

Pathwise and probabilistic analysis in the context of Schramm-Loewner Evolutions (SLE)



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To my parents and grandparents

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Abstract

On the pathwise analysis side, this thesis contains results between Rough Path Theory and Schramm-Loewner Evolutions (SLE). In the study of Rough Differential Equations questions such as continuity of the solutions, methods of approximations of solutions and their uniqueness/non-uniqueness depending on the behavior of parameters of the equation, appear naturally. We adapt these type of questions to the study of the backward and forward Loewner differential equation in the upper half-plane, conformal welding homeomorphism and the SLE traces. On the probabilistic analysis side, we study a coordinate change of the Loewner equation in which we obtain via a random time change, a stochastic dynamics on a specific line in the upper half-plane, that is depending only on the argument of the points. In this setting, we analyze this dynamics and obtain results about related objects.

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Chapter 1

Introduction

The Loewner equation (also known as the Loewner evolution) was introduced by Charles Loewner in 1923 in [33] and it played an important role in the proof of the Bieberbach Conjecture [9] by Louis de Branges in 1985 in [13]. In 2000, Oded Schramm introduced in [42] a stochastic version of the Loewner equation, the Stochastic Loewner Evolution (SLE_κ). The SLE_κ describes the evolution of a curve in terms of a driving function that is chosen to be $\sqrt{\kappa}B_t$, with $\kappa \geq 0$ a real parameter and B_t , $t \in [0, \infty)$, a real-valued standard Brownian motion. This is a one-parameter family of random planar fractal curves that are the only possible conformally invariant scaling limits of interfaces of a number of discrete models that appear in planar Statistical Physics. In several cases, it was proved that indeed the interfaces converge to the SLE_κ curves. For instance, it was proved in [27] that the scaling limit of loop-erased random walk (with the loops erased in a chronological order) converges in the scaling limit to SLE_κ , with $\kappa = 2$. Moreover, other two dimensional discrete models from Statistical Mechanics including Ising model clus-

ter boundaries, Gaussian free field interfaces, percolation on the triangular lattice at critical probability, and uniform spanning trees converge in the scaling limit to SLE_κ for values of $\kappa = 3$, $\kappa = 4$, $\kappa = 6$ and $\kappa = 8$, respectively, in the series of works [51], [45], [50] and [27]. In fact, the use of Loewner equation along with the techniques of stochastic calculus provided tools to perform a rigorous analysis of the scaling limits of the discrete models. Moreover, in this framework, a precise meaning of the passage to the scaling limit could be established and used to prove the conformal invariance of the limits. We refer to [25] for a detailed study of the object and many of its properties.

Rough Path Theory was introduced in 1998 by Terry Lyons in [34]. The theory provides a deterministic platform to study stochastic differential equations which extends both Young's integration and stochastic integration theory beyond regular functions and semi-martingales. Also, Rough Path Theory provides a method of constructing solutions to differential equations driven by paths that are not of bounded variation but have controlled roughness. Step by step, we introduce the ingredients and terminology necessary to characterize the roughness of a path and to give precise meaning to natural objects that appear in the study of rough paths. We also give precise meaning to the notion of solution of a differential equation with rough driver.

Recently, many papers were written at the interface between the aforementioned domains. The paper of Brent Werness [59] defines the expected signature for the SLE_κ traces, that is the expected values of iterated integrals of the path against itself. This approach provides new ideas about how one can use a version of Green's formula for rough paths and a certain observable for SLE_κ to compute the first three elements of the expected signature of SLE_κ in the regime $\kappa \in [0, 4]$. An extension to

this computation is provided in [10], where the authors show ways of computing the fourth grading of the signature (and do it explicitly for $SLE_{8/3}$, where the required observable is known), along with several other parts of the higher grading. From a different perspective, in [15] the authors question the existence of the trace for a general class of processes (such as semimartingales) as a driving function in the Loewner differential equation. These ideas are also developed in a Rough Path Theory flavour. More recently, Peter Friz with Huy Tran in [16] revisited the regularity of the SLE_κ traces and obtained a clear result using Besov spaces type analysis. Also, Atul Shekhar, Huy Tran and Yilin Wang studied the continuity of the traces generated by Loewner chains driven by bounded variation drivers in [48].

The work in this thesis is centered around the study of the backward Loewner Differential Equation and various quantities of interest of Loewner theory in both a probabilistic and pathwise manner. The probabilistic part is motivated by a study of a radius independent SDE obtained by a time change from the centered backward Loewner differential equation. By expressing the imaginary part of the backward Loewner flow in terms of the angular dynamics, we obtain a change of coordinates in which the Loewner dynamics can be redefined. This approach is more amenable to Stochastic Analysis techniques since the angular part is a one-dimensional diffusion.

On the pathwise side, this work addresses natural questions in the study of (rough) differential equations as in Rough Path Theory such as uniqueness/non-uniqueness of solutions starting from the origin, continuity of solutions (or related quantities) in the natural parameters of the system as well as numerical approximations of the solutions to the Loewner differential equation. Given the nature of the backward Loewner differential equation, i.e. the singularity of the drift when start-

ing at the origin, the backward Loewner differential equation cannot be studied with the typical techniques of Rough Path Theory. That is because this theory provides tools and techniques for studying rough differential equations with bounded vector fields, while in SLE theory one needs to consider the backward Loewner differential equation started from the origin, i.e. from a singularity.

While the uniqueness/non-uniqueness of solutions and the sequential continuity of the welding homeomorphisms chapters are closely linked with the structure of the backward Loewner differential equation and extensions to the boundary of the conformal maps that satisfy this differential equation, the chapter on numerical approximations opens a new road in the study of singular rough differential equations that is not applicable only to the backward Loewner differential equation near the singularity, but rather to a wider class of equations. In this specific direction, we identify how the symmetries of the vector fields and the roughness of the drivers influence the behavior of the Taylor approximation series.

This thesis is divided in several chapters. In Chapter 2 and Chapter 3 we give an introduction to both SLE_κ Theory and Rough Path Theory.

The fourth chapter addresses the existence and uniqueness/non-uniqueness of solutions to the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ started from the origin with applications to the structural behaviour of the backward SLE_κ traces. In general, when studying the stochastic version of the Loewner differential equation as a differential equation, the uniqueness/non-uniqueness of the solutions to the backward Loewner differential equation starting from the origin is a natural question. These type of questions appear also in the study of Rough Differential Equations (RDEs) in Rough Path Theory. In the framework of Rough Path The-

ory, the uniqueness/non-uniqueness of solutions depends on the regularity of the vector field compared with the roughness of the driving process. For more details, we refer to [17]. As mentioned earlier, the backward Loewner differential equation started from the singularity cannot be interpreted directly as a RDE due to the singularity of the vector field. However, questions about uniqueness/non-uniqueness of solutions started from the singularity can be asked. In this chapter, we adapt these type of questions to the study of the backward Loewner differential equation in the upper half-plane and we use the information provided by the extension of the conformal maps to the boundary to perform the analysis. One main feature of the analysis is that uniqueness/non-uniqueness of solutions to the backward Loewner equation driven by Brownian Motion, started from the singularity, can be understood using some property of the dynamics on the boundary. The analysis consists of two parts: in the first part we study the existence of solutions to the backward Loewner differential equation starting from the origin by constructing a solution, while in the second part we question the uniqueness/non-uniqueness of solutions in this setting. In order to show the existence of the solutions as well as to study their uniqueness/non-uniqueness, we use the fact that the conformal maps describing SLE_κ can be continuously extended to the boundary. When performing this extension, we obtain a Bessel process of dimension $d(\kappa) = 1 - \frac{4}{\kappa}$, for $\kappa > 0$. The Bessel processes are studied extensively in a number of papers and monographs. For more details about the various properties and characterizations of these processes, we refer to [39] and also to [19]. As a main ingredient in our analysis of the uniqueness/non-uniqueness of solutions, we use the phase transition in the boundary behavior of the Bessel process along the real line for the extended maps, i.e. absorption at the origin of the Bessel process for $\kappa \leq 4$, and reflection at the origin

for $\kappa > 4$. In this manner, when comparing our method with the typical forward flow analysis, using the backward flow we recover also new structural information about the SLE_κ traces via Excursion Theory of this underlying Bessel process, such as a characterization of a subspace of its double points for $\kappa > 4$.

In the fifth chapter, we study the welding homeomorphism induced on the real line by the backward Loewner differential equation via identifying two points on the real line if they hit the origin under the backward flow simultaneously. In this manner we define a (random) real function by seeing the points on the left part of the origin as function of the points on the right part of the origin. We study the continuity of the welding homeomorphism induced by the backward Loewner differential equation with respect to the parameter κ for $\kappa \in (0, 4]$. For more details about the welding homeomorphism induced by the backward Loewner differential equation, we refer to [41].

The problem of continuity of welding homeomorphisms in the parameter κ , for $\kappa \in (0, 4]$, is in the same spirit as the problem of the continuity of the traces generated by the Loewner differential equation with respect to perturbations of the driving term. The problem of continuity of traces generated by Loewner chains was studied in the context of chains driven by bounded variation drivers in [48], where the continuity of the traces generated by the Loewner chains was established. Also, the question appeared in [30], where the Loewner chains were driven by Hölder-1/2 functions with norm bounded by σ , with $\sigma < 4$. In this context, the continuity of the corresponding traces was established with respect to the uniform topologies on the space of drivers and on the space of simple curves in $\mathbb{H}$. Another paper that addressed a similar problem is [47], in which the condition $\|U\|_{1/2} < 4$ is avoided at the cost of assuming some conditions on the limiting trace. Some stronger continuity

results are obtained in [15] under the assumption that the driver has finite energy, in the sense that $\dot{U}$ is square integrable. The question appeared naturally also when considering the solution of the corresponding welding problem in [5]. In this paper it is proved that the trace obtained when solving the corresponding welding problem is continuous in a natural sense.

In the context of SLE_κ traces, the problem was studied in [56], where the a.s. continuity in κ of the SLE_κ traces was proved for any $\kappa < 8(2 - \sqrt{3})$.

In this chapter, we prove the sequential continuity of the welding homeomorphisms in the parameter κ induced by the backward Loewner differential equation when driven by $\sqrt{\kappa}B_t$, for $\kappa \in (0, 4]$, a.s. everywhere except at most countably many points on the real axis, with respect to the Euclidean norm. The restriction to this interval is a consequence of the connection between the SLE_κ traces and the welding homeomorphisms in this regime. We also prove the sequential continuity in the parameter $\kappa \neq 8$ of the SLE_κ traces for sequences $\kappa_n \rightarrow \kappa$ that respect a technical condition, as well as of a certain algorithm that approximates these traces.

In Chapter 6 we provide a closed-form expression for a law obtained in the study of the backward Loewner flow. In the last years, ergodic properties of the Loewner flow were studied in [60], in which the ergodic properties of the tip of the SLE_κ trace are computed in capacity time parametrization. In our analysis, we perform a random time change of the backward Loewner differential equation.

After performing the random time change, we obtain a one-dimensional diffusion, that describes, via a time change, the cotangent of the argument of the points in the in $\mathbb{H}$ under the backward Loewner flow.

$$dT_u = -4 \frac{T_u}{1 + \kappa T_u^2} du - dB_u, \quad T_0 = 0. \quad (1.0.1)$$

We use this diffusion process in order to characterize the law of the arguments of the backward Loewner flow. In order to achieve this, we show the convergence in a random ergodic average sense of the law of the diffusion T_u described by (1.0.1) to its stationary measure. The density of this measure has the closed-form expression

$$\rho(T) = C (\kappa T^2 + 1)^{-4/\kappa} ,$$

where C is a normalizing constant. We also find a sequence of random times on which the law of the argument of points under the backward Loewner flow converges to this stationary measure in the total variation sense, solving in this manner a dual Skorokhod-like Embedding Problem. For $\kappa = 4$, we recover a precise construction of the random times on which this law is uniform. Moreover, we show that these laws are part of a family of distributions that are stationary for the process T_u , for various choices of $\kappa \in [0, 8)$. We also show that this analysis provides a phase transition in the behavior of the backward Loewner flow at $\kappa = 8$, from the integrability of the density of the stationary measure for $\kappa < 8$ (positive recurrence of the process T_u) to non-integrability of the density of this quantity for $\kappa \geq 8$ (null-recurrence of the process T_u) .

The work in Chapter 7 is situated at the interface between SLE Theory and Rough Path Theory. We show that the backward Loewner differential equation started away from singularity is a Rough Differential Equation. Furthermore, we show the continuity of the solutions of this equation in the parameters of the system. We prove also results about truncated Taylor approximations of the forward Loewner differential equation near the origin. Questions about numerical approximations of Stochastic Differential Equations are of broad interest, see [12]. In the last chapter, we do this analysis for the forward Loewner differential equation near the singularity.

Chapter 2

Introduction to SLE_κ

2.1 The Loewner differential equation and SLE_κ

An important object in the study of the Loewner differential equation is the $\mathbb{H}$ -compact hull that is a bounded set in $\mathbb{H}$ such that its complement in $\mathbb{H}$ is simply connected. To every compact $\mathbb{H}$ -hull, that we typically denote with K we associate a canonical conformal map $g_K : \mathbb{H} \setminus K \rightarrow \mathbb{H}$ that is called the *mapping out function* of K .

Using the Riemann Mapping Theorem, we get uniqueness by imposing the *hydrodynamic normalization at infinity* for g_t (i.e. we require that $g_t(z)$ has no constant term and its complex derivative is 1). The mapping near infinity is of the form

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}), \quad |z| \rightarrow \infty.$$

The coefficient a_K that appears in the expansion at infinity of the mapping has the traditional name *halfplane capacity*. Throughout the Thesis we use the notation $\text{hcap}(K)$ for the halfplane capacity a_K .

We start by introducing the Chordal Loewner Theory which establishes a one-to-one correspondence between continuous valued paths $(U_t)_{t \geq 0}$ and an increasing families $(K_t)_{t \geq 0}$ of compact $\mathbb{H}$ -hulls having a certain local growth property. Throughout the thesis, we typically denote the family of growing compact hulls with K_t and the complements with $H_t := \mathbb{H} \setminus K_t$. Firstly, we define the radius of a hull to be

$$\text{rad}(K) = \inf\{r \geq 0 : K \subset r\mathbb{D} + x \text{ for some } x \in \mathbb{R}\}.$$

Definition 2.1.1. *Let $(K_t)_{t \geq 0}$ be a family of increasing $\mathbb{H}$ -hulls, i.e. K_s is contained in K_t whenever $s < t$. For $K_{t+} = \bigcap_{s>t} K_s$ and for $s < t$, set $K_{s,t} = g_{K_s}(K_t \setminus K_s)$. We say that $(K_t)_{t \geq 0}$ has the local growth property if*

$$\text{rad}(K_{t,t+h}) \rightarrow 0 \text{ as } h \rightarrow 0, \text{ uniformly on compacts in } t.$$

The first connection between the family of growing compact $\mathbb{H}$ -hulls and the real-valued path $(U_t)_{t \geq 0}$ is done in the following proposition.

Proposition 2.1.2 (Proposition 7.1 of [6]). *Let $(K_t)_{t \geq 0}$ be an increasing family of compact $\mathbb{H}$ -hulls having the local growth property. Then, $K_{t+} = K_t$ for all t . Moreover, the mapping $t \mapsto \text{hcap}(K_t)$ is continuous and strictly increasing on $[0, \infty)$. Moreover, for all $t \geq 0$, there is a unique $U_t \in \mathbb{R}$ such that $U_t \in \bar{K}_{t,t+h}$, for all $h > 0$, and the process $(U_t)_{t \geq 0}$ is continuous.*

The process $(U_t)_{t \geq 0}$ is called the *Loewner transform* of $(K_t)_{t \geq 0}$. The following map $t \mapsto \text{hcap}(K_t)/2$ is a non-decreasing homeomorphism on $[0, T)$ and by choosing τ to be the inverse of this homeomorphism, we obtain a new family of hulls K'_t in a new parametrization such that $\text{hcap}(K'_t) = 2t$. This is the canonical parametrization that we use throughout the thesis. We use the standard terminology for this, i.e

parametrization by halfplane capacity.

In the following proposition, we introduce the Loewner differential equation starting from the family of growing compact hulls. The main idea is that the local growth property of the hulls gives a description in terms of a specific differential equation for the associated mapping out functions. Throughout the thesis, we are mostly interested in the cases where the hull K_t is the hole-filling $\gamma[0, t]$. As such, although we state the theorems in the general form (i.e. for compact $\mathbb{H}$ -hulls), we draw in the associated pictures the hulls as simple curves in $\mathbb{H}$.

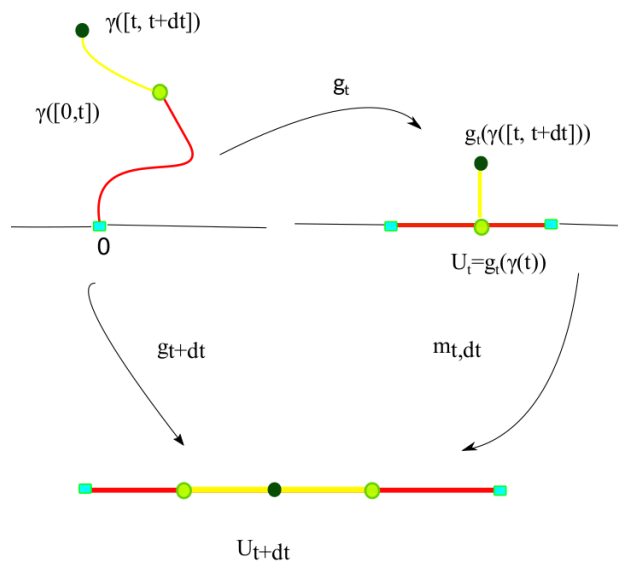


FIGURE 2.1.1. An illustration of the dynamics captured by the Loewner differential equation.

Proposition 2.1.3 (Proposition 7.3 of [6]). *Let $(K_t)_{t \geq 0}$ be a family of increasing compact hulls in $\mathbb{H}$ satisfying the local growth property and that are parametrized by the halfplane capacity. Let $(U_t)_{t \geq 0}$ be its Loewner transform. Set $g_t = g_{K_t}$ and $T(z) = \inf\{t \geq 0 : z \in K_t\}$. Then, for all $z \in \mathbb{H}$, the function $(g_t(z) : t \in [0, T(z)])$*

is differentiable with respect to t and satisfies the Loewner differential equation

$$\dot{g}_t(z) = \frac{2}{g_t(z) - U_t}.$$

Moreover, if $T(z) < \infty$ then $g_t(z) - U_t \rightarrow 0$ as $t \rightarrow T(z)$.

The reverse situation is also true, i.e. from the driving function U_t , we recover the family of growing compact $\mathbb{H}$ -hulls. We have this result in the following Theorem. Note that in [25], the result is stated for Loewner differential equation driven by measures on the real line. We state it only for the particular choice of measure on the real line $\mu_t = 2\delta_{U_t}$.

Theorem 2.1.4 (Theorem 4.6 of [25]). *For all $z \in \mathbb{H} \setminus \{\zeta_0\}$, there is a unique time $T(z) \in (0, \infty]$ and a unique continuous map $(g_t(z) : t \in [0, T(z)))$ in $\mathbb{H} \setminus K_t = H_t$ such that, for all $t \in [0, T(z))$ we have $g_t \neq U_t$ and*

$$g_t(z) = z + \int_0^t \frac{2}{g_s(z) - U_s} ds,$$

and such that $|g_t(z) - U_t| \rightarrow 0$ as $t \rightarrow T(z)$ whenever $T(z) < \infty$. Set $\zeta_0 = 0$ and define

$$H_t = \{z \in \mathbb{H} : T(z) > t\}.$$

Then, for all $t \geq 0$ H_t is open and $g_t : H_t \rightarrow \mathbb{H}$ is conformal onto $\mathbb{H}$. Moreover, the family of sets $K_t = (z \in \mathbb{H} : T(z) \leq t)$ is an increasing family of compact $\mathbb{H}$ -hulls having the local growth property with $hcap(K_t) = 2t$, and $g_{K_t} = g_t$, for all t . Moreover, the driving function U_t is the Loewner transform of $(K_t)_{t \geq 0}$.

We emphasize that the compact hull K_t is implicitly defined from the Loewner differential equation and its driver U_t by $K_t = (z \in \mathbb{H} : T(z) \leq t)$. The process

$(g_t(z) : t \in [0, T(z)))$ is called *the maximal solution starting from z* and $T(z)$ is called *lifetime of the solution*.

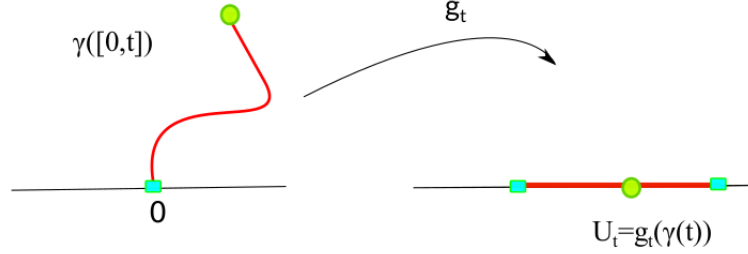


FIGURE 2.1.2. The conformal map that removes the curve grown up to time t .

The SLE_κ driving function In this subsection, we give a brief sketch of the proof that given the principles that our curves should satisfy, then the right candidate for the driver in Loewner equation is $\sqrt{\kappa}B_t$ where $\kappa \in \mathbb{R}_+$. When SLE_κ was defined by Oded Schramm, the purpose of it was to be the suitable candidate for a family of random curves in a domain $D \subset \mathbb{C}$ that respected two principles. The principles were motivated by the study of Schramm to give a precise meaning of the scaling limits of planar loop erased random walk with loops erased in the chronological order. When studying the planar loop erased random walk, Schramm realized that the scaling limit (if existed any) should manifest some *Domain Markov Property* and *Conformal Invariance*, that are presented in the following. Note that, chordal SLE_κ law can be defined in every simply connected domain, but we do not insist on the general definition in this thesis. This concept is used to make precise the following proprieties.

- [Domain Markov] Given the curve $\gamma[0, T]$, then the law of $\gamma[T, \infty]$ is a chordal SLE_κ in the unbounded component of $\mathbb{H} \setminus \gamma[0, T]$, from $\gamma(t)$ to ∞ .
- [Conformal Invariance] The chordal SLE_κ law in the unbounded component

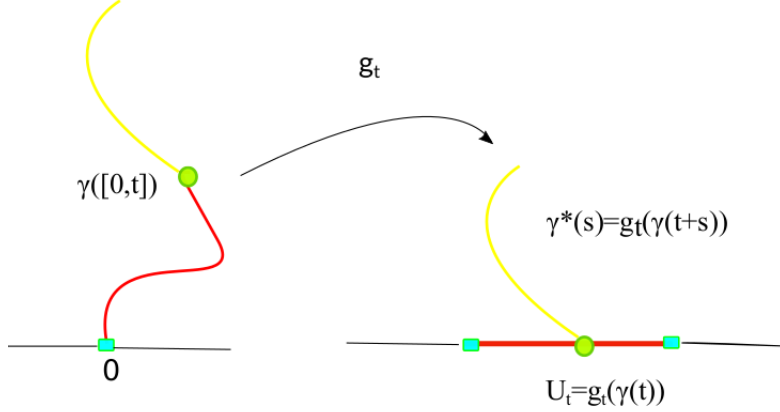


FIGURE 2.1.3. Conformal invariance of the curve γ .

of $\mathbb{H} \setminus \gamma[0, t]$ from $\gamma(t)$ to ∞ is the pull-back of the chordal SLE_κ law in $\mathbb{H}$ from U_t to ∞ .

If we require these two properties and a certain symmetry of the law on curves with respect to the imaginary axes, then the driving function in Loewner equation should be a $\sqrt{\kappa}B_t$. In the following, we give a sketch of the proof for this result of Oded Schramm that marks the introduction of SLE_κ .

Sketch of the proof. Given the Domain Markov Property, the curve from U_t to $+\infty$ that is denoted by $\gamma^*(s)$ has the law of a chordal SLE_κ curve in $\mathbb{H}$ from U_t to $+\infty$ that is independent of $\gamma[0, t]$. Note that $\gamma^*(s)$ is determined by the driving function $U_{t+s} - U_t$, $s \geq 0$, and $\gamma[0, t]$ by the driving function U_s , $0 \leq s \leq t$. We also have by the domain Markov property that $\gamma^*(s)$ is independent of $\gamma[0, t]$ means that the driving function after time t should be independent of the driving function up to time t , i.e. U_t is a Markov process.

Moreover, by conformal invariance (since g_t^{-1} is a conformal map), we have

that $\gamma^*[0, t]$ is identical in law to $\gamma[0, t]$, so U_s , for $0 \leq s \leq t$, should have the same law as $U_{t+s} - U_t$, $0 \leq s \leq t$. Thus, the increments of the U_t process have the same distribution and are independent, hence are i.i.d. Moreover, by writing $dU_t = bdt + \sigma dB_t$ we obtain that a and b are forced to be constants. Thus, the conformal invariance and the Domain Markov property force the driving process U_t to be a Brownian motion with drift. If we use our last consideration, i.e. the law of the SLE_κ curve should be invariant under reflections about the imaginary axis, we obtain that $b = 0$ leaving $dU_t = \sigma dB_t$ for some constant $\sigma > 0$ (we typically denote this by $\sqrt{\kappa}$, due to historical reasons).

□

2.2 SLE_κ definition, existence of the trace and first properties

The Schramm-Loewner Evolution (SLE_κ) is a one-parameter (usually denoted by $\kappa \in \mathbb{R}_+$) family of random planar growth processes constructed as a solution to Loewner equation when the driving term is a re-scaled Brownian motion. Thus, when studying the SLE_κ , in the upper half-plane, we study the corresponding families of conformal maps in the formats

- (i) Considering $f_t(z) = g_t^{-1}(z)$, where the $g_t^{-1}(z)$ is the inverse in the z variable, we obtain the partial differential equation version for the chordal SLE_κ in the upper half-plane

$$\partial_t f(t, z) = -\partial_z f(t, z) \frac{2}{z - \sqrt{\kappa} B_t}, \quad f(0, z) = z, z \in \mathbb{H}. \quad (2.2.1)$$

(ii) Forward differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t g(t, z) = \frac{2}{g(t, z) - \sqrt{\kappa} B_t}, \quad g(0, z) = z, z \in \mathbb{H}. \quad (2.2.2)$$

(iii) Time reversal differential equation version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t, z) = \frac{-2}{h(t, z) - \sqrt{\kappa} B_t}, \quad h(0, z) = z, z \in \mathbb{H}. \quad (2.2.3)$$

There are connections between these three formulations for studying families of conformal maps. The family of conformal maps satisfying (2.2.1) is related with the family of conformal maps satisfying (2.2.2) by the fact that at each instance of time $t \in [0, T]$ the map $z \rightarrow g_t(z)$ is the inverse of the map $z \rightarrow f_t(z)$. In other words, the maps $f_t(z)$ “grow” the curve in the reference domain, while $g_t(z)$ maps conformally the slit domain obtained by the growing of the curve up to time $t \in [0, T]$ to the reference domain. The connection between the family of maps $h_t(z)$ and $g_t(z)$ in this case is captured in the following lemma.

Lemma 2.2.1 (Lemma 5.5 of [22]). *Let $h_t(z)$ be the solution to the backward Loewner differential equation with driving function $\sqrt{\kappa} B_t$ and let $f_t(z)$ be the solution of the partial differential equation version of the Loewner differential equation with the same driver. Then, for any $t \in \mathbb{R}_+$, the function $z \rightarrow f_t(z + \sqrt{\kappa} B_t) - \sqrt{\kappa} B_t$ and $z \rightarrow h_t(z)$ have the same distribution.*

Rohde and Schramm proved in [44] a fundamental result about the existence of the trace for the SLE_κ process for all values of $\kappa \neq 8$. In order to discuss the result, we introduce the notions and definitions that we use. We say that a continuous path

$(\gamma_t)_{t \geq 0}$ in $\bar{\mathbb{H}}$ generates a family of increasing compact $\mathbb{H}$ -hulls K_t if $H_t = \mathbb{H} \setminus K_t$ is the unbounded component of $\mathbb{H} \setminus \gamma[0, t]$, for all $t \geq 0$. One of the important results in the SLE_κ literature is the following.

Theorem 2.2.2 (Theorem 4.1 of [44]). *Let $(K_t)_{t \geq 0}$ be a SLE_κ for $\kappa \neq 8$. We denote g_t to be the Loewner flow and U_t be the Loewner transform. Then, $\hat{g}_t^{-1}(z) = g_t^{-1}(z + U_t) : \mathbb{H} \mapsto H_t$ extends continuously to $\bar{\mathbb{H}}$ for all $t \geq 0$, almost surely. Moreover, if we set*

$$\gamma_t = \hat{g}_t^{-1}(U_t),$$

then γ_t is continuous and generates $(K_t)_{t \geq 0}$ almost surely.

Remark 2.2.3. The same result holds for $\kappa = 8$ as it was showed in [27] using a different approach.

Next, we introduce the SLE_κ trace.

Definition 2.2.4. *Let g_t be the SLE_κ . The SLE_κ trace is defined via*

$$\gamma(t) := \lim_{y \rightarrow 0} \hat{g}_t^{-1}(iy),$$

where $\hat{g}_t^{-1}(iy) = g_t^{-1}(iy + \sqrt{\kappa}B_t)$.

We remark that from the above theorem, the limit is shown to exist and to be continuous in the parameter t (that we interpret as time). We use the following scaling of the SLE_κ traces and of the Loewner maps.

Proposition 2.2.5 (Proposition 6.5 of [25]). *Let g_t be SLE_κ and let us take $r > 0$. Then $\hat{g}_t(z) := r^{-1}g_{r^2t}(rz)$ has the same distribution as chordal SLE_κ , i.e. if γ is an SLE_κ path, then $\hat{\gamma}(t) := r^{-1}\gamma_{r^2t}$ has the same distribution as γ .*

2.3 Forward Bessel processes and the phases of SLE_κ

In this subsection, we analyze the phases of SLE_κ in terms of the parameter κ . We show that there exists two phase transitions at $\kappa = 4$ and $\kappa = 8$.

Let $(g_t(x) : t \in [0, \tau(x)))$, $x \in \mathbb{R} \setminus \{0\}$ be the SLE_κ . Let us consider the extension of the conformal maps to the real line and consider $g_t(x) - U_t$ with $x \in \mathbb{R}$. Thus, we have that $g_t(x) - U_t \rightarrow 0$ as $t \rightarrow \zeta(x)$, whenever $\zeta(x) < \infty$. We introduce the notation

$$D = \frac{2}{\kappa}, \quad \tau(x) = \zeta(x\sqrt{\kappa})$$

and set

$$X_t(x) = \frac{g_t(x\sqrt{\kappa}) - U_t}{\sqrt{\kappa}},$$

for $t \in [0, \zeta(x))$.

In our setting, B_t is a standard Brownian motion that starts from 0 and we have that for $X_t(x) \neq 0$ for $t \in [0, \tau(x))$ and

$$X_t(x) = x + B_t + \int_0^t \frac{D}{X_s(x)} ds,$$

i.e. $X_t(x)$ is a Bessel process.

The Bessel processes manifest some phase transition that we capture in the following proposition.

Proposition 2.3.1 (Proposition 10.1 of [6]). *Let $x, y \in (0, \infty)$ with $x < y$. Then*

► *For $D \in (0, 1/4]$, we have that*

$$\mathbb{P}(\tau(x) < \tau(y) < \infty) = 1,$$

where $X_t(x)$ and $X_t(y)$ are driven by the same Brownian motion.

► For $D \in (1/4, 1/2)$, we have that

$$\mathbb{P}(\tau(x) < \infty) = 1.$$

$$\mathbb{P}(\tau(x) < \tau(y) < \infty) = \phi\left(\frac{y-x}{x}\right),$$

where ϕ is given by

$$\phi(\theta) = C(\kappa) \int_0^\theta \frac{du}{u^{2-4a}(1-u)^{2a}},$$

$$\text{where } C(\kappa) = \frac{\Gamma(2/\kappa)}{\Gamma(8/\kappa-1)\Gamma(1-4/\kappa)}.$$

► For $D \in [1/2, \infty)$, we have that

$$\mathbb{P}(\tau(x) < \infty) = 0,$$

and moreover for $D \in [1/2, \infty)$, we have $X_t(x) \rightarrow \infty$ as $t \rightarrow \infty$ almost surely.

These results give an interpretation of the phase transition of the SLE_κ processes in terms of studying the Loewner flow on the real line obtained from the continuous extension of the conformal maps as a Bessel process.

Proposition 2.3.2 (Proposition 10.3 of [6]). *Let γ be an SLE_κ trace. Then we have the following behaviors in terms of κ :*

► For $\kappa \in (0, 4]$, we have that $\gamma(0, \infty) \cap \mathbb{R} = \{0\}$, almost surely.

- For $\kappa \in (4, 8)$ and all $x, y \in (0, \infty)$, γ hits $[x, \infty)$ and

$$\mathbb{P}(\gamma \text{ hits } [x, x+y]) = \phi\left(\frac{y}{x+y}\right).$$

- When $\kappa \in [8, \infty)$, then $\mathbb{R} \subseteq \gamma[0, \infty)$, almost surely.

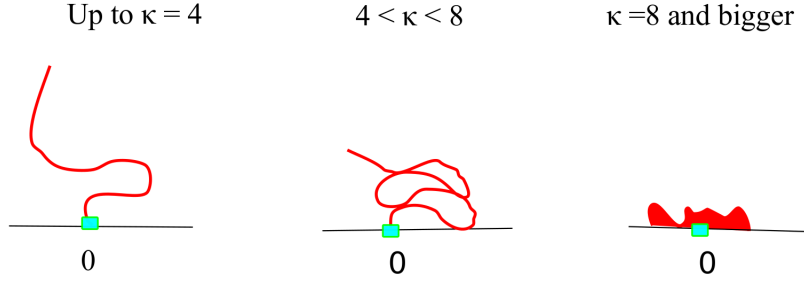


FIGURE 2.3.1. The phases of SLE_κ .

Concerning the long term behavior of the SLE_κ traces, we state the following result.

Theorem 2.3.3 (Theorem 11.1 of [6]). *Let $(\gamma_t)_{t \geq 0}$ be a SLE_κ curve. Then γ is almost surely transient for all κ , i.e. $|\gamma_t| \rightarrow \infty$ as $t \rightarrow \infty$.*

An important result is the following theorem, which gives a characterization of the behavior of the SLE_κ traces for $\kappa \in \mathbb{R}_+$. The result is proved using the phase transition in the behavior of the forward Bessel process as shown above.

Theorem 2.3.4 (Theorem 11.2 of [6]). *Let $(\gamma_t)_{t \geq 0}$ be a SLE_κ . Then, we have the following behaviors when the parameter κ changes:*

- For $\kappa \in [0, 4]$, $(\gamma_t)_{t \geq 0}$ is almost surely a simple path.
- For $\kappa \in (4, 8)$, $\bigcup_{t \geq 0} K_t = \mathbb{H}$, almost surely and for each $z \in \bar{\mathbb{H}} \setminus \{0\}$, $(\gamma_t)_{t \geq 0}$ does not hit z almost surely.

► For $\kappa \in [8, \infty)$, $\gamma[0, \infty) = \bar{\mathbb{H}}$, *almost surely*.

In this thesis, we will give, between others, a different perspective on the phase transition of the SLE_κ traces as we vary the parameter κ . In our proof, we will use the backward Loewner differential equation and when extending the conformal mappings to the boundary, we will obtain a Bessel process in a different regime of dimensions than the one obtained in the forward case. In our setting, we will obtain new structural information about the SLE_κ traces.

Chapter 3

Introduction to Rough Path Theory

In this chapter we introduce the basic notions in Rough Path Theory such as p -variation, Young's Theory of Integration, the notion of signature of a path and the underlying tensor algebra, the definition of a Rough Path and concepts related to Rough Differential Equations. The main references for the material presented in this chapter are [35] and [17].

Let $X_{[s,t]}$ denote the restriction of the path X to the compact interval $[s, t]$. We introduce the notion of p -variation.

Definition 3.0.1. *Let (E, d) be a metric space. The p -variation of a path $X : [0, T] \rightarrow E$ is defined by*

$$||X_{[0,T]}||_{p-var} := \sup_{\mathcal{D}=(t_0,t_1,\dots,t_n)\subset[0,T]} \left(\sum_{i=0}^{n-1} d(X_{t_i}, X_{t_{i+1}})^p \right)^{\frac{1}{p}},$$

where the supremum is taken over all finite partitions of the interval $[0, T]$.

Unless stated, the metric space E is a finite dimensional real vector space V with dimension d and basis vectors $e_1, \dots, e_d$. Throughout the thesis, we use the

notation $X_{s,t} = X_t - X_s$. We introduce gradually the notions used in Rough Path Theory. Let us first define $\Delta_T = \{(s,t) | 0 \leq s \leq t \leq T\}$. We introduce next the fundamental notion of control.

Definition 3.0.2. *A control on $[0, T]$ is a non-negative continuous function*

$$\omega : \Delta_T \rightarrow [0, \infty)$$

for which

$$\omega(s, t) + \omega(t, u) \leq \omega(s, u),$$

for all $0 \leq s \leq t \leq u \leq T$, and $\omega(t, t) = 0$, for all $t \in [0, T]$.

For a path $X : [0, T] \rightarrow E$ of finite p -variation, define $\omega_X(s, t) := \|X_{[s,t]}\|_{p-var}^p$. To check that this quantity indeed defines a control we need to check both properties along with the continuity (the continuity is proved in section 1.2.2 of [35] and Proposition 5.8 of [17]). Moreover, this specific control is used to provide a reparametrization of the interval $[0, T]$ such that X becomes Hölder continuous with exponent $\frac{1}{p}$. To see this, consider the function $\tau : [0, T] \rightarrow [0, T]$ to be the inverse of the function $t \rightarrow \omega_X(0, t) \frac{T}{\omega_X(0, T)}$. Using this inverse function, we obtain that $d(X_{\tau(s)}, X_{\tau(t)})^p \leq \frac{\omega_X(0, T)}{T}(t - s)$.

3.1 Young integration

In what follows next, let us consider E to be a Banach space and X, Y to be continuous functions (paths) with values in a Banach spaces. A first extension of the usual Riemann-Stieltjes integral is given by Young's Theory of Integration. Using

the results of Young, we can make sense of $\int_0^t Y_s dX_s$ provided that X and Y have finite p -variation and finite q -variation respectively with $\frac{1}{p} + \frac{1}{q} > 1$. Note that for a path X of bounded variation (i.e. $p = 1$) Young's theory of integration applies. To be precise, we have the result

Theorem 3.1.1 (Young, Theorem 1.16 of [35]). *Let V and W be Banach spaces and $X : [0, T] \rightarrow V$ and $Y : [0, T] \rightarrow \mathbf{L}(V, W)$ be two paths of finite p -variation and q -variation respectively with $\frac{1}{p} + \frac{1}{q} > 1$. Then, the limit*

$$\lim_{|\mathcal{D}| \rightarrow 0, \mathcal{D} \subset [0, t]} \sum_{i=0}^{n-1} Y_{t_i}(X_{t_i, t_{i+1}})$$

exists for all $t \in [0, T]$ and we define $\int_0^t Y_s dX_s$ as this limit. Furthermore, as a path in W , $\int_0^\cdot Y_s dX_s$ has finite p -variation.

Note that the Theorem holds independent of the sequence of partitions chosen provided that the size of the mesh $|\mathcal{D}| = \max_{0 \leq i \leq n-1} (t_{i+1} - t_i) \rightarrow 0$.

3.2 Differential equations driven by signals with finite p -variation for $p < 2$

In the classical theory of ordinary differential equations, the Cauchy-Peano Theorem asserts that when X has bounded variation, if the vector field f is a continuous function, then the differential equation $dY_t = f(Y_t) dX_t$ has solutions. However, this result does not guarantee uniqueness of solutions. Picard-Lindelöf Theorem tells us that in order to have uniqueness of the solution, we have to impose more conditions on the function f , i.e. we need f to be Lipschitz continuous. In order to extend these results to signals with finite p -variation for $1 \leq p < 2$, presumably we need to

use a class of vector fields smoother than the continuous ones. For example, in order to make sense of the integral $\int f(Y)dX$, we need $f(Y)$ to have finite q -variation for some q such that $\frac{1}{p} + \frac{1}{q} > 1$. It is clear that not every continuous function does that, but we can restrict to the specific class of functions with this property.

The main result of this section, presented in [35] is a version of the classical Picard's Fix Point Theorem in the context of differential equations driven by signals with finite p -variation for some $p < 2$. In order to state it, we need to introduce the notion of $Lip(\gamma)$ function. For this we consider again the metric space E to be a Banach space.

Definition 3.2.1. *Let V and W be two Banach spaces. Let $k \geq 0$ be an integer. Let $\gamma \in (k, k+1]$ be a real number. Let F be a closed subset of V . Let $f : F \rightarrow W$ be a function. For each integer $j = 1, \dots, k$ let $f^j : F \rightarrow \mathbf{L}(V^{\otimes j}, W)$ be a function which takes its values in the space of j -linear mappings from V to W . The collection $(f = f^0, f^1, \dots, f^k)$ is an element of $Lip(\gamma, F)$ if the following condition holds.*

There exists a constant M such that, for each $j = 0, \dots, k$,

$$\sup_{x \in F} |f^j(x)| \leq M$$

and there exists a function $R_j : V \times V \rightarrow \mathbf{L}(V^{\otimes j}, W)$ such that, for each $x, y \in F$ and each $v \in V^{\otimes j}$, we have

$$f^j(y)(v) = \sum_{l=0}^{k-j} \frac{1}{l!} f^{j+l}(x)(v \otimes (y-x)^{\otimes l}) + R_j(x, y)(v),$$

and

$$|R_j(x, y)| \leq M |x - y|^{\gamma-j}.$$

The smallest M for which the inequalities hold for all j is called the $Lip(\gamma, F)$ -norm of f .

Following [35], when V is finite dimensional we obtain that there exist for all closed F a continuous extension operator $Lip(\gamma, F) \rightarrow Lip(\gamma, V)$. Thus, in this manner we obtain $Lip(\gamma, V) = Lip(\gamma)$, i.e. bounded continuous functions on V which are k -times continuously differentiable with bounded derivatives on V and whose k -th differential is Hölder continuous with parameter $\gamma - k$. With this definition at hand, we can state the version of Picard's Fix Point Theorem for differential equations driven by signals with finite p -variation for some $p < 2$.

Theorem 3.2.2 (Picard, Theorem 1.28 of [35]). *Let p and γ be such that $1 \leq p < 2$ and $p < \gamma$. Assume that X has finite p -variation and that f is $Lip(\gamma)$. Then, the differential equation $dY_t = f(Y_t)dX_t$, admits a unique solution, i.e. for every $\zeta \in W$ there exist a unique path $Y : [0, T] \rightarrow W$ of finite p -variation which satisfies $Y_0 = \zeta$ and*

$$Y_t = \int_0^t f(Y_s)dX_s.$$

The technique used to prove this result is similar with the classical proof of the Picard's Fix Point Theorem. We consider F that sends a path $Y : [0, T] \rightarrow W$ to a new path defined via $F(Y_t) = \zeta + \int_0^t f(Y_s)dX_s$ and we prove that is a contraction under suitable conditions.

3.3 The signature of a path

In this section, we introduce one of the fundamental ingredients of the theory, the collection of its iterated integrals, i.e. the *signature of the path*. This object appears

naturally when one uses iteration in order to solve linear differential equations. Moreover, the signature of a path has very interesting algebraic properties.

In order to simplify the computations and to introduce the signature of a path in a clear manner, throughout this section we consider the driver X to be of bounded variation. For this specific class of drivers, we consider the linear differential equation $dY_t = \bar{f}(Y_t)dX_t$, where $\bar{f} : W \rightarrow \mathbf{L}(V, W)$ is a linear map.

The collection of iterated integrals appears naturally when one tries to apply an iterative procedure in order to find a solution to the ordinary differential equation via fix point arguments. To make this precise, we remark that $\bar{f}(Y_t)dX_t$ can be understood in two ways. Every linear map $\bar{f} \in \mathbf{L}(W, \mathbf{L}(V, W))$ in a natural way also induces a linear map $\bar{f} : V \rightarrow \mathbf{L}(W)$, (we call it the same using a slight abuse of notation). Another abuse of notation that we use is that we have elements of $\mathbf{L}(W)$ act on W on the right instead of the left as this will simplify notation in the following.

We start with the constant path $Y_t^0 = F(Y^0)_t = Y_0(I + \bar{f}(\int_0^t dX_s))$. By re-iterating the procedure, we obtain that

$$Y_t^2 = F(Y^1)_t = Y_0 + \int_0^t \bar{f}(Y_s^1)dX_s = Y_0 \left(I + \bar{f} \int_0^t dX_s + \bar{f}^{\otimes 2} \int_0^t \left(\int_0^s dX_u \right) \otimes dX_s \right).$$

After k steps, we obtain that

$$Y_t^n = Y_0 \left(\sum_{k=0}^n \bar{f}^{\otimes k} \int_{0 < u_1 < \dots < u_k < t} dX_{u_1} \otimes dX_{u_2} \dots \otimes dX_{u_k} \right).$$

In the next part of the section, we study the tensor vector spaces where this iterated integrals live and we state a result that ensures that the iterated integrals do converge.

Let $e_1, e_2, \dots, e_d$ be a basis for V . The space $V^{\otimes k}$ is a d^k dimensional vector space with basis elements of the form $(e_{i_1} \otimes e_{i_2} \dots \otimes e_{i_k})_{(i_1, \dots, i_k) \in \{1, \dots, d\}^k}$. We store the indices $(i_1, \dots, i_k) \in \{1, 2, \dots, d\}^k$ in a multi-index I and let $e_I = e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_k}$. From the several possible norms on the tensor space $V^{\otimes k}$, see Definition 1.25 of [35] for the so-called *admissible norms*, due to its properties, we chose the *projective* or the l_1 norm defined for any element $x = \sum_{|I|=k} \lambda_I e_I \in V^{\otimes k}$ via $\|x\| = \sum_{|I|=k} |\lambda_I|$. The two main properties of this norm are that for any $x \in V^{\otimes k}$ and $y \in V^{\otimes m}$ we have that $\|x \otimes y\| = \|x\| \|y\|$ and that for $x = \sum_{|I|=k} \lambda_I e_I$ we have

$$\|M^{\otimes k} x\| = \left\| \sum_{|I|=k} \lambda_I M^{\otimes k} e_I \right\| \leq \sum_{|I|=k} |\lambda_I| \|M\|^k = \|M\|^k \|x\|.$$

By considering the basis of $V^{\otimes k}$ given by elements of the form e_I , we can take the dual basis in $(V^{\otimes k})^*$ as e_I^* that we associate the element $e_{i_1}^* \otimes e_{i_2}^* \otimes \dots \otimes e_{i_k}^* \in (V^*)^{\otimes k}$. We further consider $T((V^*)) = \bigoplus_{k=1}^{\infty} (V^*)^{\otimes k}$. Furthermore, for any I , e_I^* is an element of $T(V^*)$, i.e. the sub-algebra of $T((V^*))$ consisting of elements with finitely many non-zero coordinates and running through all the possible multi indices with $|I| > 0$, we find a basis for $T(V^*)$ consisting of e_I . We are now ready to introduce the following result that guarantees the convergence of the Picard iterates by controlling the norm of the iterated integrals appearing naturally when performing this method.

Proposition 3.3.1 (Proposition 2.2 of [35]). *Let $X : [0, T] \rightarrow V$ be a path of bounded variation. Then it follows that*

$$\left\| \int_{0 < u_1 < \dots < u_k < t} dX_{u_1} \otimes dX_{u_2} \dots \otimes dX_{u_k} \right\| \leq \frac{\|X_{[0,t]}\|_{1-var}^k}{k!}.$$

3.4 The tensor algebra

Given that the collection of iterated integrals is such an important object in the study of the linear differential equations, it is natural to consider it as a separate object which we call the signature of a path. This object takes value in the tensor algebra $T(V)$, which is described in this subsection.

The tensor algebra $T((V)) = \bigoplus_{k \geq 0} V^{\otimes k}$ is the infinite sum of all tensor products of V . For two elements $a = (a^0, a^1, \dots)$ and $b = (b^0, b^1, \dots)$ we define

$$a + b = (a^0 + b^0, a^1 + b^1, \dots)$$

and

$$a \otimes b = (c^0, c^1, \dots),$$

where $c^k = \sum_{i=0}^k a^i \otimes b^{k-i}$. These operations together with the typical component by component multiplication by scalars $\lambda a = (\lambda a^0, \lambda a^1, \dots)$, turn $T((V))$ into a real non-commutative algebra with unit $\mathbf{1} = (1, 0, \dots)$. An important aspect of this Tensor Algebra is that a generic element $d \in T((V))$ is invertible if and only if $d^0 \neq 0$, and moreover there is an explicit way to compute the inverse

$$d^{-1} = \frac{1}{d_0} \sum_{k \geq 0} \left(\mathbf{1} - \frac{1}{d_0} d \right)^{\otimes k}. \quad (3.4.1)$$

For all $d \in T((V))$, we define the exponential and the logarithm maps via

$$\exp d = \sum_{k \geq 0} \frac{1}{k!} d^{\otimes k}. \quad (3.4.2)$$

$$\log d = \log d^0 + \sum_{k \geq 1} \frac{(-1)^k}{k} \left(\mathbf{1} - \frac{1}{d_0} d \right), \quad (3.4.3)$$

whenever $d^0 > 0$. Like the inverse mapping, the logarithm is well defined since only finitely many terms of d contribute to each term of $\log(d)$. For the exponential map in the finite dimensional case it suffices to ensure the convergence of terms of each degree according to Lemma 2.19 of [35]. An important subspace of $T((V))$ that is denoted by B_0 is the subspace for which the term of degree zero is 0. Another important object is the multiplicative subgroup of $T((V))$ which is denoted by $T((\tilde{V}))$, for which the term of degree zero is 1. Then the maps $\exp : B_0 \rightarrow T((\tilde{V}))$ and $\log : T((\tilde{V})) \rightarrow B_0$ are bijections and inverses of each other.

3.5 The signature of a path, shuffle product and group-like elements

In this subsection, we introduce the signature of a path with p -variation for some $p < 2$ and describe some computation tools that are extremely useful when computing terms in the signature.

Definition 3.5.1. *Let $X : [0, T] \rightarrow V$ be a path of finite p -variation for some $p < 2$. For $k \geq 0$, let $\tilde{S}_k([s, t]) = \int_{s < u_1 < \dots < u_k < t} dX_{u_1} \otimes \dots \otimes dX_{u_k}$. Then the signature of the path $X_{[s, t]}$ is given by $S(X_{[s, t]}) = (1, \tilde{S}_1([s, t]), \tilde{S}_2([s, t]), \dots)$.*

Note that the signature of a path X lies in the subgroup $\tilde{T}((V))$ of $T((V))$. Note that the integral in the definition is a Young integral since $p < 2$. One of the properties of the signature is that is not sensitive to the starting point or to the speed that a path is traversed. That is, $S(X_{[s, t]}) = S(Y_{[u, v]})$ for $Y_w = X_{\tau(w)} + x$,

where $\tau : [s, t] \rightarrow [u, v]$ is an increasing bijection and $x \in V$. For two paths $X_{[s,t]}$ and $Y_{[u,v]}$, we define the concatenated path $(X * Y) : [s, t + v - u] \rightarrow V$ via

$$(X * Y)_w = \begin{cases} X_w, & \text{if } w \in [s, t] \\ X_t + Y_w - Y_t, & \text{if } w \in [t, t + v - u]. \end{cases}$$

In this setting, we have one result about how the signature map behaves for the concatenated path in terms of the paths, attributed to Chen.

Theorem 3.5.2 (Chen, Theorem 2.9 and Corollary 2.13 of [35]). *For two paths $X_{[s,t]}$ and $Y_{[u,v]}$ of finite p -variation for some $p < 2$, we have that*

$$S(X * Y) = S(X) \otimes S(Y).$$

Chen's Theorem also gives that the range of the signature is closed in $T((V))$. Moreover, if we consider the reversal of the path $X : [s, t] \rightarrow E$, $X^r : [s, t] \rightarrow E$ given by

$$X_u^r = X_{s+t-u},$$

then we have the following result.

Proposition 3.5.3 (Proposition 2.14 of [35]). *Let $X : [s, t] \rightarrow V$ be a path of finite p -variation for some $p < 2$, and $X_{[s,t]}^r$ be its reversal. Then it follows that*

$$S(X^r) = S(X)^{-1}.$$

Using the previous proposition and the fact that $S(x_{[s,t]}) = \mathbf{1}$ with $x \in V$, where $x_{[s,t]}$ is understood as the constant path at x). We conclude that the range of the signature map forms a multiplicative subgroup of $\tilde{T}((V))$.

We now define another useful property when it comes to dealing with signatures

of paths. For $i, j \in \mathbb{Z}$, we define the *shuffle product* $Sh(i, j) \subset S_{i+j}$ as all the permutations σ on the set $\{1, 2, \dots, i+j\}$ which preserve the order of $\{1, 2, \dots, i\}$ and $\{i+1, \dots, i+j\}$, that is $\sigma(1) < \dots < \sigma(i)$ and $\sigma(i+1) < \dots < \sigma(i+j)$. For two multi-indexes $U = (u_1, \dots, u_i)$, and $V = (v_1, \dots, v_j)$, we define the joint multi-index $H = (h_1, \dots, h_{i+j}) = (u_1, \dots, u_i, v_1, \dots, v_j)$, and for $\sigma \in S_{i+j}$ we define $\sigma(H) = (h_{\sigma_1}, \dots, h_{\sigma(i+j)})$. Then, we define for e_U^* and $e_V^* \in T(V^*)$ their shuffle product by

$$e_U^* \sqcup e_V^* = \sum_{\sigma \in Sh(i+j)} e_{\sigma^{-1}(H)}^* \in T(V^*).$$

We further define the *group-like elements* of $\tilde{T}((V))$, denoted by $G^{(*)}$, consisting of all elements $a \in \tilde{T}((V))$ for which $f(a)g(a) = (f \sqcup g)(a) \quad \forall f, g \in T(V^*)$.

The verification that $G^{(*)}$ forms indeed a subgroup of $\tilde{T}((V))$ is given in Lemma 2.17 in [35]. One of the most important results about this group is that this group is actually the subspace where the signature takes values.

Theorem 3.5.4 (Theorem 2.15 of [35]). *For a path $X_{[s,t]}$ of finite p -variation with $p < 2$, it holds that $S(X) \in G^{(*)}$.*

We put on $T((V))$ the product topology (i.e. the topology for which a sequence of elements $(a_n)_{n \geq 0} \in T((V))$ converges to $a \in T((V))$ in the product topology if and only if it converges coordinate-wise, i.e. for every $k \geq 0$, the limit $\lim_{n \rightarrow \infty} a_n^k = a^k$. Also, all the maps and operations discussed so far, that is $+, \otimes : T((V)) \times T((V)) \rightarrow T((V))$, inversion, exp and log are continuous with respect to the product topology on $T((V)) \times T((V))$.

The (associative) tensor algebra $T((V))$ can be viewed as a Lie algebra with the Lie bracket given by $[a, b] = a \otimes b - b \otimes a$. When studying the Lie algebra $T((V))$, two

Lie sub-algebras are of interest. First the Lie sub-algebra generated by the elements $V \subset T((V))$, which we denote by $\mathcal{L}(V)$, and $\mathcal{L}((V))$ that is the completion of $\mathcal{L}(V)$ in $T((V))$ with respect to the product topology. In order to describe $\mathcal{L}(V)$ and $\mathcal{L}((V))$, we have the following procedure. For the subspace $U, W \subset T((V))$, we define $[U, W] \subset T((V))$ as the subspace spanned by all the elements of the form $[a, b]$ with $a \in U$ and $b \in W$. We define inductively for $k \in \mathbb{N}$, $L_k = [V, L_{k-1}] \subset V^{\otimes k}$, with $L_0 = 0$ and $L_1 = V$. It follows that

$$\mathcal{L}((V)) = \bigoplus_{k \geq 0} L_k$$

and naturally $\mathcal{L}(V)$ consists of those elements of $\mathcal{L}((V))$ which have finitely many non-zero terms. The elements of $\mathcal{L}((V))$ are called Lie series and those of $\mathcal{L}(V)$ are called Lie polynomials. As a complete classification of group-like elements, we have the following result.

Theorem 3.5.5 (Theorem 2.23 of [35]). *An element is group-like if and only if its logarithm is a Lie series. That is, $\log G^{(*)} = \mathcal{L}((V))$.*

3.6 Free Nilpotent Lie Group and Rough Paths

All the spaces that we have defined so far in our analysis, have also a finite dimensional version (that we call the truncated version) which consists of terms up to a fixed degree n . In order to define them precisely, we consider

$$B_n = \{a = (a^0, a^1, \dots) | a^0 = a^1 = \dots = a^n = 0\}$$

that is an ideal of $T((V))$. We define the truncated tensor algebra by the quotient $T^{(n)}(V) = T((V))/B_n$ with $\pi_n : T((V)) \rightarrow T^{(n)}(V)$ the natural quotient map. As an algebraic structure, $T^{(n)}(V)$ is an algebra where multiplication $\otimes$ is the same as in $T((V))$, but truncated at level n (i.e. all the elements with degree higher than n are automatically null in this space). Between the truncated spaces, we can define truncated versions of the log map or of the exponential map. We can also view $T^{(n)}(V) = \bigoplus_{0 \leq k \leq n} V^{\otimes k}$ - as a finite sum of finite dimensional vector spaces. Thus, when V is of dimension d , $T^{(n)}(V)$ is of dimension $1 + d + \dots + d^n$. We also remark that we consider $T^{(n)}(V)$ endowed with the product topology from $T((V))$. In this setting, as before, the operations remain continuous.

We consider also the sets of truncated group-like elements and Lie polynomials of degree n , given by $G^{(n)} = \pi_n G^{(*)}$ and $\mathcal{L}^{(n)}(V) = \pi_n \mathcal{L}(V) = \pi_n \mathcal{L}((V))$. Also, remark that $\tilde{T}^{(n)}(V) = \pi_n \tilde{T}((V))$ is a simply connected n -step nilpotent Lie group with Lie algebra $\pi_n B_0$ for which $\exp^{(n)} : \pi_n B_0 \rightarrow \tilde{T}^{(n)}(V)$ is a diffeomorphism.

We have also the following proposition.

Proposition 3.6.1 (Lemma 2.24 and Proposition 2.25 of [35]). *We have that an element $a \in T((V))$ is group-like if and only if $\pi_n a \in G^{(n)}$ for all $n \geq 0$ and if and only if $\log^{(n)} \pi_n a \in \mathcal{L}^{(n)}(V)$ for all $n \geq 0$. Furthermore, $G^{(n)}$ is a closed, simply connected n -step nilpotent Lie subgroup of $\tilde{T}^{(n)}(V)$, for which $\mathcal{L}^{(n)}(V)$ is the Lie algebra and $\exp^{(n)} : \mathcal{L}^{(n)} \rightarrow G^{(n)}$ is the diffeomorphism.*

We call $G^{(n)}$ and $\mathcal{L}((V))$ the free-step nilpotent Lie group and algebra over V . Also, we can naturally consider the notion of truncated signature

$$S^{(n)}(X_{[s,t]}) = \pi_n S^{(n)}(X_{[s,t]})$$

as an element of $G^{(n)}$. We also have that for two paths X and Y ,

$$S^{(n)}(X * Y) = S^{(n)}(X) \otimes S^{(n)}(Y)$$

and

$$S^{(n)}(X)^{-1} = S^{(n)}(X^r),$$

where X^r is the reversal of X . An important result that gives a description of elements in $G^{(n)}$ is the Chow-Rashevskii Theorem.

Theorem 3.6.2 (Chow-Rashevskii Theorem, Theorem 2.26 of [35]). *For every $a \in G^{(n)}$ there exists a piecewise linear path $X : [0, T] \rightarrow V$ such that $a = S^{(n)}(X_{[0, T]})$.*

Remarkably, each element of $G^{(n)}$ is in fact the signature of some piecewise linear path. With this result at hand, we can define the Carnot-Caratheodory (CC) norm on the group $G^{(n)}$ by $\|a\|_{CC} = \inf_{S^{(n)}(X_{[0, T]})} \|X\|_{1-var}$ which is guaranteed to be finite. Since $G^{(n)}$ is not a vector space, the CC norm is not a norm in the real sense but gives rise to a metric on $G^{(n)}$ by $d_{CC}(a, b) = \|a^{-1}b\|_{CC}$. We are now ready to define the notion of *p-rough path*. This notion builds naturally on previous results. These paths are taking values in the Lie group $T^{(n)}(V)$ and satisfy an algebraic condition (Chen's identity) and an analytic bound that is very similar with the factorial decay of the norms of the signature levels presented in Proposition 3.3.1. We now define the notion of *multiplicative functional*.

Definition 3.6.3. *Let $n \geq 1$ be an integer and let $X : \Delta_T \rightarrow T^{(n)}(V)$ be a continuous map. Denote by $X_{s,t}$ the image of the interval (s, t) by X , and write*

$$X_{s,t} = (X_{s,t}^0, \dots, X_{s,t}^n) \in \mathbb{R} \oplus V \oplus V^{\otimes 2} \dots \oplus V^{\otimes n}.$$

The function X is called *multiplicative functional of degree n in V* if $X_{s,t}^0 = 1$ and for all $(s, t) \in \Delta_t$ we have

$$X_{s,u} \otimes X_{u,t} = X_{s,t} \quad \forall s, u, t \in [0, T].$$

The multiplicative property is called *Chen's identity*. To get familiar with the object, let us take the following example.

Example 3.6.4 (Example 3.2 from [35]). Let X be a multiplicative functional of degree 1 that is for all $(s, t) \in \Delta_T$, $X_{s,t} \in T^{(1)}(V) = \mathbb{R} \oplus V$. That is $X_{s,t} = (1, X_{s,t}^1)$, with $X_{s,t}^1 \in V$. Then Chen's identity reads

$$(1, X_{s,t}^1) = (1, X_{s,u}^1) \otimes (1, X_{u,t}^1) = (1, X_{s,u}^1 + X_{u,t}^1).$$

So, in $T^{(1)}(V)$ (at 'first level') the multiplication condition in $T^{(1)}(V)$ reduces to the additivity in V of the mapping $(s, t) \rightarrow X_{s,t}^1$. This fact is equivalent to the existence of a path $X_{s,t}^1 = x_t - x_s$, for all $(s, t) \in \Delta_T$. This path is unique up to an addition of a constant element in V . Note that if X is a multiplicative functional of degree $n \geq 2$ in V , then we have that $\pi_1 X : \Delta_T \rightarrow T^{(1)}(V)$ is a multiplicative functional of degree 1. This implies that there exists a classical path $x : [0, T] \rightarrow V$, which underlies X (i.e. $X_{s,t}^1 = x_t - x_s$) but X is not the signature of X in general. Consider for example the path $X_{s,t} = (1, 0, (t - s)w)$ that is a multiplicative functional of degree 2. If we look only at the first level, then the underlying path x is a constant path whose signature is simply $(1, 0, 0)$ which is different from X . We are now ready to define the central object of Rough Path Theory.

Definition 3.6.5. A p -rough path of degree n is a map $X : \Delta_T \rightarrow \tilde{T}^{(n)}(V)$ which satisfies Chen's identity $X_{s,t} \otimes X_{t,u} = X_{s,u}$ and the following 'level dependent' analytic bound

$$\|X_{s,t}^i\| \leq \frac{w(s,t)^{\frac{i}{p}}}{\beta_p(\frac{i}{p})!},$$

where $y! = \Gamma(y+1)$ whenever y is a positive real number and β_p , is a positive constant.

The factors $(\frac{i}{p})!$ and β_p in the definition of the p -rough path are not important due to the possibility of recalling; however, they become important in the following Extension Theorem that states that there is only one way to extend a p -rough path to the entire group $\tilde{T}((V))$.

Theorem 3.6.6 (Extension Theorem, Theorem 3.7 of [35]). *Let $p \geq 1$ be a real number and $n \geq 1$ be an integer. Let $X : \Delta_T \rightarrow T^{(n)}(V)$ be a multiplicative functional with finite p -variation controlled by w . Assume that $n \geq [p]$. Then there exists a unique extension of X to a multiplicative functional $\Delta_T \rightarrow T((V))$ which has finite p -variation, i.e. for every $m \geq [p] + 1$, there exists a unique continuous function*

$$X^m : \Delta_T \rightarrow V^{\otimes m}$$

such that

$$(s, t) \rightarrow X_{s,t} = (1, X_{s,t}^1, \dots, X_{s,t}^{[p]}, \dots, X_{s,t}^m, \dots) \in T((V)),$$

is a multiplicative functional with finite p -variation controlled by w , in the sense of Definition 3.6.5, with $\beta_p = p^2 \left(1 + \sum_{r=3}^{\infty} \left(\frac{2}{r-2} \right)^{\frac{[p]+1}{p}} \right)$.

We denote the space of all p -rough paths of degree $[p]$ by $\Lambda_p(V)$. The first

remark is that for every continuous map $X : \Delta_T \rightarrow \tilde{T}((V))$, we associate the path $x_t = X_{0,t}$, and conversely for every $x \in C([0, T], \tilde{T}^{(n)}(V))$ we can define the following mapping $X : \Delta_T \rightarrow \tilde{T}((V))$ by $X_{s,t} = x_s^{-1}x_t$, which indeed satisfies Chen's identity. Thus, every map $X : \Delta_T \rightarrow \tilde{T}((V))$ that satisfies Chen's identity is completely characterized by its associated path $x : t \rightarrow X_{0,t}$. For a path with finite p -variation for $p < 2$, that is canonically a p -rough path of degree $[p] = 1$, we have the signature $S(X) : (s, t) \rightarrow S(X_{[s,t]})$, that satisfies Chen's identity. Furthermore, a modification of the factorial decay of the signature gives that the signature satisfies the required analytic bound. To sum up, we are in the position to apply Theorem 3.6.6 to obtain that the signature is the unique extension of a p -rough path for $p < 2$. Also, the Extension map is a continuous mapping as stated in the following important result.

Theorem 3.6.7 (Continuity of the Extension Map, Theorem 3.10 of [35]).

Let $X, Y \in \Lambda_p(V)$ with p -variation control w and let $\varepsilon \in (0, 1)$. Then the bound

$$\|X_{s,t}^i - Y_{s,t}^i\| \leq \varepsilon \frac{w(s, t)^{\frac{i}{p}}}{\beta_p(\frac{i}{p})!},$$

holds for all $(s, t) \in \Delta_t$ and $i \geq 1$, provided that it holds for all $1 \leq i \leq [p]$.

3.7 The spaces of p -Rough Paths and Geometric Rough Paths

We introduce a metric on $\Lambda_p(V)$ which transform the space $\Lambda_p(V)$ in a complete metric space. For $X, Y \in \Lambda_p(V)$ we define

$$d_p(X, Y) = \max_{1 \leq i \leq [p]} \sup_{\mathcal{D} \subset [0, T]} \left(\sum_{\mathcal{D}} \|X_{t_i, t_{i+1}}^i - Y_{t_i, t_{i+1}}^i\|^{\frac{p}{i}} \right)^{\frac{i}{p}}.$$

In general, this quantity might be infinite for general mappings from the space $\Delta_T \rightarrow T((V))$, but in the context of p -rough paths this is finite due to Theorem 7.4.1. Related to this notion is a notion of convergence (stronger notion) that is the *convergence in the p -variation topology*. Formally, this is defined in terms of converging sequences. A sequence $(X(n))_{n \geq 1} \in \Lambda_p(V)$ is said to converge to $X \in \Lambda_p(V)$ in p -variation topology if there exists a p -control w of X and $X(n)$ for all $n \geq 1$, and a sequence $(a(n))_{n \geq 1}$ of positive reals such that $\lim_{n \rightarrow \infty} a(n) = 0$ and

$$||X(n)_{s,t}^i - X_{s,t}^i|| \leq a(n)w(s,t)^{\frac{i}{p}},$$

for all $(s,t) \in \Delta_T$ and $1 \leq i \leq [p]$.

A path in V with finite q -variation for some $q < 2$ defines a q -rough path. Moreover, for $p \geq q$, this q -rough path can be extended by the Extension Theorem to a multiplicative functional of degree $[p]$ with finite q -variation, hence also with finite p -variation. This means that in particular, a path with bounded variation can be considered canonically as a p -rough path for every $p \geq 1$, where the extension is given by the signature, as discussed before. We are now ready to define the notion of a geometric rough path.

Definition 3.7.1. *A geometric p -rough path is a p -rough path that can be expressed as a limit of 1-rough paths in the p -variation metric.*

The space of geometric rough paths in V is denoted by $G\Omega_p(V)$.

Example 3.7.2 (Example from [35]). Let us take $V = \mathbb{R}^d$, ($d > 2$). Let $(x^i)_{i=1,\dots,d}$ be an element of V . The 2-tensors are pictured as arrays $(x^{ij})_{i,j \in 1,\dots,d}$. Let us take x_u to be a path of finite 1-variation in $\mathbb{R}^d$. We define $X_{s,t}^i = x_t^i - x_s^i$, and $X_{s,t}^{i,j} = \int \int_{s < u_1 < u_2 < t} dx_{u_1}^i dx_{u_2}^j$. Then $X_{s,t} = (1, (X_{s,t}^i)_{i=1}^d, (X_{s,t}^{i,j})_{i,j=1}^d)$, is a multiplica-

tive functional in $T^{(2)}(\mathbb{R}^d)$, with finite p -variation for some $p < 3$. This functional belongs to $G\Omega_p(\mathbb{R}^d)$ and is called the canonical extension of the path x_u . Writing $(X_{s,t}^{i,j})_{i,j=1}^d$ into its symmetric and anti-symmetric parts, we obtain

$$X_{s,t}^{i,j} = \frac{1}{2}(x_t^i - x_s^i)(x_t^j - x_s^j) + A_{s,t}^{i,j},$$

where $A_{s,t}^{i,j} = \frac{1}{2} \int \int_{s < u_1 < u_2 < t} dx_{u_1}^i dx_{u_2}^j - dx_{u_1}^j dx_{u_2}^i$. The $A_{s,t}^{i,j}$ has the geometric interpretation of the area enclosed by the path concatenated with the chord between the points given by the indices i and j . Thus, integrating the winding number over the plane gives $A_{s,t}^{i,j}$.

3.8 Rough Differential Equations and Rough Differential equations with drift

In this section, we introduce the Rough Differential Equations (RDEs). First, we need the following definition.

Definition 3.8.1. *A weakly geometric p -rough path is p -rough path which takes its values in $G^{[p]}$, the free nilpotent group of step $[p]$. The space of weakly geometric rough paths is denoted by $WG\Omega_p(V)$.*

This space is indeed related to the space of geometric rough path via the inclusion $G\Omega_p(V) \subset WG\Omega_p(V)$, and this inclusion is strict. For further details, see section 3.2.2, [35].

In the following, we work with the following notion of distance given by

$$d_{0;[0,1]}(\mathbf{x}, \mathbf{y}) := \sup_{0 \leq s < t \leq 1} d(x_{s,t}, y_{s,t}).$$

For an introduction to Rough Differential Equations we refer the reader to chapter 10 in [17]. For now let $x \in C^{1-var}([0, 1], \mathbb{R}^d)$. For the solution y to the following controlled differential equations

$$dy = V(y)dx := \sum_{i=1}^d V_i(y)dx^i, \quad y_0 \in \mathbb{R}^q,$$

we use the notation $\pi_{(V)}(0; y_0, x)$. In fact, the notation $\pi_{(V)}(s, y_s; x)$ stands for solutions of controlled ODE with vector fields $V = (V_i)_{i=1, \dots, d}$ started at time s from a point $y_s \in \mathbb{R}^q$.

We extend our study to the Rough Differential Equations with drift term, as this fits perfectly with the study of the Loewner differential equation as a Rough Differential Equation. We start with the corresponding definition of RDEs with drift.

Definition 3.8.2. *Let $p, q \geq 1$ and such that $1/p + 1/q > 1$.*

Let

$$\mathbf{x} \in C^{p-var}([0, T], G^{[p]}(\mathbb{R}^d))$$

be a weak geometric p -rough path and

$$\mathbf{h} \in C^{q-var}([0, T], G^{[p]}(\mathbb{R}^{d'}))$$

be a weakly geometric q -rough path. We say that $y \in C([0, T], \mathbb{R}^e)$ is a solution to the rough differential equation with drift driven by (x, h) along the collection of $\mathbb{R}^e$ -vector fields $((V_i)_{1 \leq i \leq d}, (W_j)_{1 \leq j \leq d'})$ and started at y_0 if there exists a sequence $(x^n, h^n) \subset C^{1-var}([0, T], \mathbb{R}^d) \times C^{1-var}([0, T], \mathbb{R}^{d'})$, such that

$$\blacktriangleright \sup_n \|S_{[p]}(x_n)\|_{p-var; [0, T]} + \|S_{[p]}(h_n)\|_{q-var; [0, T]} \leq \infty.$$

► $\lim_{n \rightarrow +\infty} d_{0;[0,T]}(S_{[p]}(x_n), \mathbf{x}) = 0$, and $\lim_{n \rightarrow +\infty} d_{0;[0,T]}(S_{[p]}(h_n), \mathbf{h}) = 0$,

and ODE solutions $y_n = \pi_{(V)}(0, y_0; (x_n, h_n))$ such that

$$y_n \rightarrow y \text{ uniformly on } [0, 1] \text{ as } n \rightarrow \infty$$

When this happens, we write the formal equation:

$$dY = V(y)d\mathbf{x} + W(y)d\mathbf{h}, y_0 \in \mathbb{R}^e,$$

which is called rough differential equation with drift.

A very important set of theorems regarding RDEs with drift are the ones that give existence, uniqueness of solutions and continuity estimates in terms of the regularity of the vector fields involved in the RDE with drift. The theorems follow the spirit of the Universal Limit Theorem (Theorem 5.3 of [35]).

The first result, is giving existence of solution along with some continuity estimates. However, this result does not guarantee uniqueness of solution.

Lemma 3.8.3 (Lemma 12.3 of [17]). *Let $V = (V_i)_{1 \leq i \leq d}$ a collection of vector fields in $Lip^{\gamma-1}(\mathbb{R}^d)$ with $\gamma \geq 1$, and $W = (W_j)_{1 \leq j \leq d'}$ a collection of vector fields in $Lip^{\beta-1}(\mathbb{R}^d)$ with $\beta > 1$. Let $x, \tilde{x}$ be two paths in $C^{1-var}([s, u], \mathbb{R}^d)$ such that*

$$S_{[\gamma]}(x)_{s,u} = S_{[\gamma]}(\tilde{x})_{s,u},$$

and $h, \tilde{h}$ be two paths in $C^{1-var}([s, u], \mathbb{R}^d)$ such that

$$S_{[\beta]}(h)_{s,u} = S_{[\beta]}(\tilde{h})_{s,u}.$$

We then have that the two corresponding RDEs with drift have solutions and moreover we have the following estimate for some $C = C(\gamma, \beta)$,

$$\begin{aligned} & |\pi_{V,W}(s, y_s; (x, h))_{s,u} - \pi_{V,W}(s, y_s; (\tilde{x}, \tilde{h}))_{s,u}| \\ & \leq C(l_h^\beta + l_x l_h^{\gamma-1} + l_x l_h + l_x^{\beta-1} l_h + l_x^\gamma) \exp(C(l_x + l_h)) \end{aligned} \quad (3.8.1)$$

where l_x and l_h are bounds for

$$|V|_{Lip^{\gamma-1}} \max \left\{ \int_s^u |dx|, \int_s^u |d\tilde{x}| \right\}$$

and respectively

$$|W|_{Lip^{\beta-1}} \max \left\{ \int_s^u |dh|, \int_s^u |d\tilde{h}| \right\}.$$

In order to provide existence and uniqueness of solution, the collection of vector fields defining the RDE with drift should be in $Lip(\gamma)$ respectively $Lip(\beta)$. See Theorem 12.10 and Theorem 12.11 in [17]. So, the fundamental difference between the two results is in the regularity of the vector fields that give the transition between existence and uniqueness and guaranteed existence only. In the classical theory of ordinary differential equations this results are in the spirit of Cauchy-Peano Theorem and Picard-Lindelöf Theorem, i.e. in the transition between just continuous vector field versus Lipschitz vector field in the space variable.

Chapter 4

Phase transition at $\kappa = 4$ and new information about the SLE_κ traces

Overview

We study the properties of the backward Loewner differential equation, as a differential equation started from the singularity. When the backward Loewner differential equation is driven by $\sqrt{\kappa}B_t$, we show that there is a phase transition at $\kappa = 4$, in terms of uniqueness/non-uniqueness of solutions starting from the origin. In the regime $\kappa > 4$, we study some features of the non-uniqueness in terms of the macroscopic excursions of the backward Bessel process. Compared with the typical proof of the phase transition in the behavior of the SLE_κ trace at $\kappa = 4$, that uses information of the behavior on the boundary for the maps satisfying the forward Loewner differential equation, we use the behavior on the boundary of a different family of mappings to obtain new structural information about the backward SLE_κ traces, by identifying some classes of double points along the backward SLE_κ traces for $\kappa \in (4, \infty)$.

4.1 Introduction

In the current literature related to the phase transition of the SLE_κ curves there is used as a main tool the forward Bessel process obtained by the extension of conformal maps to the boundary. In this work we study the backward Bessel process obtained from the extension to the real line of the conformal maps that satisfy the backward Loewner differential equation. The study of phase transition phenomena using this process is new and reveals new ways of obtaining structural information about the backward SLE_κ traces.

The chapter is divided into several sections. In the first section, we introduce the notations and setting of the problem. In the next sections we give a heuristic of the proof of the main result and we perform the analysis with all details. First, we show the uniqueness of solutions for $\kappa \leq 4$, and then the non-uniqueness of the backward Loewner differential equation started from the origin. At the end of each section, we give applications on the structure of the SLE_κ traces. In particular, this emphasizes the objective of the work in the chapter, that is double-sided. On one hand, we obtain existence and phase transition in terms of the uniqueness/non-uniqueness of solutions for a singular stochastic differential equation started from its singularity. On the other hand, we use this analysis to obtain structural information about the SLE traces that is not apparent when one works with the forward Loewner flow.

4.2 Preliminaries

In this chapter, we work with the time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t, z) = \frac{-2}{h(t, z) - \sqrt{\kappa} B_t}, \quad h(0, z) = z, z \in \mathbb{H} \quad (4.2.1)$$

We recall from the introduction the following lemma.

Lemma 4.2.1 (Lemma 5.5 of [22]). *Let $h_t(z)$ be the solution to the backward Loewner differential equation with driving function $\sqrt{\kappa} B_t$ and let $f_t(z)$ be the solution of the partial differential equation version of the Loewner differential equation with the same driver. Then, for any $t \in \mathbb{R}_+$, the function $z \rightarrow f_t(z + \sqrt{\kappa} B_t) - \sqrt{\kappa} B_t$ and $z \rightarrow h_t(z)$ have the same distribution.*

Throughout this chapter, we will use the following lemma.

Lemma 4.2.2. *The maps $h_t(z) : \mathbb{H} \rightarrow \mathbb{H} \setminus K_t$ solving the backward Loewner differential equation driven by $\sqrt{\kappa} B_t$ can be extended continuously to the boundary for all times $t \in [0, +\infty)$, a.s..*

Proof. Using Rohde-Schramm Theorem (see [44]) we obtain that the maps $g_s^{-1}(z)$ can be extended for all $s \in [0, \infty)$, for almost every Brownian path (since the SLE_κ hulls K_s are locally connected, for all $s \in [0, T]$ for almost every Brownian path). Let us consider the Loewner chain for $h_{T-s} \circ h_T^{-1}(z) - \sqrt{\kappa} B_T$

$$\partial_s (h_{T-s} \circ h_T^{-1}(z)) = \frac{2}{(h_{T-s} \circ h_T^{-1}(z)) - (\sqrt{\kappa} B_{T-s} - \sqrt{\kappa} B_T)}. \quad (4.2.2)$$

This is a forward Loewner chain with driver $\tilde{B}_s := \sqrt{\kappa} B_{T-s} - \sqrt{\kappa} B_T$ that is a

Brownian motion. The maps $g_s(z)$ also satisfy this chain. Since the Loewner chain has a unique solution a.s., we obtain that fixed $T > 0$ the maps $g_s(z) = h_{T-s} \circ h_T^{-1}(z)$. Thus, $h_{T-s}(z) = g_s \circ h_T(z)$ can be continuously extended to the boundary. Indeed since for any fixed $T > 0$, the maps $h_T(z)$ have the same distribution as $g_T^{-1}(z)$ (according to Lemma 4.2.1) and the maps $g_s(z)$ can be extended continuously to the boundary. Let us choose $(T_n)_{n \in \mathbb{N}}$ an increasing sequence of times such that $T_n \rightarrow \infty$, as $n \rightarrow \infty$. Let

$A_{T_n} := \{h_{T_n-s}(z) \text{ can be continuously extended to the boundary for all } s \in [0, T_n]\}$.

For each $T_n > 0$, the events A_{T_n} happen with probability one from the previous argument. Using an union bound argument, we obtain that $\mathbb{P}(\cup_{n \in \mathbb{N}} A_{T_n}^c) = 0$, and the conclusion follows. $\square$

In the last sections of the chapter, we will work with the square of the (shifted) conformal mappings, i.e. we will consider

$$\tilde{h}_t(z) = (h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2,$$

which is a conformal map from $\mathbb{C} \setminus [0, \infty)$ into $\mathbb{C} \setminus [0, \infty)$. We emphasize that $\sqrt{z}$ is not the principal branch and we work with the branch where we measure the values of the Argument from the positive semi-axis. These mappings can be extended continuously to the boundary using the same idea as in Lemma 4.2.2. They form a Squared Bessel process of certain dimension that we analyze in the next sections.

From Lemma 4.2.1, it follows that for fixed $t \in [0, \infty)$, $h_t(z) - \sqrt{\kappa}B_t$ is equal in distribution to $g_t^{-1}(z + \sqrt{\kappa}B_t)$. For $x > 0$, following Theorem 4 of [29], we obtain

that the mappings

$$H(x, t) = (g_t^{-1}(i\sqrt{x} + \sqrt{\kappa}B_t))^2$$

are continuously extended a.s. on $[0, \infty) \times [0, \infty)$ and they also generate a trace

$$\gamma_2(t) = H(0, t) = (g_t^{-1}(\sqrt{\kappa}B_t))^2.$$

We introduce the following definitions that we use in our analysis as well:

Definition 4.2.3. *For a one-dimensional diffusion X_t on the interval $I \subset \mathbb{R}$, let us write $H_y := \inf\{t : X_t = y\}$. A point $c \in I = [0, \infty)$ is called absorbing if*

$$\mathbb{P}^c(H_y < \infty) = 0$$

for all $y \in I \setminus \{c\}$.

A point $c \in I = [0, \infty)$ is called reflective if

$$\mathbb{P}^c(H_y < \infty) > 0$$

for some $y \in I \setminus \{c\}$.

We call a point $c \in [0, \infty)$ instantaneously reflecting if the point is reflecting and the Lebesgue measure of the time spent at point $c \in I$ is zero.

Definition 4.2.4. *A diffusion is called regular if for all $x \in \text{Int}(I)$, $y \in I$, we have that*

$$\mathbb{P}^x(H_y < \infty) > 0.$$

Definition 4.2.5 (Scale function). *For a (regular) diffusion X_t on I with the following laws $\{\mathbb{P}^x : x \in I\}$, a scale function is a continuous strictly increasing function*

$s : I \rightarrow \mathbb{R}$ such that for all $a < x < b \in I$,

$$\mathbb{P}^x(H_b < H_a) = \frac{s(x) - s(a)}{s(b) - s(a)}.$$

Next, we introduce the following definition.

Definition 4.2.6. *If $s(x) = x$ is a scale function for X_t , we say that X_t is in natural scale.*

For a diffusion X_t with drift μ and variance σ^2 , the scale function can be computed via $s(x) = \int_{x_0}^x \exp(-\int_{\eta}^y \frac{2\mu(z)}{\sigma^2(z)} dz) dy$, where x_0, η are points fixed (arbitrarily) in (a, b) . We further define

Definition 4.2.7. *The function $m(\zeta) = \frac{1}{\sigma^2(\zeta)s'(\zeta)}$ is the density of the speed measure.*

Throughout this chapter and in the next one as well, we use the notion of strong solution of one-dimensional stochastic differential equations that we introduce in the following.

Let B_t be a one-dimensional standard Brownian motion. Let us consider the Borel functions

$$\sigma : \mathbb{R} \rightarrow \mathbb{R}$$

and $b : \mathbb{R} \rightarrow \mathbb{R}$ bounded on compact sets.

Consider the equation (that we write shortly $Eq(\sigma, b)$) given by

$$dX_t = \sigma(X_t)dB_t + b(X_t)dt.$$

Definition 4.2.8. *A solution to the equation $Eq(\sigma, b)$ consists of:*

- a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$.

- an $\mathcal{F}_t$ -Brownian motion.
- an $\mathcal{F}_t$ adapted continuous process X such that

$$X_t = X_0 + \int_0^t \sigma(X_s) dB_s + \int_0^t b(X_s) ds.$$

When $X_0 = x$, we say that X_t started in x .

We are ready to introduce the notion of strong solutions.

Definition 4.2.9. We say that a solution (X, B) to $Eq(\sigma, b)$, defined on a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ is a strong solution on this space if X is adapted to the complete filtration generated by the Brownian motion B .

In order to perform the analysis, let us consider $Z_t(z) := \frac{h_t(z\sqrt{\kappa}) - \sqrt{\kappa}B_t}{\sqrt{\kappa}}$ with $z \in \bar{\mathbb{H}}$, where $h_t(z)$ are the continuous extensions to the real line of the conformal maps solving (4.2.1).

Thus, when restricting the analysis to the real line, we obtain the following SDE

$$\begin{aligned} dX_t &= \frac{-2/\kappa}{X_t} dt + dB_t \\ X_0 &= x_0 \in \mathbb{R}. \end{aligned} \tag{4.2.3}$$

This SDE is the same as the ones governing real Bessel processes started from $x_0 \in \mathbb{R}$,

$$dX_t = \frac{d-1}{2} \frac{1}{X_t} dt + dB_t,$$

with $-2/\kappa = (d-1)/2$, i.e. $d = 1 - 4/\kappa$. Thus, $d \leq 0$ for $\kappa \leq 4$ and $d > 0$ for $\kappa > 4$. We show throughout this chapter that for $\kappa \leq 4$, there is no solution to the SDE (4.2.3) starting from the origin for almost every Brownian path, since $\{0\}$ is absorbing boundary point a.s.. For $\kappa > 4$, we show that there exists more than one

strong solution to the SDE (4.2.3) that is instantaneous reflecting at zero. In this way we give meaning (abstractly) to a notion of solution for the Loewner equation started from the origin (along the real line). In [4] there is a proof that there exists a unique strong non-negative solution to (4.2.3) that starts at zero and that spends zero Lebesgue measure time there. We use this solution to construct another strong solution that is not positive and has the same characteristics (spends null Lebesgue measure time at zero).

Excursion theory for Markov processes. We start with a subsection on Excursion Theory for Markov processes. First, we introduce the notion of local time.

Definition 4.2.10. *For a stochastic process $X_t : \Omega \rightarrow \mathbb{R}$, for every $t \geq 0$ and $x \in \mathbb{R}$, the quantity*

$$L_{x,t} := \limsup_{\varepsilon \rightarrow 0+} \frac{1}{2\varepsilon} \int_0^t \mathbf{1}_{|X_s - x| \leq \varepsilon} ds$$

is called the local time at level x and time t .

When we refer to the local time of the process at zero (i.e. when $x = 0$), we simplify the notation $L_t = L(t)$.

In this subsection, we follow the approach in [8], that we refer to for more details. We define the terminology for a general real-valued Markov process M_t , keeping in mind that in our setting, we apply the techniques for the solutions to the Squared Bessel SDE.

Definition 4.2.11. *An open interval $(g, d) \subset \mathbb{R}$ is called an excursion interval for a stochastic process M_t if $M_t \neq 0$, for all $t \in (g, d)$, with $M_t = M_d = 0$.*

We fix a real number $c > 0$. If we are dealing with a process that is not identically zero, then with positive probability there exists at least one excursion interval with

length $l > c$. We introduce as in [8], the right continuous function

$$\bar{\Pi} : (0, \infty] \rightarrow (0, \infty)$$

which describes the distribution of the lengths of excursion intervals. Let us consider the natural ordering of excursion intervals with length $l > a$. Let us consider the length of the first such interval to be $l_1(a)$. The function is defined for every $a \in (0, \infty]$ as

$$\bar{\Pi}(a) = \begin{cases} \frac{1}{\mathbb{P}(l_1(a) > c)} & a \leq c, \\ \mathbb{P}(l_1(c) > a) & a > c. \end{cases}$$

We are interested in the behavior of $(M_{g+t}, 0 \leq t \leq d - g)$ corresponding to each excursion interval (g, d) .

For every $a > 0$, let us consider the set of excursions of length $\tau > a$

$$\mathcal{E}^{(a)} = \{\omega \in \Omega, \omega(0) = 0 : \tau > a \text{ and } \omega(t) \neq 0, \forall 0 < t < \tau\}.$$

Let us consider the set of all excursions $\mathcal{E} = \cup_{a>0} \mathcal{E}^{(a)}$. Following the approach in [7] for each $a > 0$ with $\bar{\Pi}(a) > 0$, we denote by $n(\cdot | \tau > a)$ the probability measure on $\mathcal{E}^{(a)}$ corresponding the law of the process M_t on the first excursion with length $l > a$, i.e. the law of the process $(M_{g_1(a)+t}, 0 \leq t \leq l_1(a))$ under $\mathbb{P}$, where we used that the assumption $\bar{\Pi}(a) > 0$ guarantees $g_1(a) < \infty$ a.s.. It is proved in [7], that there exists a unique measure n on $\mathcal{E} = \cup_{a>0} \mathcal{E}^{(a)}$ that we call the excursion measure, such that

$$n(\lambda) = \bar{\Pi}(a)n(\lambda | \tau > a)$$

for every $\lambda \in \mathcal{E}^{(a)}$.

Let us define $L^{-1}(t)$ as the inverse of the local time at zero of the process M_t up to time $t > 0$, i.e.

$$L^{-1}(t) = \inf\{s \geq 0 : L(s) > t\}.$$

Moreover, let us consider

$$L^{-1}(t-) = \inf\{s \geq 0 : L(s) \geq t\}.$$

We introduce the excursion process $e_t(\omega)$ on $\mathcal{E} \cup \{\eta\}$, where $\{\eta\}$ is an additional isolated point, associated with the stochastic process M_t

$$e_t(\omega) = (M_{s+L^{-1}(t-)}(\omega), 0 \leq s < L^{-1}(t) - L^{-1}(t-)) \text{ if } L^{-1}(t) - L^{-1}(t-) > 0$$

and $e_t(\omega) = \eta$ otherwise.

We have the following result.

Theorem 4.2.12 (Theorem 10, chapter IV of [7]). *If 0 is recurrent then, $(e_t(\omega), t \geq 0)$ is a Poisson Point Process with characteristic measure n .*

We will use that for $0 < d < 2$, for the Squared Bessel process of dimension d , the origin is recurrent. We obtain from the above result the fact that there exists an excursion decomposition for this process. We will use this in the following subsections to obtain results about the backward SLE_κ trace for $\kappa > 4$.

4.3 Main result

Before stating our main result, we introduce the following concepts.

Definition 4.3.1. For $m > 0$, an m -macroscopic excursion is an excursion of the Bessel process of length at least $m > 0$, where the length of an excursion is understood as the length of the time interval corresponding to the excursion.

Next, we introduce the notion of macroscopic hull.

Definition 4.3.2. A macroscopic hull is a compact set in $\bar{\mathbb{H}}$ with a simply connected complement, such that its intersection with the real axis is an interval $I \subset \mathbb{R}$ with strictly positive Lebesgue measure. We call the intersection of macroscopic hull with the real axis the base of the hull.

Definition 4.3.3. Let $m > 0$ be a real number. Let $\gamma(t)$ be the SLE_κ curve. If there exists two times r_1, r_2 in the time interval such that $\gamma(r_1) = \gamma(r_2)$ and $|r_1 - r_2| \geq m$, then we say that there is a self-touching of the curve after time at least m .

We also introduce the following definition.

Definition 4.3.4. Let $m > 0$. For the (backward) SLE_κ trace, an m -macroscopic double point is a double point that corresponds to self-touching of the trace after time at least $m > 0$.

The origin in $\mathbb{H}$ is a singularity for the backward Loewner differential equation, since the vector field $\frac{-2/\kappa}{Z_0}$ has infinite modulus. For almost every Brownian path, a notion of solution for $t \in [0, T]$ for this differential equation is defined by the limit $Z_t(0) := \lim_{y \rightarrow 0+} Z_t(iy)$.

We are now ready to state our main result.

Theorem 4.3.5. For $\kappa \in (0, 4]$ and for all times in a compact time interval, there is a unique solution of the backward Loewner differential equation started from the origin, a.s.. Moreover, for $\kappa \leq 4$, the backward SLE_κ trace is a simple curve a.s..

For $\kappa > 4$, there are at least two real solutions for the backward Loewner differential equation started from the origin, a.s.. Let $m > 0$ be a positive real number. For $\kappa \in (4, \infty)$, on m -macroscopic excursions from the origin of the Squared Bessel process obtained from extensions of the backward SLE_κ maps on the real line, we obtain macroscopic hulls with base depending on $m > 0$ and m -macroscopic double points of the backward SLE_κ trace.

Remark 4.3.6. Compared with the proof of the a.s. simpleness of the SLE_κ trace for $\kappa \leq 4$ that already exists in the literature, we show that this is implied by the behavior at the origin the backward Bessel process obtained from the extension of the conformal maps to the boundary.

In addition, this analysis also offers a possible answer to the question: what pathwise properties of the Brownian motion influence the behavior of the SLE_κ trace. We will study this in more detail in the last section of this chapter.

In a nutshell, in our analysis, we extend the conformal maps to the boundary and obtain that the dynamics of boundary points satisfy the SDE (4.2.3). We use known results about the solutions of the SDE (4.2.3) along with theorems about the backward Loewner evolution in order to study the start of the backward Loewner evolution from the origin. For this, we use the result from [4], in which there is a proof of the existence and uniqueness of strong positive solutions to the Bessel SDE started from the origin for $0 < d < 1$. In our case, this gives the existence and uniqueness of a strong non-negative solution for the SDE (4.2.3) on the real line for $\kappa > 4$, since $d(\kappa) = 1 - \frac{4}{\kappa}$.

Remark 4.3.7. For dimensions $d \in (0, 1]$ ($\kappa > 4$) there exists a construction of the solutions to the SDE using Excursion Theory. Using this and the previous

analysis, we obtain that beginning of m-macroscopic excursions for the SDE (4.2.3) give macroscopic hulls for the backward Loewner differential equation started from the origin. Thus, we obtain another structural information about the dynamics of the backward SLE_κ hulls, and implicitly backward SLE_κ traces. That is, the m-macroscopic excursions of the Bessel process started from the origin, create in this context, macroscopic hulls of the backward SLE_κ . In particular, when starting the backward Loewner differential equation from the beginning of m-macroscopic excursions of the Bessel processes, we obtain a macroscopic hull, a.s. For further details about the Excursion Theory construction of the Bessel processes for $d \in (0, 1]$, see [7].

Backward Loewner differential equation and the Backward Loewner trace.

In the following we show how using the a.s. existence of the forward chordal SLE_κ , we can obtain the a.s. existence of the backward Loewner trace. For $\kappa > 0$, the backward chordal SLE_κ is defined by solving the backward Loewner differential equation with the driver $\sqrt{\kappa}B_t$, $0 \leq t < \infty$. For any $t_0 > 0$, we have that the process $(\sqrt{\kappa}B_{t_0-t} - \sqrt{\kappa}B_{t_0}, 0 \leq t \leq t_0)$ has the same distribution as $(\sqrt{\kappa}B_t, 0 \leq t \leq t_0)$. Rohde and Schramm showed in [44] that for every $t_0 \in [0, T)$, $\sqrt{\kappa}B_{(t_0-t)}$, $0 \leq t \leq t_0$ generates a forward Loewner trace which we denote by $\beta_{t_0}(t_0 - t)$, $0 \leq t \leq t_0$. Then, using the identity in distribution for the Brownian motion from above, it is shown in [41] that $\sqrt{\kappa}B_t$ generates the backward SLE_κ traces β_{t_0} for $0 \leq t_0 \leq T$.

Note that β_t is a continuous function defined on $[0, t]$. The parametrizations of the backward traces β_t is different from the usual parametrization of the chordal SLE_κ trace in the sense that the backward traces β_t is a continuous function defined on $[0, t]$ such that β_0 is the tip and β_t is the root that is an element of $\mathbb{R}$. The difference with the chordal SLE_κ is that the chordal trace is parametrized such that

the root of it is β_0 . For further details about this construction, we refer to [41].

We also use in our analysis that for fixed time $T > 0$, the law of the curves $\beta_T = \beta(0, T]$ generated by the backward Loewner differential equation is the same as the forward chordal SLE_κ trace $\gamma(0, T]$ (modulo a translation with the driver $\sqrt{\kappa}B_T$).

4.4 Heuristic of the proof of the main result

- We consider a probability space $(\Omega, \mathcal{F}_t, \mathbb{P})$ equipped with the Wiener measure. We consider the backward Loewner differential equation driven by a standard Brownian motion that is defined on $(\Omega, \mathcal{F}_t, \mathbb{P})$. We consider also the extension of the conformal maps $h_t(z)$ to the real line.
- On the real line, when extending continuously the maps solving the backward Lowener differential equation with $\sqrt{\kappa}B_t$ driver, we obtain a real-valued SDE (4.2.3). This real valued SDE, has the origin as a boundary point. Moreover, the behaviour of this real-valued SDE at this boundary point dependes on the dimension $d = d(\kappa)$. For dimension $d = 0$ and $d < 0$, the origin is an absorbing boundary and for $d > 0$, zero is a reflecting boundary. For the backward flow we have $-2/\kappa = (d - 1)/2$, i.e. $d = 1 - 4/\kappa$. Thus, $d \leq 0$ for $\kappa \leq 4$ and $d > 0$ for $\kappa > 4$, i.e. zero has a different behaviour as a boundary point for the SDE (4.2.3). For $\kappa > 4$, there is a way to give meaning to a notion of (strong) solution to the SDE (4.2.3), started from $\{0\}$ for almost every Brownian path. This idea appeared in [4], where it is proved that there exists a strong non-negative solution for this process, that spends null Lebesgue measure time at the origin, for almost every Brownian path. In contrast, for $d = 0$ ($\kappa = 4$) and

$d < 0$ (i.e. $\kappa < 4$) there is no notion of solution that exits the origin along the real line a.s., so if there is a notion of solution started from the origin, the only possibility is the one constructed from the interior of $\mathbb{H}$.

- First, we show how we can define a solution for the backward Loewner differential equation started from the origin, when driven by $\sqrt{\kappa}B_t$, for all κ and for almost every Brownian path.

Then, we show that the behavior of the real-valued SDE (4.2.3) at the origin is absorbing for $d = d(\kappa) \leq 0$.

Using this, we argue that for $\kappa \in [0, 4]$, there is a unique solution a.s.. The argument is the developed in the following section.

- For $d > 0$, i.e. $\kappa > 4$, the picture differs. In this case, we prove that there are at least two real solutions with the same starting point and with the same Brownian path as a driver. We use the existence of the strong solution result for the SDE (4.2.3) of dimension $d > 0$ proved in [4], to construct a new (negative) strong solution (with the same starting point and the same Brownian path as a driver). Thus, we conclude that in this setting, the origin gets mapped to at least two different points for $t \in [0, T]$, a.s.. The result in [4], gives also the uniqueness of the strong solution on the positive semi-axis.

Extending the conformal maps $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ to the real line and considering the trace $\gamma_2(t) = (g_t^{-1}(\sqrt{\kappa}B_t))^2$, we obtain using the analysis of the Squared Bessel SDE new structural information about the SLE_κ traces.

This perspective helps us to recover the phase transition for the backward SLE_κ traces from simple for $\kappa \in [0, 4]$ to non-simple curves for $\kappa \in (4, +\infty)$ using the phase transition in terms of the behavior of the Squared Bessel SDE

at $\{0\}$, seen as boundary point for this SDE. This analysis, compared with the typical phase transition in terms of the behavior of the forward Bessel process, gives additional information about the structure of the backward SLE_κ traces (to be compared with the forward flow case, where the proof uses only information on the positive part of the real line).

4.5 Proof of the main result

Proof of Theorem 4.3.5: the existence and uniqueness of solutions for $\kappa \leq 4$.

In this section, we prove the first part of the result. First, we use Lemma 4.2.2 in order to assure that for the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$, we have that $\lim_{y \rightarrow 0} h_t(iy)$ exists for all times $t \in [0, +\infty)$, a.s.. We postpone to show the fact that the limit indeed satisfies the backward Loewner differential equation until the end of the section and we first show the uniqueness.

Application: Almost sure simpleness of the SLE_κ trace, for $\kappa \in [0, 4]$ using just the information on the boundary.

In this section, we show how the information about the dynamics of boundary points under the backward Loewner differential equation, gives information about the structure of the SLE_κ traces for $\kappa \in [0, 4]$. We recall that the SLE_κ traces are defined via a limiting procedure from the interior of the domain. Essentially, we show that the a.s. behavior of the origin for the Bessel SDE (4.2.3), gives information about the behavior of the SLE_κ traces.

We first need the following definition.

Definition 4.5.1. *The real Squared Bessel process for any $\delta \geq 0$ and $x \geq 0$ is defined as the unique strong solution for the equation $X_t = x + 2 \int_0^t \sqrt{X_s} dB_s + \delta t$.*

For this process, we have the following result

Proposition 4.5.2 (Proposition 1.5, Chapter XI of [39]). *For $\delta = 0$, the point 0 is absorbing and for $0 < \delta < 2$ the point 0 is instantaneously reflecting.*

We use this result, in order to argue that the point 0 is also absorbing for the Bessel process of dimension $d = 0$ (i.e. $\kappa = 4$). Indeed, if with positive probability the point $\{0\}$ would not be absorbing then this would imply that with positive probability we could construct solutions for the Squared Bessel SDE that are different from the identical zero solution, that is a contradiction.

In addition, we have the following general theorem from [40].

Theorem 4.5.3 (Theorem 51.2 of [40]). *Let X_t be a real valued diffusion in the natural scale on $I = [0, \infty)$ with speed measure m . Then the boundary $\{0\}$ must be absorbing if $\int_I m(dx) = \infty$.*

When computing the speed measure for the Squared Bessel process of dimension $d = d(\kappa) = 1 - \frac{4}{\kappa}$, we obtain $m_d(dy) = K_d y^{2(d-1)/(2-d)} dy$, where K_d depends on d . We obtain that for $0 < d < 2$

$$\frac{2(d-1)}{(2-d)} \in (-1, \infty).$$

Since $\frac{2(d-1)}{(2-d)} \leq -1$ for $d \leq 0$ we obtain via Theorem 4.5.3 that for $d \leq 0$ the origin is reached a.s. and the process is absorbing. Thus, when started from $x_0 = 0$ the solutions of the Squared Bessel process of dimension $d > 0$, 'escape' the origin and reflect when reaching the origin back, while the process is absorbed at the origin for $d \leq 0$.

We showed that the Squared Bessel process has an absorbing origin for $d \leq 0$, i.e. $\kappa \leq 4$. Since the Bessel process is the square root of the Squared Bessel process

we obtain that the same result for the Bessel process, that we use in the following.

For $0 < d < 2$, the process is also reaching the origin, a.s. but the origin is not absorbing anymore.

Let us consider the maps $g_t(z)$ solving forward Loewner differential equation, and let us assume that they generate a trace (when the diver is $\sqrt{\kappa}B_t$ this happens with probability 1). Let us consider fixed $s < t$, and let us consider $g_{s,s+t} = g_{s+t} \circ g_s^{-1}$. Then the Loewner chain $g_t^{(s)} = g_{s,s+t}$ has driving function $U_t^{(s)} = U_{s+t}$. When $U_t = \sqrt{\kappa}B_t$, the process $U_{s+t} - U_s$ is again a Brownian motion. Then, the Loewner chain for $g_t^{(s)}(z)$ is

$$\partial_t g_{s,s+t}(z) = \frac{2}{g_{s,s+t}(z) - \sqrt{\kappa}B_{t+s}}, \quad g_{s,s}(z) = z.$$

Thus, the format of the equation is the same. We have the following definition.

Definition 4.5.4. *Suppose g_t is a Loewner chain with driving function U_t , and let $\hat{f}_t(z) = g_t^{-1}(z + U_t)$. Suppose that for each t the limit*

$$\gamma(t) = \lim_{y \rightarrow 0+} \hat{f}_t(iy)$$

exists and the function $t \rightarrow \gamma(t)$ is continuous. Then the Loewner chain g_t is generated by the curve γ .

We use the notation $\gamma(t)$ for the trace generated by $g_t(z)$ and $\gamma^{(s)}(t)$ for the trace generated by $g_t^{(s)}(z)$ for $s > 0$.

Lemma 4.5.5 (Lemma 4.34 of [25]). *Suppose $g_t(z)$ is generated by the curve γ . Then γ is a simple curve with $\gamma(0, \infty) \in \mathbb{H}$ if and only if for all $s \geq 0$, $\gamma^{(s)}(0, \infty) \cap \mathbb{R} = \emptyset$.*

The SLE_κ traces $\gamma^{(s)}(t)$ have the scaling property. The argument is identical

to the one for the forward Loewner differential equation for $s = 0$. Using the deterministic Lemma 4.5.5, we see that in general if the Loewner chain is generated by a curve γ , then γ is not simple if and only if there exists s such that $\gamma^{(s)}(0, \infty) \cap \mathbb{R} \neq \emptyset$. For SLE_κ traces we obtain that γ is a.s. non simple if and only if with probability one there exists s such that $\gamma^{(s)}(0, 1) \cap \mathbb{R} \neq \emptyset$. Using the scaling property of the SLE_κ traces, we obtain that probability of events is either 0 or 1. Indeed define for $t > 0$, $E_t = \{\text{there exists } s \text{ such that } \gamma^{(s)}(0, t) \cap \mathbb{R} \neq \emptyset\}$. By scaling the sets E_t have the same probability, that we denote by p . Then $p = \mathbb{P}(\cap_t E_t)$ and $\cap_t E_t \in \mathcal{F}_{0+}$ where $\mathcal{F}_{0+} = \cap_{t>0} \mathcal{F}_t$. By Blumenthal's zero-one law, $\mathcal{F}_{0+}$ contains only nullsets and their complements. Hence, $p \in \{0, 1\}$.

Thus, it is enough to show that such $s > 0$ exists with positive probability, i.e. we have that the SLE_κ trace $\gamma(t)$ is a.s. non-simple if and only if there exists s such that $\gamma^{(s)}(0, 1) \cap \mathbb{R} \neq \emptyset$ with positive probability. In the next proof, we show that the last statement is equivalent to the statement that the origin is non-absorbing for the Bessel process.

In order to show this, we need the following observation. Fix $s > 0$, a deterministic time. We consider the Loewner chain $h_{s,s+t}$, i.e.

$$\partial_t h_{s,s+t}(z) = \frac{-2}{h_{s,s+t}(z) - \sqrt{\kappa} B_{s+t}}, \quad h_{s,s}(z) = z.$$

For any deterministic time shift $s > 0$, when considering the continuous extension to the boundary, we obtain the same Bessel process with dimension $d = 1 - \frac{4}{\kappa}$ since the format of the shifted chain is the same with the one for $s = 0$. This gives us that for any deterministic shift of time $s > 0$, we obtain a Bessel process that has an absorbing origin. Thus, for a.e. Brownian Motion for all times $s \in [0, 1]$ on these paths the corresponding Bessel process is absorbing. This excludes the possibility

of random shifts of starting times $s \in [0, 1]$ on which the Bessel process obtained from the continuous extension of the conformal maps to the real line started from those (random) times has any other solution than $X = 0$, with positive probability.

For the backward Loewner differential equation, we have that on the real line the dynamics of points is governed by the following SDE

$$dZ_r = \frac{-2/\kappa}{Z_r} dr + d\tilde{B}_r, \quad Z_0 = x \in \mathbb{R} \setminus \{0\}. \quad (4.5.1)$$

with dimensions $d \leq 0$, where $\tilde{B}_r = B_1 - B_{1-r}$. On the positive part of the real axis this is the equation of a Bessel process. In addition, in our analysis we have as a driver the same collection of Brownian paths, the dynamics on the negative part of the real axis is given by

$$\begin{aligned} d\tilde{X}_t &= \frac{-2}{\kappa \tilde{X}_t} dt + dB_t, \\ \tilde{X}_0 &= x_0 < 0. \end{aligned} \quad (4.5.2)$$

Since $\tilde{X}_t$ is negative (until the first hitting time of zero), multiplying by -1 we obtain the following SDE

$$\begin{aligned} dX_t &= \frac{-2}{\kappa X_t} dt - dB_t, \\ X_0 &= x_0 > 0, \end{aligned} \quad (4.5.3)$$

with $X_t = -\tilde{X}_t$. Thus, the dynamics on the negative part of the real axis, when coupling with the same Brownian drivers, is equivalent to a dynamics on the positive side of the real axis with the driver $-B_t$ (that is still a Brownian motion).

If with positive probability the SLE_κ trace were to be not simple, then we obtain that with positive probability $\exists \tau(\omega) < 1$, such that $\gamma^{(s)}(\tau) \cap \mathbb{R} \neq \emptyset$. Then, when considering the SDE (4.5.1), we obtain that the collection of Brownian Motion paths such that the origin is reflective for the equation (4.5.1) has positive measure. Thus, when studying the maps $h_r(z)$ driven by $\sqrt{\kappa}B_{1-r}, r \in [0, 1]$ we obtain a contradiction with the fact that $\{0\}$ is a.s. absorbing boundary : $Z_t \equiv 0$ for all $t > 0$, for the backward Loewner differential equation for almost every Brownian path. Indeed if with positive probability the trace would touch the real line in $[0, 1]$, then we could find a positive mass of paths on which there exists times $r(\omega)$ such that $Z_{r(\omega)} \neq 0$, that gives a contradiction with the previous argument that shows that for almost every Brownian path for all times $t \in [0, 1]$ on these paths the only possible solution for the Bessel process from the origin is $Z_t \equiv 0$.

This shows that the information on the boundary is enough to determine the behavior of the SLE_κ trace. Indeed, since for $\kappa \in [0, 4]$, for almost every Brownian path, the boundary behavior of the origin is absorbing, we obtain as a corollary that the the only possible notion of solution for the backward Loewner differential equation started from the origin is the one constructed before from 'inside the domain'.

Next, we finish the proof of the first part of the main theorem by showing that the mappings $\lim_{y \rightarrow 0+} h_t(iy)$ satisfy the backward Loewner equation for $\kappa \leq 4$ on compact time intervals, a.s.. Let us consider the sequence of squared mappings

$$\hat{Z}_t^n = \left(\frac{h_t(i/n) - \sqrt{\kappa}B_t}{\sqrt{\kappa}} \right)^2.$$

We perform the analysis for this sequence but we can redo it for any other choice of

sequence in the same manner. Then, the mappings $\hat{Z}_t^n$ satisfy the equation

$$d\hat{Z}_t^n = d(\kappa)dt + 2\sqrt{\hat{Z}_t^n}dB_t. \quad (4.5.4)$$

where $d(\kappa) = 1 - \frac{4}{\kappa}$. There are two possible extensions of $z^{1/2}$ to the positive real line (i.e. $+x^{1/2}$ and $-x^{1/2}$), depending on which direction the real line is approached.

For $\kappa \leq 4$, for all $n \in \mathbb{N}$, $\hat{Z}_t^n$ and $\hat{Z}_t$ do not touch the real line for all times $t \in [0, T]$ a.s.. We choose the principal branch of square root (i.e. the arguments of the complex numbers are measured from the positive semi-axis counterclockwise).

We show that $\lim_{n \rightarrow \infty} \hat{Z}_t^n$ satisfies the equation (4.5.4). In order to show this, it suffices to show that $\lim_{n \rightarrow \infty} 2 \int_0^t \sqrt{\hat{Z}_s^n} dB_s = 2 \int_0^t \sqrt{\hat{Z}_s} dB_s$, a.s..

Furthermore, in order to show this, it suffices to show that

$$\left\| 2 \int_0^t \sqrt{\hat{Z}_s^n} dB_s - 2 \int_0^t \sqrt{\hat{Z}_s} dB_s \right\|_{L^2}^2 \rightarrow 0$$

as $n \rightarrow \infty$. By Itô isometry, we have that

$$\left\| 2 \int_0^t \sqrt{\hat{Z}_s^n} dB_s - 2 \int_0^t \sqrt{\hat{Z}_s} dB_s \right\|_{L^2}^2 = \mathbb{E} \left(2 \int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right).$$

We need to show that $\mathbb{E} \left(2 \int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right) \rightarrow 0$. One of the ingredients used to show this is following deterministic result.

Lemma 4.5.6 (Lemma 2.4 of [53]). *Suppose γ is the curve generated by the Loewner differential equation with the driving function $\lambda(t)$. Then for all $z \in [0, t]$, we have that*

$$|\operatorname{Re} z| \leq \sup_{0 \leq s \leq r \leq t} |\lambda(r) - \lambda(s)|.$$

and

$$\operatorname{Im} z \leq 2\sqrt{t}.$$

We will apply the Lemma for $\lambda(t) = \sqrt{\kappa}B_t$ and we use the modulus of continuity of Brownian motion to argue that the real value of the solution to the backward Loewner Differential Equation when running time on a compact time interval, is finite a.s.. We emphasize also that in the proof of the Lemma these deterministic estimates (modulo some finite terms that depend on the starting point i/n and vanish in the limit) are first obtained for $\hat{Z}_s(i/n)$ for fixed $n < \infty$, so we will use them also for them.

Let us define

$$A_t^n := 2 \int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds.$$

In order to apply Lebesgue Dominated Convergence for $\mathbb{E} \left(2 \int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right)$, we need to argue that $A_t^n \rightarrow A_t^\infty$ a.s..

For this, we apply Lebesgue Dominated Theorem for $\int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds$.

We know that $\left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 \rightarrow 0$ pointwise by the fact that for $s \in [0, t]$, a.s. $\hat{Z}_s^n \rightarrow \hat{Z}_s$, as $n \rightarrow \infty$.

The a.s. integrability of $\left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2$ is a consequence of triangle inequality, after that splitting $\hat{Z}_s^n$ and $\hat{Z}_s$ into real and imaginary parts, and applying Lemma 4.5.6 for $\lambda(t) = \sqrt{\kappa}B_t$ (using that $\sup_{s \in [0, T]} B_s$ is finite a.s.). Thus, from Lebesgue Dominated Convergence, $A_t^n \rightarrow 0$ a.s., i.e. for almost every ω we have that $2 \int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \rightarrow 0$. In order to argue that $\mathbb{E} \left(2 \int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right) \rightarrow 0$, we just need the integrability of $\left(2 \int_0^t \left| \sqrt{\hat{Z}_s^n} - \sqrt{\hat{Z}_s} \right|^2 ds \right)$. This is again a consequence of triangle inequality, splitting into real and imaginary parts and applying

Lemma 4.5.6 (using the fact that $\mathbb{E}[\sup_{s \in [0, T]} B_s]$ is finite).

Thus using the Lebesgue dominated convergence theorem we obtain that the limit $\hat{Z}_s$ satisfies the equation for $[0, t]$ a.s.. Since for $\kappa \leq 4$, the solution takes values in $\mathbb{C} \setminus [0, \infty)$, we have that $\sqrt{\hat{Z}_t}$ (obtained by halving the angles and considering square root of the modulus of $\hat{Z}_t$) satisfies the backward Loewner differential equation.

Thus, combining all the tools from this section we obtain an existence and uniqueness result for the solutions of the (backward) SLE_κ for $\kappa \in [0, 4]$ for almost every Brownian path, as well as structural information about the backward SLE_κ traces using information on the boundary.

Proof of Theorem 4.3.5: the non-uniqueness when $\kappa > 4$.

In this section, we prove that in the case $\kappa > 4$, almost surely there are at least two types of solutions for the backward Loewner differential equation started from the origin (i.e. solutions obtained from the same starting point and the same Brownian motion path).

In order to show this, we consider again the maps $h_t(z)$ solving the backward Loewner differential equation driven by the Brownian driver $\sqrt{\kappa}B_t$, for $t \in [0, T]$. We extend continuously the conformal mappings to the real line. We show that we can give meaning to a notion of solution to SDE (4.2.3) that spends null Lebesgue measure time at the origin. Moreover, in the case $d > 0$ (i.e. $\kappa > 4$) we use this strong solution to construct a different solution, that gives the non-uniqueness in this regime.

The main result that we use is proved in [4] and is phrased as follows:

We use indistinguishably the notations $L_a^Z(t)$ and $L_{t,a}^Z$ for the local time of a

process Z_t at the point $a \in \mathbb{R}_+$. Let us introduce the principal value correction for $d \in (0, 1]$, i.e.

$$Z_t = Z_0 + B(t) + \frac{d-1}{2}k(t), Z_0 \geq 0,$$

where

$$k(t) = P.V. \int_0^t \frac{1}{Z_s} ds := \int_0^\infty a^{d-2} (L_a^Z(t) - L_0^Z(t)) da.$$

In [4] there is a proof that the existence of a weak solution spending zero time at the origin implies the existence and uniqueness of a non-negative strong solution spending zero time at the origin for the above equation. Note that this analysis allows us to view the strong solutions on the real line starting from the origin as functions of the Brownian motion paths.

We consider the following version of the previous equation (since the dynamics is defined on both the positive part of the real line and on the negative part of the real line). We consider the following :

$$\tilde{k}(t) = P.V. \int_0^t \frac{dt}{Z_t} = \int_{-\infty}^\infty \text{sgn}(a) |a|^{d-2} (L_a^Z(t) - L_0^Z(t)) da. \quad (4.5.5)$$

Furthermore, we use the result from [4] to see this positive solution as a unique positive strong solution for the 'extended' SDE

$$Z_t = Z_0 + B(t) + \frac{d-1}{2}\tilde{k}(t), Z_t \geq 0. \quad (4.5.6)$$

Indeed this holds, since $Z_t \geq 0$, gives $k(t) = \tilde{k}(t)$. Note that this is a strong solution and thus can be thought as a measurable function $\phi : C(\mathbb{R}_+) \rightarrow C(\mathbb{R}_+)$ such that $\rho(t) := \phi(B_t(\omega))$ solves (4.5.5). Note that this solution is unique, non-negative and starts from the origin, according to the result from [4].

The next step is to consider $\tilde{B}_t(\omega) = -B_t(\omega)$, and as before $Z_0 = 0$. We consider

$$\tilde{\rho}(t) := -\phi(-B_t(\omega)). \quad (4.5.7)$$

We obtain that

$$-\phi(-B_t(\omega)) = \frac{d-1}{2}\tilde{k}(t) + B_t(\omega). \quad (4.5.8)$$

Thus, $\tilde{\rho}(t)$ is another solution to the equation (4.5.6), with the same starting point and the same Brownian motion path. Note that the two solutions are different since one is positive and the other one is negative. Thus, for all times $t \in [0, T]$, for $\kappa > 4$, we have at least two different real solutions with the same starting point and with the same Brownian driver. In particular, at each fixed time $t > 0$, for $\kappa > 4$ we obtain for the backward Loewner differential equation started from the origin at least two solutions, a.s..

4.6 Excursion Theory of the Bessel processes and behavior of the SLE_κ traces

SLE_κ traces and the Squared Bessel processes.

In this subsection, we discuss applications of the analysis developed so far on the SLE_κ traces for $\kappa > 4$. We consider, as in [29], the definition of the SLE_κ trace using the mappings $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$, where $h_t(z) : \mathbb{H} \rightarrow \mathbb{H} \setminus K_t$ are solving the backward Loewner differential equation (4.2.1).

The family of maps $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ are conformal maps from $\mathbb{C} \setminus [0, \infty)$ to $\mathbb{C} \setminus [0, \infty) \setminus \tilde{K}_t$, where $\tilde{K}_t$ is the SLE_κ hull corresponding to the maps

$$(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2.$$

Using the fact that the hulls of the backward SLE are locally connected, we obtain that the maps $(h_t(\sqrt{z}) - \sqrt{\kappa}B_t)^2$ can be extended continuously to the real line.

When studying the extensions of the conformal maps satisfying the backward Loewner differential equation (4.2.1), we obtain on the real line two SDEs (that are both running on $\mathbb{R}_+$). These SDEs are obtained by applying Itô's formula for $x \rightarrow x^2$.

These SDEs are the one satisfied by Squared Bessel processes. Following [39], by using the Yamada-Watanabe Theorem (see [39]), we obtain that the Squared Bessel process of any dimension, starting from $x \geq 0$ has a strong unique solutions a.s..

Thus, in our case, we have the following SDEs. We call the first one, the upper

Squared Bessel SDE and the second one the lower Squared Bessel SDE:

$$d\tilde{Z}_t = \left(1 - \frac{4}{\kappa}\right) dt + 2\sqrt{\tilde{Z}_t} dB_t \quad (4.6.1)$$

and

$$d\bar{Z}_t = \left(1 - \frac{4}{\kappa}\right) dt - 2\sqrt{\bar{Z}_t} dB_t. \quad (4.6.2)$$

We couple these processes with the same Brownian driver $B_t(\omega)$, in the sense that we drive (4.6.1) and (4.6.2) with $B_t(\omega)$. First, we restrict our attention to the solution of (4.6.1). The analysis for the lower Squared Bessel process (i.e. the solution to $d\tilde{Z}_t = \left(1 - \frac{4}{\kappa}\right) dt - 2\sqrt{\tilde{Z}_t} dB_t$) will be performed in the same manner.

As in [29], we denote the SLE_κ trace in this new setting by $\gamma_2(t) := (g_t^{-1}(\sqrt{\kappa}B_t))^2$. Note that, by conformal invariance $\gamma_2(t)$ has the same law as the backward SLE_κ trace in $\mathbb{H}$ (that modulo a shift has the same law with the forward SLE_κ trace in $\mathbb{H}$).

For $d(\kappa) = 1 - \frac{4}{\kappa} > 0$, there is a unique positive strong solution for this process starting from the origin such that $\tilde{Z}_t \geq 0$, a.s.. Also, for $2 > d(\kappa) = 1 - \frac{4}{\kappa} > 0$ the Squared Bessel process is recurrent. Thus, we can apply elements of Excursion Theory developed in the previous section.

The m-macroscopic excursions of the real Squared Bessel process and application to the study of the backward SLE_κ traces.

Fix $m > 0$. In this section, we analyze the backward Loewner flow during a m-macroscopic excursion from the origin of the (upper) Squared Bessel process on $\mathbb{R}$. The same analysis holds for the (lower) Squared Bessel process.

Let us consider the collection of m-macroscopic excursions of the solution to the real Squared Bessel SDE. Let us consider the backward Loewner differential

equation from the origin, for time $t \in [0, \infty)$. For the Squared Bessel process which is obtained by extending the conformal maps on the real line, we have the following result (see Section 4.6 of [40]), for $0 < d < 2$

$$\int_0^\infty \mathbf{1}_{\{\tilde{Z}_t=0\}} dt = 0.$$

Thus, the origin is instantaneously reflecting for this process for all the times $[0, \infty)$. Among these times, there exist a set of times which correspond to beginning of macroscopic excursions of the Squared Bessel process from the origin. In order to see this, we prove the following result:

$$\mathbb{P}(\text{there is an excursion of length at least } m) = 1.$$

Since the Bessel process for $d > 0$ is not identically zero, with positive probability there is at least one excursion interval with length $l > m$. The choice of this constant $m > 0$ is arbitrary and the modification of this constant does not change the result. Following the approach in [8], let us consider the event

$$\Lambda_t = \{\text{all the excursion intervals with right-end point } d < t \text{ have length } l \leq m\}.$$

If the first return to $\{0\}$ of the process (that is a stopping time) is infinite, then the Markov process has an excursion interval of length $l > m$. If this stopping time is finite, then we apply the Markov property and we get that $\mathbb{P}(\Lambda_{3t}) \leq \mathbb{P}(\Lambda_t)^2$ and by iteration we get that $\mathbb{P}(\Lambda_{3^n t}) \leq \mathbb{P}(\Lambda_t)^{2^n}$ for $n \in \mathbb{N}$. Thus, we get that $\lim_{s \rightarrow \infty} \mathbb{P}(\Lambda_s) = 0$.

We further investigate the continuous extensions $(h_t(\sqrt{z}) - \sqrt{\kappa} B_t)^2$, for times

$t \in [0, \infty)$. Among these times, there will be times $r = r(\omega)$ (that depend on the path of the Brownian motion) that are beginning of m-macroscopic excursions of the real Squared Bessel process, started from the origin. Let us consider

$$\tilde{h}_r(z) := h_{s+r} \circ h_r^{-1}(z),$$

for $s \geq 0$. In particular, we obtain that the family of extended conformal maps $(\tilde{h}_r(0) - \sqrt{\kappa}\hat{B}_r)^2$, where $\hat{B}_r := B_{t+r} - B_r$ for times in $[r(\omega), r(\omega) + m)$, map the origin to the real line. In order to see this, let us assume that the image of the origin under the maps $(\tilde{h}_r(0) - \sqrt{\kappa}\hat{B}_r)^2$ is mapped in $\mathbb{H}$. Then, for any $\tilde{m} \in (0, m)$ we would have two images of the origin, for the extended conformal maps, one in $\mathbb{H}$ and the unique strong solution of the Squared Bessel SDE at time $r(\omega) + \tilde{m}$.

Using Lemma 4.2.2, we obtain that for all times $t \in [0, \infty)$, for almost every Brownian path, the maps $h_t(z)$ (and their functional inverses) are continuously extended to the boundary. This gives that the maps $\tilde{h}_r(z)$ are continuously extended to the boundary (since they are obtained from compositions of maps that are continuously extended to the boundary). Hence, since the maps $\tilde{h}_r(z)$ should map the origin to a unique point, we obtain a contradiction. Thus, the two images of the origin should coincide. Since the backward SLE_κ trace grows from the origin, we obtain that when starting the backward Loewner flow from the set of m-macroscopic excursions of the underlying Bessel, we obtain points of intersection of the trace with the real line. Using that a.s. there is an excursion interval of length at least $m > 0$ as shown before, we obtain that the closure of backward SLE_κ hulls are macroscopic hulls whenever we have m-macroscopic excursions of the Squared Bessel process. Moreover, since the Squared Bessel process is recurrent and the backward SLE_κ trace starts from the origin, we obtain also double points (that correspond to self

touchings of the curve after at least $m > 0$ time).

Moreover, from topological considerations, we obtain at the end of a m -macroscopic excursion of the unique strong solution of the Squared Bessel process, a closed 'large bubble' of the backward SLE trace formed from the outer boundary of a portion of the curve, where $t(\omega)$ is the beginning of a large excursion that collects the self intersections of the backward SLE_κ trace with itself and with real line, that happened on $[t(\omega), t(\omega) + m)$. In contrast with the forward flow approach, this approach not only recovers the phase transition in the behavior of the SLE_κ traces, but also it provides structural information about the backward SLE_κ traces.

The same analysis holds also for the solution to the SDE (4.6.2). When driving them simultaneously with $B_t(\omega)$ and $-B_t(\omega)$ we obtain the complete picture of the evolution of the backward Loewner hulls in $\mathbb{C} \setminus [0, \infty)$.

Brownian motion and the behavior of the SLE_κ trace for $\kappa \in (4, \infty)$.

In addition, this analysis also offers a possible answer to the question: what pathwise properties of the Brownian motion influence the behavior of the backward SLE_κ trace? The previous analysis suggests that the classes of points along a Brownian driver that allow the process to escape the origin (and to further reflect at the origin), give the possibility of the non-simpleness of the backward SLE_κ traces for $\kappa > 4$. Compared with the case when the parameter $\kappa \in [0, 4]$, where the solution along the positive real line is stuck at the origin, in the case $\kappa \in (4, \infty)$ there is a unique notion of strong solution along the real line. Equivalently, when the parameter $\kappa \in [0, 4]$ is not large enough to allow the solution of the backward Bessel SDE to escape the origin, there is no solution along the real line.

In general, the change in behavior of the backward Bessel process started from

the origin, gives also the following additional information: the collection of points that represent the beginning of macroscopic excursions of the real Bessel process of dimensions $d \in (0, 1)$ that is obtained from the extensions of the conformal maps to the real line, give also the formation of macroscopic bubbles for the backward SLE_κ hulls.

Chapter 5

Continuity in κ of the welding homeomorphism and of the SLE_κ traces

Overview

We prove that for $\kappa \in (0, 4]$ the welding homeomorphism of the real line induced by the backward Loewner differential equation when driven by $\sqrt{\kappa}B_t$ is a.s. sequentially continuous in κ for $\kappa \in (0, 4]$, everywhere except at most countably many points on the real axis. The proof uses properties of the real Bessel process obtained from the continuous extension of the family of conformal maps satisfying the backward Loewner differential equation on the real line, such as the Lamperti relation, and the monotonicity of the drifts. For $\kappa \in \mathbb{R}_+ \setminus \{8\}$, we also give a proof for the sequential continuity of SLE_κ traces and the algorithm that simulates them. The sequential continuity for $\kappa \in [0, 8(2 - \sqrt{3}))$ is not a new result, however the technique to show this is new (and works for all $\kappa \neq 8$). A byproduct of this analysis

and of this new technique is the sequential continuity of the traces generated by an algorithm that we will present in the last section.

5.1 Introduction

Given a dynamical system that depends on parameters, a natural question to be addressed is the continuity of the outputs of that system in its parameters. We investigate the continuity in the parameter κ of a natural random function associated with the backward Loewner differential equation: the conformal welding homeomorphism for $\kappa \in (0, 4]$. We further investigate the continuity in the parameter κ of the SLE_κ traces and of the algorithm that generates them for $\kappa \in \mathbb{R}_+^* \setminus \{8\}$. We emphasize that whenever we refer to the sequential continuity of the traces and the algorithm that generates them throughout this chapter, we consider $\kappa \in \mathbb{R}_+^* \setminus \{8\}$.

The chapter is divided in several parts. In the first section we introduce versions of Loewner differential equations that we work with and we state the main results. In the second one, we provide as in [41] the setting for studying the extensions of the maps in the backward Loewner differential equation. In the third section we introduce the setting needed for the proof of the result. Next, we present a heuristic illustrating the pointwise convergence of the welding homeomorphisms induced on the real line by the backward SLE_κ , and in the final section we give a complete proof of this result using known results about Bessel processes which we use as a priori information in our analysis.

5.2 Preliminaries and main result

In this chapter, we work with the time reversal differential equation (backward) version for chordal SLE_κ in the upper half-plane

$$\partial_t h(t, z) = \frac{-2}{h(t, z) - \sqrt{\kappa} B_t}, \quad h(0, z) = z, \quad z \in \mathbb{H}. \quad (5.2.1)$$

Given the Loewner differential equation, the conformal welding homeomorphism is a homeomorphism of intervals of the real line given by the following rule: two points x and y situated at different sides of the origin are to be identified if they hit zero simultaneously under the backward Loewner differential equation when coupled with the same Brownian motion. The conformal welding homeomorphism induced by the backward SLE_κ for $\kappa < 4$ was studied in [41] and [46]. In [41] it is proved that the backward SLE_κ trace $\beta(t)$ exists and that is a continuous function of $t \in [0, T]$. We use the notation $\beta(0, T]$ to denote the backward Loewner trace on the interval $(0, T]$ and $\gamma(0, T]$ to denote the forward Loewner trace on the interval $(0, T]$. The proof uses that the backward Loewner differential equation is driven by $\sqrt{\kappa}(B_T - B_{T-r})$ for $r \in [0, T]$ and that $\sqrt{\kappa}(B_T - B_{T-r})$ has the same distribution as $\sqrt{\kappa}B_t$. Moreover, $\beta(0, T] - \sqrt{\kappa}B_T$ has the same distribution as $\gamma(0, T]$, where $\gamma(0, T]$ is the forward SLE_κ trace. Thus, one can use the backward SLE_κ traces to study the SLE_κ curves. In particular, for $\kappa \in (0, 4]$, the backward SLE_κ trace $\beta(t)$ is a simple curve for almost every Brownian driver. In this case, under the backward Loewner flow on the real line, points from opposite sides of the singularity hit the origin and afterwards get mapped to the simple trace $\beta(t)$ for almost every Brownian path. Note that so far we have considered finite time intervals, and consequently the welding homeomorphism is defined on finite intervals of the real

line. However, in [41] it is proved that the backward Loewner differential equation induces a conformal welding homeomorphism on the whole real line as well. This corresponds to allowing the hitting time of the origin under backward Loewner flow on the real line to be $T = \infty$. In this chapter, we take the same perspective and study the welding homeomorphism induced by the backward Loewner differential equation on the whole real line.

We prove that for $\kappa_i \rightarrow \kappa \in (0, 4]$ the welding homeomorphism, which we denote by $\phi^\kappa(x, \omega)$, converges a.s. pointwise everywhere except at most countably many points on the real axis.

We are ready to present our main result.

Theorem 5.2.1. *The welding homeomorphism on the real line induced by the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ is sequentially continuous in κ , for $\kappa \in (0, 4]$, a.s., everywhere except at most countably many points on the real axis.*

Remark 5.2.2. We emphasize that by sequentially continuous in κ , we mean the following:

Given a sequence $\kappa_j \rightarrow \kappa$, there is a nullset that depends on this sequence, outside of which the welding homeomorphism converges pointwise everywhere except at most countably many points on the real axis.

5.3 The setting needed for the proof of Theorem 5.2.1

Let $\mathcal{F}_t = \sigma(B_s, s \in [0, t])$ and let us consider the probability space $(\Omega, \mathcal{F}_t, \mathbb{P}_B)$, where $\mathbb{P}_B$ denotes the Wiener measure. The conformal maps $h_t(z)$ satisfying (5.2.1) can be continuously extended to the boundary with probability one. In what follows next,

we work with these extensions to the real line of the mappings $h_t(z)$. By performing the shift $Z_t(x) := \frac{h_t(x\sqrt{\kappa}) - \sqrt{\kappa}B_t}{\sqrt{\kappa}}$, we obtain

$$\begin{aligned} dZ_t &= \frac{-2/\kappa}{Z_t} dt + dB_t, \\ Z_0 &= x \in \mathbb{R}^*. \end{aligned} \tag{5.3.1}$$

Note that the nullsets of the Brownian paths outside of which the SLE_κ exists and $h_t(z)$ is continuous up to the boundary, depend on κ .

In what follows we consider the Bessel processes of various dimensions to be functions of the Brownian motion driver.

Thus, in our analysis, we naturally obtain the following coupling: we drive the backward Loewner differential equation with the same Brownian motion as the solutions to the Bessel SDE (5.3.1) on the real line.

5.4 Proof of the Theorem 5.2.1

The techniques needed for the proof of the (sequential) continuity in κ of the welding homeomorphisms in the parameter κ , for $\kappa \in (0, 4]$ are borrowed from the conformal welding framework that is developed in [41], [46]. The analysis is divided into several steps. In the following, we present the heuristics of the argument along with a detailed list of techniques used in the proof of the result.

Heuristics of the argument and details on the techniques used in the proof.

- An important element is the Lamperti relation (that can be found in [39] or in [19]). It states that the first hitting time of zero by a Bessel process has a certain integral representation. In particular, this gives that for the first hitting

time of zero of a Bessel process of dimension $d = d(\kappa)$ we have $T_x^\kappa \stackrel{(d)}{=} \frac{x^2}{2Z_{-\mu(\kappa)}}$, where $Z_{-\mu(\kappa)}$ is a Gamma random variable with index $\mu(\kappa)$ which is an explicit continuous function of κ .

- Another useful tool in the proof is the following order of the hitting times of zero for a fixed sample path of the Brownian motion driver. Let us consider Bessel processes driven by the same sample path of the Brownian motion. Using that the drift depends monotonically on the dimension, we obtain that for a fixed starting point one of the processes (the one with smaller value of the drift) will hit the origin before the other processes with different drifts, a.s.. Thus, in the coupled picture that we consider, we will get that for $\kappa_1 \leq \kappa_2$ and any $x_0 \in \mathbb{R}$ we have $T_{x_0}^{\kappa_1} \leq T_{x_0}^{\kappa_2}$ for $\kappa_1 \leq \kappa_2$ a.s.. We want to understand how these stopping times depend on κ for fixed $x_0 \in \mathbb{R}$. For this, we consider the Laplace transforms of these hitting times. We further use that if $T_{x_0}^{\kappa_i}$, $i \in \mathbb{N}$, are coupled random variables, such that $T_{x_0}^{\kappa_i} \xrightarrow{(d)} T_{x_0}^\kappa$, as $\kappa_i \rightarrow \kappa$ and $T_{x_0}^{\kappa_i} \nearrow T_{x_0}^\kappa$ or $T_{x_0}^{\kappa_i} \searrow T_{x_0}^\kappa$, then $T_{x_0}^{\kappa_i} \rightarrow T_{x_0}^\kappa$, a.s..

Proof of Theorem 5.2.1. We first introduce a probabilistic version of Lemma 3 from [31].

Lemma 5.4.1. *For any $\kappa \in (0, 4]$, with probability one, there are no two points $0 < x_0 < y_0$ or $0 > x_0 > y_0$ such that $T_{x_0}^\kappa = T_{y_0}^\kappa$.*

Proof. Let us consider the starting points $0 < x_0 < y_0$. For all Brownian paths, the function $\delta(t) = y(t) - x(t)$ is increasing in t , since

$$\frac{d}{dt}\delta(t) = \frac{d}{dt}(y(t) - x(t)) = 2 \frac{y(t) - x(t)}{(y(t) - \sqrt{\kappa}B_t)(x(t) - \sqrt{\kappa}B_t)}.$$

Thus, with probability one there are no two points at the same side of the singularity that can hit the origin in the same time. $\square$

Combining the previous lemma with the existence of the welding homeomorphism, we obtain that when starting the backward Loewner differential equation from any point $x_0 > 0$ on $\mathbb{R}$, we obtain only one point on the negative real axis that will hit simultaneously with $x_0 > 0$, for almost every Brownian path.

Let us fix a point $x > 0$ and a parameter $\kappa \in [0, 4]$. Let us start the reverse backward Loewner differential equation from x_0 and consider it until the first hitting time of the origin. We know that the dynamics of the reverse Loewner equation restricted to the real line is given by the Bessel SDE for dimensions $d(\kappa) = 1 - \frac{4}{\kappa}$. We recall that we consider some $\kappa \in (0, 4]$ that corresponds to $d(\kappa) \leq 0$. We consider a family of Bessel processes with indices $\kappa_i \in (0, 4]$, $i \in \mathbb{N}$, that are coupled with the same Brownian paths.

Let us consider a fixed value $\kappa \in (0, 4]$ and a sequence of parameters $\kappa_i \rightarrow \kappa$, as $i \rightarrow \infty$.

The next lemma is a result about the ordering of hitting times for the Bessel process for the same starting point, same drivers and sequence of different parameters. The argument uses that the time of hitting zero is finite a.s. for dimension $d = 0$ (i.e. $\kappa = 4$) in case of the real Bessel processes. Moreover, since the drift for smaller values is a smaller negative number, then if the Bessel process of dimension $d = 0$ hits the origin in finite time, the process of lower dimension should hit in a shorter time due to the ordering of the drifts (for more details, see [19]). Thus, the hitting time of zero is finite, for $\kappa \in (0, 4]$ a.s.. In the following lemma, we use the indexed notation $Z_t^{\kappa_i}(x_0)$ for the solution of the SDE (5.3.1) on the real line in order to keep track of the sequence of parameters κ_i . We know from [39], chapter XI,

that for the SDE (5.3.1), for $\kappa \in \mathbb{R}_+$, for almost every Brownian path there exists a unique strong solution, until the first hitting time of zero.

Lemma 5.4.2. *Let us consider $Z_t^{\kappa_i}(x_0)$ to be a collection of Bessel processes started from fixed $x_0 > 0$ coupled as above. Let $(\kappa_i)_{i \in \mathbb{N}}$ be a strictly increasing sequence of values inside $(0, 4]$. Then for all starting points $x_0 > 0$, it holds with probability one that $T_{x_0}^{\kappa_i} \leq T_{x_0}^{\kappa_j}$ for any $i \leq j$.*

Proof. Let $a(\kappa_i) := \frac{-2}{\kappa_i}$, for any $i \in \mathbb{N}$. Let us consider the Bessel processes $Z_t^{\kappa_i}(x_0)$ given as solutions to the SDE

$$\begin{aligned} dZ_t^{\kappa_i} &= \frac{a(\kappa_i)}{Z_t^{\kappa_i}} dt + dB_t, \\ Z_0^{\kappa_i} &= x_0 > 0. \end{aligned} \tag{5.4.1}$$

Let us consider the Bessel processes to be driven by the same collection of Brownian paths. Since $a(\kappa_i) = \frac{-2}{\kappa_i}$ we obtain that if $\kappa_i \leq \kappa_j$, then $a(\kappa_i) \leq a(\kappa_j)$. Thus, using the fact that the latter drift dominates, we obtain that for almost every Brownian path $T_{x_0}^{\kappa_i} \leq T_{x_0}^{\kappa_j}$. $\square$

In the natural coupling that we consider, by looking at the difference of these hitting times, we obtain a random variable

$$0 \leq [T_{x_0}^{\kappa_i} - T_{x_0}^{\kappa_j}](\omega).$$

The Lamperti relation gives the law of the hitting time for a fixed value of the parameter and for a fix starting point $x_0 > 0$. Thus, by Lamperti relation $T_{x_0}^{\kappa_j} \stackrel{d}{=} \frac{x_0}{2Z_{\mu(\kappa_j)}^{\kappa_j}}$, where $Z_{\mu(\kappa_i)}$ is a Gamma random variable with index $\mu(\kappa)$ that is an explicit continuous function of κ . For a fixed value of the starting point $x_0 > 0$,

let us consider a sequence of Bessel processes with this starting point driven by the same Brownian driver. Let us assume that the corresponding parameters κ_i lie in $[0, 4]$ and κ_i converges to κ . We split $(\kappa_i)_{i \in \mathbb{N}}$ into two subsequences: $\kappa_i^- \rightarrow \kappa$ converging from below and $\kappa_i^+ \rightarrow \kappa$ converging from above. By the lemma above, we have that

$$T_{x_0}^{\kappa_i^-} \leq T_{x_0}^{\kappa} \leq T_{x_0}^{\kappa_i^+}, \text{ a.s..}$$

Taking the Laplace transforms (denoted by $\mathcal{L}$) of these random variables, we obtain the following

$$\mathcal{L}(T_{x_0}^{\kappa_i^-}) \leq \mathcal{L}(T_{x_0}^{\kappa}) \leq \mathcal{L}(T_{x_0}^{\kappa_i^+}).$$

Thus, using the convergence of the laws in the parameter κ , we obtain via Lévy's continuity theorem that

$$\lim_{\kappa_i \rightarrow \kappa} \mathcal{L}(T_{x_0}^{\kappa_i^+}) = \mathcal{L}(T_{x_0}^{\kappa}).$$

The same holds for the sequence converging from below. Furthermore, we use that if $T_{x_0}^{\kappa_i}$, $i \in \mathbb{N}$, are coupled random variables such that as $\kappa_i \rightarrow \kappa$, $T_{x_0}^{\kappa_i} \xrightarrow{(d)} T_{x_0}^{\kappa}$, and $T_{x_0}^{\kappa_i} \nearrow T_{x_0}^{\kappa}$ or $T_{x_0}^{\kappa_i} \searrow T_{x_0}^{\kappa}$, then $T_{x_0}^{\kappa_i} \rightarrow T_{x_0}^{\kappa}$, a.s..

Note that since the value of κ is chosen arbitrarily, this argument works for any value $\kappa \in (0, 4]$.

The next ingredient that we need is Lemma 5.4.1. We use Lemma 5.4.1 to argue that for any fixed value κ_i , there is only one point on the left side of the singularity that will hit the origin in time $T_{x_0}^{\kappa_i}$ a.s.. We know from the above that the hitting time is sequentially continuous in κ_i for fixed value of the starting point.

Note that since in our analysis we have as a driver the same collection of Brow-

nian paths, the dynamics on the negative part of the real axis is given by

$$\begin{aligned} d\tilde{X}_t &= \frac{-2}{\kappa\tilde{X}_t}dt + dB_t, \\ \tilde{X}_0 &= x_0 < 0. \end{aligned} \tag{5.4.2}$$

Since $\tilde{X}_t$ is negative (until the first hitting time of zero), multiplying by -1 we obtain the following SDE

$$\begin{aligned} dX_t &= \frac{-2}{\kappa X_t}dt - dB_t, \\ X_0 &= x_0 > 0, \end{aligned} \tag{5.4.3}$$

with $X_t = -\tilde{X}_t$. Thus, the dynamics on the negative part of the real axis, when coupling with the same Brownian drivers, is equivalent to a dynamics on the positive side of the real axis with the driver $-B_t$ (that is still a Brownian motion). In particular, the Lamperti relation that we have used for the positive part of the real line, holds also for the dynamics on $\mathbb{R}_-$.

Let us consider the backward Loewner flow starting from $x_0 > 0$ and driven by $\sqrt{\kappa}B_t$. Let us take a sequence $\kappa_i \rightarrow \kappa$. In [41], there is a proof of the existence of the welding homeomorphism for $\kappa \in (0, 4]$. The argument uses the existence of the SLE_κ trace and the fact that the trace is simple a.s. for $\kappa \in (0, 4]$. Thus, we consider in our analysis welding homeomorphisms generated by the SLE_κ trace for $\kappa \in (0, 4]$.

Lemma 5.4.3 (Lemma 5 of [3]). *Let X_t be a Bessel process of dimension $d = d(\kappa) = 1 - \frac{4}{\kappa}$, started from $x \geq 0$. Then, for any fixed $d \leq 2$ and almost every Brownian path, the hitting time of the origin $T_x^\kappa(\omega)$ is continuous in $x \geq 0$.*

We will use this lemma for sequences of dimensions. Since the result gives an a.s. continuity of the hitting times in the starting point for a fixed dimension, when considering sequences of dimensions we obtain the same conclusion outside a different nullset formed by the countable union of both the nullsets outside of which the processes are defined and the nullsets where the result of Lemma 5.4.3 holds. Specifically, in our case we use this lemma for real Bessel processes of dimensions $d_i = d(\kappa_i) = 1 - \frac{4}{\kappa_i}$, $i \in \mathbb{N}$ with $d_i < 2, \forall i \in \mathbb{N}$, that are (strong) solutions to the real-valued SDE (5.3.1).

We also consider as an ingredient in our analysis the following theorem.

Theorem 5.4.4 (Dini's Theorem). *Let M be a compact topological space, and let us consider monotonic sequence of continuous functions $f_n : M \rightarrow \mathbb{R}$ which converge pointwise to a continuous function $f : M \rightarrow \mathbb{R}$. Then this convergence is uniform.*

The final step needed in the proof is the following lemma.

Lemma 5.4.5. *Let us consider an increasing sequence of real parameters $\kappa_i \rightarrow \kappa$ with $0 < \kappa_i \leq 4$. Let us consider a sequence of Bessel processes of dimension $d = d(\kappa_i) = 1 - \frac{4}{\kappa_i}$, with the same starting point $x_0 > 0$. For almost every Brownian motion, we can define $y_i(\omega) < 0$ such that $T_{y_i}^{\kappa_i}(\omega) = T_{x_0}^{\kappa_i}(\omega)$.*

Then, for almost every Brownian path, we can find a convergent subsequence $y_{m_i}(\omega) \rightarrow \hat{y}(\omega)$. Moreover, we obtain that a.s.

$$T_{y_{m_i}(\omega)}^{\kappa_i} \rightarrow T_{\hat{y}(\omega)}^{\kappa}.$$

Proof. For the sequence $\kappa_i \rightarrow \kappa$, we have a sequence of starting points to the left of the origin $y_i = y_i(\omega)$ such that $T_{y_i}^{\kappa_i} = T_{x_0}^{\kappa_i}$. We claim that the sequence y_i is

a.s. bounded. For contradiction, let us assume that with positive probability the sequence y_i is not bounded. Then, with positive probability, we can find a subsequence that is diverging to $-\infty$. This is contradicting the fact that the hitting times of the origin for $y_i(\omega)$ are bounded by $T_{x_0}^{\kappa_i} \leq T_{x_0}^\kappa = T < +\infty$, almost surely, by the monotonicity of the drifts. Indeed, if with positive probability there exists a divergent subsequence, then with positive probability the hitting times of the origin for this subsequence would have $T_{y_i}^\kappa \rightarrow \infty$. In order to see this, we recall that the Lamperti relation gives that $T_{y_i}^\kappa \stackrel{(d)}{=} \frac{y_i}{Z_\kappa}$, where Z_κ is a Gamma random variable with index depending on κ . Thus, for $T < +\infty$ as above, we have that on the subsequence $y_{m_i} \rightarrow \infty$, it holds that $\mathbb{P}\left(T_{y_{m_i}}^\kappa > T\right) = \mathbb{P}\left(\frac{y_{m_i}}{Z_\kappa} > T\right) > 0$, contradicting the fact that $T_{x_0}^\kappa = T < +\infty$, almost surely.

Thus, the sequence $y_{\kappa_i}(\omega)$ is almost surely bounded. From Bolzano-Weierstrass Theorem, we have a.s. a convergent subsequence $y_{m_i}(\omega) \rightarrow \hat{y}(\omega)$.

In order to apply Dini's Theorem for each Brownian path, we consider the splitting

$$|T_{y_i}^{\kappa_i} - T_{\hat{y}}^\kappa| \leq |T_{y_i}^{\kappa_i} - T_{y_i}^\kappa| + |T_{y_i}^\kappa - T_{\hat{y}}^\kappa|.$$

In order to estimate the first term, we use Dini's Theorem for the family of functions

$$f_j(y) := T_y^{\kappa_j}.$$

Note that in order to apply Dini's Theorem for each path of the Brownian driver, we need that for almost every Brownian path the functions $f_j(y)$ are defined on a compact set and are monotonically increasing continuous real valued functions that converge pointwise to a continuous function. Indeed, by the Bolzano-Weierstrass argu-

ment we can restrict for any $j \in \mathbb{N}$ the functions $f_j(y)$ on compact sets $\{y_1, y_2, \dots, \hat{y}\}$. In this manner we obtain different compacts for each Brownian paths. All these sets are compact sets since for almost every Brownian path they are closed (they contain only countably many points) and bounded (by Bolzano-Weierstrass). By Heine-Borel Theorem we obtain that the functions are indeed defined on compacts for almost every Brownian path. Moreover, they are monotonically increasing a.s. by the ordering of the drifts and they are a.s. continuous at each point by Lemma 5.4.3.

Using the result from Lemma 5.4.3, we have that $T_y^{\kappa_i}$ and T_y^κ are a.s. sequentially continuous real valued functions at $\hat{y}$.

We have that

$$|T_{y_i}^{\kappa_i} - T_{y_i}^\kappa| \leq \sup_{y \in \{y_1, y_2, \dots, \hat{y}\}} |T_y^{\kappa_i} - T_y^\kappa|.$$

From Dini's Theorem, we know that $\exists i_0(x_0, \omega, \kappa) > 0$ such that for all $i > i_0$,

$$\sup_{y \in \{y_1, y_2, \dots, \hat{y}\}} |T_y^{\kappa_i} - T_y^\kappa| \leq \varepsilon.$$

For the second term, we estimate using the sequential continuity in parameter y_i for a fixed $\kappa \in (0, 4]$, as implied by Lemma 5.4.3. From this, we obtain that for almost all Brownian paths there exists an index $i_1 = i_1(\omega, \kappa) > 0$ such that for all $i \geq i_1$, $|T_{y_i}^\kappa - T_{\hat{y}}^\kappa| \leq \varepsilon/2$. Taking $i(x_0, \omega) = \max(i_0(x_0, \omega, \kappa), i_1(\omega, \kappa))$ we obtain the desired conclusion.

By applying Lemma 5.4.5 together with Lemma 5.4.1, we obtain that a.s. $\hat{y}(\omega)$ is the unique point with this value of the hitting time. Indeed, otherwise we would have two points on the same side of the singularity that will hit the origin in the simultaneously, contradicting Lemma 5.4.1.

Note that for any other sequence $\tilde{\kappa}_i \rightarrow \kappa$, we obtain a converging subsequence in the same manner as described before using Bolzano-Weierstrass Theorem for almost every Brownian path. Using the sequential continuity in both variables proved in Lemma 5.4.5 along with Lemma 5.4.1, we prove that for almost every Brownian path all the subsequences have the unique limit $\tilde{y}(\omega)$ (which depends on the path). $\square$

Performing the same analysis for a countable collection of starting points we obtain the point-wise convergence argument on a dense set of points on the real line. Since the functions $T_y^{\kappa_i}$ are monotonic in y a.s. for all i , and since the limit T_y^κ is also monotone, we obtain that they have at most countably many discontinuities. Moreover, since $T_{y_j}^{\kappa_j} \rightarrow T_{y_j}^\kappa$ on a countable set of points y_j and by monotonicity of the functions, we have that

$$T_{y_m}^{\kappa_j} \leq T_y^{\kappa_j} \leq T_{y_n}^{\kappa_j}$$

for $y_m \rightarrow y$ from below and $y_n \rightarrow y$ from above. Thus, we have that

$$T_x^\kappa = \lim_{j \rightarrow \infty} \lim_{m \rightarrow \infty} T_{y_n}^{\kappa_j} \leq \liminf_{j \rightarrow \infty} T_y^{\kappa_j} \leq \limsup_{j \rightarrow \infty} T_y^{\kappa_j} \leq \lim_{j \rightarrow \infty} \lim_{n \rightarrow \infty} T_{y_n}^{\kappa_j} = T_x^\kappa,$$

at points of continuity of T_x^κ . We have the desired conclusion on the set of continuity points of T_x^κ , a.s.. Since the function is monotone, the set of discontinuities is at most countable.

Note that in our analysis we used a.s. properties of the Bessel processes. In several steps of the proof we considered only countably many changes in either the starting point or the dimension of the process. Thus, the previous estimates hold outside a countable union of nullsets that remain a nullset.

Thus, outside the nullset that is the countable union of all the nullsets where the backward SLE_{κ_i} traces are defined for $(\kappa_i)_{i \in \mathbb{N}}$, along with the nullsets corresponding

to the analysis of the Bessel processes described in the previous paragraph, we obtain the point-wise convergence of the welding homeomorphism at the value $\kappa \in (0, 4]$. Finally, since this value of κ was chosen arbitrarily, we obtain the almost sure point-wise convergence of the welding homeomorphism induced by the backward SLE_κ for $\kappa \in (0, 4]$.

5.5 On the sequential continuity of the SLE_κ traces for $\kappa \in \mathbb{R}_+^* \setminus \{8\}$ and stability in κ of an approximation algorithm for these traces

We give a proof of the sequential continuity of the SLE_κ traces in κ for $\kappa \in \mathbb{R}_+^* \setminus \{8\}$ along some sequences that respect a technical condition. The proof is based on the algorithm introduced in [53]. We consider $\kappa \neq 0$ and $\kappa \neq 8$, to avoid some technical difficulties with some derivative estimate on the SLE maps that we use in our analysis. We also prove the stability of this algorithm in the parameter κ for $\kappa \in \mathbb{R}_+^* \setminus \{8\}$, in the sense that we show that when interpolating $\sqrt{\kappa_1}B_t$ and $\sqrt{\kappa_2}B_t$ with κ_1 and κ_2 close to each other, the curves generated by the algorithm are close in the sup-norm.

The continuity of the SLE_κ is proven in [56] using direct methods from differential equations to estimate the difference between the SLE_κ traces as the parameter $\kappa \in [0, 8(2 - \sqrt{3}))$ is varied. In this section we show a sequential continuity result using the approximation of the Brownian drivers with square root interpolations at fixed time steps for all values of $\kappa \in \mathbb{R}_+^* \setminus \{8\}$.

A byproduct of this analysis is that the algorithm designed in [53] is a.s. sequentially continuous (along the sequences that respect the technical condition) in κ , for

$$\kappa \in \mathbb{R}_+^* \setminus \{8\}.$$

The algorithm is based on square root interpolation of the Brownian driver. More precisely, when the backward Loewner differential equation is driven by a real valued function of the form $\lambda(t) = c\sqrt{t} + d$ with some real constants c and d , one can solve the Loewner equation explicitly (see [21]) and obtain the solution

$$g_t^{-1}(z + \lambda(t)) = \left(z + 2\sqrt{t}\sqrt{\frac{\alpha}{1-\alpha}}\right)^{1-\alpha} \left(z - 2\sqrt{t}\sqrt{\frac{\alpha}{1-\alpha}}\right)^{\alpha} + d.$$

Moreover, the Loewner equation driven by this function generates a trace that is a straight segment in the upper half-plane starting at $d \in \mathbb{R}$ and that makes an angle $\alpha\pi$ with the positive real axis where

$$\alpha = \frac{1}{2} - \frac{1}{2} \frac{c}{\sqrt{16 + c^2}}.$$

In the algorithm, we consider a fixed time step partition of the interval $[0, 1]$, $t_k = \frac{k}{n}$ for $0 \leq k \leq n$. Then Loewner equation is solved with the driving function $\lambda(t + t_k)$ for $0 \leq t \leq \frac{1}{n}$. The algorithm is designed to approximate the driving function $\lambda(t)$ in the forward Loewner differential equation by $\lambda^n(t)$. The function $\lambda^n(t)$ is chosen such that it has the same values as $\lambda(t)$ at t_k . Formally, λ^n is defined as $\lambda^n(t) = \sqrt{n}(\lambda(t_{k+1}) - \lambda(t_k))\sqrt{t - t_k} + \lambda(t_k)$ on $[t_k, t_{k+1}]$.

The explicit mappings obtained on each element of the partition are then composed with each other and the resulting trace is compared with the SLE_{κ} trace. The choice of this particular interpolation of the Brownian driver is motivated by the fact that the equation can be resolved explicitly on each element of the partition.

Definition 5.5.1. A function $\phi(x) : \mathbb{R} \rightarrow \mathbb{R}$ is a sub-power function, if for any $\nu > 0$,

$$\lim_{x \rightarrow \infty} x^{-\nu} \phi(x) = 0.$$

We recall from the introduction that we use the parametrization by capacity of the SLE curves, i.e. the parametrization in which $hcap(K_t) = 2t$. In this section, we prove the following result.

Theorem 5.5.2. Let $\kappa \in \mathbb{R}_+^* \setminus \{8\}$. Let us consider $\kappa_n \rightarrow \kappa$, such that $|\kappa_n - \kappa| \leq \frac{\phi(n)}{\sqrt{n}}$, as $n \rightarrow \infty$, for $\phi : \mathbb{R} \rightarrow \mathbb{R}$ a sub-power function. The SLE_{κ_n} traces on finite time intervals, parametrized by capacity, are a.s. converging to SLE_{κ} , for $\kappa \in \mathbb{R}_+^* \setminus \{8\}$, with respect to the sup-norm. Moreover, the curves generated by the square-root interpolation algorithm described in the introduction are sequentially continuous along the sequences that respect the technical condition above, in κ for $\kappa \in \mathbb{R}_+^* \setminus \{8\}$, a.s.

Remark 5.5.3. Since we are interested in the sequential continuity of the traces only, all the properties and estimates related with the SLE_{κ} traces that hold with probability one, for fixed $\kappa \in \mathbb{R}_+^* \setminus \{8\}$, will remain true with probability one outside the nullset formed from the union of the nullsets obtained when considering the sequence $\kappa_n \rightarrow \kappa$. Note that the union of nullsets depends on the sequence.

Since the driver is a $\sqrt{\kappa}B_t$, the subpower function that appears in the statement of the main result can be tracked through the analysis as in [53]. Following the proof, we obtain a subpower function of the form $c\sqrt{\log n}^{c'}$ for some constants c and c' .

An essential result that we use is Theorem 2.2 from [53]. The result is a sup-norm convergence of the approximated trace to the trace generated by the Loewner differential equation driven by a weakly 1/2-Hölder driver. We emphasize that

the result is a deterministic result that holds for driving functions U_t and maps that respect some assumptions. We show at the end of this section that these assumptions hold a.s. for the SLE_κ maps and driver.

Let us first introduce a list of κ dependent parameters: Let $-\infty < r < r_c := \frac{1}{2} + \frac{4}{\kappa}$ and let $\beta = \beta(r, \kappa) = -1 + \frac{\kappa}{4 + \kappa - r\kappa}$. For $r_+ = 4(-2 + \sqrt{8 + \kappa})/\kappa$ we have

$$\beta_+(r_+, \kappa) = \beta_+(\kappa) = -1 + \frac{\kappa}{12 + \kappa - 4\sqrt{8 + \kappa}}.$$

Note that $r_+ < r_c$ and

$$\beta_+ = -1 + \frac{\kappa}{12 + \kappa - 4\sqrt{8 + \kappa}} < \beta(r_c) = 1.$$

Definition 5.5.4. *A function λ is called weakly Hölder-1/2 if there exists a subpower function ϕ such that*

$$\text{osc}(\lambda, \delta) := \sup\{|\lambda(t) - \lambda(s)| : s, t \in [0, 1], |t - s| \leq \delta\} \leq \sqrt{\delta} \phi\left(\frac{1}{\delta}\right)$$

for all $\delta > 0$.

Theorem 5.5.5 (Theorem 2.2. of [53]). *Suppose the driving function of the Loewner chain $g_t(z)$ is weakly Hölder-1/2 with sub-power function ϕ and suppose that for $f_t(z) = g_t^{-1}(z)$, the estimate $|f'_t(iy)| \leq c_0 y^{-\beta}$ is satisfied for $y \leq y_0$. Then, the curve generated by the Loewner equation can be approximated by the curves obtained by interpolating with square-root functions the Brownian driver, in the sense that there exists a sub-power function $\tilde{\phi}(n)$ that depends on ϕ , c_0 and β , such that for all*

$n \geq \frac{1}{y_0^2}$ and $t \in [0, 1]$ we have that

$$|\gamma^n(t) - \gamma(t)| \leq \frac{\tilde{\phi}(n)}{n^{\frac{1}{2}(1-\sqrt{\frac{1+\beta}{2}})}},$$

where γ^n is the curve generated in the algorithm described on the previous page.

Proof of Theorem 5.5.2. Throughout this section, the precise subpower function that we use is changing from line to line. We do not track these changes in order to not burden the reader, since in our analysis all that is needed is that this is a subpower function.

Let us consider $(\kappa_n)_{n \in \mathbb{N}}$, such that $\kappa_n \rightarrow \kappa$, with $\kappa \in \mathbb{R}_+^* \setminus \{8\}$. Let us define the following subsets of $\mathbb{H}$ as in [53].

$$A_{n,c,\phi} = \left\{ x + iy \in \mathbb{H} : |x| \leq \frac{\phi(n)}{\sqrt{n}}, \frac{1}{\sqrt{n}\phi(n)} \leq y \leq \frac{c}{\sqrt{n}} \right\}.$$

We also consider the following two lemmas from [53] as well as the following proposition from [54] that we use for sequences of parameters $\kappa_n \rightarrow \kappa$, with $\kappa \in \mathbb{R}_+^* \setminus \{8\}$.

Lemma 5.5.6 (Lemma 3.2 of [53]). *There exists a subpower function ψ depending only on ϕ, c_0 and β such that for $n \geq 1$ and $0 \leq k \leq n-1$, there exists $s \in [0, \frac{2}{n}]$ such that $\gamma_k(s) \in A_{n,2\sqrt{2},\psi}$.*

The estimate $|(f_t^\kappa)'(y_0)| \leq c(\omega)y^{-\beta(\kappa)}$ with $\beta(\kappa) < 1$, $\forall \kappa \neq 8$ can be applied for a sequence κ_n such that $\kappa_n \rightarrow \kappa$, with $\kappa \in \mathbb{R}_+^* \setminus \{8\}$. In this way, we have the estimates on SLE_{κ_n} traces for values κ_n (with different subpower functions that depend on κ_n) outside $\cup_{\kappa_n} \mathcal{N}^{\kappa_n}$.

Lemma 5.5.7 (Lemma 3.3 of [53]). *There exists a subpower function $\tilde{\psi}$ depending only on ϕ, c_0 and β such that $\gamma_k^n(r)$ is in the box $A_{n, 2\sqrt{2}, \tilde{\psi}}$ for $n \geq 1$, $0 \leq k \leq n-1$ and $r \in [\frac{1}{n}, \frac{2}{n}]$.*

This lemma is applied to the trace obtained when approximating the SLE_κ trace for the fixed value κ .

Proposition 5.5.8 (Proposition 3.8 of [54]). *Let (g_t) be a Loewner chain with the driver $\lambda(t)$ that satisfies the conditions*

- $\text{osc}(\lambda; \delta) = \sup\{|\lambda(t) - \lambda(s)| : s, t \in [0, 1], |t - s| \leq \delta\} \leq \sqrt{\delta}\phi(1/\delta)$.
- $|\hat{f}'_t(iy)| \leq c_0 y^{-\beta}$ for all $0 \leq y \leq y_0$, $t \in [0, 1]$.

Then, there exists a subpower function ϕ_1 such that if $0 \leq t \leq t + s \leq 1$ and $s \in [0, y^2]$, we have that

$$|\gamma(t + s) - \gamma(t)| \leq \phi_1 \left(\frac{1}{y} \right) \frac{2}{1 - \beta} y^{1-\beta} \quad (5.5.1)$$

for $0 \leq s \leq y^2 \leq y_0^2$.

Recall that we consider a sequence $\kappa_n \rightarrow \kappa$ such that $|\kappa_n - \kappa| \leq \frac{\phi(n)}{\sqrt{n}}$, with $\kappa \in \mathbb{R}_+^* \setminus \{8\}$.

We have that

$$|\gamma^{\kappa_n}(t) - \gamma^\kappa(t)| \leq |\gamma^{\kappa_n}(t) - \gamma_\kappa^n(t)| + |\gamma_\kappa^n(t) - \gamma^\kappa(t)|, \quad (5.5.2)$$

where $\gamma_\kappa^n(t)$ is the trace obtained from interpolating with square root terms the driver $\sqrt{\kappa}B_t$. The last term in the inequality is estimated by Theorem 5.5.5. For the first term, let $z = \gamma^{\kappa_n}(s)$, $w = \gamma^{\kappa, n}(r)$. We estimate, for $r \in [\frac{1}{n}, \frac{2}{n}]$,

$$|\gamma^{\kappa_n}(s + t_k) - \gamma^{\kappa,n}(r + t_k)| \leq |f^{\kappa_n}_{t_k}(z) - f^{\kappa_n}_{t_k}(w)| + |f^{\kappa_n}_{t_k}(w) - f^{\kappa,n}_{t_k}(w)| \quad (5.5.3)$$

We apply the previous Lemmas 5.5.6 and 5.5.7 for sequences $\kappa_n \rightarrow \kappa$ that respect our technical condition. The sizes of the boxes depend on κ_n via the dependence of the subpower function that we choose on $\beta = \beta(\kappa_n)$ and on ϕ (that depends also on κ_n , since the driver is $\sqrt{\kappa_n}B_t$). However, since the constant $c = 2\sqrt{2}$ is fixed, the upper level of the boxes remains the same as we consider $\kappa_n \rightarrow \kappa$, only their width and lower level changes.

We consider $\kappa_n \rightarrow \kappa$ by choosing for each n the largest box that contains both points z and w in order to estimate the hyperbolic distance between them, i.e. we make use of the fact that the upper height of the boxes coincides and we work on $A_{n,2\sqrt{2},\xi(\kappa_n)}$ with $\xi(\kappa_n) = \max(\psi(\kappa_n), \tilde{\psi}(\kappa))$. This is a dynamical version (as we vary the index n) of the analysis in [53] that is performed for fixed κ . For each fixed n , the estimates work in the same manner.

As in [53], we estimate **the first term** in (5.5.3) using

$$|f^{\kappa_n}_{t_k}(z) - f^{\kappa_n}_{t_k}(w)| \leq (2\text{Im}z)|(f^{\kappa_n}_{t_k})'(z)| \exp(4d_{\mathbb{H},hyp}(z, w)),$$

$$\text{where } d_{\mathbb{H},hyp}(z, w) = \text{Arccosh} \left(1 + \frac{|z-w|^2}{2\text{Im}z\text{Im}w} \right).$$

For a generic point $u = x + iy$, we use the almost sure estimate that holds outside of $\cup_{\kappa_n} \mathcal{N}^{\kappa_n}$.

$$|(f^{\kappa_n}_{t_k})'(z)| \leq c(\omega, \kappa_n) y^{-\beta(\kappa_n)}.$$

For the second term along with these estimates, we use Lemma 2.3 of [56]. For

this we estimate the distance ε between the two driving terms: the square root interpolation of the $\sqrt{\kappa}B_t$ and $\sqrt{\kappa_n}B_t$ with

$$|\lambda_\kappa^n(t) - \sqrt{\kappa_n}B_t| \leq |\lambda_\kappa^n(t) - \sqrt{\kappa}B_t| + |\sqrt{\kappa}B_t - \sqrt{\kappa_n}B_t|.$$

Thus, by considering an appropriate mesh of the interval where κ takes values in, we obtain, combining the estimates with the ones in [53], that

$$|\lambda_\kappa^n(t) - \sqrt{\kappa_n}B_t| \leq \frac{\phi(n)}{\sqrt{n}} + |\sqrt{\kappa_n} - \sqrt{\kappa}| \sup_{t \in [0,1]} |B_t| \leq \frac{\phi(n)}{\sqrt{n}} + c \frac{\phi(n)}{\sqrt{n}}.$$

The next result that we use is the following lemma that appeared in [56].

Lemma 5.5.9 (Lemma 2.3 of [56]). *Let $0 < T < \infty$. Suppose that for $t \in [0, T]$, $f_t^{(1)}$ and $f_t^{(2)}$ satisfy the backward Loewner differential equation with drivers $W_t^{(1)}$ and $W_t^{(2)}$. Suppose that $\varepsilon = \sup_{s \in [0, T]} |W_s^{(1)} - W_s^{(2)}|$. Then, if $u = x + iy \in \mathbb{H}$, then*

$$|f_T^{(1)}(u) - f_T^{(2)}(u)| \leq \varepsilon \exp \left[\frac{1}{2} \left[\log \frac{I_{Ty} |(f_T^{(1)})'(u)|}{y} \log \frac{I_{Ty} |(f_T^{(2)})'(u)|}{y} + \log \log \frac{I_{T,y}}{y} \right] \right],$$

where $I_{T,y} = \sqrt{4T + y^2}$.

Thus, in our case this gives that

$$|f_{t_k}^{\kappa_n}(w) - f_{t_k}^{\kappa,n}(w)| \leq \varepsilon \exp \left[\frac{1}{2} \left[\log \frac{I_{t_k,y} |(f_{t_k}^{\kappa_n})'(w)|}{y} \log \frac{I_{t_k,y} |(f_{t_k}^{\kappa,n})'(w)|}{y} + \log \log \frac{I_{t_k,y}}{y} \right] \right],$$

where $I_{t_k,y} = \sqrt{4t_k + y^2}$ with $\varepsilon = \sup_{t \in [0, t_k]} |\lambda_\kappa^n(t) - \sqrt{\kappa_n}B_t| \leq \frac{2\phi(n)}{\sqrt{n}} + c \frac{2\phi(n)}{\sqrt{n}}$.

These estimates are used for points inside the boxes $A_{n,c,\phi}$. Thus, for $y = \text{Im}u \in [\frac{1}{\sqrt{n}\phi(n)}, \frac{2\sqrt{2}}{\sqrt{n}}]$ we have that

$$\frac{I_{t_k,y}}{y} \leq 2\sqrt{2}\sqrt{n}\phi(n),$$

for $\phi(n)$ some sub-power function of n . We use the estimates

$$|(f_{t_k}^{\kappa_n})'(u)| \leq cy^{-\beta(\kappa_n)} \leq c_0\phi(n)^{\beta(\kappa_n)}\sqrt{n}^{\beta(\kappa_n)}.$$

and

$$|(f_{t_k}^{\kappa,n})'(w)| \leq C(1/y + 1) \leq 2C\phi(n)\sqrt{n}$$

Note that the second estimate holds true for any conformal map of $\mathbb{H}$. Combining these estimates, we obtain that

$$\begin{aligned} |f_{t_k}^{\kappa_n}(w) - f_{t_k}^{\kappa,n}(w)| &\leq (1+c)\frac{\phi(n)}{\sqrt{n}} \exp \left[\sqrt{\frac{1+\beta(\kappa_n)}{2}} \log(c\phi(n)\sqrt{n}) + \log \log 2\sqrt{2n}\phi(n) \right] \\ &=: \frac{\phi(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_n)}{2}}}}. \end{aligned} \quad (5.5.4)$$

Thus, using the lemmas before for the sequence of parameters κ_n we obtain that

$$|\gamma^{\kappa_n}(s+t_k) - \gamma^{\kappa,n}(r+t_k)| \leq \frac{\phi_1(n)}{\sqrt{n}^{1-\beta(\kappa_n)}} + \frac{\phi(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_n)}{2}}}}. \quad (5.5.5)$$

for all $r \in [\frac{1}{n}, \frac{2}{n}]$. Using Proposition 5.5.8, for the sequence of parameters κ_n we obtain that

$$|\gamma^{\kappa_n}(s+t_k) - \gamma^{\kappa,n}(r+t_k)| \leq \frac{\phi_2(n)}{\sqrt{n}^{1-\sqrt{\frac{1+\beta(\kappa_n)}{2}}}}. \quad (5.5.6)$$

for all $r \in [t_{k+1}, t_{k+2}]$ and $0 \leq k \leq n-2$ and hence for all $r \in [0, 1]$.

In order to estimate the first term in (5.5.2), i.e. the term $|\gamma^{\kappa_n}(t) - \gamma^{\kappa,n}(t)|$, we consider the following estimate from [54].

Proposition 5.5.10 (Proposition 4.2 in [54]). *Let $\beta = \beta(\kappa) < 1$ for $\kappa \neq 8$. Suppose*

$\beta > \max\{0, \beta_+\}$ where β_+ is an explicit quadratic function of κ . With probability one, there exists $y_0(\omega) > 0$, such that for all $t \in [0, 1]$ and all $y < y_0$,

$$|f'_t(iy)| \leq y^{-\beta}. \quad (5.5.7)$$

This implies that for constants c_3 and c_4 , β' depending on κ ,

$$\sum_{m=n}^{\infty} \sum_{j=1}^{2^m} \mathbb{P} \left[|\hat{f}'_{(j-1)2^{-2m}}(i2^{-m})| \geq 2^{\beta'm} \right] \leq \frac{c_3}{2^{nc_4}}. \quad (5.5.8)$$

As in [53], this implies (via the change $2^m \rightarrow \sqrt{m}$), that there exists $c_5 > 0$ and $\beta \in (\beta', 1)$ such that

$$\mathbb{P} \left[|\hat{f}'_t(iy)| \leq y^{-\beta} \text{ for all } 0 \leq y \leq \frac{1}{\sqrt{n}}, t \in [0, 1] \right] \geq 1 - \frac{c_3}{n^{c_4/2}}. \quad (5.5.9)$$

Note that the estimate (5.5.9) holds for fixed value of the parameter κ . A weaker form of this estimate along with the modulus of continuity of Brownian motion is used in order to prove the existence of the SLE_{κ} trace for fixed value of $\kappa \neq 8$ in [44]. Taking the limit $n \rightarrow \infty$, the estimate holds outside of a nullset that depends on the fixed value κ . Thus, the estimate in (5.5.9) implies the existence of the SLE_{κ} trace outside of a nullset that depends on κ .

Thus, we can use the estimate

$$\mathbb{P} \left[|\hat{f}'_t(iy)| \leq y^{-\beta(\kappa_n)} \text{ for all } 0 \leq y \leq \frac{1}{\sqrt{n}}, t \in [0, 1] \right] \geq 1 - \frac{c_3(\kappa_n)}{n^{c_4(\kappa_n)/2}}. \quad (5.5.10)$$

Moreover, it follows from [26] that there exists constants c_1 (depending on κ) and

c_2 such that

$$\mathbb{P} \left[\text{osc}(\sqrt{\kappa} B_t, \frac{1}{m}) \geq c_1 \sqrt{\frac{\log m}{m}} \right] \leq \frac{c_2}{n^2}. \quad (5.5.11)$$

Using (5.5.6), we obtain the following estimate for

$$\mathbb{P} \left[\|\gamma^{k_n} - \gamma^{\kappa, n}\|_{[0,1],\infty} \leq \frac{c_6(\log m)^{c_7}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta(\kappa_n)}{2}}}} \text{ for all } m \geq n \right] \geq 1 - \left(\frac{c_2(\kappa_n)}{n^2} + \frac{c_3(\kappa_n)}{n^{c_4(\kappa_n)/2}} \right).$$

In [53] it is proved the following result about the SLE_κ trace $\gamma(t)$ for a generic fixed κ .

$$\mathbb{P} \left[\|\gamma^\kappa(t) - \gamma^m(t)\|_{[0,1],\infty} \leq \frac{c_1(\log(m))^{c_2}}{\sqrt{m}^{1-\sqrt{\frac{1+\beta(\kappa)}{2}}}} \text{ for all } m \geq n \right] \geq 1 - \frac{c_3}{n^{c_4}},$$

where the constants c_1, c_2, c_3 and c_4 depend on κ . We use this to estimate the second term in (5.5.2), i.e. the term $|\gamma^{\kappa, n}(t) - \gamma^\kappa(t)|$.

By letting $n \rightarrow \infty$ we obtain the desired conclusion.

The sequential continuity in κ for $\kappa \neq 8$ of the curves generated by the algorithm is obtained by estimating

$$|\gamma_{\kappa_n}^n(t) - \gamma^{\kappa, n}(t)| \leq |\gamma^{\kappa, n}(t) - \gamma^\kappa(t)| + |\gamma^\kappa(t) - \gamma^{\kappa_n}(t)| + |\gamma^{\kappa_n}(t) - \gamma_{\kappa_n}^n(t)|. \quad (5.5.12)$$

The first and the last term can be directly estimated using Theorem 5.5.5, since these are terms that compare the SLE_{κ_n} and SLE_κ traces with the corresponding approximated traces. The middle term is estimated using the analysis performed in the proof so far, and the conclusion follows.

□

Remark 5.5.11. The algorithm uses estimates on the derivative of the conformal maps. We remark that the derivative of the composition of the conformal maps obtained when solving Loewner equation on each element of the partition of the time interval $[t_k, t_{k+1}]$ (where $t_k = \frac{k}{n}$, $0 \leq k \leq n$) with $c\sqrt{t} + d$ with $c, d \in \mathbb{R}$, is not easy to estimate directly. That is why we used in our proof the Rohde-Schramm estimate on the derivative of the Loewner map : $|(f_t^\kappa)'(y_0)| \leq c(\omega)y^{-\beta(\kappa)}$ with $\beta(\kappa) < 1$, $\forall \kappa \neq 8$. This estimate is obtained in [44].

Chapter 6

Explicit stationary measure and Skorokhod Embedding type problem for the backward SLE_κ flow

Overview

We study a one-dimensional SDE that we obtain by performing a random time change of the backward Loewner dynamics in $\mathbb{H}$. The stationary measure for this SDE has a closed-form expression. We show the convergence towards its stationary measure for this SDE, in the sense of random ergodic averages. The precise formula of the density of the stationary law gives a phase transition at the value $\kappa = 8$ from integrability to non-integrability, that happens at the same value of κ as the change in behavior of the SLE_κ trace from non-space filling to space-filling curve. Using convergence in total variation for the law of this diffusion towards stationarity, we identify families of random times on which the law of the arguments of points under the backward SLE_κ flow converge to a closed form expression measure. For $\kappa = 4$,

this gives precise characterization for the random times on which the law of the arguments of points under the backward SLE_κ flow converge to the uniform law.

6.1 Introduction

In this chapter we consider a time change of the backward differential equation. The main reason for doing this is that when considering the backward Loewner differential equation in the upper half-plane and investigating its real and imaginary dynamics, we obtain naturally a system of coupled differential equations. These systems are usually hard to analyze. The purpose of the work in this chapter is to transform this coupled system of equations by making use of techniques of Stochastic Analysis into an one-dimensional radial-independent diffusion process that is easier to analyze. In turn, the imaginary part of the backward Loewner differential equation can be expressed in terms of this diffusion process making this approach a veritable change of coordinates.

The chapter is divided into several sections. In the first part, we introduce the basic notions that we use throughout. In the second section, we describe the random time change of the backward Loewner differential equation that we use in our analysis. In the third section, we build step by step the setting required to investigate the problem and in the fourth section we adapt the convergence in the sense of ergodic average for the SDE (4.2.3) to uniformly large random times in order to conclude.

6.2 A time change and the main result

We consider the probability space $(\Omega, \sigma(B_s, s \in [0, +\infty)), \mathbb{P})$, where $\mathbb{P}$ denotes the Wiener measure. On this space, we consider the standard Brownian motion B_t . Throughout the proof we use the Lévy's characterization of Brownian motion for a local martingale that we obtain in our analysis, in order to obtain a new Brownian motion. Before going to the proof of the result in this section, we define a time change of the (backward) Loewner differential equation that is going to be the framework of study for our problem. First, we rewrite the equation

$$\partial_t h(t, z) = \frac{-2}{h(t, z) - \sqrt{\kappa} B_t}, \quad h(0, z) = z, \quad z \in \mathbb{H}, \quad (6.2.1)$$

in the format

$$dz_t = \frac{-2}{z_t} dt - \sqrt{\kappa} dB_t, \quad z_0 \in \mathbb{H}, \quad (6.2.2)$$

where we make the identification $z_t = x_t + iy_t = h_t(z) - \sqrt{\kappa} B_t$. Throughout this chapter, we choose $z_0 = i$. Furthermore, we split this flow in the upper half-plane into its real and imaginary components. This gives

$$dx_t = \frac{-2x_t}{x_t^2 + y_t^2} dt - \sqrt{\kappa} dB_t, \quad dy_t = \frac{2y_t}{x_t^2 + y_t^2} dt. \quad (6.2.3)$$

We consider $D_t = x_t/y_t$. Applying Itô's formula for the function $f(x, y) = \frac{x}{y}$, we obtain that

$$dD_t = -\frac{\sqrt{\kappa} dB_t}{y_t} - \frac{4D_t}{x_t^2 + y_t^2} dt.$$

We study the random time change given by

$$\tilde{u}(s) = \int_0^s \frac{dt}{y_t^2},$$

Note that since $y_0 = 1$, and y_t is a.s. increasing, we have that the function $t \rightarrow dt/y_t^2$ is integrable. We consider also

$$c(u) = \inf\{t \geq 0 : \tilde{u}(s) \geq u\}. \quad (6.2.4)$$

By setting

$$\tilde{B}(u) = \int_0^u \frac{dB_{c(r)}}{y_{c(r)}}, \quad (6.2.5)$$

we obtain that T_u defined as $T_u = \frac{D_{c(u)}}{\sqrt{\kappa}}, u \geq 0$ satisfies the following stochastic differential equation

$$dT_u = -4 \frac{T_u}{1 + \kappa T_u^2} du + d\tilde{B}_u. \quad (6.2.6)$$

The continuous local martingale $\tilde{B}_u$ has quadratic variation u . Thus, by Lévy's characterization, we obtain that $\tilde{B}_u$ is a Brownian motion. Thus, when performing this random time change we obtain a one-dimensional stochastic differential equation. We emphasize that the filtration that we consider when performing the random time change it is changing from $\mathcal{F}_t$ (the natural filtration of the standard Brownian motion up to time $t \in [0, +\infty)$) to $\mathcal{F}_{\tilde{u}(t)}$, where $\tilde{u}(t)$ is the random time change described before.

We introduce next a list of useful definitions for the work in this chapter:

Definition 6.2.1 (Stationary distribution). *Let X_t be a one-dimensional Markov process. A stationary distribution for X_t is a probability distribution ψ on $\mathbb{R}$ such that if $X_0 \sim \psi$, then X_t has distribution ψ for all $t \geq 0$.*

In particular, one can obtain from this definition that

$$\frac{d}{dt}\mathbb{E}[f(X_t)|X_0 \sim \psi] = 0,$$

with $f \in L^1(\mu)$, where $\mu(dx)$ is the density of the stationary distribution of X_t .

Definition 6.2.2. *For a one-dimensional diffusion X_t , let $P_t f(x) = \mathbb{E}^x[f(X_t)]$ be its transition semigroup. A one-dimensional diffusion is said to converge strongly to its stationary measure μ if for any compact $K \in \mathbb{R}$, for $f \in L^\infty(\mathbb{R})$ we have that $\sup_{x \in K} |P_t f(x) - \mu(f)| \rightarrow 0$, as $t \rightarrow \infty$, where $\mu(f) = \int_{\mathbb{R}} f(y)\mu(dy)$.*

The stationary distribution of a diffusion is obtained using the Kolmogorov forward equation that we introduce next.

Definition 6.2.3 (Kolmogorov forward equation). *Let $C_0^2(\mathbb{R})$ be the space of differentiable functions with continuous derivatives up to the second order and that vanish at infinity. Let X_t be a diffusion on $\mathbb{R}$ whose transition semigroup admits a density, with generator*

$$Af(y) = a(y)\frac{\partial^2 f(y)}{\partial y^2} + b(y)\frac{\partial f(y)}{\partial y}.$$

Let $p_t(x, y)$ be the density of the transition semigroup (i.e. the function $p_t(x, y)$ in $P_t f(x) = \mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}} f(y)p_t(x, y)dy$, $f \in C_0^2(\mathbb{R})$). Then $p_t(x, y)$ satisfies the Kolmogorov forward equation

$$\frac{\partial p_t(x, y)}{\partial t} = \frac{\partial^2 (a(y)p_t(x, y))}{\partial y^2} + \frac{\partial (b(y)p_t(x, y))}{\partial y}.$$

In order to compute the stationary distribution, we will consider the solution $p_t(x, y)$ of the Kolmogorov forward equation for which $\frac{\partial p_t(x, y)}{\partial t} = 0$. Next, we introduce a special class of functions that will appear in our analysis.

Definition 6.2.4. For $z \in \mathbb{D}$ and for $\alpha, \beta, \gamma \in \mathbb{C}$ with γ not a nonpositive integer, the hypergeometric function is defined as the following convergent series

$${}_2F_1(\alpha, \beta, \gamma; z) = 1 + \sum_{k=1}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k,$$

where $(\cdot)_k$ is the rising Pochhammer symbol, which is defined by:

$$(\cdot)_k = \begin{cases} 1, & \text{if } k = 0, \\ \frac{\Gamma(\cdot+k)}{\Gamma(\cdot)}, & \text{if } k > 0. \end{cases} \quad (6.2.7)$$

Remark 6.2.5. It can be shown that for complex arguments z with $|z| > 1$ the function can be analytically continued along any path in the complex plane that avoids the branching points 1 and infinity. These functions are extremely well studied (see [28]).

Remark 6.2.6. Let us consider α, β, γ as before. The hypergeometric functions appear as solutions to the hypergeometric equations of the form

$$z(1-z)\phi''(z) + [\gamma - (\alpha + \beta + 1)z]\phi'(z) - \alpha\beta\phi(z) = 0 \quad (6.2.8)$$

for $z \in \mathbb{C} \setminus \{1\}$.

In this section, we will study differential equations that are related with the hypergeometric differential equations. The solutions of these equations can be written in terms of hypergeometric functions.

We are now ready to state the main result.

Theorem 6.2.7. *For $\kappa > 0$, let us consider the random time change*

$$\tilde{u}(S, \omega) = \int_0^S \frac{dt}{y_t^2(\omega)},$$

where y_t is the imaginary part of the chordal backward SLE_κ . We consider the process T_u started with initial value $T_0 = 0$ given by the stochastic differential equation

$$dT_u = -4 \frac{T_u}{1 + \kappa T_u^2} du + d\tilde{B}_u, \quad (6.2.9)$$

where $\tilde{B}_u$ is given by (6.2.5). Then, this process has a stationary distribution on with density

$$\rho(T) = C (\kappa T^2 + 1)^{-4/\kappa},$$

where $T \in \mathbb{R}$ and C is a normalizing constant. Let us consider functions $f \in L^1(\mu)$, where $\mu(dx) = \frac{dx}{(1+x^2)^{4/\kappa}}$.

For $\kappa < 8$, the process T_u converge in the sense of random ergodic averages towards its stationary law, i.e.

$$\left| \frac{1}{\tilde{u}(S, \omega)} \int_0^{\tilde{u}(S, \omega)} f(T_s(\omega)) ds - \mu(f) \right| \xrightarrow{S \rightarrow \infty} 0$$

a.s., with respect to the Wiener measure.

Let (a_n) be a strictly increasing sequence such that $a_n \rightarrow \infty$ as $n \rightarrow \infty$. Moreover, let

$$s_n(\omega) = \inf \left\{ t \in [0, \infty) : \int_0^t \frac{ds}{y_s^2}(\omega) = a_n \right\}.$$

Then for any bounded measurable function $f \in L^\infty$ for $\kappa < 8$, we obtain

$$f(\text{ctg}(\arg(h_{s_n(\omega)}(i)))) = f(T_{a_n})$$

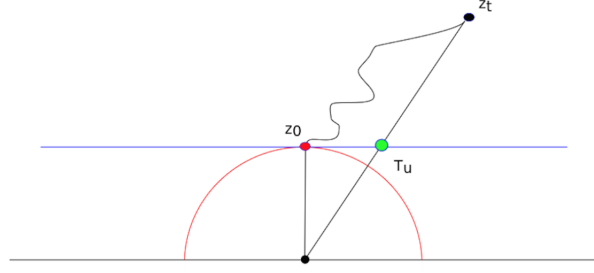


FIGURE 6.2.1. The process T_u is equivalent to a diffusion on the horizontal constant height line $y = 1$ in $\mathbb{H}$.

and as $n \rightarrow \infty$ we have that the law of $f(T_{a_n})$ converges strongly to $\mu(f)$, i.e. $f(\text{ctg}(\arg(h_{s_n(\omega)}(i)))) \rightarrow \mu(f)$.

Corollary 6.2.8. *For $\kappa = 4$, on the sequence of random times $s_n(\omega)$ the law of the backward SLE_κ flow converges to the uniform distribution. Indeed, changing the coordinates from cotangent of the argument to the argument, we obtain that the density of the stationary measure can be written as $d\tilde{\rho}(\theta) = \tilde{C} \sin^{\frac{8}{\kappa}-2}(\theta)d\theta$, for some normalizing constant $\tilde{C}$.*

Remark 6.2.9. From the definition of our process, we have that $D_{c(u)} = T_u$ and $D_S = T_{\tilde{u}(S)}$, where S is a deterministic time. Thus, we consider the format of the random ergodic average as in the main theorem, in order to relate it with the backward Loewner flow up to a fixed time.

Remark 6.2.10. We remark that there is a phase transition at the value $\kappa = 8$ in terms of the integrability of the density of the stationary distribution of the process T_u , i.e. T_u has stationary distribution that is a probability measure for all $\kappa < 8$ and is not a probability measure for all $\kappa \geq 8$.

The proof of the main result is divided in several subsections. First, we prove a useful lemma in the next section.

6.3 Technical lemma

We recall that we start with

$$h_t(i) - \sqrt{\kappa} B_t = x_t(i) + i y_t(i),$$

and we consider the process $D_t = \frac{x_t(i)}{y_t(i)}$, for $t \in [0, +\infty)$.

By Itô's formula, we have that $\frac{x_t(i)}{\sqrt{\kappa} y_t(i)}$, as t varies in $[0, +\infty)$, satisfies the SDE

$$d \frac{x_t(i)}{y_t(i)} = -4 \frac{\frac{x_t(i)}{y_t(i)}}{1 + \kappa \left(\frac{x_t(i)}{y_t(i)} \right)^2} \frac{dt}{y_t^2} + \frac{dB_t}{y_t}. \quad (6.3.1)$$

Thus, we can rephrase (6.3.1) as before, to the following SDE

$$\begin{aligned} dT_u &= -4 \frac{T_u}{1 + \kappa T_u^2} du + d\tilde{B}_u, \\ T_0 &= 0. \end{aligned} \quad (6.3.2)$$

The starting point of the diffusion is $T_0 = 0$ since $\hat{h}_0(i) = i$.

We consider the following useful lemma that gives one of the properties of the random time change $\tilde{u}(S, \omega)$ defined in the previous section.

Lemma 6.3.1. *Let $\kappa > 0$. The random time change $\tilde{u}(S, \omega)$ is bounded from below, for all $S > 0$, uniformly in ω , by*

$$\tilde{u}(S, \omega) \geq \frac{\kappa}{4} \log \left(1 + \frac{4S}{\kappa} \right).$$

Proof. Using the equation for the dynamics of y_t from the Loewner differential

equation, we obtain that $\partial_t[y_t(z)^2] = 2y_t(z)\partial_t y_t(z) = \frac{4}{\kappa} \frac{y_t(z)^2}{|z_t(z)|^2} \leq \frac{4}{\kappa}$. Thus, we obtain that

$$y_t^2 \leq y_0^2 + \frac{4t}{\kappa},$$

i.e.

$$\frac{dt}{y_t^2} \geq \frac{dt}{(y_0^2 + 4t/\kappa)}.$$

By integrating up to time S , we obtain that

$$\int_0^S \frac{dt}{y_t^2} \geq \int_0^S \frac{dt}{(y_0^2 + 4t/\kappa)} = \frac{\kappa}{4} \log \left(1 + \frac{4S}{\kappa y_0^2} \right).$$

□

6.4 The stationary measure of the diffusion process T_u

In this section, we return to studying

$$dT_u = -4 \frac{T_u}{1 + \kappa T_u^2} du + d\tilde{B}_u. \quad (6.4.1)$$

We recall that from Lemma 6.3.1, we obtain that $\tilde{u}(S, \omega) \geq \log(1 + 4S/\kappa)$, for almost all realizations of the Brownian motion. In order to link the behavior of the backward Loewner flow and the behavior of the law of solutions of the SDE (6.4.1), we need to study $\tilde{u}(S, \omega)$ as $S \rightarrow \infty$. Thus, questions about the long term behavior of the SDE (6.4.1) become natural. In this section, we show that there is an explicit stationary law for the SDE (6.4.1).

The density of the stationary measure for the process T_u can be obtained as

a time-independent solution of the Kolmogorov forward equation (see Definition 6.2.3). For the process T_u , we obtain

$$\frac{1}{2}\rho''(T) + \frac{4T}{\kappa T^2 + 1}\rho'(T) + \frac{(4 - 4\kappa T^2)}{(\kappa T^2 + 1)^2}\rho(T) = 0.$$

The solution to the above equation process is

$$\rho(T) = C_1 T (\kappa T^2 + 1)^{-4/\kappa} {}_2F_1\left(\frac{1}{2}, -\frac{4}{\kappa}; \frac{3}{2}; -T^2\kappa\right) + 2C_2 (\kappa T^2 + 1)^{-4/\kappa},$$

where C_1 and C_2 are constants to be chosen.

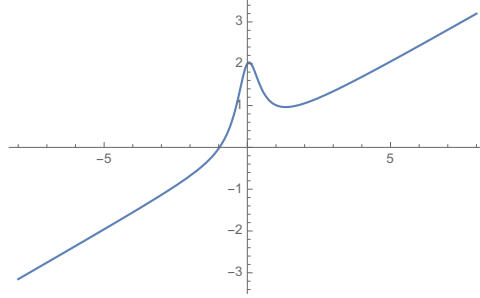


FIGURE 6.4.1. Solution of the Kolmogorov forward equation for T_u for $C_1 = C_2 = 1$ and $\kappa = 1.2$.

We observe that the explicit solution contains a function of the form

$$T (\kappa T^2 + 1)^{-4/\kappa} {}_2F_1\left(\frac{1}{2}, -\frac{4}{\kappa}; \frac{3}{2}; -T^2\kappa\right).$$

Next, we study the behavior of the Hypergeometric function near its singularities.

We use the following formula (see 15.3.7 in [1])

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-z)^a {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{z}\right) \\ &\quad + (-z)^b \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)} {}_2F_1\left(b, b-c+1, -a+b+1; \frac{1}{z}\right) \end{aligned}$$

given that $a-b \notin \mathbb{Z}$ and $z \notin (0, 1)$.

Since $T \in \mathbb{R}$, we restrict the functions to real variables. Thus, we have that

$$\begin{aligned} {}_2F_1(a, b, c; x) &= \frac{\Gamma(b-a)\Gamma(c)}{\Gamma(b)\Gamma(c-a)}(-x)^a {}_2F_1\left(a, a-c+1, a-b+1; \frac{1}{x}\right) \\ &\quad + \frac{\Gamma(a-b)\Gamma(c)}{\Gamma(a)\Gamma(c-b)}(-x)^b {}_2F_1\left(b, b-c+1, -a+b+1; \frac{1}{x}\right). \end{aligned} \quad (6.4.2)$$

For $b-a$ a nonnegative integer, doing the interchange a with b and keeping for simplicity the notation $b = a + n$, we have that (see 15.3.13 in [1])

$$\begin{aligned} {}_2F_1(a, b, c; z) &= \frac{(-z)^{-a}}{\Gamma(a+n)} \sum_{k=0}^{n-1} \frac{(a)_k (n-k-1)!}{k! \Gamma(c-a-k)} z^{-k} \\ &\quad + \frac{(-z)^{-a}}{\Gamma(a)} \sum_{k=0}^{\infty} \frac{(a+n)_k}{k! (k+n)! \Gamma(c-a-k-n)} (-1)^k z^{-k-n} (\ln(-z) + \Psi(k, n, a)), \end{aligned} \quad (6.4.3)$$

where $\Psi(k, n, a) = \psi(k+1) + \psi(k+n+1) - \psi(a+k+n) - \psi(c-a-k-n)$ and $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, for $z \in \mathbb{C} \setminus \{0, -1, -2, \dots\}$. The same holds when we restrict the functions to real variables.

For $a-b \notin \mathbb{Z}$, we apply the formula (6.4.2) for $T \rightarrow \infty$. Let us consider $m = \frac{4}{\kappa}$. Using the series representation of the hypergeometric function, we obtain that for a function $f(T) = {}_2F_1(\frac{1}{2}, -m, \frac{3}{2}; -T^2\kappa)$ as $T \rightarrow 0$, we have that

$$f(T) = \sum_{k=0}^{\infty} \frac{(-m)_k (-1)^k T^{2k}}{k! (2k+1)}.$$

In particular, the leading term is a constant. We use this and the formula (6.4.2) to study the asymptotic behavior of $f(T)$ as $T \rightarrow \infty$.

Since in general for p, q in ${}_2F_1(p, q, r; x)$ are critical exponents at infinity, the asymptotic behavior as $T \rightarrow \infty$ of the function

$$f(T) = {}_2F_1\left(\frac{1}{2}, -m, \frac{3}{2}; -T^2\kappa\right)$$

depends if $\operatorname{Re} m > -\frac{1}{2}$ or $\operatorname{Re} m < -\frac{1}{2}$. In our case, $\operatorname{Re} m = m = \frac{4}{\kappa} > -\frac{1}{2}$. Thus, as $T \rightarrow \infty$, we obtain that

$$f(T) = \frac{1}{2m+1}.$$

Finally, for $\kappa > 0$, we have

$$\lim_{T \rightarrow \infty} T (\kappa T^2 + 1)^{-4/\kappa} {}_2F_1\left(\frac{1}{2}, -\frac{4}{\kappa}, \frac{3}{2}; -T^2\kappa\right) \sim \frac{T}{1+8\kappa},$$

and

$$\lim_{T \rightarrow -\infty} T (\kappa T^2 + 1)^{-4/\kappa} {}_2F_1\left(\frac{1}{2}, -\frac{4}{\kappa}, \frac{3}{2}; -T^2\kappa\right) \sim \frac{T}{1+8\kappa}.$$

Since we are searching for the solution that is a density of a probability distribution, the first constant needs to be zero, since otherwise the solution will also contain negative values.

For $a - b \in \mathbb{Z}$, we apply the formula (6.4.3) and as $T \rightarrow -\infty$, we have that

$$\lim_{T \rightarrow -\infty} T (\kappa T^2 + 1)^{-4/\kappa} {}_2F_1\left(\frac{1}{2}, -\frac{4}{\kappa}, \frac{3}{2}; -T^2\kappa\right) \sim T \frac{\log(\kappa T^2) + C}{1+8\kappa},$$

where C is a constant obtained from the formula (6.4.3). Thus, the constant C_1 should be zero, since otherwise, as before, we obtain a negative solution that is not

a probability density.

Finally, we obtain an integrable probability density on $\mathbb{R}$ for $\kappa \in [0, 8)$ that has the form

$$\rho(T) = C (\kappa T^2 + 1)^{-4/\kappa} .$$

where $C = \left(\int_{-\infty}^{+\infty} (\kappa T^2 + 1)^{-4/\kappa} dT \right)^{-1}$.

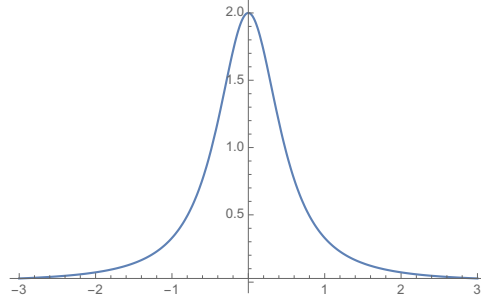


FIGURE 6.4.2. The stationary measure for the process T_u for $\kappa = 3$.

Remark 6.4.1. Changing the coordinates from cotangent of the argument to the argument, we obtain that the density of the stationary measure can be written as $d\tilde{\rho}(\theta) = \tilde{C} \sin^{\frac{8}{\kappa}-2}(\theta) d\theta$. This function changes its behaviour at $\kappa = 4$, being concave for $\kappa < 4$, while being convex for $\kappa > 4$. At $\kappa = 4$ the function is constant. Thus, for $\kappa = 4$, we obtain a uniform distribution on $[0, \pi]$.

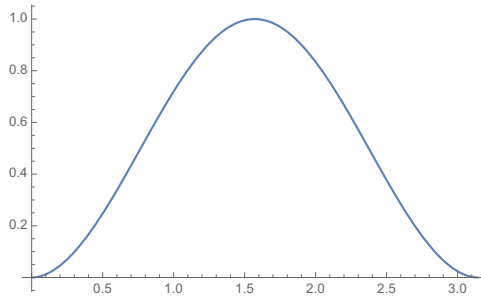


FIGURE 6.4.3. Density of the stationary measure for the argument via change of variable from the stationary measure of T_u for $\kappa = 2.3$.

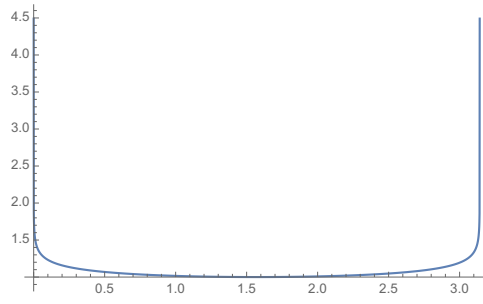


FIGURE 6.4.4. Density of the stationary measure for the argument via change of variable from the stationary measure of T_u for $\kappa = 4.5$.

6.5 Proof of the main result

In this section, we give the proof of our main result. The analysis consists of two parts. First, we use a result from [40] that gives convergence to the stationary distribution of the laws of a one-dimensional diffusion as time goes to infinity, in the sense of ergodic averages, for almost every Brownian path. We stress that this result gives the convergence towards the stationary law for large deterministic times.

As in Chapter 4, we use that for a diffusion X_t with drift μ and variance σ^2 , the scale function can be computed via $s(x) = \int_{x_0}^x \exp(-\int_{\eta}^y \frac{2\mu(z)}{\sigma^2(z)} dz) dy$, where x_0, η are points fixed (arbitrarily) in (a, b) .

Definition 6.5.1. *The function $m(\zeta) = \frac{1}{\sigma^2(\zeta)s'(\zeta)}$ is the density of the speed measure.*

For the process T_u , we obtain the following:

$$s(x) = e^{-2 \int_0^x \frac{-4/\kappa u}{1+u^2} du} = e^{8/\kappa \int_0^x \frac{u}{1+u^2} du} = e^{4/\kappa \int_1^{1+x^2} \frac{1}{l} dl} = e^{4/\kappa \log(1+x^2)} = (1+x^2)^{4/\kappa}.$$

Using the definition of the speed measure we obtain that $m(x) = \frac{2}{(1+x^2)^{4/\kappa}}$. The first result that we use is the following theorem, that gives the a.s. convergence of the

ergodic averages up to a deterministic time. Using the previous computations, we obtain that the process T_u satisfies the conditions of Theorem 6.5.2 for $\kappa < 8$.

Theorem 6.5.2 (Theorem 53.1, [40]). *Let T_s be a recurrent diffusion on the real line in natural scale, such that $\int_{\mathbb{R}} \mu(dx) < \infty$, where $\mu(dx)$ is the stationary measure of the diffusion, that is up to a constant equal to the speed measure. Let $f \in L^1(\mu)$ and $f \geq 0$, Then*

$$\frac{1}{t} \int_0^t f(T_s) ds \xrightarrow{\text{a.s.}} \mu(f),$$

as $t \rightarrow \infty$.

Proof of the first part of the main result. Let $Z_u(f) := \frac{1}{u} \int_0^u f(T_s) ds$. By Theorem 6.5.2, we have that for a.e. ω and $\forall \varepsilon > 0$, $\exists R(\omega) \in \mathbb{R}_+$ such that

$$\sup_{u > R(\omega)} |Z_u(\omega) - \mu(f)| \leq \varepsilon.$$

Since the convergence in Theorem 6.5.2 holds a.s., we have the following: For almost every sample ω of the Brownian motion, the limit $\lim_{u \rightarrow \infty} Z_u \rightarrow \mu(f)$. Since this is an a.s. result about the limit it holds on every sequence of times that goes to infinity. Picking the times $\tilde{u}(S, \omega)$, we have that from Lemma 6.3.1, that a.s. $\tilde{u}(S, \omega) \rightarrow \infty$ as $S \rightarrow \infty$. Then a.s., $Z_{\tilde{u}(S, \omega)} \rightarrow \mu(f)$, as $S \rightarrow \infty$.

□

6.6 The phase transition at $\kappa = 8$ and the law of the backward Loewner flow at asymptotic large times

In this section we show how stronger notions of convergence (convergence in total variation norm) for Stochastic Differential Equations towards their stationary measure give information about the law of the backward SLE trace at certain times. For $\kappa = 4$, these times are exactly the times when the law of $\text{Arg}(h_t(i) - \sqrt{\kappa}B_t)$ converges to the uniform distribution. This analysis is a Skorokhod type embedding problem for the backward SLE_κ flow. In the setting of the Skorokhod Embedding Problem, given a stochastic process and a fixed distribution, a stopping time should be constructed in order to obtain the fixed distribution when evaluating the process at those stopping times. In this section, we discuss a similar problem in the context of Loewner differential equation. We show how to construct random times on which the law of the argument of points under this flow converge to the uniform law for $\kappa = 4$. For more details on Skorokhod problems for other classical processes, see [38]. Moreover, we show that this belongs to a general family of laws that are obtained in the same manner for the SLE_κ trace for other values of $\kappa < 8$.

Let us consider the dynamics of the imaginary value of the backward Loewner differential equation. The random time change that we study is a natural process that is written in terms of the imaginary value of the backward Loewner flow. Moreover, it is a stochastic process that is a.s. increasing. Thus, it is natural to consider the hitting times of constant levels of this process.

For the diffusion process T_u defined by the SDE (6.4.1) there are stronger convergence results towards the stationary measure (at sequences of large deterministic times) than the convergence in the sense of ergodic averages.

We use the main result from [18], that we state below. Let us consider the following solution of a one-dimensional SDE:

$$X_t = x + B_t - \frac{1}{2} \int_0^t b(X_s) ds,$$

where $b : \mathbb{R} \rightarrow \mathbb{R}$ is a continuous function. When imposing conditions on the behavior of the function $b(x)$, we obtain various convergence results for the law of the stochastic process X_t towards its stationary measure in stronger senses than the ergodic convergence.

It is proved in [18] that for one-dimensional diffusion processes, if $|b(x)| \sim \frac{C}{x}$ with $C > 1$, as $|x| \rightarrow \infty$, then we have that there exists $\gamma > 0$ and $t_0 > 0$ such that for any compact set K , there exists $C(K) > 0$ such that for any $f \in L^\infty$ and $t \geq t_0$, we have

$$\sup_{x \in K} |P_t f(x) - \mu(f)| \leq C(K) \frac{1}{t^\gamma} \|f\|_\infty, \quad (6.6.1)$$

where P_t is the transition semigroup of the one-dimensional diffusion X_t .

Proof of the second part of Theorem 6.2.7. For the diffusion process T_u , we meet the required conditions on the drift for $\kappa < 8$ since the drift $\frac{-4/\kappa T}{1+T^2} = -\frac{1}{2} \frac{8/\kappa T}{1+T^2}$ and $\frac{8/\kappa T}{1+T^2} \sim \frac{8}{\kappa T}$ with $\frac{8}{\kappa} > 1$, for $\kappa < 8$.

For $n \in \mathbb{N}$ let us define

$$a_n := \log \left(1 + \frac{4n}{\kappa} \right),$$

and let us consider

$$s_n(\omega) = \inf \left\{ t \in [0, \infty) : \int_0^t \frac{ds}{y_s^2}(\omega) = a_n \right\}.$$

Thus, in our case, we take $t = a_n$, and from the Lemma 6.3.1, we have that

$$\sup_{x \in K} |P_{a_n}(f(T_u)) - \mu(f)| \leq C(K) \frac{1}{a_n^\gamma} \|f\|_\infty \leq C(K) \frac{1}{(\log(1 + 4n/\kappa))^\gamma} \|f\|_\infty.$$

Thus, $\sup_{x \in K} |P_{a_n}(f(T_u)) - \mu(f)| \rightarrow 0$ as $n \rightarrow \infty$.

Finally, in terms of the backward Loewner differential equation, since by definition $T_{a_n} = \frac{X_{s_n(\omega)}(i)}{Y_{s_n(\omega)}(i)}$, we obtain that as $n \rightarrow \infty$, on the random sequence $s_n(\omega)$, for $f \in L^\infty(\mathbb{R})$, the law of the limiting random variable $\lim_{n \rightarrow \infty} f\left(\frac{X_{s_n(\omega)}(i)}{Y_{s_n(\omega)}(i)}\right)$ converges to the stationary law of the process T_u that is explicitly computed. In particular, on this sequence of times, we obtained the convergence in the total variation sense towards the uniform distribution for $\kappa = 4$. Thus, this method provides us with a solution to a Skorokhod type embedding problem for the backward Loewner differential equation, in the sense that it identifies a sequence of random times along the backward SLE_κ flow started from $y_0 = 1$ on which the argument of points is uniformly distributed for $\kappa = 4$. In fact this is a closed-form expression family of distributions that are recovered using the stationary law of the diffusion process T_u for $\kappa < 8$. $\square$

Scaling and conjecture. Our main conjecture is the following:

Conjecture 6.6.1. *For $\kappa < 8$, the density of the law of the cotangent of the argument of the tip of the SLE_κ trace at any fixed deterministic time is*

$$\mu(dx) = \frac{Cdx}{(1+x^2)^{4/\kappa}},$$

where C is a normalizing constant.

The conjecture is supported also by the phase transition at $\kappa = 8$ that we

described in the previous sections. We use this section to describe the main strategy to prove this conjecture in a future work.

We consider the scaling of the SLE_κ maps

$$\frac{h_t(i)}{\sqrt{t}} = h_1(i/\sqrt{t}) \quad (6.6.2)$$

in distribution, as processes for $t \in [0, +\infty)$.

We can recover the same identity in distribution as processes for the shifted mappings also:

$$\frac{h_t(i) - \sqrt{\kappa}B_t}{\sqrt{t}} = h_1(i/\sqrt{t}) - \sqrt{\kappa}B_1.$$

Since the SLE_κ trace is defined a.s. with respect to the initial Wiener measure, its tip is a well defined random variable. Then, the limit $\lim_{t \rightarrow \infty} ctg(arg(\hat{h}_1(i/\sqrt{t})))$ exists for almost every Brownian motion and is therefore a well-defined random variable.

Using that $\text{Im } \hat{h}_1(0) > 0$ a.s. (since the event that $\gamma_1 \in \mathbb{R}$ has zero probability), along with the continuity of the cotangent and argument in $\mathbb{C}$ at z for $|z| \neq 0$, we obtain that

$$\lim_{t \rightarrow \infty} ctg(arg(\hat{h}_1(i/\sqrt{t}))) = ctg(arg(\lim_{t \rightarrow \infty} \hat{h}_1(i/\sqrt{t}))) = ctg(arg(\hat{h}_1(0))). \quad (6.6.3)$$

In order to study the distribution at a fixed capacity time t of $ctg(arg(\hat{h}_t(0)))$, we use the scaling (6.6.2) along with the fact that the law of $ctg(arg(\frac{\hat{h}_t(i)}{\sqrt{t}}))$ is the same as the law of the cotangent of $ctg(arg(\hat{h}_t(i)))$. Thus, we can identify the distribution of $\lim_{t \rightarrow \infty} ctg(arg(\hat{h}_t(i)))$ with the distribution of $\lim_{t \rightarrow \infty} ctg(arg(\hat{h}_t(i/\sqrt{t})))$.

In the previous parts of the chapter we studied $\frac{x_t(i)}{y_t(i)}$. We would like to use this pre-

vious work in order to prove the conjecture stated earlier. The $\lim_{t \rightarrow \infty} \text{ctg}(\arg(\hat{h}_t(i)))$ does not exist a.s., however as a random variable we would like to show that its distribution converges to the stationary distribution of T_u started from $T_0 = 0$.

A possible strategy: analysis on the events $E_{t,l}$ described in [54]. Let us consider the following quantities $r_c = \frac{1+4a}{2}$, $\lambda = r \left(1 + \frac{1}{2a}\right) - \frac{r^2}{4a}$, $\zeta = \lambda - \frac{r}{2a}$, where $a = \frac{2}{\kappa}$. Let us consider the following SDE that appears in [54].

$$dJ_t = -r_c \tanh J_t dt + dW_t, J_0 = 0,$$

where W_t is a standard Brownian motion.

Let

$$L_t = t - \int_0^t \frac{2ds}{\cosh^2 J_s}.$$

Using Itô's and Girsanov Theorems, in [54] it is proved that $N_t = e^{a\lambda L_t} e^{a\zeta t} [\cosh J_t]^r$ is a martingale satisfying $dN_t = r[\tanh J_t] N_t dW_t$.

Let $\mathbb{P}_*$ denote the probability measure obtained from $\mathbb{P}$ by weighting with the martingale N_t .

Let us consider the events $E_{t,l}$ as in [54]. These events are described in Lemma 5.10 of [54].

Lemma 6.6.2 (Lemma 5.10 of [54]). *For each $l, t \geq 0$, let $E_{t,l}$ be the events that following estimates hold for all $0 \leq s \leq t$.*

$$|J_s| \leq l \log(\min\{s+2, t-s+2\}),$$

$$|L_s - s\beta| \leq l\sqrt{s} \log(s+2),$$

$$|L_t - L_s - (t - s)\beta| \leq l \leq \sqrt{t - s} \log(t - s + 2),$$

Then,

$$\lim_{l \rightarrow \infty} \inf_{t > 0} \mathbb{P}_*(E_{t,l}) = 1.$$

One of the feature of the events is that they have probability one in the limit when the parameter $l \rightarrow \infty$, in the weighted measure. The whole analysis concerning these events is done in Section 5 of [54]. We extract from this analysis the fact that on these events we have the following estimates

$$e^{as - c \log F(s,t)} \leq y_{e^{2as}} \leq ce^{as}, \quad (6.6.4)$$

where $F(s, t) = 2 + \min\{s, t - s\}$.

Specifically, on these events under the measure $\mathbb{P}_*$ we can control $\log(\tilde{u}(S, \omega))$ via the control on $y_{e^{2as}}(\omega)$ in (6.6.4). However, the estimates are not good enough to control $\tilde{u}(S, \omega)$. The upper bound in the estimate is true for any choice of driver and is not specific to the Brownian driver. The main body of future work will be to obtain a sharp upper bound in (6.6.4). If a better control in (6.6.4) such that $\tilde{u}(S, \omega)$ behaves asymptotically deterministic, then the stationary law of the process T_u can be identified with the law of the tip of the SLE_κ trace at a fixed time. The result will follow from the total variation convergence result for the diffusion T_u . If this conjecture turns out to be true, then we will create a new observable that can be used to study further convergence of scaling limits of lattice models from Statistical Physics to SLE_κ curves.

Chapter 7

Loewner differential equation and Rough Path Theory

7.1 Introduction

In this chapter we study the backward Loewner differential equation in $\mathbb{H}$ started from $z_0 \in \mathbb{H}$ with $|z_0| = \delta > 0$. We show that this differential equation is a Rough Differential Equation and we establish the continuity of its solutions in the p -variation topology. In addition, in a different direction, we study some features of the truncated Taylor approximation series associated with the forward Loewner differential equation obtained from extending the conformal maps to the real line. The Stochastic Taylor approximation is a popular approximation technique for solutions of stochastic differential equations. We refer the reader to [23] as well as [17].

For $T > 0$, let us define $\Delta_T = \{(s, t) | 0 \leq s \leq t \leq T\}$. We collect all the iterated integrals in the following way. We consider for

$$X : \Delta_T \rightarrow T((\mathbb{R}))$$

the collection of iterated integrals appearing in our approximation as

$$(s, t) \rightarrow \mathbf{X}_{s,t} = (1, X_{s,t}^1, \dots, X_{s,t}^{[p]}, \dots, X_{s,t}^m, \dots) \in T((V)).$$

We will show in the following section that the pair $dX_t = (dt, dB_t)$ can be lifted to a p -rough path for $p > 2$. We use the Itô lift as well as the geometric p -rough path lift in order to define the iterated integrals.

7.2 The pair $X_t = (t, B_t)$ as a p -rough path for $p > 2$

In this section we show that the pair $X_t = (t, B_t)$ can be lifted to a p -rough path for $p > 2$ and moreover the lift is uniquely determined by the first two levels.

The main ingredient that we use is the Extension Theorem, Theorem 3.7 of [35], that we state in the following.

Theorem 7.2.1 (Extension Theorem, Theorem 3.7 of [35]). *Let $p \geq 1$ be a real number and $n \geq 1$ be an integer. Let $X : \Delta_T \rightarrow T^{(n)}(V)$ be a multiplicative functional with finite p -variation controlled by w . Assume that $n \geq [p]$. Then there exists a unique extension of X to a multiplicative functional $\Delta_T \rightarrow T((V))$ that has finite p -variation, i.e. for every $m \geq [p] + 1$, there exists a unique continuous function*

$$X^m : \Delta_T \rightarrow V^{\otimes m}$$

such that $(s, t) \rightarrow \mathbf{X}_{s,t} = (1, X_{s,t}^1, \dots, X_{s,t}^{[p]}, \dots, X_{s,t}^m, \dots) \in T((V))$, is a multiplicative functional with finite p -variation controlled by the same w , i.e.

$$\|X_{s,t}^i\| \leq \frac{w(s, t)^{i/p}}{\beta(\frac{i}{p})!} \quad \forall i \geq 1, \quad \forall (s, t) \in \Delta_T,$$

$$\text{with } \beta_p = p^2 \left(1 + \sum_{r=3}^{\infty} \left(\frac{2}{r-2} \right)^{\frac{|p|+1}{p}} \right).$$

The metric $\|\cdot\|$ on $T((V))$ is the projective norm defined for

$$x = \sum_{|I|=k} \lambda_I e_I \in V^{\otimes k}$$

via

$$\|x\| = \sum_{|I|=k} |\lambda_I|.$$

Thus, the bound $\|X_{s,t}^i\| \leq \frac{w(s,t)^{i/p}}{\beta(\frac{i}{p})!}$, $\forall i \geq 1$, $\forall (s,t) \in \Delta_T$, gives control on the sum of i -iterated integrals. In the last part, for the real numbers $\varepsilon \geq 0$ and $\alpha > 0$ we will consider pairs $s, t \in [0, \varepsilon^\alpha]$ and we will estimate the corresponding iterated integrals.

In order to check that the pair $X_t = (t, B_t)$ can be lifted to a p -rough path for any $p > 2$, we have to check according to the definition, the algebraic identities (Chen's relations) and the analytic bounds. The algebraic identities follow from Fubini's Theorem which Riemann-Stieltjes and Itô integrals satisfy. Since we consider iterated integrals of a pair consisting of a bounded variation path with some other p -variation path for any $p > 2$ the algebraic requirements are straightforward. We check the analytic bounds in the following. For the pair $X_t = (t, B_t)$, we have that the second component has clearly finite p -variation for $p > 2$ since it is of bounded variation. Using standard result in Rough Path Theory (see [17]) we have that B_t has finite p -variation for $p = 2 + q$ for any $q > 0$.

We show that there exists a control $\omega(s, t)$ such that

$$\left| \int \int_{s < u_1 < u_2 < t} dB_{u_1} du_2 \right| \leq C_1 \omega(s, t)^{2/p}$$

$$\left| \int \int_{s < u_1 < u_2 < t} du_1 dB_{u_2} \right| \leq C_2 \omega(s, t)^{2/p}$$

and

$$\left| \int \int_{s < u_1 < u_2 < t} dB_{u_1} dB_{u_2} \right| \leq C_3 \omega(s, t)^{2/p}$$

where C_1, C_2 and C_3 are a.s. finite random constants.

Since the Brownian path is weakly $1/2$ -Hölder continuous, we will take as a control the function $w(s, t) = (t - s)$. In particular, we know from standard properties of Brownian motion that it is of finite p -variation for $p = 2 + q$ for any $q > 0$. For any pair $s, t \in \Delta_T$, the first iterated integral has finite $p/2$ -variation since it is a Riemann-Stieltjes integral due to the fact that u_2 is of bounded variation. Moreover, the integral is controlled by $w(s, t) = (t - s)^{2/p}$, since we have $(t - s) = (t - s)^{2/p}(t - s)^{1-2/p}$. For the second iterated integral, for any pair $s, t \in \Delta_T$ we use the integration by parts for the Itô integrals. Since one of the terms is of finite variation, the formula gives

$$\int_s^t (u_2 - s) dB_{u_2} = (t - s)B_t - \int_s^t (B_{u_2} - B_s) du_2. \quad (7.2.1)$$

We further have that

$$\left| \int_s^t (u_2 - s) dB_{u_2} \right| \leq |(t - s)B_t| + \left| \int_s^t (B_{u_2} - B_s) du_2 \right| \quad (7.2.2)$$

The control for the second term in the right hand side of (7.2.2) is obtained as in the previous case, since it is an example of a Riemann-Stieltjes integral bounded by $\omega(s, t) = (t - s)^{2/p}$. The first term in the right hand side of (7.2.2) is bounded directly by $\omega(s, t) = (t - s)^{2/p}$ using that $(t - s)$ is a bounded variation function that in particular has finite p -variation for any $p > 2$, since $(t - s)B_t \leq (t - s)^{2/p}(t - s)^{1-2/p}B_t$.

Using that B_t is an a.s. finite random variable, we obtain the required estimate.

For the last iterated integral, we obtain via direct computations

$$\begin{aligned}
\left| \int_s^t (B_r - B_s) dB_r \right| &= \left| \int_s^t B_r dB_r - B_s(B_t - B_s) \right| \\
&= \left| \frac{B_t^2 - B_s^2}{2} - \frac{t-s}{2} - B_s B_t + B_s^2 \right| \\
&= \left| \frac{B_t^2}{2} + \frac{B_s^2}{2} - \frac{2B_s B_t}{2} - \frac{t-s}{2} \right| \\
&= \left| \frac{(B_t - B_s)^2}{2} - \frac{t-s}{2} \right| \\
&< C_3(t-s)^{2/p}.
\end{aligned} \tag{7.2.3}$$

The last inequality holds since $|B_t - B_s| \leq C\omega(s, t)^{1/p}$ and thus

$$|B_t - B_s|^2 \leq C^2 \omega(s, t)^{2/p},$$

where C is a random constant that is a.s. finite. Finally, $(t-s) \leq (t-s)^{1-\frac{2}{p}}(t-s)^{2/p}$.

Taking

$$c_m(\omega) := \max(C_1(\omega)^{1/2}, C_2(\omega)^{1/2}, C_3(\omega)^{1/2}, C_0(\omega), C_1(\omega)),$$

where $C_0(\omega)$ and $C_1(\omega)$ are the a.s. finite constants obtained by bounding the increments of the path (t, B_t) at the first level, we define the control

$$w_1(s, t) = c_m(\omega)(t-s).$$

Note that the choice of the constants β_p and $(\frac{i}{p})!$ does not change the analysis. Their choice is usually related with the fact that they simplify computations when

Extension Theorem is applied.

With the previous estimates and the choice of control $w_1(s, t) = c_m(\omega)(t - s)$, we check the conditions of Theorem 7.2.1. In this manner, we obtain the unique Itô lift up to any order $n \geq 2$ for the path $X_t = (t, B_t)$.

Note that using the Itô lift we do not obtain a geometric p -rough path. We use this lift and the bounds on iterated integrals from the Extension Theorem when we study the truncated Taylor approximation to the forward Loewner differential equation extended to the real line, in the next section, where we apply the bounds from the Extension Theorem for the time interval $[0, \varepsilon^\alpha]$ for the Itô iterated integrals of $X_t = (t, B_t)$.

Remark 7.2.2 (Geometric Rough Path lift of $X_t = (t, B_t)$). When discussing continuity properties of the solutions to the backward Loewner differential equation with respect to the parameter κ , we need that the pair $X_t = (t, B_t)$ to be a geometric rough path. Thus, we need to consider a different lift from the Itô one. Since t is of bounded variation, the pair X_t is a Young pairing. Since B_t is one-dimensional Brownian motion, then there is a canonical lift to a geometric rough path (for higher dimensions, it is shown in [49] that the Stratonovich lift of the Brownian motion is a geometric rough path). We use this lift to see the pair $X_t = (t, B_t)$ as a geometric p -rough path for $p > 2$. For further details, see Section 9.4 in [17].

7.3 Universal Limit Theorem and the continuity of the solution to the backward Loewner differential equation

In this section, we work with the backward Loewner differential equation.

$$\partial_t h(t, z) = \frac{-2}{h(t, z) - \sqrt{\kappa} B_t}, \quad h(0, z) = z, z \in \mathbb{H} \quad (7.3.1)$$

By performing the identification $h_t(z) - \sqrt{\kappa} B_t = Z_t$, we obtain the following dynamics in $\mathbb{H}$, that we consider throughout this section

$$dZ_t = \frac{-2}{Z_t} dt - \sqrt{\kappa} dB_t, \quad Z_0 = z_0 \in \mathbb{H}.$$

We consider the backward Loewner differential equation started from $Z_0 \in \mathbb{H}$ with $|Z_0| = \delta$ for some $\delta > 0$.

We have the following remark.

Remark 7.3.1. In the case of the chordal SLE_κ the Itô lift or the Stratonovich lift of the iterated integrals produce the same solution. Indeed, when considering the Itô-Stratonovich correction for a time-homogenous diffusion

$$dZ_t = \mu(Z_t)dt + \sigma(Z_t)dB_t$$

we have

$$\int_0^T \sigma(Z_t) \circ dB_t = \frac{1}{2} \int_0^T \frac{d\sigma(Z_t)}{dx} \sigma(Z_t) dt + \int_0^T \sigma(Z_t) dB_t$$

For the backward/forward chordal SLE_κ we have that $dZ_t = \frac{\pm 2}{Z_t} dt - \sqrt{\kappa} dB_t$. The

important remark is that $\sigma(Z_t)$ is a constant, thus when studying these equations we have that

$$\int_0^T \sigma(Z_t) \circ dB_t = \int_0^T \sigma(Z_t) dB_t.$$

In this section we work with the geometric rough path lift of $X_t = (t, B_t)$ and in the next section we work with the Itô lift of the same path, in order to exemplify results about both of the choices.

We further define the notion of solution to a Rough Differential Equation.

Definition 7.3.2. *Let $f : W \rightarrow L(V, W)$ be a $Lip(\gamma - 1)$ function and let us consider*

$$X \in G\Omega_p(V) \text{ and } \zeta \in W.$$

Set $f_\zeta(\cdot) = f(\cdot + \zeta)$. Define $h : V \oplus W \rightarrow \text{End}(V \oplus W)$ via

$$\begin{bmatrix} Id_V & 0 \\ f_\zeta(y) & 0 \end{bmatrix}.$$

We call $Z \in G\Omega_p(V \oplus V)$ a solution to the differential equation $dY_t = f(Y_t)dX_t$, $Y_0 = \zeta > 0$ if the following conditions hold

- $Z = \int h(Z)dZ,$
- $\Pi_V(Z) = X$, where $\Pi_V(\cdot)$ is the projection map.

The main result that we use in this section is the Universal Limit Theorem.

Theorem 7.3.3 (Universal Limit Theorem, Theorem 5.3 in [35]). *Let $p \geq 1$ and let $\gamma > p$ be real numbers. Let $f : W \rightarrow L(V, W)$ be a $\text{Lip}(\gamma)$ function. For all $X \in G\Omega_p(V)$ and all $\zeta \in W$, the equation*

$$dY_t = f(Y_t)dX_t, \quad Y_0 = \zeta$$

admits a unique solution $Z = (X, Y) \in G\Omega_p(V \oplus W)$ in the sense of the Definition 7.3.2. The solution depends continuously on X and ζ and the mapping

$$I_f : G\Omega_p(V) \times W \rightarrow G\Omega_P(W)$$

which sends (X, ζ) to Y is the unique extension of the Itô map which is continuous in the p -variation topology.

First, we recall the following definition from the introduction.

Definition 7.3.4. *A sequence $(X(n))_{n \geq 1} \in \Lambda_p(V)$ is said to converge to $X \in \Lambda_p(V)$ in p -variation topology if there exists a p -control w of X and $X(n)$ for all $n \geq 1$, and a sequence $(a(n))_{n \geq 1}$ of positive reals such that $\lim_{n \rightarrow \infty} a(n) = 0$ and*

$$\|X(n)_{s,t}^i - X_{s,t}^i\| \leq a(n)w(s, t)^{\frac{i}{p}},$$

for all $(s, t) \in \Delta_T$ and $1 \leq i \leq [p]$.

In this section we prove the following theorem.

Theorem 7.3.5. *For $\delta > 0$, the backward Loewner differential equation driven by $\sqrt{\kappa}B_t$ with $\kappa \in \mathbb{R}_+$, $\kappa \neq 0$, started from z_0 with $|z_0| = \delta > 0$ is a well defined Rough Differential Equation that has a unique solution. This unique solution is a.s.*

continuous with respect to the starting point $z_0 \in \mathbb{H}$ and $\sqrt{\kappa}B_t$ in the p -variation topology for $p \in (2, 3]$.

7.4 Proof of Theorem 7.3.5

First, we recall the following result.

Theorem 7.4.1 (Continuity of the Extension Map, Theorem 3.10 of [35]). *Let us consider $X, Y \in \Lambda_p(V)$ with p -variation control w and let $\varepsilon \in (0, 1)$. Then the bound*

$$\|X_{s,t}^i - Y_{s,t}^i\| \leq \varepsilon \frac{w(s, t)^{\frac{i}{p}}}{\beta_p(\frac{i}{p})!},$$

holds for all $(s, t) \in \Delta_t$ and $i \geq 1$, provided that it holds for all $1 \leq i \leq \lfloor p \rfloor$.

Lemma 7.4.2. *For $\kappa > 0$, let us consider $\kappa_n \rightarrow \kappa$, as $n \rightarrow \infty$. Then,*

$$(t, \sqrt{\kappa_n}B_t) \rightarrow (t, \sqrt{\kappa}B_t)$$

in the p -variation topology as $\kappa_n \rightarrow \kappa$, for $p \in (2, 3]$.

Proof of Lemma 7.4.2. Let us consider the sequence $\sqrt{\kappa_n} \rightarrow \sqrt{\kappa}$ as $n \rightarrow \infty$, for $\kappa > 0$. Since in the first component there are no changes, we focus directly on the second component of the paths $(t, \sqrt{\kappa_n}B_t)$ and $(t, \sqrt{\kappa}B_t)$. Without loss of generality we can choose an increasing sequence. Using the control $\omega(s, t) = \sqrt{\kappa}(t - s)$ and the sequence $a_\kappa(n) = \frac{\sqrt{\kappa} - \sqrt{\kappa_n}}{\sqrt{\kappa}}$ in the Definition 7.3.4 we obtain the convergence of in the p -variation topology for $p \in (2, 3]$ of the paths $(t, \sqrt{\kappa_n}B_t)$ and $(t, \sqrt{\kappa}B_t)$. Indeed, we have that $\omega(s, t) = \sqrt{\kappa}(t - s)$ is a control for both $\sqrt{\kappa}B_{s,t}$ as well as $\sqrt{\kappa_n}B_{s,t}$ and $|\sqrt{\kappa}B_{s,t} - \sqrt{\kappa_n}B_{s,t}| \leq |\sqrt{\kappa}B_{s,t}| |1 - \frac{\sqrt{\kappa_n}}{\sqrt{\kappa}}|$.

From Theorem 7.4.1, we see that the bound in the Definition 7.3.4 implies that there exists a sequence $\varepsilon(n)$ with $\lim_{n \rightarrow \infty} \varepsilon(n) \rightarrow 0$ such that the bound in the Definition 7.3.4, holds for all pairs $s, t \in \Delta_T$ and $i \geq 1$, when n is sufficiently large. $\square$

Proof of Theorem 7.3.5. We consider the geometric p -rough path for lift for $X_t = (t, B_t)$. In order to prove that the backward Loewner differential equation started from $z_0 \in \mathbb{H}$ with $|Z_0| = \delta$ is a Rough Differential Equation with a unique solution, we show that the vector fields $V_1(Z)$ and $V_2(Z)$ are indeed $Lip(\gamma)$ vector fields for $\gamma > 2$. The problematic vector field is $V_1(z)$ since the second one is clearly $Lip(\gamma)$ for $\gamma \geq 4$.

In order to show that indeed the first vector field is $Lip(\gamma)$ for $\gamma > 3$ we use that

$$\frac{d}{dz} \frac{1}{z} = \frac{-1}{z^2},$$

$$\frac{d}{dz} \frac{-1}{z^2} = \frac{2}{z^3},$$

$$\frac{d}{dz} \frac{2}{z^3} = \frac{-6}{z^4}$$

and

$$\frac{d}{dz} \frac{-6}{z^4} = \frac{24}{z^5}.$$

Thus, the $Lip(4)$ norm of the vector field $1/z$ is bounded for $|z| > \delta > 0$. Note that, in general, $\frac{d^n}{dz^n} \frac{1}{z} = \frac{c(n)}{z^{n+1}}$, where the function $c(n) = (-1)^n n!$. Thus, the $Lip(\gamma)$ norm of the vector fields is bounded for any finite γ . In our analysis we restrict to $p \in (2, 3]$, thus checking $\gamma > 3$ is enough. However, we can show the result for any $p > 2$ and any $\gamma > p$ using the bound on the vector fields.

In addition, we observe that the imaginary part of the backward Loewner differential equation Y_t , we have

$$dY_t = \frac{2Y_t}{X_t^2 + Y_t^2} dt.$$

In particular, $Y_t \geq Y_0$ for all $t \geq 0$. Thus, we have that $Y_t > 0$ for all $t \in [0, T]$. Since the imaginary part Y_t increases the vector field remains bounded for all times $t \in [0, T]$. Using the Stratonovich lift of the pair (t, B_t) and the bounds on $Lip(3)$ norms of the vector fields, we obtain that indeed the backward Loewner differential equation started from z_0 with $|z_0| = \delta > 0$ is a Rough Differential equation. In particular, the fact that the vector fields of the backward Loewner differential equation are $Lip(\gamma)$ for $\gamma \geq 3$ assures via Theorem 7.3.3 that the solution exists and is unique. Let us consider the paths $X_t^{\kappa_n} = (t, \sqrt{\kappa_n} B_t)$ and $X_t^\kappa = (t, \sqrt{\kappa} B_t)$. Using Lemma 7.4.2, for $\kappa_n \rightarrow \kappa$ as $n \rightarrow \infty$ we obtain that $X_t^{\kappa_n} \rightarrow X_t^\kappa$ in the p -variation topology. Next, by applying Theorem 7.3.3 we obtain that the backward Loewner flow started from $\delta > 0$, for any $\delta > 0$ is continuous in both starting point and the path $X_t = (t, \sqrt{\kappa} B_t)$ in the p -variation topology, and the conclusion follows. $\square$

Remark 7.4.3. 1) In [56], the authors prove the continuity of the SLE_κ traces for $\kappa \in [0, 8(2 - \sqrt{3}))$ in the uniform topology. Although never written in their paper explicitly, we believe that their argument can be adapted also to prove the continuity of the solution to the backward Loewner differential equation started from $\delta > 0$ in the parameter κ for $\kappa \neq 8$.

2) The p -variation topology in which we recover the continuity is stronger than the uniform topology. Thus, our result is new and provides a stronger notion of convergence.

7.5 Truncated Taylor approximation for the forward Loewner differential equation

In this section, we perform an analysis in a different direction. Let us consider the forward chordal SLE_κ in the upper half-plane

$$\partial_t g(t, z) = \frac{2}{g(t, z) - \sqrt{\kappa} B_t}, \quad g(0, z) = z, \quad z \in \mathbb{H}. \quad (7.5.1)$$

By performing the identification $g_t(z) - \sqrt{\kappa} B_t = Z_t$, we obtain the following dynamics in $\mathbb{H}$,

$$dZ_t = \frac{2/\kappa}{Z_t} dt + dB_t, \quad Z_0 = z_0.$$

By extending the conformal mappings $g_t(z) - \sqrt{\kappa} B_t$ to the real line, we obtain the following dynamics on the real line

$$dZ_t = \frac{2/\kappa}{Z_t} dt - dB_t \quad Z_0 = \varepsilon > 0. \quad (7.5.2)$$

To sum up, in this section we are interested in the following differential equation on the real line

$$dZ_t = \frac{2/\kappa}{Z_t} dt - dB_t \quad Z_0 = \varepsilon > 0, \quad (7.5.3)$$

where $Z_t(x) = g_t(x) - \sqrt{\kappa} B_t$ is a real-valued process. In this section, we choose to work with the Itô lift of the pair (t, B_t) . The first order differential operators $\frac{2/\kappa}{Z} \frac{d}{dz}$ and $\frac{d}{dz}$ are associated to the vector fields of (7.5.3). Note that since our process Z_t is one-dimensional (and therefore there is a unique notion of directional derivative), we abuse the notation and use for the $\mathbb{R}$ -differentiation the notation $\frac{d}{dz}$, in order to match it with the notation of the functions that is acting on. The Bessel SDE

(7.5.3) can not be interpreted as a Rough Differential Equation anymore since the singularity of the vector field corresponding to the drift. However, we know from standard results in Stochastic Analysis that the Bessel SDE has a unique strong solution for $x_0 \geq 0$ up to the first hitting time (see [39], chapter XI). We consider the one-step Taylor approximation to the unique strong solution of the Bessel Stochastic Differential Equation started around the singularity.

A natural question regarding the Taylor approximation series associated to the forward Loewner differential equation extended to the real line is what should be the size of the time step one is allowed to take in order to assure that the series is a.s. not divergent. In practice, one can only compute a fixed number $r < \infty$ of steps in a numerical method and consequently generate a finite number of iterated integrals. Most numerical methods that one is interested in practice use derivatives of the vector fields in order to approximate the solution. We will be focusing on the truncated Taylor approximation up to some fixed finite $r > 0$.

In general, for a Rough Differential Equation of the form

$$dZ_t = V(Z_t)dX_t,$$

with $Z_0 = z_0 > 0$ with $X_t : [0, T] \rightarrow \mathbb{R}^d$ a finite p -variation path for any $p > 2$, we use the following compact notation for the first r terms of the Taylor approximation.

Definition 7.5.1. *Given the continuously differentiable vector fields $(V_1, \dots, V_d)$ on $\mathbb{R}^e$, and a multiplicative functional with finite p -variation, $\mathbf{X} \in T((\mathbb{R}^d))$, we define*

$$\mathcal{E}_{(V)}^r(Z, \mathbf{X}_{0,t}) := \sum_{k=1}^r \sum_{i_1, \dots, i_k} V_{i_1} \dots V_{i_k} \mathbf{Id}(Z) \mathbf{X}_{0,t}^{k, i_1, \dots, i_k}$$

as the increment of the step r -truncated Taylor approximation on the interval $[0, t]$.

The notation $V_{i_1} \dots V_{i_k} \mathbf{Id}(Z)$ stands for the composition of differential operators associated with the vector fields

$$V_{i_1} \circ V_{i_2} \circ \dots \circ V_{i_k} \circ \mathbf{Id}(Z),$$

and $\mathbf{X}_{0,t}^{k,i_1,\dots,i_k}$ stands for the terms obtained from the iterated integrals

$$\int_{0 < s_1 < \dots < s_k < t} dX_{s_1} \otimes \dots \otimes dX_{s_k}.$$

Definition 7.5.2. Let $\alpha > 0$. We call a one time step r -truncated Taylor approximation for a differential equation started from $y_0 = 0$ weakly α stable if for pairs consisting of initial point and time step used in the approximation $(y_0^\varepsilon, t^\varepsilon) = (\varepsilon, \varepsilon^\alpha)$ we have that the approximation converges as $\varepsilon \rightarrow 0$ to the initial value $y_0 = 0$, a.s..

In this section, we prove the following result

Theorem 7.5.3. The one time step r -truncated Taylor approximation for the Bessel process started from $\varepsilon > 0$ obtained by continuously extending the conformal maps solving the forward Loewner differential equation to the real line it is a.s. weakly α stable for time steps $t < \varepsilon^{4+\delta}$, for any $\delta > 0$.

7.6 Proof of Theorem 7.5.3

In order to prove the main result, we first use the following lemma.

Lemma 7.6.1. At fixed level r there are 2^r terms obtained from all the possible ways of composing the vector fields $V_1 = \frac{2/\kappa}{Z} \frac{d}{dz}$ and $V_2 = \frac{d}{dz}$.

Proof. We prove this using induction. For $r = 1$, there are $2 = 2^1$ possible terms obtained from either of the vector fields. For $r = 2$, the possible compositions are

$V_1 \circ V_1, V_2 \circ V_2, V_2 \circ V_1$ and $V_1 \circ V_2$ given 2^2 possibilities. Let us assume that at level $k > 0$ there are 2^k possible combinations. To obtain all the possible compositions at level $k + 1$, we have to consider $V_1 \circ V^k$ and $V_2 \circ V^k$ where V^k are all the possible compositions at level k . Thus, at level $k + 1$ we obtain in total $2^k + 2^k = 2^{k+1}$ possibilities, and the argument follows by induction. $\square$

Proof of the Theorem 7.5.3. Given the format of the forward Loewner differential equation extended on the real line, we have that the vector fields that can appear are either $V_1 = \frac{2/\kappa}{Z} \frac{d}{dz}$ or $V_2 = \frac{d}{dz}$. For fixed values of m and $n = r - m$ we have, by definition m time entries in the iterated integrals and m times the vector field $V_1 = \frac{2/\kappa}{Z} \frac{d}{dz}$ and n Brownian motion entries together with n times the vector field $V_2 = \frac{d}{dz}$. We also note that in order for $V_{i_1} \dots V_{i_r} \mathbf{Id}(z)$ to be non zero then $V_{i_r} = \frac{1}{Z}$, otherwise $V_{i_r} = 1$ and applying any other choice of $V_{i_{r-1}}$ will give the derivative of a constant that is zero.

Then, we provide the following rules when considering the composition of vector fields $V_{i_1} \dots V_{i_r} \mathbf{Id}(z)$ (up to some absolute constants that change, but we avoid keeping their dependence in our analysis). These rules are specific to the structure of the vector fields of the Loewner differential equation and they give a way to transform the composition of the differential operators associated with vector fields in the left into multiplication with the function on the right up to some constants (that we do not keep track of since they do not influence the analysis)

$$\blacktriangleright \frac{2/\kappa}{Z} \frac{d}{dz} \longleftrightarrow \frac{-2/\kappa}{Z^2}$$

$$\blacktriangleright \frac{d}{dz} \longleftrightarrow \frac{-1}{Z}$$

in the following sense: $V_1 \circ V^k = \frac{-2/\kappa}{Z^2} f(Z)$ and $V_2 \circ V^k = \frac{-1}{Z} f(Z)$, where we have that $f(z) = V_{i_1} \circ V_{i_2} \circ \dots \circ V_{i_k} \circ \mathbf{Id}(Z)$ for any $k > 0$.

To illustrate this, we consider

$$V_1 \circ V_1 \circ \mathbf{Id}(Z) = \frac{1}{Z} \frac{d}{dz} \frac{1}{Z}.$$

Then, $\frac{d}{dz} \frac{1}{Z} = \frac{-1}{Z^2}$ and $V_1 \circ V_1 \circ \mathbf{Id}(Z) = \frac{1}{Z^3}$ up to some constants. In general, the analysis is similar. Indeed, the result of the composition is obtained by iteration using either of the vector fields applied to the initial $\frac{2/\kappa}{Z}$ value (since otherwise we would obtain zero since we apply differential operators to a constant), and since the differential operators either act by differentiation or by differentiation and multiplication with $\frac{1}{Z}$ up to some constants, the rules hold.

Using these rules, independent of the order of applications of the vector fields we have that for $Z_0 = \varepsilon > 0$

$$|V_{i_1} \dots V_{i_r} \mathbf{Id}(Z_0)| = O\left(\frac{1}{\varepsilon^n} \frac{1}{\varepsilon^{2m-1}}\right) = O\left(\frac{1}{\varepsilon^{2m-1+n}}\right).$$

We consider $\Delta_T = \Delta_{\varepsilon^\alpha}$ for $\alpha > 0$. Thus, at any fixed level k , $k = m + n$, $k \in [0, r]$, we have

$$\sum_{k=1}^r \sum_{i_1, \dots, i_k} V_{i_1} \dots V_{i_k} \mathbf{Id}(Z) \mathbf{X}_{0,t}^{k, i_1, \dots, i_k} = \sum_{k=1}^r 2^k \frac{C(k)}{\varepsilon^{2m+n-1}} \int \dots \int_{0 \leq s_1 \leq s_2 \dots \leq \varepsilon^\alpha} dX_{s_1} \dots dX_{s_k}.$$

For each $k \in [0, r]$, there are terms that are zero (since they correspond to a particular order of application of the vector fields as explained before).

Using the estimate on the iterated integrals for the control $w_1(s, t) = c_m(\omega)(t - s)$ for $s = 0$ and $t = \varepsilon^\alpha$ from Theorem 7.2.1, we obtain that

$$\sum_{k=1}^r 2^k \frac{C(k)}{\varepsilon^{2m+n-1}} \int \dots \int_{0 \leq s_1 \leq s_2 \dots \leq \varepsilon^\alpha} dX_{s_1} \dots dX_{s_k} \leq \sum_{k=1}^r \frac{\tilde{C}(k)}{\varepsilon^{2m+n-1}} \varepsilon^{\frac{(m+n)\alpha}{2+\delta}},$$

where $\tilde{C}(k)$ is a constant obtained by merging all the constants obtained from the iterated integrals estimates and from the operation of composition of the vector fields. Thus, we obtain that

$$\frac{\alpha(m+n) - (2+\delta)(2m+n-1)}{2+\delta} > 0 \Leftrightarrow \alpha > \frac{(2+\delta)(2m+n-1)}{m+n}$$

In order to guarantee the control of all terms in the Taylor approximation, we need

$$\alpha > \limsup_{m,n \rightarrow \infty} \frac{(2+\delta)(2m+n-1)}{m+n} = 4 + \tilde{\delta},$$

with $\tilde{\delta} = 2\delta$ and the conclusion of the theorem follows. □

Remark 7.6.2. Note that for pairs of the form $(y_0, t^\varepsilon) = (u, \varepsilon^\alpha)$ for some fixed constant $u > 0$, the convergence holds for any value of $\alpha > 0$. However, when dealing with singular SDEs, one is interested in considering starting the equation closer to the singularity. The estimates that we use show that the time step one needs to take has to be small enough to be able to control the output of the approximation as $\varepsilon \rightarrow 0$, for almost every Brownian path.

Remark 7.6.3 (The typical estimates from Rough Path Theory do not work for the non-truncated Taylor approximation in our setting). We can show that the usual estimates on the iterated integrals from Rough Path Theory are not enough to ensure convergence for the Taylor approximation in the case of singular differential equations. Let us consider the absolute value of the r -level Taylor approximation remainder

$$R_r(t, Z_t, \mathbf{X}) = \pi(0, Z_0; X)_{0,t} - \mathcal{E}_{(V)}(Z_0, \mathbf{X}_{0,t}^{(r-1)}).$$

Using the previous analysis, we have that the remainder is equal to

$$\sum_{\tilde{r} \geq r}^{\infty} 2^{\tilde{r}} C(\tilde{r}) \frac{1}{\varepsilon^{2m+n-1}} \int \dots \int_{0 \leq s_1 \leq s_2 \dots \leq \varepsilon^{\alpha}} dX_{s_1} \dots dX_{s_{\tilde{r}}}.$$

Using the estimate on the Extension Theorem 7.2.1 for the iterated integrals we have that

$$\sum_{\tilde{r} \geq r}^{\infty} 2^{\tilde{r}} C(\tilde{r}) \frac{1}{\varepsilon^{2m+n+1}} \int \dots \int_{0 \leq s_1 \leq s_2 \dots \leq \varepsilon^{\alpha}} dX_{s_1} \dots dX_{s_{\tilde{r}}} \leq \sum_{\tilde{r} \geq r}^{\infty} 2^{\tilde{r}} C(\tilde{r}) \frac{1}{\varepsilon^{2m+n-1}} \varepsilon^{\frac{(m+n)\alpha}{2+\delta}}$$

Thus, the estimates do not work since the series corresponding to the upper bound diverges. Indeed, the factor $C(\tilde{r})$ grows like $\frac{\tilde{r}!}{(\frac{\tilde{r}}{p})!}$ for $p > 2$. The $\tilde{r}!$ term is obtained taking $\tilde{r}$ derivatives of the function $\frac{1}{z}$ while the term $(\frac{\tilde{r}}{p})!$ comes from the bound on the iterated integrals.

However, the two factors cancel when $p = 1$, i.e. when we consider Loewner Equation driven by bounded variation paths. The same analysis can be done then, and a control on the reminder for the one-step Taylor approximation can be obtained.

Appendix A

A.1 Pathwise estimate for Brownian Motion

Proposition A.1.1. *Let B_t be a standard one-dimensional Brownian motion.*

Let $M_n = \max_{j=0, \dots, 2^n-1} \sup_{0 \leq t \leq 2^{-n}} |B_{j2^{-n}+t} - B_{j2^{-n}}|$. Then

$$\lim_{n \rightarrow \infty} \frac{M_n}{\sqrt{2^{-n} \log(2^n)}} = \sqrt{2}.$$

Proof. For X a standard normally distributed random variable we obtain via direct estimates of the density the following bounds

$$e^{-\frac{(a+1)^2}{2}} \leq \sqrt{2\pi} \mathbb{P}(X \geq a) \leq e^{-\frac{a^2}{2}}.$$

Using the independence of increments and the scaling property of Brownian motion,

we have for $c > 0$ that

$$\begin{aligned}
\mathbb{P}(M_n \leq c2^{-\frac{n}{2}}\sqrt{n}) &\leq \prod_{j=0}^{2^n-1} \mathbb{P}(|B_{(j+1)2^{-n}} - B_{j2^{-n}}| \leq c2^{-n/2}\sqrt{n}) \\
&= |\mathbb{P}(|B_1| \leq c\sqrt{n})|^{2^n} \\
&\leq |1 - (\sqrt{2/\pi}e^{-(c\sqrt{n}+1)^2/2})|^{2^n} \\
&\leq \exp(-2^n(\sqrt{2/\pi}e^{-(c\sqrt{n}+1)^2/2})).
\end{aligned}$$

Let us set $C_n = \sqrt{2\log 2} - 2n^{-1/2}$. Then we can form the events $A_n := \{M_n \leq C_n 2^{-n/2}\sqrt{n}\}$. By applying the previous estimate, we obtain that

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n \leq C_n 2^{-n/2}\sqrt{n}) \leq \infty,$$

and hence by applying Borel-Cantelli we obtain that a.s.

$$\liminf_{n \rightarrow \infty} \frac{M_n}{\sqrt{2^{-n}n}} \geq \sqrt{2\log 2}.$$

In the opposite direction, we use the reflection principle for one-dimensional Brownian Motion, i.e. $\mathbb{P}(\sup_{0 \leq s \leq t} B_s \geq a) = 2\mathbb{P}(B_t \geq a)$. For $c > \sqrt{2\log 2}$, we have that

$$\begin{aligned}
\mathbb{P}(M_n \geq c2^{-\frac{n}{2}}\sqrt{n}) &\leq \mathbb{P}(\sup_{0 \leq t \leq 2^{-n}} |B_t| \geq c2^{-n/2}\sqrt{n}) \\
&\leq 2^n 4\mathbb{P}(B_1 \geq c\sqrt{n}) \\
&\leq (4\sqrt{2\pi})2^n e^{-c^2 n/2} \\
&\leq (4\sqrt{2\pi}) \exp(-n(c - \sqrt{2\log 2})^2/2).
\end{aligned}$$

If we choose $C_n = \sqrt{2 \log 2} + n^{-1/4}$, we get that

$$\sum_{n=1}^{\infty} \mathbb{P}(M_n \geq C_n 2^{-n/2} \sqrt{n}) \leq \infty,$$

and hence by Borel-Cantelli, we have with probability 1 that

$$\limsup_{n \rightarrow \infty} \frac{M_n}{\sqrt{2^{-n} n}} \leq \sqrt{2 \log 2}.$$

□

A.2 Estimate for the mapping-out function and the Distortion Theorem

Proposition A.2.1. *Let $S \subset \delta H$ be a measurable set on the Martin boundary of H . Then*

$$\lim_{y \rightarrow \infty, x/y \rightarrow 0} \pi y \mathbb{P}_{x+iy}(\hat{B}_{T(H)} \in S) = \text{Leb}(g_K(S)).$$

Proof. Consider the map $g_K(x + iy) = u + iv$. Then as $y \rightarrow \infty$ and $x/y \rightarrow 0$, we have that $u/y \rightarrow 0$ and $v/y \rightarrow 1$. Using the conformal invariance for of the Brownian motion and the (known) formula for the density of the harmonic measure in $\mathbb{H}$, we have that

$$\mathbb{P}_{x+iy}(\hat{B}_{T(H)} \in S) = \mathbb{P}_{u+iv}(B_{T(\mathbb{H})} \in g_K(S)) = \int_{g_K(S)} \frac{v}{\pi((t-u)^2 + v^2)} dt.$$

□

A second result that we use is the Distortion Theorem for univalent functions.

For this, we consider $\mathcal{S}$ to be the set of univalent functions on the unit disk that fix the origin and have $f'(0) = 1$. Note that Riemann Mapping Theorem tells us that we can study the simply connected domains by considering the study of univalent functions on $\mathbb{D}$. In the following, we present the following Distortion Theorem. Its proof relies on an estimate that the reader can find in [27], Proposition 3.20.

Theorem A.2.2. *For $f \in \mathcal{S}$ and $z \in \mathbb{D}$, we have that*

$$\frac{1 - |z|}{(1 + |z|)^3} \leq |f'(z)| \leq \frac{1 + |z|}{(1 - |z|)^3}.$$

Proof. Due to the fact that $\mathcal{S}$ is closed under conjugation by rotations, it is enough to prove the estimate for $x \in (0, 1)$ on the real line. Using Koebe 1/4 Theorem, we prove that for $f \in \mathcal{S}$, we have $|f'| \neq 0$ (for the precise statement, look at Corollary 3.19 in [27]). Since the derivative of a holomorphic function is holomorphic and $f'(0) = 1$, we can define an analytic function $\log f'$. i.e. we find h analytic such that $e^h = f'$. Using that $\log z = \log |z| + i \arg(z)$, we have that

$$x \partial_x \operatorname{Re}(h(x)) = \operatorname{Re} \left(\frac{x f''(x)}{f'(x)} \right).$$

Thus by the technical estimate in Proposition 3.20 in [27] we obtain that

$$\frac{2x - 4}{1 - x^2} \partial_x \log |f'(x)| \leq \frac{2x + 4}{1 - x^2},$$

and since $\log |f'(0)| = 0$ we can integrate the inequalities and we obtain the bounds.

□

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