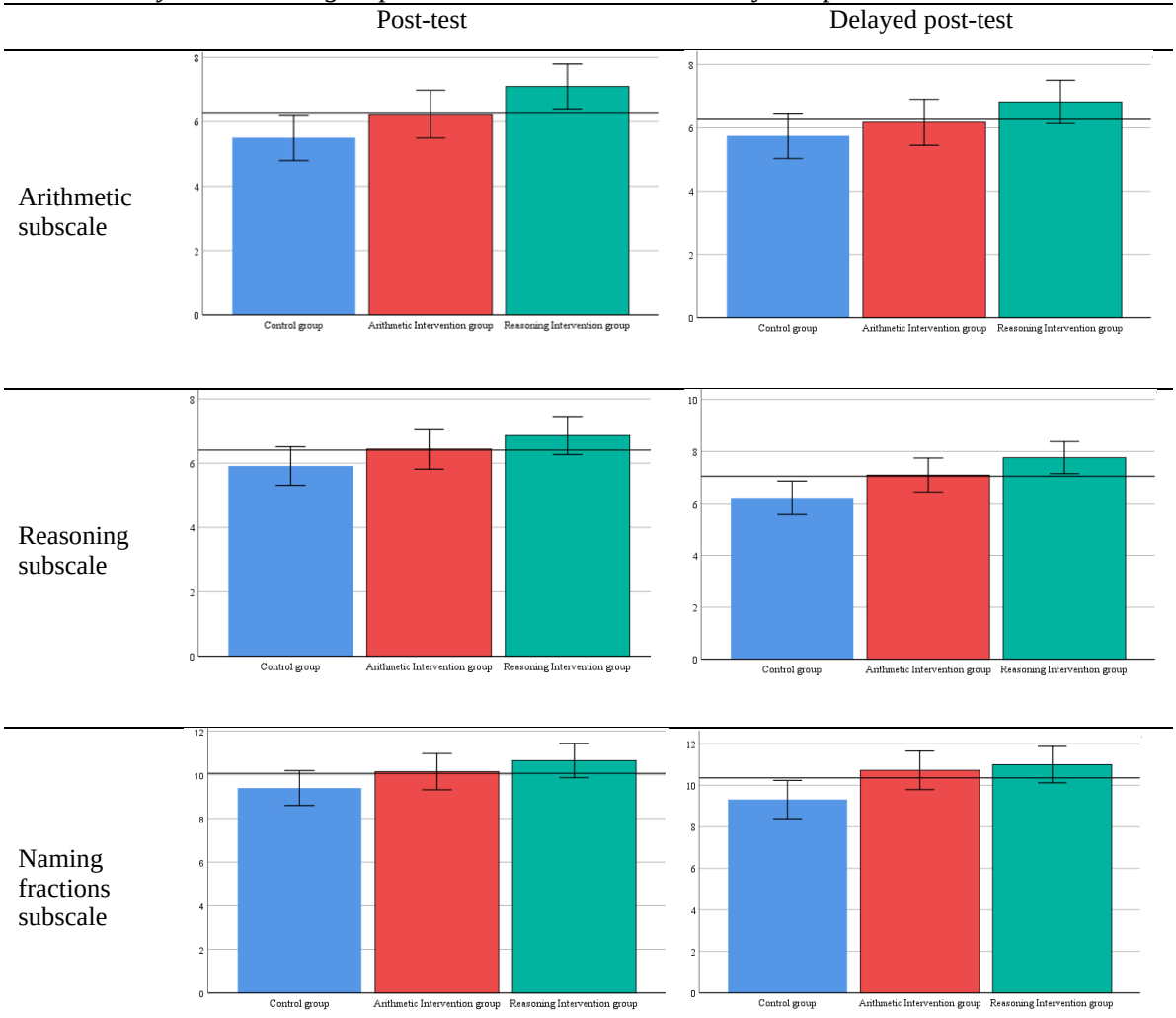
In Chapter 2, it was established that reasoning about quantities represented by fractions and doing arithmetic with fractions can be seen as separate, although related skills. It was therefore considered useful to analyse whether the two types of intervention had differential effects on the separate subscales, which assessed attainment in reasoning, arithmetic and fraction naming. In view of the finding from the longitudinal study, that the reasoning measure is a better predictor than the arithmetic measure of learner's outcomes after receiving instruction on fractions, it was predicted that the reasoning intervention would have a larger overall impact on all three measures. It was further predicted that the reasoning intervention would have a stronger impact on the reasoning subscale than on the arithmetic subscale, and that the arithmetic intervention would have a stronger impact on

the arithmetic subscale than on the reasoning subscale scale. It was not possible to make predictions about the differential impact of the interventions on the fraction naming subscale, as both groups were taught about how to use fractions to represent quantities in the course of the interventions.

In order to test these predictions, six analyses of covariance were carried out. The first two analyses tested the impact of the two interventions on the arithmetic subscale at the immediate and the delayed post-test, controlling for the children's performance in the Raven's Matrices and in the arithmetic subscale of the pre-test. The third and fourth analyses tested the impact of the two interventions on the reasoning subscale at the immediate and the delayed post-test, controlling for the children's performance in the Raven's Matrices and in the reasoning subscale of the pre-test. The fifth and sixth analyses tested the impact of the two interventions on the fraction naming subscale at the immediate and the delayed post-test, controlling for the children's performance in the Raven's Matrices and in the fraction naming subscale of the pre-test. The details of these analyses are presented in Appendix 8, and Figure 41 displays the raw means on each of these subscales at the immediate and the delayed post-test by group allocation. Error bars at 95% confidence interval and the overall mean for all groups are displayed to support visual inspection of the figures.

**Figure 41**

*Raw means by intervention group. Covariates: Non-verbal ability and pre-test overall score*




*The horizontal line in the graphs represents the overall mean.

In all the comparisons, the reasoning intervention group's means are above the overall means, and the control group's means are below the overall means. In contrast, the arithmetic intervention group's means are generally closer to the overall mean.

Table 21 summarises the results of the planned comparisons between each intervention group and the control group in the six analyses of covariance. Inspection of the *p* values shows that the reasoning intervention group differed significantly from the control group in all comparisons, whereas the arithmetic group's performance was

statistically significantly better than that of the control group in one of the six comparisons. Inspection of Cohen's *d* values shows that all were positive, indicating that the intervention groups' means were consistently higher than those of the control group. Surprisingly, in the immediate post-test the reasoning intervention group's advantage over the control group was larger in the arithmetic subscale than in the reasoning subscale, but in the delayed post-test the advantage was larger in the reasoning subscale, as expected. For the arithmetic intervention the results were not as predicted either: although the advantage over the control group was larger in the arithmetic than in the reasoning subscale at the immediate post-test, at the delayed post-test the reverse was observed.

**Table 21**
*Comparisons between each intervention group and the control group in the 6 subscales*

| | ANCOVA | | | Pairwise comparisons | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | Group effect | | | Control | | Reasoning intervention | | | Arithmetic intervention | | |
| | *F*-ratio | df | $\eta^2$ | *M* | *SD* | *M* | *SD* | Cohen's d | *M* | *SD* | Cohen's d |
| Arithmetic immediate post-test | 4.60* | 2,96 | 0.09 | 5.51 | 3.11 | 7.00** | 3.34 | 0.46 | 6.27 | 3.41 | 0.23 |
| Arithmetic delayed post-test | 2.33 | 2,91 | 0.05 | 5.78 | 3.25 | 6.74* | 3.21 | 0.30 | 6.28 | 3.4 | 0.15 |
| Reasoning immediate post-test | 2.27 | 2,96 | 0.05 | 5.91 | 2.81 | 6.78* | 2.33 | 0.33 | 6.48 | 3.15 | 0.19 |
| Reasoning delayed post-test | 5.99** | 2,91 | 0.12 | 6.25 | 2.72 | 7.69** | 2.67 | 0.53 | 7.06 | 3.45 | 0.26 |
| Fraction naming immediate post-test | 2.45 | 2,96 | 0.05 | 9.40 | 3.46 | 10.56* | 3.08 | 0.35 | 10.12 | 3.89 | 0.2 |
| Fraction naming delayed post-test | 3.84* | 2,91 | 0.08 | 9.41 | 4.02 | 10.89* | 3.41 | 0.4 | 10.66* | 4.1 | 0.31 |

*$p < .05$. **$p < .01$

The proposed hypothesis was that children with a high level of cognitive ability would be able to transfer their knowledge from quantitative reasoning to arithmetic

knowledge, more than arithmetic knowledge to quantitative reasoning. It was reasoned that two groups divided by the median were more appropriate than three groups divided by standard deviation, as some groups had very few cases.

Table 22 shows the means of each subscale by intervention group divided into two groups of ability split at the median.

**Table 22**

*Means by group by non-verbal ability split at the median*

Arithmetic subscale

| | Pre | Post | Delayed post | Pre | Post | Delayed post |
|---|---|---|---|---|---|---|
| Intervention group | Equal or below the median | | | Above the median | | |
| Control | 4.33 (n=15) | 5.67 (n=15) | 6.00 (n=15) | 5.30 (n=20) | 5.40 (n=20) | 5.59 (n=20) |
| Arithmetic | 3.89 (n=19) | 4.89 (n=18) | 4.71 (n=17) | 5.57 (n=14) | 8.21 (n=14) | 8.07 (n=14) |
| Reasoning | 4.33 (n=18) | 5.78 (n=18) | 5.83 (n=18) | 5.17 (n=18) | 8.22 (n=18) | 7.29 (n=17) |

Reasoning subscale

| | Pre | Post | Delayed post | Pre | Post | Delayed post |
|---|---|---|---|---|---|---|
| Intervention group | Equal or below the median | | | Above the median | | |
| Control | 4.80 (n=15) | 5.67 (n=15) | 6.20 (n=15) | 5.25 (n=20) | 6.10 (n=20) | 6.29 (n=20) |
| Arithmetic | 4.37 (n=19) | 5.61 (n=18) | 5.53 (n=17) | 5.07 (n=14) | 7.71 (n=14) | 9.07 (n=14) |
| Reasoning | 4.28(n=18) | 6.11 (n=18) | 7.00 (n=18) | 5.72 (n=18) | 7.44 (n=18) | 8.41 (n=17) |

Naming subscale

| | Pre | Post | Delayed post | Pre | Post | Delayed post |
|---|---|---|---|---|---|---|
| Intervention group | Equal or below the median | | | Above the median | | |
| Control | 7.20 (n=15) | 8.60 (n=15) | 8.80 (n=15) | 8.65 (n=20) | 10.00 (n=20) | 9.94 (n=20) |
| Arithmetic | 7.58 (n=19) | 9.56 (n=18) | 9.53 (n=17) | 10.71 (n=14) | 11.14 (n=14) | 12.21(n=14) |
| Reasoning | 7.56 (n=18) | 9.67 (n=18) | 9.94 (n=18) | 8.61 (n=18) | 11.44 (n=18) | 11.88(n=17) |

In brief, as expected, the impact of the reasoning intervention was larger than the impact of the arithmetic intervention when the control group is used as the baseline comparison group. However, the impact was not as specific as anticipated: the reasoning

intervention had a significant impact on the arithmetic subscale on both post-testing occasions, and the arithmetic intervention had an effect on the reasoning subscale in the delayed post-test, although it did not have a statistically significant impact on the arithmetic scale. On the whole, both interventions were more effective for the higher ability group.

## 5.3 Conclusions

Consistent with the hypothesis, the intervention study demonstrated the effect of quantitative reasoning training on fractions concepts. These results provide support for the hypothesis that quantitative reasoning acts as a foundation for learning fractions concepts. It is important to note that children participating in the intervention groups received extra sessions on maths (eight sessions of 30 minutes each = four hours in total), which could be easily incorporated by teachers in regular maths time. A four-hour training was able to improve their performance in a test of fractions knowledge independently designed for older children, and their performance remained at delayed post-test. The quantitative intervention aimed to improve children's understanding about multiplicative relations; children who received this training outperformed children from the control group. These results suggest a direct connection between children's understanding of multiplicative reasoning and their fractions learning. Thus, schools could benefit from including activities that aim to promote multiplicative reasoning.

Four main conclusions can be drawn from the results of this intervention study. First, children aged 9 years old can learn about fractions. Children from all the three groups, including the children that did not receive any intervention, improved their

fractions scores from the pre-test to the post-test, which shows that children of this age can learn about fractions.

The second conclusion is that quantitative reasoning training yields the best performance in fractions learning. This learning remained stable after several weeks.

The third conclusion is that arithmetic training improved children's performance only in the long run, and it is not clear if this effect may vanish over time.

The fourth conclusion is that non-verbal ability acts as a significant moderator in children's learning in slightly longer terms, but not for immediate learning.

An unexpected result was related to the arithmetic intervention. It was hypothesised that, given the specific connection between arithmetic with whole numbers and arithmetic with fractions, an intervention on arithmetic would predict how well they learn the new rules for calculating with fractions. The arithmetic intervention had a stronger relation with reasoning about fractions than with the arithmetic of fractions.

In summary, this study showed that quantitative reasoning and arithmetic knowledge have an effect on children's learning about fractions, by means of an effective quantitative reasoning training.

# Chapter 6 Conclusions and Limitations

This section starts with a summary of the findings in each study and then explores their implications for teaching and learning fractions around two main themes: the difficulties of learning fractions and two types of mathematical knowledge. The final section discusses the limitations and future research. Finally, concluding remarks are presented.

## 6.1 Summary of the main findings

The aim of this study was to test a theoretical model in which quantitative reasoning and arithmetic knowledge provide the foundations for children's fractions learning. Results from the studies in this thesis contribute to our understanding of mathematics learning specifically about fractions and build upon existing research. In order to evaluate and reflect on the contributions of the different studies, the findings are summarised by considering how well they support the different hypotheses.

### 6.1.1 Distinction between arithmetic knowledge and quantitative reasoning

The distinction between arithmetic knowledge and quantitative reasoning is the basis of this thesis. A Confirmatory Factor Analysis with two latent variables showed the first key finding: quantitative reasoning and arithmetic knowledge should be seen as two empirically different mathematical abilities. Thus, the first hypothesis was supported for claiming that these abilities can be measured and that these measures stand as two different constructs, independently from non-verbal ability, establishing that quantitative reasoning and arithmetic knowledge are not only theoretically different mathematical abilities but also statistically different, thus allowing them to stand as different precursors for learning fractions. This finding expands previous knowledge on the measurement of mathematical abilities: firstly, measures of quantitative reasoning and arithmetic knowledge were

developed and analysed, and now consisting of reliable scales which can be useful for further research or educational purposes, expanding from Nunes and her colleagues (2015) who developed and analysed a scale of quantitative reasoning for younger children. The systematic analyses of the measures also provided insights on levels of difficulty that these concepts represent for students at the age of 9. Finally, the findings provide important evidence for the empirical separability of quantitative reasoning and arithmetic knowledge, which has not been studied before.

### 6.1.2 Quantitative reasoning and arithmetic knowledge as longitudinal predictors

The longitudinal study offers a specific connection between fractions learning with quantitative reasoning and arithmetic over time, having controlled for non-verbal reasoning, which is crucial because it allows us to establish whether these factors make unique contributions to the prediction of children's subsequent fraction learning.

Two different models were tested in the prediction of children's learning of fractions. The difference between these two models was the last step in the model after controlling for non-verbal ability and either quantitative reasoning or arithmetic knowledge. The second key finding from this study is the relative importance of quantitative reasoning and arithmetic knowledge to children's learning of fractions. The second hypothesis was supported by claiming that quantitative reasoning demonstrated to be a stronger predictor of fractions learning; arithmetic knowledge also made a significant contribution, but weaker than quantitative reasoning. Thus, both mathematical abilities are important for learning fractions. It is noteworthy that the contribution of quantitative reasoning was the strongest, even more so than non-verbal ability.

The regression equations that combined all three factors (non-verbal ability, arithmetic knowledge and quantitative reasoning) explained 74% of the variance in the

children's learning of fractions, measured 12 months later. This is important because it explains a large part of variance in children's learning, without taking into account other cognitive abilities such as working memory.

This finding expands previous knowledge on the relative importance of quantitative reasoning and arithmetic knowledge by expanding the results from Nunes and her colleagues (2012a), who also found that quantitative reasoning was the strongest predictor of mathematical achievement after 3 and 5.5 years. Their results were specifically on whole numbers; to the best of my knowledge this is the first study to assess the longitudinal connection of quantitative reasoning and arithmetic knowledge to fractions.

### 6.1.3 Evidence of quantitative reasoning and arithmetic knowledge acting as a foundation for learning fractions

It was stated that the research strategy of this study needed a longitudinal and an intervention study to test a theoretical model that aims to assess the foundations for learning fractions, because each design has a weakness in claiming to be the foundation. The longitudinal study demonstrated that quantitative reasoning is the stronger predictor of children's learning of fractions, but it could be possible that there was an unknown latent variable not measured which was driving the learning. An intervention study provides evidence that the longitudinal study cannot, which is the specific cause–effect relationship (Bradley and Bryant, 1983).

The third step of this research study was an intervention study. The intervention study had two aims: the first was to test the specific cause–effect relationship that quantitative reasoning and arithmetic knowledge, as products of an intervention, may have on children's learning of fractions.

Three groups of children in Year 4 participated in the intervention study; children within each class of their school were randomly assigned to one of the three groups. Two groups received an intervention study designed to improve their quantitative reasoning or their arithmetic knowledge; the third group participated only in the testing, acting as a control group. This study found that an intervention on quantitative reasoning about fractions had an effect on children's learning of fractions at an immediate post-test and at a delayed post-test, taken six weeks after the immediate post-test, statistically differing outperforming the control group. The arithmetic intervention only had a significant impact at the delayed post-test. This evidence supports the third hypothesis of this thesis, confirming that quantitative reasoning is a foundational factor for children's learning of fractions. This finding expands previous knowledge on the effects of interventions that aim at teaching children quantitative reasoning. Nunes and her colleagues found that interventions that aimed at improving quantitative reasoning had strong effects on children's mathematical attainment with whole numbers (Nunes et al., 2007; Nunes, Bryant, Evans, Bell, & Barros, 2012b). This study has expanded this knowledge to fractions and to assessing the difference on the effects of an intervention on quantitative reasoning and on arithmetic knowledge.

### 6.1.4 Differences in non-verbal ability moderate the impact of an intervention

The second aim of the intervention study was to test the effect that individual differences in non-verbal ability may have on the impact of the interventions.

Cognitive abilities have long been proposed in the psychology literature as a basis for individual differences in learning, and thus results in different levels of impact of any intervention that aims to develop learning and understanding as a function of the level of cognitive ability. However, very few studies have aimed to study this kind of interaction.

Results from the effect of a moderator term of intervention group by non-verbal ability, assessing the impact that the interventions had on children's scores on the fractions knowledge test taken immediately after they finished the intervention or six weeks after, showed no significant interaction in the immediate post-test; there was no significant interaction. In the delayed post-test, taken six weeks after the immediate post-test, the interaction was a positive one, confirming our hypothesis that non-verbal ability plays a role in the effect of an intervention, favouring the children with higher non-verbal ability. Children with higher scores on non-verbal ability will benefit more from these interventions on fractions in the long term.

It seems reasonable to think that children with low non-verbal ability need more time to benefit from the intervention. The intervention programme took 8 weeks; this time may have been too short for the children in the lower ability group to benefit as substantially from the intervention. Other studies with students at risk in mathematics tend to have longer intervention periods: for example Fuchs and colleagues (2014) designed a 12-week intervention, with a weekly half-hour session.

This finding expands knowledge on how general cognitive abilities may moderate the effects of an intervention to improve fractions knowledge. Fuchs and colleagues (2014) found that individual differences in working memory moderate fractions interventions. This study found that individual differences in non-verbal ability moderate a fractions intervention in the long term.

## 6.2 Implications for teaching and learning

This study addressed two basic distinctions in the teaching and learning of mathematics in primary school. One of these is the distinction between whole and rational numbers, the difficulties of learning fractions; and the other is that between two

mathematical abilities, quantitative reasoning and arithmetic knowledge. This section of the concluding chapter focuses first on what was learned in this study about fractions, and then on what was learned about the two mathematical abilities.

### 6.2.1 Difficulties of learning fractions

There is agreement in mathematics education that teaching children about fractions has led to disappointing results in many Western countries, to put it mildly (Behr, Lesh, Post, and Silver, 1983; Kerslake, 1986; Siegler, Fazio, Bailey, and Zhou, 2013). In the UK, Kerslake (1986) concluded that children find it difficult to relate their knowledge of equivalent fractions to addition of fractions. In the US, the Rational Number Project found similarly that some children struggled to understand equivalence of fractions even after instruction (Behr, Harel, Post, and Lesh, 1992; Behr, Lesh, Post, and Silver, 1983).

Fractions are a great challenge for children, and although fractions are part of our everyday lives, some children go through school without mastering the difficulties they present (Nunes and Bryant, 1996). Different reasons why children may find it difficult have been studied. The main difficulty may be because fractions are a new type of number, whose origins for understanding are different from the origins of whole numbers. Whole numbers come from additive relations, by putting things together and taking them apart (Nunes and Bryant, 1996); fractions come from quotients (Kieren, 1993), from dividing. Children learn about whole numbers from a very young age: they learn to count, to represent those numbers and to operate with them; relations between whole numbers are based on addition and subtraction, 2=1+1 or 3=2+1 and so on. Whole numbers present different properties, which are the only rules that children have encountered until they are faced with fractions. Fractions have a different origin: they come from division, and present different properties to whole numbers.

Some of the differences are: the order of fractions depends on the relation between numerator and denominator; different fractions may represent equivalent quantities and the same fraction may represent different quantities as they might refer to different wholes; the number of fractions between fractions or between two whole numbers is infinite, whereas there are no natural numbers between two whole numbers. The equivalence of fractions is a difficult concept for children to grasp, and different studies have found that children find this very challenging (Behr et al., 1984; Kerslake, 1986), although necessary to understand and operate with fractions.

Studies on misconception of fractions and whole number bias found that children, adults and even expert adults such as mathematicians have a tendency to use natural number properties in tasks with rational numbers, including in cases where this is inappropriate (Vamvakoussi et al., 2012; Vamvakoussi and Vosniadou, 2010; Van Hoof, Verschaffel, Ghesquière, and Van Dooren, 2017). Normally developing children tend to operate with fractions as they do with whole numbers; but children with maths difficulties present an even lower understanding of rational numbers (Van Hoof et al., 2017); if this is because they apply what is known to them, it is reasonable to consider that known concrete situations in which fractions are required would provide better results to introducing fractions.

In multiplicative reasoning, relations between quantities are based on one-to-many correspondences and proportional relations (Nunes and Bryant, 2015; Nunes et al., 2007; Vergnaud, 1988). This kind of situation demands understanding a ratio between two quantities. Children can reason about quantities smaller than a unit, contextualising fractions and providing a basis to use and operate with a new number.

The great mathematician and mathematics educator Hans Freudenthal suggested that, even though arithmetic is not easy, most children leave school with a reasonable knowledge of arithmetic. In contrast, this is not so when it comes to fractions and he asked why. This was his conjecture:

> In its traditional teaching [i.e. teaching of fractions] the concrete context is no more than a ceremony which is hurried through in a jiffy. If afterwards the abstract theory of fractions has to be applied, it comes too late, on too high a level, and is not connected to any previous experience on a level where fractions should have been introduced. What is the reason for this change of attitude of the teacher [who spent years giving examples of arithmetic with whole numbers]? Is the patience of the schoolmaster exhausted when fractions turn up? I believe the answer is rather that the schoolmaster himself does not know fractions in a concrete context, and that for this reason he is not able to teach them in a more responsible way than he is used to do. (Freudenthal, 1968, p. 4)

Freudenthal's conjecture is not just his explanation for why the concrete context for fractions is "hurried through", but also a challenge for those who argue, as I do, that reasoning about quantities is essential for children to understand the referential as well as the analytical meaning of numbers.

In order to design the interventions, it was necessary to consider the connections between fractions and different contexts and how children could develop their action schemas in preparation for learning about fractions as numbers.

The quantitative reasoning was based first on understanding that a fraction was a new number, then analysing the relations between quantities smaller than a unit. For example, children explored and discussed why it is not possible to count a quantity smaller

than a unit as a whole if it is smaller than one. Another important aspect that encompasses the fraction concept which was discussed was to establish that a fraction means that the parts are equal, uneven parts of a whole are not a fraction, but at the same time a half could represent different quantities, for example half of a small pie or half of a big pie and still be a half. All this seems obvious to an adult, but not necessarily to a child who has only encountered whole numbers and their properties. Children thought about and discussed different relations between quantities in concrete situations. Another example of a situation presented was children sharing the biscuits in a package: sometimes they knew how many biscuits each child received and had to find the total number of biscuits in the package; or sometimes they knew the total number of biscuits and had to find how many each child received. This kind of discussion made them consider the relation between quantities, in a direct relation and inverse relation. They also shared pizzas and bars of chocolates. It is important to note that they were not sharing actual food; this was discussed and showed in PowerPoint. In different situations they were asked to imagine that the teacher was going to share a chocolate bar through the week, or that the class visited a pizzeria and pizzas were not provided all at once so they had to share, and so on. All exercises in the intervention aiming at improving quantitative reasoning, referring to activities and concrete situations with quantities that the children were familiar with and required them to consider relations between the quantities, multiplicative relations.

The design of the arithmetic intervention aimed at improving understanding of how to operate with fractions. First a fraction was introduced as a new number and what the meaning of that number was: the result of a division. A connection to concrete and familiar situations was also considered and children discussed fractions on a number line, which most of them are familiar with; they reasoned about equivalence of fractions and rules on

how to operate with fractions, in which children used their previous knowledge of four arithmetic operations: multiplication, division, addition and subtraction. It is important to note that even if they are familiar with a number line, it is still an abstract concept; they do not encounter number lines naturally in their world.

Understanding fractions, being able to use the concept and to operate with fractions, is difficult for children: what is the foundation for mastering fractions or to make a smooth transition from whole numbers to fractions? Two types of mathematical ability are considered as foundations for learning about fractions.

### 6.2.2 Two types of mathematical knowledge

There is little doubt that the mathematics education literature recognises the need to distinguish two types of mathematical knowledge, but the best way to characterise this distinction is not yet quite clear. In curricula and textbooks there is often a distinction between teaching arithmetic and applying it, or teaching algebra and using it in applied situations. In England, the English National Curriculum makes a distinction by suggesting that "pupils should develop their ability to solve a wider range of problems, including increasingly complex properties of numbers and arithmetic, and problems demanding efficient written and mental methods of calculation. With this foundation in arithmetic, pupils are introduced to the language of algebra as a means for solving a variety of problems." (National Curriculum in England, 2013, p. 30). This distinction is stronger when it comes to Key Stage Mathematics assessments, which reflect the English National Curriculum. Two tests are designed, one that assesses arithmetic and one that assesses reasoning (Standard and Testing Agency, 2016; Standard and Testing Agency, 2019). The arithmetic test items include different sums, for example find the missing addend or the result of a multiplication. The reasoning assessment includes contextualised items. This distinction is not different in other

countries; a report by the OECD (2016) about PISA inequalities describes the difference between learning and applying as students being asked to mechanically learn simple mathematical procedures and then be given practical problems to apply these mathematical procedures.

The difference between teaching arithmetic and applying it has been summarised under different names. For example, Freudenthal above: defines this difference as abstract and concrete. This distinction would be embraced by most mathematics teachers as teaching a general abstract concept from several more concrete and realistic examples. Others have described this difference as procedural and conceptual knowledge: referring to teaching the procedures for solving a sum or understanding the concepts of fractions, such as ordering fractions in order of magnitude. This thesis proposes the distinction as quantitative reasoning versus arithmetic knowledge.

These distinctions do not necessarily replace each other, as they are not quite the same. The boundaries are not the same: for example, procedural and conceptual knowledge are differentiated by procedural referring to the steps taken to answer a sum following arithmetical rules, while conceptual refers to mental models containing mathematical meaning and relational representations.In this thesis, the distinction between quantitative reasoning and arithmetic knowledge is proposed: both are correlated and both include conceptual and procedural aspects to them; children may use reasoning in both, either to find out about a relation between quantities which was unknown, or reasoning to solve an arithmetical sum, but these abilities differ in their number meaning.

Quantitative reasoning refers to quantities which, as stated by Thompson (1993), are more concrete. One can consider and reason about quantities without numbers; only when quantities are measured do they have a numerical value. This number meaning is

representing a quantity, referring to the representational meaning of a number. However, numbers don't need a quantity to exist. Numbers are an impressive cultural tool that help us model the world; but they have meaning on their own, defined by the numerical system. Arithmetic as defined by Guedj (1997) is the science of numbers, in their analytical meaning and how they behave in four operations: addition, subtraction, multiplication and division.

In this sense, although it was proposed to reformulate the distinction of conceptual knowledge and procedural knowledge to quantitative reasoning and arithmetic knowledge, this study acknowledged that conceptual knowledge relies more on quantitative reasoning and procedural knowledge relies more on arithmetic skills. The findings of this study are in agreement with those that aimed to distinguish between conceptual and procedural knowledge, in three types of studies: (a) studies that explored clusters with residuals of each kind of knowledge that is not shared with the other, providing evidence that there is unique variance not shared between these two types of knowledge; (b) studies that aimed to find independent contribution from these two types of knowledge, showing that they could provide independent variance; and (c) studies that aimed to test the distinction by developing measures and analysing their factor structure using confirmatory factor analyses.

Results from this study are in agreement with two studies that explored clusters of types of knowledge in the fractions domain: Hallett et al. (2010) and Hecht and Vagi (2012), who found that conceptual and procedural knowledge show independent residual scores, evidencing that they are two different abilities. Then the residuals supported different clusters, showing that children may have a combination of skills, for example, high conceptual and low procedural or high in both. These studies show evidence to

believe that conceptual and procedural knowledge are distinct abilities; quantitative reasoning and arithmetic knowledge are also empirically different.

Findings are also aligned with results presented by the longitudinal study carried out by Nunes and her colleagues (Nunes et al., 2012b), who aimed to find the independent contribution of quantitative reasoning and arithmetic skills to mathematical knowledge. They found that quantitative reasoning and arithmetic skills can provide independent contributions to mathematical achievement. If they can provide independent contributions after controlling for each other, it suggests that these abilities are different from each other.

Lenz and colleagues (2019) aimed to develop two different measures which aimed to assess conceptual and procedural knowledge. They carried out confirmatory factor analyses which showed that conceptual and procedural knowledge are distinct abilities. The present study replicated the results, but specifically with quantitative reasoning and arithmetic knowledge.

This distinction is of great importance for educational decisions. Even though quantitative reasoning and arithmetic knowledge are related, it is possible to measure them separately by using distinct sets of items. This allowed us to establish that both mathematical abilities could make independent contributions to children's mathematical learning and to test that one could be of greater impact. Quantitative reasoning showed itself a better predictor of fractions learning, even after controlling for non-verbal ability and arithmetic knowledge. An intervention on quantitative reasoning provided better results in a fractions test compared to an arithmetic intervention or a control group, thus giving evidence that quantitative reasoning is the foundation factor for learning fractions.

## 6.3 Limitations and future research

The main strength of this research is the use of a combination of methods, which allows us to make strong conclusions providing secure evidence. The evidence provided helps in gaining understanding of the factors that predict children's learning of fractions longitudinally and the impact of an intervention study. This also has important implications for teaching and learning fractions.

This research has three main limitations. The first limitation is the relatively small sample in the studies. However, power calculations were carried out and there is no reason to believe that a larger sample would yield different results, though studies with larger samples would provide stronger evidence. In this study the combination of research methods also allowed us to make strong conclusions.

Another limitation is that the intervention study training was carried out by a researcher and not by a teacher within a class; the children were taken outside the classroom in small groups. Teachers know their students better though, and including activities that aim to promote quantitative reasoning in the class should yield even better results. Some may argue that teachers prefer to teach arithmetic (Brown, 1981), although once teachers are familiar with the concepts of quantitative reasoning and activities that promote understanding of multiplicative relations, incorporating them in their maths class should not be time-consuming. Stokes et al. (2018) showed that teachers can positively impact students' scores using quantitative reasoning interventions.

The third limitation, which is also a message for educators, is based on the results of the moderation of non-verbal ability with an intervention. It seems that children with high non-verbal ability were the ones who benefited the most from the interventions, which implies that educators should consider that children with initial low non-verbal

abilities would need more time to reach the same competence in fractions as their peers with a higher level of non-verbal abilities. This could be because the number of sessions of intervention was not enough for the children with lower non-verbal ability to benefit; it would be important to consider why this was the case, and how this could be adjusted in order to benefit all children, closing the gap between children with lower non-verbal ability and children with higher non-verbal ability. However, generally, all children benefited from the interventions, although it was the quantitative reasoning intervention which showed a higher impact on fractions learning.

A consideration for future research is that a combination of both interventions against either a control group or quantitative reasoning alone may provide even better results. Thus, a further research question is: if we use a combination of teaching quantitative reasoning and arithmetic knowledge, would it be better than just quantitative reasoning?

### 6.4 Concluding remarks

This research study tested a theoretical model in which two mathematical abilities were proposed as important factors for learning fractions.

First, it was established that these two mathematical abilities can be measured and are not just theoretically different, but also empirically different abilities. Second, quantitative reasoning and arithmetic knowledge are longitudinal and independent predictors of children's fractions achievement, after controlling for non-verbal ability. Quantitative reasoning is the strongest predictor. Third, results from this study showed that the impact of a quantitative reasoning intervention, compared to the impact of an arithmetic intervention, to a control group yield the children to better results in a fractions test. Generally, all children in the quantitative reasoning intervention group improved, but

children with higher non-verbal ability took more advantage of the intervention in the long term, as assessed by the delayed post-test.

Finally, these results together show that quantitative reasoning acts as a foundation factor for learning about fractions. The implication is that educators must aim explicitly to include quantitative reasoning in their mathematics teaching.

# 7. References


Bailey, D. H., Siegler, R. S., and Geary, D. C. (2014). Early predictors of middle school fraction knowledge. *Developmental science*, 17(5), 775-785.

Ball, D. L. (1990). Prospective elementary and secondary teachers' understanding of division. *Journal for Research in Mathematics Education*, 132-144.

Baron, R. M. and Kenny, D. A. (1986). The moderator–mediator variable distinction in social psychological research: Conceptual, strategic, and statistical considerations. *Journal of personality and social psychology*, 51(6), 1173.

Barros, R. (2007). *Individual differences between chronological age and mathematical ability.* Master of Science Research, Oxford Brookes University.

Behr, M. J., Lesh, R., Post, T., and Silver, E. A. (1983). Rational number concepts. *Acquisition of mathematics concepts and processes*, 91-126.

Behr, M. J., Wachsmuth, I., Post, T. R., and Lesh, R. (1984). Order and Equivalence of Rational Numbers: A Clinical Teaching Experiment. *Journal for Research in Mathematics Education*, 15(5), 323-341. doi: 10.2307/748423

Bland, J. M. and Altman, D. G. (1997). Statistics notes: Cronbach's alpha. *British Medical Journal*, 314(7080), 572.

Booth, J. L. and Newton, K. J. (2012). Fractions: Could they really be the gatekeeper's doorman? *Contemporary Educational Psychology*, 37(4), 247-253.

Bradley, L. and Bryant, P. E. (1983). Categorizing sounds and learning to read: A causal connection. *Nature*, 419-421.

Brown, M. (1981). Number concepts. In K. M. Hart (ed.), *Children's understanding of mathematics: 11-16* (pp. 23-47). Windsor, UK: NFER-Nelson.

Bump, W. M. (1992). The Homogeneity of Regression Assumption in the Analysis of Covariance.

Byrne, B. M. (1994). *Structural equation modeling with EQS and EQS/Windows: Basic concepts, applications, and programming*: Sage.

Byrne, B. M. (2001). Structural equation modeling: Perspectives on the present and the future. *International Journal of Testing*, 1(3-4), 327-334.

Campos, T. M. M. and Silva, A. da F. G. (2009). Conhecimento profissional docente de professoras das séries iniciais da educação básica acerca da equivalência de números racionais na representação fracionária em um processo de formação continuada. *Revemat: revista eletrônica de educação matemática*, 4(1), 114-127.

Cartwright, N. (2009). What are randomised controlled trials good for? *Philosophical Studies*, 147(1), 59. doi: 10.1007/s11098-009-9450-2

Clausem-May, T., Vappula, H., and Ruddock, G. (2004). *Progress in Maths 10*. London: NFER-Nelson.

Cohen, J. (1988). *Statistical power analysis for the behavioural sciences* (2nd edn). New York: Academic Press.

Confrey, J. (2008). *A Synthesis of the Research on Rational Number Reasoning: A Learning Progressions Approach to Synthesis.* Paper presented at the 11th International Congress of Mathematics Instruction, Monterrey, Mexico.

Confrey, J. and Maloney, A. (2010). *The construction, refinement, and early validation of the equipartitioning learning trajectory*. Paper presented at the Proceedings of the 9th International Conference of the Learning Sciences. Vol. 1, Chicago, Illinois.

Cramer, K., Behr, M., Post, T.R, and Lesh, R. (2009). *Rational Number Project: Initial Fraction Ideas*. Kendall Hunt Publishing Co.

Cramer, K., Monson, D., Ahrendt, S., Wyberg, T., Pettis, C., and Fagerlund, C. (2019). Reconstructing the unit on the number line: Tasks to extend fourth graders' fraction understandings. *Investigations in Mathematics Learning*, 11(3), 180-194.

Cronbach, L. J. (1951). Coefficient alpha and the internal structure of tests. *Psychometrika*, 16(3), 297-334.

Cronbach, L. J. (1957). The two disciplines of scientific psychology. *American psychologist*, 12(11), 671.

De Smedt, B., Janssen, R., Bouwens, K., Verschaffel, L., Boets, B., and Ghesquière, P. (2009). Working memory and individual differences in mathematics achievement: A longitudinal study from first grade to second grade. *Journal of experimental child psychology*, 103(2), 186-201. doi: https://doi.org/10.1016/j.jecp.2009.01.004

Department for Education (2014). *The national curriculum in England: complete framework for key stages 1 to 4*. Available at: https://www.gov.uk/government/publications/national-curriculum-in-england-framework-for-key-stages-1-to-4 (Accessed 11 March 2019).

Faul, F., Erdfelder, E., Buchner, A., and Lang, A.-G. (2009). Statistical power analyses using G* Power 3.1: Tests for correlation and regression analyses. *Behavior research methods*, 41(4), 1149-1160.

Freudenthal, H. (1968). Why to teach mathematics so as to be useful. *Educational Studies in Mathematics*, 1(1), 3-8. doi: 10.1007/BF00426224

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Hamlett, C. L., Cirino, P. T., . . . Changas, P. (2013). Improving at-risk learners' understanding of fractions. *Journal of Educational Psychology*, 105(3), 683.

Fuchs, L. S., Schumacher, R. F., Long, J., Namkung, J., Malone, A. S., Wang, A., . . . Changas, P. (2016). Effects of intervention to improve at-risk fourth graders' understanding, calculations, and word problems with fractions. *Elementary School Journal*, 116(4), 625-651.

Fuchs, L. S., Schumacher, R. F., Sterba, S. K., Long, J., Namkung, J., Malone, A., . . . Siegler, R. S. (2014). Does working memory moderate the effects of fraction intervention? An aptitude–treatment interaction. *Journal of Educational Psychology*, 106(2), 499.

Geary, D. C. (2011). Cognitive predictors of achievement growth in mathematics: A 5-year longitudinal study. *Developmental Psychology*, 47(6), 1539.

Guedj, D. (1997). *Numbers: the universal language*. Harry N. Abrams.

Hallett, D., Nunes, T., and Bryant, P. (2010). Individual differences in conceptual and procedural knowledge when learning fractions. *Journal of Educational Psychology*, 102(2), 395.

Hallett, D., Nunes, T., Bryant, P., and Thorpe, C. M. (2012). Individual differences in conceptual and procedural fraction understanding: The role of abilities and school experience. *Journal of experimental child psychology*, 113(4), 469-486. doi: http://dx.doi.org/10.1016/j.jecp.2012.07.009

Hart, K. M., Brown, M., Kerslake, D., Kuchemann, D., and Ruddock, G. (1985). *Chelsea diagnostic mathematics tests*. NFER-Nelson.

Hattie, J. (2012). *Visible learning for teachers: Maximizing impact on learning*. Routledge.

Hecht, S. A. (1998). Toward an information-processing account of individual differences in fraction skills. *Journal of Educational Psychology, 90*(3), 545.

Hecht, S. A., Close, L., and Santisi, M. (2003). Sources of individual differences in fraction skills. *Journal of experimental child psychology, 86*(4), 277-302.

Hecht, S. A., and Vagi, K. J. (2012). Patterns of strengths and weaknesses in children's knowledge about fractions. *Journal of experimental child psychology, 111*(2), 212-229.

Howe, C., Nunes, T., and Bryant, P. (2011). Rational number and proportional reasoning: Using intensive quantities to promote achievement in mathematics and science. *International Journal of Science and Mathematics Education, 9*(2), 391-417.

Inhelder, B., and Piaget, J. (1958). The growth of logical thinking, New York (Basic Books) 1958.

Jastak, S., and Wilkinson, G. S. (1984). *The Wide Range Achievement Test-Revised: WRAT-R*: Jastak Associates.

Jastrzębski, J., Ciechanowska, I., and Chuderski, A. (2018). The strong link between fluid intelligence and working memory cannot be explained away by strategy use. *Intelligence, 66*, 44-53.

Jordan, N. C., Hansen, N., Fuchs, L. S., Siegler, R. S., Gersten, R., and Micklos, D. (2013). Developmental predictors of fraction concepts and procedures. *Journal of experimental child psychology, 116*(1), 45-58.

Kerslake, D. (1986). *Fractions: Children's strategies and errors*. Slough, UK: NFER-Nelson.

Kieren, T. E. (1976). 'On the mathematical, cognitive and instructional foundations of rational numbers', in Richard A. Lesh (ed.), *Number and Measurement: Papers from a Research Workshop*, ERIC/SMEAC. Columbus, pp. 101–144.

Kieren, T. E. (1993). Rational and fractional numbers: From quotient fields to recursive understanding. *Rational numbers: An integration of research*, 49-84.

King, J. P. (2010). *Mathematics in 10 lessons: The grand tour*. Prometheus Books.

Krowka, S. K. and Fuchs, L. S. (2017). Cognitive Profiles Associated with Responsiveness to Fraction Intervention. *Learning Disabilities Research and Practice*, 32(4), 216-230. doi: 10.1111/ldrp.12146

Kyttälä, M. and Lehto, J. E. (2008). Some factors underlying mathematical performance: The role of visuospatial working memory and non-verbal intelligence. *European Journal of Psychology of Education*, 23(1), 77-94.

Mack, N. K. (1995). Confounding whole-number and fraction concepts when building on informal knowledge. *Journal for Research in Mathematics Education*, 422-441.

Mamede, E. (2015). *Exploring young children's reasoning and naming of fractions.* Paper presented at the British Society for Research into Learning Mathematics 35, Reading, UK.

Mamede, E. and Nunes, T. (2008). *Building on children's informal knowledge in the teaching of fractions.* Paper presented at the Proceedings of the 32nd Conference of the International Group for the Psychology of Mathematics Education and 30th North American Chapter of the International Group for the Psychology of Mathematics Education.

Namkung, J. M. and Fuchs, L. S. (2016). Cognitive predictors of calculations and number line estimation with whole numbers and fractions among at-risk students. *Journal of Educational Psychology*, 108(2), 214.

Nillas, L. (2003). Division of fractions: Preservice teachers' understanding and use of problem-solving strategies. *Mathematics Educator*, 7(2), 96-113.

Nunes, T. (2008). *Understanding rational numbers.* Proceedings of the Conference of European Association for Research on Learning and Instruction–EARLI.

Nunes, T. (2019). *Procedural and conceptual knowledge and the Ornithorhynchus paradox*. Paper presented at the EARLI Conference, Aachen, Germany.
Nunes, T. and Bryant, P. (1996). *Children doing mathematics*. Wiley-Blackwell.
Nunes, T. and Bryant, P. (1997). *Learning and teaching mathematics: An international perspective*. Psychology Press.
Nunes, T. and Bryant, P. (2008). Rational numbers and intensive quantities: Challenges and insights to pupils implicit knowledge. *Anales de psicología*, 24(2), 262-270.
Nunes, T. and Bryant, P. (2009a). Children's understanding of fractions. *Revista Contrapontos*, 8(3), 509-517.
Nunes, T. and Bryant, P. (2009b). Paper 3: Understanding rational numbers and intensive quantities. *Key understandings in mathematics learning*. Nuffield Foundation.
Nunes, T. and Bryant, P. (2015). The Development of Mathematical Reasoning. *Handbook of Child Psychology and Developmental Science*. John Wiley and Sons, Inc.
Nunes, T., Bryant, P., Barros, R., and Sylva, K. (2012a). The relative importance of two different mathematical abilities to mathematical achievement. *British Journal of Educational Psychology*, 82(1), 136-156.
Nunes, T., Bryant, P., Evans, D., and Barros, R. (2015). Assessing Quantitative Reasoning in Young Children. *Mathematical Thinking and Learning*, 17(2-3), 178-196.
Nunes, T., Bryant, P., Evans, D., Bell, D., & Barros, R. (2012b). Teaching children how to include the inversion principle in their reasoning about quantitative relations. *Educational Studies in Mathematics, 79*(3), 371-388. doi: 10.1007/s10649-011-9314-5
Nunes, T., Bryant, P., Evans, D., Bell, D., Gardner, S., Gardner, A., and Carraher, J. (2007). The contribution of logical reasoning to the learning of mathematics in primary school. *British Journal of Developmental Psychology*, 25(1), 147-166.
Nunes, T., Bryant, P., Pretzlik, U., Bell, D., Evans, D., and Wade, J. (2007). La compréhension des fractions chez les enfants. *Activité humaine et conceptualisation. Questions à Gérard Vergnaud*, 255-262.
Nunes, T., Bryant, P., Pretzlik, U., and Hurry, J. (2006). Fractions: difficult but crucial in mathematics learning. *Teaching and Learning Research Programme (TLRP) Research Briefing*.
Nunes, T., Desli, D., and Bell, D. (2003). The development of children's understanding of intensive quantities. *International Journal of Educational Research*, 39(7), 651-675. doi: http://dx.doi.org/10.1016/j.ijer.2004.10.002
Obersteiner, A., Van Dooren, W., Van Hoof, J., and Verschaffel, L. (2013). The natural number bias and magnitude representation in fraction comparison by expert mathematicians. *Learning and Instruction*, 28, 64-72.
Philpott, S. (1936). The Meaning of G and other Factors in Terms of Rates of Output. *British Journal of Psychology*, 27(2), 196.
Piaget, J. (1952). The child's concept of number. New York.
Piaget, J., Inhelder, B., and Szeminska, A. (1960). The child's conception of geometry. New York: Harper and Row.
Post, T. R., Wachsmuth, I., Lesh, R., and Behr, M. J. (1985). Order and equivalence of rational numbers: A cognitive analysis. *Journal for Research in Mathematics Education*, 18-36.
Raven, J. (2000). The Raven's Progressive Matrices: change and stability over culture and time. *Cognitive psychology*, 41(1), 1-48.

Raven, J. C., Court, J. H., and Raven, J. (1996). Raven's Standard Progressive Matrices and Vocabulary Scales. London: H.K. Lewis and Co. Ltd.

Rittle-Johnson, B., Siegler, R. S., and Alibali, M. W. (2001). Developing conceptual understanding and procedural skill in mathematics: An iterative process. *Journal of Educational Psychology*, 93(2), 346.

Robertson, G. J. (2010). Wide-Range Achievement Test. *Corsini Encyclopedia of Psychology*, 1-2.

Saxe, G. B., Diakow, R., and Gearhart, M. (2013). Towards curricular coherence in integers and fractions: A study of the efficacy of a lesson sequence that uses the number line as the principal representational context. *ZDM*, 45(3), 343-364.

Saxe, G. B., Shaughnessy, M. M., Shannon, A., Langer-Osuna, J. M., Chinn, R., and Gearhart, M. (2007). Learning about fractions as points on a number line. *The learning of mathematics: Sixty-ninth yearbook*, 221-237.

Schulz, K. F. and Grimes, D. A. (2002). Sample size slippages in randomised trials: exclusions and the lost and wayward. *The Lancet*, 359(9308), 781-785.

Siegler, R. S., Fazio, L. K., Bailey, D. H., and Zhou, X. (2013). Fractions: The new frontier for theories of numerical development. *Trends in cognitive sciences*, 17(1), 13-19.

Siegler, R. S. and Pyke, A. A. (2013). Developmental and individual differences in understanding of fractions. *Developmental Psychology*, 49(10), 1994.

Siegler, R. S., Thompson, C. A., and Schneider, M. (2011). An integrated theory of whole number and fractions development. *Cognitive psychology*, 62(4), 273-296.

Simon, M. A. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, 233-254.

Star, J. R., and Stylianides, G. J. (2013). Procedural and conceptual knowledge: Exploring the gap between knowledge type and knowledge quality. *Canadian Journal of Science, Mathematics and Technology Education*, 13(2), 169-181.

Standards and Testing Agency (2016). Key Stage 1 Mathematics: sample test materials, mark schemes and test administration instructions. Department for Education. Retrieved Feb. 2017 from https://www.gov.uk/government/publications/2016-key-stage-1-mathematics-sample-test-materials-mark-schemes-and-test-administration-instructions

Steffe, L. P. (1994). Children's multiplying schemes. *The development of multiplicative reasoning in the learning of mathematics*, 1, 3-39.

Steffe, L. P. (2001). A new hypothesis concerning children's fractional knowledge. *Journal of Mathematical Behavior*, 20(3), 267-307.

Stokes, L., Hudson-Sharp, N., Dorsett, R., Rolfe, H., Anders, J., George, A., Buzzeo, J., and Munro-Lott, N. (2018). Mathematical Reasoning: Evaluation report and executive summary. Education Endowment Foundation. Retrieved 3 November 2020 from http://educationendowmentfoundation.org.uk/projects-and-evaluation/projects/mathematical-reasoning/

Strand, S. (2004). Consistency in reasoning test scores over time. *British Journal of Educational Psychology*, 74(4), 617-631.

Streefland, L. (1993). Fractions: A realistic approach. *Rational numbers: An integration of research*, 289-325.

Streefland, L. (1997). Charming fractions or fractions being charmed. *Learning and Teaching Mathematics–An International Perspective*, 347-372.

Tavakol, M. and Dennick, R. (2011). Making sense of Cronbach's alpha. *International journal of medical education*, 2, 53.

Thompson, P. W. (1993). Quantitative reasoning, complexity, and additive structures. *Educational Studies in Mathematics*, 25(3), 165-208.

Thompson, P. W. and Saldanha, L. A. (2003). Fractions and multiplicative reasoning. In W. G. M. Jeremy Kilpatrick, Deborah Schifter (ed.), *A Research Companion to Principles and Standards for School Mathematics*, pp. 95-113. National Council of Teachers of Mathematics.

Tolman, R. C. (1917). The measurable quantities of physics. *Physical Review*, 9(3), 237-253.

Torbeyns, J., Schneider, M., Xin, Z., and Siegler, R. S. (2015). Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents. *Learning and Instruction*, 37, 5-13. doi: https://doi.org/10.1016/j.learninstruc.2014.03.002

Vamvakoussi, X., Van Dooren, W., and Verschaffel, L. (2012). Naturally biased? In search of reaction time evidence for a natural number bias in adults. *Journal of Mathematical Behavior*, 31(3), 344-355.

Vamvakoussi, X. and Vosniadou, S. (2010). How many decimals are there between two fractions? Aspects of secondary school students' understanding of rational numbers and their notation. *Cognition and Instruction, 28*(2), 181-209.

Van Hoof, J., Lijnen, T., Verschaffel, L., and Van Dooren, W. (2013). Are secondary school students still hampered by the natural number bias? A reaction time study on fraction comparison tasks. *Research in Mathematics Education*, 15(2), 154-164.

Van Hoof, J., Verschaffel, L., Ghesquière, P., and Van Dooren, W. (2017). The natural number bias and its role in rational number understanding in children with dyscalculia. Delay or deficit? *Research in Developmental Disabilities*, 71, 181-190.

Vergnaud, G. (1979). The acquisition of arithmetical concepts. *Educational Studies in Mathematics*, 10(2), 263-274.

Vergnaud, G. (1988). Multiplicative structures. *Number concepts and operations in the middle grades*, Vol. 2, pp. 141-151.

Vergnaud, G. (1997). The nature of mathematical concepts. In T. Nunes and P. Bryant (eds.), *Learning and teaching mathematics: An international perspective* (pp. 5-28). Hove, UK: Psychology Press.

Vergnaud, G. (2009). The Theory of Conceptual Fields. *Human Development*, 52(2), 83-94. doi: 10.1159/000202727

Vukovic, R. K., Fuchs, L. S., Geary, D. C., Jordan, N. C., Gersten, R., and Siegler, R. S. (2014). Sources of individual differences in children's understanding of fractions. *Child development*, 85(4), 1461-1476.

Vygotsky, L. S. (1978). *Mind in Society: Development of Higher Psychological Processes*. Cambridge, MA: Harvard University Press.

Wechsler, D. (1991). *WISC-III: Wechsler intelligence scale for children: Manual*. Psychological Corporation.

# Appendices

**Appendix 1: CUREC Application**

**University of Oxford**

**CENTRAL UNIVERSITY RESEARCH ETHICS COMMITTEE (CUREC)**

CUREC/1A CHECKLIST FOR THE SOCIAL SCIENCES AND HUMANITIES

The University of Oxford places a high value on the knowledge, expertise, and integrity of its members and their ability to conduct research to high standards of scholarship and ethics. The research ethics clearance procedures have been established to ensure that the University is meeting its obligations as a responsible institution. They start from the presumption that all members of the University will take their responsibilities and obligations seriously and will ensure that their research on human subjects is conducted according to the established principles and good practice in their fields and in accordance, where appropriate, with legal requirements. Since the requirements of research ethics review will vary from field to field and from project to project, the University accepts that different guidelines and procedures will be appropriate. Please check the CUREC website to ensure that you have the correct form for your project.

This form does not cover research governance, satisfactory methodology, or the health and safety of employees and students. As principal investigator, it is your responsibility to ensure that requirements in these areas are met. Please carry out a risk assessment of the project, in consultation with all researchers involved, using the checklist and CUREC's other documentation.

The use of an asterisk in this form indicates a phrase defined in the glossary. The glossary and further information on the University's research ethics procedures are available from the CUREC website:

**www.admin.ox.ac.uk/curec**

This form is designed largely for research that falls within the Divisions of Social Sciences and Humanities and which does not involve a high level of risk to the subjects. Elite interviews, fieldwork and oral history are included in the CUREC process. Please take a moment to read through it and if you have any questions or doubts as to whether it is the appropriate form, please review Section A or consult the CUREC website.

***Note on anonymised data:*** If you are using previously collected anonymised data about people which neither you nor anyone else involved in your study can trace back to the individuals who provided them (e.g. census data, administrative data, secondary analysis), you do not need to obtain ethical approval for your study. Please refer to the definition of *personal data in the glossary and FAQ no. 6 for further guidance.

***Note on audit:*** If you are conducting research on behalf of or at the request of a service provider that matches the definition of *audit in the glossary, you do not need to obtain ethical approval for your study.

**SECTION A**

| | **Yes** | **No** |
|---|---|---|
| ***1) Are you using research methodologies commonly used in biomedical or behavioural laboratory sciences?*** | | ✓ |
| ***2) Is there a significant risk that the research will induce anxiety, stress or other harmful psychological states in participants that might persist beyond the duration of any test or interview in which they are participating?*** | | ✓ |
| ***3) Will the research involve human participants recruited by means of their status as present or past NHS patients or their relatives or carers or present or past NHS staff?*** | | ✓ |
| ***4) Does the research involve *human participants aged 16 and over who do not have *capacity to consent for themselves? See the Mental Capacity Act 2005*** | | ✓ |
| ***5) Is the study to be funded by the US National Institutes of Health or another US federal funding agency?*** | | ✓ |

**Office use only: IDREC Ref. No. _____**

**Date of confirmation that checklist accepted on behalf of IDREC: // //**

If you answered 'yes', please stop work on this checklist and

- **for questions 1 and 2**, complete CUREC/1 instead (available from www.admin.ox.ac.uk/curec);
- **for questions 3 and 4**, submit your proposal to the appropriate NHS ethics committee (see www.nres.npsa.nhs.uk and www.admin.ox.ac.uk/rso/clinical for further information);
- **for question 5**, or **if you answered 'yes' to questions 1, 2 or 4 and your research will take place outside the EU**, submit your proposal to OXTREC, which uses separate documentation (www.tropicalmedicine.ox.ac.uk/oxtrec).

If you have answered 'no' to all questions in Section A, please complete Sections B-E. This form and any supporting materials should be typewritten.

**SECTION B**

| | |
|---|---|
| ***Principal investigator/ supervisor/student researcher (title and name):*** | Professor Nunes, Dr Lars-Erik Malmberg and Mrs Rossana Barros |
| ***Name of supervisor (STUDENT RESEARCH PROJECTS ONLY):*** | Nunes and Lars-Erik Malmberg |
| ***Department or institute:*** | Oxford University, Department of Education, 15 Norham Gardens, Oxford, OX2 6PY |
| ***Address for correspondence (if different):*** | |
| ***Email and phone contact:*** | Terezinha.nunes@education.ox.ac.uk<br>rossana.barros@education.ox.ac.uk |
| ***Title of research project:*** | Factors contributing to mathematical achievement: Reasoning and Arithmetic Knowledge in Fractions Skills |
| ***Brief description of research methods and goals plus description of the nature of participants (including the criteria for inclusion/exclusion, method of recruitment), explanation of how professional guidelines and/or CUREC protocol(s) will be applied (if relevant) and expected use to which the results/data will be put. Please describe how you will obtain informed consent. Approx 400 words.*** | |

This project aims to study the development of children's reasoning and arithmetical knowledge of fractions and their contribution to mathematical achievement using a combination of longitudinal and intervention methods. Fractions are important in education and daily life and prove to be challenging for children, who sometimes do not master the reasoning and underlying relations in fractions.

Our aim is to recruit 5 state primary schools in Oxfordshire and surrounding counties for each study. In agreement with the methods for recruiting participants (Section 4 P25): First, I will send an invitation letter to schools (see appendix 1 and 2) and with their approval I will send letters to guardians or parents of all children in Year 4 to obtain their permission, adhering to section 5 of P25 (see appendix 3 and 4); the letters will be sent through the teachers. All data collected will be anonymised and strictly confidential. Data collected will be kept in a computer with password protection to which only the research team (student and supervisors) will have access.

*The longitudinal study* will describe whether reasoning and arithmetical skills develop independently of each other or not, the effect of instruction and the relations between these skills and mathematical achievement. For the longitudinal study, children aged between 9 and 10 years will be recruited and tested at three times ($t_1$: June 2015, $t_2$: October 2015, $t_3$: July 2016) over 12 months. Children will be measured on mathematical reasoning and arithmetical skills in fractions on each testing occasion and on control measures. The Reasoning and Arithmetic skills of Fractions Test will be adapted from previous research on fractions (see appendix 1). It will be presented to the children on PowerPoint and on paper, on which they will write their answers. The researcher will give the instructions orally for each question. The task is composed of 40 questions; 40 sub items measure reasoning and 47 measure arithmetic skills. The control measures will be: at Time 1 (i) intelligence test (Raven's Progressive Matrices), at Time 2 (ii) arithmetical knowledge task (British Ability Scale: number skills subtest adapted) and at $t_3$ the standardised mathematical test, Progress in Maths 9 and 10 (NFER, 2004).

*The intervention study:* Children will be randomly assigned within each class to one of three groups: Reasoning intervention, Arithmetic intervention and an Unseen Control group. All children will be tested as a group at three points: pre, post and delayed post using the same measures as for the longitudinal study. The intervention will consist of 8 half-an-hour sessions, one per week, which will be given in small groups of 5 children between February and July 2016. Children's discussions of their solutions will be tape-recorded and transcripts will be identified by a number rather than children's names. Permission for tape-recording will be obtained from parents and children. Teachers will receive a questionnaire in which they will be asked about the mathematics topics taught during the intervention period to assess overlaps with the intervention. If the intervention proves to be effective, all teachers will be invited to a half-day workshop and will receive the materials in a CD along with a research brief. To maintain confidentiality it will not be possible for parents or teachers to have access to the results of individual children.

| | |
|---|---|
| ***List actual or probable location(s) where project will be conducted, if known:*** | 10 primary schools in Oxfordshire and surrounding counties |
| ***Anticipated duration of project:*** | 31 months |
| ***Anticipated start date:*** | 23 / 03 / 2015 |
| ***Anticipated end date:*** | 30 / 09 / 2017 |
| ***Name and status (e.g. 3rd year undergraduate; post-doctoral research assistant) of others taking part in the project:*** | |
| ***Please indicate what training on research ethics the researchers involved with this study have received, e.g. the title of the online or in-person course, and date completed (online training available at www.admin.ox.ac.uk/rso/integrity/#Training):*** | Research ethics training day for staff on 2 February 2011<br>Online course (currently taking) |

**SECTION C**

Methods to be used in the study (**tick** as many as apply: this information will help the committee understand the nature of your research and may be used for audit).

| | Plea |
|---|---|
| ***Interview*** | |
| ***Questionnaire*** | |
| ***Analysis of existing records*** | |
| ***Participant performs verbal/paper and pencil/computer based task*** | ✓ |
| ***Measurement/recording of motor behaviour*** | |
| ***Audio recording of participant*** | ✓ |
| ***Video recording or phtotography of participant*** | |
| ***Physiological recording from participant*** | |
| ***Participant observation*** | |
| ***Covert observation*** | |
| ***Systematic observation*** | |
| ***Observation of specific organisational practices*** | |
| ***Other (please specify)*** | |

**SECTION D**

Have you read one or more of the following professional guidelines and do you undertake to use the principles listed there as a guide for your own work? Please note that this is not intended to be an exhaustive list. Links to the guidelines listed below are included on the CUREC website.

| | Please tick |
|---|---|
| ***British Society of Criminology: Code of Ethics for Researchers in the Field of Criminology [britsoccrim.org/ethical.htm]*** | |
| ***British Educational Research Association Ethical Guidelines for Educational Research [www.bera.ac.uk/publications/guidelines/]*** | ✓ |
| ***Academy of Management's Professional Code of Ethics [www.aomonline.org]*** | |
| ***Association of American Geographers Statement on Professional Ethics [www.aag.org/Info/ethics.htm]*** | |
| ***Oral History Society of the UK Ethical Guidelines [www.oralhistory.org.uk/ethics/index.php]*** | |
| ***American Political Science Association (APSA) Guide to Professional Ethics in Political Science (Section H) [www.apsanet.org/content_9350.cfm]*** | |
| ***British Psychological Society Code of Ethics and Conduct [www.bps.org.uk/the-society/code-of-conduct/code-of-conduct_home.cfm]*** | ✓ |
| ***Ethics Guidelines of the Association of Social Anthropologists of the UK and Commonwealth [www.theasa.org/ethics/guidelines.htm]*** | |

| | |
|---|---|
| ***Social Research Association: Ethical Guidelines [www.the-sra.org.uk/ethical.htm]*** | |
| ***Statement of Principles of Ethical Research Practice from the Socio-Legal Studies Association [www.slsa.ac.uk]*** | |
| ***Statement of Ethical Practice for the British Sociological Association [www.britsoc.co.uk/equality/Statement+Ethical+Practice.htm]*** | |
| ***Other professional guidelines (please specify):*** | |

**SECTION E**

Please put a tick in the yes/no column as appropriate to indicate your response.

| | | |
|---|---|---|
| ***1) Will you obtain informed consent according to good practice in your discipline before participation?*** | **Yes** | **No** |
| | ✓ | |
| ***2) Will you ensure that *personal data collected directly from participants or via a *third party is held and processed in accordance with the provisions of the Data Protection Act?*** | **Yes** | **No** |
| | ✓ | |
| ***3) Does the research involve as participants *people whose ability to give free and informed consent is in question? (This includes those under 18 and vulnerable adults.)*** | **Yes** | **No** |
| | ✓ | |
| ***4) As a consequence of taking part in the research, will participants be at serious risk of rendering themselves liable to criminal prosecution (e.g. by providing information on drug abuse or child abuse)?*** | **Yes** | **No** |
| | | ✓ |
| ***5) Does the research involve the *deception of participants, as part of the investigation/experiment?*** | **Yes** | **No** |
| | | ✓ |
| If any of your answers above are in a shaded box, please indicate whether those aspects of your project are fully covered by the following. | | |
| ***6) Research protocol(s) which has/ve received IDREC/CUREC approval?*** | **Yes** | **No** |
| *If yes, please give protocol number(s): Non-invasive methods, children (Protocol 25)* | ✓ | |
| ***7) Professional guidelines that you will be following, as noted under Section D?*** | **Yes** | **No** |
| | ✓ | |

If any of your answers in Section E are in a shaded box and are not covered by a protocol or by professional guidelines, please complete CUREC/2, available to download from the CUREC website. Then submit both this form (you need not complete section F) and the CUREC/2 to the Social Sciences and Humanities IDREC.

If all your answers in Section E are in the unshaded boxes or your answers in shaded boxes are covered by a protocol or professional guidelines, complete Section F and submit this form and any accompanying documents to the Social Sciences and Humanities IDREC (see notes and address below).

**SECTION F**

**Complete this section only if you do not need to submit form CUREC/2.**

I understand my responsibilities as principal researcher/supervisor/student researcher as outlined in the CUREC glossary and guidance on the CUREC website.

I declare that the answers above accurately describe my research as presently designed and that I will submit a new checklist should the design of my research change in a way which would alter any of the above responses so as to require completion of CUREC/2 (involving full scrutiny by an IDREC). I will inform the relevant IDREC if I cease to be the principal researcher on this project and supply the name and contact details of my successor if appropriate.

**Signed by principal researcher/supervisor/student researcher:**
Nunes and Rossana Barros
**Date:** 06/03/2015
**Print name** (block capitals)… NUNES, LARS ERIC-MALMBERG and ROSSANA BARROS
**Signed by supervisor:**……(for student projects)
**Date:**……
**Print name** (block capitals)……

I understand the questions and answers that have been entered above describing the research, and I will ensure that my practice in this research complies with these answers, subject to any modifications made by the principal researcher properly authorised by the CUREC system.

**Signed by associate/other researcher:** …….

**Print name** (block capitals)……

**Date** ……

I have read the research project application named above. On the basis of the information available to me, I:

(i) consider the principal researcher/supervisor/student researcher to be aware of her/his ethical responsibilities in regard to this research;

(ii) consider that any ethical issues raised have been satisfactorily resolved or are covered by relevant professional guidelines and/or CUREC approved protocols, and that it is appropriate for the research to proceed without further formal ethical scrutiny at this stage (noting the principal researcher's obligation to report should the design of the research change in a way which would alter any of the above responses so as to require completion of a CUREC/2 full application);

(iii) am satisfied that the proposed project has been/will be subject to appropriate *peer review and is likely to contribute something useful to existing knowledge and/or to the education and training of the researcher(s) and that it is in the *public interest.

(iv) [FOR DEPARTMENTS/FACULTIES WITH A DEPARTMENTAL RESEARCH ETHICS COMMITTEE (DREC) OR EQUIVALENT BODY - PLEASE DELETE IF NOT APPLICABLE] confirm that this checklist (and associated research outline) has been reviewed by the Department's Research Ethics Committee (DREC)/equivalent body, and attach the associated report from that body.

**Signed:**……..
**(Head of department or nominee e.g Chair of DREC, Director of Graduate Studies for postgraduate student projects)**
**Print name** (block capitals)……
**Date:**……

**Please send an electronic copy and a signed paper copy of this completed checklist, together with any supporting documentation, to the following addresses, keeping a copy for yourself:**

Secretary of the Social Sciences and Humanities IDREC Email: ethics@socsci.ox.ac.uk
University of Oxford Social Sciences Division
Hayes House, George Street

Oxford, OX1 2BQ

**Forms may be sent by email (without signature), where both the note of submission from the researcher and the note of endorsement from the supervisor/Head of Department are sent from a University of Oxford email address.**

IDRECs and/or CUREC will review a sample of completed checklists and may ask for further details of any project.

**FINAL CHECK**

To prevent delay please check each of the following before submitting the application.

Have you completed Sections A-E? ☒

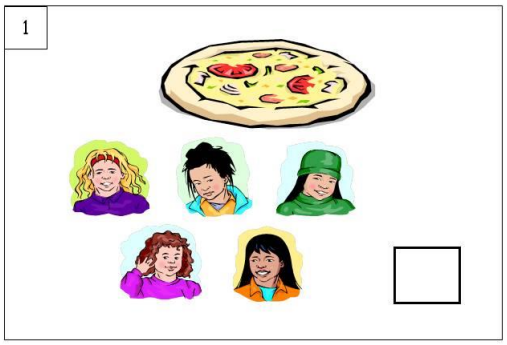
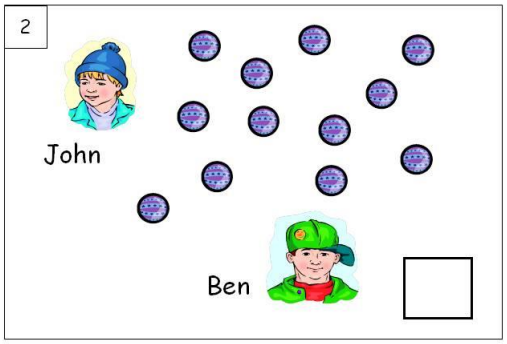
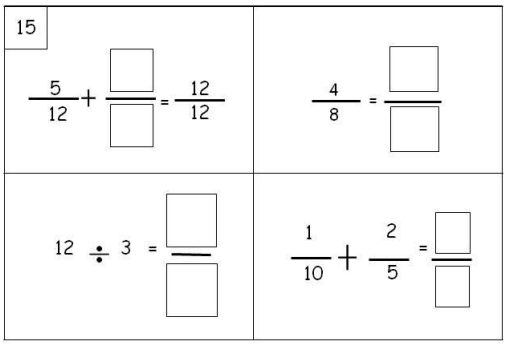
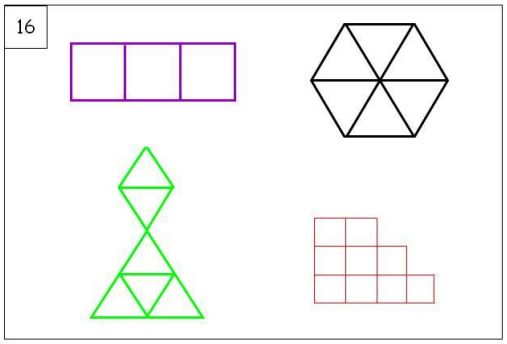
Have you defined all technical terms and abbreviations used? ☒

Have you included any supporting documentation, including – as appropriate – questionnaires and participant information, consent forms/form or note of procedure for recording oral consent, advertisements and surveys to be used? ☒

Are all pages (including appendices and attachments) numbered? ☒

Are all relevant declarations in Section F complete and any necessary authorisations obtained (by email or by signing the form)? ☒

**CUREC FORM** Appendix 1: Examples of Reasoning and Arithmetic Fraction test

| Children's booklet | Instructions from researcher: |
|---|---|
| <br><br>1 | (1)This pizza has to be shared equally between 5 girls. What fraction of the pizza does each girl get? Write your answer in the box.<br><br>(2)<br>a) John wins 1/3 of these marbles. Draw a ring around the right number of marbles for John.<br><br>b) Ben wins 2/3 of the marbles. How many marbles does he get? Write your answer in the box. |
| <br><br>8 | (15) Fill in the boxes with the appropriate answers.<br><br>(16) Shade in two-thirds of each of these shapes. |

**CUREC FORM** Appendix 2: Letter to schools to take part in the Longitudinal study

**University of Oxford**
**Department of Education**
15 Norham Gardens, Oxford OX2 6PY
*Director*: Ernesto Macaro
Tel: +44 (0)1865 284893

[Head teacher name]
[School name and address]
[Date]

Dear [Head teacher name],

Investigating ways of improving children's understanding of Fractions

In June 2015, I am starting a study about children's understanding of Fractions and its contribution to mathematical achievement as part of my doctoral training. I am writing to invite your school to participate. Fractions are important not just in mathematics but also in everyday life but it is a difficult topic to teach and many people continue to have difficulty with fractions after primary school. We want to study how children's understanding of fractions develops over time and the impact this may have in their future learning of mathematics. We have developed a set of diagnostic assessments that can be used with children in Year 4 and Year 5 of Primary School to explore their understanding in this area. This will give children extra practice and help with their thinking about fractions and problem solving, both important elements in the curriculum. This type of study is invaluable for understanding children's learning over long periods of time.

In practical terms, I would like to work with your classes of Year 4 children and follow them over one year, assessing them three times. The children will be assessed on fractions, general cognitive ability, a standardized mathematical achievement test and arithmetical skills test. The three times will be in June 2015, January 2016 and June 2016. I would consult with the class teachers and arrange how best to fit with their timetable to cause minimal disruption. I will seek parental consent and each child would be asked also if they wanted to participate and they would be free to withdraw at any time if they wished to do so. Once the study has been completed I will provide a report of our findings to schools and parents are welcome to have copies of this.

I have the required clearance to work with children in schools and the information collected will be confidential in accordance with the UK data protection requirements. The data will be used only for research purposes and will be kept in locked cabinets in the research office in the University. Access to the information is restricted to the research student and two supervisors and children's names will be replaced by a number to protect their identity. No information about individual children will be made available to anyone other than the research team.

The research will be carried out by a research student from the Department of Education, University of Oxford, and has been approved by the University Ethics Committee. I will contact you within a few days to ask whether you would like your school to participate in this project or you may prefer to complete the attached form to indicate initial interest. If you have any questions you would like to ask before replying, please do not hesitate to contact me or my supervisor by emailing rossana.barros@education.ox.ac.uk or Professor Nunes .nunes@education.ox.ac.uk.

Thank you for your time and consideration of this project. I hope it may be of interest to you and if so, I look forward to hearing from you.

Yours truly,
Rossana Barros

**To be returned to:**

Ms Rossana Barros/Professor Nunes

University of Oxford

Department of Education

15 Norham Gardens, Oxford OX2 6PY

**Title of Project:** Factors contributing to mathematical achievement: Fractions Skills.

School's name: ……

Head teacher: ……..

I would like to:

☐ Receive more information regarding the project before we decide if we would like to participate.

☐ Sign up to participate in the project

Number of Year 4 classes: .....

Date……

**CUREC FORM** Appendix 3: Letter to schools to take part in the Intervention study

**University of Oxford**
**Department of Education**
15 Norham Gardens, Oxford OX2 6PY
*Director*: Ernesto Macaro
Tel: +44 (0)1865 284893

[Head teacher name]
[School name and address]
[Date]
Dear [Head teacher name],

Investigating ways of improving children's understanding of Fractions

In February 2016, I am starting a Mathematics study about Fractions and their contribution to mathematical achievement as part of my doctoral training and I was wondering if this is a project which you might be interested in supporting with your school's participation.

Understanding rational numbers is important not just in mathematics but also in everyday life. I along with my supervisor have developed a set of lessons that can be used with children in Year 4 of Primary School to improve their understanding in this area and we aim to assess its effectiveness in this project. I want to study how children's understanding of fractions impacts their future learning of mathematics. I feel this is a valuable intervention and would very much like to trial it with your children. This teaching programme draws on research about the cognitive basis of understanding rational number, and should have benefits for mathematical achievement.

In practical terms, I would like to work with children in Year 4 between February and July 2016. During that time I would carry out 8 weekly sessions of half an hour each between March and May, in which I would work with children in small groups teaching them about different aspects of fractions. The children who agree to participate would be randomly allocated in one of three groups; one group would receive sessions on reasoning about fractions, the second group would learn about arithmetic skills to solve fractions; whilst the third group (unseen children) could receive the fraction tasks and activities through their teachers, which would be sent to the teacher later in the year. Teachers will also be invited to attend an afternoon workshop for teachers of participating schools where I will present the results and discuss the use of the materials. I would like to assess the children before delivering the sessions and immediately after finishing the sessions, and finally 6 weeks after we finished, to assess the stability of their learning. All assessments would be carried out as a whole class activity.

I would consult with the class teachers and arrange how best to fit with their timetable to cause minimal disruption. I will seek parental consent and each child would be asked also if they wanted to participate and they would be free to withdraw at any time if they wished to do so.

At the start of 2016, I would meet with the Year 4 teachers to discuss any specific circumstances relevant to their children. I would then work half an hour per week for 8

sessions with their class in small groups during terms 4 and 5, 2016. There would then be an option, if teachers would like it, for class teachers to attend a workshop and use the Fractions programme with their new Year 4 class, with materials used in the study.

I have the required clearance to work with children in schools and the information collected will be confidential in accordance with the UK data protection requirements. The data will be used only for research purposes and will be kept in locked cabinets in the research office in the University. Access to the information is restricted to the research student and two supervisors and children's names will be replaced by a number to protect their identity. No information about individual children will be made available to anyone other than the research team.

I will provide a report of our findings to schools once the study has been completed and parents are welcome to have copies of this.

The research will be carried out by a research student from the Department of Education, University of Oxford, and has been approved by the University Ethics Committee. I will contact you within a few days to ask whether you would like your school to participate in this project or you may prefer to complete the attached form to indicate initial interest. If you have any questions you would like to ask before replying, please do not hesitate to contact me or my supervisor by emailing rossana.barros@education.ox.ac.uk or Professor Nunes .nunes@education.ox.ac.uk.

Thank you for your time and consideration of this project. I hope it may be of interest to you and if so, I look forward to hearing from you.

Yours truly,
Rossana Barros

**To be returned to:**

Ms Rossana Barros/Professor Nunes

University of Oxford

Department of Education

15 Norham Gardens, Oxford OX2 6PY

**Title of Project:** Factors contributing to mathematical achievement: Fractions Skills.

School's name: ……

Head teacher: ……..

I would like to:

☐ Receive more information regarding the project before we decide if we would like to participate.

☐ Sign up to participate in the project

Number of Year 4 classes: .....

Date……

# Improving children's understanding of fractions

Research study to improve mathematical achievement



Department of Education
University of Oxford

UNIVERSITY OF OXFORD

## Important information

The research will be carried out by a doctoral student from the Department of Education, University of Oxford, who has the required clearance to work with children in school. All the information collected will be confidential in accordance with the UK data protection requirements. The information collected will only be used only for the purpose of the study and children will only be identifiable by a number. No information about individual children will be made available to anyone.

All sessions will be carried out as a whole class.

The ethics committee of Oxford University has approved this study. With your consent, we would like your child to be part of this study.

Please don't hesitate to contact us if you have any questions.

## Contact Us

Rossana Barros / Prof. Terezinha Nunes
Department of Education,
University of Oxford
15 Norham Gardens, Oxford

rossana.barros@education.ox.ac.uk

# Improving children's understanding of fractions

## Invitation

to take part in a study

FURTHER INFORMATION

Please contact Rossana Barros

E: rossana.barros@education.ox.ac.uk
T: 01865 274027

P: Fill in the slip on the back and return it to your child's teacher.
Please reply by 15th May.

Dear Parents,

I am writing to ask for your permission to allow your child to take part in a Mathematics study about Fractions. Your child's teacher has kindly agreed to cooperate with the project and now we are asking for your permission for your child to take part.

I would like to work with children as a whole class on 3 occasions, overall lasting 7 hours during which we'll assess your child's understanding of reasoning about fractions and their arithmetical skills; as well as their mathematical achievement. Your child will be asked if they want to participate and they are free to withdraw at any time if they wish to do so.

I hope that your child will want to take part in the research, but before you decide, it is important that you understand what it will involve. Please read through the information on this pamphlet and contact me if you have any questions.

I would be very grateful if, after reading this pamphlet, you could please fill in the slip and return it to your child's teacher. Please reply by 15th May 2015.

Rossana Barros
Oxford University, Department of Education

## Project background and aims

Fractions are an important part of learning mathematics and of everyday life, for example when we cook we may use ¾ cups of flour. There is however consistent evidence that many children have great difficulties in understanding fractions. Understanding of fractions also can predict math achievement.

We aim to evaluate how reasoning and arithmetical skills about fractions correlate over time and their contribution to mathematical achievement.

We believe this information will be most valuable for future teaching and educational research.

It will give your child practice and thinking about fractions and problem solving, two important topics included in the National Curriculum.

## How will it work?

In practical terms we ask for 7 hours spread across during two academic years, Year 4 and Year 5. Your child's participation will be of invaluable help to educational research.

### Timeline

- Consent forms (May 15)
- First assessments are carried out (Jun 15)
- Second assessments are carried out (Jan 16)
- Final assessments are carried out (Jun 16)
- School receives general results which can be shared with parents (Oct 16)

**To be returned to your child's teacher**

**Title of Project:** Factors contributing to mathematical achievement: Fractions Skills.

Child's name: .....

Class teacher: .....

I agree to my child's participation in this study

Parent signature: ..... Date .....

**Appendix 5: Parent consent letter/pamphlet for Intervention study**

# Improving children's understanding of fractions

Research study to improve mathematical achievement



Department of Education
University of Oxford

UNIVERSITY OF OXFORD

## Important information

The research will be carried out by a doctoral student from the Department of Education, University of Oxford, who has the required clearance to work with children in schools. All the information collected will be confidential in accordance with the UK data protection requirements. The information collected will be used only for the purpose of the study and children will only be identifiable by a number. No information about individual children will be made available to anyone.

All intervention work will be carried out with small groups of children in the school premises.

The ethics committee of Oxford University has approved this study. With your consent, we would like your child to be part of this study.

Please don't hesitate to contact us if you have any questions.

## Contact Us

Rossana Barros / Prof. Terezinha Nunes
Department of Education,
University of Oxford
15 Norham Gardens, Oxford

rossana.barros@education.ox.ac.uk

Improving children's understanding of fractions

**Invitation**

to take part in a study

FURTHER INFORMATION

Please contact Rossana Barros

E: rossana.barros@education.ox.ac.uk
T: 01865274027

P.S. Fill in the slip on the back and return it to your child's teacher.
Please reply by 15th December.

Dear Parents,
I am writing to ask for your permission to allow your child to take part in a Mathematics study about Fractions. Your child's teacher has kindly agreed to cooperate with the project and now we are asking for your permission for your child to take part.

I would like to work with children in small groups on 8 weekly sessions of 30 minutes each, which are designed to be fun for the children to do. Your child will be asked if they want to participate and they are free to withdraw at any time if they wish to do so.

I hope that your child will want to take part in the research, but before you decide, it is important that you understand what it will involve. Please read through the information on this pamphlet and contact me if you have any questions.

I would be very grateful if, after reading this pamphlet, you could please fill in the slip and return it to your child's teacher. Please reply by 15th January.

Rossana Barros
Oxford University, Department of Education

## Project background and aims

Fractions are an important part of learning mathematics and of everyday life, for example when we cook we may use ¾ cups of flour. There is however consistent evidence that many children have great difficulties in understanding fractions. Understanding of fractions can also predict math achievement.

We have developed a set of lessons to improve their understanding in this area and we aim to assess its impact in their future learning of mathematics.

We feel this is a valuable intervention as the National Curriculum includes fractions and problem solving and would very much like your support by giving your permission for your child to take part. This teaching programme draws on research about the cognitive basis of understanding rational number, so would have benefits for mathematical achievement.

## How will it work?

A set of 8 sessions and assessments will be implemented by a trained research student with a small group at school.

The project will be implemented and the intervention evaluated. The whole class would be randomly allocated to one of the following groups:

- Reasoning about Fractions
- Arithmetical skills of Fractions
- No teaching from the research student

Each of the first two groups will receive 8 half-an-hour sessions, one per week, between March and May 2016. We expect that these sessions, 30 minutes on average, would be arranged with your child's class teacher to fit best with their timetable.

Children will be assessed at three testing points: to measure improvement from pre-intervention-tests to post-tests and a delayed post-test to assess the stability of their learning.

In practical terms we ask for 10 hours spread across 3 terms during Year 4. We believe your child could benefit from these sessions as they are designed to help children understand a difficult area of mathematics. Your child's participation will be of invaluable help to educational research.

### Timeline

- Consent forms (Dec 15)
- First assessments are carried out (Feb 16)
- 8 -30 minute- sessions by researcher (Mar – May 16)
- Second assessments are carried out (May 16)
- Final assessments are carried out (Jul 16)
- School receives general results which can be shared with parents (Oct 16)

**To be returned to your child's teacher**

**Title of Project:** Factors contributing to mathematical achievement: Fractions Skills.

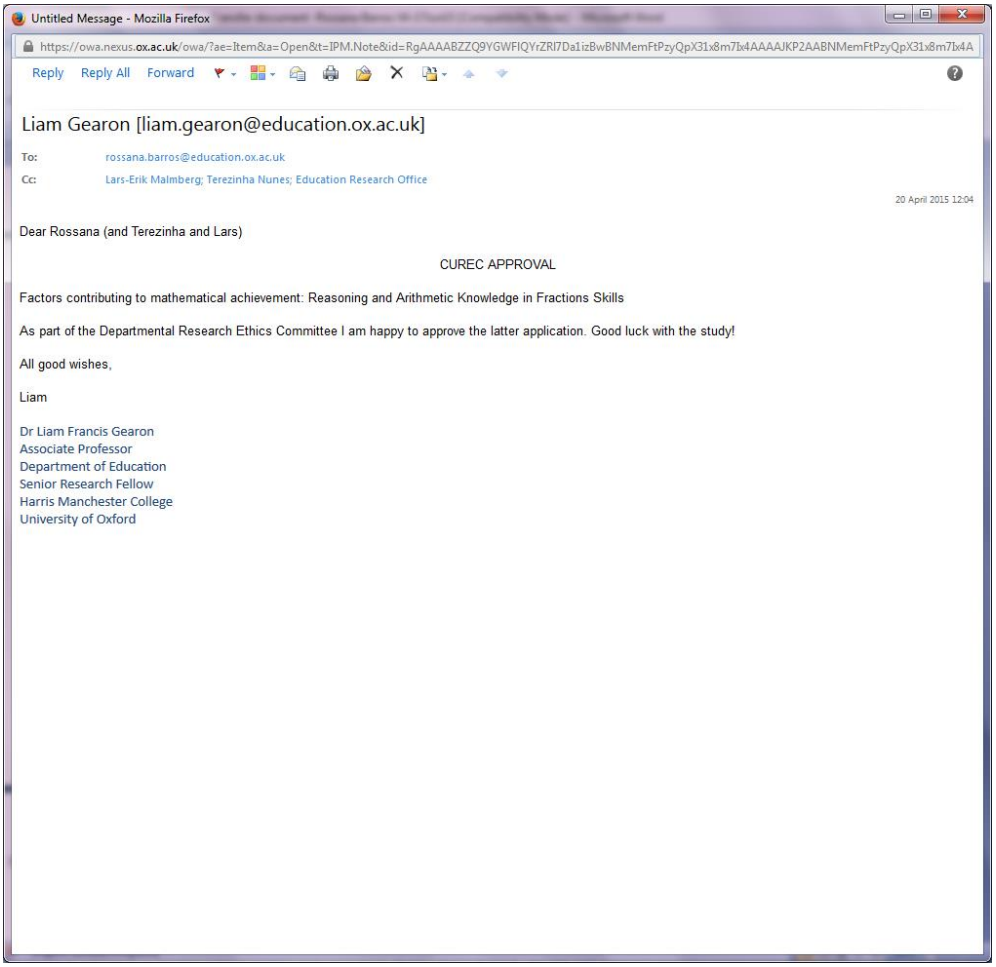
Child's name: .....

Class teacher: .....

I agree to my child's participation in this study

Parent signature: ..... Date .....

## Appendix 2: CUREC Approval



Untitled Message - Mozilla Firefox

https://owa.nexus.ox.ac.uk/owa/?ae=Item&a=Open&t=IPM.Note&id=RgAAAABZZQ9YGWFIQYrZRl7Da1izBwBNMemFtPzyQpX31x8m7lx4AAAAJKP2AABNMemFtPzyQpX31x8m7lx4A

Reply Reply All Forward

Liam Gearon [liam.gearon@education.ox.ac.uk]

To: rossana.barros@education.ox.ac.uk

Cc: Lars-Erik Malmberg; Terezinha Nunes; Education Research Office

20 April 2015 12:04

Dear Rossana (and Terezinha and Lars)

CUREC APPROVAL

Factors contributing to mathematical achievement: Reasoning and Arithmetic Knowledge in Fractions Skills

As part of the Departmental Research Ethics Committee I am happy to approve the latter application. Good luck with the study!

All good wishes,

Liam

Dr Liam Francis Gearon
Associate Professor
Department of Education
Senior Research Fellow
Harris Manchester College
University of Oxford

Appendix 3: Risk assessment approval

**RE: Fieldwork Safety and Risk Assessment form**

Jackie Bridges
**Sent:**25 March 2015 11:21
**To:** Rossana Barros Baertl

Dear Rossana

I am writing to inform you that your form has been approved. Please note that you are not insured for travel insurance.

Best wishes

Jackie
Jackie Bridges – Higher Degrees Administrative Assistant
University of Oxford | Department of Education
15 Norham Gardens, Oxford OX2 6PY
Tel: 01865 274187
Email: jackie.bridges@education.ox.ac.uk
Web: www.education.ox.ac.uk

**Appendix 4: Quantitative Reasoning and Arithmetic Knowledge tests**

Please email if you would like to request access to the tests.

**Appendix 5: Fractions Test**

Implemented at:

Longitudinal study Time 3

Intervention study Pre-test, Post-test and Delayed Post-test

Please visit the Chelsea Diagnostic Mathematics Tests to request permission to use.

**Appendix 6: CSMS Test**

Please visit the Chelsea Diagnostic Mathematics Tests to request permission to use.

**Appendix 7 Chapter 5 Intervention Study Secondary Analyses**

## Multiple regressions

A set of extra analyses was carried out to verify whether the interpretation of the results is adequate. Multiple regression analyses will be carried out using the covariates.

In order to carry out multiple regression analyses assumptions should be tested: (1) *Normality and homoscedasticity assumptions*, these assumptions were revised for the analyses of variance and met. (2) *Assumption of absence of multicollinearity*, this assumption was verified only for the covariates. For the regression analyses, the relation between the variables included in the model and the dependent variable or outcome measure will be analysed. If the variables are highly correlated to each other, it would mean that the variables are multicollinear. In order to test the absence of multicollinearity between the variables in the model, one can use correlations analyses.

The information about the relationships between the variable, group allocation and control variables (non-verbal ability and pre-test) with dependent variables (post-test and delayed post-test scores) were provided within the Chapter on Intervention Study. The correlations also presented within the Chapter on Intervention Study showed that the variable group allocation is not correlated with the control variables, nor with the outcome measures, which would indicate there is no multicollinearity.

The results from the analyses carried out to test the assumptions to use multiple regression techniques showed that it is appropriate to use the statistical analyses proposed in this study with the data collected.

Hierarchical fixed-order multiple regression analyses will test whether group allocation has a specific relation to the outcome measure, after controlling for non-verbal ability and pre-test score.

Table 23 shows the results of the multiple regression in which group allocation was the last step after controlling for non-verbal ability and pre-test scores. Group allocation made a significant contribution to the post-test score. Interestingly the β coefficient for group allocation is higher than non-verbal ability. These results showed that the intervention had an effect on the post-test score.

**Table 23**
*Multiple regression dependent variable post-test*

| Regression | Steps | Variables | $R^2$ | $R^2$ change | β coefficient | B | Standard error of B |
|---|---|---|---|---|---|---|---|
| Controls | 1 | Non-verbal ability | 0.12 | 0.12* | 0.10 | 0.12 | 0.06 |
| | 2 | Pre-test score | 0.69 | 0.58* | 0.80 | 0.91 | 0.06 |
| Group allocation | 3 | Group | 0.73 | 0.03* | 0.18 | 1.84 | 0.55 |

* Significant at $p<0.05$, N=103

Table 24 shows the results of a multiple regression in which the dependent variable was the delayed post-test. It first controlled for non-verbal ability and pre-test scores; the last step entered to the model was group allocation. Group allocation also made a significant contribution to the delayed post-test score. The β coefficient for Group allocation remained higher than non-verbal ability.

**Table 24**
*Multiple regression dependent variable delayed post-test*

| Regression | Steps | Variables | R Square | R Square change | β coefficient | B | Standard error of B |
|---|---|---|---|---|---|---|---|
| Controls | 1 | Non-verbal ability | 0.14 | 0.14* | 0.11 | 0.14 | 0.07 |
| | 2 | Pre-test score | 0.68 | 0.54* | 0.78 | 0.96 | 0.07 |
| Group allocation | 3 | Group | 0.72 | 0.04* | 0.19 | 2.19 | 0.63 |

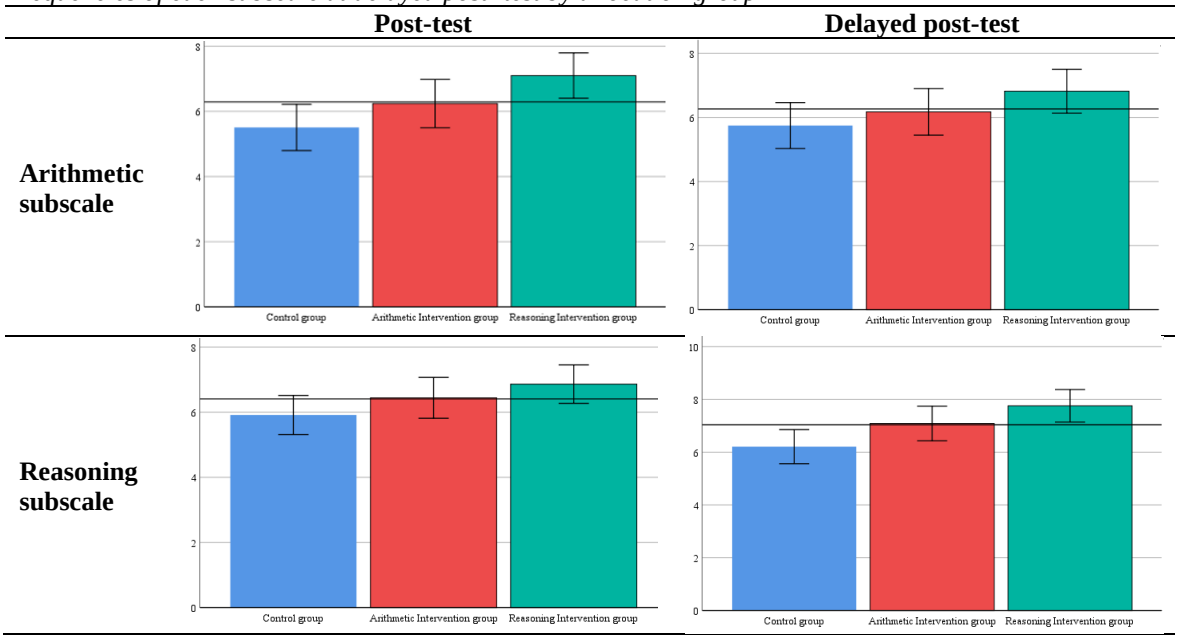
* Significant at $p<0.05$, N=98

**Appendix 8 Covariance analyses by subscale**

Six analyses of covariance were carried out to test predictions related to the subscales: Arithmetic, Reasoning and Naming Fractions. The first two analyses tested the impact of the two interventions on the arithmetic subscale at the immediate and the delayed post-test, controlling for the children's performance in the Raven's Matrices and in the arithmetic subscale of the pre-test. The third and fourth analyses tested the impact of the two interventions on the reasoning subscale at the immediate and the delayed post-test, controlling for the children's performance in the Raven's Matrices and in the reasoning subscale of the pre-test. The fifth and sixth analyses tested the impact of the two interventions on the fraction naming subscale at the immediate and the delayed post-test, controlling for the children's performance in the Raven's Matrices and in the fraction naming subscale of the pre-test. The following sections present the details of these analyses.

**Figure 42**

*Frequencies of each subscale at delayed post -test by allocation group*



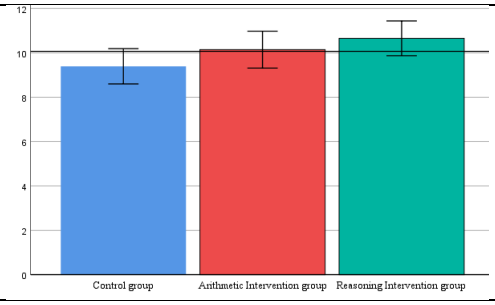
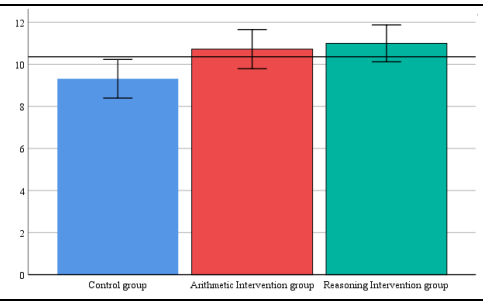
| | | |
|---|---|---|
| **Naming fractions subscale** |  |  |

**Table 25**

*Mean and standard deviation of each subscale at post -test by allocation group*

| Measures | Arithmetic Intervention Group (N= 32) | | Quantitative Reasoning Intervention Group (N=36) | | Control Group (N=35) | | ANCOVA | |
|---|---|---|---|---|---|---|---|---|
| | M | SD | M | SD | M | SD | F (2,98) | $\eta^2$ |
| Arithmetic subscale | 6.34 | 3.44 | 7.00 | 3.34 | 5.51 | 3.11 | 5.04* | .09 |
| Reasoning subscale | 6.53 | 3.19 | 6.78 | 2.33 | 5.91 | 2.81 | 2.50 | .05 |
| Naming subscale | 10.25 | 3.88 | 10.56 | 3.08 | 9.40 | 3.46 | 2.53 | .05 |

*$p < .05$

***Arithmetic subscale**:* Quantitative reasoning intervention group showed a statistically significant difference with the control (p= .002). The arithmetic intervention group didn't show a significant difference with the control group.

***Reasoning subscale:*** Quantitative reasoning intervention group showed a statistically significant difference with the control (p= .028). The arithmetic intervention group didn't show a significant difference with the control group.

***Naming subscale:*** Quantitative reasoning intervention group showed a statistically significant difference with the control (p= .027). The arithmetic intervention group didn't show a significant difference with the control group.

**Table 26**
*Mean and standard deviation of each subscale at delayed post -test by allocation group*

| Measure | Arithmetic Intervention Group (N= 31) | | Quantitative Reasoning Intervention Group (N=35) | | Control Group (N=32) | | ANCOVA | |
|---|---|---|---|---|---|---|---|---|
| | M | SD | M | SD | M | SD | F (2,93) | $\eta^2$ |
| Arithmetic subscale | 6.23 | 3.44 | 6.74 | 3.21 | 5.78 | 3.25 | 2.34 | .05 |
| Reasoning subscale | 7.13 | 3.49 | 7.69 | 2.67 | 6.25 | 2.72 | 5.93* | .11 |
| Naming subscale | 10.74 | 4.14 | 10.89 | 3.41 | 9.41 | 4.02 | 3.90* | .08 |

*$p$ <.05

***Arithmetic subscale****:* Quantitative reasoning intervention group showed a statistically significant difference with the control (p= .035). The arithmetic intervention group didn't show a significant difference with the control group.

***Reasoning subscale:*** Quantitative reasoning intervention group showed a statistically significant difference with the control (p= .001). The arithmetic intervention group didn't show a significant difference with the control group.

***Naming subscale:*** Quantitative reasoning intervention group showed a statistically significant difference with the control (p= .010). The arithmetic intervention group also showed a significant difference with the control group (p=.035).