

Free product decompositions in images of certain free products of groups

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Abstract

Let F be the free product of n groups and let R be a normal subgroup generated (as a normal subgroup) by m elements of F , where $m < n$. The Main Theorem gives sufficient conditions for families of fewer than $n - m$ subgroups in certain quotients of F/R to generate their free product. This leads to a more direct proof of a result of the first author, that if G is a group having a presentation with n generators and m relators, where $m < n$, then any generating set for G contains $n - m$ elements that freely generate a free subgroup of G . Another consequence is that an n -generator one-relator group cannot be generated by fewer than $n - 1$ subgroups each having a non-trivial abelian normal subgroup.

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1. Introduction

In [8], the second author proved the following result:

Theorem 1. *Let G be a group which has a presentation with n generators x_1, \dots, x_n and m relators, where $m < n$, and let S be any generating set for G . Then some subset of $n - m$ elements of S freely generates a free group.*

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The history of this result dates back to 1930, when Magnus [4] published his Freiheitssatz, which is essentially the case of Theorem 1 in which $S = \{x_1, \dots, x_n\}$ and $m = 1$. In 1978 the first author [5] generalized the result of Magnus by proving the case of Theorem 1 in which $S = \{x_1, \dots, x_n\}$ and m is any integer less than n . The proof of Theorem 1 was indirect, relying on another result of the first author [6].

Here we shall give a direct proof of a considerably more general result. Roughly speaking, the improvement consists of the replacement of the elements x_i by subgroups, of the members of S by suitably small subgroups, and of the hypothesis that S generates G by a weaker hypothesis. Before stating our Main Theorem we illustrate its power by stating two further consequences.

Theorem 2. *Let $G = F/R$, where F is a free product of groups A_1, \dots, A_n and R is generated as a normal subgroup of F by m elements and where $m < n$. Suppose also that each group A_i has a non-trivial torsion-free abelian image. Let B_1, \dots, B_{n-m-1} be torsion-free subgroups that generate G . Then some subgroup B_i has a non-cyclic free subgroup.*

Theorem 3. *Let G be a torsion-free group having a presentation with n generators x_1, \dots, x_n and one relation, where $n \geq 2$.*

- (a) *Suppose that H is a subgroup of G that can be generated by fewer than $n - 1$ subgroups each having a non-trivial abelian normal subgroup. Then, for some x_i with non-trivial image \bar{x}_i in G , the subgroups $\langle \bar{x}_i \rangle$ and H of G generate their free product.*
- (b) *If G can be generated by subgroups B_1, \dots, B_{n-1} , each having a non-trivial abelian normal subgroup, then G is the free product $B_1 * \dots * B_{n-1}$ and all but one of the subgroups B_i are cyclic.*

All of the above results concern subgroups of groups of the form F/R where F is a free product of n groups and R is generated as a normal subgroup by fewer than n elements. Our Main Theorem makes strong assertions about families of subgroups not usually of F/R but of certain images of F/R . We begin by describing the condition that will be imposed on the subgroups.

We say that a finitely generated torsion-free group is *linked* if it lies in the smallest class \mathcal{X} of groups containing all infinite cyclic groups and having the property that $\langle A, B \rangle \in \mathcal{X}$ whenever $A, B \in \mathcal{X}$ and $\langle A, B \rangle$ is not the free product of A, B . Every such group G can be constructed by finitely many operations of taking joins of previously constructed subgroups A, B whose join is not the free product of A, B , starting from cyclic groups; the minimal number of operations required is called the *height* of G .

We call an arbitrary group linked if every finitely generated subgroup is contained in a linked finitely generated subgroup.

The class of linked groups is rather extensive. It contains all torsion-free 2-generator groups that are not free and all torsion-free groups having no free subgroups of rank 2. We shall show that it also contains every torsion-free group having a non-trivial subnormal subgroup with no free subgroups of rank 2. This, and other properties of linked groups, are discussed in Section 2 below.

Now we shall describe the images of free products with relations to which our results apply. The following notation will be used in the Main Theorem.

Hypothesis. Let F be the free product of a finite family \mathcal{A} of groups. Write $n = |\mathcal{A}|$, let $m < n$, and let R be a normal subgroup of F generated (as a normal subgroup) by m elements of F .

By a *filter* of normal subgroups of a group G we understand a family \mathcal{F} of normal subgroups with the property that for any $K_1, K_2 \in \mathcal{F}$, there is a subgroup $K_3 \in \mathcal{F}$ with $K_3 \leq K_1 \cap K_2$. We call a family \mathcal{F} of normal subgroups of F a *strong special filter* if it has the following three properties:

- (i) \mathcal{F} is a filter consisting of normal subgroups of F containing R ;
- (ii) for each $K \in \mathcal{F}$ the group ring $\mathbb{Z}(F/K)$ can be embedded in a skew-field Q_K ;
- (iii) if $K \in \mathcal{F}$ and if L is a normal subgroup of F such that $R \leq L \leq K$ and K/L is a torsion-free abelian group then $L \in \mathcal{F}$.

By a theorem of Kropholler, Linnell and Moody [2], the group ring of a soluble (or even elementary amenable) torsion-free group over a right Ore domain is itself a right Ore domain, and so can be embedded in a skew-field. Therefore the family consisting of all normal subgroups $K \geq R$ with F/K soluble and torsion-free is a strong special filter (and similarly with ‘soluble’ replaced by ‘elementary amenable’).

In our Main Theorem we shall be concerned with filters of normal subgroups of F , but we do not require the full force of condition (iii): it will suffice that conditions (i), (ii) hold and that, for each $K \in \mathcal{F}$, we have $L \in \mathcal{F}$ for just one subgroup L associated with K . This subgroup, denoted by $s(K)$, will be defined in Section 3; it depends on the choice of the skew-field Q_K . We call a family \mathcal{F} of normal subgroups of F a *special filter* (or an *s-filter* for short) if it satisfies (i), (ii) and the following condition:

- (iii)' if $K \in \mathcal{F}$ then $s(K) \in \mathcal{F}$.

Main Theorem. Let \mathcal{F} be an *s-filter* and $G = F/R_{\mathcal{F}}$, where $R_{\mathcal{F}} = \bigcap_{K \in \mathcal{F}} K$. For $A \in \mathcal{A}$ write \bar{A} for the image of A in G . Suppose that for each $A \in \mathcal{A}$ with $\bar{A} = 1$ the abelianization of A is not a torsion group. Let \mathcal{B} be a family of linked subgroups of G , set $J = \langle B \mid B \in \mathcal{B} \rangle$, and suppose that for each A in \mathcal{A} with $\bar{A} \neq 1$, the subgroups \bar{A} and J do not generate their free product in G . Then $|\mathcal{B}| \geq n - m$, and there are $n - m$ members of \mathcal{B} that generate in G their free product.

Among the noteworthy special cases are the case in which F/R is residually (soluble and torsion-free) and \mathcal{F} consists of all subgroups K with $R \leq K \triangleleft F$ and F/K soluble and torsion-free (so that $G = F/R$), and the case when \mathcal{B} generates G (so that the condition that J and certain subgroups of G do not generate their free product becomes automatic). We conclude this Introduction by showing that Theorems 1–3 are immediate consequences of the Main Theorem.

Assume the hypotheses of Theorem 1; thus F is free on a set $\{x_1, \dots, x_n\}$ and R is generated as a normal subgroup of F by m elements. Let $\mathcal{A} = \{\langle x_i \rangle \mid i = 1, \dots, n\}$, and let \mathcal{F} be the *s-filter* of all normal subgroups K of F containing R such that F/K is torsion-free and soluble. Let $h \mapsto \bar{h}$ be the map $F/R \rightarrow F/R_{\mathcal{F}}$. Let S be any generating set for F/R . By the Main Theorem, there is a subset \mathcal{X} of $\{\langle \bar{s} \rangle \mid s \in S, \bar{s} \neq 1\}$ with $|\mathcal{X}| \geq n - m$ that generates its free product; it follows that the set $\{s \mid \langle \bar{s} \rangle \in \mathcal{X}\}$ freely generates a free group.

Next assume the hypotheses of Theorem 2. Define \mathcal{F} exactly as above and let \mathcal{B} be the family consisting of the images in $F/R_{\mathcal{F}}$ of the subgroups B_i . Since $|\mathcal{B}| < n - m$, the Main Theorem implies that not all groups in \mathcal{B} are linked, so that (e.g., by Proposition 1 below) one of them has a free subgroup of rank 2, and Theorem 2 follows.

Finally we show that Theorem 3 holds, with the requirement that subgroups have non-trivial abelian normal subgroups replaced by the much weaker requirement that they are linked. Let G be isomorphic to F/R , where F is free of rank n and R is generated (as a normal subgroup) by one element of F . Since $\mathbb{Z}G$ can be embedded in a skew-field by a theorem of J. Lewin and T. Lewin [1], the family $\{R\}$ is an s -filter. Assertion (a) and the fact that $G = B_1 * \cdots * B_{n-1}$ in (b) now follow directly from the Main Theorem, and the rest of (b) follows since G can be generated by n elements.

2. Linked groups

Let G be a non-trivial torsion-free group. For $a, b \in G \setminus \{1\}$ write $a \sim b$ if and only if a, b are contained in some (finitely generated) linked subgroup. Clearly \sim is an equivalence relation on $G \setminus \{1\}$. The union of $\{1\}$ and the equivalence class containing a is called the linked component of G containing a .

Lemma 1. *Let G be a non-trivial torsion-free group.*

- (a) *Each linked component is a subgroup of G .*
- (b) *If G_1, G_2 are distinct linked components of G then $\langle G_1, G_2 \rangle = G_1 * G_2$.*
- (c) *Let $\{G_i \mid i \in I\}$ be a family of linked subgroups of G such that $G = \bigcup_{i \in I} G_i$ and $\langle G_i, G_j \rangle = G_i * G_j$ if $i \neq j$. Then the subgroups G_i are exactly the linked components of G .*

Proof. (a) follows immediately from the definition of the equivalence relation \sim .

Suppose that G_1, G_2 are linked components and that G_1, G_2 do not generate their free product. Then there are finitely generated linked subgroups $H_1 \leq G_1$ and $H_2 \leq G_2$ such that $\langle H_1, H_2 \rangle \neq H_1 * H_2$. Thus $\langle H_1, H_2 \rangle$ is a linked subgroup and it is contained in one linked component. It follows that $G_1 = G_2$, and assertion (b) follows.

(c) It is sufficient to prove that each finitely generated linked subgroup H is contained in some G_i . If H is cyclic this is clear. Otherwise, we may write $H = \langle A, B \rangle \neq A * B$ where A, B are finitely generated linked subgroups with heights less than the height of H . By induction we can suppose that there are indices i, j such that $A \leq G_i, B \leq G_j$. If $G_i \neq G_j$ then $\langle G_i, G_j \rangle = G_i * G_j$ so that $\langle A, B \rangle = A * B$, and the result follows from this contradiction. \square

Lemma 2. *Let G be a torsion-free group with a non-trivial linked subgroup H such that $\langle g, H \rangle \neq \langle g \rangle * H$ for all $g \in G \setminus \{1\}$. Then G is linked.*

Proof. Fix $h_0 \in H \setminus \{1\}$, and let $g \in G \setminus \{1\}$. By hypothesis, H has a finitely generated linked subgroup A containing h_0 such that $\langle g, A \rangle \neq \langle g \rangle * A$. Clearly $\langle g, A \rangle$ is linked, so that $g \sim h_0$. Thus G has a single equivalence class and the result follows. \square

Since H is not a subnormal subgroup of the free product $H * K$ unless $H = 1$ or $K = 1$, Lemma 2 has the following immediate consequence.

Proposition 1. *Let G be a torsion-free group.*

- (a) *If G has a non-trivial subnormal linked subgroup then G is linked.*
- (b) *If G has a non-trivial subnormal subgroup which contains no free subgroups of rank 2, then G is linked.*

Lemma 3. *Let G be a group and \mathcal{K} be a filter of normal subgroups of G such that $\bigcap_{K \in \mathcal{K}} K = 1$. Let $\mathcal{L} = \{L \triangleleft G \mid G/L \text{ is torsion-free}\}$. Let B be a finitely generated linked subgroup of G . Then \mathcal{K} contains a subgroup K_0 such that the image of B in G/L is linked whenever $L \in \mathcal{L}$ and $L \leq K_0$.*

Proof. We argue by induction on the height of B . The result is clear if B is cyclic. Assume therefore that $B = \langle D_1, D_2 \rangle$, where each of D_1, D_2 is linked and B is not the free product of D_1, D_2 . By induction there are subgroups $K_1, K_2 \in \mathcal{K}$ such that $D_1 L/L$ is linked for all $L \in \mathcal{L}$ with $L \leq K_1$ and $D_2 L/L$ is linked for all $L \leq K_2$. Since B is not the free product of D_1, D_2 , we can find a non-trivial element w of $D_1 * D_2$ that maps to 1 in B . Now w is a product of a sequence Y of elements coming alternately from $D_1 \setminus \{1\}$ and $D_2 \setminus \{1\}$. Let K_3 be a subgroup in \mathcal{K} such that all distinct elements of Y are non-congruent modulo K_3 ; then for each $L \leq K_3$ the images of w in $D_1 L/L * D_2 L/L$ and BL/L are respectively non-trivial and trivial. We conclude that BL/L is linked for each $L \in \mathcal{L}$ with $L \leq K_1 \cap K_2 \cap K_3$, and the result follows. \square

3. Further preliminary results

Our notation for conjugates and commutators in a group G is as follows: we write $a^b = b^{-1}ab$ and $[a, b] = a^{-1}b^{-1}ab$. We write $[G, G]$ for the derived subgroup of a group G and we set $G^{(0)} = G$, $G^{(n+1)} = [G^{(n)}, G^{(n)}]$ for $n \geq 0$. We write X^G for the normal subgroup of a group G generated by a subset X .

Let G be a group, let $K \triangleleft G$ and let V be a right $\mathbb{Z}(G/K)$ -module. It is convenient to regard the split extension of V by G/K as the group of matrices

$$\begin{pmatrix} G/K & 0 \\ V & 1 \end{pmatrix} = \left\{ \begin{pmatrix} gK & 0 \\ v & 1 \end{pmatrix} \mid g \in G, v \in V \right\},$$

with multiplication defined in the obvious manner. We may regard V as a $\mathbb{Z}G$ -module. If $\delta: G \rightarrow V$ is a derivation (that is, if $\delta(g_1 g_2) = \delta(g_1)g_2 + \delta(g_2)$ for all $g_1, g_2 \in G$) then the map

$$g \mapsto \begin{pmatrix} gK & 0 \\ g\delta & 1 \end{pmatrix}$$

is a group homomorphism with kernel containing $[K, K]$; if δ is an inner derivation (that is, δ is the map $g \mapsto v_0(g - 1)$ for some element v_0 of V) then the kernel of this map is K . To examine the kernels of certain such maps in more detail we shall use the Magnus representation, as extended by Shmelkin and the first author; all we require can be deduced from the following result from [7]:

Lemma 4. Let $F = A_1 * \cdots * A_n$ be the free product of the groups A_1, \dots, A_n and let $H = F/R$, where $A_i \cap R = 1$ for $i = 1, \dots, n$. Let T be the free right $\mathbb{Z}H$ -module with basis $\{t_1, \dots, t_n\}$. Let

$$\varphi: F \rightarrow \begin{pmatrix} H & 0 \\ T & 1 \end{pmatrix}$$

be the homomorphism defined on the free factors A_i of F by

$$a \mapsto \begin{pmatrix} aR & 0 \\ t_i(a-1) & 1 \end{pmatrix} \quad \text{for } a \in A_i.$$

Then $\ker \varphi = [R, R]$.

We may extend Lemma 4 as follows.

Lemma 5.

(a) If the requirement that $R \cap A_i = 1$ for each i is omitted in Lemma 4 then

$$\ker \varphi = [R, R] \langle (A_i \cap R)^F \mid 1 \leq i \leq n \rangle.$$

(b) The conclusion of Lemma 4 remains true if the hypothesis on T is replaced by the requirement that $\{t_2, \dots, t_n\}$ is a basis of T and $t_1 = 0$.

Proof. Assertion (a) follows from standard arguments and assertion (b) from the formula

$$\begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} aR & 0 \\ t_i(a-1) & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ t_1 & 1 \end{pmatrix} = \begin{pmatrix} aR & 0 \\ (t_i - t_1)(a-1) & 1 \end{pmatrix}. \quad \square$$

Lemma 6. Let $F = D_1 * D_2$ and $G = F/R$, where $D_1 \cap R = D_2 \cap R = 1$. Suppose that $\mathbb{Z}G$ is contained in a skew-field Q and let V be a vector space over Q and $v_1, v_2 \in V$. Let

$$\varphi: F \rightarrow \begin{pmatrix} G & 0 \\ V & 1 \end{pmatrix}$$

be the group homomorphism defined by

$$\varphi(x_1) = \begin{pmatrix} x_1 R & 0 \\ v_1(x_1 - 1) & 1 \end{pmatrix}, \quad \varphi(x_2) = \begin{pmatrix} x_2 R & 0 \\ v_2(x_2 - 1) & 1 \end{pmatrix}$$

for all $x_1 \in D_1, x_2 \in D_2$. Then either $\ker \varphi = [R, R]$ or $\ker \varphi = R$, and in the latter case $v_1 = v_2$.

Proof. The conclusion holds if $v_1 = v_2 = 0$. Therefore, interchanging D_1, D_2 if necessary, we may assume that $v_1 \neq 0$.

Let W be a vector space over Q with basis $\{w_1, w_2\}$ and define

$$\sigma: F \rightarrow \begin{pmatrix} G & 0 \\ W & 1 \end{pmatrix}$$

by

$$x_1 \mapsto \begin{pmatrix} x_1 R & 0 \\ w_1(x_1 - 1) & 1 \end{pmatrix}, \quad x_2 \mapsto \begin{pmatrix} x_2 R & 0 \\ w_2(x_2 - 1) & 1 \end{pmatrix}$$

for all $x_1 \in D_1, x_2 \in D_2$. From Lemma 4 we have $\ker \sigma = [R, R]$.

Let $\tau_1, \tau_2: W \rightarrow V$ be the linear maps defined by

$$\tau_1(w_1) = \tau_1(w_2) = \tau_2(w_1) = v_1, \quad \tau_2(w_2) = v_2.$$

Then $\dim \ker \tau_1 = 1$ and $\dim \ker \tau_2 \leq 1$. The maps τ_1, τ_2 and the identity map $G \rightarrow G$ induce two group homomorphisms

$$\lambda_1, \lambda_2: \begin{pmatrix} G & 0 \\ W & 1 \end{pmatrix} \rightarrow \begin{pmatrix} G & 0 \\ V & 1 \end{pmatrix}.$$

Thus $\lambda_2 \sigma = \varphi$, and

$$\lambda_1 \sigma(x) = \begin{pmatrix} xR & 0 \\ v_1(x - 1) & 1 \end{pmatrix}$$

for all $x \in D_1 \cup D_2$, and hence for all $x \in F$. Thus $\ker \lambda_1 \sigma = R$.

Clearly $[R, R] \leq \ker \varphi \leq R$. Suppose that $\ker \varphi \neq [R, R]$ and choose $r \in \ker \varphi \setminus [R, R]$. Let

$$\sigma(r) = \begin{pmatrix} 1 & 0 \\ w & 1 \end{pmatrix}.$$

Since $\lambda_1 \sigma(r) = 1$ and $\lambda_2 \sigma(r) = \varphi(r) = 1$ we have $w \in \ker \tau_1 \cap \ker \tau_2$. We conclude that $\ker \tau_1 = \ker \tau_2 = wQ$, and so $\lambda_1 = \lambda_2$ and $\varphi = \lambda_1 \sigma$, as required. \square

Lemma 7. Let G be a group and \mathcal{L} be a filter of normal subgroups such that $\bigcap_{L \in \mathcal{L}} L = 1$. Let \mathcal{B} be a finite set of subgroups of G . Suppose that for each $L \in \mathcal{L}$ there are n subgroups from \mathcal{B} whose images in G/L generate their free product in G/L . Then there are n subgroups from \mathcal{B} which generate their free product in G .

Proof. Let $\mathcal{B}_1, \dots, \mathcal{B}_q$ be the subfamilies of \mathcal{B} with n elements. For each i write F_i for the free product of the groups in \mathcal{B}_i and $F_{i,L}$ (for $L \in \mathcal{L}$) for the free product of the family $\{BL/L \mid B \in \mathcal{B}_i\}$. If the conclusion of the lemma is false then for each i there is an element $w_i \in F_i \setminus \{1\}$ whose image in G is trivial. The word w_i is a product of a sequence of non-trivial elements of $\bigcup_{B \in \mathcal{B}_i} B$ with no two consecutive terms from the same member of \mathcal{B}_i ; choose $L_i \in \mathcal{L}$ such that the distinct elements in this sequence are non-trivial and non-congruent modulo L_i , and choose $M \in \mathcal{L}$ with $M \leq \bigcap_{i=1}^q L_i$. Then for each i the image of w_i in $F_{i,M}$ is non-trivial, and since the image of w_i in G/M is trivial, the map from $F_{i,M}$ to G/M is not injective. The result follows from this contradiction. \square

Lemma 8. Let $R_1 \geq R_2 \geq \dots$ be a series of normal subgroups of a free product $E = A * B$ and suppose that

$$\bigcap_{i=1}^{\infty} (A \cap R_i) = \bigcap_{i=1}^{\infty} (B \cap R_i) = 1.$$

Suppose that for infinitely many indices i we have

$$R_{i+1} \leq [R_i, R_i](R_i \cap A)^E (R_i \cap B)^E.$$

Then $\bigcap_{i=1}^{\infty} R_i = 1$.

Proof. Fix some index k and set $A_k = A/(A \cap R_k)$, $B_k = B/(B \cap R_k)$. Write $E_k = A_k * B_k$ and write \bar{R}_i for the image of R_i in E_k . We have $A_k \cap \bar{R}_k = B_k \cap \bar{R}_k = 1$ and so \bar{R}_k is free by the Kurosh subgroup theorem (see [3, IV.1.10]). It follows that $\bigcap_{k=1}^{\infty} (\ker E \rightarrow E_k) = 1$ (see [3, I.3.4]). By hypothesis we have $\bar{R}_{i+1} \leq [\bar{R}_i, \bar{R}_i]$ for infinitely many indices $i \geq k$, and hence

$$\bigcap_{i=1}^{\infty} \bar{R}_i \leq \bigcap_{n=1}^{\infty} \bar{R}_k^{(n)} = 1.$$

Therefore

$$\bigcap_{i=1}^{\infty} R_i \leq \bigcap_{k=1}^{\infty} (\ker E \rightarrow E_k) = 1. \quad \square$$

Now we discuss the subgroup $s(K)$ which occurs in property (iii) of s -filters, and place it in the context in which we shall use it. Let F and R be as in the Hypothesis from the Introduction. Let K be a normal subgroup of F such that $R \leq K$ and such that $\mathbb{Z}(F/K)$ can be embedded in a skew-field Q_K . Consider the right vector space V over Q_K with basis $\{t_A \mid A \in \mathcal{A}\}$ and the homomorphism

$$\varphi: F \rightarrow \begin{pmatrix} F/K & 0 \\ V & 1 \end{pmatrix},$$

which is defined on the free factors A of F by

$$a \mapsto \begin{pmatrix} aK & 0 \\ t_A(a-1) & 1 \end{pmatrix} \quad \text{for } a \in A.$$

By Lemma 5(a) we have $\ker \varphi = [K, K](\langle A \cap K \rangle^F \mid A \in \mathcal{A})$. For each $r \in R$ the element $\varphi(r)$ has the form

$$\begin{pmatrix} 1 & 0 \\ u(r) & 1 \end{pmatrix}.$$

Let U be the subspace of V spanned by $\{u(r) \mid r \in R\}$, let

$$\psi : F \rightarrow \begin{pmatrix} F/K & 0 \\ V/U & 1 \end{pmatrix}$$

be the map induced by φ , and define $s(K) = \ker \psi$. It is clear that $R \ker \varphi \leq s(K) \leq K$ and that $K/s(K)$ is a torsion-free abelian group. We note that $s(K)$ depends not only on K but also on the choice of Q_K .

Let \mathcal{F} be an s -filter and $K \in \mathcal{F}$. We define $K_0 = K$, $K_{i+1} = s(K_i)$ for $i \geq 0$ and $L_K = \bigcap_{i=0}^{\infty} K_i$. The family $\mathcal{C}_K = \{K_0, K_1, \dots\}$ is an s -subfilter of \mathcal{F} ; we call it the *chain generated by K* . Note that by construction $K \cap A = K_i \cap A$ for all $A \in \mathcal{A}$ and for $i = 0, 1, \dots$.

4. Proof of the Main Theorem

Assume the hypotheses of the Main Theorem. We begin with some reductions.

Lemma 9. *It suffices to prove the Main Theorem under the following additional hypotheses:*

- (a) *all groups in \mathcal{B} are finitely generated linked groups;*
- (b) *if $R \leq L \triangleleft F$ and F/L is torsion-free, and if $L \leq K$ for some $K \in \mathcal{F}$, then*
 - (i) *the image in F/L of each group $B \in \mathcal{B}$ is linked, and*
 - (ii) *for each group $A \in \mathcal{A}$ with non-trivial image in G , the image in F/L of A is non-trivial and moreover the images in F/L of A and J do not generate their free product in F/L ;*
- (c) *the s -filter \mathcal{F} is the chain $\{K = K_0, K_1, \dots\}$ generated by some normal subgroup K .*

Proof. For each $A \in \mathcal{A}$ write \bar{A} for the image of A in G .

Suppose that the hypotheses of the Main Theorem hold, and that the conclusion is false. For each group $A \in \mathcal{A}$ with $\bar{A} \neq 1$ and each $B \in \mathcal{B}$ we choose a finitely generated subgroup $M_{B,A}$ of B such that some non-trivial element of $\langle M_{B,A} \mid B \in \mathcal{B} \rangle * \bar{A}$ maps to 1 in G . If $|\mathcal{B}| \geq n - m$ then for each subset \mathcal{X} of \mathcal{B} with $|\mathcal{X}| \geq n - m$ we also choose a non-trivial element $w_{\mathcal{X}}$ of the free product of the groups in \mathcal{X} such that the image of $w_{\mathcal{X}}$ in G is trivial.

Now choose for each $B \in \mathcal{B}$ a finitely generated linked subgroup H_B that contains $\langle M_{B,A} \mid A \in \mathcal{A}, \bar{A} \neq 1 \rangle$ and such that the elements $w_{\mathcal{X}}$ (if any) lie in the free product of the groups H_B with $B \in \mathcal{X}$. The hypotheses but not the conclusion of the theorem hold with \mathcal{B} replaced by $\{H_B \mid B \in \mathcal{B}\}$.

Therefore we may assume in proving the Main Theorem that (a) holds. By Lemma 3 we can choose a subgroup $K_0 \in \mathcal{F}$ such that the images in F/L of the groups of \mathcal{B} are linked whenever $R \leq L \triangleleft F$, F/L is torsion-free and $L \leq K_0$. For each $A \in \mathcal{A}$ with $\bar{A} \neq 1$ choose an element $w_A \in \bar{A} * J$ whose image in G is non-trivial; then w_A is a product of a sequence Y_A of non-trivial elements of $\bar{A} \cup J$. Choose $K_1 \in \mathcal{F}$ such that all distinct elements in all of these sequences are non-trivial and non-congruent modulo K_1 . If we replace the s -system \mathcal{F} by $\{K \in \mathcal{F} \mid K \leq K_0 \cap K_1\}$, then (a), (b) hold.

Now we assume that (a), (b) hold. For each $K \in \mathcal{F}$ let L_K be the intersection of the chain $\mathcal{C}_K = \{K = K_0, K_1, \dots\}$ generated by K . The set $\mathcal{L} = \{L_K \mid K \in \mathcal{F}\}$ is a filter of normal subgroups of F and $\bigcap_{K \in \mathcal{F}} L_K = R_{\mathcal{F}}$. Since conditions (a), (b) hold for \mathcal{F} , it follows that the hypotheses of the Main Theorem and conditions (a), (b) also hold for each chain \mathcal{C}_K . Thus if the theorem holds for s -filters that are chains, then we can conclude that for each $K \in \mathcal{F}$ there are $n - m$ subgroups

of \mathcal{B} whose images in F/L_K generate their free product in F/L_K . It follows by Lemma 7 that there are $n - m$ subgroups of \mathcal{B} whose images in $F/F_{\mathcal{F}}$ generate their free product in $F/F_{\mathcal{F}}$. \square

From now on we assume that the additional hypotheses (a)–(c) of Lemma 9 hold. Write $\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2$, where \mathcal{A}_1 contains all subgroups A with non-trivial images in G and $\mathcal{A}_2 = \mathcal{A} \setminus \mathcal{A}_1$ and write $G_i = G/K_i$, $Q_i = Q_{K_i}$ for each i . As $K \cap A = K_i \cap A$ for all $A \in \mathcal{A}$ and all $i \geq 0$, we can replace all groups A from \mathcal{A}_1 by their images in G and also identify them with their images in G_i . By hypothesis, the abelianization of each group $A \in \mathcal{A}_2$ is not a torsion group and so for each $A \in \mathcal{A}_2$ there is a non-zero group homomorphism $v_A: A \rightarrow \mathbb{Q} \subset Q_i$. Let V_i be the right vector space over Q_i with basis $\{t_{A,i} \mid A \in \mathcal{A}\}$ and define a group homomorphism

$$\theta_{i+1}: F \mapsto \begin{pmatrix} G_i & 0 \\ V_i & 1 \end{pmatrix}$$

by specifying its restrictions $\theta_{i+1}|_A$ to the free factors as follows:

$$\begin{aligned} a &\mapsto \begin{pmatrix} a & 0 \\ t_{A,i}(a-1) & 1 \end{pmatrix} \quad \text{for } a \in A \in \mathcal{A}_1, \\ a &\mapsto \begin{pmatrix} 1 & 0 \\ t_{A,i}(v_A(a)) & 1 \end{pmatrix} \quad \text{for } a \in A \in \mathcal{A}_2. \end{aligned}$$

We note that the subspace of V_i spanned by the bottom left-hand entries of the images of the elements of F contains all elements $t_{A,i}$, and so is equal to V_i and has dimension n .

Let R be generated as a normal subgroup of F by the set $\{r_1, \dots, r_m\}$. The elements $\theta_{i+1}(r_j)$ have the form

$$\begin{pmatrix} 1 & 0 \\ u_j & 1 \end{pmatrix}$$

and so they all lie in the subgroup

$$\begin{pmatrix} 1 & 0 \\ U_i & 1 \end{pmatrix},$$

where U_i is the subspace of V_i spanned by all elements u_j . Write $W_i = V_i/U_i$. Then the kernel of the map

$$\psi_{i+1}: F \rightarrow \begin{pmatrix} G_i & 0 \\ W_i & 1 \end{pmatrix}$$

induced by θ_{i+1} contains R , and (as $t_{A,i} \in U_i$ for $A \in \mathcal{A}_2$) it is equal to K_{i+1} . Therefore we may identify G_{i+1} with $\psi_{i+1}(F)$. The map ψ_{i+1} induces a map

$$\sigma_{i+1}: G \rightarrow \begin{pmatrix} G_i & 0 \\ W_i & 1 \end{pmatrix}.$$

Denote by σ_0 the canonical homomorphism $G \rightarrow G_0$. By construction we have

$$\sigma_{i+1}(g) = \begin{pmatrix} \sigma_i(g) & 0 \\ \beta_i(g) & 1 \end{pmatrix},$$

where $\beta_i : G \rightarrow V_i$ is a derivation.

Lemma 10. *For any finitely generated linked subgroup D of G there is an index $i(D)$ such that $\sigma_i|_D$ is injective and $\beta_i|_D$ is an inner derivation for all $i \geq i(D)$.*

Proof. We argue by induction on the height of D . If $D = \langle d_0 \rangle$ is cyclic then choose $i_0 = i(D)$ such that $\sigma_{i_0}(d_0) \neq 1$. For $i \geq i_0$ the map σ_i is injective on D and defining

$$w_{D,i} = \beta_i(d_0)(\sigma_i(d_0) - 1)^{-1}$$

we have $\beta_i(d) = w_{D,i}(\sigma_i(d) - 1)$ for all $d \in D$.

Suppose then that $D = \langle D_1, D_2 \rangle \neq D_1 * D_2$, where D_1, D_2 are finitely generated linked subgroups of smaller height, and that for $i \geq i_0 = \max(i(D_1), i(D_2))$ the maps $\sigma_i|_{D_1}$ and $\sigma_i|_{D_2}$ are injective and there are $y_1, y_2 \in V_i$ such that $\beta_i(d_1) = y_1(d_1 - 1)$, $\beta_i(d_2) = z_i(d_2 - 1)$ for all $d_1 \in D_1, d_2 \in D_2$. Let S_i be the kernel of the homomorphism $D_1 * D_2 \rightarrow \langle \sigma_1(D_1), \sigma_2(D_2) \rangle \leq G_i$ for each $i \geq i_0$. As $\sigma(D_1) \cap S_i = \sigma_2(D_2) \cap S_i = 1$, by Lemma 6 either we have $S_{i+1} = [S_i, S_i]$ or we have $S_{i+1} = S_i$ and $y_i = z_i$. If the former holds for infinitely many values of i then

$$\ker(D_1 * D_2 \rightarrow G) \leq \bigcap_{j=1}^{\infty} S_{i_0}^{(j)};$$

since S_{i_0} is free by the Kurosh subgroup theorem we conclude that the map from $D_1 * D_2$ to G is injective, and this is a contradiction. Therefore there is an index $i(D) \geq i_0$ such that $S_{i+1} = S_i$ and $y_i = z_i$ for all $i \geq i(D)$, as required. \square

Using Lemma 10 and replacing K by some K_i if necessary, we may suppose the following additional property:

(d) $\sigma_i|_B$ is injective and $\beta_i|_B$ is an inner derivation for each $i \geq 0$ and each $B \in \mathcal{B}$.

We identify $B \in \mathcal{B}$ with its image in G_i and let $\beta_i(b) = w_{B,i}(b - 1)$ for $b \in B$. Denote by Y_i the subspace of W_i spanned by $\{w_{B,i} \mid B \in \mathcal{B}\}$.

Lemma 11. $Y_i = W_i$ for all sufficiently large indices i .

Proof. It is enough to prove that for each $A \in \mathcal{A}_1$ we have $t_{A,i} \in Y_i$ for all sufficiently large i . Fix $A \in \mathcal{A}_1$. Set $\bar{W}_i = W_i/Y_i$ and consider the homomorphism

$$\bar{\sigma}_{i+1} : G \rightarrow \begin{pmatrix} G_i & 0 \\ \bar{W}_i & 1 \end{pmatrix}$$

induced by σ_{i+1} . Consider also the homomorphism $\tau : C = A * J \rightarrow \langle A, J \rangle \leq G$. Set $S_{i+1} = \ker \sigma_{i+1} \tau$, $R_{i+1} = \ker \bar{\sigma}_{i+1} \tau$. By construction $S_{i+1} \leq R_{i+1} \leq S_i$. If $t_{A,i} \in Y_i$ then $\sigma_i \tau = \bar{\sigma}_{i+1} \tau$ and $R_{i+1} = S_i$. If $t_{A,i} \notin Y_i$ and $\bar{t}_{A,i}$ is the image of $t_{A,i}$ in \bar{W}_i then

$$\bar{\sigma}_{i+1}(\tau(a)) = \begin{pmatrix} a & 0 \\ \bar{t}_{A,i}(a-1) & 1 \end{pmatrix} \quad \text{for all } a \in A,$$

and

$$\bar{\sigma}_{i+1}(\tau(b)) = \begin{pmatrix} \sigma_i(b) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{for all } b \in J,$$

so that by Lemma 6 we have

$$R_{i+1} = [S_i, S_i](S_i \cap J)^C \leq [R_i, R_i](R_i \cap J)^C.$$

If the latter holds for infinitely many indices, then by Lemma 9 we have $\bigcap_{i=1}^{\infty} R_i = 1$ and τ is injective, contradicting the hypothesis of the theorem. Therefore $t_{A,i} \in Y_i$ for all sufficiently large i , as required. \square

Using Lemma 11 we may suppose in addition to (a)–(d) that we have

(e) $Y_i = W_i$ for all $i \geq 0$.

Now we resume the proof of the theorem. We recall that $W_i = V_i/U_i$, $\dim V_i = |\mathcal{A}| = n$ and $\dim U_i \leq m$. Therefore $\dim Y_i = \dim W_i \geq n - m$ and, in particular, $|\mathcal{B}| \geq n - m$. So there is a subset \mathcal{B}_0 of \mathcal{B} such that $|\mathcal{B}_0| = n - m$ and for an infinite set Ω of indices i the family $\{w_{B,i} \mid B \in \mathcal{B}_0\}$ is linearly independent over Q_i . We claim that the subgroups \mathcal{B}_0 generate their free product in G . Write H for the free product of the groups $B \in \mathcal{B}_0$ and consider the homomorphism $\tau : H \rightarrow \langle B \mid B \in \mathcal{B}_0 \rangle \leq G$. Let $S_{i+1} = \ker \sigma_{i+1} \tau$. Note that S_1 is a free group. We have $B \cap S_{i+1} = 1$ and

$$\sigma_{i+1}(\tau(b)) = \begin{pmatrix} b & 0 \\ t_{B,i}(b-1) & 1 \end{pmatrix}, \quad b \in B, \quad B \in \mathcal{B}_0.$$

By Lemma 4 we have $S_{i+1} = [S_i, S_i]$ for all $i \in \Omega$. Therefore

$$\bigcap_{i=1}^{\infty} S_i \leq \bigcap_{j=1}^{\infty} S_1^{(j)} = 1$$

and τ is injective. This concludes the proof of the Main Theorem.

References

- [1] J. Lewin, T. Lewin, An embedding of the group algebra of a torsion-free one-relator group in a field, *J. Algebra* 52 (1978) 39–74.
- [2] P.H. Kropholler, P.A. Linnell, J.A. Moody, Applications of a new K -theoretic theorem to soluble group rings, *Proc. Amer. Math. Soc.* 104 (1988) 675–684.

- [3] R.C. Lyndon, P.E. Schupp, *Combinatorial Group Theory*, Springer, Berlin, 1977.
- [4] W. Magnus, Über diskontinuierliche Gruppen mit einer definierenden Relation. (Der Freiheitssatz), *J. Reine Angew. Math.* 163 (1930) 141–165.
- [5] N.S. Romanovskii, Free subgroups of finitely presented groups, *Algebra Logic* 16 (1978) 62–68.
- [6] N.S. Romanovskii, A generalized theorem on freedom for pro- p -groups, *Siberian Math. J.* (1986) 267–280.
- [7] N.S. Romanovskii, On Shmelkin embeddings for abstract and profinite groups, *Algebra Logic* 38 (1999) 598–612.
- [8] J.S. Wilson, On growth of groups with few relators, *Bull. London Math. Soc.* 36 (2004) 1–2.