

# Applying Bussgang's Theorem to Fixed-gain OFDM-based Relay Networks: A Profile Decay Analysis

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**Abstract**—In recent years, Bussgang's theorem has proven useful when studying and optimizing peak-power constrained OFDM-based amplify-and-forward (AF) networks. For the variable-gain (VG) scenario, Bussgang's theorem can be applied at the relay without taking into consideration the instantaneous behavior of the channel. This significantly improves the mathematical tractability of performance evaluation. For fixed-gain (FG) relaying, this is not necessarily the case, and we must also ensure that a sufficient number of *significant* channel taps are present in the channel response if we wish to ignore the channel's instantaneous behavior. In this paper, we determine conditions that the power-decay profile of the channel's  $l$  tap response must satisfy so that, in the limit of large  $l$  the FG Bussgang parameters become independent of the channel's instantaneous behavior. We conclude our analysis by considering some examples (exponential and power-law decay profiles). For exponential profiles we find that exact calculation of the Bussgang parameters always requires instantaneous channel knowledge, regardless of the number of channel taps present; while for power-law profiles this is not necessarily the case, provided the decay is not too steep.

**Index Terms**—Nonlinear Distortion, OFDM, Relay, Fixed-gain, Decay Profile, Outage probability.

## I. INTRODUCTION

OFDM-based systems exhibit a large PAPR in their transmit waveforms and are thus susceptible to the effects of nonlinear amplifier distortion. To study nonlinear amplification in these systems, Bussgang's theorem can be employed. For a memoryless nonlinear amplifier with complex Gaussian input<sup>1</sup>, Bussgang's theorem allows us to write the amplifier's output as a scaled version of its input plus an affine distortion term which is uncorrelated with the input. The input scaling factor and the distortion's variance are referred to as Bussgang parameters.

When combined with the effects of channel fading, even larger PAPR may be produced at the relay of OFDM-based networks. It is thus particularly important that we understand how nonlinear amplification might affect the performance of OFDM-based relaying [1] systems. In recent years, Bussgang's theorem has proven remarkably successful for this purpose. In [2], the outage probability of a two-hop cooperative OFDM

VG relay system in the presence of relay nonlinearities is approximated, while [3] focuses on the bit-error rate of such a system when nonlinear amplifications occur at the source. Outage and symbol error rate expressions are obtained for FG systems subject to nonlinear amplification at the relay in [4]. Outage probability expressions and power allocation strategies are obtained in [5] and [6], respectively, for two-way AF networks with nonlinearities at the relay. In [7], one-way networks are studied when the network operated in a distortion limited scenario.

In [7], it is observed that for FG systems the first hop channel should contain a sufficient number of *significant* channel taps if Bussgang's theorem is to be applied without considering the channel's instantaneous behavior. In that work, a non decaying channel profile is considered. This paper generalizes the model considered in [7] by associating a decay profile with the channel's response so that we can more accurately model practical scenarios. We then determine precisely what it means for there to be a sufficient number of significant channel taps in this more general setting. To the best of the authors' knowledge this is the first time such a study has been performed.

The main contributions of this work are given as follows:

- 1) For an arbitrary channel decay profile  $\delta(i)$ , we show in Lemma 1 that as the number of channel taps  $l$  grows without bound the instantaneous behavior of the channel can be ignored when calculating the Bussgang parameters provided

$$\lim_{l \rightarrow \infty} \frac{\sum_{j=0}^{l-1} \delta(j)^4}{\left(\sum_{j=0}^{l-1} \delta(j)^2\right)^2} = 0. \quad (1)$$

- 2) For an exponentially decaying profile we show that the above limit is never zero, while for a power-law decay profile it may be zero, provided the profile does not decay too fast.

In section II we discuss the system model. Section III determines conditions under which the Bussgang parameters can be calculated without considering the instantaneous behavior of the channel. Section III also considers some specific

<sup>1</sup>Through the central limit theorem, the complex Gaussianity of OFDM-based waveforms occurs as the number of subcarriers grows large.

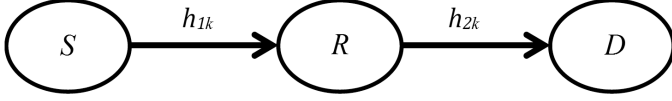


Fig. 1. Relay network.  $h_{ik}$  is the channel coefficient for  $i$ th hop on the  $k$ th subcarrier.

decay profile scenarios. Section V provides some numerical examples while section VI concludes the paper.

Throughout this manuscript, we use  $x := y$  and  $y =: x$  to denote that  $x$  is defined to be equal to  $y$ .

## II. SYSTEM MODEL

Consider a linear [8], [9] two-hop, time-division duplexing (TDD) AF OFDM network operating over  $n$  subcarriers, Fig. 1. The impulse response for hop  $\beta \in \{1, 2\}$  is assumed to be  $l$  taps long, quasi-static, and given by the time-domain (TD) vector

$$\tilde{H}_\beta = \sqrt{\frac{1}{\delta^2(l)}} \begin{bmatrix} \delta(0)\tilde{h}_{\beta,0} & \delta(1)\tilde{h}_{\beta,1} & \cdots & \delta(l-1)\tilde{h}_{\beta,l-1} \end{bmatrix}^T, \quad (2)$$

where

$$\delta^r(l) = \sum_{j=0}^{l-1} \delta(j)^r, \quad (3)$$

the  $i$ th entry of  $\tilde{H}_\beta$  corresponds to the  $i$ th channel tap,  $\tilde{h}_{\beta i} \sim \mathcal{C}(0, \mu_\beta) \forall i \in \{0, \dots, l-1\}$ ,  $\delta(\cdot)$  is the channel decay profile and the coefficient  $\sqrt{1/\delta^2(l)}$  has been selected to ensure the per-subcarrier variance is also  $\mu_\beta$ . Here, we consider two decay profiles: an exponentially decaying profile [10]

$$\delta(i) = e^{-\kappa i}, \quad (4)$$

or a power-law decay profile [11]

$$\delta(i) = (i+1)^{-\kappa}, \quad (5)$$

where in both (4) and (5)  $\kappa \geq 0$ . After taking the Fourier transform (FT) of  $\tilde{H}_\beta$ , the frequency response for the  $k$ th subcarrier of hop  $\beta \in \{1, 2\}$  is given by  $h_{\beta k} \sim \mathcal{CN}(0, \mu_\beta)$ ,  $k \in \{1, \dots, n\}$ . Note, in this model different subcarriers are necessarily correlated.

The relaying protocol takes place over two time slots. The first time-slot, amplification model, and second time-slot are detailed in the following subsections.

### A. First Time Slot

Node  $S$  constructs an OFDM symbol vector comprised of  $n$  symbols, which we denote by the frequency-domain (FD) vector  $X_S = [x_{S1}, \dots, x_{Sn}]^T$ , where  $p_S := \mathbb{E}[|x_{Sk}|^2]$ . It is assumed that the symbol  $x_{Sk}$  is chosen uniformly and independently from a quadrature phase-shift keying (QPSK) constellation.

The vector  $X_S$  is processed with an inverse FT, after which a cyclic prefix (CP) of suitable length is appended to mitigate inter-block interference (IBI). We assume that CP insertion and

removal is performed entirely at the source and destination. Consequently, the CP must be more than  $2l$  sample periods long to mitigate IBI. The TD OFDM block at  $S$  is given by

$$\tilde{X}_S = [\tilde{x}_{S,n-2l-1} \cdots \tilde{x}_{S,n-1} \tilde{x}_{S0} \tilde{x}_{S1} \cdots \tilde{x}_{S,n-1}]^T, \quad (6)$$

where  $\tilde{x}_{S,i}$  is the  $(i+1)$ th entry of  $\tilde{X}_S := \mathbf{F}^{-1}X_S$  (the inverse FFT of  $X_S$ ). The  $k$ th sample of the OFDM symbol received by the relay is given by

$$\tilde{y}_{Rk} = \sqrt{\frac{1}{\delta^2(l)}} \sum_{j=0}^{l-1} \delta(j) \tilde{h}_{1j} \tilde{x}_{S,k-j} + \tilde{v}_{Rk}, \quad (7)$$

where  $\delta^2(l)$  is given by (3),  $\tilde{x}_{S,k-j}$  is the  $k-j$ th element of  $\tilde{X}_S$ , and it is assumed that if the subscript  $k-j < 0$  then  $\tilde{x}_{S,k-j}$  is the  $(k-j) \bmod (n+2l)$ th element of the previous OFDM symbol, and  $\tilde{v}_{Rk} \sim \mathcal{CN}(0, \tilde{n}_0)$  is an additive noise term. In the FD, the received signal on the  $k$ th carrier at the relay is given by

$$y_{Rk} = h_{1k}x_{Sk} + v_{Rk}, \quad (8)$$

where  $h_{1k} \sim \mathcal{CN}(0, \mu_1)$  is the channel response for the  $k$ th subcarrier and  $v_{Rk} \sim \mathcal{CN}(0, n_0)$  is an additive noise term for the  $k$ th subcarrier.

### B. Relay Amplification Model

Once  $\tilde{y}_{Rk}$ , (7), has been received, the relay performs the amplification process and then transmits the resultant signal. Mathematically, this process takes place over two distinct steps.

a) *Step One*: The relay applies the fixed-gain amplification factor  $G$  to the  $k$ th subcarrier. This is given by [4]

$$G = \sqrt{\frac{\sigma_R^2}{p_A \mu_1 + n_0}}, \quad (9)$$

where  $\sigma_R^2$  is selected arbitrarily by the relay and represents the average power (averaged over the channel's state, source's symbol and noise received at the relay) of the signal after amplification by  $G$ .

b) *Step Two*: The relay passes the amplified TD waveform through an SEL [5, eq. (7)]

$$F(G\tilde{y}_{Rk}) = \begin{cases} G\tilde{y}_{Rk}, & |G\tilde{y}_{Rk}|^2 \leq p_{maxR} \\ \arg(G\tilde{y}_{Rk}) \sqrt{p_{maxR}}, & |G\tilde{y}_{Rk}|^2 > p_{maxR} \end{cases},$$

to clip the signal so that its maximum power is  $p_{maxR}$ . As an immediate consequence of the central limit theorem, this TD signal converges in distribution to a non stationary<sup>2</sup> ZMCG variable as the number of subcarriers grows large. The average input power over a quasi-static TD block into the relay's SEL will be given by

$$\sigma_R^2 := \mathbb{E} \left[ |G\tilde{y}_{Rk}|^2 \mid \{\tilde{h}_{10}, \dots, \tilde{h}_{1l-1}\} \right], \quad (10)$$

<sup>2</sup>The variance of the signal will be a function of the quasi-static channel coefficients, which vary in a block fading manner.

where  $\tilde{h}_{1k}$  is obtained from the  $k$ th element of (2). We can now apply Bussgang's theorem [12] and write the transmitted signal at the relay as

$$x_{Rk} = \zeta_R G y_{Rk} + d_{Rk}. \quad (11)$$

The term  $\zeta_R$  is given by [5, eq. (9)]

$$\zeta_R = 1 - e^{-\frac{p_{maxR}}{\sigma_R^2}} + \sqrt{\frac{\pi p_{maxR}}{4\sigma_R^2}} \operatorname{erfc}\left(\frac{p_{maxR}}{\sigma_R^2}\right), \quad (12)$$

and  $d_{Rk}$  is well approximated by a ZMCG variable with total (conditional) variance [5, eq. (11)]

$$\eta_R = \sigma_R^2 \left(1 - e^{-\frac{p_{maxR}}{\sigma_R^2}}\right) - \zeta_R^2 \sigma_R^2. \quad (13)$$

Throughout the rest of this paper, we refer to  $\zeta_R$  and  $\eta_R$  as Bussgang parameters. In section III, we determine a condition under which

$$\sigma_R^2 \rightarrow \sigma_R^2$$

so that the Bussgang parameters become independent of the instantaneous channel coefficients. This will be done in section III.

### C. Second Time Slot

In the second time-slot, the relay transmits  $x_{Rk}$  (11) on the  $k$ th subcarrier. The received signal at the destination is then

$$y_{Dk} = h_{2k} x_{Rk} + v_{Dk}, \quad (14)$$

where  $v_{Dk} \sim \mathcal{CN}(0, n_0)$  is a  $k$ th carrier noise term at  $D$ . Note, to obtain (14) the destination must first remove the CP from the received TD block and perform an FT on this block. The  $k$ th element of the FT's output vector will then be given by (14). The end-to-end  $k$ th subcarrier SNDR is then

$$\lambda_k = \frac{p_S |h_{1k}|^2 G^2 \zeta_R^2 |h_{2k}|^2}{|h_{2k}|^2 \left(G^2 \zeta_R^2 n_0 + \eta_R\right) + n_0}. \quad (15)$$

## III. APPLYING BUSSGANG'S THEOREM TO FIXED-GAIN RELAYING

In the previous section, it was shown that  $\zeta_R$  and  $\eta_R$  will be a function of  $\sigma_R^2$ , and we alluded to the fact that  $\sigma_R^2 \rightarrow \sigma_R^2$  under certain conditions. In what follows (Lemma 1) we determine a sufficient condition for this to occur.

From (7) and (10), it can be shown that  $\sigma_R^2$  is given by

$$\sigma_R^2 = G^2 \left( p_S \left( \frac{1}{\bar{\delta}^2(l)} \sum_{j=0}^{l-1} \delta(j)^2 |\tilde{h}_{1j}|^2 \right) + n_0 \right), \quad (16)$$

where  $\bar{\delta}^2(l)$  is given by (3). It is easy to show that  $\mathbb{E}[\sigma_R^2]$  is  $\sigma_R^2$ . The following lemma, determines a condition under which the probability that  $\sigma_R^2$  deviates from  $\sigma_R^2$  goes to zero as  $l \rightarrow \infty$ .

**Lemma 1:**  $\sigma_R^2$  converges weakly to  $\mathbb{E}[\sigma_R^2] = \sigma_R^2$ , i.e.,

$$\lim_{l \rightarrow \infty} \mathbb{P} \left[ \left| \sigma_R^2 - \sigma_R^2 \right| \geq a \right] = 0, \quad \forall a > 0,$$

if

$$\lim_{l \rightarrow \infty} \Delta(l) = 0, \quad (17)$$

where

$$\Delta(l) := \frac{\bar{\delta}^4(l)}{(\bar{\delta}^2(l))^2}. \quad (18)$$

*Proof:* From (16), it is easy to see that the condition under which weak convergence in  $\sigma_R^2$  will be held is identical to the condition that

$$h_{\delta,l} := \frac{1}{\bar{\delta}^2(l)} \sum_{j=0}^{l-1} \delta(j)^2 |\tilde{h}_{1j}|^2$$

will converge weakly to  $\mathbb{E}[h_{\delta,l}]$ . Applying Chebychev's inequality [13], we find that

$$\begin{aligned} \mathbb{P} \left[ \left| h_{\delta,l} - \mathbb{E}[h_{\delta,l}] \right| \geq a \right] &\leq \frac{\mathbb{V}[h_{\delta,l}]}{a^2} \\ &= \frac{\bar{\delta}^4(l) \mathbb{V} \left[ |\tilde{h}_{1j}|^2 \right]}{(\bar{\delta}^2(l) a)^2}. \end{aligned} \quad (19)$$

By noting that  $\mathbb{V} \left[ |\tilde{h}_{1j}|^2 \right]$  and  $a$  are independent of  $l$ , the stated result follows immediately from (19). ■

Equation (17) of Lemma 1 provides us with a condition under which the Bussgang parameters become independent of the instantaneous channel coefficients. When this is the case, performance evaluation of the network will be significantly less difficult. In the following two subsections, we will discuss Lemma 1 in relation to our two decay profiles (exponential and power-law) and determine whether and/or when (17) holds.

### A. Exponential profile

For the exponential profile (4), we have

$$\begin{aligned} \Delta(l) &= \frac{\sum_{j=0}^{l-1} e^{-4\kappa j}}{\left( \sum_{j=0}^{l-1} e^{-2\kappa j} \right)^2} \\ &\rightarrow \frac{(1 - e^{-2\kappa})^2}{1 - e^{-4\kappa}}. \end{aligned} \quad (20)$$

Thus, we find that when  $\kappa > 0$ ,  $\lim_{l \rightarrow \infty} \Delta(l) > 0$ , and in this scenario the Bussgang parameters will always be dependent on the instantaneous channel coefficients. It is only when  $\kappa = 0$  (i.e., the flat profile) that the Bussgang parameters become independent of the instantaneous state of the channel in the limit.

### B. Power-law profile

For the power-law profile, from (4)

$$\Delta(l) = \frac{\sum_{i=1}^l i^{-4\kappa}}{\left( \sum_{i=1}^l i^{-2\kappa} \right)^2}. \quad (21)$$

We now present the following lemma, which gives us the conditions under which (21) goes to zero.

**Lemma 2:** For the power-law decay profile, with  $\Delta(l)$  given by (21),  $\Delta(l) \rightarrow 0$  if and only if  $\kappa \leq 1/2$ .

*Proof:* To prove sufficiency, we bound  $\Delta(l)$  from above and show that this bound is zero for  $\kappa \leq 1/2$ . Because  $i^{-r\kappa}$ ,  $r, \kappa \geq 0$ , is a decreasing function of  $i$ , it is easy to see that

$$i^{-2\kappa} \geq \int_{x=i}^{i+1} x^{-2\kappa} dx \quad \text{and} \quad i^{-4\kappa} \leq \int_{x=i-1}^i x^{-4\kappa} dx. \quad (22)$$

Consequently,

$$\begin{aligned} \sum_{i=1}^l i^{-2\kappa} &\geq \sum_{i=1}^l \int_{x=i}^{i+1} x^{-2\kappa} dx = \int_{x=1}^{l+1} x^{-2\kappa} dx \\ &= \frac{1}{1-2\kappa} (l+1)^{1-2\kappa} - \frac{1}{1-2\kappa} \end{aligned} \quad (23)$$

and

$$\begin{aligned} \sum_{i=1}^l i^{-4\kappa} &= 1 + \sum_{i=2}^l i^{-4\kappa} \\ &\leq 1 + \sum_{i=2}^l \int_{x=i-1}^i x^{-4\kappa} dx \\ &= 1 + \int_{x=1}^l x^{-4\kappa} dx \\ &= 1 + \frac{1}{1-4\kappa} l^{1-4\kappa} - \frac{1}{1-4\kappa}. \end{aligned} \quad (24)$$

Substituting (23) and (25) into (21), we obtain the upper bound

$$\Delta(l) \leq \frac{(1-2\kappa)^2 + \frac{(1-2\kappa)^2}{1-4\kappa} l^{1-4\kappa} - \frac{(1-2\kappa)^2}{1-4\kappa}}{\left((l+1)^{1-2\kappa} - 1\right)^2}. \quad (26)$$

Taking the limit of (26), we obtain

$$\lim_{l \rightarrow \infty} \Delta(l) \leq 0, \quad \text{for } 0 \leq \kappa < 1/2. \quad (27)$$

Of course, from (21) we will always have  $\Delta(l) \geq 0$ . Thus the inequality in (27) can be strengthened to an equality. For the critical point  $\kappa = 1/2$ , considering the right-hand side of (26) it can be shown that

$$\lim_{\kappa \rightarrow 0.5} \frac{(1-2\kappa)^2 + \frac{(1-2\kappa)^2}{1-4\kappa} l^{1-4\kappa} - \frac{(1-2\kappa)^2}{1-4\kappa}}{\left((l+1)^{1-2\kappa} - 1\right)^2} = 0, \quad \forall l. \quad (28)$$

This proves the sufficiency of the statement.

To prove the necessity, for  $\kappa > 1/2$  we can write  $\lim_{l \rightarrow \infty} \Delta(l)$  in terms of the Riemann zeta function [14]

$$\zeta(s) := \sum_{i=1}^{\infty} \frac{1}{i^s}, \quad s > 1,$$

so that we have

$$\lim_{l \rightarrow \infty} \Delta(l) = \frac{\zeta(4\kappa)}{\zeta(2\kappa)^2} > 0 \quad \text{for } \kappa > 1/2. \quad (29)$$

This completes the proof.  $\blacksquare$

In this section, we have shown that the degree to which the Bussgang parameters can be approximated using the average channel state is determined by (18), and if this goes to zero the instantaneous state of the channel can be ignored in the limit of large  $l$ . Furthermore, we were able to show that

- 1) if the profile decays exponentially then an exact calculation of the Bussgang parameters will always require instantaneous channel knowledge for  $\kappa > 0$ ;
- 2) if the profile has a power-law decay then when  $\kappa \leq 1/2$ , in the limit of large  $l$ , the instantaneous channel coefficients can be ignored, and the Bussgang parameters will be determined entirely by the average state of the channel.

In the next section, we will discuss the implications of our result to real world channels.

#### IV. THE IMPLICATIONS OF LEMMA 1 FOR REAL-WORLD CHANNELS

The analysis in the previous section was performed from a purely theoretical perspective. It is now important that we discuss some subtleties regarding the applicability of this analysis to real world channels. Specifically, we will now discuss under which settings it makes sense from a practical perspective to consider an infinite number of channel taps.

First, we must point out that it is reasonable to assume that infinitely long power-delay profiles may occur in practical settings because electromagnetic waves emitted from a source may continue to reverberate within an environment indefinitely, provided reflectors are present. However, it can be seen that if  $\Delta(l) \rightarrow 0$  (see (18)) then, because  $\overline{\delta^4}(l) \not\rightarrow 0$ ,

$$\overline{\delta^2}(l) \rightarrow \infty, \quad (30)$$

and with reference to (2), it can be seen that the channel's impulse response will go to the zeros vector. Another way of interpreting this is that when  $\Delta(l) \rightarrow 0$ , the *finite* energy in the transmit waveform necessarily gets spread across an *infinite* time period. Thus,  $\Delta(l) \rightarrow 0$  is clearly not representative of a real world channel, and this discussion tells us that the conditions required for Bussgang's theorem to be applicable without considering the channel's instantaneous state result in a channel that will not be realized in practice.

One might be tempted to claim that this problem arises purely from the normalization that takes place in (2), and if we removed the normalization this problem would disappear. However, this is not true, and if it were performed the channel would become unstable in the frequency domain (i.e., the average channel coefficients become infinitely large). Again, this would not represent a practical channel.

While this discussion appears somewhat anticlimactic, it is important to note that in certain highly reflective environments it has been shown that the power-law decay exponent may be less than 0.5 [15] over a short duration (i.e., for a finite number of specular clusters within the environment). This adds credit to our theoretical analysis, and suggests that certain environments possess the appropriate properties (over finite

intervals) that allow accurate modeling of the performance of FG networks whilst considering only the average statistics of the channel when calculating the Bussgang parameters.

## V. NUMERICAL EXAMPLES

The outage probability  $\underline{P}_o(\gamma_{th})$  of the end-to-end link is given by the probability that the instantaneous SNDR (15) drops below an SNDR threshold  $\gamma_{th}$ ,

$$\underline{P}_o(\gamma_{th}) := \mathbb{P}[\underline{\lambda}_k \leq \gamma_{th}].$$

To demonstrate our results, we plot the *relative* outage probability between  $\underline{P}_o(\gamma_{th})$  and the outage probability that is attained when  $\underline{\sigma}_R^2 = \sigma_R^2$ ,  $P_o(\gamma_{th})$ , which is given by

$$P_d(\gamma_{th}) := \mathbb{P}[\lambda_k \leq \gamma_{th}],$$

where

$$\lambda_k = \frac{p_S |h_{1k}|^2 G^2 \zeta_R^2 |h_{2k}|^2}{|h_{2k}|^2 (G^2 \zeta_R^2 n_0 + \eta_R) + n_0}. \quad (31)$$

and

$$\zeta_R^2 = \zeta_R^2|_{\sigma_R^2 = \sigma_R^2} \quad \text{and} \quad \eta_R = \eta_R|_{\sigma_R^2 = \sigma_R^2}.$$

The relative outage probability is given by

$$P_d(\underline{P}_o(\gamma_{th}), P_o(\gamma_{th})) := \left| \frac{\underline{P}_o(\gamma_{th}) - P_o(\gamma_{th})}{\underline{P}_o(\gamma_{th})} \right|, \quad (32)$$

and

$$P_d(\underline{P}_o(\gamma_{th}), P_o(\gamma_{th})) = \epsilon$$

implies that  $\underline{P}_o(\gamma_{th})$  and  $P_o(\gamma_{th})$  are a factor of  $1 - \epsilon$  (or  $1 + \epsilon$ ) away from each other.

Fig. 2 shows a plot of (32) as a function of channel taps when an exponentially decaying profile (4) is considered for different values of  $\kappa$ . Fig. 3 shows a similar plot when a power-law profile (4) is considered. For both figures, the input power to the relay's SEL  $\sigma_R^2$  is equal to the relay's maximum transmit power  $p_{maxR}$ . For the exponential profile, we can see that for  $\kappa > 0$  all the plots are bounded above zero. This observation agrees with the theory that was presented in section III-A. For the power-law profile, we can see that when  $\kappa > 0.5$  the plot appears to be bounded above zero, while when  $\kappa \leq 0.5$  the plots continue to decay with  $l$ . Again, this observation agrees with the theory that was presented in section III-B.

## VI. CONCLUSION

In this paper, we were able to show that Bussgang's theorem can be applied at the relay of FG networks without considering instantaneous channel state, provided the channel profile decay rate is not too great. This observation is critical when analytically evaluating the performance of such networks because, if instantaneous channel knowledge is required for the Bussgang parameters, mathematical performance expressions become difficult to resolve analytically.

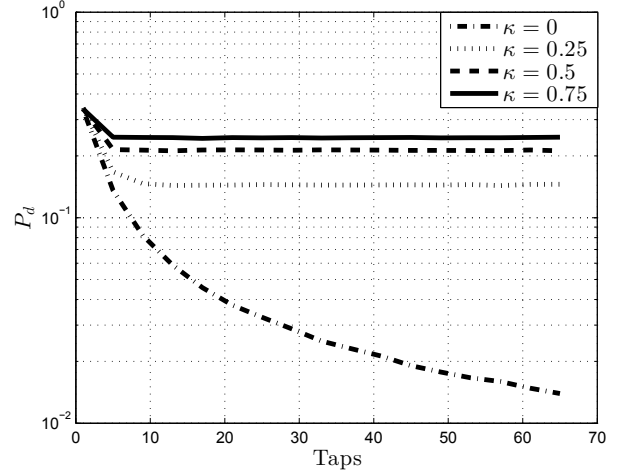


Fig. 2. Plot of  $P_d$ , (32) as a function of channel taps when the power delay profile decays exponentially (4). We set  $\mu_1 = \mu_2 = n_0 = 1$ ,  $p_A/n_0 = \sigma_R^2/n_0 = p_{maxR}/n_0 = 500$  and  $\gamma_{th} = 10$ .

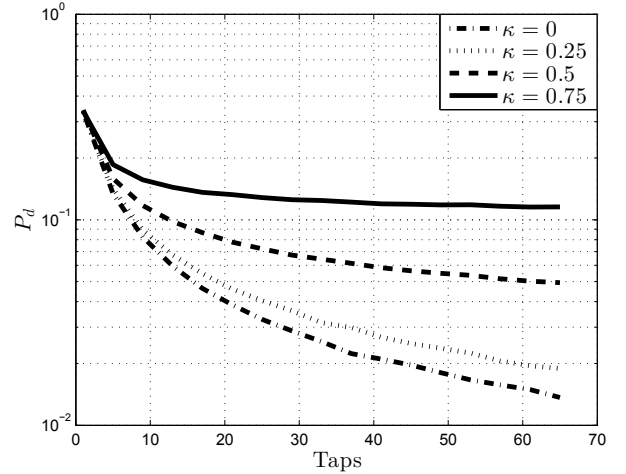


Fig. 3. Plot of  $P_d$ , (32) as a function of channel taps when the power delay profile decays according to a power-law (5). We set  $\mu_1 = \mu_2 = n_0 = 1$ ,  $p_A/n_0 = \sigma_R^2/n_0 = p_{maxR}/n_0 = 500$  and  $\gamma_{th} = 10$ .

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