Calculus and Ricci curvature in non-smooth semi-Riemannian geometry

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Abstract

This thesis presents several results that connect the synthetic approach—via optimal transport on metric measure spaces—with the analytic approach—via distributional methods on low-regularity manifolds—in the study of non-smooth semi-Riemannian geometry, encompassing both Riemannian and Lorentzian metric tensors of low regularity.

In a preliminary Chapter 2, we collect relevant definitions and background that the results in this thesis require.

In Chapter 3, we present the results from a collaboration with Andrea Mondino, where we established the equivalence of distributional and synthetic lower Ricci curvature bounds for weighted Riemannian manifolds with continuous metric tensor that admits L^2_{loc}-Christoffel symbols and a weight in $C^0 \cap W^{1,2}_{\mathrm{loc}}$.

In Chapter 4, we study infinitesimal Hilbertianity (namely, the property that the synthetic Sobolev space $W^{1,2}_w$ is Hilbert) on manifolds endowed with a discontinuous Riemannian metric g. In dimensions $d \geq 3$, we provide examples where $g, g^{-1} \in L^\infty_{\mathrm{loc}} \cap W^{1,p}_{\mathrm{loc}}$ for $p \in [1, d-1]$ which fail to be infinitesimally Hilbertian.

In Chapter 5, we prove that strongly causal and causally simple spacetimes arising from manifolds with continuous Lorentzian metric tensors are infinitesimally Minkowskian, the Lorentzian analogue of infinitesimal Hilbertianity.

Contents

Chapter 1

Introduction

1.1 Motivation

In this thesis, we aim to shed new light on the connections between two approaches to non-smooth geometry: the synthetic and the analytic approach. In this case, the word "synthetic" refers to definitions that do not rely on underlying differentiable structures, whereas "analytic" definitions are based on computations using differentiation. The difference between synthetic and analytic definitions can instructively be illustrated through two definitions of convexity (see [140]): A function $\varphi\colon \mathbb{R} \to \mathbb{R}$ is convex if:

(i) For all $x \in \mathbb{R}$, it holds $\varphi''(x) \geq 0$.

(ii) For all $x, y \in \mathbb{R}$, $\lambda \in [0,1]$, it holds $\varphi(\lambda x + (1-\lambda)y) \leq \lambda\varphi(x) + (1-\lambda)\varphi(y)$.

Here, (i) corresponds to the *analytic* definition, based on a pointwise inequality involving second order derivatives. Thus, at least classically, the definition only makes sense if φ is twice differentiable. The *synthetic* definition (ii) is harder to check, but has the advantage of requiring no regularity assumptions on φ. Given a function $\varphi \in C^2$, the two definitions are equivalent, and complement each other in the sense that (i) is a local criterion and (ii) tells us something global about the shape of the graph of the function. Moreover, definition (ii) can be seen as a sensible way to extend convexity to less regular functions. Strikingly, functions satisfying (ii) automatically enjoy more regularity than a priori assumed. By Alexandrov's regularity theorem, convex functions are almost everywhere twice differentiable and the second derivative is non-negative wherever it exists, which can be seen as an analytic generalisation of (i) and hints compatibility of the analytic and synthetic approach to convexity beyond the case that $\varphi \in C^2$.

The issue of compatibility will be central, as we shift our attention to non-smooth semi-Riemannian geometry, a setting in which several analytic and synthetic approaches and definitions have been made.

As a result, the question whether two such approaches are compatible is not only natural but also fundamental. Indeed, a positive answer would open up a vast tool set for problems in non-smooth semi-Riemannian geometry, as results from the synthetic setting would hold as soon as the analytic criterion is satisfied and vice versa. A negative answer, however, could suggest to change or adapt the approaches or definitions, or point out that in those instances special care is required.

In Chapter 3 of this thesis, we will present one example for which the question of compatibility has a positive answer. We establish that, under fairly general assumptions, the analytic and synthetic approaches to lower Ricci curvature bounds on non-smooth Riemannian manifolds are equivalent.

Later, in Chapter 4, we will discuss a notion for which the compatibility is not always given. We analyse infinitesimal Hilbertianity, a concept from the synthetic first order calculus that is supposed to detect the Hilbertian analytic structure, on manifolds with potentially discontinuous Riemannian metric tensors g, with $g, g^{-1} \in L^\infty_{\mathrm{loc}}$. By constructing a counterexample in dimension $d \geq 3$, we will point out that up to additional $W^{1,p}_{\mathrm{loc}}$-Sobolev regularity of g for $p \in [1, d-1]$, infinitesimal Hilbertianity may fail.

The issue of compatibility remains a key motivation for the final part of the thesis, Chapter 5, devoted to the emerging and rapidly developing field of non-smooth Lorentzian geometry. There, we demonstrate the compatibility between the classical and synthetic first-order calculus on non-smooth Lorentzian manifolds, assuming continuity of the Lorentzian metric.

Overall, the results of this thesis both extend connections between analytic and synthetic notions to settings of lower regularity, and demonstrate that relaxing analytic assumptions, specifically assuming lower regularity, may result in a loss of compatibility with the synthetic framework.

1.2 A brief historical introduction

Before studying compatibility, we aim to recall the history leading to the field of synthetic calculus and lower Ricci curvature bounds and shed light on their analytic origins in smooth semi-Riemannian geometry. This summary has been inspired by [3, 112, 35, 124, 133, 84].

Riemannian geometry and Ricci curvature

A semi-Riemannian manifold is given by a smooth manifold M equipped with an everywhere non-degenerate (smooth) $(0,2)$-tensor $g \in (T^*M)^{\otimes 2}$. Throughout this thesis, we are interested in Riemannian manifolds and Lorentzian manifolds. In the case of a Riemannian manifold, we ask the metric tensor g to be everywhere positive definite. The metric g induces a distance d_g on M, given by the infimum of the length of piecewise smooth curves, i.e.

$$\mathsf{d}_g(x,y) := \inf \Big\{ \int_0^1 \sqrt{g(\dot\gamma_t,\dot\gamma_t)}\,\mathrm{d}t, \gamma : [0,1] \to M \text{ piecewise smooth from } x \text{ to } y \Big\}, \tag{1.2.1}$$

and a volume measure locally given by $\mathrm{dvol}_g = \sqrt{\det g}\,\mathrm{d}x$.

Curvature plays a fundamental role in geometry as it provides an infinitesimal measurement of how much a Riemannian manifold differs from the Euclidean space. The Riemannian curvature tensor is a $(0,4)$-tensor that encapsulates a substantial amount of geometric information, rendering it impractical for direct use. Consequently, it is commonly employed to derive alternative curvature measures that are more amenable to interpretation. Sectional curvature for instance can be computed for two linearly independent vectors $v, w \in T_pM$, for any $p \in M$ and measures how much the distance between two geodesics differs from the Euclidean case.

We will focus on the Ricci curvature, which can be thought of as the Laplacian of g. It is a symmetric bilinear form on $TM \otimes TM$ that occurs in the Taylor expansion of the volume form as follows: In normal coordinates, the volume form is locally given by

$$\mathrm{vol}_g(x) = 1 - \sum_{i,j} \frac{1}{6} \mathrm{Ric}_{ij} x^i x^j + O(|x|^3). \tag{1.2.2}$$

In [139], the Ricci curvature and its connection with the volume measure are discussed in further detail, using Jacobi fields. It is also shown that the Ricci curvature naturally occurs in Bochner's formula, which states that for a function $f \in C^\infty(M)$, one has

$$\frac{1}{2}\Delta|\nabla f|^2 = \langle \nabla\Delta f, \nabla f\rangle + |\nabla^2 f|^2 + \mathrm{Ric}(\nabla f, \nabla f). \tag{1.2.3}$$

When studying Riemannian manifolds, it is sensible to assume bounds on the (Ricci) curvature, since these ensure a good control on the geometry and the topology of the manifold, as discussed below. For a real number $K \in \mathbb{R}$, we say that (M, g) has Ricci curvature bounded below by K if for any $v \in TM$, it holds $\mathrm{Ric}(v,v) \geq K g(v,v)$. Lower Ricci curvature bounds together with upper bounds on the dimension have remarkable implications, such as the volume comparison proved by Bishop in 1964 [18] (and improved by Gromov in 1980 to a monotonicity statement [72, 74]), stating that if (M, g) satisfies $\mathrm{Ric} \geq K$, then the volume of a metric ball of radius $r > 0$ is at most the volume of the metric ball with radius r in the space of constant sectional curvature $\frac{K}{n-1}$. Further strong implications are an estimate on the Laplacian of any distance function proved by Calabi in 1958 [28], and the Cheeger-Gromoll splitting theorem [43]. Ricci curvature bounds are the crucial ingredient to several further fundamental results in geometric analysis, as pointed out in [88].

One challenge about the set of manifolds with Ricci curvature bounded below by some constant K is that it is not compact in the C^∞-topology. Nevertheless, in [72], Gromov proved that the set $\mathcal{M}_{K,n,D}$ of manifolds with Ricci curvature bounded below by K, dimension bounded above by n, and diameter bounded above by D is precompact with respect to the Gromov-Hausdorff topology. The Gromov-Hausdorff topology is induced by the Gromov-Hausdorff distance, a measurement of how far two metric spaces are away from being isometric. Gromov-Hausdorff limits of sequences of n-manifolds with Ricci curvature bounded from below by some real number K are called *Ricci limits*. The study of Ricci limits was initiated by Cheeger and Colding in [39, 40, 41, 42], driven by the following questions:

- Are analytic and geometric properties stable under convergence?
- Is there an intrinsic definition that reflects how Ricci limit spaces satisfy lower Ricci curvature bounds?
- What do general spaces with lower Ricci curvature bounds (Ricci limits) look like?

Metric measure spaces and synthetic lower Ricci curvature bounds

Alexandrov used a purely metric characterisation of sectional curvature bounds on smooth Riemannian manifolds (through Toponogov's Theorem), to develop a theory of metric spaces with sectional curvature bounds (see [2]). In [73], Gromov stated that no such theory of singular spaces with Ricci curvature bounded from below existed yet, and suggested that one should try to develop such a theory for *metric measure spaces*. This gap was later on again pointed out by Cheeger and Colding in [40], who predicted that a synthetic definition of lower Ricci curvature bounds would require a way to localise the Bishop-Gromov-inequality.

It turns out that the missing ingredient to find a synthetic definition of lower Ricci curvature bounds was optimal transport. The optimal transport problem was first studied by Monge in 1781 and further developed by Kantorovich in 1942. In our setting, we consider the following optimisation problem: Given a Polish space (X, d) and two probability measures μ, ν on X, what is the "cheapest" way to transport μ to ν? More rigorously: what is

$$\min \left\{ \int_{X \times X} \mathsf{d}^2(x, y) \mathrm{d}\pi(x, y) \right\},$$

where π is a probability measure on $X \times X$ such that its first (resp. second) marginal equals μ (resp. ν)? The cases when X is the Euclidean space or a Riemannian manifold turned out to be of central interest. Through the above problem, one can define the Wasserstein 2-distance W_2 on the finite second moment space $\mathcal{P}_2(X)$, which is given by

$$\mathcal{P}_2(X) := \left\{ \mu \in \mathcal{P}(X) : \int_X \mathsf{d}^2(x, x_0) \mathrm{d}\mu(x) < \infty, \text{ for some } x_0 \in X \right\}.$$

In order to generalise lower Ricci curvature bounds, one considers a *metric measure space*, which is a triple $(X, \mathsf{d}, \mathfrak{m})$, where (X, d) is a Polish metric space and $\mathfrak{m}$ is a σ-finite Borel measure. It turned out that a crucial object to characterise lower Ricci curvature bounds is the Boltzmann-Shannon entropy, given by

$$\mathrm{Ent}(\mu|\mathfrak{m}) := \int_X \rho \log \rho \, \mathrm{d}\mathfrak{m},$$

for probability measures $\mu = \rho\mathfrak{m}$, where $\rho, \rho \log \rho \in L^1(\mathfrak{m})$. Indeed, after being conjectured by Otto and Villani [110], Cordero-Erasquin, McCann, and Schmuckenschläger [48], and Sturm and von Renesse [141] proved that in a smooth Riemannian manifold, Ricci curvature bounded below by $K \in \mathbb{R}$ is equivalent to the Boltzmann-Shannon entropy being K-convex along W_2-geodesics.

Lott and Villani [92, 91], and Sturm [133, 134] independently used this characterisation to define lower Ricci curvature bounds for metric measure spaces. More precisely, they introduced the curvature-dimension condition $\mathsf{CD}(K, \infty)$, where $K \in \mathbb{R}$ is the lower bound for the Ricci curvature. This condition was then refined to the $\mathsf{CD}(K, N)$-condition, where $N \in [1, \infty]$ is an upper bound of the dimension, using the dimensional counterparts of the Boltzmann-Shannon entropy, the so-called Rényi's entropies. In the following, we will review some key developments around $\mathsf{CD}(K, N)$-spaces.

In [108], Ohta verified that Finsler manifolds (a generalisation of Riemannian manifolds based on norms rather than inner products) that satisfy lower Ricci curvature bounds enter the class of $\mathsf{CD}(K, N)$-spaces. Petrunin [114] proved that n-dimensional Alexandrov spaces, i.e., metric spaces that have synthetic sectional curvature bounded from below by K are indeed $\mathsf{CD}((n-1)K, n)$-spaces, which confirms the understanding that sectional curvature bounds are stronger than Ricci curvature bounds beyond the smooth case. Furthermore, the class of $\mathsf{CD}(K,N)$-spaces is stable under pointed measured Gromov-Hausdorff convergence [133, 134, 92, 139, 67], that is a variant of the Gromov-Hausdorff convergence that takes a reference point and a reference measure into account.

Whilst lower Ricci curvature bounds on smooth Riemannian manifolds are a local property, the $\mathsf{CD}(K,N)$-condition is not. Moreover, only the global condition is stable under pointed measured Gromov-Hausdorff convergence, which made the globalisation problem a fundamental issue. The search for globalisation properties lead to the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ introduced by Bacher and Sturm in [12], a slightly weaker condition than the $\mathsf{CD}(K,N)$-condition. Soon after, a result by Rajala [116] showed that without additional assumptions, a local $\mathsf{CD}(K,N)$-condition may not imply a global one. Rajala and Sturm then introduced the *essential non-branching* condition [117], a condition that states that almost all geodesics are non-branching. Here almost all should be understood in the sense of test plans, i.e., probability measures on the space of continuous curves that, roughly speaking, respect the reference measure (see Chapter 2 for the precise and technical definition). In [34], Cavalletti and Mondino verified several functional and geometric inequalities for $\mathsf{CD}^*(K,N)$-spaces under the essential non-branching assumption after establishing a localisation technique and a disintegration theorem for optimal transport plans in [33]. These include the Lévy-Gromov isoperimetric inequality, the Brunn-Minkowski inequality, the Poincaré inequality, and a Sobolev inequality. Moreover, Cavalletti, Maggi, and Mondino proved a quantitative isoperimetric inequality in [30]. The globalisation, as well as the equivalence of the $\mathsf{CD}(K,N)$-condition and the $\mathsf{CD}^*(K,N)$-condition was obtained by Cavaletti and Milman, both under the essential non-branching assumption [32].

As the above results demonstrate, $\mathsf{CD}(K,N)$-spaces proved to generalise Riemannian manifolds with lower Ricci curvature bounds in many aspects, yet the curvature-dimension condition did not appear as an accurate description of Ricci limit spaces. Ricci limits of manifolds with lower curvature bounds going to zero satisfy a splitting theorem [39]. However, the $\mathsf{CD}(0,N)$-space $(\mathbb{R}^n, \|\cdot\|, \mathcal{L}^n)$, where $\|\cdot\|$ is any (non-Euclidean) norm on $\mathbb{R}^n$, does generally not satisfy the splitting theorem.

The first order calculus on metric measure spaces turned out to be a crucial ingredient in the search for a criterion that distinguishes Riemannian-like structures from Finslerian structures. Initial approaches to the generalised first order calculus were previously made by Cheeger [38] to generalise Rademacher's theorem to PI-spaces, Hajłasz [76] to prove generalised Sobolev embedding theorems, and Shanmugalingam [125]. Thereafter, Ambrosio, Gigli, and Savaré introduced a Sobolev calculus based on the notion of test plans in [7] that they proved to be equivalent to the approaches in [38] and [125]. They used the Cheeger energy Ch, which generalises the Dirichlet energy to metric measure spaces, to define a synthetic Sobolev space $W^{1,2}_w(X)$, and a Laplace operator. In this framework, the heat flow is defined as the L^2-gradient flow of the Cheeger energy.

In [62], Gigli studied conditions under which the Laplacian and the heat flow are linear and introduced the notion of *infinitesimal Hilbertianity* (after [8]). A metric measure space is infinitesimally Hilbertian if the associated synthetic Sobolev space $W^{1,2}_w$ is a Hilbert space, or equivalently if the Cheeger energy is a quadratic form. Then, the Laplacian is automatically linear and the theory on Dirichlet forms provides various tools to analyse the Laplacian and the heat flow. Examples of infinitesimally Hilbertian metric measure spaces include geodesically complete weighted Riemannian manifolds [93], locally $\mathrm{CAT}(\kappa)$-spaces equipped with any Borel measure [54], and sub-Riemannian manifolds equipped with any Borel measure [86].

In [8], Ambrosio, Gigli, and Savaré introduced the *Riemannian curvature-dimension condition* $\mathsf{RCD}(K,\infty)$ for $K \in \mathbb{R}$, which combines the $\mathsf{CD}(K,\infty)$-condition with infinitesimal Hilbertianity. Examples of RCD-spaces include Ricci limits (as the $\mathsf{RCD}(K,\infty)$-property is stable [67]), and cones over RCD-spaces (see for instance [78]). Subsequently, several properties of $\mathsf{RCD}(K,\infty)$-spaces were established. In particular, it was shown that $\mathsf{RCD}(K,\infty)$-spaces are essentially non-branching [117]. Another key discovery was the identification of the heat flow as an EVI_K gradient flow [6], allowing to deduce strong contractivity properties.

This identification was crucial to bridge the gap to the Eulerian approach to curvature-dimension conditions [9] based on (1.2.3). Such an Eulerian approach was proposed by Bakry and Émery [14], Bakry [13], and Bakry, Gentil and Ledoux [15]. It is known as the Bakry-Émery condition $\mathrm{BE}(K,\infty)$ and it consists of a generalisation of the Bochner inequality to metric measure spaces based on Γ-calculus. Using the heat flow contraction and the $\mathrm{BE}(K,\infty)$-condition, Gigli was able to develop a second order calculus on $\mathsf{RCD}(K,\infty)$-spaces [63], leading to a measure valued Ricci curvature tensor.

The refinement to the $\mathsf{RCD}(K,N)$ condition for finite N was proposed by Gigli in [61]. Analogously, the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ combined with infinitesimal Hilbertianity was defined to be the reduced Riemannian curvature-dimension condition $\mathsf{RCD}^*(K,N)$.

The class of RCD-spaces displays various striking properties. Gigli proved a splitting theorem for $\mathsf{RCD}(0,N)$-spaces [64]. Furthermore, RCD-spaces are rectifiable [101], have almost everywhere Euclidean tangents [66], have constant dimension [23], and their boundary satisfies structural regularity properties [22].

The metric measure space $(\mathbb{R}^n, \|\cdot\|, \mathcal{L}^n)$, where $\|\cdot\|$ denotes an arbitrary norm, is an $\mathsf{RCD}(0,N)$-space if and only if $\|\cdot\|$ arises from an inner product. Hence, in that case infinitesimal Hilbertianity indeed singles out Riemannian structures among Finslerian ones. The reduced Riemannian curvature-dimension condition $\mathsf{RCD}^*(K,N)$ was studied by Erbar, Kuwada, and Sturm [55] and Ambrosio, Mondino, and Savaré [10], both papers including proofs of the equivalence of the $\mathsf{RCD}^*(K,N)$-condition and the $\mathrm{BE}(K,N)$-condition. Using the globalisation result by Cavalletti and Milman from [32] allowed to conclude that the $\mathsf{RCD}(K,N)$-condition and the $\mathsf{RCD}^*(K,N)$-condition are equivalent, showing that the Lagrangian and the Eulerian approaches to lower Ricci curvature bounds are equivalent on infinitesimally Hilbertian spaces.

Let us mention that it is still an open problem whether the Riemannian curvature-dimension condition is sufficient to describe Ricci limits. As pointed out by De Philipps, Mondino, and Topping [136], the cone over $\mathbb{R}P^2$ is an $\mathsf{RCD}(0,3)$-space [127], but not a topological manifold, so by a result of Simon [126] and Simon-Topping [127], it is not a non-collapsed Ricci limit. However, the study of RCD-spaces has become of independent interest.

Lorentzian geometry

We now drift away from the Riemannian setting. Let (M,g) be a smooth semi-Riemannian manifold such that g has signature $(-,+,\dots,+)$. We then call (M,g) a *Lorentzian manifold*. Lorentzian manifolds play an important role in physics, more precisely in general relativity. The simplest example of a Lorentzian manifold is given by the Minkowski space, which is $(\mathbb{R}^{d+1}, \mathrm{diag}(-1,1,\dots,1))$. In this case the 0-th coordinate represents the time direction, whereas all other coordinates represent space directions. In a Lorentzian manifold (M,g), we say that a vector $v \in TM$ is *timelike* if $g(v,v) < 0$, *lightlike/null* if $v \neq 0$ and $g(v,v) = 0$, *spacelike* if $v = 0$ or $g(v,v) > 0$, and *causal* if v is null or timelike. Note that at each point, the set of timelike vectors in the tangent space is the interior of a double cone. Bearing in mind the physical interpretation, we are mainly interested in causal directions, as they represent directions in which the spacial velocity is smaller than or equal to the time velocity, hence a trajectory satisfying that particles do not move faster than the speed of light (here normalised to be 1). We will work with time orientable Lorentzian manifolds, these are manifolds which admit a smooth timelike vector field, which indicates the future cone at each point. A causal vector $v \in TM$ is called *future directed* if $g(X, v) < 0$, otherwise it is called *past directed*. Time orientable manifolds are called *spacetimes*.

Lorentzian signature suggests the definition of a causal structure on the manifold, as opposed to the metric structure from the Riemannian case. We define the relations $\ll$ and $\leq$ as follows: For two points $x, y \in M$ we write $x \ll y$ if there exists a timelike and future directed curve from x to y, i.e., a continuous and piecewise smooth curve $\gamma\colon [0,1] \to M$ such that $\dot{\gamma}_t$ is timelike and future directed for almost every t. We write that $x \leq y$ if $x = y$ or if there exists a causal future directed curve from x to y, that is a continuous and piecewise smooth curve $\gamma : [0,1] \to M$ such that $\dot{\gamma}_t$ is causal and future directed for almost every $t \in [0,1]$. The Lorentzian analogue to a distance is given by the time separation, defined via

$$\ell_g(x, y) := \sup\Big\{ \int_0^1 \sqrt{-g(\dot{\gamma}_t, \dot{\gamma}_t)}\, \mathrm{d}t, \gamma \text{ causal and future directed from } x \text{ to } y \Big\}. \tag{1.2.4}$$

Then ℓ_g takes values in $\{-\infty\} \cup [0, \infty]$. In the literature it is sometimes employed to consider $\tau_g := \max(\ell_g, 0)$. An important property of the time separation, hinting that Lorentzian manifolds behave very differently to Riemannian manifolds, is the reverse triangle inequality: For $x, y, z \in M$, with $\ell_g(x, y), \ell_g(y, z) < \infty$, it holds

$$\ell_g(x, z) \geq \ell_g(x, y) + \ell_g(y, z).$$

The curvature tensor can be computed just as in the Riemannian case. In general relativity, Ricci curvature plays a central role, as it occurs in the Einstein equations: Given a constant $\Lambda \in \mathbb{R}$ and a $(0,2)$-tensor T, the Einstein equations are given by

$$\mathrm{Ric} + Rg + \Lambda g = T,$$

where R denotes the scalar curvature, $\Lambda \in \mathbb{R}$ is called the cosmological constant and T denotes the stress-energy tensor, which measures the flux of energy and momentum in the spacetime. If $T = 0$, this gives the vacuum Einstein equations.

Motivated by their central role in general relativity, the Einstein equations have been thoroughly studied from a classical PDE point of view. The standard local existence result assumes the metric g to be of Sobolev regularity H^s_{loc} for $s > \frac{5}{2}$ (see for instance [118]), but there have been advances in even lower regularity [80].

The significance of Ricci curvature in general relativity goes beyond the Einstein equations: In Hawking's singularity theorem, a lower bound on the Ricci curvature on timelike vectors is one of the crucial assumptions for the spacetime to be incomplete, which can be interpreted as a mathematical justification for a "big bang" to exist. There are further singularity theorems with similar flavour, like the Penrose and Hawking-Penrose singularity theorems, which all contain curvature assumptions (see for instance [129] for a survey on that topic).

As in the Riemannian case, timelike lower Ricci curvature bounds have far-reaching geometrical and topological implications, as for instance demonstrated by the Lorentzian splitting theorem [57, 59].

Several physically relevant phenomena such as cosmic strings [137] and impulsive gravitational waves [111] (see also [115]) can be modeled with singular Lorentzian manifolds. The intrinsic interest in a Lorentzian analogue of metric geometry, as well as the quest to study the Einstein equations and singularity theorems in low regularity from a geometric point of view naturally raised the question, whether an analogue of the theory on metric measure spaces as generalised Riemannian manifolds and a synthetic definition of Ricci curvature bounds can be developed for Lorentzian manifolds.

Lorentzian length spaces and synthetic Lorentzian geometry

After the pioneering work by Kronheimer-Penrose [81] and Busemann [26], Kunzinger-Sämann [84] proposed the notion of Lorentzian (pre-)length spaces, a synthetic approach to Lorentzian geometry. They also studied timelike sectional curvature bounds through timelike triangle comparison.

In 2020, Cavalletti and Mondino then developed optimal transport tools for Lorentzian pre-length spaces in [35].

McCann [95] and Mondino-Suhr [105] proved that in a smooth, globally hyperbolic Lorentzian manifold, timelike lower Ricci curvature bound are equivalent to a displacement convexity condition of the Boltzmann-Shannon entropy along Lorentzian optimal transport geodesics. This motivated the definition of synthetic timelike lower Ricci curvature bounds and upper dimension bounds. For $K \in \mathbb{R}$ and $N \in (0, \infty)$ the timelike curvature-dimension condition $\mathsf{TCD}(K, N)$ on measured Lorentzian pre-length spaces was introduced by Cavalletti and Mondino in [35]. In the same paper, they also introduced a timelike measure contraction property TMCP (a broader generalisation of timelike lower Ricci curvature bounds) and proved stability of both under a certain notion of convergence. In [105], Mondino and Suhr furthermore introduced a synthetic definition of upper Ricci curvature bounds which lead to a synthetic definition of solutions to the Einstein equations. Multiple theorems about Lorentzian manifolds generalise to synthetic Lorentzian spaces that satisfy synthetic timelike curvature bounds, such as Hawking's singularity theorem for TMCP-spaces [35].

In [16], the eight authors rigorously developed a first order calculus on *metric measure spacetimes*, a synthetic approach to Lorentzian geometry compatible with measured Lorentzian pre-length spaces. They consist of a set M, with a Polish topology τ on M, a time separation ℓ (playing the role of (1.2.4)) and a non-negative Borel measure $\mathfrak{m}$ on M. They study causal functions, i.e., functions that are increasing along causal curves and introduce a weak notion of Lorentzian modulus of the gradient, the maximal weak subslope. This is used to introduce the notion of *infinitesimal Minkowskianity*, a criterion which serves to distiguish genuine Lorentzian structures from Lorentzian-Finslerian ones, providing a Lorentzian analogue to infinitesimal Hilbertianity.

Further developments in this rapidly expanding field include a Lorentzian splitting theorem for Lorentzian length spaces with non-negative timelike sectional curvature [17], a definition of Lorentzian Hausdorff measures and dimension [96], the study of time functions on Lorentzian length spaces [24], a definition of timelike lower Ricci curvature bounds on Finsler spacetimes [21], a new Lorentzian isoperimetric-type inequality under timelike lower Ricci bounds [36], and the null distance [128]. In the flavour of the celebrated singularity theorems on smooth spacetimes, related developments in the non-smooth case include the study of cones and related singularity theorems [1], the relationship between spacetime inextendability and synthetic curvature blow-up established in [70], singularity theorems under variable timelike lower Ricci bounds [20], an extension of the Penrose singularity theorem to continuous spacetimes satisfying a synthetic null energy condition [31], and the development of a Lorentzian Gromov-Hausdorff convergence and of a Gromov pre-compactness theorem [100, 27, 104].

Semi-Riemannian metrics of low regularity

A large class of metric measure spaces arises from Riemannian manifolds (M, g), for which the metric tensor g is non-smooth (in particular below C^2). These occur in gluing constructions for example. If $g \in C^0$, we can still define the Riemannian distance d_g as before (see (1.2.1)) and naturally obtain the metric measure space $(M, \mathsf{d}_g, \mathrm{vol}_g)$. The length structure of the metric measure space has been thoroughly studied by Burtscher in [25] in the case that g is continuous.

If g is only measurable, with $g, g^{-1} \in L^\infty_{\mathrm{loc}}$, the definition of a distance is not as straightforward as in the continuous case. However, such manifolds have been relevant in the context of potential theory for uniformly elliptic operators and have been studied by Norris [107], Saloff-Coste [121], Sturm [131], and De Cecco and Palmieri [51]. In particular, De Cecco and Palmieri [51] and Norris [107] defined a distance on such manifolds.

In [53], De Giorgi formulated questions on manifolds with Riemannian metrics of low regularity, such as under which regularity assumption the metric slope of a Lipschitz function coincides almost everywhere with the norm of its gradient. This led to his definition of quasi-Riemannian manifolds and to the formulation of several conjectures on them. A related question, arising from the results in [102] on manifolds with continuous Riemannian metrics, is whether for non-continuous metrics, the associated metric measure space is infinitesimally Hilbertian.

In Lorentzian signature, manifolds of low regularity have been of central interest, for instance in the context of singularity theorems (e.g. [69, 29]). Metric measure spacetimes arising from Lorentzian manifolds (M, g) with $g \in C^0$ have been studied (see for example [122]). The time separation function ℓ_g can be defined similarly to the smooth case (see (1.2.4)). We can always find a complete Riemannian metric h on M that induces a Polish topology τ_h on M and together with the volume measure, we get that $(M, \tau_h, \ell_g, \mathrm{vol}_g)$ is a naturally obtained metric measure spacetime.

An important family of semi-Riemannian metrics of low regularity is the Geroch-Traschen class [60], see also [130], whose elements are all the semi-Riemannian metrics such that $g, g^{-1} \in L^\infty_{\mathrm{loc}}$ and $g \in W^{1,2}_{\mathrm{loc}}$. These are the minimal regularity assumptions to define distributional (Ricci) curvature (see [75, 69]) that have been studied in both Riemannian [56] and Lorentzian signature [130].

This thesis will present recent results bridging the analytic approach based on Sobolev calculus and distribution theory with the synthetic approach in both Riemannian and Lorentzian signature.

1.3 Main contributions

In this section, we will briefly summarise the main contributions presented in this thesis and relate them to the history.

1.3.1 The equivalence of synthetic and distributional lower Ricci curvature bounds

Chapter 3 is based on a collaboration with Andrea Mondino [102], where we study the relationship between synthetic and distributional lower Ricci curvature bounds on manifolds with non-smooth Riemannian metrics.

Let (M, g) be a smooth n-dimensional Riemannian manifold without boundary. One can locally compute the coefficients of the Ricci curvature tensor in terms of the metric tensor

g and its first two derivatives. More precisely, denoting by Γ_{ij}^k the Christoffel symbols of g

$$\Gamma_{ij}^k = \frac{1}{2} g^{kl}(\partial_j g_{li} + \partial_i g_{jl} - \partial_l g_{ij}), \quad i, j, k = 1, \dots, n,$$

the Ricci curvature can be written locally as

$$\mathrm{Ric}_{ij} = \partial_k \Gamma_{ij}^k - \partial_i \Gamma_{kj}^k + \Gamma_{ji}^k \Gamma_{lk}^l - \Gamma_{jl}^k \Gamma_{ik}^l, \quad i, j = 1, \dots, n, \tag{1.3.1}$$

where ∂_i is the shorthand notation for $\frac{\partial}{\partial x^i}$, and where we used the Einstein convention that repeated indices are summed.

Given a smooth function V on M, one defines the weighted measure $\mathrm{d}\mu = e^{-V} \mathrm{dvol}_g$. Given $N \in [n, \infty]$, the Bakry-Émery N-Ricci curvature tensor of the weighted Riemannian manifold (M, g, μ) is defined by

$$\mathrm{Ric}_{\mu,N} := \mathrm{Ric} + \mathrm{Hess} V - \frac{1}{N-n} \nabla V \otimes \nabla V. \tag{1.3.2}$$

In case that $N = n$, we adopt the convention that V must be constant and that $\mathrm{Ric}_{\mathrm{vol}_g, n} = \mathrm{Ric}$. We also adopt the standard convention that $1/\infty = 0$, so that for $N = \infty$ the last adding term on the right hand side of (1.3.2) disappears. For $K \in \mathbb{R}$, we say that the Bakry-Émery N-Ricci curvature tensor is bounded below by K, if $\mathrm{Ric}_{\mu,N}(X, X) \geq K g(X, X)$ for all smooth vector fields X on M.

Since the local expression of the Ricci tensor involves two derivatives of the metric tensor g, and the Bakry-Émery N-Ricci curvature tensor involves the Ricci tensor and the Hessian of the weight function V, some care is needed if g or V are not twice differentiable. We will compare two approaches to Ricci curvature lower bounds for weighted manifold of regularity below C^2: the distributional and the synthetic approaches.

Via the distribution theory on smooth manifolds (cf. [87, 75, 69]), one can define a distributional Ricci curvature and generalise the notion of Ricci curvature lower bounds. This approach is suitable for handling the case of a smooth manifold M endowed with a continuous metric g having Christoffel symbols in L^2_{loc}, as it is apparent from (1.3.1). Moreover, assuming $V \in C^0 \cap W^{1,2}_{\mathrm{loc}}$ suffices to define a distributional Bakry-Émery N-Ricci curvature tensor of the corresponding weighted space, as one can check using (1.3.2). Note that this approach assumes the space to be smooth: all the non-smoothness is encoded in the metric tensor and, possibly, in the weighted measure. Let us also mention [89] for a notion of Ricci curvature lower bounds for continuous metrics conformal to smooth ones, based on the concept of viscosity solutions for non-linear elliptic partial differential equations.

The synthetic approach to lower Ricci curvature bounds is to drop any underlying smoothness assumption and work with the metric measure space (M, d_g, μ). Since the conditions $\mathsf{CD}(K,\infty)$ and $\mathsf{CD}(K,N)$ provide a definiton of Ricci curvature lower bounds for manifolds with metrics of regularity below C^2, a natural question to ask is whether distributional and synthetic Ricci curvature lower bounds continue to agree on metrics with regularity below C^2.

First advances to this question have been made in 2022 by Kunzinger, Oberguggenberger and Vickers [82]. They proved that:

Theorem (Kunzinger-Oberguggenberger-Vickers)**.** *Let M be a compact connected manifold equipped with a C^1-Riemannian metric g. If the distributional Ricci curvature is bounded below by a real number K, then $(M, \mathsf{d}_g, \mathrm{vol}_g)$ satisfies the $\mathsf{CD}(K,\infty)$-condition. Conversely, given a compact and connected manifold M equipped with a $C^{1,1}$-Riemannian metric g that satisfies one additional assumption (see [82], (6.8)), then if $(M, \mathsf{d}_g, \mathrm{vol}_g)$ satisfies the $\mathsf{CD}(K,\infty)$-condition, the distributional Ricci curvature is bounded below by K.*

Their proof is based on a regularisation scheme via convolution of the metric g, combined with convergence results previously established in [69].

In Chapter 3, we will present the following original equivalence result from January 2024 obtained in [102].

Theorem 1 (see Theorem 3.6.3)**.** *Let M be a smooth manifold, g a continuous Riemannian metric with Christoffel symbols in L^2_{loc}, and $V \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$ a function on M. Define the weighted measure μ as $\mathrm{d}\mu := e^{-V}\mathrm{dvol}_g$. Let $N \in [n,\infty]$ and $K \in \mathbb{R}$. The following are equivalent:*

(i) *(M, d_g, μ) is a $\mathsf{CD}(K,N)$ space.*

(ii) *The distributional Bakry-Émery N-Ricci curvature tensor is bounded below by K and Asumption 3.6.1 is satisfied.*

The significant improvements to the previous result in [82] are:

- We do not need any compactness assumption on M.
- We dropped the regularity from C^1 (resp. $C^{1,1}$) to $C^0 \cap W^{1,2}_{\mathrm{loc}}$.
- We cover the weighted case and the case that $N < \infty$.

Assumption 3.6.1 is a volume growth and density assumption of technical nature. It is used to deduce the CD-condition from distributional Ricci curvature bounds. In January

2025, we proved Proposition 3.6.6 and Proposition 3.6.7, which state that Assumption 3.6.1 is implied by the lower distributional Ricci curvature bound if:

(i) The manifold is unweighted and $g \in W^{1,p}_{\mathrm{loc}}$ for some $p > n$.

(ii) The manifold is weighted and both the weight V and the metric tensor g are C^1.

In June 2024, Kunzinger, Ohanyan and Vardabasso uploaded the following independently obtained equivalence result [83]:

Theorem (Kunzinger-Ohanyan-Vardabasso)**.** *Let M be a smooth connected manifold equipped with a $C^{0,1}_{\mathrm{loc}}$-Riemannian metric g. If $(M, \mathsf{d}_g, \mathrm{vol}_g)$ is a $\mathsf{RCD}(K,\infty)$-space, then M has distributional Ricci curvature bounded below by K. Conversely, if M satisfies $\mathrm{Ric} \geq K$ in the distributional sense, then $(M, \mathsf{d}_g, \mathrm{vol}_g)$ is a $\mathsf{RCD}(K, \dim M)$-space.*

Note that this result does not need a volume growth and density assumption, but only holds for the unweighted case and for locally Lipschitz continuous metric tensors. Then, (i) in the above discussion of the volume growth and density condition (Assumption 3.6.1) shows that Theorem 1 is the most general equivalence result obtained so far.

Let us quickly comment on the significance of Theorem 1. Firstly, it supports the compatibility of the synthetic and analytic approaches to beyond the smooth setting. Secondly, note that many proofs in smooth Riemannian geometry use the exponential map, Jacobi fields and related notions that require smoothness of the metric g. Our result now allows to apply the rich theory on RCD-spaces to manifolds with low regularity Riemannian metrics that satisfy distributional lower Ricci curvature bounds, as the smooth tools cannot be applied in that setting. Furthermore, our result was used in [49] as an ingredient of a technical proposition that contributes to proving topological restrictions of $\mathsf{RCD}(0,N)$-spaces.

In the following, we discuss the organisation of Chapter 3 and the main steps to prove Theorem 1.

The **main ingredient** of the proof is the equivalence of the Riemannian curvature-dimension condition and the Bakry-Émery-condition that states that for a metric measure space $(M, \mathsf{d}, \mathfrak{m})$, the $\mathsf{RCD}^*(K,N)$-condition (resp. the $\mathsf{RCD}(K,\infty)$-condition) implies the $\mathrm{BE}(K,N)$-condition (resp. the $\mathrm{BE}(K,\infty)$-condition), the latter being a synthetic Bochner inequality (see also Definition 3.4.3) for metric measure spaces, stating that

$$\mathbf{\Gamma}_2(f,f) \geq \Big(K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2\Big)\mathfrak{m}, \tag{1.3.3}$$

for all synthetic test functions $f \in \mathrm{TestF}(M)$ (for details see Section 3.4). Here, one can think of $\mathbf{\Gamma}_2(f,f)$ as a synthetic version of the quantity $\mathrm{Ric}(\nabla f, \nabla f) + |\nabla^2 f|^2$. The equivalence was obtained by Ambrosio-Gigli-Savaré in [9] for the case $N = \infty$ and by Erbar-Kuwada-Sturm [55] and Ambrosio-Mondino-Savaré [10] for the case $N < \infty$ (see Theorem 3.4.4).

We remark that due to a deep result by Cavalletti-Milman [32] (see also [90]), we know that $\mathsf{RCD}(K, N)$ and $\mathsf{RCD}^*(K, N)$ are equivalent, hence the $\mathsf{RCD}(K, N)$-condition implies the $\mathrm{BE}(K, N)$-condition.

The **first step** is to establish the equivalence for the case $N = \infty$.

Gigli used the equivalence of $\mathsf{RCD}(K, \infty)$ and $\mathrm{BE}(K, \infty)$ (proved previously in [9]) to develop a synthetic second order calculus building on (1.3.3) for $\mathsf{RCD}(K, \infty)$-spaces in [63]. This lead to the definition of a synthetic measure valued Ricci curvature tensor $\mathbf{Ric}$, which satisfies

$$\mathbf{Ric}(X, X) \geq |X|^2 \mathfrak{m} \tag{1.3.4}$$

in the sense of measures for all synthetic test vector fields X (see Theorem 3.4.20, [63, Theorem 3.6.7]).

A manifold with a low regularity metric, however, has distributional Ricci curvature bounded from below by K if

$$\mathrm{Ric}(X, X) \geq K g(X, X) \text{ in } \mathcal{D}', \tag{1.3.5}$$

for all classical test vector fields $X \in C^\infty(M)$.

This suggests the following **strategy**: For a manifold M with a $C^0 \cap W^{1,2}_{\mathrm{loc}}$-Riemannian metric, we compare the synthetic calculus and test objects on the metric measure space $(M, \mathsf{d}_g, \mathrm{vol}_g)$ with the classical calculus and test objects on manifolds to establish the equivalence between synthetic and distributional lower Ricci curvature bounds.

As a starting point, we recall useful approximation results using the mollification on manifolds in Section 3.1.

Thereafter, we recall useful notions from calculus on weighted Riemannian manifolds and then formulate a distributional version of Bochner's formula for weighted Riemannian manifolds in Section 3.2.

In Section 3.3, we compare the classical theory of Sobolev spaces on manifolds with the synthetic first order calculus on metric measure spaces. We observe that for manifolds M with continuous Riemannian metrics g, the synthetic Sobolev space $W^{1,2}_w(M)$ and the classical Sobolev space $W^{1,2}(M)$ are isometrically isomorphic (see Corollary 3.3.15), which in particular implies that $W^{1,2}_w(M)$ is a Hilbert space, making the induced metric measure space infinitesimally Hilbertian. This implies that in our setting, the $\mathsf{CD}(K,N)$-condition and the $\mathsf{RCD}(K,N)$-condition are equivalent.

In Section 3.4, we compare the second order calculus on $\mathsf{RCD}(K,\infty)$-spaces developed in [63] with the classical second order calculus on manifolds with low regularity metrics. Using the compatibility established in Section 3.3, we derive that in our setting $\mathbf{Ric}(X,X)$, as in (1.3.3) indeed equals the classical $\mathrm{Ric}(X,X)$ (see Proposition 3.4.21) in a weak sense and establish that several classical Sobolev $W^{2,2}$-functions, in particular C_c^∞-functions, can be approximated with synthetic test functions up to the second order (Proposition 3.4.22).

Section 3.5 includes the proof that RCD implies distributional lower Ricci curvature bounds. As mentioned before, the first step is to prove equivalence in the case $N=\infty$. This is achieved by proving that one can approximate C_c^∞-vector fields well enough with gradients of synthetic test functions $f \in \mathrm{TestF}(M)$ to obtain (1.3.5) from (1.3.4). Preliminary approximation results can be found in Subsection 3.5.1 and the proof of the implication $\mathsf{RCD}(K,\infty) \implies \mathrm{Ric} \geq Kg$ in $\mathcal{D}'$ is presented in Subsection 3.5.2, see Theorem 3.5.11. We conclude the first step by establishing that a lower bound on the distributional Ricci curvature tensor implies that the distributional Ricci curvature tensor is a distribution of order 0, thus a measure. This is done in Theorem 3.5.13.

This is crucial for the **second step:** The case $N<\infty$. As before, the strategy is to prove that the $\mathrm{BE}(K,N)$-condition (1.3.3) implies that the distributional Bakry-Émery N-Ricci tensor is bounded below by K, i.e.,

$$\mathrm{Ric}_{\mu,N} \geq Kg \text{ in } \mathcal{D}'. \tag{1.3.6}$$

Note that for finite N, the distributional Bakry-Émery N-Ricci tensor (see (1.3.2)) contains a nonlinearity, which needs to be handled delicately. The fact that $\mathsf{RCD}(K,N)$ implies $\mathsf{RCD}(K,\infty)$ allows us to use Theorem 3.5.11 with Theorem 3.5.13 to infer that the distributional Ricci tensor $\mathrm{Ric} = \mathrm{Ric}_{\mathrm{vol}_g,\infty}$ (defined as in (1.3.1)) is a measure, rather than a distribution of order 1. This allows us to locally approximate vector fields with gradients of smooth functions, which we choose in a way that (1.3.3) translates into (1.3.6) in the sense of measures. Another approximation argument based on the Besicovitch covering theorem (Lemma 3.5.18) allows us to globalise and conclude that $\mathsf{RCD}(K,N)$ implies $\mathrm{Ric}_{\mu,N} \geq Kg$ in $\mathcal{D}'$ in Theorem 3.5.20.

Finally, the reverse implication is proved in Section 3.6 using that the $\mathrm{BE}(K,N)$-condition, together with a volume growth condition (2.2.1) implies the $\mathsf{RCD}(K,N)$-condition. Passing from distributional Ricci bounds to the $\mathrm{BE}(K,N)$ condition is immediate through approximation with the compatibility results established in Section 3.3. This is done in Theorem 3.6.3. Thanks to a result by Ketterer on L^p-Ricci curvature bounds [79], we can use the approximation results in Section 3.1 to drop Assumption 3.6.1 whenever the metric and weight are in C^1, or in the unweighted case if $g \in W^{1,p}_{\mathrm{loc}}$ for some $p > n$, (see Proposition 3.6.6 and Proposition 3.6.7).

1.3.2 The infinitesimal structure of manifolds with non-continuous Riemannian metrics

In Chapter 4, we will present the results from the solo-authored paper [119], where we analyse metric measure spaces arising from manifolds endowed with non-continuous Riemannian metrics, i.e., where the metric tensor g satisfies merely $g, g^{-1} \in L^\infty_{\mathrm{loc}}$.

The project emerged from the question, whether Theorem 1 can be extended to metric tensors, that are of regularity below C^0. A crucial ingredient to prove Theorem 1 is the first order calculus on metric measure spaces arising from manifolds equipped with non-smooth Riemannian metrics [102, Section 4], here Subsection 3.3.2.

Recall that a smooth d-manifold M, equipped with a continuous Riemannian metric g can naturally be seen as a metric measure space, where the measure is induced by the volume form (locally given via $\mathrm{dvol}_g = \sqrt{\det g}\,\mathrm{d}\mathcal{L}^d$) and the Riemannian distance d_g is defined as the infimum over lengths of piecewise smooth curves (see (1.2.1)).

However, there are several relevant classes of Riemannian metrics that are non-continuous, for instance the Geroch-Traschen's one [60], i.e., those Riemannian metrics g such that $g, g^{-1} \in L^\infty_{\mathrm{loc}}(M)$ and $g \in W^{1,2}_{\mathrm{loc}}(M)$. These regularity assumptions are minimal to compute the distributional Ricci curvature on M (see [75], [69]) and have been studied in both Riemannian (see for example [83]) and Lorentzian signature (see [130]). Moreover, in the context of potential theory for uniformly elliptic operators, the even broader class of (Riemannian) metrics g such that $g, g^{-1} \in L^\infty_{\mathrm{loc}}(M)$ has been studied in [121], [131], [107], and [51].

If the metric tensor is merely measurable, Norris (cf. [107]) and De Cecco-Palmieri (cf. [51]) defined a distance d_g overcoming the difficulty that piecewise smooth curves γ might lie on a null set where g is not well-defined (see Section 4.1).

Chapter 4 is driven by the following questions, which arise naturally from the results on smooth Riemannian manifolds and compare analytic and geometric approaches to first order calculus:

Question 1 (see also [53]). *Under which regularity assumptions on g does (M, d_g, g) satisfy that for any Lipschitz continuous function $f\colon M \to \mathbb{R}$ its slope equals the norm of its gradient almost everywhere, i.e., $|Df|(x) = \limsup_{y\to x} \frac{|f(y)-f(x)|}{\mathsf{d}_g(x,y)} = |\nabla_g f|_g(x)$ almost everywhere?*

This question has been posed by De Giorgi in [53] as the positive answer would be a necessary condition for (M, d_g, g) to be a quasi-Riemannian manifold, as defined in [53], see also Definition 4.1.6. A related question:

Question 2. *Under which regularity assumptions on g does the metric speed of "almost every" absolutely continuous curve equal the norm of its derivative, i.e., $\lim_{h\to 0} \frac{\mathsf{d}_g(\gamma_t,\gamma_{t+h})}{|h|} = |\dot\gamma_t|_g$ for almost every $t \in [0,1]$?*

In this case "almost every" curve is to be interpreted in the sense of test plans, i.e. probablilty measures $\boldsymbol{\pi}$ on $C([0,1], M)$ such that for each time $t \in [0,1]$, the "evaluation" of $\boldsymbol{\pi}$ at time t is absolutely continuous with respect to vol_g (see Subsection 2.2.1 for the precise definition).

Question 3. *Under which regularity assumptions on g is $(M, \mathsf{d}_g, \mathrm{vol}_g)$ infinitesimally Hilbertian?*

This particular question refers back to the original purpose of infinitesimal Hilbertianity to identify the Riemannian-like structures among metric measure spaces.

It is already known that assuming $g \in C^0$ is sufficient for the second and third question to have positive answers (see [25], [102] and Section 3.3). Moreover, it is known that for $g \in C^0$ and $f \in C^1$, the slope $|Df|$ of f equals the norm of its gradient [102].

In [132], Sturm proved the following theorem:

Theorem (Sturm). *For any Lipschitz manifold M and Riemannian metric g on M such that $g, g^{-1} \in L^\infty_{\mathrm{loc}}$, there exists a Riemannian metric $\tilde g$ on M such that $\tilde g, \tilde g^{-1} \in L^\infty_{\mathrm{loc}}$, $\mathsf{d}_g = \mathsf{d}_{\tilde g}$ and such that $\tilde g < g$ in the sense of forms.*

Sturm embedded this result in the context of diffusion processes, but it indeed shows that there exist regularity assumptions on Riemannian metrics such that Questions 1 and 2 may have a negative answer. An example by De Cecco and Palmieri in [52] shows that there exists a non-continuous Riemannian metric g on $\mathbb{R}^2$ with $g, g^{-1} \in L^\infty_{\mathrm{loc}}$ such that on a set of positive measure, the tangents are normed spaces with a non-Euclidean norm. This hints that Question 3 may have a negative answer in low regularity.

Assuming additional $W^{1,p}_{\mathrm{loc}}$-Sobolev regularity, with growing exponent p, can be seen as a bridge from L^∞_{loc}-metrics to C^0-metrics. Indeed, for $p < p'$, it holds $W^{1,p'}_{\mathrm{loc}} \subset W^{1,p}_{\mathrm{loc}}$

and by the Sobolev embedding theorem, a Riemannian metric g on a manifold M with $g \in W^{1,p}_{\mathrm{loc}}$, if $p > \dim M$, is automatically continuous. This suggests that additional Sobolev regularity could be a discriminant for Questions 1 to 3.

In Chapter 4, we discuss a related example to [52] that facilitates the study of infinitesimal Hilbertianity, and that allows a refinement to demonstrate that in at least 3 dimensions the above questions still may have negative answers if we additionally assume Sobolev regularity of the metric tensor.

Questions 1 and 2 are then addressed in Subsubsection 4.2.3.1, where we prove:

Theorem 2. *For $d \geq 3$, $p \in [1, d-1]$ there exists a d-dimensional manifold M and a Riemannian metric g on M with $g, g^{-1} \in L^\infty$, $g \in W^{1,p}_{\mathrm{loc}}(M)$ and such that:*

(i) There exists a test plan that gives positive measure to curves for which the metric speed is strictly smaller than the norm of the derivative for almost every time $t \in [0,1]$.

(ii) There exists a function $f \in C^1(M)$ for which the minimal weak upper gradient and the metric slope are strictly larger than the norm of the gradient on a set of positive measure, i.e., $|Df|, |Df|_w > |\nabla_g f|_g$ on a set of positive vol_g-measure.

The minimal weak upper gradient $|Df|_w$ is a generalisation of the modulus of the gradient of functions on metric measure spaces, which is relevant in the computation of the Cheeger energy and the definition of the synthetic Sobolev space $W^{1,2}_w$.

Let us comment on the difference between Sturm's Theorem, the example in [52], and our result: Sturm started with a given Riemannian metric g (possibly of low regularity), which induces the distance d_g and then constructed a metric $\tilde{g}$ that has the same distance but is strictly smaller as a quadratic form. This is achieved by selecting a dense set of points $\{x_n\}$ and ensuring that $\tilde{g}$ closely approximates g on a set of curves that connect the x_n to each other.

In [52], the authors construct a Riemannian metric on $\mathbb{R}^2$ that coincides with the Euclidean metric on a dense set of lines parallel to the coordinate axes and four times the Euclidean metric on "most" other points. This construction yields Finslerian tangents.

Our approach (see Section 4.2.3) is related to the one in [52], with the main difference that we only use one coordinate direction in which the Riemannian metric is small. We construct one specific non-continuous Riemannian metric g on $\mathbb{R}^d$ such that there exist many points at which short straight lines parallel to the d-th coordinate direction have Euclidean length and short straight lines in the other directions have (almost) twice the

Euclidean length. Beyond characterising the Lipschitz slopes of smooth functions and the metric speed of curves, this behaviour explicitly influences the properties of minimal weak upper gradients. This relationship is crucial to conclude that the space lacks infinitesimal Hilbertianity. In particular, this example allows to refine the construction to obtain this behaviour from a metric g that has additional Sobolev regularity.

Question 3 is addressed in Subsection 4.2.3.2, where we conclude:

Theorem 3. *For $d \geq 3$, $p \in [1, d-1]$, there exists a d-dimensional manifold (M, g) with $g, g^{-1} \in L^\infty_{\mathrm{loc}}$ and $g \in W^{1,p}_{\mathrm{loc}}(M)$ such that the metric measure space $(M, \mathsf{d}_g, \mathrm{vol}_g)$ is not infinitesimally Hilbertian.*

We quickly remark that in the case $d = 2$, our constructions, as well as the one in [52], allow to deduce statements analogous to Theorems 2 and 3, where we only have $g, g^{-1} \in L^\infty$ (see Propositions 4.2.1 and 4.2.6).

In the following, we present the organisation of Chapter 4. In Section 4.1, we discuss how the Riemannian distance d_g is defined if g is not continuous, as done by Norris [107] and De Cecco-Palmieri [51]. We recall under what assumptions their definitions are equal and formulate Assumption 4.1.1, that will be assumed throughout the entire chapter.

Thereafter, in Section 4.2, we construct examples where Questions 1 to 3 have negative answers. We will start with a Riemannian metric g on $\mathbb{R}^d$, with $d \geq 2$, $g, g^{-1} \in L^\infty_{\mathrm{loc}}$ that resembles the example in [52]. We will first quickly comment on Questions 1 and 2 in Subsection 4.2.1. Thereafter, in Subsection 4.2.2, we will thoroughly study the first order calculus by Ambrosio-Gigli-Savaré on these spaces and conclude that the space is not infinitesimally Hilbertian, addressing Question 3 (see Proposition 4.2.6). In Subsection 4.2.3, we will refine the previous example to a Riemannian metric g on $\mathbb{R}^d$, $d \geq 3$ such that $g, g^{-1} \in L^\infty \cap W^{1,d-1}_{\mathrm{loc}}$ that proves Theorems 2 and 3.

These results point out that the compatibility of the synthetic and classical first order calculus on manifolds with non-continuous metrics is rather subtle, and not true in general. Consequently, manifolds equipped with non-continuous Riemannian metrics, even those satisfying additional Sobolev regularity, require careful treatment, when simultaneously considering the distributional perspective and the metric measure space framework. In particular, the result points out that Geroch-Traschen metrics may not induce infinitesimally Hilbertian metric measure spaces showing that the minimal regularity assumptions to define distributional Ricci curvature are not sufficient for a compatible first order calculus. If one tried to extend Theorem 1 in at least 3 dimensions to non-continuous metrics, Theorem 3 indicates that more work than in Chapter 3 would be necessary.

1.3.3 Infinitesimal Minkowskianity of manifolds with continuous Lorentzian metrics

In the last part of the thesis, we will leave the Riemannian setting and turn to Lorentzian geometry. Chapter 5 is based on a solo authored paper draft [120].

A vast class of metric measure spacetimes arises from manifolds M endowed with a non-smooth Lorentzian metric g. They have been studied in [46, 122, 69, 29]. Such manifolds serve to describe various phenomena from general relativity, examples being the inside or outside of a star, spacetimes with conical singularities, and cosmic strings ([137, 138]). In Chapter 5, we study the case that g is continuous. A physically relevant example is provided by Penrose's impulsive gravitational waves (see for instance [111] and [71, Chapter 20]).

A $(d+1)$-manifold M, for $d \geq 1$, with a continuous Lorentzian metric g on M that is time orientable naturally enters the class of metric measure spacetimes as follows: The time separation $\ell_g \colon M^2 \to \{-\infty\} \cup [0, \infty]$ is defined as

$$\begin{aligned}&\ell_g(x,y)\\ &:= \sup\left\{\int_0^1 |\dot\gamma_t|_g \,\mathrm{d}t : \ \gamma \in C^{0,1}([0,1], M), \text{ causal and future directed, } \gamma_0 = x, \gamma_1 = y\right\}.\end{aligned}$$

Analogously to the Riemannian setting, the Lorentzian metric g induces a volume measure that is locally given by $\mathrm{dvol}_g = \sqrt{-\det g}\mathrm{d}\mathcal{L}^{d+1}$. We endow M with the manifold topology τ inherited by the charts and obtain the metric measure spacetime $(M, \tau, \ell_g, \mathrm{vol}_g)$.

Both metric measure spacetimes and manifolds endowed with continuous Lorentzian metrics can be seen as generalisations of smooth Lorentzian manifolds. Hence, as in the Riemannian case, we aim to study the compatibility between these two approaches. More precisely, the objective of this chapter is to compare the classical first order calculus on manifolds with the first order calculus on metric measure spacetimes developed in [16]. This work can be seen as a Lorentzian counterpart to [25] and [102, Section 4] (Section 3.3).

The main result of Chapter 5 is:

Theorem 4. *Let M be a $(d+1)$-dimensional manifold and g be a continuous Lorentzian metric on M such that (M, g) is a causally simple spacetime. Then the metric measure spacetime $(M, \tau, \ell_g, \mathrm{vol}_g)$ is infinitesimally Minkowskian.*

This can be seen as a Lorentzian version of Corollary 3.3.15 and [25]. It verifies the compatibility of the synthetic and the classical first order calculus within a class of manifolds that contains multiple relevant models from general relativity as pointed out in the history section. In particular, it can be applied to non-expanding impulsive gravitational waves, which is relevant to study the Lorentzian splitting theorem in low regularity [103].

The assumption that (M, g) is causally simple is of technical nature. Essentially, strong causality is relevant to the proofs, see Remark 5.2.3.

While the idea of the proof is similar to the Riemannian one, the Lorentzian setting contains more subtleties and difficulties. In the Riemannian case, it was sufficient to compute the minimal weak upper gradient for very regular C^1-functions (Subsection 3.3.2). Thereafter, the rich theory from [7] allowed us to pass to lower regularities and identify the minimal weak upper gradient of all synthetic Sobolev functions. Such a theory has not yet been developed in the Lorentzian setting, so more advanced techniques are required to work with general causal functions.

We refer to Section 2.3 for the relevant background on synthetic Lorentzian theory after [16]. In Section 5.1, we establish some measure theoretic tools that will be required for the proof of Theorem 4. The central result in that section is a Lebesgue differentiation theorem for singular measures (Lemma 5.1.2) which is obtained using the Besicovitch covering theorem.

In Section 5.2, we study how a manifold with a continuous Lorentzian metric enters the setting of metric measure spacetimes and compare the first order calculus on metric measure spacetimes with the classical one.

One first important step towards proving Theorem 4 is to verify that for any sufficiently regular causal curve $\gamma\colon [0,1] \to M$, and almost every $t \in [0,1]$, the causal speed $|\dot{\gamma}_t|$ which is almost everywhere given by $|\dot{\gamma}_t| = \lim_{s \searrow t} \frac{\ell(\gamma_t,\gamma_s)}{|s-t|}$ essentially coincides with the Lorentzian length of the derivative $|\dot{\gamma}_t|_g = \sqrt{-g(\dot{\gamma}_t, \dot{\gamma}_t)}$. This step is contained in Corollary 5.2.13 and can be seen as a Lorentzian counterpart of [25, Corollary 3.8].

To understand the maximal weak subslope of a causal function, a first relevant observation is that a causal function f is locally of bounded variation and allows a measure valued distributional derivative $\mathcal{D}f$. This has been observed in the smooth case in [16] and can similarly be obtained in the continuous case.

The distributional derivative can now be written as the sum of the singular and the absolutely continuous part (with respect to the Lebesgue measure on a chart) $\mathcal{D}f = Df\,\mathrm{d}\mathcal{L}^{d+1} + \mathcal{D}f^s$. Using the results from Section 5.1 allows us to localise away from $\mathcal{D}f^s$ so at any point of differentiability of f, we can identify the maximal weak subslope as the Lorentzian g-norm of the differential of f. The proof is provided in Proposition 5.2.19, where it is derived from Corollary 5.2.13 in conjunction with the equality case of the Lorentzian Cauchy-Schwartz inequality.

This allows to verify the parallelogram identity for weak subslopes and deduce Theorem 4.

1.4 Statement of originality

I declare that all the work in my thesis is my own, except where explicitly stated otherwise. Chapter 3 contains the results from a joint work with Andrea Mondino. Section 1.3.1 (part of the introduction) contains an introduction to those results and a summary of the main ideas.

Chapter 2 contains preliminary definitions and results collected from several standard references in the field.

All other chapters contain my own work.

Chapter 2

Preliminaries

This chapter is dedicated to summarising the basic notions and materials that will be used throughout this thesis. This includes an introduction to distributions on manifolds, a summary of the theory on metric measure spaces and synthetic lower Ricci curvature bounds, and an introduction to first order calculus on metric measure spacetimes.

2.1 Distributional calculus on manifolds with semi-Riemannian metrics of low regularity

In this section, we recall some elements of the theory of distributions on manifolds, including the distributional Riemann tensor; for more details, we refer to [69], [75], [82], and [87].

Let M be a smooth real manifold of dimension n without boundary, and let (E, M, π) be a vector bundle over M. To keep notation short, we will simply write E instead of (E, M, π).

In this thesis, we will always consider smooth manifolds M and fix a smooth atlas $\mathbb{A} := (V_\alpha, \psi_\alpha)_\alpha$. The definitions and results in this thesis do not depend on $\mathbb{A}$ specifically, but on the smooth structure, i.e., the class of atlases of M that are smoothly equivalent to $\mathbb{A}$.

For $0 \leq k \leq \infty$, we denote by $\Gamma^k(M, E)$ (resp. $\Gamma^k_c(M, E)$) the space of C^k-sections (resp. with compact support). In the case $k = \infty$, we often drop the superscript to simplify the notation. We denote $T^r_s(M) := (\bigotimes_{i=1}^r TM) \otimes (\bigotimes_{i=1}^s T^*M)$ and $\mathcal{T}^r_s := \Gamma(M, T^r_s(M))$. We will also write $\mathfrak{X}(M)$ for $\mathcal{T}^1_0(M)$ and $\mathfrak{X}^*(M)$ for $\mathcal{T}^0_1(M)$. For some vector bundle F and $V \subset M$ open, we say that a sequence ω_j converges to ω in $\Gamma^k_c(V, F)$ if there exists a compact set $K \subset V$ such that $\operatorname{supp} \omega_j, \operatorname{supp} \omega \subset K$ and $\nabla^l \omega_j \to \nabla^l \omega$ uniformly on K for all $l \leq k$.

We denote by $\mathrm{Vol}(M)$ the vector bundle of 1-densities, i.e., the one dimensional vector bundle with transition functions $\Psi_{\alpha\beta}(x) = |\det D(\psi_\beta \circ \psi_\alpha^{-1})(x)|$, where ψ_α, ψ_β denote local charts into $\mathbb{R}^n$. A section $\omega \in \Gamma_c^k(M, \mathrm{Vol}(M))$ is called a *test volume.* The *space of distributions of order k on M* is defined as the topological dual of $\Gamma_c^k(M, \mathrm{Vol}(M))$ (see [75, Section 3.1]), i.e.,

$$\mathcal{D}'^{(k)}(M) := \Gamma_c^k(M, \mathrm{Vol}(M))'.$$

The space of distributional (r,s)-tensor fields of order k is given by

$$\mathcal{D}'^{(k)}\mathcal{T}_s^r(M) := \Gamma_c^k(M, T_r^s(M) \otimes \mathrm{Vol}(M))'.$$

It is known that

$$\mathcal{D}'^{(k)}\mathcal{T}_s^r(M) \cong \mathcal{D}'(M) \otimes_{C^k(M)} \mathcal{T}_s^r(M) \cong L_{C^k(M)}(\mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s; \mathcal{D}'(M)),$$

where the latter denotes the $C^k(M)$-module of $C^k(M)$-multilinear maps from $\mathfrak{X}^*(M)^r \times \mathfrak{X}(M)^s$ to $\mathcal{D}'^{(k)}(M)$ (cf. [75]).

Definition 2.1.1 ([87], Action of a vector field on a distribution)**.** *For a scalar distribution $T \in \mathcal{D}'(M)$ and a vector field X, we define the action of X on T via*

$$\langle X(T), \omega \rangle_{\mathcal{D}',\mathcal{D}} := -\langle T, \mathcal{L}_X \omega \rangle_{\mathcal{D}',\mathcal{D}},$$

where ω is a test volume and $\mathcal{L}_X \omega$ denotes the Lie derivative of ω in the direction of X.

In the following, we will quickly the results of the discussion in [75], Chapter 3.1.4, as they will allow is to locally identify distributions on manifolds with distributions on $\mathbb{R}^n$.

Let (V_α, ψ_α) be an atlas for M and E be a vector bundle of dimension n'. We denote by $\Psi_\alpha : \pi^{-1}(V_\alpha) \to \psi_\alpha(V_\alpha) \times \mathbb{R}^{n'}$ the local trivialisations of E and by $g_{\alpha\beta} : V_\alpha \cap V_\beta \to \mathrm{GL}(n', \mathbb{R})$ the transition functions. For a vector bundle isomorphism $\Phi : E \to F$ along the diffeomorphism $\phi : M \to N$, we define the map $\Phi_* : \Gamma_c(M, E) \to \Gamma_c(N, F)$ via $\Phi_* : u \mapsto \Phi \circ u \circ \phi^{-1}$ and $\Phi^\wedge : \Gamma_c'(M, E) \to \Gamma_c'(N, F)$ by $\langle \Phi^\wedge T, u \rangle := \langle T, \Phi_*^{-1} u \rangle$, where we wrote Φ_*^{-1} for $(\Phi^{-1})_* = (\Phi_*)^{-1}$. For the local isomorphisms Ψ_α this yields

$$\begin{gathered}\Psi_\alpha^\wedge : \mathcal{D}'(V_\alpha, E) \to \mathcal{D}'(\psi_\alpha(V_\alpha), \mathbb{R}^{n'}) \\ \langle \Psi_\alpha^\wedge(T), u \rangle := \langle T, (\Psi_\alpha)_*^{-1} u \rangle.\end{gathered}$$

So to any distribution $T \in \mathcal{D}'(M,E)$, we may assign the family $(T_\alpha)_\alpha := \Psi_\alpha^\wedge(T|_{V_\alpha}) \in \mathcal{D}'(\psi_\alpha(V_\alpha), \mathbb{R}^{n'})$. satisfying

$$\langle T, u\rangle = \langle T_\alpha, u_\alpha\rangle = \sum_{i=1}^{n'} \langle T_{\alpha\ i}, u_\alpha^i\rangle,$$

for $\operatorname{supp}(u) \subset\subset V_\alpha$.

Theorem 2.1.2 ([75], Theorem 3.1.9)**.** *Elements of $\mathcal{D}'^{(k)}(M,E)$ can be identified with families $(T_\alpha)_\alpha$ of distributions $T_\alpha \in \mathcal{D}'(\psi_\alpha(V_\alpha), \mathbb{R}^{n'})$ satisfying the following transformation law*

$$T_\alpha = (\Psi_\alpha \circ \Psi_\beta^{-1})^\wedge(T_\beta).$$

Using a partition of unity, it is now clear that one can localise and reduce the computations to distributions on $\mathbb{R}^n$.

The rest of this section is tailored to non-smooth Riemannian metrics.

Definition 2.1.3 ([87], Definition 4.1)**.** *A distribution $g \in \mathcal{D}'\mathcal{T}_2^0(M)$ is called a generalised semi-Riemannian metric if it satisfies*

$$\begin{aligned} &g(X,Y) = g(Y,X) \quad \text{in } \mathcal{D}'(M) \quad \text{and} \\ &g(X,Y) = 0 \quad \forall Y \in \mathfrak{X}(M) \quad \implies \quad X = 0. \end{aligned}$$

In this thesis, we will consider distributional semi-Riemannian metrics with additional regularity assumptions, that require the following the definition.

Definition 2.1.4. *Let M be a smooth manifold, $k, s, r \in \mathbb{N}_0$ and $p \in [1, \infty]$. We say that a distribution $T \in \mathcal{D}'\mathcal{T}_s^r(M)$ is of regularity $W_{\mathrm{loc}}^{k,p}(M)$, if for every local trivialisation (V_α, ψ_α), it holds that $T_\alpha := \Psi_\alpha^\wedge(T|_{V_\alpha}) \in \mathcal{D}'(\psi_\alpha(V_\alpha), \mathbb{R}^{(r+s)n})$ is a regular distribution and $T_\alpha \in W_{\mathrm{loc}}^{k,p}(\psi_\alpha(V_\alpha), \mathbb{R}^{(r+s)n})$.*

Note that whenever one would like to take a $W^{k,p}$-norm, it will depend on the specific chart we have chosen. However, the local finiteness does not depend on the chart, as it is preserved under smooth coordinate transformations.

In order to generalise the Riemann curvature tensor, it is appropriate to consider a Riemannian metric that is locally given as a positive definite matrix such that the coefficients g_{ij} and the coefficients of the inverse matrix g^{ij} are both in $L_{\mathrm{loc}}^\infty \cap H_{\mathrm{loc}}^1$ (cf. [75, Definition 5.2.1]).

The class of Riemannian metrics satisfying these regularity assumptions is called the *Geroch-Traschen* class [60, 130]. This condition is automatically met if $g \in C^0(M)$ and admits L^2_{loc}-Christoffel symbols (see Lemma 3.2.8).
In order to define a distributional covariant derivative, one generalises the first Koszul formula. For smooth vector fields X, Y, Z, one can define $\nabla^\flat_X Y \in \mathcal{D}'\mathcal{T}^0_1(M)$ via

$$\begin{aligned}\langle \nabla^\flat_X Y, Z\rangle := \frac{1}{2}\big(&X(g(Y,Z)) + Y(g(X,Z)) - Z(g(X,Y)) \\ &- g(X,[Y,Z]) - g(Y,[X,Z]) + g(Z,[X,Y])\big),\end{aligned}$$

in $\mathcal{D}'(M)$. Raising the index gives the distributional covariant derivative $\nabla_X Y := (\nabla^\flat_X Y)^\sharp$ and a computation in local coordinates yields that indeed

$$(\nabla_X Y)_s = \delta^s_i (X^k \partial_k Y^i) + \Gamma^s_{ik} X^i Y^k \text{ in } \mathcal{D}'(\mathbb{R}^n).$$

For a 1-form θ, the distributional covariant derivative is defined via

$$\langle \nabla_X \theta, Z\rangle := X(\langle \theta, Z\rangle) - \langle \theta, \nabla_X Z\rangle \text{ in } \mathcal{D}'(M),$$

which, when spelled out in local coordinates, coincides with the classical covariant derivative of a 1-form. We have all the ingredients to define the distributional Riemann tensor. Here, it is crucial that the coefficients of the covariant derivatives are in L^2_{loc}, as the expression of the Riemann tensor involves a quadratic term in the Christoffel symbols. Following the notation of [87, Definition 3.3], we have that

$$\begin{aligned}\langle \mathrm{Riem}(X,Y)Z, \theta\rangle = &X\langle \nabla_Y Z, \theta\rangle - Y\langle \nabla_X Z, \theta\rangle - \langle \nabla_Y Z, \nabla_X \theta\rangle_{L^2} \\ &+ \langle \nabla_X Z, \nabla_Y \theta\rangle_{L^2} - \langle \nabla_{[X,Y]} Z, \theta\rangle\end{aligned}$$

in $\mathcal{D}'(M)$. All the quantities on the right hand side can be computed locally. Thus, in local coordinates, the Riemann tensor can be written as

$$R^l_{ijk} = \partial_i \Gamma^l_{jk} - \partial_j \Gamma^l_{ik} + \Gamma^s_{kj}\Gamma^l_{is} - \Gamma^s_{ki}\Gamma^l_{js}, \tag{2.1.1}$$

in the sense of distributions on $\mathbb{R}^n$. Note that in local coordinates, the components of X, Y, Z and θ "are part of the test function". If g is a smooth Riemannian metric, the expression (2.1.1) coincides with the classical Riemann tensor (cf. [112]). For a metric g with $g, g^{-1} \in L^\infty_{\mathrm{loc}}$ admitting L^2_{loc}-Christoffel symbols, let e_i be a $L^\infty \cap W^{1,2}_{\mathrm{loc}}$-local orthonormal frame in $L^\infty(TM)$. Note that this exists by an argument similar to the proof of Lemma 3.2.8. For two smooth vector fields X, Y the distributional Ricci tensor evaluated at X and Y is then locally given by

$$\mathrm{Ric}(X,Y) = \sum_i g(e_i, \mathrm{Riem}(e_i, X)Y).$$

It is immediate to check that Ric is a symmetric bilinear form that does not depend on the choice of e_i (cf. [82, 75]) and that can be expressed in local coordinates and the components are given by

$$\mathrm{Ric}_{jk} = R^p_{pjk} = \partial_p \Gamma^p_{jk} - \partial_j \Gamma^p_{pk} + \Gamma^s_{kj}\Gamma^p_{ps} - \Gamma^s_{kp}\Gamma^p_{js}, \tag{2.1.2}$$

which coincides with the classical expression in the smooth case. This expression respects smooth coordinate transformations and can hence via Theorem 2.1.2 be lifted to a distribution $\mathrm{Ric} \in \mathcal{D}'\mathcal{T}^0_2(M)$. We next define distributional lower bounds of the Ricci curvature. We say that a one-density is non-negative if it can be written as $\phi \mathrm{vol}_g \in \mathrm{Vol}(M)$ for some non-negative $\phi \in C^\infty(M)$.

Definition 2.1.5. *Let $K \in \mathbb{R}$. We say that a Riemannian metric g with $g, g^{-1} \in L^\infty_{\mathrm{loc}}$ and L^2_{loc}-Christoffel symbols satisfies* Ric $\geq K$ *in the* distributional sense *if, for all $X \in \mathfrak{X}(M)$, it holds* $\mathrm{Ric}(X,X) \geq Kg(X,X)$ *in $\mathcal{D}'(M)$, i.e., for all non-negative test volumes ω it holds,*

$$\langle \mathrm{Ric}(X,X), \omega \rangle_{\mathcal{D}',\mathcal{D}} \geq \int Kg(X,X)\omega.$$

We conclude this section by recalling how to perform convolutions on a manifold. Fix a function $\rho \in C^\infty_c(\mathbb{R}^n)$ such that $\operatorname{supp}\rho \subset B_1(0)$, $\rho \geq 0$, and $\int_{\mathbb{R}^n} \rho = 1$. Define $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$. For a smooth manifold M, fix an atlas (U_α, ψ_α) such that $(U_\alpha)_\alpha$ is locally finite and each U_α is relatively compact, and fix a partition of unity η_α subordinate to the atlas. Choose a family of smooth cutoff functions χ_α such that $\operatorname{supp}\chi_\alpha \subset U_\alpha$ and $\chi_\alpha = 1$ on $\operatorname{supp}\eta_\alpha$. For any tensor $T \in L^1_{\mathrm{loc}}(T^r_s(M))$, define

$$T * \rho_\varepsilon(x) := \sum_\alpha \chi_\alpha(x)\psi^*_\alpha\big((\psi_{\alpha *}(\eta_\alpha T)) * \rho_\varepsilon\big)(x). \tag{2.1.3}$$

Note that all these computations work similarly for all semi-Riemannian metrics g that satisfy $g, g^{-1} \in L^\infty_{\mathrm{loc}}$, $g \in W^{1,2}_{\mathrm{loc}}$.

2.2 Metric measure spaces

We will split this section on metric measure spaces into two parts. First, we recall the Cheeger energy and the related first order calculus on metric measure spaces. Thereafter, we will quickly recall the relevant notions on optimal transport to define synthetic lower Ricci curvature bounds. The structure of this section has been inspired by [102].

2.2.1 First order calculus on metric measure spaces

In this subsection we recall the generalised first order calculus on metric measure spaces, as it was introduced by Cheeger in [38] and further analysed by Ambrosio, Gigli, and Savaré [7].

Throughout the section, (X, d) is a Polish, i.e., complete and separable metric space. The slope (or local Lipschitz constant) of a real valued function $F\colon X \to \mathbb{R}$ is defined by

$$|DF|(x) := \limsup_{y\to x} \frac{|F(x) - F(y)|}{\mathsf{d}(x, y)},$$

if $\{x\}$ is not isolated, and 0 otherwise.

We endow (X, d) with a non-negative σ-finite Borel measure, obtaining the metric measure space $(X, \mathsf{d}, \mathfrak{m})$. Throughout the rest of this work, we assume that there exists a bounded Borel Lipschitz map $V : X \to [0, \infty)$ such that (cf. [7, Chapter 4])

$$V \text{ is bounded on each compact set } K \subset X \text{ and } \int_X e^{-V^2}\, \mathrm{d}\mathfrak{m} \le 1. \tag{2.2.1}$$

Definition 2.2.1. *A curve $\gamma \in C^0([0, 1], X)$ is said to be* absolutely continuous *if there exists a $g \in L^1((0, 1))$ such that for all $0 \le s < t \le 1$, it holds $\mathsf{d}(\gamma_s, \gamma_t) \le \int_s^t g(r)\,\mathrm{d}r$. The* metric speed *of an absolutely continuous curve $\gamma \in C^0([0, 1], X)$ is given by*

$$|\dot\gamma_t| := \lim_{h\to 0} \frac{\mathsf{d}(\gamma_{t+h}, \gamma_t)}{|h|},$$

which exists for almost every $t \in [0, 1]$.

We next recall the notion of test plan and weak upper gradient. We will use the conventions of [63], after [7].

Given a topological space U, we denote by $\mathcal{P}(U)$ the space of Borel probability measures on U. Given two topological spaces U_1, U_2, a Borel measurable map $T : U_1 \to U_2$, and a Borel measure μ on U_1, the *pushforward* measure $T_\#\mu$ on U_2 is the Borel measure defined by $T_\#\mu(B) = \mu(T^{-1}(B))$ for any Borel set $B \subset U_2$.

Given a $t \in [0, 1]$, the evaluation map $e_t : C^0([0, 1], X) \to X$ is defined via $e_t(\gamma) := \gamma_t$.

Definition 2.2.2. *Let $\boldsymbol{\pi} \in \mathcal{P}(C^0([0, 1], X))$. We say that $\boldsymbol{\pi}$ is a* test plan *if there exists a constant $C(\boldsymbol{\pi}) > 0$ such that*

$$(e_t)_\#\boldsymbol{\pi} \le C(\boldsymbol{\pi})\mathfrak{m} \text{ for all } t \in [0, 1]$$

and

$$\int \int_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty.$$

We use the convention that if γ is not absolutely continuous, then $\int_0^1 |\dot\gamma_t|^2 \,\mathrm{d}t = \infty$.

Definition 2.2.3. *Given $f\colon X \to \mathbb{R}$ a $\mathfrak{m}$-measurable function, then a $\mathfrak{m}$-measurable function $G\colon X \to [0,\infty]$ is called a weak upper gradient of f, if*

$$\int |f(\gamma_1) - f(\gamma_0)| \mathrm{d}\boldsymbol{\pi}(\gamma) \le \int \int_0^1 G(\gamma_t)|\dot\gamma_t| \,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty, \quad \text{for all test plans } \boldsymbol{\pi}. \tag{2.2.2}$$

We say that f is in the Sobolev class $S^2(X,\mathsf{d},\mathfrak{m})$ if there exists $G \in L^2(\mathfrak{m})$ such that (2.2.2) *holds.*

The discussion in [7, Proposition 5.9 and Definition 5.11] shows the existence of a weak upper gradient $|Df|_w$ such that $|Df|_w \le G$ $\mathfrak{m}$-a.e. for all other weak upper gradients G. We will call it the *minimal weak upper gradient* of f.

Definition 2.2.4. *The Sobolev space $W^{1,2}_w(X,\mathsf{d},\mathfrak{m})$ is defined as $L^2 \cap S^2(X)$ and becomes a Banach space with the norm*

$$\|f\|^2_{W^{1,2}_w(X)} = \|f\|^2_{L^2} + \||Df|_w\|^2_{L^2}.$$

For simplicity, we will from now on write $W^{1,2}_w(X)$ instead of $W^{1,2}_w(X,\mathsf{d},\mathfrak{m})$.

Definition 2.2.5. *The* Cheeger energy *is defined in the class of $\mathfrak{m}$-measurable functions by*

$$\mathrm{Ch}(f) := \begin{cases} \frac{1}{2}\int_X |Df|^2_w \,\mathrm{d}\mathfrak{m} & \text{if } f \text{ has a weak upper gradient in } L^2(X,\mathfrak{m}), \\ \infty & \text{otherwise,} \end{cases}$$

with proper domain $D(\mathrm{Ch}) = \{f : X \to [0,\infty], \mathfrak{m}\text{−measurable}, \ \mathrm{Ch}(f) < \infty\}$.

Note that $D(\mathrm{Ch}) = S^2(X,\mathsf{d},\mathfrak{m})$. A metric measure space is called *infinitesimally Hilbertian* if Ch is a quadratic form or, equivalently, if $W^{1,2}_w(X)$ is a Hilbert space [8, 62]. In this case, we can define the associated Dirichlet form $\mathcal{E}\colon D(\mathrm{Ch})^2 \to \mathbb{R}$, via $\mathcal{E}(f,f) = 2\mathrm{Ch}(f)$ [8, Chapter 4.3].

Proposition 2.2.6 ([8], Definition 4.12 and Proposition 4.14)**.** *For any $f,g \in D(\mathrm{Ch})$, it holds*

$$\mathcal{E}(f,g) = \int_X G(f,g)\,\mathrm{d}\mathfrak{m},$$

where

$$G(f,g) = \lim_{\varepsilon \searrow 0} \frac{|D(f+\varepsilon g)|^2_w - |Df|^2_w}{2\varepsilon} \in L^1(X,\mathfrak{m}).$$

By the theory of Dirichlet forms (cf. [19]), there exists an associated Laplace operator $\Delta\colon D(\Delta) \to L^2(X, \mathfrak{m})$, given by

$$-\int_X \Delta f h \,\mathrm{d}\mathfrak{m} = \int_X G(f,h)\,\mathrm{d}\mathfrak{m}.$$

Here $D(\Delta) = \{f \in L^2(\mathfrak{m}) : \Delta f \text{ exists}, \ \Delta f \in L^2\}$ is dense in L^2. Moreover, it induces a linear semi-group $(H_t)_{t\geq 0}$, $H_t\colon L^2(X) \to L^2(X)$ such that $H_{t+s} = H_s H_t$, for all $f \in L^2$, we have that $\lim_{t\to 0} H_t f = f$, and for all $t > 0$, we have that $H_t f \in D(\Delta)$ and $\frac{\mathrm{d}}{\mathrm{d}t} H_t f = \Delta H_t f$. $(H_t)_{t\geq 0}$ is called *heat flow* semi-group. Write f_t for $H_t(f_0)$. By [7, (4.26)], we have that

$$\mathrm{Ch}(f_t) \leq \inf\left\{\mathrm{Ch}(g) + \frac{1}{2t}\int_X |g - f_0|^2 \,\mathrm{d}\mathfrak{m} : \ g \in D(\mathrm{Ch})\right\}.$$

Another useful fact is the maximum principle for the heat flow [8, Proposition 2.14]: If $f \in L^2$, $f \leq C$, ($f \geq C$) $\mathfrak{m}$-a.e., then $H_t f \leq C$ ($H_t f \geq C$) $\mathfrak{m}$-a.e. in X.

2.2.2 Optimal transport and curvature-dimension conditions

In this subsection, we recall the definitions of several (Riemannian) curvature-dimension conditions, starting with some elements of optimal transport theory (cf. [5]). As before, (X, d) is a Polish metric space. Recall that $\mathcal{P}(X)$ denotes the set of Borel probability measures on X. Denote by $\mathcal{P}_2(X)$ the space of probability measures with finite second moment:

$$\mathcal{P}_2(X) := \left\{\mu \in \mathcal{P}(X) : \ \int d^2(x, x_0)\,\mathrm{d}\mu(x) < \infty \text{ for some } x_0 \in X\right\}.$$

For $\mu \in \mathcal{P}(X)$ and $\nu \in \mathcal{P}(Y)$, we define the set $\Gamma(\mu,\nu)$ as the set of all transport plans $\gamma \in \mathcal{P}(X \times Y)$, i.e., the set of Borel probability measures on $X \times Y$ such that $\pi^X_\# \gamma = \mu$ and $\pi^Y_\# \gamma = \nu$. Here π^X and π^Y denote the natural projections from $X \times Y$ onto X and Y respectively.

For $\mu, \nu \in \mathcal{P}_2(X)$, the *quadratic Wasserstein distance* W_2 is defined as

$$W_2(\mu,\nu) := \sqrt{\inf_{\gamma \in \Gamma(\mu,\nu)} \int \mathsf{d}^2(x,y)\,\mathrm{d}\gamma(x,y)}.$$

The Wasserstein distance W_2 is a distance on $\mathcal{P}_2(X)$ and turns it into a complete and separable metric space. Moreover, if (X, d) is a geodesic space, then so is $(\mathcal{P}_2(X), W_2)$. Given a metric measure space $(X, \mathsf{d}, \mathfrak{m})$, we denote by $\mathcal{P}_2^a(X) \subset \mathcal{P}_2(X)$ the set of measures with finite second moment that are absolutely continuous with respect to $\mathfrak{m}$.

In order to introduce curvature-dimension conditions, we will need the following distortion coefficients. For $\kappa \in \mathbb{R}$ and $\theta \geq 0$, let

$$\mathfrak{s}_\kappa(\theta) := \begin{cases} \frac{1}{\sqrt{\kappa}} \sin(\sqrt{\kappa}\theta) & \text{if } \kappa > 0, \\ 0 & \text{if } \kappa = 0, \\ \frac{1}{\sqrt{-\kappa}} \sinh\big(\sqrt{-\kappa}\theta\big) & \text{if } \kappa < 0. \end{cases}$$

Moreover, for $t \in [0,1]$, set

$$\sigma_\kappa^{(t)}(\theta) := \begin{cases} \frac{\mathfrak{s}_\kappa(t\theta)}{\mathfrak{s}_\kappa(\theta)} & \text{if } \kappa\theta^2 \neq 0 \text{ and } \kappa\theta^2 < \pi^2, \\ t & \text{if } \kappa\theta^2 = 0, \\ +\infty & \text{if } \kappa\theta^2 \geq \pi^2. \end{cases} \tag{2.2.3}$$

For $K \in \mathbb{R}$, $N \in [1,\infty)$, denote $\tau_{K,N}^{(t)}(\theta) := t^{\frac{1}{N}} (\sigma_{K/(N-1)}^{(t)}(\theta))^{\frac{N-1}{N}}$.

Define the convex and continuous function U_∞ on $[0,\infty)$ as

$$U_\infty(z) := z \log(z).$$

The Boltzmann-Shannon entropy functional $\mathcal{E}_\infty \colon \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$ is defined by

$$\mathcal{E}_\infty(\mu) := \begin{cases} \int_X U_\infty(\rho)\,\mathrm{d}\mathfrak{m}, & \text{if } \mu \ll \mathfrak{m}, \text{ with } \mu = \rho\mathfrak{m} \text{ and } U_\infty(\rho) \in L^1(X,\mathfrak{m}) \\ \infty, & \text{else.} \end{cases}$$

Next, we recall the synthetic notion of Ricci curvature bounded below for a metric measure space, pioneered by Sturm [133, 134] and Lott-Villani [92]. For the finite dimensional case, we will mainly use the reduced curvature-dimension condition of Bacher-Sturm [12].

Definition 2.2.7. *We say that a metric measure space $(X,\mathsf{d},\mathfrak{m})$ has Ricci curvature bounded from below by $K \in \mathbb{R}$ provided the functional $\mathcal{E}_\infty \colon \mathcal{P}(X) \to \mathbb{R} \cup \{+\infty\}$ is K-geodesically convex on $(\mathcal{P}_2^a(X), W_2)$, that means that for any $\mu_0, \mu_1 \in \mathcal{P}_2^a(X)$, there exists a Wasserstein geodesic $(\mu_t)_{t\in[0,1]}$ such that for all $t \in [0,1]$,*

$$\mathcal{E}_\infty(\mu_t) \leq (1-t)\mathcal{E}_\infty(\mu_0) + t\mathcal{E}_\infty(\mu_1) - \frac{K}{2} t(1-t) W_2^2(\mu_0,\mu_1).$$

In this case we say that $(X,\mathsf{d},\mathfrak{m})$ satisfies the curvature-dimension condition $\mathsf{CD}(K,\infty)$*, or that $(X,\mathsf{d},\mathfrak{m})$ is a $\mathsf{CD}(K,\infty)$-space.*

Definition 2.2.8. *Given two numbers $K \in \mathbb{R}$ and $N \in [1,\infty)$, we say that a metric measure space $(X,\mathsf{d},\mathfrak{m})$ satisfies the* curvature-dimension condition $\mathsf{CD}(K,N)$ *if for each pair $\mu_0,\mu_1 \in \mathcal{P}_2^a(X)$, there exists an optimal transport coupling $\boldsymbol{\gamma}$ and a Wasserstein geodesic $(\mu_t = \rho_t\mathfrak{m})_{t\in[0,1]} \subset \mathcal{P}_2^a(X)$ connecting $\mu_0 = \rho_0\mathfrak{m}$ and $\mu_1 = \rho_1\mathfrak{m}$ such that for all $t \in [0,1]$, $N' \geq N$, it holds*

$$\int_X \rho_t^{-\frac{1}{N'}}\,\mathrm{d}\mu_t \geq \int_{X\times X} \Big[\tau_{K,N'}^{(1-t)}(\mathsf{d}(x_0,x_1))\rho_0(x_0)^{-\frac{1}{N'}} + \tau_{K,N'}^{(t)}(\mathsf{d}(x_0,x_1))\rho_1(x_1)^{-\frac{1}{N'}}\Big]\mathrm{d}\boldsymbol{\gamma}(x_0,x_1).$$

We say that $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* reduced curvature-dimension condition $\mathsf{CD}^*(K, N)$ *if for each pair* $\mu_0, \mu_1 \in \mathcal{P}_2^a(X)$ *with bounded supports there exists an optimal transport coupling* $\boldsymbol{\gamma}$ *and a Wasserstein geodesic* $(\mu_t = \rho_t \mathfrak{m})_{t\in[0,1]} \subset \mathcal{P}_2^a(X)$ *connecting* $\mu_0 = \rho_0 \mathfrak{m}$ *and* $\mu_1 = \rho_1 \mathfrak{m}$ *such that for all* $t \in [0, 1]$, $N' \geq N$, *it holds*

$$\int_X \rho_t^{-\frac{1}{N'}} \,\mathrm{d}\mu_t \geq \int_{X\times X} \left[\sigma_{K/N'}^{(1-t)}(\mathsf{d}(x_0, x_1))\rho_0(x_0)^{-\frac{1}{N'}} + \sigma_{K/N'}^{(t)}(\mathsf{d}(x_0, x_1))\rho_1(x_1)^{-\frac{1}{N'}}\right] \mathrm{d}\boldsymbol{\gamma}(x_0, x_1).$$

In order to single out the "Riemannian" like structures of the "possibly Finslerian" CD spaces, Ambrosio-Gigli-Savaré [8] (see also [6]) in the $N = \infty$ case, and Gigli [62] in the $N < \infty$ case introduced the Riemannian curvature-dimension condition.

Definition 2.2.9. *Let* $K \in \mathbb{R}$. *We say that* $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* Riemannian curvature-dimension condition $\mathsf{RCD}(K, \infty)$ *if it is infinitesimally Hilbertian and if it satisfies the* $\mathsf{CD}(K, \infty)$*-condition.*
For $N \in [1, \infty)$, *we say that* $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* Riemannian curvature-dimension condition $\mathsf{RCD}(K, N)$ *if it is infinitesimally Hilbertian and if it satisfies the* $\mathsf{CD}(K, N)$*-condition. We say that* $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* reduced Riemannian curvature-dimension condition $\mathsf{RCD}^*(K, N)$ *if it is infinitesimally Hilbertian and if it satisfies the* $\mathsf{CD}^*(K, N)$*-condition.*

Remark 2.2.10. *In the case of weighted Riemannian manifolds with continuous metric and continuous weight, the resulting measure will always be* σ*-finite in which case the* $\mathsf{RCD}^*(K, N)$ *and* $\mathsf{RCD}(K, N)$*-condition coincide, see [32, 90].*

Let us recall the following useful results:

Lemma 2.2.11 ([55], Lemma 3.2 and Theorem 3.17)**.** *If the metric measure space* $(X, \mathsf{d}, \mathfrak{m})$ *satisfies the* $\mathsf{RCD}^*(K, N)$*-condition, then it also satisfies the* $\mathsf{RCD}^*(K', N')$*-condition for any* $K' \leq K$ *and* $N' \geq N$. *Moreover, it satisfies the* $\mathsf{RCD}(K, \infty)$*-condition.*

Theorem 2.2.12 ([8], Theorem 6.2)**.** *Let* $(X, \mathsf{d}, \mathfrak{m})$ *be a* $\mathsf{RCD}(K, \infty)$ *space and* $f \in W^{1,2}_w(X)$. *Then*

$$|D(H_t f)|_w^2 \leq e^{-2Kt}(H_t|Df|_w^2).$$

Remark 2.2.13. *By [133, Theorem 4.24], the condition* (2.2.1) *is automatically satisfied in every* $\mathsf{CD}(K, \infty)$ *space, by choosing the function* $V\colon X \to \mathbb{R}$, $V(x) := C\mathsf{d}(x, x_0)$ *for a suitable* $C > 0$ *and any base point* $x_0 \in X$.

2.3 Calculus on metric measure spacetimes

This section contains an introduction to one approach to synthetic Lorentzian geometry: metric measure spacetimes. We will briefly summarise relevant aspects of the theory on metric (measure) spacetimes following [16]. When working in the synthetic Lorentzian setting, we will adapt the infinity conventions from [16] which are given by

$$\begin{aligned} +\infty - (+\infty) &:= +\infty, \\ -\infty - (-\infty) &:= +\infty, \\ \pm\infty + z &:= \pm\infty, \quad \text{for } z \in [-\infty, +\infty], \\ (\pm\infty) \cdot 0 &:= 0. \end{aligned}$$

Given a set M and a function $\ell\colon M \times M \to \{-\infty\} \cup [0, \infty]$, we call ℓ a *time separation* if for all $x, y, z \in M$ it holds

$$\ell(x, x) = 0, \quad \min(\ell(x, y) + \ell(y, z), \ell(y, z) + \ell(x, y)) \leq \ell(x, z).$$

Define the transitive relations chronology $\ll$ and causality $\leq$ via

$$\ll := \ell^{-1}((0, +\infty]), \ \leq := \ell^{-1}([0, +\infty]).$$

We call a subset A of M *achronal*, if no two points in A are related by $\ll$.

Definition 2.3.1. *We call (M, ℓ) a* metric spacetime *if ℓ is a time separation and $\leq$ is a partial order. We call (M, τ, ℓ) a* Polish metric spacetime *if τ is a Polish topology on the metric spacetime (M, ℓ) and if $\{\ell \geq 0\}$ is a closed set in M^2.*

Definition 2.3.2 ([16], Definition 3.1)**.** *A function $f\colon M \to \overline{\mathbb{R}}$ is called a* causal function *if f is causally order-preserving, i.e., for all points $x, y \in M$ with $x \leq y$ it holds $f(x) \leq f(y)$. We denote by $\mathrm{Dom}(f)$ the domain of f, i.e. the set of all $x \in M$ such that $f(x)$ is a real number.*

Definition 2.3.3 ([16], Definition 2.26)**.** *A map $\gamma\colon [0, 1] \to M$ is called a* left continuous causal path *if it is causal, i.e., for $s \leq t$ it holds $\gamma_s \leq \gamma_t$, and left continuous, i.e., for all $t \in (0, 1]$ the limit $\gamma_{t-} := \lim_{s \nearrow t} \gamma_s$ exists and equals γ_t. The space of all left continuous causal paths is denoted by $\mathrm{LCC}([0, 1], M)$.*

Lemma 2.3.4 ([16], Lemma 3.2)**.** *Every causal function f has a representative that is continuous outside a countable union of achronal sets.*

Definition 2.3.5 ([16], Theorem 2.23 - Definition 2.25). *Let $\gamma\colon [0,1] \to M$ be a causal path. Then there exists a maximal Borel measure $|\dot{\boldsymbol{\gamma}}|$ on $[0,1]$ such that for all $0 \le s \le t \le 1$, it holds $|\dot{\boldsymbol{\gamma}}|([s,t]) \le \ell(\gamma_s, \gamma_t)$. Denote its Lebesgue decomposition with respect to $\mathcal{L}^1$ as $|\dot{\boldsymbol{\gamma}}| = |\dot{\gamma}|\mathcal{L}^1 + |\dot{\boldsymbol{\gamma}}|^\perp$. Then, the function $|\dot{\gamma}|$ is called the* causal speed *of γ. It is for almost every $t \in [0,1]$ given by*

$$|\dot{\gamma}_t| = \lim_{s \nearrow t} \frac{\ell(\gamma_s, \gamma_t)}{t-s} = \lim_{s \searrow t} \frac{\ell(\gamma_t, \gamma_s)}{s-t}. \tag{2.3.1}$$

Definition 2.3.6 ([16] Definition 2.48). *A* metric measure spacetime *is a quadruple $(M, \tau, \ell, \mathfrak{m})$, where (M, τ, ℓ) is a Polish metric spacetime and $\mathfrak{m}$ a non-negative and non-zero Radon measure on M.*

For $t \in [0,1]$, the evaluation map $e_t\colon \mathrm{LCC}([0,1], M) \to M$ is, analogously to the metric case, defined via $e_t(\gamma) := \gamma_t$.

Definition 2.3.7 ([16], Definition 3.7). *A measure $\boldsymbol{\pi} \in \mathcal{P}(\mathrm{LCC}([0,1], M))$ is called a* test plan *if it has bounded compression, i.e., if there exists a constant $C > 0$ such that for all $t \in [0,1]$ it holds*

$$(e_t)_\# \boldsymbol{\pi} \le C\mathfrak{m}.$$

Definition 2.3.8 ([16], Definition 3.11). *Let $(M, \tau, \ell, \mathfrak{m})$ be a metric measure spacetime and $f : M \to \overline{\mathbb{R}}$ a causal function. A Borel function $G : M \to [0, \infty]$ is called a* weak subslope *of f if for every test plan $\boldsymbol{\pi} \in \mathcal{P}(\mathrm{LCC}([0,1], M))$, it holds*

$$\int [f(\gamma_1) - f(\gamma_0)]\, \mathrm{d}\boldsymbol{\pi}(\gamma) \ge \int \int_0^1 G(\gamma_t)|\dot{\gamma}_t|\, \mathrm{d}t \mathrm{d}\boldsymbol{\pi}(\gamma). \tag{2.3.2}$$

By Theorem 3.14 in [16], there exists a unique weak subslope G that is no smaller than any other weak subslope. We will call it the maximal weak subslope *and denote it by $|\mathrm{d}f|$.*

Definition 2.3.9 ([16], Definition 1.4). *We call $(M, \tau, \ell, \mathfrak{m})$* infinitesimally Minkowskian *if for any two causal functions $f, g : M \to \overline{\mathbb{R}}$, it holds*

$$2|\mathrm{d}(f+g)|^2 + 2|\mathrm{d}f|^2 = |\mathrm{d}g|^2 + |\mathrm{d}(2f+g)|^2 \;\; \mathfrak{m}-\text{a.e.} \tag{2.3.3}$$

Given a test plan $\boldsymbol{\pi}$, we denote by $\rho_{\boldsymbol{\pi}}$ the density of the pushforward of the measure $|\dot{\gamma}_t| \mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t$ under the "full" evaluation map $\mathsf{e}(\gamma, t) = \gamma_t$. Define $\mathrm{Vis}_{\boldsymbol{\pi}} := \{\rho_{\boldsymbol{\pi}} > 0\}$. We define $\mathrm{Vis}(M)$ as the essential union of the sets $\mathrm{Vis}_{\boldsymbol{\pi}}$, as $\boldsymbol{\pi}$ varies over all test plans (see [68, Proposition 3.1.9]). We will use that a property holds $\mathfrak{m}$-almost everywhere on

$\mathrm{Vis}(M)$ if and only if it holds $\mathfrak{m}$-almost everywhere on $\mathrm{Vis}_{\boldsymbol{\pi}}$ for every test plan $\boldsymbol{\pi}$. Finally denote $\mathrm{Invis}(M) := M \setminus \mathrm{Vis}(M)$.

Lemma 2.3.10 ([16], Lemma 3.19)**.** *Let $(M, \tau, \ell, \mathfrak{m})$ be a metric measure spacetime and $A \subset M$ be an achronal set. Then $\mathfrak{m}(A \setminus \mathrm{Invis}(M)) = 0$.*

Lemma 2.3.11 ([120])**.** *Let $(M, \tau, \ell, \mathfrak{m})$ be a metric measure spacetime and f be a causal function. Then for $\mathfrak{m}$-almost every $p \in \{|f| < \infty\} \cap \mathrm{Vis}(M) = \mathrm{Dom}(f) \cap \mathrm{Vis}(M)$, there exists an open neighbourhood U of p such that $\mathfrak{m}(U \setminus \{|f| < \infty\}) = \mathfrak{m}(U \setminus \mathrm{Dom}(f)) = 0$.*

Proof. By Lemma 2.3.4, we can find a representative $\tilde{f}$ of f and achronal sets $\{A_n\}_{n\in\mathbb{N}}$ such that $\tilde{f}$ is continuous outside $\bigcup_{n\in\mathbb{N}} A_n$. We will from now on write f for $\tilde{f}$. Now Lemma 2.3.10 yields that $\mathfrak{m}$-almost every point in $\{|f| < \infty\} \cap \mathrm{Vis}(M)$ is in $F := \{|f| < \infty\} \cap \mathrm{Vis}(M) \setminus \bigcup_{n\in\mathbb{N}} A_n$. Pick a point $p \in F$. As f is continuous and finite at p, we get that there exists an open neighbourhood U of p such that f is finite on U. $\square$

Chapter 3

On the equivalence of distributional and synthetic Ricci curvature lower bounds

This chapter is based on a joint paper with Andrea Mondino [102]. For an introduction to these results, we refer to Subsection 1.3.1. As mentioned in the introduction, the main result is the following equivalence:

Theorem 1 (see Theorem 3.6.3)**.** *Let M be a smooth manifold, g a continuous Riemannian metric with Christoffel symbols in L^2_{loc}, and $V \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$ a function on M. Define the weighted measure μ as $\mathrm{d}\mu := e^{-V}\mathrm{dvol}_g$. Let $N \in [n, \infty]$ and $K \in \mathbb{R}$. The following are equivalent:*

(i) (M, d_g, μ) *is a* $\mathsf{CD}(K, N)$ *space.*

(ii) *The distributional Bakry-Émery N-Ricci curvature tensor is bounded below by K and Assumption 3.6.1 is satisfied.*

We quickly remark that, for C^0-Riemannian metrics, the natural class of differentiability of the manifolds is C^1. However, by a classical result of Whitney [142] (see also [106, Sect. 4]), a C^1-manifold always possesses a C^∞-sub-atlas, unique up to diffeomorphisms; we will choose some such sub-atlas whenever convenient.

Notation

Consider a smooth manifold M. When equipped with a Riemannian metric, we will always denote the metric by g. Moreover, we denote by d_g the distance and by vol_g the volume form induced by g.

In local coordinates g_{ij} will denote the coefficients of g as a matrix, g^{ij} the coefficients of

the inverse matrix, and $|g|$ the determinant of $(g_{ij})_{ij}$. We will write Γ for the Levi-Civita connection of g and denote its coefficients by Γ_{ij}^k. In this chapter, we will use the Einstein summation convention.
The covariant derivative with respect to Γ will be denoted by either ∇ or ∇_c, with the only exception that $\nabla f = g^{ij}\partial_i f \partial_{x^j} = (df)^\sharp$ always denotes the gradient field of f and never its covariant derivative (which in that particular case equals the differential df).
For a vector field V and a differentiable function f, we define $V(f) := df(V)$ to be the action of V on f. For a vector field V and a k-form $\omega \in \Omega^k(M)$ on M, where $k \geq 1$, we denote by $\iota_V \omega \in \Omega^{k-1}(M)$ the contraction of the first entry of ω with V.

Given a topological Hausdorff space X, we denote by $\mathrm{Meas}(X)$ the Banach space of finite Radon measures on X equipped with the total variation norm $\|\cdot\|_{TV}$. Given a finite dimensional vector space V with an inner product $\langle \cdot, \cdot \rangle$, we denote by $|\cdot|_{HS}$ the Hilbert-Schmidt norm on the space of linear operators on V.

3.1 Some regularisation results and their application to Sobolev Riemannian metrics

In this section, we present some variants of the convergence results obtained in [69, 29], which will be key to obtain L^p-convergence of the Ricci curvature tensor under regularisation of the Riemannian metric. More precisely, we will follow the strategy of [29, Lemma 3.3] with the following difference: we will work with L^p-functions instead of locally Lipschitz functions, and we will keep track of the exponents. The following lemma generalises [69, Lemma 4.7] to the L^p-case.

Lemma 3.1.1. *Let $f \in L^p_{loc}(\mathbb{R}^n)$, for some $p \in [1, \infty)$. Let $\rho \in C_c^\infty(B_1^{euc}(0), [0, \infty))$ be a rotationally symmetric standard mollifier and define $\rho_\varepsilon(x) := \frac{1}{\varepsilon^n}\rho(\frac{x}{\varepsilon})$. Then, for any compact set $K \subset \mathbb{R}^n$, it holds that*

$$\varepsilon \|\partial_j \rho_\varepsilon * f\|_{L^p(K)} \to 0, \qquad \text{as } \varepsilon \to 0.$$

Proof. Note that, for any constant $c \in \mathbb{R}$, integration by parts gives that $\partial_j \rho_\varepsilon * c = 0$.

Hence, using the definition of ρ_ε, Jensen's inequality, and Fubini's theorem, we get that

$$\begin{aligned}
\varepsilon\Big(\int_K |\partial_j \rho_\varepsilon * f|^p\Big)^{\frac{1}{p}} &= \varepsilon\Big(\int_K \Big|\int_{B_\varepsilon(0)} \partial_j\rho_\varepsilon(y)\big(f(y-x)-f(x)\big)dy\Big|^p dx\Big)^{\frac{1}{p}} \\
&= \Big(\int_K \Big|\int_{B_\varepsilon(0)} \frac{\varepsilon}{\varepsilon^{n+1}}\partial_j\rho\Big(\frac{y}{\varepsilon}\Big)\big(f(y-x)-f(x)\big)\,dy\Big|^p dx\Big)^{\frac{1}{p}} \\
&\leq C(n)\Big(\int_K \frac{1}{\varepsilon^n}\int_{B_\varepsilon(0)} |\partial_j\rho\Big(\frac{y}{\varepsilon}\Big)|^p|f(y-x)-f(x)|^p\,dydx\Big)^{\frac{1}{p}} \\
&\leq C(n,\rho)\Big(\frac{1}{\varepsilon^n}\int_{B_\varepsilon(0)}\int_K |f(y-x)-f(x)|^p\,dydx\Big)^{\frac{1}{p}}.
\end{aligned}$$

The last integral converges to 0, since

$$\lim_{h\to 0}\|f-f(\cdot-h)\|_{L^p(K)} = 0.$$

□

Lemma 3.1.2 (A Friedrichs type lemma)**.** *Let $p\in[2,\infty)$ and $a\in C^0\cap W^{1,p}_{loc}(\mathbb{R}^n)$ and $f\in L^p_{loc}(\mathbb{R}^n)$. Then, for any compact set K, it holds that*

$$\|(a*\rho_\varepsilon)(\rho_\varepsilon*f)-\rho_\varepsilon*(af)\|_{W^{1,\frac{p}{2}}(K)}\to 0, \qquad \textit{as } \varepsilon\to 0.$$

Proof. We will proceed as in the proof of [69, Lemma 4.8], see also [47, Section 2]. Fix a compact set K. We need to estimate the $L^{\frac{p}{2}}(K)$-norm of

$$\begin{aligned}
h_\varepsilon(x) &= ((af)*\partial_j\rho_\varepsilon - (\partial_j\rho_\varepsilon * a)(f*\rho_\varepsilon) - (\rho_\varepsilon * a)(f*\partial_j\rho_\varepsilon))(x) \\
&= \underbrace{\big(((a-a(x))(f-f(x)))*\partial_j\rho_\varepsilon\big)(x)}_{=:F_1(x)} - \underbrace{\big((a*\partial_j\rho_\varepsilon)((f-f(x))*\rho_\varepsilon)\big)(x)}_{=:F_2(x)} \\
&- \underbrace{\big(((a-a(x))*\rho_\varepsilon)(f*\partial_j\rho_\varepsilon)\big)(x)}_{=:F_3(x)}.
\end{aligned}$$

Notice that the second line follows by the fact that $c*\rho_\varepsilon = c$ and $c*\partial_j\rho_\varepsilon = 0$, for every constant $c \in \mathbb{R}$. We start by estimating F_1.

$$
\begin{aligned}
\|F_1\|_{L^{\frac{p}{2}}(K)} &= \Big(\int_K \big| (((a-a(x))(f-f(x))) * \partial_j \rho_\varepsilon)(x) \big|^{\frac{p}{2}} dx \Big)^{\frac{2}{p}} \\
&= \Big(\int_K \Big| \int_{B_\varepsilon(0)} ((a(x-y)-a(x))(f(x-y)-f(x)))\, \partial_j \rho_\varepsilon(y) dy \Big|^{\frac{p}{2}} dx \Big)^{\frac{2}{p}} \\
&= \Big(\int_K \Big| \frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \frac{1}{\varepsilon}((a(x-y)-a(x))(f(x-y)-f(x)))\, \partial_j \rho\Big(\frac{y}{\varepsilon}\Big) dy \Big|^{\frac{p}{2}} dx \Big)^{\frac{2}{p}} \\
&\le C(n) \Big(\int_K \frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \Big| \frac{1}{\varepsilon}((a(x-y)-a(x))(f(x-y)-f(x)))\, \partial_j \rho\Big(\frac{y}{\varepsilon}\Big) \Big|^{\frac{p}{2}} dy dx \Big)^{\frac{2}{p}} \\
&\le C(n,\rho) \Big(\frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \int_K \Big| \frac{1}{\varepsilon}((a(x-y)-a(x))(f(x-y)-f(x))) \Big|^{\frac{p}{2}} dx dy \Big)^{\frac{2}{p}} \\
&\le C(n,\rho) \Big(\Big(\frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \int_K \Big| \frac{1}{\varepsilon}(a(x-y)-a(x)) \Big|^{p} dx dy \Big)^{\frac{1}{2}} \\
&\Big(\frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \int_K \Big| (f(x-y)-f(x)) \Big|^{p} dx dy \Big)^{\frac{1}{2}} \Big)^{\frac{2}{p}}.
\end{aligned}
$$

It is a classical fact of Sobolev functions that

$$
\frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \int_K \Big| \frac{1}{\varepsilon}(a(x-y)-a(x)) \Big|^{p} dx dy \le C(n) \|\nabla a\|^p_{L^p(B_\varepsilon(K))}. \tag{3.1.1}
$$

Indeed, by the density of C^1-functions in $W^{1,p}$, we can assume without loss of generality that $a \in C^1$. Using the fundamental theorem of calculus, Hölder's inequality, and Fubini's theorem, we estimate:

$$
\begin{aligned}
\frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \int_K \Big| \frac{1}{\varepsilon}(a(x-y)-a(x)) \Big|^{p} dx dy &\le \frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \int_K \Big| \frac{1}{\varepsilon} \int_0^{|y|} \nabla a\Big(x - t\frac{y}{|y|}\Big) dt \Big|^p dx dy \\
&\le \frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \frac{|y|^{p-1}}{\varepsilon^p} \int_K \int_0^{|y|} \Big| \nabla a\Big(x - t\frac{y}{|y|}\Big) \Big|^p dt dx dy \\
&\le \frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \frac{|y|^{p-1}}{\varepsilon^p} \int_0^{|y|} \int_K \Big| \nabla a\Big(x - t\frac{y}{|y|}\Big) \Big|^p dx dt dy \\
&\le \frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \frac{\varepsilon^p}{\varepsilon^p} \|\nabla a\|^p_{L^p(B_\varepsilon(K))} dy \le C(n) \|\nabla a\|^p_{L^p(B_\varepsilon(K))},
\end{aligned}
$$

proving (3.1.1). Hence, for $\varepsilon \le 1$, we get that

$$
\begin{aligned}
\|F_1\|_{L^{\frac{p}{2}}(K)} &\le C(n,p) \|\nabla a\|_{L^p(B_1(K))} \Big(\Big(\frac{1}{\varepsilon^n} \int_{B_\varepsilon(0)} \int_K \Big| (f(x-y)-f(x)) \Big|^p dx dy \Big)^{\frac{1}{2}} \Big)^{\frac{2}{p}} \\
&\le C(n,p) \|\nabla a\|_{L^p(B_1(K))} \Big(\sup_{|y| \le \varepsilon} \|f - f(\cdot - y)\|^p_{L^p(K)} \Big)^{\frac{1}{p}} \to 0,
\end{aligned}
$$

as $\varepsilon \to 0$. Furthermore,

$$\begin{aligned}
\|F_2\|_{L^{\frac{p}{2}}(K)} &\leq \|(a * \partial_j \rho_\varepsilon)\|_{L^p(K)} \|((f - f(x)) * \rho_\varepsilon)\|_{L^p(K)} \\
&\leq \|\nabla a\|_{L^p(B_1(K))} \Big(\int_K \Big| \int_{B_\varepsilon(0)} \rho_\varepsilon(y)((f(x-y) - f(x))dy \Big|^p dx \Big)^{\frac{1}{p}} \\
&\leq \|\nabla a\|_{L^p(B_1(K))} \Big(\int_{B_\varepsilon(0)} \rho_\varepsilon(y) \int_K \Big| ((f(x-y) - f(x)) \Big|^p dxdy \Big)^{\frac{1}{p}} \\
&\leq \|\nabla a\|_{L^p(B_1(K))} \Big(\sup_{|y| \leq \varepsilon} \|f - f(\cdot - y)\|^p_{L^p(K)} \Big)^{\frac{1}{p}} \to 0,
\end{aligned}$$

as $\varepsilon \to 0$. We next estimate F_3. To this aim, we start by estimating

$$\begin{aligned}
\|(a - a(x)) * \rho_\varepsilon\|_{L^p(K)} &= \Big(\int_K \Big| \int_{B_\varepsilon(0)} \rho_\varepsilon(y)(a(x-y) - a(x))dy \Big|^p dx \Big)^{\frac{1}{p}} \\
&\leq \Big(\int_K \int_{B_\varepsilon(0)} \rho_\varepsilon(y) |(a(x-y) - a(x))|^p dydx \Big)^{\frac{1}{p}} \\
&= \Big(\int_{B_\varepsilon(0)} \rho_\varepsilon(y) \int_K |(a(x-y) - a(x))|^p dxdy \Big)^{\frac{1}{p}}. \qquad (3.1.2)
\end{aligned}$$

For $|y| \leq \varepsilon$, it is a classical fact for Sobolev functions that

$$\Big(\int_K |(a(x-y) - a(x))|^p dx \Big)^{\frac{1}{p}} \leq \varepsilon \|\nabla a\|_{L^p(B_1(K))}. \qquad (3.1.3)$$

Indeed, by the density of C^1-functions in $W^{1,p}$, we can assume without loss of generality that $a \in C^1$. Denote $v = \frac{y}{|y|} \in \mathbb{R}^n$. Using the fundamental theorem of calculus, Hölder's inequality, and Fubini's theorem, we estimate:

$$\begin{aligned}
\Big(\int_K |(a(x-y) - a(x))|^p dx \Big)^{\frac{1}{p}} &= \Big(\int_K \Big| \int_0^{|y|} \nabla a(x - tv)dt \Big|^p dx \Big)^{\frac{1}{p}} \\
&\leq \Big(\int_K |y|^{p-1} \int_0^{|y|} |\nabla a|^p (x - tv)dtdx \Big)^{\frac{1}{p}} \\
&\leq \Big(\varepsilon^{p-1} \int_0^{|y|} \int_K |\nabla a|^p (x - tv)dxdt \Big)^{\frac{1}{p}} \\
&\leq \Big(\varepsilon^{p-1} \int_0^{|y|} \|\nabla a\|^p_{L^p(B_1(K))} dt \Big)^{\frac{1}{p}} \leq \varepsilon \|\nabla a\|_{L^p(B_1(K))}.
\end{aligned}$$

Combining the estimates above with Hölder's inequality, we obtain

$$\|F_3\|_{L^{\frac{p}{2}}(K)} \leq \|(a - a(x)) * \rho_\varepsilon\|_{L^p(K)} \|\partial_j \rho_\varepsilon * f\|_{L^p} \leq \varepsilon \|\nabla a\|_{L^p(B_1(K))} \|\partial_j \rho_\varepsilon * f\|_{L^p} \to 0,$$

as $\varepsilon \to 0$, by Lemma 3.1.1. $\square$

Lemma 3.1.3. *Let $p \in [2,\infty)$, $a \in C^0 \cap W^{1,p}_{loc}(\mathbb{R}^n)$ and $f \in L^p_{loc}(\mathbb{R}^n)$. Let ρ_ε be as in the previous lemma. Let $a_\varepsilon \in C^\infty(\mathbb{R}^n)$ be such that $a_\varepsilon \to a$ locally uniformly and in $W^{1,p}_{loc}$; moreover, assume that, for each $K \subset \mathbb{R}^n$ compact subset there exists $C_K > 0$ such that*

$$\|a_\varepsilon - a\|_{L^p(K)} \leq C_K \varepsilon. \tag{3.1.4}$$

Then, for any compact set $K \subset \mathbb{R}^n$, it holds that

$$\|a_\varepsilon(\rho_\varepsilon * f) - \rho_\varepsilon * (af)\|_{W^{1,\frac{p}{2}}(K)} \to 0, \quad \text{as } \varepsilon \to 0.$$

Proof. Write

$$a_\varepsilon(\rho_\varepsilon * f) - \rho_\varepsilon * (af) = \underbrace{(a_\varepsilon - \rho_\varepsilon * a)(\rho_\varepsilon * f)}_{(I)} + \underbrace{(\rho_\varepsilon * a)(\rho_\varepsilon * f) - \rho_\varepsilon * (af)}_{(II)}.$$

We only need to estimate the $W^{1,p}$-norm of term (I), as Lemma 3.1.2 yields the desired estimate for term (II). Write

$$\partial_j((a_\varepsilon - \rho_\varepsilon * a)(\rho_\varepsilon * f)) = (\partial_j(a_\varepsilon - \rho_\varepsilon * a))(\rho_\varepsilon * f) + (a_\varepsilon - \rho_\varepsilon * a)(\partial_j(\rho_\varepsilon * f)).$$

We start estimating the first term:

$$\begin{aligned}\|(\partial_j(a_\varepsilon - \rho_\varepsilon * a))(\rho_\varepsilon * f)\|_{L^{\frac{p}{2}}(K)} &\leq \|\partial_j(a_\varepsilon - \rho_\varepsilon * a)\|_{L^p(K)} \|(\rho_\varepsilon * f)\|_{L^p(K)} \\ &\leq \left(\|\partial_j(a_\varepsilon - a)\|_{L^p(K)} + \|\partial_j(a - \rho_\varepsilon * a)\|_{L^p(K)}\right) \|(\rho_\varepsilon * f)\|_{L^p(K)} \to 0,\end{aligned}$$

as $\varepsilon \to 0$. Moreover, for the second term, we use the assumption (3.1.4) together with (3.1.2) and (3.1.3) and get that

$$\|a_\varepsilon - (a * \rho_\varepsilon)\|_{L^p(K)} \leq \|a_\varepsilon - a\|_{L^p(K)} + \|a - (a * \rho_\varepsilon)\|_{L^p(K)} \leq C(K)\varepsilon.$$

Hence,

$$\begin{aligned}\|(a_\varepsilon - \rho_\varepsilon * a)(\partial_j(\rho_\varepsilon * f))\|_{L^{\frac{p}{2}}(K)} &\leq \|(a_\varepsilon - \rho_\varepsilon * a)\|_{L^p(K)} \|(\partial_j(\rho_\varepsilon * f)\|_{L^p(K)} \\ &\leq C_K \varepsilon \|(\partial_j(\rho_\varepsilon * f)\|_{L^p(K)} \to 0,\end{aligned}$$

as $\varepsilon \to 0$, by Lemma 3.1.1. □

Let us point out the subtlety that af is not in $W^{1,\frac{p}{2}}$, however, the above result tells us that the approximations are close in $W^{1,\frac{p}{2}}$, which includes a higher level of weak differentiation than one would expect from the mere assumption that f is only in L^p.

We will now apply these results to Riemannian metrics $g \in C^0 \cap W^{1,p}_{loc}$, for some $p \in [2,\infty)$ to obtain a statement on the behaviour of the Ricci curvature tensor under mollification.

Proposition 3.1.4. *Let M be a smooth manifold and let g be a Riemannian metric of regularity $g \in W^{1,p}_{loc}$ (in the sense of Definition 2.1.4), for some $p \in [2, \infty)$. Then, for each compact set $K \subset M$, it holds*

$$\|\mathrm{Ric}[g_\varepsilon] - \rho_\varepsilon * \mathrm{Ric}[g]\|_{L^{\frac{p}{2}}(K)} \to 0, \quad \textit{as } \varepsilon \to 0. \tag{3.1.5}$$

Proof. Fix a compact set $K \subset M$. Using a partition of unity, we may assume that K is contained in one coordinate patch. Through Theorem 2.1.2, we can use the pushforward with a coordinate map $\psi : V \to \mathbb{R}^n$ for some $V \supset K$ and work with distributions on $\mathbb{R}^n$. To simplify the presentation, we will now write K, V for $\psi(K), \psi(V) \subset \mathbb{R}^n$ and Ric for $\Psi^\wedge\mathrm{Ric} \in \mathcal{D}'(V, \mathbb{R}^{n\times n})$ with a slight abuse of notation.

Note that as g is continuous, we get that $g_\varepsilon^{-1} \to g^{-1}$ locally uniformly and in $W^{1,p}_{\mathrm{loc}}$. The inverse of the metrics are given by

$$g_\varepsilon^{-1} = \frac{\mathrm{adj}(g_\varepsilon)}{\det g_\varepsilon}, \text{ and } g^{-1} = \frac{\mathrm{adj}(g)}{\det g}.$$

Now, $\det : \mathbb{R}^{n\times n} \to \mathbb{R}$ and $\mathrm{adj} : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ are (componentwise) polynomials. The locally uniform convergence of g_ε to g ensures that $\det(g_\varepsilon)$ is bounded away from zero and that all components of g_ε are bounded on K for ε small enough. It follows that the map $\mathrm{Inv} : A \in \mathrm{Gl}(n, \mathbb{R}) \subset \mathbb{R}^{n\times n} \mapsto A^{-1} \in \mathrm{Gl}(n, \mathbb{R}) \subset \mathbb{R}^{n\times n}$ is locally Lipschitz continuous on the set $\{g_\varepsilon\}_{\varepsilon>0} \cup \{g\} \subset C^0(\mathbb{R}^n, \mathbb{R}^{n\times n})$. The local Lipschitz constant depends only on n, the lower bound of det and the upper bound on the components on K. Hence, we have that there exists a constant $c_K > 0$ such that

$$|g_\varepsilon^{-1}(x) - g^{-1}(x)| \leq c_K |g_\varepsilon(x) - g(x)| \text{ for all } x \in K. \tag{3.1.6}$$

Using that $g_\varepsilon = \rho_\varepsilon * g$ together with $g \in W^{1,p}_{\mathrm{loc}}$ and (3.1.3), it follows that there exists a constant $C_K > 0$ such that

$$\left\|g_\varepsilon^{-1} - g^{-1}\right\|_{L^p} \leq C_K \varepsilon. \tag{3.1.7}$$

Recalling (2.1.2), it suffices to prove that for each i, j, k, i', j', k', it holds

$$\Gamma[g_\varepsilon]^k_{ij} - \rho_\varepsilon * \Gamma^k_{ij} \to 0 \text{ in } W^{1,\frac{p}{2}}(K), \text{ and} \tag{3.1.8}$$

$$\Gamma[g_\varepsilon]^k_{ij}\Gamma[g_\varepsilon]^{k'}_{i'j'} - (\rho_\varepsilon * \Gamma^k_{ij})(\rho_\varepsilon * \Gamma^{k'}_{i'j'}) \to 0 \text{ in } L^{\frac{p}{2}}(K). \tag{3.1.9}$$

Note that each Christoffel symbol Γ is a sum of products of the inverse metric and a derivative of the metric, i.e. $\Gamma[g_\varepsilon] \sim g_\varepsilon^{-1} D g_\varepsilon$ and $\rho_\varepsilon * \Gamma \sim \rho_\varepsilon * (g^{-1} D g)$. Then, (3.1.9) follows from the fact that both $g_\varepsilon^{-1} D g_\varepsilon$ and $\rho_\varepsilon * \Gamma \sim \rho_\varepsilon * (g^{-1} D g)$ converge to $g^{-1} D g$ in $L^{\frac{p}{2}}(K)$. Using Lemma 3.1.3 with $a = g^{-1}$, $a_\varepsilon = g_\varepsilon^{-1}$ and $f = Dg$, we obtain (3.1.8). This finishes the proof. $\square$

Remark 3.1.5. *As noted after Lemma 3.1.3, the conclusion of Proposition 3.1.4 is remarkable as* $\mathrm{Ric}[g]$ *is not a regular distribution and not in* $L^{\frac{p}{2}}$*, yet the two approximations are close in* $L^{\frac{p}{2}}$*.*

3.2 A weak formulation of Bochner's formula for metrics of low regularity

3.2.1 Smooth calculus on manifolds

In this subsection, we will recall some basic calculus concepts from smooth Riemannian geometry. More details can be found in [112].

Lemma 3.2.1. *Let g be a smooth Riemannian metric on M and f, h be smooth functions on M. The following equations hold in local coordinates:*

(i) $\nabla f = g^{ik}(\partial_i f)\partial_{x^k}$.

(ii) $\mathrm{vol}_g = \sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n$.

(iii) $\Delta f = \partial_j g^{ij}\partial_i f + g^{ij}\partial_{ij} f + \frac{1}{2}\mathrm{tr}(g^{-1}\partial_i g)g^{ij}\partial_j f = \frac{1}{\sqrt{|g|}}\partial_i(\sqrt{|g|}g^{ij}\partial_j f)$.

(iv) $\langle \nabla f, \nabla h\rangle_g = \partial_i f g^{ij}\partial_j h$.

Note that these expressions still hold for $g \in C^0(M)$, $\Gamma^k_{ij} \in L^2_{\mathrm{loc}}$. The Hessian is given by

$$\mathrm{Hess} f(\partial_{x^i}, \partial_{x^j}) = \partial_{ij} f - \Gamma^s_{ij}\partial_s f.$$

For all smooth functions ϕ, h, and f, it holds

$$2\mathrm{Hess} f(\nabla\phi, \nabla h) = \langle \nabla\phi, \langle \nabla f, \nabla h\rangle_g\rangle_g + \langle \nabla h, \langle \nabla f, \nabla\phi\rangle_g\rangle_g - \langle \nabla f, \langle \nabla h, \nabla\phi\rangle_g\rangle_g. \quad (3.2.1)$$

Now let X be a smooth vector field and ω a smooth 1-form. Then the musical isomorphisms $\sharp : T^*M \to TM$ and $\flat : TM \to T^*M$ are defined via

$$X^\flat(Y) = \langle X, Y\rangle_g, \text{ and } \omega(Y) = \langle \omega^\sharp, Y\rangle_g,$$

for all vector fields Y. Note that the musical isomorphisms come from the smooth setting, but also make sense for C^0-Riemannian metrics.

3.2.2 Bochner's formula for vector fields on smooth weighted manifolds

In this subsection we briefly recall the generalisation of Bochner's formula to the case of a smooth weighted manifold. Denote by Δ_H the Hodge-Laplacian and recall that for $f \in C^\infty(M)$ it holds $\Delta_H f = -\Delta f$. Then:

Proposition 3.2.2. *For a smooth manifold M with a smooth Riemannian metric g and any smooth vector field X, it holds*

$$-\frac{1}{2}\Delta_H|X|^2 = -\langle(\Delta_H X^\flat)^\sharp, X\rangle + |\nabla X|^2_{\mathrm{HS}} + \mathrm{Ric}(X,X). \tag{3.2.2}$$

Now let (M,g) be a smooth Riemannian manifold and let μ be a measure on M such that $\mathrm{d}\mu = h^2\mathrm{dvol}_g$ for a smooth and positive function h. As usual, we write for a C^1-function u, that $(\nabla u)^i = g^{ij}\partial_j u$. We can now define the associated divergence operator div_μ by

$$\mathrm{div}_\mu(X) = \frac{1}{h^2}\mathrm{div}(h^2X) = \frac{1}{h^2}\Big(h^2\partial_i X^i + 2h\partial_i h X^i + \frac{1}{2}h^2X^i\mathrm{tr}(g^{-1}\partial_i g)\Big) \tag{3.2.3}$$

and the weighted Laplace operator Δ_μ

$$\Delta_\mu := \mathrm{div}_\mu \circ \nabla = \frac{1}{h^2}\mathrm{div}(h^2\nabla) = \Delta + \frac{2\langle\nabla h, \nabla\rangle}{h}.$$

The divergence operator satisfies

$$\int_M f\mathrm{div}_\mu(X)\,\mathrm{d}\mu = -\int_M X(f)\,\mathrm{d}\mu, \tag{3.2.4}$$

for any function $f \in C_c^\infty(M)$ and any smooth vector field X. It then follows that for $u, v \in C^\infty(M)$ such that u or v are compactly supported, we have that

$$\int_M u\Delta_\mu v\,\mathrm{d}\mu = -\int_M \langle\nabla u, \nabla v\rangle_g\,\mathrm{d}\mu = \int_M \Delta_\mu uv\,\mathrm{d}\mu.$$

Sometimes it is notationally convenient to write $h^2 = e^{-V}$ for some smooth function V.

Definition 3.2.3. *Let $N \geq n$ and V, h as above. The generalised Bakry-Émery N-Ricci tensor is defined by*

$$\mathrm{Ric}_{\mu,N} := \mathrm{Ric} + \nabla^2 V - \frac{1}{N-n}\nabla V \otimes \nabla V.$$

In the case $n = N$, we use the convention that the only admissible V is constant. For $N = \infty$, we get

$$\mathrm{Ric}_{\mu,\infty} := \mathrm{Ric} + \nabla^2 V = \mathrm{Ric} + \mathrm{Hess}[-2\log h].$$

We will now generalise the Hodge star operator and the codifferential to this setting. Recall the Hodge star operator $* : \Omega^p(M) \to \Omega^{n-p}(M)$ in a Riemannian manifold without weight, where $*\beta$ is defined via $\alpha\wedge*\beta = \langle\alpha,\beta\rangle\mathrm{vol}_g$ for all $\alpha \in \Omega^p(M)$. This shall motivate our next definition.

Definition 3.2.4. *The weighted Hodge star operator $*_\mu : \Omega^p(M) \to \Omega^{n-p}(M)$ is defined as h^2*, where $*$ denotes the classical Hodge star operator. It satisfies*

$$\alpha \wedge *_\mu \beta = \langle \alpha, \beta \rangle_g \mu,$$

where we interpret the n-form $\alpha \wedge *_\mu \beta$ as a measure on M.

It directly follows that for all $p \in \{0, \dots, n\}$ and $\beta \in \Omega^p(M)$, it holds

$$*_\mu^{-1} \beta = *^{-1} \left(\frac{1}{h^2} \beta \right).$$

As in the unweighted case, we get that $\delta_\mu := (-1)^p *_\mu^{-1} d *_\mu$ is adjoint to d in the $L^2(M, \mu)$-inner product, i.e.

$$\int_M \langle \mathrm{d}\alpha, \beta \rangle_g \, \mathrm{d}\mu = (-1)^p \int_M \langle \alpha, *_\mu^{-1} d *_\mu \beta \rangle_g \, \mathrm{d}\mu,$$

for all compactly supported $(p-1)$-form α and p-form β.

Definition 3.2.5. *The above operator $\delta_\mu : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ is called the weighted codifferential. We denote by $\Delta_{\mu,H} : \Omega^\bullet(M) \to \Omega^\bullet(M)$ the weighted Laplace-Beltrami operator, which is defined by*

$$\Delta_{\mu,H} := \mathrm{d}\delta_\mu + \delta_\mu d.$$

We now aim to generalise (3.2.2) to the weighted case. In local normal coordinates, we can compute that

$$-\frac{1}{2}\Delta_{\mu,H}|\alpha|^2 = \frac{1}{2}\Delta_\mu|\alpha|^2 = \frac{1}{2}\Delta|\alpha|^2 + \frac{1}{h}\langle \nabla h, \nabla |\alpha|^2 \rangle = -\frac{1}{2}\Delta_H|\alpha|^2 + \frac{2}{h}\partial_j h \alpha_k \partial_j \alpha_k$$

and

$$\begin{aligned} \mathrm{Ric}_{\mu,\infty}(\alpha^\sharp, \alpha^\sharp) &= \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp) + \mathrm{Hess}[-2\log h](\alpha^\sharp, \alpha^\sharp) \\ &= \mathrm{Ric}(\alpha^\sharp, \alpha^\sharp) - 2\alpha_j \alpha_k \Big(\frac{h\partial_{jk}^2 h - \partial_j h \partial_k h}{h^2} \Big). \end{aligned} \tag{3.2.5}$$

Finally, we get that

$$\begin{aligned} -\langle \alpha, \Delta_{\mu,H}\alpha \rangle &= -\langle \alpha, \Delta_H \alpha \rangle + \frac{2}{h}(\partial_j h \partial_j \alpha_k \alpha_k - \partial_k h \partial_j \alpha_k \alpha_j) + \frac{2}{h}\partial_j \alpha_k \partial_k h \alpha_j + \partial_j \left(\frac{2\partial_k h}{h} \right) \alpha_k \alpha_j \\ &= -\langle \alpha, \Delta_H \alpha \rangle + \frac{2}{h}\partial_j h \partial_j \alpha_k \alpha_k - \frac{2}{h}\partial_k h \partial_j \alpha_k \alpha_j + \frac{2}{h}\partial_j \alpha_k \partial_k h \alpha_j + \alpha_k \alpha_j \mathrm{Hess}[2\log h]_{kj} \\ &= -\langle \alpha, \Delta_H \alpha \rangle + \frac{2}{h}\partial_j h \partial_j \alpha_k \alpha_k + \alpha_k \alpha_j \mathrm{Hess}[2\log h]_{kj}. \end{aligned}$$

Recalling (3.2.2), this yields

Proposition 3.2.6. *Let (M, g, μ) be a weighted Riemannian manifold with smooth metric and smooth weight. Then, for every smooth 1-form α, it holds*

$$-\frac{1}{2}\Delta_{\mu,H}|\alpha|^2 = -\langle \alpha, \Delta_{\mu,H}\alpha\rangle + \mathrm{Ric}_{\mu,\infty}(\alpha^\sharp, \alpha^\sharp) + |\nabla\alpha|^2_{HS}.$$

In other terms, for every smooth vector field X, it holds

$$-\frac{1}{2}\Delta_{\mu,H}|X|^2 = -\langle X, (\Delta_{\mu,H}X^\flat)^\sharp\rangle + \mathrm{Ric}_{\mu,\infty}(X, X) + |\nabla X|^2_{HS}. \tag{3.2.6}$$

3.2.3 A Bochner formula for weighted manifolds of lower regularity

We can generalise Definition 3.2.3 to the distributional case:

Definition 3.2.7. *Let M be a smooth manifold and $N \in [n, \infty]$. Let g be a C^0-Riemannian metric with L^2_{loc} Christoffel symbols and let $h \in C^0(M, (0, \infty)) \cap W^{1,2}_{\mathrm{loc}}(M)$, $V \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$ such that $h^2 = e^{-V}$. We define the distributional Bakry-Émery N-Ricci-curvature tensor of the weighted manifold $(M, g, h^2\mathrm{vol}_g)$ as*

$$\mathrm{Ric}_{\mu,N} = \mathrm{Ric} + \nabla^2_g V - \frac{1}{N-n}\nabla V \otimes \nabla V \in \mathcal{D}'\mathcal{T}^0_2. \tag{3.2.7}$$

The case $N = \infty$ can also be rewritten as:

$$\mathrm{Ric}_{\mu,\infty} := \mathrm{Ric} - 2\mathrm{Hess}[\log h] \in \mathcal{D}'\mathcal{T}^0_2. \tag{3.2.8}$$

In the following, we will integrate by parts, to get a version of Bochner's formula that only involves first derivatives of g and h. First we make the following useful observation: Assume g to be a smooth metric on the manifold M. Then the Christoffel symbols (of the second kind) are given by

$$\Gamma^k_{ij} = \frac{1}{2}g^{kl}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}).$$

We can recover the Christoffel symbols of the first kind by multiplication with g, i.e.

$$\Gamma_{ij,l} = \frac{1}{2}(\partial_i g_{jl} + \partial_j g_{il} - \partial_l g_{ij}) = g_{lk}\Gamma^k_{ij} = g(\nabla_{\partial_i}\partial_j, \partial_l). \tag{3.2.9}$$

Now note that

$$(\Gamma_{il,j} + \Gamma_{jl,i}) = \partial_l g_{ij}. \tag{3.2.10}$$

This follows directly from the definition of the Levi-Civita connection, hence the last identity holds in a weak sense for the distributional Levi-Civita connection for g of lower regularity. More precisely:

Lemma 3.2.8. *Let g be a C^0-Riemannian metric such that g admits L^2_{loc}-Christoffel symbols (of the second kind). Then $g \in W^{1,2}_{\mathrm{loc}}(M)$.*

Proof. As g is in C^0, it follows from (3.2.9), that the Christoffel symbols of the first kind are in L^2_{loc} as well. Then using (3.2.10) integration by parts and the linearity of the Christoffel symbols of the first kind in terms of g, we get that all partial derivatives of g are in L^2_{loc}. $\square$

Recall that (3.2.6) holds on a smooth manifold. Using that for functions $\Delta_\mu = -\Delta_{\mu,H}$ and testing with a function $\phi \in C^\infty_c(M)$, we obtain

$$\int_M \left(-\frac{1}{2}\Delta_\mu |X|^2 - \langle (\Delta_{\mu,H} X^\flat)^\sharp, X\rangle + |\nabla X|^2 + \mathrm{Ric}_{\mu,\infty}(X,X)\right)\phi \,\mathrm{d}\mu = 0.$$

We will again integrate by parts, where necessary, to spell this out in local coordinates. By potentially using a partition of unity, we can assume that ϕ is supported in one coordinate patch. We have that

$$\int \frac{1}{2}\langle \nabla |X|^2, \nabla\phi\rangle_g \,\mathrm{d}\mu = \int -\frac{1}{2}\Delta_\mu |X|^2 \phi \,\mathrm{d}\mu$$

and

$$\begin{aligned}
\int_M \mathrm{Ric}_{\mu,\infty}(X,X)\phi \,\mathrm{d}\mu &= \int X^j X^k (\partial_p \Gamma^p_{jk} - \partial_j \Gamma^p_{pk} + \Gamma^s_{kj}\Gamma^p_{ps} - \Gamma^s_{kp}\Gamma^p_{js})\phi h^2 \sqrt{|g|}\, dx^1 \dots dx^n \\
&\quad + 2\int X^j X^k (\partial_j h \partial_k h - h\partial^2_{jk} h + h\partial_s h \Gamma^s_{kj})\phi \sqrt{|g|} dx^1 \dots dx^n \\
&= \int X^j X^k (\Gamma^s_{kj}\Gamma^p_{ps} - \Gamma^s_{kp}\Gamma^p_{js})\phi h^2 \sqrt{|g|}\, dx^1 \dots dx^n \\
&\quad - \int \Gamma^p_{jk}\partial_p (X^j X^k \phi h^2 \sqrt{|g|})\, dx^1 \dots dx^n \\
&\quad + \int \Gamma^p_{pk}\partial_j (X^j X^k \phi h^2 \sqrt{|g|})\, dx^1 \dots dx^n \\
&\quad + 2\int X^j X^k (\partial_j h \partial_k h + h\partial_s h \Gamma^s_{kj})\phi \sqrt{|g|} dx^1 \dots dx^n \\
&\quad + 2\int \partial_j h \partial_k (X^j X^k h\phi \sqrt{|g|}) dx^1 \dots dx^n.
\end{aligned}$$

The above holds for any smooth g and h. Using that each function $h \in C^0 \cap W^{1,2}_{\mathrm{loc}}$ and each metric $g \in C^0$ with L^2_{loc}-Christoffel symbols can be approximated in the $C^0 \cap W^{1,2}_{\mathrm{loc}}$ topology by smooth objects, we get:

Proposition 3.2.9. *Let M be a smooth manifold, g a C^0-Riemannian metric with L^2_{loc} Christoffel symbols and $h \in C^0(M,(0,\infty)) \cap W^{1,2}_{\mathrm{loc}}(M), V \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$ such that*

$h^2 = e^{-V}$.

Then, for every smooth vector field X, it holds:

$$-\frac{1}{2}\Delta_\mu |X|^2 - \langle (\Delta_H X^\flat)^\sharp, X\rangle + |\nabla X|^2 + \mathrm{Ric}_{\mu,\infty}(X,X) = 0 \quad \text{in } \mathcal{D}'(M). \tag{3.2.11}$$

For what will come next, it is convenient to investigate the other terms of the formula more closely:

$$\langle (\Delta_H X^\flat)^\sharp, X\rangle = |dX^\flat|^2 + |\delta_\mu X^\flat|^2 = |dX^\flat|^2 + |\mathrm{div}_\mu(X)|^2 \text{ in } \mathcal{D}'(M).$$

Note that if locally $X = X^i\partial_{x^i}$ then $X^\flat = g_{ij}X^i dx^j$ and hence

$$dX^\flat = d(g_{ij}X^i dx^j) = \partial_k(g_{ij}X^i)dx^k \wedge dx^j = \sum_{j<k}(\partial_k(g_{ij}X^i) - \partial_j(g_{ik}X^i))dx^k \wedge dx^j.$$

It follows that

$$|dX^\flat|^2 = \sum_{j<k}\sum_{l<m} g^{jl}g^{km}(\partial_k(g_{ij}X^i) - \partial_j(g_{ik}X^i))(\partial_m(g_{pl}X^p) - \partial_l(g_{pm}X^p)).$$

We also know that

$$|\mathrm{div}_\mu(X)|^2 = \frac{1}{h^4}\Big(h^2\partial_i X^i + 2h\partial_i h X^i + \frac{1}{2}h^2 X^i \mathrm{tr}(g^{-1}\partial_i g)\Big)^2.$$

Recalling Lemma 3.2.8 and arguing by density, we get that the following formula holds for all $g \in C^0$ with L^2_{loc} Christoffel symbols, all $\phi \in C^0_c \cap W^{1,2}_{\mathrm{loc}}(M)$ and all positive $h \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$:

$$\begin{aligned}
&\int X^j X^k(\Gamma^s_{kj}\Gamma^p_{ps} - \Gamma^s_{kp}\Gamma^p_{js})\phi h^2\sqrt{|g|}\,dx^1\ldots dx^n - \int \Gamma^p_{jk}\partial_p(X^jX^k\phi h^2\sqrt{|g|})\,dx^1\ldots dx^n\\
&\quad + \int \Gamma^p_{pk}\partial_j(X^jX^k\phi h^2\sqrt{|g|})\,dx^1\ldots dx^n\\
&\quad + 2\int X^jX^k(\partial_j h\partial_k h + h\partial_s h\Gamma^s_{kj})\phi\sqrt{|g|}dx^1\ldots dx^n + 2\int \partial_j h\partial_k(X^jX^kh\phi\sqrt{|g|})dx^1\ldots dx^n\\
&= -\frac{1}{2}\int \langle \nabla |X|^2, \nabla\phi\rangle_g h^2\sqrt{|g|}\,dx^1\ldots dx^n\\
&\quad + \int \Big(\partial_i X_i + X_i\frac{1}{2}\,\mathrm{tr}\big(g^{-1}\partial_i g\big) + \frac{2}{h}\partial_i h X^i\Big)^2 h^2\phi\sqrt{|g|}\,dx^1\ldots dx^n\\
&\quad + \int \sum_{j<k}\sum_{l<m} g^{jl}g^{km}(\partial_k(g_{ij}X^i) - \partial_j(g_{ik}X^i))(\partial_m(g_{pl}X^p) - \partial_l(g_{pm}X^p))\phi h^2\sqrt{|g|}\,dx^1\ldots dx^n\\
&\quad - \int \Big(\sum_{i,j,r,s}(\partial_i X^s + \Gamma^r_{ip}X^p)(\partial_j X^r + \Gamma^s_{jq}X^q)g^{ij}g_{sr}\Big)\phi h^2\sqrt{|g|}\,dx^1\ldots dx^n.
\end{aligned} \tag{3.2.12}$$

3.3 First order calculus

In this section we recall the definition of the first order quadratic Sobolev space H^2_1, using local charts and distributional weak derivatives. Thereafter we will show that in the case of a manifold M with a C^0-Riemannian metric g, the synthetic Sobolev space $W^{1,2}_w(M)$ (as defined in Section 2.2) coincides with H^2_1. The results will be useful in the establishing the equivalence of distributional and synthetic Ricci curvature lower bounds.

3.3.1 L^p and Sobolev spaces on weighted Riemannian manifolds of low regularity

In this subsection, we recall the language of Sobolev spaces on manifolds. We will summarise facts from [77, Ch. 2], adapting to the case of weighted manifolds of low regularity. Let (M, g) be a Riemannian manifold (with g a smooth metric for now). Let $h : M \to (0, \infty)$ be a C^0-function and define the measure μ via $\mathrm{d}\mu := h^2 \mathrm{dvol}_g$. For an integer $k \geq 0$ and $u \in C^\infty(M)$ define $\nabla^k_c u$ to be the k-th covariant derivative ($\nabla^0_c u := u$). Notice that $\nabla^k_c u$ is a $(0, k)$-tensor. We will only be interested in the cases $k \leq 2$, so we write down the explicit formulas for the covariant derivatives in local coordinates:

$$(\nabla^1_c u)_i = (du)_i = \partial_i u \quad \text{and} \quad (\nabla^2_c u)_{ij} = \partial_{ij} u - \Gamma^k_{ij} \partial_k u.$$

By definition, we have that

$$|\nabla^k_c u|^2 = g^{i_1 j_1} \dots g^{i_k j_k} (\nabla^k_c u)_{i_1 \dots i_k} (\nabla^k_c u)_{j_1 \dots j_k},$$

where $i_1 < \ldots < i_k$ and $j_1 < \ldots < j_k$. Let $p \in [1, \infty)$. Denote by $C^p_k(M, \mu)$ the space of smooth functions $u \in C^\infty(M)$ such that $|\nabla^j_c u| \in L^p(M, \mu)$ for any $j = 0, \ldots k$.

Definition 3.3.1. *The Sobolev space $H^p_k(M, \mu)$ is the completion of $C^p_k(M, \mu)$ with respect to the norm*

$$\|u\|_{H^p_k(M,\mu)} = \sum_{j=0}^{k} \left(\int_M |\nabla^j_c u|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

Proposition 3.3.2. *$H^2_k(M, \mu)$ is a Hilbert space with the inner product*

$$\langle u, v \rangle = \sum_{m=0}^{k} \int_M \left(g^{i_1 j_1} \dots g^{i_m j_m} (\nabla^m_c u)_{i_1 \dots i_m} (\nabla^m_c v)_{j_1 \dots j_m} \right) h^2 \, \mathrm{dvol}_g.$$

Note that the inner product induces an equivalent norm on $H^2_k(M, \mu)$. In the case $p = 2$, we will work with the norm induced by the inner product. We can generalise these definitions to the following:

Definition 3.3.3. *Let $s, r \geq 0$ and $p \geq 1$. Define $C_k^p(T_s^r(M), \mu)$ the space of smooth sections $u \in \mathcal{T}_s^r(M)$ such that $|\nabla_c^j u|_g \in L^p(M, \mu)$ for $j = 0, \ldots, k$. The Sobolev space $H_k^p(T_s^r(M), \mu)$ is defined to be the closure of $C_k^p(T_s^r(M), \mu)$ under the norm*

$$\|u\|_{H_k^p(T_s^r(M),\mu)} := \sum_{j=0}^{k} \left(\int_M |\nabla_c^j u|^p \, \mathrm{d}\mu \right)^{\frac{1}{p}}.$$

Moreover, for all such s, r, we define the space $\mathring{H}_k^p(T_s^r(M), \mu)$ as the closure of the set of smooth compactly supported (r, s)-tensors under the $H_k^p(T_s^r(M), \mu)$-norm.

We notice that these definitions make sense for $k \leq 2$ and $p = 2$ if $g \in C^0(M)$ with L^2_{loc}-Christoffel symbols. The following criterion for weak convergence will be useful later. We will repeatedly make use of the following elementary result.

Lemma 3.3.4. *Let M be a smooth manifold, g be a continuous Riemannian metric and h a continuous positive function. Moreover let $p \in [1, \infty)$. Define the measure μ by $\mu := h^2 \mathrm{dvol}_g$. Then $C_c^\infty(M)$ is dense in $L^p(M, \mu)$.*

Lemma 3.3.5. *Let (M, g, μ) be a weighted Riemannian manifold with a C^0-Riemannian metric g with L^2_{loc}-Christoffel symbols and endowed with a measure μ defined via $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$ for a function $h \in C^0(M, (0, \infty))$. Let $(V_\alpha, \psi_\alpha)_\alpha$ be an atlas for M such that $\overline{V_\alpha}$ is compact, for all α.*
Let f_i be a sequence in $L^2(M, \mu)$ and let $f \in L^2(M, \mu)$.
Then $f_i \rightharpoonup f$ in $L^2(M, \mu)$ if and only if $(\psi_\alpha)_* f_i \rightharpoonup (\psi_\alpha)_* f$ in $L^2(\psi_\alpha(V_\alpha), (\psi_\alpha)_\# \mu)$, for all α.
If $f, f_i \in H_k^2(M, \mu)$ $(k \leq 1)$ and $\partial^\beta f_i \rightharpoonup \partial^\beta f$ in $L^2(\psi_\alpha(V_\alpha), (\psi_\alpha)_\# \mu)$ for all α and all $\beta \in \mathbb{N}^n$, $|\beta| \leq k$, then $f_i \rightharpoonup f$ in $H_k^2(M, \mu)$.

Proof. For the first part, note that it suffices to prove convergence of the inner product on a dense subset of $L^2(M, \mu)$. Using Lemma 3.3.4, we can pick a function $\varphi \in C_c(M)$. Let $(V_j)_{j=1}^l$ be a finite cover of $\operatorname{supp} \varphi$ and pick a partition of unity (ρ_j) subordinate to that cover. Now

$$\int_M \varphi f_i h^2 \, \mathrm{dvol}_g = \int_M \sum_{j=1}^{l} \rho_j \varphi f_i h^2 \, \mathrm{dvol}_g = \sum_{j=1}^{l} \int_{\psi_j(V_j)} (\psi_j)_* (\rho_j \varphi f_i) h^2 \sqrt{|g|} \, \mathrm{d}\mathcal{L}^n.$$

If $(\psi_j)_* f_i \rightharpoonup (\psi_j)_* f$, it is easy to see that the above term converges. For the other direction it suffices to note that every function $\varphi \in L^2(\psi_j(V_j), (\psi_j)_* \mu)$ for some j can be extended to a function $\Phi \in L^2(M, \mu)$ via pullback and extension by 0.
For the second part, we simply use the first part and the definition of the $H_k^2(M, \mu)$-inner product. $\square$

Now, we can generalise the validity of the identity (3.2.12) to Sobolev vector fields:

Lemma 3.3.6 (Distributional Ricci curvature for $H_1^2(TM,\mu)$-fields)**.** *Let g be a C^0-Riemannian metric whose Christoffel symbols are in L^2_{loc} and μ be the measure defined by $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$ for a positive $h \in C^0(M,(0,\infty)) \cap W^{1,2}_{\mathrm{loc}}(M)$. Let $X \in H_1^2 \cap L^\infty_{loc}(TM,\mu)$. Then* (3.2.12) *holds.*

Proof. This follows from the density of $C_1^2(TM,\mu)$ in $H_1^2(TM,\mu)$ and the fact that $\phi, |\nabla\phi|_g \in L^1 \cap L^\infty(M)$ have compact support. $\square$

3.3.2 Cheeger energy on a manifold with continuous Riemannian metric and weight

In this subsection, we examine the Cheeger energy in the setting of a smooth n-manifold M endowed with a continuous Riemannian metric g and a continuous weight h^2 on the volume form. The distance induced by g is given by

$$\mathsf{d}_g(x,y) = \inf\Big\{ \int_0^1 |\dot{\gamma_t}|_g \,\mathrm{d}t : \gamma \text{ piecewise } C^\infty, \gamma_0 = x, \gamma_1 = y \Big\}. \tag{3.3.1}$$

We also consider the Riemannian volume form given locally by $\sqrt{|g|}dx^1 \wedge \ldots \wedge dx^n$ and the associated volume measure $\mathrm{dvol}_g = \sqrt{|g|}\mathrm{d}\mathcal{L}^n$. We will consider the metric measure space (M, d_g, μ), where $\mathrm{d}\mu = h^2\mathrm{dvol}_g$ for a positive and continuous h. It is easily checked that μ is a σ-finite Borel measure. We assume that (2.2.1) holds, as we will work with $\mathsf{CD}(K,\infty)$-manifolds (cf. Rem. 2.2.13).

To begin with, we will collect some properties of the metric space arising from a manifold with a continuous Riemannian metric g. The even more general setting of Lipschitz manifolds, with emphasis on potential theory for uniformly elliptic operators, has previously been studied by Norris [107], Saloff-Coste [121], Sturm [131], De Cecco and Palmieri [51]. In that case, the term Lipschitz manifold refers to a topological manifold M with Lipschitz charts, endowed with a Riemannian metric tensor g, such that the coefficients of g and g^{-1} are in L^∞_{loc}. This shall not be confused with metric tensors whose coefficients are locally Lipschitz continuous. More in the spirit of the present section is the work by Burtscher [25]. For the reader's convenience, in the following we will focus on our original results, while trying to be as self-contained as possible.

Proposition 3.3.7 (cf. [112] Thm. 5.3.8, and [25] Thm. 4.5)**.** *Let f be a Lipschitz function on (M,d_g) and $(V_\alpha,\psi_\alpha)_\alpha$ be an atlas for M. Then for each α, $f \circ (\psi_\alpha)^{-1}$ is locally Lipschitz.*

Proof. We show that $|\cdot|_{euc}$ and d_g are locally equivalent. Pick a point $p \in M$, α such that $p \in V_\alpha$ and fix an $\varepsilon > 0$ such that $\overline{B_\varepsilon^{euc}(\psi_\alpha(p))} \subset \psi_\alpha(V_\alpha)$. From now on, we will drop the index α. Take two arbitrary x, y such that $\psi(x), \psi(y) \in B_{\varepsilon/2}^{euc}(\psi(p))$ and let $c : [0,1] \to M$ be a piecewise C^∞-curve from x to y. We note that we can find a $\lambda \geq 1$ such that $\frac{1}{\lambda}\|v\|_{euc} \leq g(v,v) \leq \lambda\|v\|_{euc}$ for all $q \in \psi^{-1}(B_\varepsilon^{euc}(\psi(p)))$ and all $v \in T_qM$. We now distinguish three cases:

1. c is a straight line in the Euclidean metric. Then

$$|x-y|_{euc} = L_{euc}(c) = \int_0^1 |\dot{c}|_{euc} \geq \frac{1}{\lambda}\int_0^1 |\dot{c}|_g \,\mathrm{d}t \geq \frac{1}{\lambda}\mathsf{d}_g(x,y).$$

2. If c is a general curve that lies in $\psi^{-1}(B_\varepsilon^{euc}(\psi(p)))$ then

$$L_g(c) = \int_0^1 |\dot{c}|_g \,\mathrm{d}t \geq \lambda \int_0^1 |\dot{c}|_{euc} \,\mathrm{d}t \geq \lambda|x-y|_{euc}.$$

3. If c leaves $\psi^{-1}(B_\varepsilon^{euc}(\psi(p)))$ then there is a minimal t_0 such that $c(t_0) \notin \psi^{-1}(B_\varepsilon^{euc}(\psi(p)))$ and hence

$$L_g(c) \geq \int_0^{t_0} |\dot{c}|_g \,\mathrm{d}t \geq \lambda \int_0^{t_0} |\dot{c}|_{euc} \,\mathrm{d}t \geq \lambda\frac{\varepsilon}{2} \geq \frac{\lambda}{2}|x-y|_{euc}.$$

This yields that the two metrics are equivalent in $\psi^{-1}(B_{\varepsilon/2}^{euc}(\psi(p))) =: U_p$. Hence, for a d_g-Lipschitz function f and any $p \in M$, there exists a constant L_{U_p} such that f is locally L_{U_p}-Lipschitz with respect to the induced Euclidean metric via ψ on U_p. □

In [25], a mollification argument yields:

Proposition 3.3.8 ([25] Proposition 4.1 and Theorem 3.15)**.** *The metric* (3.3.1) *turns M into a length space.*

The Hopf-Rinow theorem directly implies:

Corollary 3.3.9. *If M is compact, (M, d_g) is a geodesic space.*

A standard fact on the slope of functions is that if $f \in C^1(M)$, then $|Df|(x) = |\nabla f|_g(x)$ for every x. The next proposition is also classical, so the proof is omitted.

Proposition 3.3.10. *Let $\gamma : [0,1] \to M$ be an absolutely continuous curve. Then the metric speed $|\dot{\gamma}_t|$ coincides a.e. with $|\dot{\gamma}_t|_g = \sqrt{\langle\dot{\gamma}_t, \dot{\gamma}_t\rangle_g}$, where $\dot{\gamma}_t$ denotes the (a.e. existing) derivative.*

We will show that if $f \in C^1$, then $|\nabla f|_g$ is the minimal weak upper gradient. The next lemma will be used in the proof of Proposition 3.3.12.

Lemma 3.3.11. *Let M be a smooth manifold equipped with a C^0-Riemannian metric g and a measure μ defined via $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$, for some $h \in C^0(M, (0, \infty))$. Let $k \in L^1_{\mathrm{loc}}(M, \mu)$ and fix some coordinate patch (V_m, ψ_m). Let $x \in \psi_m(V_m)$ and $\theta > 0$, such that $B^{euc}_{4\theta}(x) \subset \psi_m(V_m)$. For $\delta \in (0, \theta) \cap \mathbb{Q}$, $\dot\gamma \in \mathbb{Q}^n \cap B^{euc}_\theta(0)$, we define*

$$F_{\dot\gamma,\delta,x} : t \to \frac{1}{\mathcal{L}^n(B_\delta(x))} \int_{B_\delta(x)} |\dot\gamma|_{g(y+t\dot\gamma)} k(y + t\dot\gamma) \, \mathrm{d}\mathcal{L}^n(y), \quad \forall t, \ |t| < 1.$$

Then for $\mathcal{L}^n$-a.e. $x \in \psi_m(V_m)$ and all $\delta \in (0, \theta) \cap \mathbb{Q}$, $\dot\gamma \in \mathbb{Q}^n \cap B^{euc}_\theta(0)$, we have that $t = 0$ is a Lebesgue point of $F_{\dot\gamma,\delta,x}$.

Proof. As g and h are continuous, we get that $(\psi_m)_* k \in L^1_{\mathrm{loc}}(\psi_m(V_m), \mathcal{L}^n)$. Fix $w \in \psi_m(V_m)$, and choose $\theta > 0$ accordingly. Moreover, fix δ and $\dot\gamma$. For any $z \in B^{euc}_\theta(w)$, the Lebesgue differentiation theorem yields that $\mathcal{L}^1$-almost every $t \in (-1, 1)$ is a Lebesgue point of $F_{\dot\gamma,\delta,z}$. In other words that means that for $\mathcal{L}^1$-almost every y on the line $\{z + t\dot\gamma, |t| < 1\}$, $t = 0$ is a Lebesgue point of $F_{\dot\gamma,\delta,y}$. The set of non-Lebesgue points on that line shall be denoted by $N_{\dot\gamma,\delta,z}$, and it holds that $\mathcal{L}^1(N_{\dot\gamma,\delta,z}) = 0$. Let $H_{\dot\gamma,w}$ be the $(n-1)$-dimensional hyperplane through w that is orthogonal to $\dot\gamma$, and intersected with $B^{euc}_\theta(w)$. Then, by Fubini's Theorem, it holds

$$\mathcal{L}^n\Big(\bigcup_{z \in H_{\dot\gamma,w}} N_{\dot\gamma,\delta,z} \Big) = \int_{H_{\dot\gamma,w}} \mathcal{L}^1(N_{\dot\gamma,\delta,z}) \mathrm{d}\mathcal{L}^{n-1}(z) = 0.$$

Hence for $\mathcal{L}^n$-almost every $y \in B^{euc}_\theta(w)$, $t = 0$ is a Lebesgue point of $F_{\dot\gamma,\delta,y}$. Note that, by construction, $\dot\gamma$ and δ vary in a countable set. This shows the claim in an open neighbourhood around w. As w was arbitrary, the proof is complete. □

Proposition 3.3.12. *Let M be a smooth manifold, g a C^0-Riemannian metric on M and μ a measure on M defined by $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$, where $h \in C^0(M, (0, \infty))$. Let f be a C^1-function. Then*

$$|Df|_w(x) = |\nabla f|_g(x), \quad \textit{for } \mu\textit{-a.e. } x.$$

Proof. First, we observe that $|\nabla f|_g$ is a weak upper gradient because for any absolutely continuous curve γ, we have

$$|f(\gamma_1) - f(\gamma_0)| = \left| \int_0^1 \langle \nabla f, \dot\gamma_t \rangle_g \big|_{\gamma_t} \, \mathrm{d}t \right| \leq \int_0^1 |\nabla f|_g(\gamma_t) |\dot\gamma_t|_g \, \mathrm{d}t.$$

To see that it is minimal, we argue by contradiction. Suppose that $|\nabla f|_g$ is not minimal. Then there exists an $\varepsilon \in (0, \frac{1}{4})$ such that $\mu(\{|\nabla f|_g - |Df|_w > \varepsilon\}) > 0$. We can then find an open set V such that $\overline{V}$ is compact, is contained in one coordinate patch and

$\mu(\{|\nabla f|_g - |Df|_w > \varepsilon\} \cap V) > 0$. We denote by $\psi : V \to \mathbb{R}^n$ the associated coordinate chart. By the continuity of g and h, there exists a constant $C = C(g, h, V)$ such that for each Borel set $E \subset V$, $\frac{1}{C}\mathcal{L}^n(\psi(E)) \le \mu(E) \le C\mathcal{L}^n(\psi(E))$. For almost all $x \in \psi(V)$ take $\theta, \dot\gamma, \delta$ as in Lemma 3.3.11. It still holds $\mathcal{L}^n(\{|\nabla f|_g - |Df|_w > \varepsilon\} \cap V) > 0$. By the Lebesgue differentiation theorem and Lemma 3.3.11 with $k = |Df|_w$, we get that there exists an $x \in \{|\nabla f|_g - |Df|_w > \varepsilon\} \cap V$ such that x is a Lebesgue point of $|Df|_w$, and for all $\dot\gamma, \delta$ as above, $t = 0$ is a Lebesgue point of $F_{\dot\gamma,\delta,x}$. Set

$$\eta = \frac{1}{101} \min\left(1, \frac{1}{\|\nabla f\|_{L^\infty}}, \frac{1}{2n^3\|g\|_{L^\infty}}\right)^2.$$

Choose $\dot\gamma \in \mathbb{Q}^n$ such that and

$$\frac{99}{100} \le |\dot\gamma|_g(x) \le \frac{101}{100} \quad \text{and} \quad |\langle \nabla f, \dot\gamma\rangle_g(x) - |\nabla f|_g(x)|\dot\gamma|_g(x)| \le \frac{1}{4}\eta\varepsilon. \tag{3.3.2}$$

Note that if $t = 0$ is a Lebesgue-point of $F_{x,\dot\gamma,\delta}$ then it is a Lebesgue-point of $F_{x,\alpha\dot\gamma,\delta}$ for all $\alpha \in \mathbb{Q}$.

Now choose $q \in \mathbb{Q} \cap (0, \theta)$ such that for all $\delta \in (0, q)$, it holds

$$\frac{1}{\mathcal{L}^n(B_\delta(x))} \int_{B_\delta(x)} (|Df|_w(y) - |Df|_w(x)) < \eta\varepsilon, \tag{3.3.3}$$

$$||\nabla f|_g(y) - |\nabla f|_g(z)| \le \frac{1}{2}\eta\varepsilon, \ \forall y, z \in B_{3\delta}(x), \tag{3.3.4}$$

$$|g_{ij}(y) - g_{ij}(z)| \le \frac{1}{2}\eta\varepsilon, \ \forall 1 \le i, j \le n, \ y, z \in B_{3\delta}(x). \tag{3.3.5}$$

The first inequality can be achieved because x is a Lebesgue point of $|Df|_w$ and the remaining inequalities follow from the continuity of g and ∇f. We can now compute that for $s \le q$ and $y \in B_q(x)$,

$$|f(y - s\dot\gamma) - f(y + s\dot\gamma)| = \int_{-s}^{s} \langle \nabla f, \dot\gamma\rangle_g(y + t\dot\gamma)\, \mathrm{d}t.$$

Using (3.3.2), (3.3.4), and (3.3.5), we get that for $t \in [-s, s]$ it holds

$$\begin{aligned} |\langle \nabla f, \dot\gamma\rangle_g(y + t\dot\gamma) - |\nabla f|_g|\dot\gamma|_g(x)| \le & |\langle \nabla f(y + t\dot\gamma), \dot\gamma\rangle_{g(y+t\dot\gamma)-g(x)}| \\ & + |\langle \nabla f(y + t\dot\gamma) - \nabla f(x), \dot\gamma\rangle_{g(x)}| \\ & + |\langle \nabla f, \dot\gamma\rangle_g(x) - |\nabla f|_g(x)|\dot\gamma|_g(x)| \le \frac{3\varepsilon}{100}. \end{aligned}$$

Hence, for all $\delta \in (0, q)$, $y \in B_\delta(x)$,

$$|(|f(y - s\dot\gamma) - f(y + s\dot\gamma)|) - (2s|\nabla f|_g(x)|\dot\gamma|_g(x))| \le \frac{6s\varepsilon}{100}. \tag{3.3.6}$$

Moreover, again by (3.3.2), (3.3.4), and (3.3.5), we get that

$$\Big|\frac{1}{|B_q(x)|}\int_{B_q(x)}|\nabla f|_g(y)\mathrm{d}\mathcal{L}^n(y)-|\nabla f|_g(x)\Big|\le\frac{1}{100}\varepsilon, \tag{3.3.7}$$

$$\Big|\frac{1}{|B_q(x)|}\int_{B_q(x)}|\nabla f|_g|\dot\gamma|_g(y)\mathrm{d}\mathcal{L}^n(y)-|\nabla f|_g|\dot\gamma|_g(x)\Big|\le\frac{1}{100}\varepsilon, \tag{3.3.8}$$

and (3.3.3) yields that

$$\Big|\frac{1}{|B_q(x)|}\int_{B_q(x)}|Df|_w(y)\mathrm{d}\mathcal{L}^n(y)-|Df|_w(x)\Big|\le\frac{1}{100}\varepsilon, \tag{3.3.9}$$

thus,

$$\frac{1}{|B_q(x)|}\int_{B_q(x)}(|\nabla f|_g-|Df|_w)(y)\mathrm{d}\mathcal{L}^n(y)>\frac{9\varepsilon}{10}. \tag{3.3.10}$$

Using that $t=0$ is a Lebesgue point of $F_{\dot\gamma,q,x}$, and the fact that g and $|\nabla f|_g$ are continuous, we can choose $s\in(0,q)$ small enough such that

$$\begin{aligned}\Bigg|\frac{1}{|B_q(x)|}&\int_{-s}^{s}\int_{B_q(x)}|\nabla f|_g(y+t\dot\gamma)|\dot\gamma|_{g(y+t\dot\gamma)}\,\mathrm{d}\mathcal{L}^n(y)\mathrm{d}t\\&-2s\frac{1}{|B_q(x)|}\int_{B_q(x)}|\nabla f|_g(y)|\dot\gamma|_g(y)\mathrm{d}\mathcal{L}^n(y)\Bigg|\le\frac{2s\varepsilon}{100},\end{aligned} \tag{3.3.11}$$

and

$$\begin{aligned}\Bigg|\frac{1}{|B_q(x)|}&\int_{-s}^{s}\int_{B_q(x)}|Df|_w(y+t\dot\gamma)|\dot\gamma|_{g(y+t\dot\gamma)}\,\mathrm{d}\mathcal{L}^n(y)\mathrm{d}t\\&-2s\frac{1}{|B_q(x)|}\int_{B_q(x)}|Df|_w(y)|\dot\gamma|_g(y)\mathrm{d}\mathcal{L}^n(y)\Bigg|\le\frac{2s\varepsilon}{100}.\end{aligned} \tag{3.3.12}$$

Now, we have all the ingredients to construct a test plan that yields a contradiction. For $y\in B_q(x)$ define $\gamma_y:[0,1]\to\psi_m(V_m)$, $t\mapsto y+2(t-\frac{1}{2})s\dot\gamma$ and define a test plan $\boldsymbol{\pi}$ as

$$\mathrm{d}\boldsymbol{\pi}(\gamma):=\begin{cases}\frac{1}{|B_q(x)|}\mathrm{d}\mathcal{L}^n(y) & \text{if } \gamma=\gamma_y,\ \text{for some } y\in B_q(x),\\ 0 & \text{otherwise.}\end{cases}$$

Now, we can directly compute that

$$\int|f(\gamma_1)-f(\gamma_0)|\mathrm{d}\boldsymbol{\pi}(\gamma)=\frac{1}{|B_q(x)|}\int_{B_q(x)}|f(y-s\dot\gamma)-f(y+s\dot\gamma)|\,\mathrm{d}\mathcal{L}^n(y). \tag{3.3.13}$$

Moreover,

$$\begin{aligned}\iint_0^1&|\nabla f|_g(\gamma_t)|\dot\gamma_t|_{g(\gamma_t)}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\&=\frac{1}{|B_q(x)|}\int_{B_q(x)}\int_0^1|\nabla f|_g\Big(y+2\Big(t-\frac{1}{2}\Big)s\dot\gamma\Big)2s|\dot\gamma|_{g((\gamma_y)_t)}\,\mathrm{d}t\,\mathrm{d}\mathcal{L}^n(y)\\&=\frac{1}{|B_q(x)|}\int_{B_q(x)}\int_{-s}^{s}|\nabla f|_g(y+t\dot\gamma)|\dot\gamma|_{g((\gamma_y)_t)}\,\mathrm{d}t\,\mathrm{d}\mathcal{L}^n(y),\end{aligned} \tag{3.3.14}$$

and similarly

$$\iint_0^1 |Df|_w(\gamma_t)|\dot\gamma_t|_{g(\gamma_t)}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)$$
$$= \frac{1}{|B_q(x)|}\int_{B_q(x)}\int_{-s}^{s} |Df|_w(y+t\dot\gamma)|\dot\gamma|_{g((\gamma_y)_t)}\,\mathrm{d}t\,\mathrm{d}\mathcal{L}^n(y).$$

We then have that

$$\begin{aligned}
&\iint_0^1 |\nabla f|_g(\gamma_t)|\dot\gamma_t|_{g(\gamma_t)}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) - \iint_0^1 |Df|_w(\gamma_t)|\dot\gamma_t|_{g(\gamma_t)}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\\
&\quad= \frac{1}{|B_q(x)|}\int_{B_q(x)}\int_{-s}^{s}(|\nabla f|_g - |Df|_w)(y+t\dot\gamma)|\dot\gamma|_{g(y+t\dot\gamma)}\,\mathrm{d}t\mathrm{d}\mathcal{L}^n(y)\\
&\quad\overset{(3.3.12),(3.3.11)}{\geq} 2s\frac{1}{|B_q(x)|}\int_{B_q(x)}(|\nabla f|_g - |Df|_w)|\dot\gamma|_g(y)\,\mathrm{d}\mathcal{L}^n(y) - \frac{4s\varepsilon}{100}\\
&\quad\overset{(3.3.10)}{>} \frac{99}{100}\frac{9s\varepsilon}{10} - \frac{4s\varepsilon}{100} \geq \frac{3s\varepsilon}{2} > 0.
\end{aligned}\tag{3.3.15}$$

The combination of the above estimates gives:

$$\begin{aligned}
\int |f(\gamma_1)-f(\gamma_0)|\mathrm{d}\boldsymbol{\pi}(\gamma) &\overset{(3.3.13),(3.3.6)}{\geq} 2s|\nabla f|_g(x)|\dot\gamma|_g(x) - \frac{6s\varepsilon}{100}\\
&\overset{(3.3.8)}{\geq} \frac{2s}{|B_q(x)|}\int_{B_q(x)}|\nabla f|_g|\dot\gamma|_g(y)\mathrm{d}\mathcal{L}^n(y) - \frac{s\varepsilon}{10}\\
&\overset{(3.3.11)}{\geq} \frac{1}{|B_q(x)|}\int_{-s}^{s}\int_{B_q(x)}|\nabla f|_g(y+t\dot\gamma)|\dot\gamma|_{g(y+t\dot\gamma)}\,\mathrm{d}\mathcal{L}^n(y)\mathrm{d}t - \frac{3s\varepsilon}{20}\\
&\overset{(3.3.14)}{=} \iint_0^1 |\nabla f|_g(\gamma_t)|\dot\gamma_t|_{g(\gamma_t)}\,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) - \frac{3s\varepsilon}{20}\\
&\overset{(3.3.15)}{>} \iint_0^1 |Df|_w(\gamma_t)|\dot\gamma_t|_{g(\gamma_t)}\,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) - \frac{3s\varepsilon}{20} + \frac{3s\varepsilon}{2}.
\end{aligned}$$

This yields a contradiction, as $|Df|_w$ fails to satisfy the condition of a weak upper gradient for the chosen test plan $\boldsymbol{\pi}$. $\square$

In the following, we will investigate $W^{1,2}_w(X)$ for $(X,\mathsf{d},\mathfrak{m}) = (M,\mathsf{d}_g,\mu)$, with $\mathrm{d}\mu = h^2\mathrm{dvol}_g$.

A mollification argument together with Rademacher's theorem and Proposition 3.3.12 shows that:

Proposition 3.3.13. *Let (M,d_g,μ) be the metric measure space arising from a smooth manifold M a continuous Riemannian metric g and a measure given by $\mathrm{d}\mu = h^2\mathrm{dvol}_g$ for a function $h \in C^0(M,(0,\infty))$. Let f be a locally d_g-Lipschitz function. Then f is differentiable μ-almost everywhere. If in addition $f, |\nabla f|_g \in L^2(M,\mu)$ then $|Df|_w = |\nabla f|_g$ almost everywhere, moreover $f \in W^{1,2}_w(M)\cap H^2_1(M,\mu)$, and $\|f\|_{W^{1,2}_w(M)} = \|f\|_{H^2_1(M,\mu)}$.*

Proof. Throughout this proof, we will fix a locally finite cover $(B_i, \psi_i)_{i=1}^{\infty}$ of M such that each B_i is a regular coordinate ball. As f is locally d_g-Lipschitz, we can apply Proposition 3.3.7 (or Theorem 4.5 in [25]), to see that for each $i \geq 1$, $(\psi_i)_* f$ is locally Lipschitz continuous with respect to the Euclidean metric on $\psi_i(B_i)$. By Rademacher's theorem, we have that for each i, $(\psi_i)_* f$ is differentiable $\mathcal{L}^n$-almost everywhere in $\psi_i(B_i)$. As $(\psi_i)_{\#}\mu$ is absolutely continuous with respect to $\mathcal{L}^n$ on $\psi_i(B_i)$, we get that $(\psi_i)_* f$ is differentiable μ-almost everywhere in $\psi_i(B_i)$. As M is the countable union of the B_i, we get that f is differentiable μ-almost everywhere in M, hence the function $|\nabla f|_g$ is well-defined in $L^{\infty}_{loc}(M, \mu)$.

Assume $f, |\nabla f|_g \in L^2(M, \mu)$. We will prove that $|\nabla f|_g$ is a weak upper gradient for f. Let $\varepsilon > 0$. Fix a partition of unity ζ_i subordinate to the cover B_i and proceed as follows. For each $i = 1, 2, \ldots$, choose $0 < \delta_i \leq \operatorname{dist}^{euc}(\operatorname{supp} \zeta_i, \partial \psi_i(B_i))$ such that

$$\begin{aligned}
&\|\rho_{\delta_i} * (\psi_i)_* \zeta_i f - (\psi_i)_* \zeta_i f\|_{C^0(\psi_i(B_i))} \leq \frac{\varepsilon}{2^i} \quad \text{and} \\
&\|\rho_{\delta_i} * (\psi_i)_* \zeta_i f - (\psi_i)_* \zeta_i f\|_{W^{1,2}(\psi_i(B_i),(\psi_i)_{\#}\mu)} \\
&\leq \frac{\varepsilon}{2^i(1 + n^2(\|(\psi_i)_* g\|_{L^\infty(\psi_i(B_i))} + \|(\psi_i)_* g^{-1}\|_{L^\infty(\psi_i(B_i))}))}.
\end{aligned}$$

This is possible by classical results about L^p-spaces. Define $f_\varepsilon = \sum_{i=1}^{\infty} \rho_{\delta_i} * (\psi_i)_* \zeta_i f$. As the cover B_i is locally finite, f_ε is well-defined. That produces a sequence $(f_\varepsilon)_{\varepsilon > 0}$. Note that by construction, we have that for every point $p \in M$, it holds

$$|f_\varepsilon(p) - f(p)| \leq \varepsilon.$$

Moreover,

$$\begin{aligned}
\|f_\varepsilon - f\|_{H^2_1(M,\mu)} \leq & n^2 \sum_{i=1}^{\infty} (\|g\|_{L^\infty(\psi_i(B_i))} + \|g^{-1}\|_{L^\infty(\psi_i(B_i))}) \\
& \cdot \|\rho_{\delta_i} * (\psi_i)_* \zeta_i f - (\psi_i)_* \zeta_i f\|_{W^{1,2}(\psi_i(B_i),(\psi_i)_{\#}\mu)} \leq \varepsilon.
\end{aligned}$$

Fix a test plan $\boldsymbol{\pi}$. For all $\varepsilon \in (0, 1]$, it holds that

$$\begin{aligned}
\int |f_\varepsilon(\gamma_0) - f_\varepsilon(\gamma_1)| \, \mathrm{d}\boldsymbol{\pi}(\gamma) &= \int \left| \int_0^1 \langle \nabla f_\varepsilon(\gamma_t), \dot{\gamma}_t \rangle_{g(\gamma_t)} \, \mathrm{d}t \right| \mathrm{d}\boldsymbol{\pi}(\gamma) \\
&\leq \int \int_0^1 |\nabla f_\varepsilon(\gamma_t)|_{g(\gamma_t)} |\dot{\gamma}_t|_{g(\gamma_t)} \, \mathrm{d}t \, \mathrm{d}\boldsymbol{\pi}(\gamma). \qquad (3.3.16)
\end{aligned}$$

As $\varepsilon \leq 1$, we have that $|f_\varepsilon(x) - f_\varepsilon(x')| \leq 2 + |f(x) - f(x')|$, for any $x, x' \in M$. Now we can use that $\boldsymbol{\pi}$ is a probability measure together with the Hölder inequality, to get that

$$\begin{aligned}\left(\int |f(\gamma_0) - f(\gamma_1)| + 2\,\mathrm{d}\boldsymbol{\pi}(\gamma)\right)^2 &\leq 8 + 2\int |f(\gamma_0) - f(\gamma_1)|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma) \\ &\leq 8 + 4\int |f(\gamma_0)|^2 + |f(\gamma_1)|^2\,\mathrm{d}\boldsymbol{\pi}(\gamma) \\ &= 8 + 4\int |f|^2\,\mathrm{d}(e_0)_\#\boldsymbol{\pi}(\gamma) + 4\int |f|^2\,\mathrm{d}(e_1)_\#\boldsymbol{\pi}(\gamma) \\ &\leq 8 + 8C(\boldsymbol{\pi})\|f\|_{L^2}^2.\end{aligned}$$

Hence, we can use dominated convergence theorem to infer that

$$\int |f_\varepsilon(\gamma_0) - f_\varepsilon(\gamma_1)|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \to \int |f(\gamma_0) - f(\gamma_1)|\,\mathrm{d}\boldsymbol{\pi}(\gamma), \tag{3.3.17}$$

as $\varepsilon \to 0$. Moreover, we have that for all $k \in L^2(M,\mu)$, it holds

$$\int\int_0^1 |k|(\gamma_t)|\dot{\gamma}_t|_{g(\gamma_t)}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \left(\int\int_0^1 |k|^2(\gamma_t)\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\right)^{\frac{1}{2}}\left(\int\int_0^1 |\dot{\gamma}_t|^2_{g(\gamma_t)}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma)\right)^{\frac{1}{2}}.$$

By the definition of test plans, we have that

$$\int\int_0^1 |\dot{\gamma}_t|^2_{g(\gamma_t)}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma) < \infty,$$

and, using Fubini's theorem,

$$\int\int_0^1 |k|^2(\gamma_t)|\,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) = \int_0^1\int |k|^2\,\mathrm{d}(e_t)_\#\boldsymbol{\pi}\mathrm{d}t \leq C(\boldsymbol{\pi})\|k\|_{L^2}^2.$$

Now we can use the L^2-convergence of $\nabla f_\varepsilon \to \nabla f$ and choose $k = |\nabla f_\varepsilon - \nabla f|_g$ to see that

$$\int\int_0^1 |\nabla f_\varepsilon(\gamma_t)|_{g(\gamma_t)}|\dot{\gamma}_t|_{g(\gamma_t)}\,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) \to \int\int_0^1 |\nabla f(\gamma_t)|_{g(\gamma_t)}|\dot{\gamma}_t|_{g(\gamma_t)}\,\mathrm{d}t\,\mathrm{d}\boldsymbol{\pi}(\gamma), \tag{3.3.18}$$

as $\varepsilon \to 0$. The combination of (3.3.17) and (3.3.18) allows passing to the limit as $\varepsilon \to 0$ in (3.3.16), and obtain

$$\int |f(\gamma_0) - f(\gamma_1)|\,\mathrm{d}\boldsymbol{\pi}(\gamma) \leq \int\int_0^1 |\nabla f(\gamma_t)|_{g(\gamma_t)}|\dot{\gamma}_t|_{g(\gamma_t)}\,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma),$$

hence, $|\nabla f|_g$ is a weak upper gradient. In particular, we get that for all Lipschitz functions f, it holds $\|f\|_{W^{1,2}_w(M)} \leq \|f\|_{H^2_1(M,\mu)}$. As $f \in H_1^2(M,\mu)$, we can find a sequence $f_p \in H_1^2(M,\mu) \cap C^1(M)$ such that $f_p \to f$ in $H_1^2(M,\mu)$ as $p \to \infty$. This implies that $f_p \to f$ in $W^{1,2}_w(M)$ and by Proposition 3.3.12, $|\nabla f_p|_g = |Df_p|_w \to |\nabla f|_g$ in $L^2(M,\mu)$, hence $|\nabla f|_g = |Df|_w$ almost everywhere, and $\|f\|_{W^{1,2}_w(M)} = \|f\|_{H^2_1(M,\mu)}$. □

By [7, Rem. 5.5], the slope $|Df|$ of a d_g-Lipschitz function f is a weak upper gradient. Thus, if $f \in L^2(M,\mu)$ is a d_g-Lipschitz function such that $|Df| \in L^2(M,\mu)$, then Proposition 3.3.13 implies that:

$$|Df|_w = |\nabla f|_g \leq |Df|. \tag{3.3.19}$$

This observation is crucial for the following lemma.

Lemma 3.3.14. *Let M be a smooth manifold, g a C^0-Riemannian metric on M, d_g the distance induced by g, vol_g the volume form, and h be a positive continuous function. Define the measure μ by $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$. Then $W^{1,2}_w(M) \subset H^2_1(M,\mu)$ and for each $f \in W^{1,2}_w(M)$, it holds $|Df|_w = |\nabla f|_g$ a.e..*

Proof. Let $f \in W^{1,2}_w(M)$. By [8, (2.22)], we can find a sequence of Lipschitz functions $(f_l)_{l=1}^{\infty}$ such that $f_l \to f$ in L^2 and $|Df_l| \to |Df|_w$ in L^2. Using (3.3.19), we get that $(f_l)_{l=1}^{\infty}$ is a bounded sequence in $H^2_1(M,\mu)$. As this space is reflexive, we can find a subsequence that converges weakly in $H^2_1(M,\mu)$. As $f_l \to f$ strongly in L^2, we get that $f_l \rightharpoonup f$ in $H^2_1(M,\mu)$. As we now know that $f \in H^2_1(M,\mu)$, we can approximate it with a different sequence $f_k \in H^2_1(M,\mu) \cap C^2_1(M,\mu)$, i.e. $\lim_{k\to\infty} \|f_k - f\|_{H^2_1(M,\mu)} = 0$. Then f_k is a Cauchy sequence in $H^2_1(M,\mu)$. As for $C^2_1(M,\mu)$-functions $\|\cdot\|_{W^{1,2}_w(M)} = \|\cdot\|_{H^2_1(M,\mu)}$ (by Proposition 3.3.12), it is a Cauchy sequence in $W^{1,2}_w$. Hence $|\nabla f_k|_g = |Df_k|_w \to |Df|_w$ in L^2, which in particular shows that $|Df|_w = |\nabla f|_g$ almost everywhere. $\square$

We can now finally identify the space $W^{1,2}_w$ with the classical Sobolev space H^2_1:

Corollary 3.3.15. *Let M be a smooth manifold with a C^0-Riemannian metric g. Consider the metric measure space (M, d_g, μ), where $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$ for a continuous positive function h. Then $W^{1,2}_w(M) = H^2_1(M,\mu)$ and the Dirichlet form associated to the Cheeger energy on (M,d_g,μ) is a quadratic form given by the $L^2(M,\mu)$-inner product of the gradients, more precisely for all functions $f, h \in W^{1,2}_w(M) = H^2_1(M,\mu)$ it holds*

(i) $|Df|_w = |\nabla f|_g$.

(ii) $\mathcal{E}(f,h) = \int_M \langle \nabla f, \nabla h\rangle_g \mathrm{d}\mu$

Moreover, every $f \in D(\mathrm{Ch})$ with $|Df|_w \leq 1$ μ-a.e. admits a 1-Lipschitz representative μ-a.e. with respect to d_g.

Proof. We prove that $H^2_1(M,\mu) \subset W^{1,2}_w(M)$. At first we notice that by Proposition 3.3.12, $C^2_1(M,\mu) \subset W^{1,2}_w(M)$ and for each $f \in C^2_1(M,\mu)$, we have that $\|f\|_{H^2_1(M,\mu)} = \|f\|_{W^{1,2}_w(M)}$. Now, fix $f \in H^2_1(M,\mu)$ and approximate it with a sequence $f_k \in C^2_1(M,\mu)$. This is then a Cauchy sequence in $H^2_1(M,\mu)$ and hence in $W^{1,2}_w(M)$, by the norm equality we proved in Lemma 3.3.14. Hence, f_k converges in $W^{1,2}_w(M)$ to some $\tilde{f}$. It follows that $f_k \to \tilde{f}$ in $L^2(M,\mu)$, hence, $\tilde{f} = f \in W^{1,2}_w(M)$. We get $W^{1,2}_w(M) = H^2_1(M,\mu)$ from Lemma 3.3.14.

For the second part, we use that by Proposition 2.2.6, the associated Dirichlet form $\mathcal{E}$ is defined on $(H^2_1(M,\mu))^2$ and is of the form

$$\mathcal{E}(f,k) = \int_M G(f,k)\,\mathrm{d}\mu,$$

where

$$G(f,k) = \lim_{\varepsilon\searrow 0} \frac{|D(f+\varepsilon k)|_w^2 - |Df|_w^2}{2\varepsilon} = \langle \nabla f, \nabla k\rangle_g \in L^1(X,\mathfrak{m}). \tag{3.3.20}$$

For the last part, without loss of generality by localising to a suitable coordinate patch and by using a partition of unity, we can assume that f has compact support in an open set $U \subset \mathbb{R}^n$ endowed with a continuous Riemannian metric g such that for every $x \in U$ it holds that $C^{-1}\,\mathrm{Id}_n \le g \le C\,\mathrm{Id}_n$, for some constant $C \ge 1$ uniform on U. It follows that f lies in $W^{1,\infty}(\mathbb{R}^n)$ and thus, by the classical result in $\mathbb{R}^n$, f has a Lipschitz representative $\mathcal{L}^n$-a.e. that we identify with f. Hence, f is Lipschitz also in (M,g) and, by Proposition 3.3.13 and the assumption on f, it holds that $|\nabla f|_g = |Df|_w \le 1$ μ-a.e. Let $\rho_\delta(x) := \frac{1}{\delta^n}\rho(\frac{x}{\delta})$, $\delta > 0$, be standard mollifiers in $\mathbb{R}^n$ and consider $f_\delta := f \star \rho_\delta$. It is easily seen that

$$|\nabla f_\delta|_g \le 1 + \theta(\delta) \quad \text{and} \quad f_\delta \to f \quad \text{uniformly as } \delta \to 0. \tag{3.3.21}$$

Here $\lim_{\delta\to 0}\theta(\delta) = 0$. Let $x, y \in M$, $x \ne y$. By Proposition 3.3.8, for every $\varepsilon \in (0, \mathsf{d}_g(x,y)/2)$ there exists a rectifiable curve $\gamma : [0,1] \to M$ parametrised by arc-length such that

$$\mathrm{length}_g(\gamma) \le \mathsf{d}_g(x,y) + \varepsilon, \quad |\dot\gamma| = \mathrm{length}_g(\gamma),\ \mathcal{L}^1\text{-a.e. on } [0,1]. \tag{3.3.22}$$

Then

$$\begin{aligned} |f_\delta(x) - f_\delta(y)| &= \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} f_\delta(\gamma_t)\,\mathrm{d}t \right| \le \int_0^1 |\dot\gamma_t|_g\, |\nabla f_\delta|_g\,\mathrm{d}t \\ &\overset{(3.3.21),(3.3.22)}{\le} (1+\theta(\delta))(\mathsf{d}_g(x,y)+\varepsilon). \end{aligned}$$

Passing to the limit as $\varepsilon \to 0$, we get that f_δ is $1+\theta(\delta)$-Lipschitz with respect to d_g. Passing further to the limit as $\delta \to 0$ and recalling the second in (3.3.21) we conclude that f is 1-Lipschitz with respect to d_g. □

3.4 Second order calculus

In this section, we will specialise the second order calculus developed in [63, Chapter 3] to the setting of a smooth manifold with a C^0-Riemannian metric with L^2_{loc}-Christoffel symbols and $C^0 \cap W^{1,2}_{\mathrm{loc}}$-weight on the measure.

3.4.1 Some elements of the theory for general $\mathsf{RCD}(K,\infty)$-spaces

Let $(M, \mathsf{d}, \mathfrak{m})$ be an infinitesimally Hilbertian metric measure space. Recall that

$$D(\Delta) := \{f \in L^2(\mathfrak{m}) : \Delta f \in L^2(\mathfrak{m})\} \subset W^{1,2}_w(M).$$

Denote by $\mathrm{Const}(M)$ the space of constant functions on M. Following the notation of [63, Chapter 3], we define the space of test functions

$$\mathrm{TestF}(M) := \big\{f \in D(\Delta) \cap L^\infty(M, \mathfrak{m}) : \ |\nabla f| \in L^\infty(M, \mathfrak{m}) \text{ and } \Delta f \in W^{1,2}_w(M)\big\},$$

test vector fields

$$\begin{aligned}\mathrm{TestV}(M) := \Big\{ \sum_{i=1}^n h_i \nabla f_i : \ & n \in \mathbb{N} \ f_i \in \mathrm{TestF}(M), \\ & h_i \in \mathrm{TestF}(M) \cup \mathrm{Const}(M)\} \ 1 \leq i \leq n \Big\},\end{aligned}$$

and test forms

$$\begin{aligned}\mathrm{TestForm}_k(M) :=& \big\{\text{linear combinations of forms of the kind} \\ & f_0 df_1 \wedge \ldots \wedge df_k : \ f_1, \ldots f_k \in \mathrm{TestF}(M), f_0 \in \mathrm{TestF} \cup \mathrm{Const}(M)\big\},\end{aligned}$$

for $1 \leq k \leq n$. Set $\mathrm{TestForm}_0(M) = \mathrm{TestF}(M)$. For the notion of gradients and differentials in metric measure spaces, we refer to [63, Chapter 2]. We will apply this theory to weighted manifolds with continuous metrics and weights. In that case, the non-smooth notions of [63] coincide with the classical gradients and differentials on manifolds.

All expressions are well-defined, because $f \in \mathrm{TestF}(M)$ implies that $f \in W^{1,2}_w$, so the differential and the gradient are well-defined. A regularization result due to Savaré [123] (see also [63, (3.1.5)]), yields that if $f \in L^2 \cap L^\infty(M, \mathfrak{m})$ then for all $t > 0$, $f_t := H_t f \in \mathrm{TestF}(M)$. In the next proposition we recall two important density results proved in [63, Proposition 2.2.5] for general metric measure spaces. In the setting of a smooth manifold with a continuous metric and a continuous weight, we will prove an even stronger result in Lemma 3.5.10.

Proposition 3.4.1. $\mathrm{TestForm}_1(M)$ *is dense in* $L^2(T^*M)$ *and* $\mathrm{TestV}(M)$ *is dense in* $L^2(TM)$.

In [63] it is shown that for $f, g \in \mathrm{TestF}(M)$, then $\langle \nabla f, \nabla g\rangle \in W^{1,2}_w(M)$, which allows to define a notion of Hessian $H[f] : [\mathrm{TestF}(M)]^2 \to L^2(M)$ of a function $f \in \mathrm{TestF}(M)$ as

$$H[f](g,h) = \frac{1}{2}\big(\langle \nabla\langle\nabla f, \nabla g\rangle, \nabla h\rangle + \langle \nabla\langle\nabla f, \nabla h\rangle, \nabla g\rangle - \langle \nabla\langle\nabla g, \nabla h\rangle, \nabla f\rangle\big).$$

Definition 3.4.2. *The space $D(\mathbf{\Delta}) \subset W^{1,2}_w(M)$ is the space of functions $f \in W^{1,2}_w(M)$ such that there exists a measure $\nu \in \mathrm{Meas}(M)$ satisfying*

$$\int h \,\mathrm{d}\nu = -\int \langle \nabla h, \nabla f\rangle \,\mathrm{d}\mathfrak{m},$$

for all $h : M \to \mathbb{R}$ Lipschitz with bounded support. In this case the measure ν is unique and denoted by $\mathbf{\Delta} f$.

By [63, Lemma 3.2.6], we have that if $(M, \mathsf{d}, \mathfrak{m})$ is $\mathsf{RCD}(K, \infty)$, and $X, Y \in \mathrm{TestV}(M)$, then $\langle X, Y\rangle \in D(\mathbf{\Delta})$. Define

$$D_{W^{1,2}_w}(\Delta) = \{f \in W^{1,2}_w(M),\ \Delta f \in W^{1,2}_w(M)\}.$$

We are now able to define the measure valued operator $\mathbf{\Gamma}_2 : [\mathrm{TestF}(M)]^2 \to \mathrm{Meas}(M)$ given by

$$\mathbf{\Gamma}_2(f,g) := \frac{1}{2}\mathbf{\Delta}\langle\nabla f, \nabla g\rangle - \frac{1}{2}\big(\langle \nabla\Delta f, \nabla g\rangle + \langle \nabla f, \nabla \Delta g\rangle\big).$$

In the next definition we recall the Bakry-Émery condition $\mathrm{BE}(K, N)$, the reader is referred to [9, 10, 55] for more details.

Definition 3.4.3. *Let $K \in \mathbb{R}$ and $N \in [1, \infty]$. We say that $(M, \mathsf{d}, \mathfrak{m})$ satisfies the BE(K, N) condition if for every $f \in D_{W^{1,2}_w}(\Delta)$ and every $\phi \in D(\Delta)$ with $\phi, \Delta\phi \in L^\infty$, it holds*

$$\int_M \frac{1}{2}\Delta\phi\langle\nabla f, \nabla f\rangle - \phi\langle \nabla\Delta f, \nabla f\rangle \,\mathrm{d}\mathfrak{m} \geq \int_M \Big(K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2\Big)\phi \,\mathrm{d}\mathfrak{m}.$$

In that case, we have that for $f \in \mathrm{TestF}(M)$, it holds

$$\mathbf{\Gamma}_2(f,f) \geq \Big(K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2\Big)\mathfrak{m}.$$

The following important fact was proved for $N = \infty$ in [8, 9] (see also [65, 6]) and for $N \in [1, \infty)$ in [55, Section 4] and [11, Section 12]:

Theorem 3.4.4. *Let $(X, \mathsf{d}, \mathfrak{m})$ be a metric measure space, let $K \in \mathbb{R}$ and $N \in [1, \infty]$. The following are equivalent:*

(i) $(X,\mathsf{d},\mathfrak{m})$ *is a* $\mathsf{RCD}^*(K,N)$*-space or, in case* $N=\infty$, $(X,\mathsf{d},\mathfrak{m})$ *is a* $\mathsf{RCD}(K,\infty)$*-space.*

(ii) $(X,\mathsf{d},\mathfrak{m})$ *satisfies* (2.2.1)*, the* $\mathsf{BE}(K,N)$*-condition, and it is infinitesimally Hilbertian. Moreover, every* $f\in D(\mathrm{Ch})$ *with* $|Df|_w\le 1$ $\mathfrak{m}$*-a.e. admits a* 1*-Lipschitz representative* $\mathfrak{m}$*-a.e. with respect to* d.

3.4.2 Application to smooth manifolds with lower regularity Riemannian metrics and weights

Let M be a smooth manifold. Take a Riemannian metric $g\in C^0$ that admits L^2_{loc}-Christoffel symbols and induces the distance d_g. Define the measure μ via $\mathrm{d}\mu:=h^2\mathrm{dvol}_g$ for a $h\in C^0(M;(0,\infty))\cap W^{1,2}_{\mathrm{loc}}(M,\mu)$. Moreover, we assume that (M,d_g,μ) is a $\mathsf{CD}(K,\infty)$-space. Thus (2.2.1) is satisfied (see Remark 2.2.13) and by Corollary 3.3.15, (M,d_g,μ) is an $\mathsf{RCD}(K,\infty)$-space.

We will now specialise some concepts from [63] to this particular case. Using integration by parts, [8, Proposition 2.14 (iv)] and Lemma 3.3.5 yields:

Proposition 3.4.5. *Let* $f\in W^{1,2}_w(M)$ *and* H_t *denote the heat flow of the Cheeger energy. Then* $f_t:=H_tf\to f$ *in* $W^{1,2}_w(M)$ *as* $t\to 0$.

Proposition 3.4.6. *The space* $L^2(T^*M)$ *(as defined in [63, Definition 2.2.1]) is isometric to* $H^2_0(T^*M,\mu)$ *and the space* $L^2(TM)$ *(as defined in [63, Definition 2.3.1]) is isometric to* $H^2_0(TM,\mu)$.

We will not give a proof as it directly follows from the (quite technical) definitions and from [63, Proposition 2.2.5]; in the following we will identify the isomorphic spaces with each other.
We will now turn to a generalisation of the **divergence**. For a compactly supported $X\in H^2_1(TM,\mu)$ we have that $\mathrm{div}(X)=\big(\partial_iX^i+X^i\frac{2}{h}\partial_ih+X^i\frac{1}{2}\,\mathrm{tr}(g^{-1}\partial_ig)\big)\in L^2(M,\mu)$ satisfies (3.2.4). Hence, any compactly supported and smooth vector field X is in $D(\mathrm{div})$, according to [63, Definition 2.3.11].

Next, we look at a generalisation of the **Hessian**. The identity (3.2.1) motivates the following definition of the space $W^{2,2}(M)$.

Definition 3.4.7 ([63], Definition 3.3.1)**.** *The space* $W^{2,2}(M)\subset W^{1,2}_w(M)$ *is the space of all functions* $f\in W^{1,2}_w(M)$ *with the following property: There exists* $A\in L^2((T^*)^{\otimes 2}M)$

such that for any $h_1, h_2, \phi \in \mathrm{TestF}(M)$ it holds

$$2\int \phi A(\nabla h_1, \nabla h_2)\,\mathrm{d}\mu$$
$$= -\int \langle \nabla f, \nabla h_1\rangle \mathrm{div}(\phi\nabla h_2) + \langle \nabla f, \nabla h_2\rangle \mathrm{div}(\phi \nabla h_1) + \phi\langle \nabla f, \nabla\langle \nabla h_1, \nabla h_2\rangle\rangle\,\mathrm{d}\mu. \tag{3.4.1}$$

We will call A the Hessian of f and denote it as $\mathrm{Hess}(f)$. The space $W^{2,2}(M)$ is endowed with the norm $\|\cdot\|_{W^{2,2}(M)}$ defined via

$$\|f\|^2_{W^{2,2}(M)} = \|f\|^2_{L^2(M)} + \|df\|^2_{L^2(T^*M)} + \|\mathrm{Hess} f\|^2_{L^2((T^*)^{\otimes 2}M)}.$$

We next investigate such a space $W^{2,2}(M)$ in case of a smooth manifold M with a C^0-Riemannian metric g with L^2_{loc}-Christoffel symbols and a C^0 weighted measure μ.

Recall that, for every $\varphi \in C^\infty_c(M) \subset L^2 \cap L^\infty(M,\mu) \cap D(\Delta)$, it holds $H_t\varphi \in \mathrm{TestF}(M)$. Moreover, for each function $\varphi \in D(\Delta)$, we know that $\Delta H_t\varphi = H_t\Delta\varphi \to \Delta\varphi$ in L^2. Then, Proposition 3.4.5 yields that $H_t\varphi \to \varphi$ in $H^2_1(M,\mu)$ and Theorem 2.2.12 gives that $\||\nabla H_t\varphi|\|_{L^\infty} \le e^{-2Kt}\||\nabla\varphi|\|_{L^\infty}$. Hence, it follows that for each $f \in W^{2,2}(M)$, (3.4.1) also holds with $\phi, h_1, h_2 \in C^\infty_c(M)$, which together with $A \in L^2$ implies that $f \in H^2_2(M,\mu)$. This fact is summarised in the following proposition.

Proposition 3.4.8. *$H^2_2(M,\mu) \supset W^{2,2}(M)$ and for all $f \in W^{2,2}(M)$, it holds $\|f\|_{H^2_2(M,\mu)} = \|f\|_{W^{2,2}(M)}$.*

Moreover, it holds that $\mathrm{TestF}(M) \subset W^{2,2}(M)$, see [63, Theorem 3.3.8].

A consequence of Proposition 3.4.5, integration by parts in the sense of (3.4.1), Lemma 3.3.5, and Mazur's lemma is:

Proposition 3.4.9. *Let M be a smooth manifold, g a C^0-Riemannian metric that admits L^2_{loc}-Christoffel symbols and μ a measure on M defined by $\mathrm{d}\mu = h^2\mathrm{dvol}_g$ for a $h \in C^0(M,(0,\infty)) \cap W^{1,2}_{\mathrm{loc}}(M)$. Then $\mathrm{TestF}(M)$ is $H^2_2(M,\mu)$-dense in $L^\infty \cap H^2_2(M,\mu)$.*

Definition 3.4.10 ([63], Definition 3.3.17)**.** *The space $H^{2,2}(M)$ is defined as the $W^{2,2}$-closure of $\mathrm{TestF}(M)$.*

By Proposition 3.3.18 in [63], we have that $H^{2,2}(M)$ coincides with the $W^{2,2}$-closure of $D(\Delta_\mu)$, so we get:

Proposition 3.4.11. *$H^{2,2}(M) = \overline{H^2_2(M,\mu) \cap L^\infty(M,\mu)}^{H^2_2(M,\mu)}$ and it coincides with the $W^{2,2}$-closure of $D(\Delta_\mu)$.*

Now, we can turn to the abstract definition of the **covariant derivative**:

Definition 3.4.12 ([63] Definition 3.4.1)**.** *The Sobolev space $W_C^{1,2}(TM) \subset L^2(TM)$ is defined as the space of all $X \in L^2(TM)$ such that there exists $T \in L^2(T^{\otimes 2}M)$ such that for every $h_1, h_2, \varphi \in \mathrm{TestF}(M)$ it holds*

$$\int \varphi T : (\nabla h_1 \otimes \nabla h_2)\,\mathrm{d}\mu = -\int \langle X, \nabla h_2\rangle \mathrm{div}(\varphi \nabla h_1) - \varphi \mathrm{Hess}(h_2)(X, \nabla h_1)\,\mathrm{d}\mu.$$

In this case we call the tensor T the covariant derivative of X and denote it by ∇X. We endow $W_C^{1,2}(TM)$ with the norm $\|\cdot\|_{W_C^{1,2}(TM)}$ defined by

$$\|X\|^2_{W_C^{1,2}(TM)} := \|X\|^2_{L^2(TM)} + \|\nabla X\|^2_{L^2(T^{\otimes 2}M)}.$$

We denote by $H_C^{1,2}(TM)$ the closure of $\mathrm{TestV}(M)$ *in $W_C^{1,2}$.*

This definition makes sense, as in [63] it is proved that test vector fields are indeed in $W_C^{1,2}(TM)$.

Proposition 3.4.13. $\mathrm{TestV}(M) \subset H_1^2 \cap L^\infty(TM,\mu)$, *and hence $H_C^{1,2}(TM)$ is a subspace of $H_1^2(TM,\mu)$.*

We note that by our previous computations we get that for $X \in H_C^{2,1}(TM)$, it holds $\nabla X = (\nabla_c X)^\sharp$ in the classical $H_1^2(TM,\mu)$-sense. For $X \in W_C^{1,2}(TM)$ and $Z \in L^2(TM)$, we define $\nabla_Z X$ via

$$\langle \nabla_Z X, Y\rangle_g = \nabla X : (Z \otimes Y).$$

A computation shows that in the case of smooth vector fields, this coincides with the smooth covariant derivative. As for C^1-vector fields, we have that $\nabla_Z X = Z^i \partial_i X^s + \Gamma^s_{ij} X_j Z_i$, this holds by density for all vector fields $X \in H_1^2 \cap L^\infty(TM,\mu)$ and motivates the following definition of **Lie bracket**:

Definition 3.4.14 ([63], Definition 3.4.8)**.** *For $X, Y \in H_C^{1,2}(TM)$, we define $[X,Y] \in L^1(TM)$ via*

$$[X,Y] := \nabla_X Y - \nabla_Y X.$$

Note that this again coincides with the Lie bracket in the smooth case and the local expressions carry over by density. The generalised **differential** is defined as follows:

Definition 3.4.15 ([63], Definitions 3.5.1, 3.5.5)**.** *The space $W_d^{1,2}(\Lambda^k T^*M) \subset L^2(\Lambda^k T^*M)$ is the space of k-forms $\omega \in L^2(\Lambda^k T^*M)$ such that there exists a $(k+1)$-form $\eta \in L^2(\Lambda^{k+1}T^*M)$ for which the identity*

$$\begin{aligned}\int \eta(X_0,\dots,X_k)\,\mathrm{d}\mu = &\int \sum_i (-1)^{i+1} \omega(X_0,\dots,\hat{X}_i.\dots,X_k)\,\mathrm{d}\mu \\ &+ \int \sum_{i<j} (-1)^{i+j} \omega([X_i,X_j],X_0,\dots,\hat{X}_i,\dots,\hat{X}_j,\dots,X_k)\,\mathrm{d}\mu,\end{aligned}$$

holds for any $X_0, \dots, X_k \in \mathrm{TestV}(M)$*. In this case we call* η *the exterior differential of* ω *and we will denote it as* $\mathrm{d}\omega$*. We endow* $W_d^{1,2}(\Lambda^k T^*M)$ *with the norm* $\|\cdot\|_{W_d^{1,2}(\Lambda^k T^*M)}$ *given by*

$$\|\omega\|^2_{W_d^{1,2}(\Lambda^k T^*M)} := \|\omega\|^2_{L^2(\Lambda^k T^*M)} + \|\mathrm{d}\omega\|^2_{L^2(\Lambda^{k+1} T^*M)}.$$

Moreover, we define $H_d^{1,2}(\Lambda^k T^*M)$ *as the* $W_d^{1,2}$*-closure of* $\mathrm{TestForm}_k(M)$*.*

Again, this definition makes sense as in [63] it is proved that test forms are contained in $W_d^{1,2}(\Lambda^k T^*M)$. As $\mathrm{TestV}(M) \subset H_1^2(TM, \mu)$, we get that $H_1^2(M, \mu) \subset W_d^{1,2}(\Lambda^0 T^*M)$, on the intersection the coefficients need to coincide with the ones from the classical differential, and the norms are equivalent on the intersection. The cases of $k = 0, 1$ will be of particular interest.
Next, we recall the definition of the **codifferential**:

Definition 3.4.16 ([63], Definition 3.5.11)**.** *The space* $D(\delta_k) \subset L^2(\Lambda^k T^*M)$ *is the space of* k*-forms* ω *for which there exists a form* $\delta\omega \in L^2(\Lambda^{k-1} T^*M)$ *called the codifferential of* ω*, such that*

$$\int \langle \delta\omega, \eta \rangle \, \mathrm{d}\mu = \int \langle \omega, \mathrm{d}\eta \rangle \, \mathrm{d}\mu$$

for all $\eta \in \mathrm{TestForm}_{k-1}(M)$*. In the case* $k = 0$*, we set* $D(\delta_0) = L^2(\mu)$ *and define* δ *to be identically zero.*

For 1-forms, note that $\omega \in D(\delta_1)$ if and only if $\omega^\sharp \in D(\mathrm{div})$ and, in this case, $\delta\omega = -\mathrm{div}(\omega^\sharp)$. In [63], it is shown that for each k, $\mathrm{TestForm}_k(M) \subset D(\delta_k)$. Hence, the following definition makes sense:

Definition 3.4.17 ([63], Def. 3.5.13)**.** *The space* $W_H^{1,2}(\Lambda^k T^*M)$ *is defined as* $W_d^{1,2}(\Lambda^k T^*M) \cap D(\delta_k)$ *endowed with the norm* $\|\cdot\|_{W_H^{1,2}(\Lambda^k T^*M)}$ *defined by*

$$\|\omega\|^2_{W_H^{1,2}(\Lambda^k T^*M)} := \|\omega\|^2_{W_d^{1,2}(\Lambda^k T^*M)} + \|\delta\omega\|^2_{L^2(\Lambda^{k-1} T^*M)}.$$

The space $H_H^{1,2}(\Lambda^k T^*M)$ *is defined as the* $W_H^{1,2}$*-closure of* $\mathrm{TestForm}_k(M)$*.*

It is not hard to check that $\mathrm{TestForm}_k(M) \subset W_H^{1,2}(\Lambda^k T^*M)$.

Definition 3.4.18 ([63], Defintion 3.6.3)**.** *The space* $H_H^{1,2}(TM) \subset L^2(TM, \mu)$ *is the space of vector fields* $X \in L^2(TM)$ *such that* $X^\flat \in H_H^{1,2}(T^*M)$ *equipped with the norm* $\|X\|_{H_H^{1,2}(TM)} := \|X^\flat\|_{H_H^{1,2}(T^*M)}$*.*

Definition 3.4.19 ([63], Definition 3.5.14)**.** *Given $k \in \mathbb{N}$, the domain $D(\Delta_{H,k}) \subset H^{1,2}_H(\Lambda^k T^*M)$ of the Hodge Laplacian is defined as the set of $\omega \in H^{1,2}_H(\Lambda^k T^*M)$ for which there exists an $\alpha \in L^2((\Lambda^k T^*M)$ such that*

$$\int \langle \alpha, \eta \rangle \, \mathrm{d}\mu = \int \langle d\omega, \mathrm{d}\eta \rangle \, \mathrm{d}\mu + \int \langle \delta\omega, \delta\eta \rangle \, \mathrm{d}\mu \quad \forall \eta \in H^{1,2}_H(\Lambda^k T^*M).$$

In this case α is unique and we denote it by $\Delta_{H,k}\omega$.

Note that

$$\int \langle \Delta_{H,k}\omega, \omega \rangle \, \mathrm{d}\mu = \int (\langle d\omega, d\omega \rangle + \langle \delta\omega, \delta\omega \rangle) \, \mathrm{d}\mu.$$

In [63, Proposition 3.6.1], it is shown that $\mathrm{TestForm}_1(M) \subset D(\Delta_{H,1})$. Moreover, $\Delta f = -\Delta_{H,0} f$ for all $f \in \mathrm{TestF}(M)$. Finally, we recall the generalised **Ricci curvature tensor**:

Theorem 3.4.20 ([63], Theorem 3.6.7)**.** *There is a unique and continuous map $\boldsymbol{Ric} : [H^{1,2}_H(TM)]^2 \to \mathrm{Meas}(M)$ such that, for every $X, Y \in \mathrm{TestV}(M)$, it holds:*

$$\boldsymbol{Ric}(X,Y) = \boldsymbol{\Delta}\frac{\langle X, Y \rangle}{2} + \Big(\frac{1}{2}\langle X, (\Delta_H Y^\flat)^\sharp \rangle + \frac{1}{2}\langle Y, (\Delta_H X^\flat)^\sharp \rangle - \nabla X : \nabla Y\Big)\mu.$$

This map is bilinear, symmetric and satisfies:

$$\boldsymbol{Ric}(X,X) \geq K g(X,X)\mu.$$

Moreover, setting $X = Y$, we get that

$$\boldsymbol{Ric}(X,X) + (|\nabla X|^2_{HS} - \langle X, (\Delta_H X^\flat)^\sharp \rangle)\mu = \boldsymbol{\Delta}\frac{|X|^2}{2}.$$

We will now connect the measure valued Ricci tensor in the sense of Theorem 3.4.20 and the distributional Ricci tensor in the sense of Subsection 3.2.3.

Proposition 3.4.21. *Let M be a smooth manifold with a C^0-Riemannian metric g with L^2_{loc} Christoffel symbols. Consider $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$ for a positive function $h \in C^0 \cap W^{1,2}_{\mathrm{loc}}$. Assume that (M, d_g, μ) is a $\mathsf{CD}(K,\infty)$-space. For $X \in \mathrm{TestV}(M)$ we have that $\boldsymbol{Ric}(X,X) = \mathrm{Ric}_{\mu,\infty}(X,X)$ in the sense of* (3.2.12)*.*

Proof. By the previous observations we have that for $\varphi \in C_c \cap W^{1,2}_{\mathrm{loc}}(M)$ (which we assume to be supported in one coordinate patch),

$$\begin{aligned}
\int_M \varphi \, \mathrm{d}\mathbf{Ric}(X,X) &= \int_M \varphi \, d\boldsymbol{\Delta}\frac{|X|^2}{2} + \int_M (\langle X, (\Delta_H X^\flat)^\sharp \rangle - |\nabla X|^2_{HS})\varphi \, \mathrm{d}\mu \\
&= -\frac{1}{2}\int_M \langle \nabla |X|^2, \nabla\varphi \rangle \, \mathrm{d}\mu + \int_M (\langle X, (\Delta_{\mu,H} X^\flat)^\sharp \rangle - |\nabla X|^2_{HS})\varphi \, \mathrm{d}\mu \\
&= -\frac{1}{2}\int_M \langle \nabla |X|^2, \nabla\varphi \rangle \, \mathrm{d}\mu + \int_M (|dX^\flat|^2 + |\delta_\mu X^\flat|^2 - |\nabla X|^2_{HS})\varphi \, \mathrm{d}\mu.
\end{aligned}$$

Now the last line equals the right hand side of (3.2.12) as it appears in Lemma 3.3.6 so we get that locally

$$
\begin{aligned}
&\int_M \varphi\, d\mathbf{Ric}(X,X)\\
&= \int X^j X^k (\Gamma^s_{kj}\Gamma^p_{ps} - \Gamma^s_{kp}\Gamma^p_{js})\varphi h^2\sqrt{|g|}\, dx^1\dots dx^n - \int \Gamma^p_{jk}\partial_p(X^jX^k\varphi h^2\sqrt{|g|})\, dx^1\dots dx^n\\
&\quad + \int \Gamma^p_{pk}\partial_j(X^jX^k\varphi h^2\sqrt{|g|})\, dx^1\dots dx^n\\
&\quad + 2\int X^jX^k(\partial_j h\partial_k h + h\partial_s h\Gamma^s_{kj})\varphi\sqrt{|g|}dx^1\dots dx^n + 2\int \partial_j h\partial_k(X^jX^k h\varphi\sqrt{|g|})dx^1\dots dx^n.
\end{aligned}
$$

□

As in [63], we have that for all $f \in \mathrm{TestF}(M)$ it holds

$$\mathbf{Ric}(\nabla f,\nabla f) + |\nabla^2 f|^2_{HS}\mu = \mathbf{\Delta}\frac{|\nabla f|^2}{2} - \langle\nabla f,\nabla\Delta f\rangle\mu = \mathbf{\Gamma}_2(f,f). \tag{3.4.2}$$

Proposition 3.4.22. *Let M be a smooth manifold and g a C^0-Riemannian metric with L^2_{loc}-Christoffel symbols and $h \in C^0(M,(0,\infty))\cap W^{1,2}_{\mathrm{loc}}$ such that (M,d_g,μ) is a $\mathsf{CD}(K,\infty)$-space, where $\mathrm{d}\mu = h^2\mathrm{dvol}_g$.*
Then, for all $f \in H^{2,2}(M)$, it holds that

$$\int_M |\nabla^2 f|^2 h^2\,\mathrm{dvol}_g \le \int_M |\Delta f|^2 h^2\,\mathrm{dvol}_g - K\int_M |\nabla f|^2 h^2\,\mathrm{dvol}_g.$$

If $f \in D(\Delta)$, we have that $H_t f \to f$ strongly in $H^{2,2}$.

Proof. By [63, Corollary 3.3.9], we have that

$$\int_M |\nabla^2 u|^2_{HS}\,\mathrm{d}\mu \le \int_M ((\Delta u)^2 - K|\nabla u|^2)\,\mathrm{d}\mu$$

for all $u \in D(\Delta)$. If $f \in H^{2,2}\cap D(\Delta)$, we have that $\Delta f \in L^2$ and hence $H_t\Delta f \to \Delta f$ strongly in L^2 as $t \to 0$. As the semi-group H_t is generated by Δ, we have that $\Delta H_t f = H_t\Delta f$ for all $t \ge 0$. It follows that $\Delta H_t f \to \Delta f$ strongly in L^2 as $t \to 0$. Setting $u = f - H_t f$, we infer

$$
\begin{aligned}
\left\|\nabla^2(f-H_tf)\right\|^2_{L^2((T^*)^{\otimes 2}M)} &\le -K\int_M |\nabla(f-H_tf)|^2\, h^2\mathrm{dvol}_g + \int_M |\Delta(f-H_tf)|^2\, h^2\mathrm{dvol}_g\\
&\le (|K|+1)(\|f-H_tf\|_{H^2_1(M,\mu)} + \|\Delta(f-H_tf)\|_{L^2(M,\mu)}) \to 0
\end{aligned}
$$

as $t \to 0$. □

3.5 RCD implies distributional Ricci curvature lower bounds

3.5.1 Some useful approximation results

We start the section with some local considerations, so we work in $\mathbb{R}^n$ for simplicity of presentation. Recall the following consequence of the Poincaré inequality:

Lemma 3.5.1. *Let $R > 0$, $1 \le p < \infty$ and $u \in W^{1,p}(B_R(0))$. Let $m := \frac{1}{|B_R(0)|} \int_{B_R(0)} u \, \mathrm{d}\mathcal{L}^n$. Then there exists a constant $C = C(p) > 0$ independent of R such that*

$$\|u - m\|_{L^p(B_R(0))} \le CR\|\nabla u\|_{L^p(B_R(0))}.$$

Lemma 3.5.2. *Let $\Omega \subset \mathbb{R}^n$ be open and ν be a σ-finite measure on Ω such that $C_c^\infty(\Omega)$ is dense in $L^p(\nu)$ for each $p \in [1,\infty)$. Let $f, (f_k)_{k\ge1}$ be measurable functions on Ω such that $\|f_k\|_{L^\infty}$ and $\|f\|_{L^\infty}$ are bounded and such that $f_k \rightharpoonup f$ weakly in $L^p(\nu)$ for some $p \in [1,\infty)$. Then $f_k \overset{*}{\rightharpoonup} f$ in the weak* topology of $(L^1)^* = L^\infty$.*

Proof. As ν is σ-finite, we have that indeed $(L^1)^* = L^\infty$. By the Banach-Alaoglu theorem, it follows that there is a weakly* convergent subsequence of f_k, which we will still denote by f_k. Hence, f_k converges weakly* to some $\tilde{f} \in L^\infty$. Suppose $\tilde{f} \neq f$. Then there exists an L^1-function ϕ such that

$$\int_\Omega f\phi \, \mathrm{d}\nu \neq \int_\Omega \tilde{f}\phi \, \mathrm{d}\nu.$$

As $C_c^\infty(\Omega)$ is dense in L^1, we can assume that $\phi \in C_c^\infty \subset L^q$, where $q = \frac{p}{p-1}$. This contradicts the weak convergence in L^p. □

Lemma 3.5.3. *Let $K \subset B_1(0) \subset \mathbb{R}^n$ be compact and denote $d = \mathrm{dist}(K, \partial B_1(0)) > 0$. Then there exists a $\delta_0 > 0$ and a constant $C = C(n)$ such that for all $\delta \in (0, \delta_0)$, there exist an integer $m \le C\delta^{-n}$ and points $y_1, \dots, y_m \in B_1(0)$ such that $B_{3\delta(y_i)} \subset B_{2\delta + \frac{d}{12}}(y_i) \subset B_1(0)$ for all $1 \le i \le m$, $K \subset \bigcup_{i=1}^m B_\delta(y_i)$ and for all $x \in B_1(0)$, $|\{i : x \in B_{2\delta}(y_i)\}| \le C$.*

Proof. Let $\delta_0 = \frac{d}{12}$ and fix $\delta \in (0, \delta_0)$. Now let $\{y_1, \dots, y_m\} := \frac{\delta}{2\sqrt{n}}\mathbb{Z}^n \cap B_{1-\frac{3d}{4}}(0)$. It follows that $K \subset B_{1-\frac{3d}{4}-\frac{\delta}{2}}(0) \subset \bigcup_{i=1}^m B_\delta(y_i) \subset \bigcup_{i=1}^m B_{2\delta+\frac{d}{12}}(y_i) \subset B_{1-\frac{d}{4}}(0)$. Moreover, there exists a constant $C_1(n)$ such that for each $x \in \mathbb{R}^n$, $|\frac{1}{2\sqrt{n}}\mathbb{Z}^n \cap B_2(x)| \le C_1(n)$ so for each $x \in B_1(0)$, there are at most $C_1(n)$ points in $\frac{\delta}{2\sqrt{n}}\mathbb{Z}^n \cap B_{2\delta}(x)$. Finally, there exists a constant $C_2(n)$ such that for each $0 < \lambda \le 1$, $|\frac{\lambda}{2\sqrt{n}}\mathbb{Z}^n \cap B_1(0)| \le C_2(n)\lambda^{-n}$. Taking $C(n) = \max(C_1(n), C_2(n))$ finishes the proof. □

Lemma 3.5.4. *Let $R > 0$, $\varphi \in C_c^\infty(B_R(0))$. Then there exists a constant $C = C(n, R) > 0$ and $\delta_0 > 0$ such that for each $\delta \in (0, \delta_0)$, there exists a set $\{\chi_1, \dots, \chi_m\} \subset C_c^\infty(B_R(0), [0, 1])$ and a family $\{y_1, \dots, y_m\} \subset B_{R-3\delta}(0)$ such that the following holds:*

(i) $m \leq C\delta^{-n}$,

(ii) $\sum_{i=1}^m \chi_i(x) \leq 1$ *for all* $x \in B_R(0)$ *and* $\sum_{i=1}^m \chi_i(x) = 1$ *for all* $x \in \operatorname{supp} \varphi$,

(iii) *For each* $1 \leq i \leq m$, $\|\nabla\chi_i\|_{L^\infty} \leq C\delta^{-1}$.

(iv) *For all* i, *it holds* $\chi_i \in C_c^\infty(B_{2\delta}(y_i))$.

(v) *For all* $x \in B_R(0)$, *it holds* $|\{i : x \in B_{2\delta}(y_i)\}| \leq C$.

Proof. By rescaling, we can assume $R = 1$. Then take $K = \operatorname{supp}\varphi$ and apply Lemma 3.5.3 to get δ_0. Choose $0 < \delta \leq \delta_0$ and take m, $\{y_1, \dots, y_m\}$ as in Lemma 3.5.3. It then follows that for all $i = 1, \dots, m$, $B_{3\delta}(y_i) \subset B_R(0)$, hence $y_i \in B_{R-3\delta}(0)$. Now, for each $i \in \{1, \dots, m\}$ take a function $\eta_i \in C_c^\infty(B_{2\delta}(y_i), [0, 1])$ such that $\eta_i(x) = 1$ for all $x \in B_\delta(y_i)$ and $|\nabla\eta_i| \leq C\frac{1}{\delta}$. It follows that $\sum_{i=1}^m \eta_i \in C_c^\infty(B_1(0))$ and $\sum_{i=1}^m \eta_i \geq 1$ on $\operatorname{supp}\varphi$. Now take a function $h \in C^\infty(\mathbb{R})$ such that $|h'| \leq 1$, $h(x) \geq \max(|x|, \frac{1}{4})$ and $h(x) = x$ for $x \geq 1$. Define

$$\chi_j(x) = \frac{\eta_j(x)}{h(\sum_{i=1}^m \eta_i(x))}.$$

It follows that $\sum_{i=1}^m \chi_i(x) \leq 1$ for all $x \in B_1(0)$ and $\sum_{i=1}^m \chi_i(x) = 1$ for all $x \in \operatorname{supp}\varphi$. To see (iii), note that for all $x \in B_1(0)$

$$\begin{aligned}
|\nabla\chi_j(x)| &\leq \frac{|\nabla\eta_j|(x) + |h'(\sum_{i=1}^m \eta_i(x))| \sum_{i=1}^m |\nabla\eta_i|(x)|}{h(\sum_{i=1}^m \eta_i(x))^2} \\
&\leq 16C\Big(\delta^{-1} + \sum_{i: x \in \operatorname{supp}\eta_i}^m \delta^{-1}\Big) \leq 16(1 + C(n))\delta^{-1}.
\end{aligned}$$

□

Definition 3.5.5. *Let $\Omega \subset \mathbb{R}^n$ be an open set. We denote by* $\operatorname{CompV}(\Omega)$ *the linear span of functions of the form $\rho\nabla h$, where $\rho, h \in C_c^\infty(\Omega)$. For a smooth manifold M with a C^0-Riemannian metric g, we define* $\operatorname{CompV}(M)$ *as the linear span of functions of the form $\rho\nabla h$, where $\rho, h \in C_c^\infty(M)$.*

Lemma 3.5.6. *Let $R > 0$, $B_R(0) \subset \mathbb{R}^n$. Let $X \in C_c^\infty(B_R(0); \mathbb{R}^n)$. Then there exists a constant $C = C(n, R) > 0$ and a constant $C_2 = C_2(n) > 0$ such that for all $\varepsilon > 0$ there is a $\tilde{X} = \sum_{p=1}^q h_p\nabla f_p \in \operatorname{CompV}(B_R(0))$ satisfying the following*

(i) $\left\|X-\tilde{X}\right\|_{W^{1,1}(B_R(0))} \leq C\varepsilon$ *and* $\left\|X-\tilde{X}\right\|_{L^\infty(B_R(0))} \leq C\varepsilon$,

(ii) $\left\|\tilde{X}\right\|_{W^{1,\infty}(B_R(0))} \leq C(1+\|X\|_{W^{2,\infty}(B_R(0))})$,

(iii) $q \leq C_2$,

(iv) *for all* p, *we have that* $\|h_p\|_{L^\infty(B_R(0))} \leq C$ *and* $\|f_p\|_{L^\infty(B_R(0))} \leq C\|X\|_{C^1(B_R(0))}\varepsilon$,

(v) *for all* p, *we have that* $\|\nabla h_p\|_{L^\infty(B_R(0))} \leq C(1+\|X\|_{C^2(B_R(0))})\varepsilon^{-1}$ *and* $\|\nabla f_p\|_{L^\infty(B_R(0))} \leq C(1+\|X\|_{C^1(B_R(0))})$.

Proof of (i). Assume (by potentially readjusting) that $\varepsilon \leq \min(\frac{1}{2}, \delta_0)$, where $\delta_0 > 0$ is as in Lemma 3.5.4. We will denote by DX the Jacobi matrix of X. We choose

$$\delta = \frac{\varepsilon}{8n^2(1+\|DX\|_{L^\infty}+\|D^2X\|_{L^\infty})}, \tag{3.5.1}$$

so by the mean value theorem we get that for all $y, z \in B_R(0)$ with $|y-z| \leq 4\delta$, it holds $|X(y)-X(z)| < \varepsilon$ and $|DX(y)-DX(z)| < \varepsilon$. Note that we also have $\delta < \delta_0$. For this chosen δ, take m, $\{\chi_1, \dots \chi_m\}$ and $\{y_1, \dots, y_m\}$ as in Lemma 3.5.4. Let $\psi \in C_c^\infty(B_{3\delta}(0))$ such that $\psi(B_{2\delta}(0)) = \{1\}$ and $|\nabla\psi| \leq 2\delta^{-1}$. Fix an $i \in \{1, \dots, m\}$. Let

$$A_i = \frac{1}{|B_{2\delta}(y_i)|}\int_{B_{2\delta}(y_i)} X \mathrm{d}\mathcal{L}^n \in \mathbb{R}^n,$$

and

$$\alpha_i : \mathbb{R}^n \to \mathbb{R}, x \mapsto \psi(x-y_i)\langle A_i, (x-y_i)\rangle.$$

Then $\nabla\alpha_i = A_i$ and $DA_i = 0$ on $B_{2\delta}(y_i)$. For $1 \leq l, k \leq n$, let $B_i^{lk} = \partial_k X^l(y_i) \in \mathbb{R}$. Moreover, define

$$\begin{aligned}
&\rho_i^k(x) := \psi(x-y_i)(x_k-(y_i)_k) \in C_c^\infty(B_R(0)),\\
&\eta_i^{lk}(x) := \psi(x-y_i)B_i^{lk}(x_l-(y_i)_l) \in C_c^\infty(B_R(0)) \text{ and}\\
&\beta_i^{lk}(x) := (\rho_i^k\nabla\eta_i^{lk})(x) = \psi(x-y_i)(x_k-(y_i)_k)\cdot\nabla(\psi(x-y_i)B_i^{lk}(x_l-(y_i)_l) \in \mathrm{CompV}(B_R(0)).
\end{aligned}$$

Then, for all $x \in B_{2\delta}(y_i)$, we have that

$$\begin{aligned}
\beta_i^{lk}(x) &= (x_k-(y_i)_k)B_i^{lk}e_l \in \mathbb{R}^n, \quad \text{and}\\
D\beta_i^{lk}(x) &= (\delta_m^l\delta_p^k B_i^{lk})_{m,p=1,\dots,n} \in \mathbb{R}^{n\times n}.
\end{aligned}$$

Notice that

$$\int_{B_{2\delta}(y_i)} \beta_i^{lk} d\mathcal{L}^n = 0 \in \mathbb{R}^n,$$

as β_i^{lk} is linear on $B_{2\delta}(y_i)$. Now define $\tilde{X} := \sum_{i=1}^m \chi_i \cdot (\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk}) \in \mathrm{CompV}(B_R(0))$. As $X, \tilde{X} \in W_0^{1,1}(B_R(0))$, the Poincaré inequality implies that

$$\left\|X - \tilde{X}\right\|_{W^{1,1}(B_R(0))} \leq C \cdot (1+R) \left\|DX - D\tilde{X}\right\|_{L^1(B_R(0))},$$

for some $C = C(n) > 0$. Now, as $X = \sum_{i=1}^m \chi_i X$, we get that

$$\begin{aligned}\left\|DX - D\tilde{X}\right\|_{L^1(B_R(0))} &= \int_{B_R(0)} |\sum_{i=1}^m D(\chi_i X) - D(\chi_i(\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk}))| \,\mathrm{d}\mathcal{L}^n \\ &\leq \int_{B_R(0)} \sum_{i=1}^m |\nabla\chi_i||X - (\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk})| + |\sum_{i=1}^m \chi_i(DX - D\sum_{k,l=1}^n \beta_i^{kl})| \,\mathrm{d}\mathcal{L}^n.\end{aligned}$$

For each i, we get that

$$\left\|DX - D\Big(\sum_{k,l=1}^n \beta_i^{kl}\Big)\right\|_{L^\infty(B_{2\delta}(y_i))} = \|DX - DX(y_i)\|_{L^\infty(B_{2\delta}(y_i))} \leq n^2\varepsilon. \tag{3.5.2}$$

Thus, using that $\chi_i \geq 0$, we get that

$$\int_{B_R(0)} \Big|\sum_{i=1}^m \chi_i(DX - D\sum_{k,l=1}^n \beta_i^{kl})\Big| \,\mathrm{d}\mathcal{L}^n \leq \int_{B_R(0)} \sum_{i=1}^m \chi_i \Big|(DX - D\sum_{k,l=1}^n \beta_i^{kl})\Big| \,\mathrm{d}\mathcal{L}^n \leq n^2|B_R(0)|\varepsilon.$$

For each i, we have $|\nabla\chi_i| \leq C(n,R)\delta^{-1}$ and

$$\begin{aligned}\int_{B_R(0)} |\nabla\chi_i|\Big|X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl}\Big| \,\mathrm{d}\mathcal{L}^n &= \int_{B_{2\delta}(y_i)} |\nabla\chi_i|\Big|X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl}\Big| \,\mathrm{d}\mathcal{L}^n \\ &\leq C(n,R)\delta^{-1}\left\|X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl}\right\|_{L^1(B_{2\delta}(y_i))}.\end{aligned}$$

By Lemma 3.5.1 and (3.5.2), we have that

$$\begin{aligned}\left\|X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl}\right\|_{L^1(B_{2\delta}(y_i))} &\leq 2C\delta\left\|DX - D\Big(\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl}\Big)\right\|_{L^1(B_{2\delta}(y_i))} \\ &= 2C\delta\left\|DX - D\Big(\sum_{k,l=1}^n \beta_i^{kl}\Big)\right\|_{L^1(B_{2\delta}(y_i))} \\ &\leq C\delta^{n+1}\varepsilon.\end{aligned}$$

Thus,

$$\int_{B_R(0)} |\nabla\chi_i||X - \nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{kl}| \,\mathrm{d}\mathcal{L}^n \leq C(n,R)\varepsilon\delta^n.$$

Now, as $m \leq C(n,R)\delta^{-n}$, we get that

$$\int_{B_R(0)} \sum_{i=1}^m |\nabla\chi_i||X - \Big(\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk}\Big)| \leq C(n,R)^2\varepsilon.$$

All together this gives

$$\Big\|X - \tilde{X}\Big\|_{W^{1,1}(B_R(0))} \leq C(1+R)\Big\|DX - D\tilde{X}\Big\|_{L^1(B_R(0))} \leq C(n,R)\varepsilon.$$

Moreover, note that for each $x \in B_R(0)$,

$$|X - \tilde{X}|(x) \leq \Big|\sum_{i=1}^m \chi_i(X - \nabla\alpha_i)\Big|(x) + \sum_{i=1}^m \sum_{l,k} |\chi_i\beta_i^{kl}|(x) \leq C(n)\varepsilon.$$

This concludes the proof of (i).

Proof of (ii). Next, we want to investigate the L^∞-norm of $D\tilde{X}$. We get that for a point $y \in B_R(0)$

$$\begin{aligned}|DX - D\tilde{X}|(y) &= \Big|\sum_{i=1}^m D(\chi_i X) - D\Big(\chi_i(\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk})\Big)\Big|(y)\\ &\leq \Big(\sum_{i=1}^m |\nabla\chi_i|\Big|X - (\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk})\Big|\Big)(y) + \Big|\sum_{i=1}^m \chi_i\Big(DX - D\sum_{k,l=1}^n \beta_i^{kl}\Big)\Big|(y).\end{aligned}$$

We will investigate the two sums separately. For a fixed i, we know that $|\nabla\chi_i|(y) \leq C(n,R)\delta^{-1}$ and $|\chi_i|(y) \leq 1$. It suffices to only consider $y \in B_{2\delta}(y_i)$, in which case we have that $B_{2\delta}(y_i) \subset B_{4\delta}(y)$. By our choice of δ, we have that for all $z \in B_{2\delta}(y_i)$ it holds $|X(z) - X(y)| \leq \varepsilon$. Hence,

$$\begin{aligned}&|X(y) - \nabla\alpha_i|\\ &= \Big|X(y) - \frac{1}{|B_{2\delta}(y_i)|}\int_{B_{2\delta}(y_i)} X(z)\,\mathrm{d}\mathcal{L}^n(z)\Big|\\ &\leq \frac{1}{|B_{2\delta}(y_i)|}\int_{B_{2\delta}(y_i)} |X(z) - X(y)|\,\mathrm{d}\mathcal{L}^n(z) \leq \varepsilon.\end{aligned}$$

Moreover,

$$\Big|\sum_{k,l=1}^n \beta_i^{lk}\Big|(y) \leq n^2\delta(1 + |DX|_{L^\infty}).$$

This gives that

$$\begin{aligned}|\nabla\chi_i|\Big|X - \Big(\nabla\alpha_i + \sum_{k,l=1}^n \beta_i^{lk}\Big)\Big|(y) &\leq C(n)\delta^{-1}\Big(|X(y) - \nabla\alpha_i| + \Big|\sum_{k,l=1}^n \beta_i^{lk}\Big|(y)\Big)\\ &\leq C(n)\delta^{-1}(\varepsilon + \delta|DX|_{L^\infty})\\ &\leq C(n)(1 + |DX|_{L^\infty} + |D^2X|_{L^\infty}),\end{aligned}$$

where in the last estimate we used the precise dependence of ε in terms of δ, (3.5.1). The choice of δ also gives

$$\Big|DX - D\sum_{k,l=1}^{n}\beta_i^{kl}\Big|(y) \le n^2\varepsilon.$$

Noting that $|i : y \in \mathrm{supp}\chi_i| \le C(n)$, we get that

$$\begin{aligned}|DX - D\tilde{X}|(y) &\le \sum_{i=1}^{m}|\nabla\chi_i|\Big|X - \Big(\nabla\alpha_i + \sum_{k,l=1}^{n}\beta_i^{lk}\Big)\Big|(y) + \Big|\sum_{i=1}^{m}\chi_i\Big(DX - D\sum_{k,l=1}^{n}\beta_i^{kl}\Big)\Big|(y)\\ &\le C(n)\Big(\big(1 + \|DX\|_{L^\infty} + \big\|D^2X\big\|_{L^\infty}\big) + n^2\varepsilon\Big).\end{aligned}$$

It follows that there exists a constant $C(n, R)$ such that

$$\Big\|D\tilde{X}\Big\|_{L^\infty} \le \Big\|DX - D\tilde{X}\Big\|_{L^\infty} + \|DX\|_{L^\infty} \le C(n,R)(1 + \|X\|_{C^2(B_R(0))}).$$

As $\tilde{X} \in C_c^\infty(B_R(0))$, we get that also $\Big\|\tilde{X}\Big\|_{L^\infty} \le CR(1 + \|X\|_{C^2(B_R(0))})$, which proves (ii).

Proof of (iii). For the last three statements, let us explicitly spell out how we can build $\tilde{X}$ out of finitely many (bounded by $C_2 = C_2(n)$ independent of δ) C_c^∞-functions and their gradients. Recall that by the proof of Lemma 3.5.3, we have chosen the y_i to be in $\frac{\delta}{2\sqrt{n}}\mathbb{Z}^n \cap B_R(0)$. Note that $\Big|\Big(\mathbb{Z}/_{12n\mathbb{Z}}\Big)^n\Big| = (12n)^n$. For $\xi \in \Big(\mathbb{Z}/_{12n\mathbb{Z}}\Big)^n$, let

$$Z_\xi := \{\zeta \in \mathbb{Z}^n : \zeta_s \equiv \xi_s \mod 12n, \forall 1 \le s \le n\}.$$

Define $Y_\xi := \{i : y_i \in \{y_1, \dots, y_m\} \cap \frac{\delta}{2\sqrt{n}}Z_\xi\}$. If $Y_\xi = \emptyset$, define $\chi_\xi = \alpha_\xi = \rho_\xi^k = \eta_\xi^{lk} = 0 \in C_c^\infty(B_R(0))$, where $k, l \in \{1, \dots, n\}$. Otherwise, define

$$\begin{aligned}\chi_\xi &:= \sum_{i\in Y_\xi}\chi_i \in C_c^\infty(B_R(0)),\\ \alpha_\xi &:= \sum_{i\in Y_\xi}\alpha_i \in C_c^\infty(B_R(0)),\\ \rho_\xi^k &:= \sum_{i\in Y_\xi}\rho_i^k \in C_c^\infty(B_R(0)),\ \text{for } 1 \le k \le n \text{ and}\\ \eta_\xi^{lk} &:= \sum_{i\in Y_\xi}\eta_i^{lk} \in C_c^\infty(B_R(0)),\ \text{for } 1 \le k, l \le n.\end{aligned}$$

We note that

$$B_{3\delta}(y_i) \cap B_{3\delta}(y_{i'}) = \emptyset,\ \text{if } i, i' \in Y_\xi \text{ and } i \ne i', \tag{3.5.3}$$

as then $|y_i - y_{i'}| \ge 6\sqrt{n}\delta$. Now, for all i, we have that

$$\mathrm{supp}\,\alpha_i \cup \mathrm{supp}\,\chi_i \cup \bigcup_{k=1}^{n}\mathrm{supp}\,\rho_i^k \cup \bigcup_{k,l=1}^{n}\mathrm{supp}\ \eta_i^{lk} \subset B_{3\delta}(y_i),$$

so

$$
\begin{aligned}
\chi_\xi \nabla \alpha_\xi + \sum_{k,l=1}^{n} \chi_\xi \rho_\xi^k \nabla \eta_\xi^{lk} &= \left(\sum_{i\in Y_\xi} \chi_i\right)\left(\sum_{i\in Y_\xi} \nabla \alpha_i\right) + \sum_{k,l=1}^{n} \left(\sum_{i\in Y_\xi} \chi_i\right)\left(\sum_{i\in Y_\xi} \rho_i^k\right)\left(\sum_{i\in Y_\xi} \nabla \eta_i^{lk}\right) \\
&= \sum_{i\in Y_\xi} \chi_i \nabla \alpha_i + \sum_{i\in Y_\xi} \chi_i \sum_{k,l=1}^{n} \rho_i^k \nabla \eta_i^{lk}.
\end{aligned}
$$

Hence,

$$
\tilde{X} = \sum_{\xi \in \left(\mathbb{Z}/12n\mathbb{Z}\right)^n} \left(\chi_\xi \nabla \alpha_\xi + \sum_{k,l=1}^{n} \chi_\xi \rho_\xi^k \nabla \eta_\xi^{lk} \right).
$$

These are at most $2 \cdot (12n)^n(n^2+1) =: C_2(n)$ functions, proving (iii).

Proof of (iv) and (v). To prove the last two statements, we may again fix a $\xi \in \left(\mathbb{Z}/12n\mathbb{Z}\right)^n$ as above. Take an $x \in B_R(0)$. If $x \notin B_{3\delta}(y_i)$ for some $i \in Y_\xi$, then all of the above functions and their gradients are zero when evaluated at x. Otherwise, by (3.5.3), there exists exactly one $i \in Y_\xi$ such that $x \in B_{3\delta}(y_i)$. Then we have,

$$
\begin{aligned}
&|\chi_\xi(x)| = |\chi_i(x)| \leq 1, \\
&|\nabla \chi_\xi(x)| = |\nabla \chi_i(x)| \leq C\delta^{-1} \leq C(1 + \|X\|_{C^2})\varepsilon^{-1}, \\
&|\alpha_\xi(x)| = |\alpha_i(x)| \leq C\|X\|_{C^0}\delta \leq C(1 + \|DX\|_{C^1})\delta \leq C\varepsilon,
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla \alpha_\xi(x)| = |\nabla \alpha_i(x)| &\leq |\nabla \psi(x - y_i)\langle A_i, (x - y_i)\rangle|(x) + |\psi(x - y_i)A_i|(x) \\
&\leq \delta^{-1} \cdot C\|X\|_{C^0}\delta + C\|X\|_{C^0} \leq C(1 + \|X\|_{C^0}).
\end{aligned}
$$

Moreover,

$$
|\chi_\xi \rho_\xi^k(x)| = |\chi_i \rho_i^k(x)| \leq C\delta \leq C\varepsilon
$$

and

$$
\begin{aligned}
|\nabla(\chi_\xi \rho_\xi^k)|(x) =& |\nabla(\chi_i \rho_i^k)|(x) \leq |(\nabla \chi_i)|(x)|\rho_i^k|(x) \\
&\quad + |\chi_i|(x)(|\nabla \psi(x - y_i)||(x_k - (y_i)_k)| + |\psi(x - y_i)|) \\
\leq& C(\delta^{-1} \cdot \delta) + C(\delta^{-1}\delta + 1) \leq C.
\end{aligned}
$$

Finally,

$$
|\eta_\xi^{lk}|(x) = |\eta_i^{lk}|(x) = |\psi(x - y_i)||B_i^{lk}||(x_l - (y_i)_l)| \leq C\|X\|_{C^1}\delta
$$

and

$$\begin{aligned}|\nabla\eta_\xi^{lk}|(x) = |\nabla\eta_i^{lk}|(x) &\leq |\nabla\psi(x-y_i)||B_i^{lk}||x_l-(y_i)_l| + |\psi(x-y_i)||B_i^{lk}|\\ &\leq C\delta^{-1}\|X\|_{C^1}\delta + \|X\|_{C^1} \leq C\|X\|_{C^1}.\end{aligned}$$

As x was arbitrary in $B_R(0)$, this holds for all x, and as ξ was arbitrary, this holds for all ξ, which proves the lemma. □

We now deduce a corollary about Sobolev spaces in $\mathbb{R}^n$. This is not related to our further study but it is worth mentioning.

Corollary 3.5.7. *Let $\Omega \subset \mathbb{R}^n$ be an open subset. Then* $\mathrm{CompV}(\Omega)$ *is dense in $W_0^{1,p}(\Omega)$ for all $p \in [1,\infty)$.*

We are now going to apply the previous results to manifolds. The next lemma follows from partitioning the manifold in balls and patching the approximations from Lemma 3.5.6 together.

Lemma 3.5.8. *Let M be a smooth manifold with a C^0-Riemannian metric g that admits L^2_{loc}-Christoffel symbols and a measure μ defined via $\mathrm{d}\mu = h^2\mathrm{dvol}_g$, where $h \in C^0(M,(0,\infty))$. Let X be a compactly supported smooth vector field on M and let $\varepsilon > 0$. Then there exist a constant $C = C(M,g,h,X)$ and a vector field $\tilde{X} = \sum_{p=1}^q h_p\nabla f_p \in$* $\mathrm{CompV}(M)$ *such that*

(i) $\left\|X-\tilde{X}\right\|_{H_1^1(TM,\mu)} \leq C\varepsilon$,

(ii) $\sup_M(|\tilde{X}|_g + |\nabla\tilde{X}|_g) \leq C$,

(iii) $q \leq C$,

(vi) *for all p, we have that $\|h_p\|_{L^\infty} \leq C$ and $\|f_p\|_{L^\infty} \leq C\varepsilon$,*

(v) *for all p, we have that $\||\nabla h_p|_g\|_{L^\infty} \leq C\varepsilon^{-1}$ and $\||\nabla f_p|_g\|_{L^\infty} \leq C$.*

In the next proposition we invoke the parabolic De Giorgi-Nash-Moser theory [85] to obtain $C^{0,\alpha}$ convergence of the heat flow to the initial datum. The non-triviality of the statement relies on the fact that the Riemannian metric is merely C^0 with L^2_{loc} Christoffel symbols, and the weight on the measure is merely $C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$.

Proposition 3.5.9. *Let M be a smooth manifold and g a C^0-Riemannian metric with L^2_{loc} Christoffel symbols. Let moreover $h \in C^0(M,(0,\infty)) \cap W^{1,2}_{\mathrm{loc}}(M)$. Define the measure μ on M via $\mathrm{d}\mu = h^2\mathrm{dvol}_g$. Let $K \in \mathbb{R}$ and assume that (M,d_g,μ) is an $\mathsf{RCD}(K,\infty)$-space. Denote by H_t the heat flow on M. Let $\Omega \subset M$ be open such that Ω is contained in* *one coordinate patch (U, ψ) and $\overline{\Omega}$ is compact. Then there exists an $\alpha \in (0,1)$ such that for each $\rho \in C_c^\infty(M)$, $T > 0$, and for each open $\tilde{\Omega} \subset\subset \Omega$, it holds $(\mathrm{id}_{[0,1]} \times \psi)_* H_t\rho|_{\tilde{\Omega}} \in C^{0,\alpha}([0,T), \tilde{\Omega})$.*

Proof. Throughout this proof we write ρ_t for $\psi_* H_t\rho|_\Omega$. As $\overline{\Omega}$ is compact, we get that there exists a constant $\lambda > 0$ such that $\frac{1}{\lambda} \le h^2\sqrt{|g|} \le \lambda$ and $\frac{1}{\lambda}\mathrm{Id}_n \le g^{-1} \le \lambda \mathrm{Id}_n$ on Ω. We note that ρ_t weakly solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_t = \Delta_\mu \rho_t$$

on Ω. Observing that for any two functions $f, \phi \in C_c^\infty(\Omega)$ it holds

$$\int_M \Delta_\mu f \cdot \phi \,\mathrm{d}\mu = -\int_M \langle \nabla_g f, \nabla_g \phi\rangle_g \,\mathrm{d}\mu = -\int_M h^2\sqrt{|g|} g^{ij}\partial_i f \partial_j \phi \,\mathrm{d}\mathcal{L}^n,$$

we get that ρ_t weakly solves

$$\partial_t \rho_t - \mathrm{div}(AD\rho_t) = 0, \tag{3.5.4}$$

where

$$A_{ij} = h^2\sqrt{|g|} g^{ij}. \tag{3.5.5}$$

By the previous observations, A is uniformly elliptic on Ω. Recalling that $\rho \in C_c^\infty(M)$, the result follows from the parabolic DeGiorgi-Nash-Moser theory, see for instance Theorem 1.1 in Chapter V of [85]. $\square$

We are now ready to prove the main approximation result of the section that will be key in the proof of the main theorem of this chapter.

Lemma 3.5.10. *Let M be a smooth manifold endowed with a C^0-Riemannian metric g that admits L^2_{loc}-Christoffel symbols and a continuous positive weight h. Let μ be the measure defined via $\mathrm{d}\mu = h^2 dvol_g$. Assume that the metric measure space (M, d_g, μ) satisfies the $\mathsf{CD}(K,\infty)$-condition. Then for each relatively compact, open $U \subset M$ and vector field $X \in C_c^\infty(TM)$ there exists a sequence $(W_j)_j \subset \mathrm{TestV}(M)$ such that $W_j \to X$ in $H_1^2(TU, \mu)$ and $(W_j)_j$ is bounded in $L^\infty(U, \mu)$.*

Proof. As a first observation we note that by Corollary 3.3.15, the metric measure space (M, d_g, μ) is infinitesimally Hilbertian, hence an $\mathsf{RCD}(K,\infty)$-space, which in particular enables us to apply all the theory from the previous section. Let $\varepsilon > 0$. We can find a constant $C = C(M, g, h, X)$ and functions $f_p, h_p \in C_c^\infty(M)$ where $p \in \{1, \dots q\}$ such that, denoting $\tilde{X} := \sum_{p=1}^q h_p \nabla f_p$ satisfy conditions (i)-(v) from Lemma 3.5.8. We first note that by interpolation, $\left\|X - \tilde{X}\right\|_{H_1^2(TM,\mu)} \leq C\sqrt{\varepsilon}$. We have that $C_c^\infty \subset D(\Delta_\mu) \cap W^{2,2}(M) \cap L^\infty(M)$, so for all p, it holds $f_p, h_p \subset H_2^2 \cap L^\infty(M,\mu) \subset D(\Delta)$. Thanks to Proposition 3.4.22 together with the equality of norms established in Proposition 3.4.8, we get that for each p, there exists a $t \in (0, \frac{1}{|K|}]$, such that

$$\max(\|h_p - H_t h_p\|_{H_2^2(M,\mu)}, \|f_p - H_t f_p\|_{H_2^2(M,\mu)}) < \frac{\varepsilon^2}{\max_p \left(\|f_p\|_{H_2^2(M,\mu)} + \|h_p\|_{H_2^2(M,\mu)}\right)}.$$

Fix a relatively compact, open set $U \subset M$. Then by using Proposition 3.5.9 on a finite, relatively compact cover, we can potentially decrease t, to get

$$\max(\|h_p - H_t h_p\|_{L^\infty(U,\mu)}, \|f_p - H_t f_p\|_{L^\infty(U,\mu)}) < \frac{\varepsilon^2}{\max_p \left(\|f_p\|_{H_2^2(M,\mu)} + \|h_p\|_{H_2^2(M,\mu)}\right)},$$

for all $p = 1, \dots, q$. Define $\tilde{f}_p := H_t f_p$ and $\tilde{h}_p := H_t h_p$ for all p. Moreover, recall that for each $t \in \left(0, \frac{1}{|K|}\right]$ and each $f \in W^{2,2} \cap L^\infty$, we have that

$$\begin{aligned} &\|H_t f\|_{L^\infty(M,\mu)} \leq \|f\|_{L^\infty(M,\mu)} \text{ and} \\ &\||\nabla H_t f|_g\|_{L^\infty(M,\mu)} \leq e^{-Kt} \||\nabla f|_g\|_{L^\infty(M,\mu)}, \end{aligned}$$

by Theorem 2.2.12 and the maximum principle of the heat flow. Hence,

$$\begin{aligned} &\left\|\tilde{f}_p\right\|_{L^\infty} \leq \|f_p\|_{L^\infty} \leq C\varepsilon, \\ &\left\|\tilde{h}_p\right\|_{L^\infty} \leq \|h_p\|_{L^\infty} \leq C, \\ &\left\||\nabla \tilde{f}_p|_g\right\|_{L^\infty} \leq C, \text{ and} \\ &\left\||\nabla \tilde{h}_p|_g\right\|_{L^\infty} \leq C\varepsilon^{-1}. \end{aligned}$$

Define $W := \sum_{p=1}^q \tilde{h}_p \nabla \tilde{f}_p$. We directly get that

$$\||W|_g\|_{L^\infty(M,\mu)} = \left\|\left|\sum_{p=1}^q \tilde{h}_p \nabla \tilde{f}_p\right|_g\right\|_{L^\infty(M,\mu)} \leq \sum_{p=1}^q \left\||\tilde{h}_p \nabla \tilde{f}_p|_g\right\|_{L^\infty(M,\mu)} \leq qC^2.$$

To conclude, we estimate that

$$\begin{aligned} \left\|W - \tilde{X}\right\|_{L^2(TM,\mu)} &\leq \sum_{p=1}^q \left\|\tilde{h}_p \nabla \tilde{f}_p - h_p \nabla f_p\right\|_{L^2(TM,\mu)} \\ &\leq \sum_{p=1}^q \left\|\tilde{h}_p \nabla \tilde{f}_p - \tilde{h}_p \nabla f_p\right\|_{L^2(TM,\mu)} + \left\|\tilde{h}_p \nabla f_p - h_p \nabla f_p\right\|_{L^2(TM,\mu)} \\ &\leq \sum_{p=1}^q \left\|\tilde{h}_p\right\|_{L^\infty(M,\mu)} \left\|\nabla \tilde{f}_p - \nabla f_p\right\|_{L^2(TM,\mu)} + \left\|\tilde{h}_p - h_p\right\|_{L^2(M,\mu)} \left\|\nabla \tilde{f}_p\right\|_{L^\infty(TM,\mu)} \\ &\leq C(M, g, h, X)\varepsilon. \end{aligned}$$

For the covariant derivative, we note that

$$\nabla W = \left(\nabla_c \sum_{p=1}^{q} \tilde{h}_p \nabla \tilde{f}_p\right)^\sharp = \left(\sum_{p=1}^{q} \tilde{h}_p \mathrm{Hess} \tilde{f}_p + \mathrm{d}\tilde{h}_p \otimes \nabla \tilde{f}_p\right)^\sharp = \sum_{p=1}^{q} \tilde{h}_p (\mathrm{Hess} \tilde{f}_p)^\sharp + \nabla \tilde{h}_p \otimes \nabla \tilde{f}_p.$$

Then, by similar arguments as the previous ones, we get that

$$\begin{aligned}
&\left\|\nabla W - \nabla \tilde{X}\right\|_{L^2(T^{\otimes 2}U,\mu)} \\
&\leq \sum_{p=1}^{q} \left\|\nabla(\tilde{h}_p \nabla \tilde{f}_p) - \nabla(h_p \nabla f_p)\right\|_{L^2(T^{\otimes 2}U,\mu)} \\
&\leq \sum_{p=1}^{q} \left\|\tilde{h}_p\right\|_{L^\infty(U,\mu)} \left\|\mathrm{Hess}(\tilde{f}_p - f_p)^\sharp\right\|_{L^2(T^{\otimes 2}U,\mu)} + \left\|\tilde{h}_p - h_p\right\|_{L^\infty(U,\mu)} \left\|(\mathrm{Hess}\tilde{f}_p)^\sharp\right\|_{L^2(T^{\otimes 2}U,\mu)} \\
&+ \sum_{p=1}^{q} \left\|\nabla \tilde{h}_p\right\|_{L^\infty(TU)} \left\|\nabla \tilde{f}_p - \nabla f_p\right\|_{L^2(TU,\mu)} + \left\|\nabla(\tilde{h}_p - h_p)\right\|_{L^2(TU,\mu)} \left\|\nabla \tilde{f}_p\right\|_{L^\infty(TU,\mu)} \\
&\leq C(M, U, g, h, X)\varepsilon.
\end{aligned}$$

Finally, we estimate that

$$\|X - W\|_{H^2_1(TU,\mu)} \leq \left\|X - \tilde{X}\right\|_{H^2_1(TU,\mu)} + \left\|\tilde{X} - W\right\|_{H^2_1(TU,\mu)} \leq C(M, U, g, h, X)\sqrt{\varepsilon}.$$

□

3.5.2 The case $N = \infty$

We are now able to prove the first main result:

Theorem 3.5.11. *Let M be a smooth manifold and $g \in C^0(M)$ be a metric with L^2_{loc} Christoffel symbols and let $h \in C^0(M, (0,\infty)) \cap W^{1,2}_{\mathrm{loc}}(M)$ be such that (M, d_g, μ), with $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$ is a $\mathsf{CD}(K,\infty)$-space. Then*

$$\mathrm{Ric}_{\mu,\infty} \geq Kg$$

in the distributional sense.

Proof. From Corollary 3.3.15, we get that (M, d_g, μ) is indeed an $\mathsf{RCD}(K,\infty)$-space, so all the observations from the previous chapter apply. Our aim is to show that $\mathrm{Ric}_g \geq Kg$ distributionally, so we fix a smooth vector field $X \in \mathcal{T}^1_0$ and a test volume $\omega = \phi h^2 \mathrm{vol}_g$, where $\phi \in C_c \cap W^{1,2}_{\mathrm{loc}}(M)$. Using a smooth cut-off function, we can assume that X is compactly supported. We can (using a partition of unity) again assume that ϕ is supported in one coordinate patch. We will therefore directly work in an open, relatively compact set $U \subset \mathbb{R}^n$. From (3.2.8), we know that

$$
\begin{aligned}
&\int_U \mathrm{Ric}_{\mu,\infty}(X,X)\omega \\
&= \int_U X^j X^k (\Gamma^s_{kj}\Gamma^p_{ps} - \Gamma^s_{kp}\Gamma^p_{js})\phi h^2 \sqrt{|g|}\, dx^1 \dots dx^n \\
&- \int_U \Gamma^p_{jk}\partial_p (X^j X^k \phi h^2 \sqrt{|g|})\, dx^1 \dots dx^n + \int_U \Gamma^p_{pk}\partial_j (X^j X^k \phi h^2 \sqrt{|g|})\, dx^1 \dots dx^n \\
&+ 2\int_U X^j X^k (\partial_j h \partial_k h + h\partial_s h \Gamma^s_{kj})\phi\sqrt{|g|} dx^1 \dots dx^n + 2\int_U \partial_j h \partial_k (X^j X^k h\phi\sqrt{|g|}) dx^1 \dots dx^n \\
&= \int_U X^j X^k (\Gamma^s_{kj}\Gamma^p_{ps} - \Gamma^s_{kp}\Gamma^p_{js})\phi h^2 \sqrt{|g|}\, dx^1 \dots dx^n - \int \Gamma^p_{jk}\partial_p (X^j) X^k \phi h^2 \sqrt{|g|}\, dx^1 \dots dx^n \\
&- \int_U \Gamma^p_{jk} X^j \partial_p (X^k \phi h^2 \sqrt{|g|})\, dx^1 \dots dx^n + \int_U \Gamma^p_{pk}\partial_j (X^j) X^k \phi h^2 \sqrt{|g|}\, dx^1 \dots dx^n \\
&+ \int_U \Gamma^p_{pk} X^j \partial_j (X^k \phi h^2 \sqrt{|g|})\, dx^1 \dots dx^n \\
&+ 2\int_U X^j X^k (\partial_j h \partial_k h + h\partial_s h \Gamma^s_{kj})\phi\sqrt{|g|} dx^1 \dots dx^n + 2\int_U \partial_j h \partial_k X^j X^k h\phi\sqrt{|g|}) dx^1 \dots dx^n \\
&+ 2\int_U \partial_j h X^j \partial_k (X^k h\phi\sqrt{|g|}) dx^1 \dots dx^n .
\end{aligned}
$$

By the assumptions on g and h and Lemma 3.2.8, there exist functions $\Psi_{i,j,k} \in L^2(U, \mathcal{L}^n)$ and $\Phi_{j,k} \in L^1(U, \mathcal{L}^n)$ all compactly supported in U such that

$$
\int_U \mathrm{Ric}_{\mu,\infty}(X,X)\omega = \int_U X^j X^k \Phi_{j,k}\, \mathrm{d}\mathcal{L}^n + \int_U \partial_i X^j X^k \Psi_{i,j,k}\, \mathrm{d}\mathcal{L}^n.
$$

By Lemma 3.5.10, there exists a sequence $V_\delta \in \mathrm{TestV}(M)$ such that $\||V_\delta|_g\|_{L^\infty}(U) \leq C$ for some constant $C > 0$ that does not depend on δ and $\|V_\delta - X\|_{H^2_1(TU,\mu)} \to 0$ as $\delta \to 0$. Fix an $\varepsilon > 0$. Note that $V_\delta \otimes V_\delta$ is bounded in L^∞ and converges to $X \otimes X$ in L^1. By Lemma 3.5.2, $V_\delta \otimes V_\delta \xrightharpoonup{*} X \otimes X$ in $L^\infty = (L^1)^*$ and $V_\delta \xrightharpoonup{*} X$ in $L^\infty = (L^1)^*$. Hence, we can find a $\delta_0 > 0$ such that $\|V_\delta - X\|_{H^2_1(TU,\mu)} < \varepsilon$ and $\|V_\delta \otimes V_\delta - X \otimes X\|_{L^1(T^{\otimes 2}U,\mu)} < \varepsilon$ for all $\delta \in (0, \delta_0)$. As g and $h^2\sqrt{|g|}$ are uniformly bounded from above and from below on U, we get that $\|V_\delta - X\|_{W^{1,2}(\mathbb{R}^n,\mathcal{L}^n)} < C\varepsilon$ and $\|V_\delta \otimes V_\delta - X \otimes X\|_{L^1(\mathbb{R}^{n\times n},\mathcal{L}^{n\times n})} < C\varepsilon$ for all $\delta \in (0, \delta_0)$ and a $C = C(M, g, h, U) > 0$. Now we can use the weak* convergence, to choose a $\delta_1 \in (0, \delta_0)$ such that

$$
\begin{aligned}
&\left| \int_U X^j X^k \Phi_{j,k}\, \mathrm{d}\mathcal{L}^n - \int_U V^j_{\delta_1} V^k_{\delta_1} \Phi_{j,k}\, \mathrm{d}\mathcal{L}^n \right| \leq \varepsilon \text{ and} \\
&\left| \int_U \partial_i X^j X^k \Psi_{i,j,k}\, \mathrm{d}\mathcal{L}^n - \int_U \partial_i X^j V^k_{\delta_1} \Psi_{i,j,k}\, \mathrm{d}\mathcal{L}^n \right| \leq \varepsilon.
\end{aligned}
$$

From now on we will denote $V_{\delta_1} = V$ for simplicity. We get that

$$\left| \int_U \partial_i X^j V^k \Psi_{i,j,k} \,\mathrm{d}\mathcal{L}^n - \int_U \partial_i V^j V^k \Psi_{i,j,k} \,\mathrm{d}\mathcal{L}^n \right| \leq C(M,g,h,U,\phi,X)\varepsilon.$$

Now note that

$$\left| \int_M \langle V,V\rangle_g \phi\, h^2 \mathrm{dvol}_g - \int_M \langle X,X\rangle_g \phi\, h^2 \mathrm{dvol}_g \right| \leq C\|V\otimes V - X\otimes X\|_{L^1(\mathbb{R}^{n\times n})} \leq C\varepsilon,$$

where again $C = C(M,g,h,U,\phi,X)$. Finally, we use that

$$\int_M \phi h^2 \, d\mathbf{Ric}(V,V) = \int_U \mathrm{Ric}_{\mu,\infty}(V,V)\omega = \int_U V^j V^k \Phi_{j,k} \,\mathrm{d}\mathcal{L}^n + \int_U \partial_i V^j V^k \Psi_{i,j,k} \,\mathrm{d}\mathcal{L}^n. \tag{3.5.6}$$

All together, we get that

$$\begin{aligned}\int_M \mathrm{Ric}(X,X)\omega &\geq \int_M \phi h^2 \, d\mathbf{Ric}(V,V) - C\varepsilon \\ &\geq K \int_M \langle V,V\rangle_g \phi h^2 \,\mathrm{dvol}_g - C\varepsilon \\ &\geq K \int_M \langle X,X\rangle_g \phi h^2 \,\mathrm{dvol}_g - (2+|K|)C\varepsilon \\ &= K\int_M g(X,X)\omega - (2+|K|)C\varepsilon.\end{aligned}$$

Sending $\varepsilon \to 0$ yields the result. $\square$

Lemma 3.5.12. *Let $\Omega \subset \mathbb{R}^n$ be open and $M \in C_c^\infty(\Omega;\mathbb{R}^{n\times n}_{sym})$ be pointwise positive semidefinite. Let $\varepsilon > 0$. Then there exist a function $\varphi \in C_c^\infty(\Omega,[0,1])$ and vector fields $b_k \in C^\infty(\Omega,\mathbb{R}^n)$ for $k = 1,\dots,n$, such that for $M_\varepsilon := \sum_k \varphi b_k \otimes b_k$, it holds*

$$|M - M_\varepsilon|_{C^1(\mathbb{R}^n)} \leq C\varepsilon,$$

where $C = C(\Omega, \operatorname{supp} M, n)$ does not depend on ε.

Proof. First we recall a basic fact about matrices. Let $P \in \mathbb{R}^{n\times n}$ be positive definite. Then there exists a unique positive definite matrix $A \in \mathbb{R}^{n\times n}$ such that $A^T A = P$. Writing $A = (A_{ij})_{ij}$, we get that $P_{ij} = (A^T A)_{ij} = \sum_{k=1}^n A_{ki}A_{kj} = \sum_k (A_k \otimes A_k)_{ij}$, where A_k denotes the k-th row vector of A and $\otimes$ the dyadic product.
Now define $\tilde{M} := M + \varepsilon \mathrm{Id}_n \in C^\infty(\Omega;\mathbb{R}^{n\times n}_{sym})$ and note that $\tilde{M}$ is constant outside $\operatorname{supp} M$. By construction, we have that $\tilde{M}$ is positive definite everywhere in Ω and its smallest eigenvalue is bounded below by ε. Similarly, its largest eigenvalue is bounded above by $||M|_{HS}|_{C^0(\Omega)} + \varepsilon =: m$. Set

$$U := \left\{ a + ib : a \in \left(\frac{1}{2}\varepsilon, m+2\right), b \in (-1,1) \right\}$$

and $K := [\varepsilon, m+1] \subset \mathbb{R}_+ \subset \mathbb{C}$. For a matrix $B \in \mathbb{C}^{n\times n}$, we denote by $\sigma_{\mathbb{C}^{n\times n}}(B)$ the set of its complex eigenvalues. Note that for each $p \in \Omega$, $\sigma_{\mathbb{C}^{n\times n}}(\tilde{M}(p)) \subset K$. Let γ be a smooth cycle in $U \setminus K$ such that $n(\gamma, w) = 1$ for all $w \in K$, where $n(\gamma, w)$ denotes the winding number of γ around w. We denote by $s : U \to \mathbb{C}$ the unique holomorphic function which is defined by $s^2(z) = z$ and $s(w) \geq 0$ for $w \in \mathbb{R} \cap U$. Define

$$B(p) := \frac{1}{2\pi i} \int_\gamma s(z)(z\mathrm{Id}_n - \tilde{M}(p))^{-1}\, dz.$$

Using holomorphic functional calculus we infer that $B(p)$ is the unique positive square root of $\tilde{M}(p)$ for all $p \in \Omega$. As γ is a fixed curve with positive distance to K, all functions in the integral are smooth and bounded on the domain of integration. As $\tilde{M}(p)$ is a smooth function of p, so is $B(p)$ and each of its rows, which we will denote by b_k for $k = 1, \ldots, n$. Let $\varphi \in C_c^\infty(\Omega, [0,1])$ be a non-negative cut-off function such that $\varphi \equiv 1$ on $\operatorname{supp} M$. We can choose φ such that $|\nabla\varphi|$ is bounded by $C(n) \cdot \operatorname{dist}(\partial\Omega, \operatorname{supp} M)$. Now define

$$M_\varepsilon := \varphi \sum_k b_k \otimes b_k.$$

By definition, we have that

$$|M_\varepsilon - \tilde{M}|(p) = \begin{cases} 0 & \text{if } p \in \operatorname{supp} M, \\ \varphi|\varepsilon \mathrm{Id}_n| \leq C(n)\varepsilon & \text{if } p \notin \operatorname{supp} M. \end{cases}$$

Similarly,

$$|\nabla(M_\varepsilon - \tilde{M})|(p) = \begin{cases} 0 & \text{if } p \in \operatorname{supp} M, \\ \nabla\varphi|\varepsilon \mathrm{Id}_n| \leq C(\Omega, \operatorname{supp} M, n)\varepsilon & \text{if } p \notin \operatorname{supp} M. \end{cases}$$

By definition, we know that $|M - \tilde{M}|_{C^1} \leq C(n)\varepsilon$, so combining the estimates above, we get that

$$|M - M_\varepsilon|_{C^1} \leq |M - \tilde{M}|_{C^1} + |M_\varepsilon - \tilde{M}|_{C^1} \leq C\varepsilon.$$

□

Theorem 3.5.13. *Let M be a smooth manifold endowed with a C^0-Riemannian metric g that admits L^2_{loc}-Christoffel symbols and a positive function $h \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$. Define the measure μ via $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$. Suppose the distributional Bakry-Émery ∞-Ricci curvature as defined in* (3.2.8) *is bounded below by K for some $K \in \mathbb{R}$. For each coordinate patch U, there exist regular Radon measures ν_{ij} for $i, j = 1, \ldots, n$ such that for a test volume ω with* $\operatorname{supp}\omega \subset U$ *(locally written as $\omega = \varphi h^2 \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^n$) and smooth vector fields X, Y, we have that in local coordinates it holds*

$$\mathrm{Ric}_{\mu,\infty}(X, Y)\omega = \sum_{i,j} \int X^i Y^j \varphi h^2 \sqrt{|g|} \mathrm{d}\nu_{ij}(x_1, \ldots, x_n). \tag{3.5.7}$$

Proof. Using the local trivialisation $\psi : U \to \Omega \subset \mathbb{R}^n$, we may assume that we are working on the open set Ω and will express everything in local coordinates. Define

$$D_{ij} := \partial_p \Gamma^p_{ij} - \partial_i \Gamma^p_{pj} + \Gamma^s_{ij}\Gamma^p_{ps} - \Gamma^s_{jp}\Gamma^p_{is} - \frac{2}{h^2}(h\partial_{ij}h - \partial_i h \partial_j h - \Gamma^s_{ij} h \partial_s h) - Kg_{ij} \in \mathcal{D}'(\Omega).$$

By definition, D_{ij} is a distribution of order at most 1 and for any smooth vector field X and any test volume $\omega = \varphi h^2 \sqrt{|g|} dx^1 \wedge \ldots \wedge dx^n$, for a $\varphi \in C_c \cap W^{1,2}_{\text{loc}}(M)$ we get that

$$(\text{Ric}(X,X)_{\mu,\infty} - Kg(X,X))\omega = \sum_{i,j} \langle D_{ij}, X^i X^j \varphi h^2 \sqrt{|g|} \rangle_{\mathcal{D}',\mathcal{D}}.$$

Now define $D \in \mathcal{D}'(\Omega, \mathbb{R}^{n\times n}_{sym})$ via $\langle D, M\rangle_{\mathcal{D}',\mathcal{D}} := \sum_{i,j} \langle D_{ij}, M_{ij}\rangle_{\mathcal{D}',\mathcal{D}}$. Denote $B(X,\varphi) := \varphi h^2 \sqrt{|g|}(X \otimes X)$. Then

$$(\text{Ric}(X,X)_{\mu,\infty} - Kg(X,X))\omega = \sum_{ij} \langle D_{ij}, B(X,\varphi)_{ij}\rangle_{\mathcal{D}',\mathcal{D}} = \langle D, B(X,\varphi)\rangle_{\mathcal{D}',\mathcal{D}}.$$

Now let $B \in C^\infty_c(\Omega; \mathbb{R}^{n\times n}_{sym})$ be pointwise positive semidefinite and $\varepsilon > 0$. Let B_ε be as in Lemma 3.5.12. As $\text{Ric}_{\mu,\infty} - Kg \geq 0$, we get that

$$\langle D, B_\varepsilon \rangle_{\mathcal{D}',\mathcal{D}} \geq 0.$$

Moreover,

$$|\langle D, (B_\varepsilon - B)\rangle_{\mathcal{D}',\mathcal{D}}| \leq C(\Omega, \text{supp}B, g, h)|B - B_\varepsilon|_{C^1(\Omega)} \leq C(\Omega, B, n, g, h)\varepsilon.$$

Hence,

$$\langle D, B\rangle_{\mathcal{D}',\mathcal{D}} \geq -C(\Omega, B, n, g, h)\varepsilon.$$

Letting $\varepsilon \to 0$, we get that for any such B,

$$\langle D, B\rangle_{\mathcal{D}',\mathcal{D}} \geq 0. \tag{3.5.8}$$

Now fix $S \subset \Omega$ compact and let $A \in C^\infty_c(\Omega; \mathbb{R}^{n\times n}_{sym})$ (not necessarily positive semidefinite) such that $\text{supp}\, A \subset S$ and $\sup_S |A|_{HS} \leq 1$. Then A only has eigenvalues in $[-1,1]$ at each point in Ω. Let $\rho \in C^\infty_c(\Omega, [0,1])$ such that $\rho \equiv 1$ on S. It then follows that $\rho \text{Id}_n - A$ is positive semidefinite in Ω. Then by (3.5.8),

$$\langle D, A\rangle_{\mathcal{D}',\mathcal{D}} \leq \langle D, \rho \text{Id}_n\rangle_{\mathcal{D}',\mathcal{D}}.$$

In other words:

$$\sup_{A \in C^\infty_c(\Omega;\mathbb{R}^{n\times n}_{sym}), \text{supp}A \subset S, ||A|_{HS}|_{C^0} \leq 1} \langle D, A\rangle_{\mathcal{D}',\mathcal{D}} \leq \langle D, \rho \text{Id}_n\rangle_{\mathcal{D}',\mathcal{D}} < \infty.$$

As S is arbitrary, we have proven that $D \in C_c(\Omega; \mathbb{R}^{n\times n}_{sym})' \cong \big(C_c(\Omega,\mathbb{R})^{\frac{n(n+1)}{2}}\big)' \cong (C_c(\Omega,\mathbb{R})')^{\frac{n(n+1)}{2}}$. By Riesz' Theorem, $C_c(\Omega,\mathbb{R})'$ equals the space of regular Radon measures on Ω, so it follows that there exist regular Radon measures $\tilde{\nu}_{ij}$ on Ω for $i,j = 1,\dots,n$ such that

$$\langle D, M\rangle = \sum_{i,j} \int_\Omega M_{ij} \mathrm{d}\tilde{\nu}_{ij}, \quad \text{for all } M \in C_c(\Omega; \mathbb{R}^{n\times n}_{sym}).$$

Define the regular Radon measure

$$\nu_{ij} := \tilde{\nu}_{ij} + K g_{ij} \mathcal{L}^n.$$

Now for any vector field X and any compactly supported test volume $\varphi h^2 \sqrt{|g|} dx^1 \wedge, \dots, \wedge dx^n$, we have that

$$\int_\Omega X^i X^j \varphi h^2 \sqrt{|g|} g_{ij} K \mathrm{d}\mathcal{L}^n = \langle K g(X,X), \omega\rangle_{\mathcal{D}',\mathcal{D}}.$$

By the definition of $B(X,\varphi)$ and D, we get that

$$\langle \mathrm{Ric}(X,X)_{\mu,\infty}, \omega\rangle_{\mathcal{D}',\mathcal{D}} = \sum_{i,j} \int X^i X^j \varphi h^2 \sqrt{|g|} \mathrm{d}\nu_{ij}(x_1,\dots,x_n).$$

By using that $\mathrm{Ric}_{\mu,\infty}(\cdot,\cdot)$ is a symmetric bilinear form and considering $\mathrm{Ric}_{\mu,\infty}(X+Y, X+Y)$, we conclude that (3.5.7) holds. □

Remark 3.5.14. *From the proof of Theorem 3.5.13 it also follows that $\sum_{i,j} |\nu_{ij}|(S) < \infty$ for each compact subset $S \subset \Omega$. Moreover, the components $(\nu_{ij})_{ij}$ change tensorially under coordinate transformations.*

3.5.3 The case $N \in [n, \infty)$

Finally, we want to examine the case when (M, d_g, μ) is a $\mathsf{CD}^*(K,N)$ space for some $N \geq 1$. By Corollary 3.3.15, we know that (M, d_g, μ) is infinitesimally Hilbertian, hence we know that it is $\mathsf{RCD}^*(K,N)$. Then, by Theorem 3.4.4, we get that (M, d_g, μ) satisfies the $\mathsf{BE}(K,N)$-condition

$$\mathbf{\Gamma}_2(f,f) = \frac{1}{2}\mathbf{\Delta}\langle \nabla f, \nabla f\rangle - \langle \nabla \Delta f, \nabla f\rangle \mu \geq (K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2)\mu, \quad \text{for all } f \in \mathrm{TestF}(M).$$

Testing with a non-negative function $\varphi \in C_c \cap W^{1,2}_{\mathrm{loc}}(M)$, that we may assume to be supported in one coordinate patch, we obtain:

$$\begin{aligned} -\frac{1}{2}\int_M \langle \nabla\langle \nabla f, \nabla f\rangle, \nabla\varphi\rangle h^2 \sqrt{|g|} dx^1 \dots dx^n - \int_M \varphi \langle \nabla \Delta f, \nabla f\rangle h^2 \sqrt{|g|} dx^1 \dots dx^n \\ \geq \int_M (K|\nabla f|^2 + \frac{1}{N}(\Delta f)^2)\varphi h^2 \sqrt{|g|} dx^1 \dots dx^n. \end{aligned}$$

Using (3.2.12) and Proposition 3.4.21 and (3.4.2), we get that

$$
\begin{aligned}
&\int (\nabla f)^j (\nabla f)^k (\Gamma^s_{kj}\Gamma^p_{ps} - \Gamma^s_{kp}\Gamma^p_{js})\varphi h^2 \sqrt{|g|}\, dx^1 \dots dx^n \\
&\quad - \int \Gamma^p_{jk}\partial_p((\nabla f)^j (\nabla f)^k \varphi h^2 \sqrt{|g|})\, dx^1 \dots dx^n \\
&\quad + \int \Gamma^p_{pk}\partial_j((\nabla f)^j (\nabla f)^k \varphi h^2 \sqrt{|g|})\, dx^1 \dots dx^n \\
&\quad + \int \frac{2}{h^2}(\nabla f)^j (\nabla f)^k (\partial_k h \partial_j h + h\Gamma^s_{jk}\partial_s h)\varphi h^2 \sqrt{|g|}\, dx^1 \dots dx^n \\
&\quad + 2\int \partial_k h \partial_j (h(\nabla f)^j (\nabla f)^k \varphi \sqrt{|g|})\, dx^1 \dots dx^n + \int_M \varphi |\nabla^2 f|^2_{HS} h^2 \sqrt{|g|}\, dx^1 \dots dx^n \\
&\geq \int_M (K|\nabla f|^2 + \frac{1}{N}(\Delta_\mu f)^2)\varphi h^2 \sqrt{|g|}\, dx^1 \dots dx^n. \qquad (3.5.9)
\end{aligned}
$$

Now take a function $\tilde{f} \in C^2_c(M,\mu)$. It follows that $\tilde{f} \in H^{2,2}(M,\mu) \cap D(\Delta_\mu)$, so by Proposition 3.4.22 the heat flow $H_t\tilde{f} =: \tilde{f}_t$ converges strongly to $\tilde{f}$ in $H^{2,2}(M) \subset H^2_2(TM,\mu)$. Then $\tilde{f}_t \to \tilde{f}$ in $H^{2,2} \subset H^2_2$ so we can conclude that (3.5.9) holds for any $\tilde{f} \in C^2_c(M)$ and any $\varphi \in C^1_c(M)$. With Theorem 3.5.13, this gives:

Proposition 3.5.15. *Let M be a smooth manifold endowed with a C^0-Riemannian metric g that admits L^2_{loc}-Christoffel symbols and a measure μ defined via $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$, for a positive function $h \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$. If (M, d_g, μ) is a $\mathsf{CD}^*(K,N)$-space, then for all $f \in C^2_c(M)$, it holds*

$$\sum_{ij} (\nabla f)_i (\nabla f)_j \nu_{ij} + (|\nabla^2 f|^2_{HS} - K|\nabla f|^2 - \frac{1}{N}(\Delta_\mu f)^2)\mu \geq 0 \qquad (3.5.10)$$

in fixed local coordinates.

An application of Theorem 4.7 in [44] shows:

Lemma 3.5.16. *Let M be a smooth manifold, g a C^0-Riemannian metric on M and $U \subset V \subset M$ both open such that $\overline{U} \subset V$ and $\overline{V}$ is compact. Moreover denote by $g_\varepsilon = g * \rho_\varepsilon$. Denote $G = \|g\|_{C^0(\overline{V})} + \|g^{-1}\|_{C^0(\overline{V})}$. Then there exists an $\varepsilon_0 > 0$ such that for all $x \in U$ and $0 < \varepsilon < \varepsilon_0$, it holds*

$$i_\varepsilon(x) \geq C(\varepsilon, G, U, V, \rho) > 0,$$

where $i_\varepsilon(x)$ denotes the injectivity radius with respect to the metric g_ε.

Lemma 3.5.17. *Let M be a smooth manifold, g a C^0-Riemannian metric that admits L^2_{loc}-Christoffel symbols on M and $U \subset V \subset M$ both open such that $\overline{U} \subset V$, and $\overline{V}$ is compact and contained in one coordinate patch. Moreover let $g_\varepsilon := g * \rho_\varepsilon$. Denote*

$G = \|g\|_{C^0(\overline{V})} + \|g^{-1}\|_{C^0(\overline{V})}$. *For all* $p \in U$, *there exists an* $\varepsilon_0 > 0$ *such that for all* $\varepsilon \in (0, \varepsilon_0)$, *there exists a constant* $C = C(U, V, \varepsilon, G, p, \rho) > 0$ *with the following property: for all vectors* $\tilde{v} \in T_pM$ *with* $\|\tilde{v}\|_{g_\varepsilon} \le \min(1, C)$, *it holds*

$$\left|D \exp_p^{g_\varepsilon}|_{\tilde{v}}\right|_{euc} \le 3, \quad \text{and} \quad \left|D^2 \exp_p^{g_\varepsilon}|_{\tilde{v}}\right|_{euc} \le 3. \tag{3.5.11}$$

Proof. By covering and rescaling, we can assume that $U = B_R^{euc}(0) \subset \mathbb{R}^n$ and $V = B_{R+2}^{euc}(0) \subset \mathbb{R}^n$. Let $\varepsilon_0 > 0$ small enough for Lemma 3.5.16 to hold, and such that $\varepsilon_0 \le \frac{1}{2}$ and for all $\varepsilon \in (0, \varepsilon_0)$, g_ε is a Riemannian metric on $B_R(0)$ satisfying

$$\|g_\varepsilon\|_{C^0(\overline{B_R^{euc}(0)})} + \left\|g_\varepsilon^{-1}\right\|_{C^0(\overline{B_R^{euc}(0)})} \le 2G.$$

Choose $\varepsilon \in (0, \varepsilon_0)$ and denote by i_ε the injectivity radius with respect to g_ε. Fix a point $p \in U$ and $v \in T_pU$ such that $\|v\|_{euc} \le 1$. By the relative compactness of V and arguments as in the proof of Proposition 3.3.7 (or alternatively Theorem 4.5 in [25]), there exists $\lambda = \lambda(G) \ge 1$ such that g, g_ε, and $|\cdot|_{euc}$ are pairwise λ-equivalent on V and their induced metrics are pairwise λ-equivalent on $B_{1/2}^{euc}(p)$. For a curve $c : [0,1] \to M$ and W a vector field along the c, we denote by $\dot{W}$ the covariant time derivative and by W' the derivative in local coordinates. Denote $\gamma := \gamma_v^\varepsilon$ the geodesic with respect to g_ε, originating in p and tangent to v. Note that $|\dot{\gamma}(t)|_{g_\varepsilon}$ is constant in the existence domain of γ, hence we have that for all such t,

$$|\dot{\gamma}(t)|_{euc} \le \lambda^2 |\dot{\gamma}(0)|_{euc} = \lambda^2 |v|_{euc}. \tag{3.5.12}$$

For any vector $w \in T_pU$ define $J_{v,w}^\varepsilon$ to be the Jacobi field in metric g_ε along γ with $\dot{J}_{v,w}^\varepsilon(0) = \nabla_{\dot{\gamma}(0)} J_{v,w}^\varepsilon(0) = w$. Recall the Jacobi equation for the smooth metric g_ε:

$$\ddot{J}_{v,w}^\varepsilon(t) = \nabla_{\dot{\gamma}(t)}(\nabla_{\dot{\gamma}(t)} J_{v,w}^\varepsilon)(t) = R(J_{v,w}^\varepsilon(t), \dot{\gamma}(t))\dot{\gamma}(t).$$

In order to keep notation short, we write J_w for $J_{v,w}^\varepsilon$. In local coordinates, we get:

$$\begin{aligned}
(\dot{J}_w(t))^k &= (\nabla_{\dot{\gamma}(t)} J_w(t))^k = \frac{\mathrm{d}}{\mathrm{d}t} J_w^k(t) + J^j(t)\dot{\gamma}^i(t)\Gamma_{\varepsilon\, ij}^{\ k}(\gamma(t)), \text{ and} \\
(\ddot{J}_w(t))^l &= \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\mathrm{d}}{\mathrm{d}t} J_w^l(t) + J^j(t)\dot{\gamma}^i(t)\Gamma_{\varepsilon\, ij}^{\ l}(\gamma(t))\Big) \\
&+ \Gamma_{\varepsilon\, pk}^{\ l}(\gamma(t))\dot{\gamma}^p(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t} J_w^k(t) + J^j(t)\dot{\gamma}^i(t)\Gamma_{\varepsilon\, ij}^{\ k}(\gamma(t))\Big).
\end{aligned}$$

Hence, the Jacobi equation in local coordinates gives

$$\begin{aligned}
(R_\varepsilon)_{ijk}^l J_w^i(t)\dot{\gamma}^j(t)\dot{\gamma}^k(t) =& \frac{\mathrm{d}^2}{\mathrm{d}t^2} J_w^l(t) + \frac{\mathrm{d}}{\mathrm{d}t} J^j(t)\dot{\gamma}^i(t)\Gamma_{\varepsilon\, ij}^{\ l}(\gamma(t)) + J^j(t)\gamma^{i\prime\prime}(t)\Gamma_{\varepsilon\, ij}^{\ l}(\gamma(t)) \\
&+ J^j(t)\dot{\gamma}^i(t) d\Gamma_{\varepsilon\, ij}^{\ l}(\gamma(t))(\dot{\gamma}(t)) + \Gamma_{\varepsilon\, pk}^{\ l}(\gamma(t))\dot{\gamma}^p(t)\Big(\frac{\mathrm{d}}{\mathrm{d}t} J_w^k(t) \\
&+ J^j(t)\dot{\gamma}^i(t)\Gamma_{\varepsilon\, ij}^{\ k}(\gamma(t))\Big).
\end{aligned}$$

We have that

$$|D^2 g_\varepsilon| \leq C(\rho, G)\varepsilon^{-2}, \text{ and } \quad |Dg_\varepsilon^{-1}| \leq C(\rho, G)\varepsilon^{-1}, \tag{3.5.13}$$

where we may assume that $C > 1$. Moreover, recall that we can use the geodesic equation and (3.5.12) to bound $|\gamma''(t)|_{euc}$ in terms of $|v|_{euc}$, $G\varepsilon^{-1}$ and λ. Hence, all coefficients of the above linear system of ODEs are bounded by $C\varepsilon^{-2}$. Denote $Y_w^T(t) := (J^T(t), (J')^T(t))$. We get that

$$\begin{aligned} &Y_w'(t) = P(t)Y(t), \\ &Y_w(0)^T = (0^T, w^T), \end{aligned} \tag{3.5.14}$$

where P is a matrix with smooth entries whose L^∞-norm is bounded above by $C\varepsilon^{-2}$. Gronwall's inequality gives that for $u \in T_pM$

$$|Y_w(t) - Y_{w+su}(t)|_{euc} \leq |s||u|_{euc} e^{Ct\varepsilon^{-2}}.$$

Hence,

$$|J_w'(t) - J_{w+su}'(t)|_{euc} \leq |s||u|_{euc} e^{Ct\varepsilon^{-2}}.$$

Thus,

$$|J_w(t) - J_{w+su}(t)|_{euc} \leq |s||u|_{euc} \frac{\varepsilon^2}{C}(e^{Ct\varepsilon^{-2}} - 1).$$

It is known that $J_w(t) = D\exp_p^{g_\varepsilon}|_{tv}(tw)$. Hence, denoting $a := C\varepsilon^{-2}$, we get that for $0 < t \leq \frac{1}{a}$,

$$\left|D\exp_p^{g_\varepsilon}|_{tv}(w) - D\exp_p^{g_\varepsilon}|_{tv}(w + su)\right|_{euc} \leq |s||u|_{euc} \frac{1}{at}(e^{at} - 1) \leq 3|s||u|_{euc},$$

which shows local Lipschitz continuity of the first derivative at the point $\gamma(t)$. The second derivative of the exponential map exists as g_ε is smooth and by the Lipschitz continuity, it is bounded. Setting $su = -w$, and recalling that $J_0(t) = 0$ for all t, we get that

$$\left|D\exp_p^{g_\varepsilon}|_{tv}(w)\right|_{euc} \leq 3|w|_{euc}.$$

Hence the first differential is bounded at the point tv. Recall that v is arbitrary, given that the geodesic tangent to v and its variation are well defined for $t \in [0, \frac{1}{a(\varepsilon)}]$. Then the above holds for each $\tilde{v} \in (0, \frac{1}{a(\varepsilon)}]v$ and (3.5.11) holds for each $\tilde{v}$ such that

$$0 < \|\tilde{v}\|_{g_\varepsilon} \leq \frac{a}{2\lambda} \min(i_\varepsilon(p), 1).$$

By continuity of the differential and the second differential this also holds for $\|\tilde{v}\|_{g_\varepsilon} = 0$. Since p was arbitrary, taking Lemma 3.5.16 into account, we get the result. □

Lemma 3.5.18. *Let $K, K' \subset \mathbb{R}^n$ be compact sets such that $K \subset K'$. Let furthermore $\mathcal{F}$ be a family of closed balls in $\mathbb{R}^n$ such that for every point $x \in K$, there exists a radius $R_x > 0$ such that for all positive $R \leq R_x$, it holds $\overline{B_R(x)} \in \mathcal{F}$. Let $\varphi \in C_c(K', [0, \infty))$ and $\varepsilon > 0$. Then there exists a finite family of functions $\{\chi_p\}_{p=1}^q$ such that the following holds:*

(i) *For every $p = 1, \dots, q$, there exists an $x \in K$ and a positive $R \leq R_x$ such that $\chi_p \in C_c(B_R(x), [0, \infty))$ and χ_p is rotationally symmetric centered at x.*

(ii) $\sum_{p=1}^q \chi_p \leq \varphi$ *and* $\left\| \sum_{p=1}^q \chi_p - \varphi \right\|_{C^0(K)} \leq \varepsilon$.

Proof. We first recall that, by Besicovitch's covering theorem, there exists a constant $c_n \geq 1$ such that for any family $\mathcal{G}$ of closed balls in $\mathbb{R}^n$, such that the set A of their centers is bounded, there exist c_n countable subfamilies $(\mathcal{G}_h)_{h=1}^{c_n}$ such that for each $h = 1, \dots, c_n$, any two different balls in $\mathcal{G}_h$ are disjoint and $A \subset \bigcup_{h=1}^{c_n} \bigcup_{B \in \mathcal{G}_h} B$.

Claim. For every function $\varphi \in C_c(K', [0, \infty))$, there exists a finite family of functions $\{\psi_k\}_{k=1}^m$ such that for every $k = 1, \dots, m$ the following holds:

1. There exist $x \in K$ and a positive $R \leq R_x$ such that $\psi_k \in C_c(B_R(x), [0, \infty))$.
2. ψ_k is rotationally symmetric centered at x.
3. The following estimate holds:

$$\sum_{k=1}^m \psi_k \leq \varphi, \quad \text{and} \quad \left\| \sum_{k=1}^m \psi_k - \varphi \right\|_{C^0(K)} \leq \left(1 - \frac{1}{2c_n}\right) \|\varphi\|_{C^0(K)}. \tag{3.5.15}$$

Proof of the claim. If $\|\varphi\|_{C^0(K)} = 0$, we do not need any function to approximate φ, so we set $m = 0$. Otherwise, using that φ is uniformly continuous, we can find $\delta > 0$ such that for all $x, y \in K'$ with $|x - y| \leq 4\delta$, we have that $|\varphi(x) - \varphi(y)| \leq \|\varphi\|_{C^0(K)} \cdot \frac{1}{2c_n}$. For each $x \in K$ we define $\rho_x = \min(R_x, \delta)$. Now $\bigcup_{x \in K} B_{\frac{\rho_x}{2}}(x)$ is an open cover of K, so by compactness of K, there exists a finite subcover $\bigcup_{k=1}^m B_{\frac{\rho_k}{2}}(x_k)$ of K, where we write $\rho_k = \rho_{x_k}$ to keep notation short. Now we apply Besicovitch's covering theorem to find c_n finite families $\mathcal{F}_h \subset \{\overline{B_{\frac{\rho_k}{2}}(x_k)}\}_k$ such that $K \subset \bigcup_{h=1}^{c_n} \bigcup_{B \in \mathcal{F}_h} B$ and the balls in each family are pairwise disjoint. For each h denote by I_h the set of indices k such that $\overline{B_{\frac{\rho_k}{2}}(x_k)} \in \mathcal{F}_h$. Fix such an h. Note that I_h is finite, hence for each $k \in I_h$ we can find a radius $\frac{\rho_k}{2} < r_k \leq \rho_k$ such that for $k, k' \in I_h$, $k \neq k'$, we have that $B_{r_{k'}}(x_{k'}) \cap B_{r_k}(x_k) = \emptyset$. For each $k = 1, \dots, m$, define a function $\eta_k \in C_c(B_{r_k}(x_k), [0, 1])$ such that η_k is rotationally symmetric (with center x_k) and $\eta_k = 1$ on $\overline{B_{\frac{\rho_k}{2}}(x_k)}$. Define

$$\theta_k := \min_{y \in \overline{B_{r_k}(x_k)}} \varphi(y) \in [0, \infty).$$

Now define $\psi_k \in C_c(B_{r_k}(x_k), [0,\infty))$ via

$$\psi_k(x) := \frac{\theta_k}{c_n}\eta_k(x).$$

We now claim that (3.5.15) holds. Pick a point $x \in K'$. Let $L_x := \{l : x \in B_{r_l}(x_l)\}$. For each $h = 1, \ldots, c_n$, we have that $|I_h \cap L_x| \leq 1$, hence $|L_x| \leq c_n$. By definition, we have that $\theta_l \leq \varphi(x)$ for each $l \in L_x$, hence

$$\sum_{k=1}^{m} \psi_k(x) = \sum_{l \in L_x} \psi_l(x) = \sum_{l \in L_x} \frac{\theta_l}{c_n}\eta_l(x) \leq \varphi(x).$$

Now let $x \in K$. We have that for each $l \in L_x$ and $y \in \overline{B_{r_l}(x_l)}$ it holds $|x - y| \leq 2r_l \leq 2\rho_l \leq 2\delta$. Hence, for each $l \in L_x$, we have that $|\varphi(x) - \theta_l| \leq \|\varphi\|_{C^0(K)} \cdot \frac{1}{2c_n}$. By the construction of our cover, we have that there exists at least one $\lambda \in L_x$ such that $x \in B_{\frac{\rho_\lambda}{2}}(x_\lambda)$. Hence

$$\sum_{k=1}^{m} \psi_k(x) \geq \frac{\theta_\lambda}{c_n}\eta_\lambda(x) = \frac{\theta_\lambda}{c_n}.$$

Thus,

$$\begin{aligned}
0 \leq \varphi(x) - \sum_{k=1}^{m} \psi_k(x) &\leq \varphi(x) - \frac{\theta_\lambda}{c_n} \leq \varphi(x) - \frac{\varphi(x)}{c_n} + \left|\frac{\varphi(x)}{c_n} - \frac{\theta_\lambda}{c_n}\right| \\
&= \Big(1 - \frac{1}{c_n}\Big)\varphi(x) + \frac{1}{c_n}|\varphi(x) - \theta_\lambda| \\
&\leq \Big(1 - \frac{1}{c_n}\Big)\|\varphi\|_{C^0(K)} + \frac{1}{2c_n^2}\|\varphi\|_{C^0(K)} \leq \Big(1 - \frac{1}{2c_n}\Big)\|\varphi\|_{C^0(K)}.
\end{aligned}$$

This proves the claim.

Now given a function $\varphi \in C_c(K', [0,\infty))$, we define a sequence of functions $(\varphi_j)_{j=0}^{\infty} \subset C_c(K', [0,\infty))$ as follows: $\varphi_0 := \varphi$. Given φ_j, we apply the claim to find $m_j \geq 0$ and functions $\psi_k^{(j)}$, for $k = 1, \ldots, m_j$ such that

- $\psi_k^{(j)} \in C_c(B_{R_{x_{j,k}}(x_{j,k})}, [0,\infty))$ for some $x_{j,k} \in K$.
- $\psi_k^{(j)}$ is rotationally symmetric centered at $x_{j,k}$.
- The following estimate holds:

$$\sum_{k=1}^{m_j} \psi_k^{(j)} \leq \varphi_j, \quad \text{and} \quad \Bigg\|\sum_{k=1}^{m_j} \psi_k^{(j)} - \varphi_j\Bigg\|_{C^0(K)} \leq \Big(1 - \frac{1}{2c_n}\Big)\|\varphi_j\|_{C^0(K)}.$$

Then set $\varphi_{j+1} := \varphi_j - \sum_{k=1}^{m_j} \psi_k^{(j)} \geq 0$. Inductively, we get that

$$\|\varphi_j\|_{C^0(K)} \leq \Big(1 - \frac{1}{2c_n}\Big)^j \|\varphi\|_{C^0(K)}. \tag{3.5.16}$$

Now we can choose $J \geq \frac{\ln \varepsilon - \ln \|\varphi\|_{C^0(K)}}{\ln(1-\frac{1}{2c_n})}$. We choose the family of functions $\{\psi_k^{(j)} : k = 1, \ldots, m_j, j = 0, \ldots J\}$ Then

$$\varphi - \sum_{j=0}^{J} \sum_{k=1}^{m_j} \psi_k^{(j)} = \varphi_0 - \sum_{j=0}^{J} (\varphi_j - \varphi_{j+1}) = \varphi_{J+1} \geq 0.$$

By our choice of J and (3.5.16), we get that

$$\left\| \varphi - \sum_{j=0}^{J} \sum_{k=1}^{m_j} \psi_k^{(j)} \right\|_{C^0(K)} = \|\varphi_{J+1}\|_{C^0(K)} \leq \varepsilon.$$

This finishes the proof. $\square$

Lemma 3.5.19. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $h \in C^0(\mathbb{R}^n)$. Then for each $p \in \mathbb{R}^n$, we have that if p is a Lebesgue point of f, then p is also a Lebesgue point of $f \cdot h$.*

We are now able to state and prove the second main result of the chapter, namely that the $\mathsf{CD}^*(K,N)$ condition implies that the distributional N-Bakry-Émery Ricci tensor is bounded below by K, on a smooth manifold endowed with a continuous Riemannian metric with L^2_{loc}-Christoffel symbols and a $C^0 \cap W^{1,2}_{\mathrm{loc}}$-weight on the volume measure.

Theorem 3.5.20. *Let M be a smooth manifold, $g \in C^0$ a Riemannian metric that admits L^2_{loc}-Christoffel symbols and $h \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$ a positive function, $V := -2\log h \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$. Define the measure μ via $\mu := e^{-V}\mathrm{dvol}_g$. Let $K \in \mathbb{R}$ and $N \in [n, \infty)$. If (M, d_g, μ) is a $\mathsf{CD}^*(K,N)$-space, then*

$$\mathrm{Ric}_{\mu,N} = \mathrm{Ric}_{\mu,\infty} - \frac{1}{N-n} \nabla V \otimes \nabla V \geq Kg \quad \text{in } \mathcal{D}'\mathcal{T}^0_2.$$

Proof. We first notice that by Lemma 2.2.11 and Corollary 3.3.15, (M, d_g, μ) is an $\mathsf{RCD}(K,\infty)$-space, so by Theorem 3.5.13, we have that $\mathrm{Ric}_{\mu,\infty} \in \mathcal{D}'\mathcal{T}^0_2$ can be expressed via Radon measures $(\nu_{ij})_{i,j}$ as in (3.5.7). Let $U \subset W \subset M$ be open such that $\overline{W}$ is compact, $\overline{U} \subset W$ and W lies in one coordinate patch. It is enough to prove the statement for $M = U$, as we can cover the manifold with sets like U. Let $\rho \in C^\infty_c(B^{euc}_1(0), [0,\infty))$ be a rotationally symmetric standard mollifier and $\rho_\delta(x) := \frac{1}{\delta^n}\rho(\frac{x}{\delta})$, $\delta > 0$. For $0 < \delta \leq \delta_0 < \mathrm{dist}(\partial W, U)$, we define $g_\delta = \rho_\delta * g$. We assume that for all these δ, g_δ is a Riemannian metric on U and that $\max(\|g_\delta\|_{C^0(U)}, \|g_\delta^{-1}\|_{C^0(U)}) < 2\max(\|g\|_{C^0(U)}, \|g^{-1}\|_{C^0(U)})$, as this is the case for $\delta_0 > 0$ small enough. Then $g_\delta \to g$ everywhere as $\delta \to 0$. Then, by the proof of Proposition 3.3.8 there exists a $\zeta > 0$ and a $\lambda \geq 1$ such that g, g_δ and $|\cdot|_{euc}$ are pairwise λ-equivalent on U and $\mathsf{d}_g, \mathsf{d}_{g_\delta}$, and $|\cdot|_{euc}$ are pairwise λ-equivalent on $B^{euc}_\zeta(p)$ for all $p \in U$. Fix a point $p \in U$ such that

$$p \text{ is a Lebesgue point of } Dg, \text{ and of } Dg \cdot Dg. \tag{3.5.17}$$

Then, as ρ is rotationally symmetric, we get that

$$Dg_\delta(p) \to Dg(p). \tag{3.5.18}$$

Fix $\varepsilon > 0$. We can assume $p = 0$. In order to compute local coordinates with a vanishing Levi-Cevita connection with respect to g at the point p, we define the map $\psi : U \to \tilde{U} \subset \mathbb{R}^n, x \mapsto y$ via

$$y^k = \psi^k(x) := 0 + \alpha_i^k x^i + \beta_{ij}^k x^i x^j,$$

where $\alpha_i^k, \beta_{ij}^k = \beta_{ji}^k \in \mathbb{R}$ are constants. Then

$$\begin{aligned}
&(D\psi)_{ki}(x) = \partial_{x^i} y^k = \alpha_i^k + 2\beta_{ij}^k x^j,\\
&\partial_{x^l}(D\psi)_{ki}(x) = 2\beta_{il}^k, \quad \text{and}\\
&(\psi_* g)_{ij} = ((D\psi)^{-T} g (D\psi)^{-1})_{ij}.
\end{aligned}$$

Moreover, it holds

$$\begin{aligned}
(\partial_{x^l}(D\psi)^{-1})_{ij}(0) &= -((D\psi)^{-1}\partial_{x^l} D\psi (D\psi)^{-1})_{ij}(0) = -(D\psi)^{-1}_{it}\partial_{x^l} D\psi_{ts}(D\psi)^{-1}_{sj}(0)\\
&= -2(D\psi)^{-1}_{it}\beta_{sl}^t (D\psi)^{-1}_{sj}(0).
\end{aligned}$$

We assume α to be invertible and define $\alpha^{-1} =: \sigma = (\sigma_{ij})_{ij} \in \mathbb{R}^{n\times n}$. In order to be "normal coordinates" at $p = 0$, we need that at the point $\psi(0) = 0 \in \tilde{U}$, it holds

$$(\psi_* g)_{ij}(0) = (\alpha^{-T} g \alpha^{-1})_{ij}(0) = \mathrm{Id}_n. \tag{3.5.19}$$

We want the first derivative of the metric with respect to the y-coordinates to vanish at $p = 0$, hence we compute:

$$\begin{aligned}
0 &= \partial_{y^k}(\psi_* g)_{ij}(0) = \partial_{x^l}((D\psi^{-1})_{ri}(D\psi^{-1})_{qj} g_{rq})\frac{\partial x^l}{\partial y^k}(0)\\
&= (\partial_{x^l}(D\psi^{-1})_{ri}(D\psi^{-1})_{qj} g_{rq} + (D\psi^{-1})_{ri}\partial_{x^l}(D\psi^{-1})_{qj} g_{rq} + (D\psi^{-1})_{ri}(D\psi^{-1})_{qj}\partial_{x^l} g_{rq})\frac{\partial x^l}{\partial y^k}(0)\\
&= (-2\sigma_{rt}\beta_{sl}^t\sigma_{si}\sigma_{qj} g_{rq} - 2\sigma_{ri}\sigma_{qt}\beta_{sl}^t\sigma_{sj} g_{rq} + \sigma_{ri}\sigma_{qj}\partial_{x^l} g_{rq})\sigma_{lk}.
\end{aligned} \tag{3.5.20}$$

This gives $\frac{1}{2}n^2(n+1)$ equations and $\frac{1}{2}n^2(n+1)$ variables β_{ij}^k. To keep notation short, we rewrite this to

$$\zeta_{ijk}^m(g,\sigma)\beta_{ij}^k = \xi^m(Dg,\sigma), \quad m = 1,\dots,\frac{1}{2}n^2(n+1), \tag{3.5.21}$$

for some polynomials $\zeta_{ijk}^m : (\mathbb{R}^{n\times n})^2 \to \mathbb{R}$ and $\xi^m : \mathbb{R}^{n\times n\times n} \times \mathbb{R}^{n\times n}$. In order to find a solution of this system of equations, consider g_δ, $\delta \in (0,\delta_0)$. By using the exponential map, we can find a map ψ_δ defined as

$$\psi_\delta = (\exp_p^{g_\delta})^{-1} : \Omega_\delta \to B_{r_\delta}(0) \subset T_pU \cong \mathbb{R}^n, x \mapsto y_\delta,$$

where $\Omega_\delta \subset U$ is an open neighbourhood of 0 and $r_\delta > 0$. Define the distance function d_δ^p : $\Omega_\delta \to [0,\infty), y \mapsto \mathsf{d}_{g_\delta}(p,z) = \big(\sum_{i=1}^n (y_\delta^i)^2\big)^{\frac{1}{2}}$. It is known that $(\psi_\delta)_*(g_\delta)_{ij} = \delta_{ij} + O_\delta((d_\delta^p)^2)$. More precisely, this means that there exists a constant $C_{g_\delta} > 0$ such that for all $x \in \Omega_\delta$,

$$\|(\psi_\delta)_* g_\delta(x) - \mathrm{Id}_n\|_{\mathbb{R}^{n\times n}} \leq C_{g_\delta}(d_\delta^p)^2(x). \tag{3.5.22}$$

Note that $|Dg_\delta| \leq C\lambda\delta^{-1}$ for some $C > 0$. Take $\tilde{\Omega}_\delta := \Omega_\delta \cap \{x : |x|_{euc} = |x-p|_{euc} \leq \frac{\delta}{2nC},\ (d_\delta^p(x))^2 < \frac{1}{2nC_{g_\delta}}\frac{1}{4\lambda^2}\}$. Then in $\tilde{\Omega}_\delta$, we get that

$$\frac{1}{2\lambda}\mathrm{Id}_n \leq g_\delta \leq 2\lambda \mathrm{Id}_n, \quad \text{and} \quad \frac{1}{2}\mathrm{Id}_n \leq (\psi_\delta)_* g_\delta \leq 2\mathrm{Id}_n. \tag{3.5.23}$$

Using that $\|v\|_{g_\delta} = \|(\psi_\delta)_* v\|_{(\psi_\delta)_* g_\delta}$, we get that the transition map $\Psi_\delta := D\psi_\delta : T\tilde{\Omega}_\delta \to T\psi_\delta(\tilde{\Omega}_\delta)$ is bounded from above and from below in the Euclidean norm, meaning that for all $0 \neq v \in T_q\tilde{\Omega}_\delta$, we have that

$$\frac{1}{4\lambda} = \frac{\frac{1}{2}}{2\lambda} \leq \frac{\|\Psi_\delta(v)\|_{euc}}{\|v\|_{euc}} \leq \frac{2}{\frac{1}{2\lambda}} = 4\lambda. \tag{3.5.24}$$

By the inverse function theorem, we know that on $\tilde{\Omega}_\delta$

$$D\psi_\delta = (D(\exp_p^{g_\delta}))^{-1}, \quad \text{hence} \quad \partial_i D\psi_\delta = -D\psi_\delta \partial_i(D(\exp_p^{g_\delta}))D\psi_\delta.$$

Hence, using (3.5.24) and (3.5.11), we get that

$$|(D\psi_\delta)^{-1}|_{euc}(0) \leq 4\lambda \quad \text{and} \quad |D^2\psi_\delta|_{euc}(0) \leq 12\lambda^2. \tag{3.5.25}$$

Now for all $\delta > 0$, we have that

$$\zeta_{ijk}^m(g_\delta, (D\psi_\delta)^{-1})\frac{1}{2}\partial_j(D\psi_\delta)_{ik} = \xi^m(Dg_\delta, (D\psi_\delta)^{-1}), \quad m = 1,\ldots,\frac{1}{2}n^2(n+1).$$

With (3.5.24) and (3.5.25), we get that there exists a convergent subsequence $\delta' \to 0$ such that $(D\psi_{\delta'}(0), D^2\psi_{\delta'}(0)) \to (D\psi(0), D^2\psi(0))$ and

$$|D\psi|_{euc}(0) \leq 4\lambda,\ |D\psi^{-1}|_{euc}(0) \leq 4\lambda,\ \text{and}\ |D^2\psi|_{euc}(0) \leq 12\lambda^2. \tag{3.5.26}$$

By (3.5.17), (3.5.18), and the continuity of g, it follows that $(D\psi(0), D^2\psi(0))$ satisfies (3.5.19) and (3.5.20). Now we have found coefficients for our map $\psi : U \to \tilde{U}$ and we notice that for now the map is defined everywhere on U. By (3.5.19), $D\psi(0)$ must be invertible, so by the inverse function theorem we can shrink U to a neighbourhood U' of p such that ψ is a diffeomorphism on U'. Note that $D^2\psi$ is constant, so

$$|D^2\psi|_{euc}(x) \leq 12\lambda^2, \tag{3.5.27}$$

for all $x \in U'$. As $\psi_* g = \mathrm{Id}_n$, we get that the transition map $\Psi|_0 := D\psi|_0 : T_0 U' \to T_0\psi(U')$ is bounded, meaning that for all $0 \neq v \in T_0\psi(U')$, we have that

$$\frac{1}{\lambda} \leq \frac{\|\Psi(v)\|_{euc}}{\|v\|_{euc}} \leq \lambda. \tag{3.5.28}$$

Using (3.5.27), we can shrink the domain U' and find an open $\Omega \subset U'$ such that $p \in \Omega$ and the transition map $\Psi := D\psi : T\Omega \to T\psi(\Omega)$ satisfies

$$\frac{1}{2\lambda} \leq \frac{\|\Psi(v)\|_{euc}}{\|v\|_{euc}} \leq 2\lambda, \tag{3.5.29}$$

for each $v \in T_q\Omega$, $q \in \Omega$. As p satisfies (3.5.17), we can use Lemma 3.5.19, to get that for each $f \in C_c^2(M)$,

$$p \text{ is a Lebesgue point of } x \mapsto |\nabla_g^2 f(x)|_{HS}^2, \text{ and } x \mapsto (\Delta_g f(x))^2. \tag{3.5.30}$$

Here Δ_g denotes the Laplacian with respect to the volume measure induced by g. As, by the above construction, $y = \psi(x)$ defines normal coordinates at p with respect to g, we have that

$$\begin{aligned} &((\psi)_*(\nabla_g^2 f)(0))_{ij} = \partial^2_{y^i,y^j} f(0), \quad \text{and} \\ &\Delta_g f = \sum_i \partial^2_{y^i,y^i} f(0) = \sum_i ((\psi)_*(\nabla_g^2 f)(p))_{ii}. \end{aligned}$$

Now fix a smooth vector field X on U and denote $B := \psi_*(X(p)) \in \mathbb{R}^n$. Let $f : \Omega \to \mathbb{R}$ be defined by $\psi_* f := \sum_i B_i y^i + \frac{a}{2\sqrt{n}}(y^i)^2$ for some $a \in \mathbb{R}$. Then

$$\psi_* \nabla_g f(0) = B, \quad \psi_* \nabla_g^2 f(0) = \frac{a}{\sqrt{n}} \mathrm{Id}_n, \quad \text{and} \quad \psi_* \Delta_g f(0) = \sqrt{n} a.$$

Note that f is smooth because ψ is smooth. Writing $h^2 = e^{-V}$ for a $C^0 \cap W^{1,2}_{\mathrm{loc}}$-function V, we get:

$$\Delta_{g,\mu} f = \Delta_g f + \langle \nabla_g V, \nabla_g f \rangle_g.$$

Hence,

$$|\nabla_g^2 f|^2_{HS,g}(p) - \frac{1}{N}(\Delta_{g,\mu} f)^2(p) = \frac{N-n}{N} a^2 - \frac{2\sqrt{n}}{N} a \langle X, \nabla_g V \rangle_g(p) - \frac{1}{N} \langle X, \nabla_g V \rangle_g^2(p).$$

The choice $a = \frac{\sqrt{n}\langle X, \nabla_g V \rangle_g(p)}{N-n}$ minimises the right hand side and yields

$$|\nabla_g^2 f|^2_{HS,g}(p) - \frac{1}{N}(\Delta_{g,\mu} f)^2(p) = -\frac{1}{N-n} \langle X, \nabla_g V \rangle_g^2(p). \tag{3.5.31}$$

Now there is a $\tau_0 > 0$, $\tau \leq \min(\varepsilon, 1)$ such that $B_{\tau_0}(p) := B^{euc}_{\tau_0}(p) \subset \Omega$ and for all $z \in B_{\tau_0}(p)$, we have

$$|\nabla_g f(z) - X(p)|_{euc} \leq \varepsilon \quad \text{and} \quad |X(z) - X(p)|_{euc} \leq \varepsilon. \tag{3.5.32}$$

Finally note $f = \psi^* \psi_* f$, so we can bound $\|f\|_{W^{2,\infty}}$ in a neighbourhood of p in terms of $|\psi|, |D\psi|, |D\psi^{-1}|, |D^2\psi|, a$, and B. Hence, by (3.5.27) and (3.5.29), we get that

$$\|f\|_{W^{2,\infty}(B_{\tau_0}(p))} \leq C(X, V, \lambda, n, N). \tag{3.5.33}$$

Now, by (3.5.10), we have

$$\sum_{i,j} (\nabla_g f)_i (\nabla_g f)_j \nu_{ij} + (|\nabla^2_g f|^2_{HS,g} - K|\nabla_g f|^2_g - \frac{1}{N}(\Delta_{g,\mu} f)^2)\mu \geq 0.$$

Recall that $\mu = e^{-V}\sqrt{|g|}\mathcal{L}^n$, where $V, g \in C^0(U)$. Then by the triangle inequality, (3.5.30) together with Lemma 3.5.19, and by (3.5.32), we get that there exists a $\tau \in (0, \tau_0)$ such that for all $\tau' \in (0, \tau)$, it holds

$$\begin{aligned}
&\Bigg|\Big(\sum_{i,j} (\nabla_g f)_i(p) (\nabla_g f)_j(p) \nu_{ij} + (|\nabla^2_g f|^2_{HS,g}(p) - K|\nabla_g f|^2_g(p) - \frac{1}{N}(\Delta_{g,\mu} f)^2(p))\mu\Big) \\
&\quad - \Big(\sum_{i,j} (\nabla_g f)_i (\nabla_g f)_j \nu_{ij} + (|\nabla^2_g f|^2_{HS,g} - K|\nabla_g f|^2_g - \frac{1}{N}(\Delta_{g,\mu} f)^2)\mu\Big)\Bigg|_{TV} (B_{\tau'}(p)) \\
&\quad \leq C(n, U, g, V, X, N, K)\Big(\sum_{i,j} |\nu_{ij}|_{TV} + \mathcal{L}^n\Big)(B_{\tau'}(p)) \cdot \varepsilon.
\end{aligned}$$

Similarly, using again (3.5.32), we get that

$$\begin{aligned}
&\Bigg|\sum_{i,j} (\nabla_g f)_i(p) (\nabla_g f)_j(p) \nu_{ij} - (K|\nabla_g f|^2_g(p) + \frac{1}{N-n}(\langle \nabla_g f, \nabla_g V\rangle^2_g)(p))\mu \\
&\quad - \sum_{i,j} X^i X^j \nu_{ij} + (K|X|^2_g + \frac{1}{N-n}(\langle X, \nabla_g V\rangle^2_g))\mu\Bigg|_{TV} (B_{\tau'}(p)) \\
&\quad \leq C(n, U, g, V, X, N, K)\varepsilon\Big(\sum_{i,j} |\nu_{ij}|_{TV} + \mathcal{L}^n\Big)(B_{\tau'}(p)).
\end{aligned}$$

Together with (3.5.31), this yields

$$\begin{aligned}
&\Big(\sum_{i,j} X^i X^j \nu_{ij} - (K|X|^2_g + \frac{1}{N-n}(\langle X, \nabla_g V\rangle^2_g))\mu\Big)(B_{\tau'}(p)) \\
&\quad \geq -C(n, U, g, V, X, N, K)\varepsilon\Big(\sum_{i,j} |\nu_{ij}|_{TV} + \mathcal{L}^n\Big)(B_{\tau'}(p)).
\end{aligned}$$

We infer that for $\chi \in C_c(B_\tau(p), [0,\infty)])$ with χ rotationally symmetric around p, it holds

$$\int_{B_\tau(p)} \chi \, \mathrm{d}\Big(\sum_{i,j} X^i X^j \nu_{ij} - (K|X|_g^2 + \frac{1}{N-n}(\langle X, \nabla_g V\rangle_g^2))\mu\Big) \\ \geq -C(n,U,g,V,X,N,K)\varepsilon \int_{B_\tau(p)} \chi \, \mathrm{d}\big(\sum_{i,j} |\nu_{ij}|_{TV} + \mathcal{L}^n\big). \tag{3.5.34}$$

As p was chosen arbitrarily among all points satisfying (3.5.17), we have that for all $p \in U$, satisfying (3.5.17), and each $\varepsilon > 0$, there exists a $\tau_{p,\varepsilon} > 0$ such that (3.5.34) holds for $\chi \in C_c(B^{euc}_{\tau_{p,\varepsilon}}(p), [0,\infty))$ such that χ is rotationally symmetric around p. Now suppose there exists a function $\phi \in C_c(U, [0,\infty))$ such that

$$\langle \mathrm{Ric}_{\mu,\infty}(X,X) - K|X|_g^2\mu - \frac{1}{N-n}(\nabla V \otimes \nabla V)(X,X)\mu, \phi \, dx^1 \dots dx^n\rangle_{\mathcal{D}',\mathcal{D}} = \kappa < 0.$$

We can assume that $\|\phi\|_{sup} = 1$. Choose

$$0 < \varepsilon \leq \frac{-\kappa}{8C \cdot (\mathcal{L}^n(U) + \sum_{i,j} |\nu_{ij}|_{TV}(U))}, \tag{3.5.35}$$

where $C \geq 1$ is as in (3.5.34). Write

$$\mathrm{Ric}_{\mu,\infty}(X,X) := \mathfrak{r}^s_X + \mathfrak{r}^a_X,$$

where $\mathfrak{r}^s_X$ denotes the singular part and $\mathfrak{r}^a_X$ denotes the absolutely continuous part with respect to the Lebesgue measure. As $\mathrm{Ric}_{\mu,\infty}(X,X) - K|X|_g^2\mu \geq 0$, and $K|X|_g^2\mu$ is absolutely continuous with respect to the Lebesgue measure, we know that

$$\mathfrak{r}^s_X \geq 0. \tag{3.5.36}$$

Moreover, there exist two Borel sets $U_s, U_a \subset U$ such that $U_s \cap U_a = \emptyset$, $U_s \cup U_a = U$, $\mathfrak{r}^s_X(U_a) = 0$, and $\mathcal{L}^n(U_s) = 0$. Define

$$\mathcal{N} := \{p \in U : \ p \text{ does not satisfy } (3.5.17)\}.$$

Then $\mathcal{L}^n(\mathcal{N}) = 0$ and hence $\mathcal{L}^n(\mathcal{N} \cup U_s) = 0$. By the outer regularity of Borel measures, there exists an open set $O \subset U$ such that

$$(\mathcal{N} \cup U_s) \subset O, \text{ and} \\ |\mathfrak{r}^a_X|_{TV}(O) + (|K||X|_g^2 + \frac{1}{N-n}(\nabla V \otimes \nabla V)(X,X))\mu(O) \leq \frac{\varepsilon}{4}. \tag{3.5.37}$$

Denote $\operatorname{supp}\phi = K'$, $K := K' \setminus O$ and note that both K and K' are compact. Moreover, denote $\mathcal{F} := \{\overline{B_\tau(p)} : \ p \in K, 0 < \tau \leq \frac{1}{2}\tau_{p,\varepsilon}\}$. We can now apply Lemma 3.5.18 to find an $m \in \mathbb{N}$ and functions $\chi_k \in C_c(B_{\tau_k}(p_k), [0,\infty))$ for $k = 1, \dots, m$, $p_k \in K$, χ_k is rotationally symmetric centered at p_k, $\tau_k \leq \frac{\tau_{p_k,\varepsilon}}{2}$, such that $\sum_{k=1}^m \chi_k \leq \phi$ and $\|\sum_{k=1}^m \chi_k - \phi\|_{C^0(K)} \leq \varepsilon$. Denote $\psi := \sum_{k=1}^m \chi_k$. Define the signed Radon measure $\mathfrak{s}$ on U as

$$\mathfrak{s} = -K|X|_g^2\mu - \frac{1}{N-n}(\nabla V \otimes \nabla V)(X,X)\mu.$$

By (3.5.35), (3.5.37), and the bound $\|\phi - \psi\|_{C^0(K)} \leq \varepsilon$, we have that

$$\begin{aligned}
\left| \int_U (\phi-\psi)\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^a) \right| &= \left| \int_{K'} (\phi-\psi)\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^a) \right| \\
&\leq \left| \int_K (\phi-\psi)\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^a) \right| + \left| \int_O (\phi-\psi)\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^a) \right| \\
&\leq \varepsilon \cdot (|\mathfrak{s}|_{TV} + |\mathfrak{r}_X^a|_{TV})(U) + \|\phi\|_{sup}(|\mathfrak{r}_X^a|_{TV} + |\mathfrak{s}|_{TV})(O) \\
&\leq -\frac{\kappa}{8} + \frac{\varepsilon}{4} \leq -\frac{\kappa}{2}.
\end{aligned}$$

Then, using (3.5.36) and that $\phi \geq \psi$, we get that

$$\begin{aligned}
\int_U \phi\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^s+\mathfrak{r}_X^a) &= \int_U \psi\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^s+\mathfrak{r}_X^a) + \int_U (\phi-\psi)\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^a) + \int_U (\phi-\psi)\,\mathrm{d}(\mathfrak{r}_X^s) \\
&\geq \int_U \psi\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^s+\mathfrak{r}_X^a) + \frac{\kappa}{2}.
\end{aligned}$$

Finally, using (3.5.34), and $\|\psi\|_{sup} \leq \|\phi\|_{sup} \leq 1$, we get that

$$\begin{aligned}
&\int_U \psi\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^s+\mathfrak{r}_X^a) \\
&\quad= \sum_{k=1}^m \left(\sum_{i,j} \int_{B_{\tau_k}(p_k)} X^i X^j \chi_k \,\mathrm{d}\nu_{ij} - \int_{B_{\tau_k}(p_k)} \Big(K|X|_g^2 + \frac{1}{N-n}(\langle X, \nabla_g V\rangle_g^2) \Big) \chi_k \,\mathrm{d}\mu \right) \\
&\quad\geq -C(n,U,g,V,X,N,K)\varepsilon \sum_{k=1}^m \Big(\int_{B_{\tau_k}(p_k)} \sum_{i,j} \chi_k \, d|\nu_{ij}|_{TV} + \int_{B_{\tau_k}(p_k)} \chi_k \,\mathrm{d}\mathcal{L}^n \Big) \\
&\quad\geq \frac{\kappa}{8(\mathcal{L}^n(U) + \sum_{i,j} |\nu_{ij}|_{TV}(U))} \Big(\int_U \sum_{i,j} \psi \, d|\nu_{ij}|_{TV} + \int_U \psi \,\mathrm{d}\mathcal{L}^n \Big) \\
&\quad\geq \frac{\kappa}{8}.
\end{aligned}$$

But then

$$\begin{aligned}
\kappa &= \left\langle \mathrm{Ric}_{\mu,\infty}(X,X) - K|X|_g^2\mu - \frac{1}{N-n}(\nabla V \otimes \nabla V)(X,X)\mu, \phi dx^1 \dots dx^n \right\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \int_U \phi\,\mathrm{d}(\mathfrak{s}+\mathfrak{r}_X^s+\mathfrak{r}_X^a) \geq \frac{5\kappa}{8}.
\end{aligned}$$

Since $\kappa < 0$, this is a contradiction. As X was arbitrary, the proof is complete. $\square$

3.6 Distributional Ricci curvature lower bounds imply RCD

In this section, we prove the reverse implication, namely from distributional to synthetic Ricci lower bounds, under the following additional assumption:

Assumption 3.6.1. *Given a smooth manifold M equipped with a continuous Riemannian metric g that has L^2_{loc}-Christoffel symbols and a positive function $h \in C^0 \cap W^{1,2}_{\mathrm{loc}}$. Define the measure μ via $\mathrm{d}\mu = h^2 \mathrm{dvol}_g$. We assume that the metric measure space (M, d_g, μ) satisfies the volume growth condition* (2.2.1) *and that $H^2_1(M,\mu) \cap C^{0,1} \cap D_{L^\infty}(\Delta_\mu)$ is dense in $H^2_1(M,\mu) \cap D_{L^\infty}(\Delta_\mu)$ in the sense that for each function $f \in H^2_1(M,\mu) \cap D_{L^\infty}(\Delta_\mu)$ there exists a sequence $f_j \in H^2_1(M,\mu) \cap C^{0,1} \cap D_{L^\infty}(\Delta_\mu)$ such that $f_j \to f$ in $W^{1,2}_w(M)$ and $\Delta_\mu f_j \to \Delta_\mu f$ in L^2 and weakly-$*$ in L^∞.*

Note that such a condition is necessary, as all $\mathsf{RCD}(K,\infty)$ spaces satisfy it (see Remark 2.2.13 and [123], [63, (3.1.5)]). At the end of the section, we establish such volume growth and density conditions for weighted manifolds with C^1-metrics and distributional Bakry-Émery N-Ricci curvature bounded below by K for finite N, as well as unweighted manifolds with $W^{1,p}_{\mathrm{loc}}$-metrics for some $p > \dim M$.

Theorem 3.6.2. *Let M be a smooth manifold, g a continuous Riemannian metric with L^2_{loc}-Christoffel symbols and let $h \in C^0(M) \cap W^{1,2}_{\mathrm{loc}}(M)$ be a positive function. Denote $V := -2\log h \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$. Define the measure μ via $\mathrm{d}\mu := e^{-V}\mathrm{dvol}_g$. Let $N \in [n,\infty]$ and $K \in \mathbb{R}$. Assume that Assumption 3.6.1 is satisfied.*

If the distributional Bakry-Émery N-Ricci curvature tensor is bounded below by K, i.e.

$$\mathrm{Ric}_{\mu,N} \geq Kg \text{ in } \mathcal{D}'\mathcal{T}^0_2,$$

then (M, d_g, μ) satisfies the $\mathsf{RCD}^*(K,N)$-condition if $N \in [n,\infty)$, or the $\mathsf{RCD}(K,\infty)$-condition if $N = \infty$.

Proof. By Theorem 3.4.4, it suffices to prove that (M, d_g, μ) satisfies the $\mathsf{BE}(K,N)$-condition. For any smooth function $f \in C^\infty(M)$ and any non-negative, compactly supported test volume ω, we have that

$$\begin{aligned}
&\langle \mathrm{Ric}_{\mu,N}(\nabla f, \nabla f) - Kg(\nabla f, \nabla f), \omega\rangle_{\mathcal{D}',\mathcal{D}} \\
&= \left\langle \mathrm{Ric}_{\mu,\infty}(\nabla f, \nabla f) - \frac{1}{N-n}\langle \nabla V, \nabla f\rangle_g^2 - Kg(\nabla f, \nabla f), \omega \right\rangle_{\mathcal{D}',\mathcal{D}} \geq 0. \qquad (3.6.1)
\end{aligned}$$

Let $n \geq 1$ be the dimension of M and denote by Δ_g the Laplacian induced by the volume measure dvol_g.

Claim. The following inequality holds at μ-almost every point $p \in M$:

$$|\nabla^2 f|_{HS}^2 - \frac{1}{N}(\Delta_g f + \langle \nabla V, \nabla f\rangle)^2 \geq -\frac{1}{N-n}\langle \nabla V, \nabla f\rangle^2. \qquad (3.6.2)$$

Proof of the claim. Fix $p \in M$ such that p is a Lebesgue point of Dg and $(Dg)^2$. Denote $B := |\langle \nabla V, \nabla f\rangle|(p)$. For $(e_i)_{i=1}^n$, a g-orthonormal basis at T_pM, we have that

$$|\nabla^2 f|_{HS}^2(p) = \sum_{i,j=1}^n \nabla^2 f(e_i, e_j)^2 \quad \text{and} \quad \Delta_g f(p) = \sum_{i=1}^n \nabla^2 f(e_i, e_i).$$

From the Cauchy-Schwartz inequality, we get that

$$n|\nabla^2 f|_{HS}^2(p) \geq (\Delta_g f(p))^2.$$

Denote $a = |\nabla^2 f|_{HS}$. Then

$$\begin{aligned}
&|\nabla^2 f|_{HS}^2 - \frac{1}{N}(\Delta_g f + \langle \nabla V, \nabla f\rangle)^2 + \frac{1}{N-n}\langle \nabla V, \nabla f\rangle^2 \\
&\geq |\nabla^2 f|_{HS}^2 - \frac{1}{N}(|\Delta_g f| + |\langle \nabla V, \nabla f\rangle|)^2 + \frac{1}{N-n}\langle \nabla V, \nabla f\rangle^2 \\
&\geq a^2 - \frac{1}{N}(\sqrt{n}a + B)^2 + \frac{1}{N-n}B^2 \\
&= \frac{1}{N}(\sqrt{N-n}a + \frac{\sqrt{n}}{\sqrt{N-n}}B)^2 \geq 0.
\end{aligned}$$

This proves the claim.

We can now plug (3.6.2) into (3.6.1) and infer that

$$\left\langle \mathrm{Ric}_{\mu,\infty}(\nabla f, \nabla f) + |\nabla^2 f|_{HS}^2 - \frac{1}{N}(\Delta_\mu f)^2 - Kg(\nabla f, \nabla f), \omega \right\rangle_{\mathcal{D}', \mathcal{D}} \geq 0.$$

Now pick a nonnegative function $\phi \in W_w^{1,2}(M)$ with $\phi, \Delta_\mu \phi \in L^\infty(M)$ and function $f \in W_w^{1,2} \cap D_{W_w^{1,2}}(\Delta_\mu)$ with $\Delta_\mu f \in L^\infty$. By [10, Lemma 4.2], it suffices to check the Bakry-Émery-condition for such f, ϕ. We have that

$$\begin{aligned}
\int_M \phi \, \mathrm{d}\mathbf{\Gamma}_2(f,f) &= \int_M -\langle \nabla \phi, \nabla \langle \nabla f, \nabla f\rangle_g\rangle_g + (\Delta_\mu f)^2 \phi + \Delta_\mu f \langle \nabla \phi, \nabla f\rangle_g \, \mathrm{d}\mu \\
&= \int_M 2\Delta_\mu \phi \langle \nabla f, \nabla f\rangle_g + (\Delta_\mu f)^2 \phi + \Delta_\mu f \langle \nabla \phi, \nabla f\rangle_g \, \mathrm{d}\mu. \qquad (3.6.3)
\end{aligned}$$

Now fix some Lipschitz continuous $f \in W_w^{1,2}(M) \cap D(\Delta_\mu)$ with $\Delta_\mu f \in L^\infty$. As $f \in W_w^{1,2} = H_1^2(M)$, we can approximate f with smooth functions $\tilde{f}_j \in C_1^2(M)$ such that $\tilde{f}_j \to f$ in $H_1^2 = W_w^{1,2}$ as $j \to \infty$ and such that $\mathrm{Lip}(\tilde{f})$ is uniformly bounded. Integration by parts together with the strong $W_w^{1,2}$-convergence yields that $\Delta_\mu f_j \rightharpoonup \Delta_\mu f$ in $L^2(M, \mu)$. Using Mazur's lemma, we can find a sequence $f_j \in C_1^2(M)$ (of convex combinations of the $\tilde{f}_j$) such that $f_j \to f$ in $W^{1,2}_w(M)$ and $\Delta_\mu f_j \to \Delta_\mu f$ in L^2. Moreover, the bound on the Lipschitz constants implies that we can find a subsequence such that $\nabla f_j \overset{*}{\rightharpoonup} \nabla f$ in L^∞. Evaluating the last line of (3.6.3) with f_j and passing to the limit using the strong convergence yields that

$$\begin{aligned}
&\int_M 2\Delta_\mu\phi\langle\nabla f, \nabla f\rangle_g + (\Delta_\mu f)^2\phi + \Delta_\mu f\langle\nabla\phi, \nabla f\rangle_g \,\mathrm{d}\mu \\
&= \lim_{j\to\infty}\int_M 2\Delta_\mu\phi\langle\nabla f_j, \nabla f_j\rangle_g + (\Delta_\mu f_j)^2\phi + \Delta_\mu f_j\langle\nabla\phi, \nabla f_j\rangle_g \,\mathrm{d}\mu. \qquad (3.6.4)
\end{aligned}$$

Similarly, we get that

$$\int_M \phi\Big(Kg(\nabla f, \nabla f) + \frac{(\Delta_\mu f)^2}{N}\Big)\,\mathrm{d}\mu = \lim_{j\to\infty}\int_M \phi\Big(Kg(\nabla f_j, \nabla f_j) + \frac{(\Delta_\mu f_j)^2}{N}\Big)\,\mathrm{d}\mu. \quad (3.6.5)$$

Using that the assumed distributional lower Ricci curvature bound implies that

$$\begin{aligned}
&\int_M 2\Delta_\mu\phi\langle\nabla f_j, \nabla f_j\rangle_g + (\Delta_\mu f_j)^2\phi + \Delta_\mu f_j\langle\nabla\phi, \nabla f_j\rangle_g \,\mathrm{d}\mu \\
&\qquad\qquad \geq \Big(Kg(\nabla f_j, \nabla f_j) + \frac{(\Delta_\mu f_j)^2}{N}\Big) \quad \text{for all } j \in \mathbb{N},
\end{aligned}$$

we infer that the Bakry-Émery condition $\mathsf{BE}(K,N)$ is satisfied for f. As f was an arbitrary Lipschitz continuous function in $W^{1,2}_w(M) \cap D_{L^\infty}(\Delta_\mu)$ with $\Delta_\mu f \in L^\infty$ and those functions are dense in $W^{1,2}_w(M) \cap D_{L^\infty}(\Delta_\mu)$ by Asumption 3.6.1, this generally verifies the $\mathsf{BE}(K,N)$-condition. Together with the exponential bound on the volume growth from Assumption 3.6.1, this is equivalent to the $\mathsf{RCD}^*(K,N)$-condition. □

The combination of Theorem 3.5.11, Theorem 3.5.20, and Theorem 3.6.2 yields:

Theorem 3.6.3. *Let M be a smooth manifold, g a continuous Riemannian metric with L^2_{loc}-Christoffel symbols and let $h \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$ be a positive function. Denote $V := -2\log h \in C^0 \cap W^{1,2}_{\mathrm{loc}}(M)$ and define the measure μ via $\mathrm{d}\mu := e^{-V}\mathrm{dvol}_g$. Let $N \in [n,\infty]$ and $K \in \mathbb{R}$. The following are equivalent:*

(i) *(M, d_g, μ) is a $\mathsf{CD}^*(K,N)$-space.*

(ii) *The distributional Bakry-Émery N-Ricci curvature tensor is bounded below by K and Assumption 3.6.1 is satisfied.*

Remark 3.6.4. *Note that in Theorem 3.6.3 (i), the $\mathsf{CD}^*(K,N)$-condition corresponds to the $\mathsf{RCD}^*(K,N)$-condition by Corollary 3.3.15.*

Remark 3.6.5. *To define distributional lower Ricci curvature bounds, the minimal requirement is that $g, g^{-1} \in L^\infty_{loc}$ admits L^2_{loc}-Christoffel symbols; this is known in the literature as Geroch-Traschen class, after [60]. Under the assumption that $g, g^{-1} \in L^\infty_{loc}$,*

Norris ([107]) and De Cecco-Palmieri ([50], [51]) defined a distance that turns (M, d_g) into a length space, when only considering curves $\gamma : \mathrm{Lip}([0,1], M)$ for which $\mathcal{L}^1$-almost every point is outside a null set N, which includes all non-Lebesgue points of g. It is thus natural to ask whether the equivalence of distributional and synthetic Ricci curvature lower bounds can be generalized to the Geroch-Traschen class.
It does not seem immediate to extend the present result, though. For instance, it is not obvious if Proposition 3.3.10 and the identification of the slope of C^1-functions as the norm of the gradient remains true in such a higher generality. This question is studied in Chapter 4. Related to this, De Giorgi [53] introduced the notion of quasi-Riemannian metric spaces, which are Lipschitz manifolds with an elliptic metric g such that the slope of Lipschitz functions coincide almost everywhere with the norm of the gradient; moreover, he formulated several conjectures on the topic. Since such identification is crucial for our comparison of the classical and the synthetic Sobolev spaces (Corollary 3.3.15*), we assume g to be at least continuous.*

Remarks on the volume growth and density asumptions

We conclude the chapter by pointing out that, in the case of a Riemannian manifold with $g \in C^0 \cap W^{1,p}_{loc}(M)$ for some $p > n = \dim M$, one can drop the volume growth and density condition (Assumption 3.6.1) in Theorem 3.6.3 (ii). More precisely we prove that, in this case, the curvature-dimension condition immediately follows from the distributional lower Ricci curvature bound. The assumption $g \in C^0 \cap W^{1,p}(M)$, with $p > n$ is used in order to build on top of the approximation results from Section 3.1 and a result established by Ketterer [79] for Riemannian manifolds with Ricci curvature bounded below in an L^p-sense.

Proposition 3.6.6. *Let M be a smooth manifold and let $g \in C^0 \cap W^{1,p}_{loc}(M)$ be Riemannian metric on M, for some $p > n$. Suppose that the distributional Ricci curvature tensor is bounded below by K, for some $K \in \mathbb{R}$. Then Assumption 3.6.1 is satisfied. In particular, $(M, \mathsf{d}_g, \mathrm{dvol}_g)$ satisfies the $CD(K, N)$-condition.*

Proof. Fix a point $x \in M$. For each R, the set $\overline{B^{\mathsf{d}_g}_R(x)}$ is compact. For $\varepsilon \in (0,1)$, define $g_\varepsilon = \rho_\varepsilon * g$. Proposition 3.1.4 yields that, for each compact set $K \subset M$:

$$\|\mathrm{Ric}[g_\varepsilon] - \rho_\varepsilon * \mathrm{Ric}[g]\|_{L^{p/2}(K)} \to 0, \quad \text{as } \varepsilon \to 0. \tag{3.6.6}$$

For any smooth Riemannian metric $\tilde{g}$, we consider the quantity $\bar{k}[\tilde{g}](H, q, R, x)$, for $H \in \mathbb{R}, 2q > n, R > 0, x \in M$, defined as

$$\bar{k}[\tilde{g}](H,q,R,x) = \Big(\frac{R^{2q}}{\mathrm{vol}_{\tilde{g}}(B^{\mathsf{d}_{\tilde{g}}}_1(x))} \int_{(B^{\mathsf{d}_{\tilde{g}}}_R(x))} \rho^q_H \mathrm{dvol}_{\tilde{g}}\Big)^{\frac{1}{q}},$$

where $\rho_H(y) = \max(-\rho(y) + (n-1)H, 0)$ and $\rho(y)$ denotes the smallest eigenvalue of the Ricci curvature tensor at y. In other words, ρ_H denotes the Ricci curvature lying below the threshold $(n-1)H$. The convergence (3.6.6) implies that for all $R > 0$, we have that $\bar{k}[g_\varepsilon](\frac{K}{n-1}, \frac{p}{2}, R, x) \to 0$ as $\varepsilon \to 0$. Note also that $g_\varepsilon \to g$ locally uniformly, hence the distances $\mathsf{d}_{g_\varepsilon}$ converge locally uniformly to d_g and the measures $\mathrm{vol}_{g_\varepsilon}$ converge narrowly to vol_g. Then, [79, Theorem 1.2] yields that the metric measure space $(M, \mathsf{d}_g, \mathrm{dvol}_g)$ satisfies the $\mathsf{CD}(K, N)$-condition. Together with the infinitesimal Hilbertianity (Corollary 3.3.15), this implies Assumption 3.6.1. □

This proof indeed immediately shows the $\mathsf{CD}(K, N)$-condition, without using the Bakry-Émery-condition. A similar strategy was applied in [83] to establish the implication that distributional lower Ricci curvature bounds imply synthetic ones for locally Lipschitz continuous metrics. The bound on the volume growth in this case can also be obtained using the results by Chen-Wei [45] (after Petersen-Wei [113]) together with the above convergence argument.

In the presence of a weight on the volume measure, the results of [45] and [79] are not available in the literature for L^p-lower bounds on the Bakry-Émery-N-Ricci curvature tensor. For this reason, below we include a variant of Proposition 3.6.6 which allows to consider a C^1-weight, under the stronger assumption that $g \in C^1$. The proof is again by approximation, building on top of [69].

Proposition 3.6.7. *Let M be a smooth manifold and g be a C^1-Riemannian metric on M. Moreover, let $h \in C^1(M, (0, \infty))$. Let $N \in [n, \infty)$ and $K \in \mathbb{R}$. Suppose that the distributional Bakry-Émery-N-Ricci curvature tensor is bounded below by K. Then $(M, \mathsf{d}_g, h^2\mathrm{dvol}_g)$ satisfies Assumption 3.6.1.*

Proof. Fix a point $p \in M$ and a sequence $R_j \in (0, \infty)$ such that $\lim_{j\to\infty} R_j = \infty$. For each j, the set $B_j := \overline{B^{\mathsf{d}_g}_{R_j}(p)}$ is compact. For $\varepsilon \in (0, 1)$, define $g_\varepsilon = \rho_\varepsilon * g$ and $h_\varepsilon = \rho_\varepsilon * h$. Noting that [69, Lemma 4.6] holds similarly for $K < 0$, we get that, for each $j \in \mathbb{N}$, there exists an $\varepsilon_j \in (0, 1)$ such that

- for each Borel set $\Omega \subset B_j$, it holds that $\frac{1}{2}h^2\mathrm{vol}_g(\Omega) \le h^2_{\varepsilon_j}\mathrm{vol}_{g_{\varepsilon_j}}(\Omega) \le 2h^2\mathrm{vol}_g(\Omega)$;
- for each $x \in B_j$, it holds that g and g_{ε_j} are 2-equivalent; i.e., $\frac{1}{2}g \le g_{\varepsilon_j} \le 2g$;
- for each $X \in TB_j$, it holds that $\mathrm{Ric}_{\mu,\infty}[g_{\varepsilon_j}](X, X) \ge (K-1)g_{\varepsilon_j}(X, X)$.

Fix $j \in \mathbb{N}$. To keep notation short, write g_j for g_{ε_j} and h_j for h_{ε_j}. For each $R < R_j$, we can apply the weighted Bishop-Gromov theorem for $(B_j, \mathsf{d}_{g_j}, h_j^2\mathrm{dvol}_{g_j})$ together with the fact that $H := \sup_j \sup_{B_1} |h_j| < \infty$ to get constants $C, D > 0$, depending only on K, N, H, and $\mathrm{vol}_g(B_1(p))$, such that

$$h_j^2 \mathrm{vol}_{g_j}(B_R^{\mathsf{d}_{g_j}}(p)) \leq Ce^{DR^2}.$$

From the choice of ε_j, it follows that for all $R \leq \frac{1}{2}R_j$:

$$h^2 \mathrm{vol}_g(B_R^{\mathsf{d}_g}(p)) \leq 2Ce^{4DR^2}.$$

As j was arbitrary, the proof of the volume growth condition is complete.

The assumed regularity assumptions imply C^1-regularity, and hence Hölder continuity of the coefficients of the associated heat equation. Now the theory on parabolic PDEs in divergence form with Hölder continuous coefficients (see [85, Chapter V, §3]) implies that for positive times, the heat flow of a bounded L^2-function is locally Lipschitz continuous. Now, given a function $f \in W_w^{1,2} \cap D_{L^\infty}(\Delta_\mu)$, we can apply the heat flow to approximate it appropriately with a locally Lipschitz continuous function, whilst preserving the boundedness of the Laplacian due to the maximum principle.

Assume now that $f \in W_w^{1,2} \cap D_{L^\infty}(\Delta_\mu)$ additionally is locally Lipschitz. Note that in these regularity assumptions, we have that for each function $\eta \in C_c^\infty(M)$, it holds $\Delta_\mu \eta \in L^\infty$, hence ηf is compactly supported, Lipschitz and $\Delta_\mu(\eta f)$ is bounded. Using a bounded sequence η_k of smooth cut-off functions with bounded Lipschitz constant, we get that $\eta_k f \to f$ in $W^{1,2}$. Noting that $\Delta_\mu(\eta_k f) = (\Delta_\mu \eta_k) f + \langle \nabla \eta_k, \nabla f \rangle_g + \eta_k \Delta_\mu f \in L^2$, we may apply integration by parts together with the strong $W^{1,2}$-convergence of $\eta_k f$ to f, to see that $\Delta_\mu(\eta_k f) \rightharpoonup f$ in L^2. Mazur's lemma allows to find a sequence f_k (of convex combinations of the $\eta_k f$) such that $\Delta_\mu f_k \to \Delta_\mu f$ in L^2. It follows that $\Delta_\mu(\eta_k f) \overset{*}{\rightharpoonup} \Delta_\mu f$ in L^∞. Hence, we can approximate f with a compactly supported and hence (globally) Lipschitz continuous function with bounded Laplacian. □

Corollary 3.6.8. *Let M be a smooth manifold and assume one of the following two conditions hold:*

(1) Let g be a C^1-Riemannian metric and let $h \in C^1$ be a positive function. Denote $V := -2\log h \in C^1$ and define the measure μ via $\mathrm{d}\mu := e^{-V}\mathrm{dvol}_g$.

(2) Let g be a Riemannian metric such that $g \in C^0 \cap W_{loc}^{1,p}(M)$, for some $p > n$. Define the measure μ via $\mathrm{d}\mu = \mathrm{dvol}_g$.

Let $N \in [n, \infty)$ and $K \in \mathbb{R}$. Then the following are equivalent:

(i) *(M, d_g, μ) is a $\mathsf{CD}^*(K,N)$-space.*

(ii) *The distributional Bakry-Émery N-Ricci curvature tensor is bounded below by K.*

Remark 3.6.9 (Equivalent formulations of Corollary 3.6.8)**.** *Under the assumptions of Corollary 3.6.8, the* $\mathsf{CD}^*(K,N)$ *condition is equivalent to* $\mathsf{RCD}^*(K,N)$ *(which, in turn, is equivalent to* $\mathsf{RCD}(K,N)$*; see Remark 2.2.10) since the associated metric measure space is infinitesimally Hilbertian (see Corollary 3.3.15).*
Moreover in the unweighted case (i.e., assumption (1)*), the statement* (ii) *in the equivalence is in turn equivalent to* distributional Ricci curvature tensor bounded below by K.

Remark 3.6.10. *This section differs from the last section in [102], as we noticed the volume growth assumption needed to be adjusted to the volume growth and density Assumption 3.6.1 to prove that distributional lower Ricci curvature bounds imply synthetic lower Ricci curvature bounds. However, the full equivalence (see Corollary 3.6.8) is still obtained in the cases $g \in W^{1,p}_{\mathrm{loc}}$ and h constant for some $p > \dim M$ or for $g, h \in C^1$, the same regularity assumptions we assumed in [102, Corollary 7.7].*

Chapter 4

The infinitesimal structure of manifolds with non-continuous Riemannian metrics

This chapter is based on the results of the solo-authored paper [119]. We refer to Subsection 1.3.2 for an introduction to this chapter. The main results we will obtain are:

Theorem 2. *For $d \geq 3$, $p \in [1, d-1]$ there exists a d-dimensional manifold M and a Riemannian metric g on M such that $g, g^{-1} \in L^\infty$, $g \in W^{1,p}_{\text{loc}}(M)$ and such that*

(i) There exists a test plan that gives positive measure to curves for which the metric speed is strictly smaller than the norm of the derivative for almost every time $t \in [0,1]$.

(ii) There exists a function $f \in C^1(M)$ for which the minimal weak upper gradient and the metric slope are strictly larger than the norm of the gradient on a set of positive measure, i.e. $|Df|, |Df|_w > |\nabla_g f|_g$ on a set of positive vol_g-measure.

Theorem 3. *For $d \geq 3$, $p \in [1, d-1]$ there exists a d-dimensional manifold (M, g) with $g, g^{-1} \in L^\infty_{\text{loc}}$ and $g \in W^{1,p}_{\text{loc}}(M)$ such that the metric measure space $(M, \mathsf{d}_g, \mathrm{vol}_g)$ is not infinitesimally Hilbertian.*

Notation

Given a manifold M, a Riemannian metric g, and a locally Lipschitz continuous function $f : M \to \mathbb{R}$, $|Df|$ will always denote its slope, $|Df|_w$ its minimal weak upper gradient, $\nabla_g f$ its gradient with respect to g and $\nabla_{euc} f$ its differential with respect to chosen local coordinates (which will be specified).
Given an absolutely continuous curve $\gamma : [0,1] \to M$, we will for almost every $t \in [0,1]$

denote its metric speed at time t by $|\dot{\gamma}_t|$, and its derivative at time t by $\dot{\gamma}_t \in T_{\gamma_t}M$. To avoid confusion, if we take the norm of the derivative, we will always write $|\dot{\gamma}_t|_g$ for $\sqrt{g(\dot{\gamma}_t,\dot{\gamma}_t)}$ and $|\dot{\gamma}_t|_{euc}$ for $\sqrt{\langle\dot{\gamma}_t,\dot{\gamma}_t\rangle_{euc}}$ (given specified local coordinates).
When working in $\mathbb{R}^d$, e_i denotes the i-th unit vector, $\mathcal{L}^d$ the d-dim Lebesgue measure, and Id denotes the identity matrix. For $i \in \{1,\dots,d\}$ denote by $x_i : \mathbb{R}^d \to \mathbb{R}$ the projection on the i-th coordinate.
For any measure space $(X,\mathfrak{m})$ and $\mathfrak{m}$-measurable set $A \subset X$, $\mathbb{1}_A$ denotes the characteristic function of A.

4.1 The distance on manifolds with non-continuous Riemannian metrics

From now on we will consider a smooth d-dimensional manifold M with a Riemannian metric g such that $g, g^{-1} \in L^\infty_{\mathrm{loc}}(M)$. Let $U \subset M$ be open such that U lies in one coordinate patch and let $\psi : U \to \mathbb{R}^d$ be a local trivialisation (that respects our chosen smooth structure for M). Then, $g, g^{-1} \in L^\infty_{\mathrm{loc}}(M)$ means that for every such U, the components of $\psi_* g, \psi_* g^{-1}$ are locally $\mathcal{L}^d$-measurable in $\psi(U)$ and for every compact set $K \subset \psi(U)$, there exist constants $\lambda, \Lambda > 0$ such that $\lambda \mathrm{Id} \leq \psi_* g \leq \Lambda \mathrm{Id}$ $\mathcal{L}^d$-almost everywhere in K (cf. Definition 2.1.4).

Then (M,g) falls into the class of LIP-Riemannian manifolds (see for example [135, 51]). These manifolds and their metric structures, have been studied by Norris [107], and De Cecco and Palmieri [51], as well as Saloff-Coste [121] and Sturm [131] with emphasis on potential theory for uniformly elliptic operators. Note that LIP-manifolds shall not be confused with metric tensors whose coefficients are locally Lipschitz continuous.

If g is only measurable, it might not be well-defined everywhere, hence the definition of the metric d_g from the continuous case (3.3.1), cannot be used for non-continuous Riemannian metrics. However, Norris [107] and De Cecco-Palmieri [51] gave a definition of a distance d_g on M arising from g that avoids this difficulty. We will briefly summarise their results and then study the metric speed of curves, test plans, and the Cheeger energy on the corresponding metric measure space.

In the case of a Riemannian manifold with $g, g^{-1} \in L^\infty_{\mathrm{loc}}$, Norris defines the distance d via

$$\mathsf{d}(x,y) := \sup\left\{w(x) - w(y),\ w \in C^{0,1}_{\mathrm{loc}}(M) \text{ and } |\nabla_g w|_g \leq 1 \ \mathrm{vol}_g\text{−a.e.}\right\}.$$

Here, $w \in C^{0,1}_{\mathrm{loc}}(M)$ means that the function w is locally Lipschitz continuous with respect to the Euclidean distance on charts.

Moreover, for $\rho \in C_c^\infty(B_1^{\mathbb{R}^d}(0),[0,\infty))$, with $\int_{\mathbb{R}^d}\rho\,\mathrm{d}x = 1$, define $\rho_\varepsilon := \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$ and $g_\varepsilon := \rho_\varepsilon * g$ (for details on the convolution on manifolds see [75, 69]). The metrics g_ε are continuous and hence the distance $\mathsf{d}_{g_\varepsilon}$ is given as in (3.3.1). Norris then proves that for $\varepsilon \to 0$, $\mathsf{d}_{g_\varepsilon}$ converges to a distance d_0 that is locally equivalent to the underlying Euclidean metric induced from a coordinate chart; in particular, [107, Theorem 3.6] states that $\mathsf{d}_0 = \mathsf{d} =: \mathsf{d}_g$.

From the properties of g, it follows that all points outside a vol_g-null set $N_g \subset M$ are Lebesgue points of g, g^{-1}. For any vol_g-null set $N \subset M$, define the set

$$\mathrm{Lip}_N(x,y,M) := \{\gamma \in C^{0,1}_{\mathrm{loc}}([0,1],M) : \gamma(0)=x, \gamma(1)=y, \mathcal{L}^1(\{t\in[0,1]:\gamma(t)\in N\})=0\}. \tag{4.1.1}$$

Here $C^{0,1}_{\mathrm{loc}}([0,1],M)$ refers to locally Lipschitz continuous curves with respect to Euclidean distance induced by the local charts. If $\gamma \in C([0,1],M)$ satisfies $\mathcal{L}^1(\{t \in [0,1] : \gamma(t) \in N\}) = 0$, we say that γ is transversal to N and write $\gamma \perp N$.

From now on, we will make a further assumption, that will trivially hold for all the relevant example manifolds in this chapter.

Assumption 4.1.1 ([51], (1.13))**.** *The manifold (M,g) satisfies the following:*

(i) The manifold M has infinite vol_g*-volume.*

(ii) There exists a constant $h \geq 0$ such that all coordinate charts in our atlas satisfy that the smallest eigenvalue of g is uniformly bounded below h and M is complete or the interior of a complete manifold (where the completeness is with respect to the distance induced by the charts as in [51, (1.9)]).

Assumption 4.1.1 is relevant to most results in [51], in particular in establishing that two different definitions of the distance are equal. Indeed, De Cecco and Palmieri proved in [51, Theorem (2.18) and Theorem (6.1)] that under Assumption 4.1.1, it holds $\mathsf{d}_g = \boldsymbol{\delta}$, where the distance $\boldsymbol{\delta}$ is given by

$$\boldsymbol{\delta}(x,y) = \sup_{N\subset M, \mathrm{vol}_g(N)=0} \boldsymbol{\delta}_N(x,y), \tag{4.1.2}$$

where

$$\boldsymbol{\delta}_N := \inf\{L_g(\gamma), \gamma \in \mathrm{Lip}_N(x,y,M)\}, \tag{4.1.3}$$

and $L_g(\gamma) = \int_0^1 |\dot{\gamma}_t|_g \mathrm{d}t$. In [50, Theorem 4.4], it is shown that for each vol_g-null set $N_0 \subset M$, $\boldsymbol{\delta} = \boldsymbol{\delta}_0 := \sup_{N,\mathrm{vol}_g(N)=0} \boldsymbol{\delta}_{N\cup N_0}$.

Proposition 4.1.2. *Let $\gamma \in C^{0,1}_{\mathrm{loc}}([0,1] \to M)$ be a curve such that $\gamma \perp N_g$. Then, for almost every $t \in [0,1]$, it holds*

$$|\dot\gamma_t| \le |\dot\gamma_t|_g.$$

Proof. For a Lipschitz curve $\gamma \perp N_g$, and any $t \in [0,1]$, $h \neq 0$, we have that

$$\mathsf{d}_g(\gamma_t, \gamma_{t+h}) = \lim_{\varepsilon \to 0} \mathsf{d}_{g_\varepsilon}(\gamma_t, \gamma_{t+h}) \le \lim_{\varepsilon \to 0} \int_0^h |\dot\gamma_{t+\tau}|_{g_\varepsilon} \,\mathrm{d}\tau = \int_0^h |\dot\gamma_{t+\tau}|_g \,\mathrm{d}\tau,$$

where the last equality follows from the dominated convergence theorem and the fact that $\rho_\varepsilon * g \to g$ pointwise outside N_g. Now the characterisation $|\dot\gamma_t| = \lim_{h\to 0} \frac{\mathsf{d}_g(\gamma_t,\gamma_{t+h})}{|h|}$ a.e. together with the Lebesgue differentiation theorem yields the proposition. □

There are more observations that can be made on the metric speed of absolutely continuous curves. The first one is that the metric speed depends on the derivative of the curve.

Proposition 4.1.3. *Let M be a smooth manifold and g a Riemannian metric such that $g, g^{-1} \in L^\infty_{\mathrm{loc}}$. Let $\gamma : [0,1] \to M$ be a Lipschitz continuous curve and $t \in [0,1]$ such that the metric speed $|\dot\gamma_t|$ exists at t and γ is differentiable at t. Let $c : [0,1] \to M$ be another curve such that $c_t = \gamma_t$, c is differentiable at t and $\dot c_t = \dot\gamma_t$. Then the metric speed $|\dot c_t|$ of c at t exists and $|\dot c_t| = |\dot\gamma_t|$.*

Proof. We work in local coordinates in $\mathbb{R}^d$ and write $\gamma_t = c_t =: p$ and $\dot c_t = \dot\gamma_t = v$. Furthermore, we may restrict to a small neighbourhood U around p and assume that there exist $\lambda, \Lambda > 0$ such that $\lambda \mathrm{Id} \le g \le \Lambda \mathrm{Id}$. Then, by the definition of the derivative, we get that

$$\gamma_{t+h} = p + hv + a_\gamma(h),\ c_{t+h} = p + hv + a_c(h) \in \mathbb{R}^d,$$

where

$$\lim_{h \to 0} \frac{|a_\gamma(h)|_{euc} + |a_c(h)|_{euc}}{|h|} = 0. \tag{4.1.4}$$

Now, we can compute that

$$\mathsf{d}_g(\gamma_{t+h}, c_{t+h}) \le \sqrt{\Lambda}(|a_\gamma(h)|_{euc} + |a_c(h)|_{euc}), \tag{4.1.5}$$

and hence

$$0 \le \lim_{h\to 0} \frac{\mathsf{d}_g(\gamma_{t+h}, c_{t+h})}{|h|} \le \lim_{h \to 0} \sqrt{\Lambda} \frac{|a_\gamma(h)|_{euc} + |a_c(h)|_{euc}}{|h|} = 0.$$

But then

$$\limsup_{h\to 0}\left|\frac{\mathsf{d}_g(c_t,c_{t+h})}{|h|}-\frac{\mathsf{d}_g(\gamma_t,\gamma_{t+h})}{|h|}\right|\le \limsup_{h\to 0}\frac{\mathsf{d}_g(\gamma_{t+h},c_{t+h})}{|h|}=0.$$

□

Next, we establish that the metric speed depends locally Lipschitz-continuously on the derivative of a curve.

Proposition 4.1.4. *Let $\gamma^1,\gamma^2:[0,1]\to M$ be two Lipschitz continuous curves that are contained in a compact set K inside one coordinate patch V and $t\in[0,1]$ such that $\gamma^1_t=\gamma^2_t$ and such that the metric speeds $|\dot\gamma^1_t|$ and $|\dot\gamma^2_t|$ as well as their derivatives $\dot\gamma^1_t,\dot\gamma^2_t\in T_{\gamma^1_t}M$ exist. Then there exists a constant $C=C(K)$ such that*

$$||\dot\gamma^1_t|-|\dot\gamma^2_t||\le C|\dot\gamma^1_t-\dot\gamma^2_t|_g. \tag{4.1.6}$$

Proof. We may assume to be working in $\mathbb{R}^d$. Now there exist $\lambda,\Lambda>0$ such that $\lambda\mathrm{Id}\le g\le\Lambda\mathrm{Id}$ on K. Denote $v:=\dot\gamma^1_t$ and $w:=\dot\gamma^2_t$. By translation we may assume that $t=0$ and denote $p:=\gamma^1_0=\gamma^2_0$. By the previous proposition, we may assume that $\gamma^1_t=p+tv$ and $\gamma^2_t=p+tw$. But then

$$\begin{aligned}\Big|\frac{\mathsf{d}_g(\gamma^1_0,\gamma^1_h)}{|h|}-\frac{\mathsf{d}_g(\gamma^2_0,\gamma^2_h)}{|h|}\Big|&\le\frac{\mathsf{d}_g(p+hv,p+hw)}{|h|}\le\sqrt{\Lambda}\frac{\mathsf{d}_{euc}(p+hv,p+hw)}{|h|}\\&=\sqrt{\Lambda}|v-w|_{euc}\le\sqrt{\frac{\Lambda}{\lambda}}|v-w|_g=\sqrt{\frac{\Lambda}{\lambda}}|\dot\gamma^1_t-\dot\gamma^2_t|_g.\end{aligned}$$

□

Lemma 4.1.5. *Let $\boldsymbol{\pi}\in\mathcal{P}(C([0,1],M))$ be a test plan and $N\subset M$ a vol_g-null set. Then, it holds that*

$$\boldsymbol{\pi}\Big(\{\gamma\in C([0,1],M):\gamma\perp N\}\Big)=1. \tag{4.1.7}$$

Proof. We argue by contradiction. Assume that $\boldsymbol{\pi}\Big(\{\gamma\in C([0,1],M):\gamma\perp N\}\Big)<1$. Then as

$$\{\gamma\in C([0,1],M):\gamma\not\perp N\}=\bigcup_{n\in\mathbb{N}}\{\gamma\in C([0,1],M):\mathcal{L}^1(\gamma^{-1}(N))\ge\frac{1}{n}\},$$

there exists an $n\in\mathbb{N}$ and an $\varepsilon>0$ such that

$$\boldsymbol{\pi}(\{\gamma\in C([0,1],M):\mathcal{L}^1(\gamma^{-1}(N))\ge\frac{1}{n}\})=\varepsilon>0. \tag{4.1.8}$$

Denote $\{\gamma \in C([0,1],M) : \mathcal{L}^1(\gamma^{-1}(N)) \geq \frac{1}{n}\} =: B_n$ and note that for each curve $\gamma \in B_n$, it holds

$$\int_0^1 \mathbb{1}_N(\gamma_t)\,\mathrm{d}t \geq \frac{1}{n}, \tag{4.1.9}$$

hence, with Fubini's theorem, we get that

$$\int_0^1 \int_{C([0,1],M)} \mathbb{1}_N(\gamma_t)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t = \int_{C([0,1],M)} \int_0^1 \mathbb{1}_N(\gamma_t)\,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) \geq \frac{\varepsilon}{n} > 0. \tag{4.1.10}$$

By the definition of a test plan, we have that for all $t \in [0,1]$ it holds $(e_t)_\#\boldsymbol{\pi} \ll \mathrm{vol}_g$. As N is a dvol_g-null set, we get that for all $t \in [0,1]$, it holds

$$\int_M \mathbb{1}_N(y)\,\mathrm{d}(e_t)_\#\boldsymbol{\pi}(y) = 0.$$

That gives

$$\int_0^1 \int_{C([0,1],M)} \mathbb{1}_N(\gamma_t)\,\mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t = \int_0^1 \int_M \mathbb{1}_N(y)\,\mathrm{d}(e_t)_\#\boldsymbol{\pi}(y)\mathrm{d}t = 0,$$

which contradicts (4.1.10). This proves the Lemma. □

We conclude this section with the definition of a quasi-Riemannian manifold proposed by De Giorgi in [53], that we will refer to later:

Definition 4.1.6. *Given a manifold M, a distance d on M, and a measurable positive definite inner product g on TM, we call (M,d,g) a quasi-Riemannian manifold if there exists an atlas $\mathcal{A} = \{W,\psi\}$ such that the following conditions hold:*

(i) In each (W,ψ), there exist constants $\lambda,\Lambda > 0$ such that for any $x,y \in W$, it holds

$$\lambda \leq (\psi_* g)^{ij}\partial_{x^i}\mathsf{d}^W_{euc}(x,y)\partial_{x^j}\mathsf{d}^W_{euc}(x,y) \leq \Lambda.$$

(ii) For any $f \in C^0_c(W)$, it holds

$$\int_W f\mathrm{d}\mathcal{H}^d_{\mathsf{d}} = \int_W f\sqrt{\det g}\,\mathrm{d}\mathcal{L}^d,$$

where $\mathcal{H}^d_{\mathsf{d}}$ denotes the d-dimensional Hausdorff measure with respect to the distance d.

(iii) For any Lipschitz continuous $f : M \to \mathbb{R}$, and almost every $x \in M$ it holds

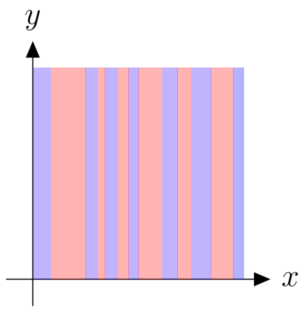
$$\limsup_{y\to x} \frac{|f(x)-f(y)|}{\mathsf{d}(x,y)} = |\nabla_g f|_g(x).$$

4.2 Construction of examples

In this section, we will construct Riemannian manifolds for with non-continuous metrics that together with the distance d_g as defined in the previous section, do not give rise to quasi-Riemannian manifolds or infinitesimally Hilbertian metric measure spaces.

Intuition in two dimensions

We will first provide a two dimensional mock example that demonstrates the intuition needed for the following two examples. We construct a Riemannian metric g on $\mathbb{R}^2$ that is given by an L^∞-function $\theta : \mathbb{R}^2 \to \{1, 2\}$ that only depends on the first coordinate times the identity matrix. More precisely, we choose $\theta^{-1}(\{1\})$ to be an open and dense set of stripes orthogonal to the x-axis with small measure. On the complement, $\theta = 2$ and hence $g = 2\mathrm{Id}$. Now, whenever, we want to move parallel to the y-axis, this will "cost" us once the Euclidean length of the movement. Indeed, we can find an arbitrarily close stripe in $\theta^{-1}(\{1\})$ along which moving upwards costs the Euclidean price.

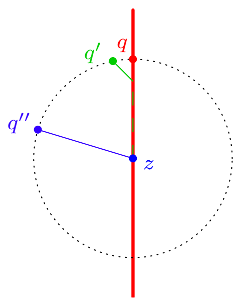


Now fix a Lebesgue point z of $\theta^{-1}(\{2\})$. The fact that z is a Lebesgue point means that a curve starting from z that is not parallel to the x-axis costs $\sqrt{2}$-times the Euclidean price. This helps us to asymptotically determine the metric speed of straight curves starting in z depending on their angle to the y-axis.

We start with curves parallel to the y-axis. If one tried to compute the distance between z and a point q that is very close to z and has the same x-coordinate as z, then the above discussion shows, that $\mathsf{d}_g(z, q) = \mathsf{d}_{euc}(z, q)$.

If we now want to compute the distance of the point z to a point q' such that there is a small angle between $q' - z$ and the y-axis, the following happens. If we choose to walk the direct straight connection from z to q' we will end up with a curve of cost $\sqrt{2}\mathsf{d}_{euc}(z, q')$. Thanks to the small angle to the y-axis, it is cheaper, to first walk along parallel to the y-axis for a bit, as that costs only the Euclidean price, and then eventually walk a short segment in a different angle, which will cost $\sqrt{2}$ times the Euclidean length. The optimal length of such a curve can be determined depending on the angle and is given by $\mathsf{d}_g(z, q') = |x(q' - z)| + |y(q' - z)|$, where x, y denote the projection on the coordinates.

Finally, if one considers a point q'' for which $q'' - z$ has a big enough angle to the y-axis, then the direct connection will be the most efficient way to walk from z to q'', giving $\mathsf{d}_g(z, q'') = \sqrt{2}\mathsf{d}_{euc}(z, q'')$.



This gives that the shape of very small metric balls converges to the shape of a metric ball induced by a non-Euclidean norm. This lets both quasi-Riemannianity and infinitesimal Hilbertianity fail.

4.2.1 An introductory example of a non-quasi Riemannian manifold

We will now construct a metric g on $\mathbb{R}^d$ for $d \geq 2$ such that $g, g^{-1} \in L^\infty$ and such that there exists a smooth function $f : \mathbb{R}^d \to \mathbb{R}$ such that $|Df|_w, |Df| > |\nabla_g f|_g$ on a set of positive measure, which implies that (M, d_g, g), where d_g is defined as in (4.1.2), is not a quasi-Riemannian manifold. In [132], Sturm proved that given a Riemannian manifold (M, g), $g, g^{-1} \in L^\infty_{\text{loc}}$, there exists a metric g' on M such that $g', g'^{-1} \in L^\infty_{\text{loc}}$, $g' < g$ on a set of positive vol_g-measure and $\mathsf{d}_g = \mathsf{d}_{g'}$, which immediately shows that Questions 1 and 2 may have negative answers. The example presented in this subsection is related to the one in [52] and specifically designed to pave the way for examples, where also infinitesimal Hilbertianity fails, that will be discussed in later subsections.

Let $q_i \in \mathbb{Q}$ be a counting of $\mathbb{Q}$. Pick a $\kappa \in (0, \frac{1}{8})$ and construct the following open set

$$O := \bigcup_{i=1}^{\infty} \Big(q_i - \frac{\kappa}{2^i}, q_i + \frac{\kappa}{2^i} \Big). \tag{4.2.1}$$

Then $\mathcal{L}^1(O) \leq 2\kappa$ and O is dense in $\mathbb{R}$. We now define

$$\begin{aligned} &\theta := (2 - \mathbb{1}_O) \in L^\infty(\mathbb{R}), \\ &g(x_1, \dots, x_d) := \theta(x_1)\text{Id} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d \times d}). \end{aligned}$$

Then $(\mathbb{R}^d, g)$ satisfies Assumption 4.1.1 and $(\mathbb{R}^d, \mathsf{d}_g, \mathrm{vol}_g)$ satisfies condition (2.2.1). We observe that

$$\begin{aligned}(2-2\kappa) \leq \int_{(0,1)} \theta \, \mathrm{d}\mathcal{L}^1 &\leq \frac{3}{2}\mathcal{L}^1\Big(\{\theta < \frac{3}{2}\} \cap (0,1)\Big) + 2\mathcal{L}^1\Big(\{\theta \geq \frac{3}{2}\} \cap (0,1)\Big) \\ &= \frac{3}{2}\Big(1 - \mathcal{L}^1\Big(\{\theta \geq \frac{3}{2}\} \cap (0,1)\Big)\Big) + 2\mathcal{L}^1\Big(\{\theta \geq \frac{3}{2}\} \cap (0,1)\Big).\end{aligned}$$

It follows that

$$\mathcal{L}^1\Big(\{\theta \geq \frac{3}{2}\} \cap (0,1)\Big) \geq 1 - 4\kappa \geq \frac{1}{2}. \tag{4.2.2}$$

Now for any point $y \in (0,1) \times \mathbb{R}^{d-1}$ consider the curve $\gamma^y : [0,1] \to \mathbb{R}^d, t \mapsto y + te_2$.

Claim. For all $t \in (0,1)$, $h \in (0, 1-t)$ it holds

$$\mathsf{d}_g(\gamma^y_t, \gamma^y_{t+h}) = h. \tag{4.2.3}$$

Proof of the claim. Fix t, h as above and fix an $\varepsilon > 0$. Note that from $g \geq \mathrm{Id}$, we get that $\mathsf{d}_g \geq \mathsf{d}_{euc}$, hence $\mathsf{d}_g(\gamma^y_t, \gamma^y_{t+h}) \geq h$. Thus, it remains to prove that $\mathsf{d}_g(\gamma^y_t, \gamma^y_{t+h}) \leq h$. There exists a $q \in \mathbb{Q} \cap (0,1)$ such that $|x_1(y) - q| \leq \varepsilon$. Let $i \in \mathbb{N}$ be such that $q_i = q$ in the chosen counting. Then define the curves

$$\begin{aligned}&\lambda_1 : [0,1] \to \mathbb{R}^d, s \mapsto \gamma^y_t + s(q - x_1(y))e_1, \\ &\lambda_2 : [0,h] \to \mathbb{R}^d, s \mapsto \gamma^y_t + (q - x_1(y))e_1 + se_2, \\ &\lambda_3 : [0,1] \to \mathbb{R}^d, s \mapsto \gamma^y_t + (q - x_1(y))e_1 + he_2 - s(q - x_1(y))e_1.\end{aligned}$$

Denote by λ the concatenation of those three curves and note that it yields a piecewise smooth curve that connects γ^y_t and γ^y_{t+h}. Now, for all $\varsigma > 0$, it holds $g_\varsigma = g * \rho_\varsigma \leq 2\mathrm{Id}$, so we get that $L_{g_\varsigma}(\lambda_1) + L_{g_\varsigma}(\lambda_3) \leq 4\varepsilon$. For $\varsigma \leq \frac{\kappa}{2^{i+2}}$ that $g_\varsigma|_{\{x_1 = q\}} \equiv \mathrm{Id}$, hence $L_{g_\varsigma}(\lambda_2) = h$. Thus,

$$\lim_{\varsigma \to 0} \mathsf{d}_{g_\varsigma}(\gamma^y_t, \gamma^y_{t+h}) \leq \lim_{\varsigma \to 0} L_{g_\varsigma}(\lambda) \leq h + 4\varepsilon.$$

As ε was arbitrary, this proves the claim.

By (4.2.2), we can choose a set $E' \subset \{\theta \geq \frac{3}{2}\} \cap (0,1)$ such that $\mathcal{L}^1(E') > 0$. Denote $E := E' \times (0,1)^{d-1}$. Define the test plan $\boldsymbol{\pi} \in \mathcal{P}(C^\infty((0,1), \mathbb{R}^d))$ via

$$\mathrm{d}\boldsymbol{\pi}(\gamma) := \begin{cases} \frac{1}{\mathcal{L}^d(E)} \mathrm{d}\mathcal{L}^d(y) & \text{if } \gamma = \gamma^y, \text{ for some } y \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Note that this is indeed a feasible test plan, as $\mathrm{d}\mathcal{L}^d \leq \mathrm{dvol}_g \leq 2^{\frac{d}{2}} \mathrm{d}\mathcal{L}^d$. But now for all $y \in E$, almost all $t \in [0,1]$, we get that $g(\gamma^y_t) \geq \frac{3}{2}\mathrm{Id}$ and $\dot{\gamma}^y_t = e_2$ and $|\dot{\gamma}^y_t|_g \geq \sqrt{\frac{3}{2}}$. But by (4.2.3), we get that the metric speed $|\dot\gamma_t^y| = 1$ for all y, t, hence we found a test plan that gives positive measure to curves whose metric speed and the norm of the derivative are not equal.

We will now give an example of a function $f \in C^\infty(\mathbb{R}^d)$ for which the norm of the gradient is strictly smaller than the minimal weak upper gradient and the metric slope on a set of points with positive measure. The function is given by $f : \mathbb{R}^d \to \mathbb{R}, x = (x_1, \dots, x_d) \mapsto x_2$. Then for all $y \in E$, we have that $|\nabla_g f|_g(y) \le \sqrt{\frac{2}{3}}$. But now for any point $y \in \mathbb{R}^d$, we have that

$$|Df|(y) = \limsup_{z \to y} \frac{|f(z) - f(y)|}{\mathsf{d}_g(z, y)} \ge \lim_{h \to 0} \frac{|f(y + he_2) - f(y)|}{\mathsf{d}_g(y + he_2, y)} = \lim_{h \to 0} \frac{|h|}{|h|} = 1 > \sqrt{\frac{2}{3}}. \tag{4.2.4}$$

This shows that for $y \in E' \times \mathbb{R}^{d-1}$, it holds $|Df|(y) > |\nabla_g f|_g(y)$.

Finally, the test plan $\boldsymbol{\pi}$ shows, that $|Df|_w > |\nabla_g f|_g$ on a set of positive measure. Indeed, we note that

$$\int_{C([0,1],\mathbb{R}^d)} |f(\gamma_1) - f(\gamma_0)| \,\mathrm{d}\boldsymbol{\pi}(\gamma) = \int_E \frac{1}{\mathcal{L}^d(E)} \,\mathrm{d}\mathcal{L}^d = 1.$$

On the other hand, we have that

$$\int_{C([0,1],\mathbb{R}^d)} \int_0^1 |\nabla_g f(\gamma_t)|_g |\dot\gamma_t| \,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) \le \int_E \int_0^1 \sqrt{\frac{2}{3}} \frac{1}{\mathcal{L}^d(E)} \,\mathrm{d}t\mathrm{d}\mathcal{L}^d = \sqrt{\frac{2}{3}}.$$

Hence, we have proved the following:

Proposition 4.2.1. *For $d \ge 2$, there exists a Riemannian metric g on $\mathbb{R}^d$ such that $g, g^{-1} \in L^\infty$ and such that*

(i) There exists a test plan that gives positive measure to curves for which the metric speed is strictly smaller than the norm of the derivative for almost every time $t \in [0,1]$.

(ii) There exists a function $f \in C^\infty(\mathbb{R}^d)$ for which the minimal weak upper gradient and the metric slope are strictly larger than the norm of the gradient on a set of positive measure, i.e., $|Df|, |Df|_w > |\nabla_g f|_g$ on a set of positive vol_g*-measure.*

4.2.2 An introductory example of non-infinitesimal Hilbertianity

In this subsection, we will more closely analyse the infinitesimal structure of a manifold similar to the previous example. This following example metric resembles the previous one, with the sole difference, that we only perturb the metric inside the interior of the unit cube $(0,1)^d \subset \mathbb{R}^d$ for $d \geq 2$ and leave it constant outside that set. We will prove that the resulting metric measure space is not infinitesimally Hilbertian.

Let $q_i \in \mathbb{Q}$ be a counting of $\mathbb{Q} \cap (0,1)$, $\kappa \in (0, \frac{1}{8})$ and define

$$O := \bigcup_{i=1}^{\infty} \Big(q_i - \frac{\kappa}{2^i}, q_i + \frac{\kappa}{2^i} \Big). \tag{4.2.5}$$

Then $\mathcal{L}^1(O) \leq 2\kappa$ and O is dense in $(0,1)$. Define

$$\begin{aligned} &\theta := (2 - \mathbb{1}_O) \in L^\infty((0,1)), \\ &g(x_1, \dots, x_d) := 2\mathbb{1}_{((0,1)^d)^c} \mathrm{Id} + \mathbb{1}_{(0,1)^d} \theta(x_1) \mathrm{Id} \in L^\infty(\mathbb{R}^d, \mathbb{R}^{d \times d}). \end{aligned}$$

Note that $(\mathbb{R}^d, g)$ satisfies Assumption 4.1.1, $(\mathbb{R}^d, \mathsf{d}_g, \mathrm{vol}_g)$ satisfies (2.2.1), and g is constant outside the precompact set $(0,1)^d$. Inside $(0,1)^d$, g the metric space arising from (M, g) will behave just as in the previous subsection, we only restrict this behaviour to a compact set to conclude the lack of infinitesimal Hilbertianity.

To see that we will first quantitatively determine the difference between the metric speed of curves and the norms of their derivatives.

We start with a preparatory lemma:

Lemma 4.2.2. *Let $f : [0,1] \to \mathbb{R}$ be a Lipschitz continuous function and $a < b \in \mathbb{R}$. Then*

$$\int_{f^{-1}((a,b))} f' \, \mathrm{d}t \in [a - b, b - a]. \tag{4.2.6}$$

Proof. We define the function $h : [0,1] \to \mathbb{R}$ as

$$h(t) := \max(a, \min(f(t), b)).$$

Then h is Lipschitz continuous and admits a weak derivative h'. Moreover, in the open set $f^{-1}((a,b)) \subset [0,1]$, we have that $f = h$, hence $h' = f'$ almost everywhere in $f^{-1}((a,b))$. Now, on $f^{-1}((-\infty, a])$ we have that $h \equiv a$.

Claim. For almost every $t \in f^{-1}((-\infty, a])$, it holds $h' = 0$.
Proof of the claim. Indeed, choose a point $t \in [0,1]$ that is a differentiability point of h and that is a Lebesgue point of both $f^{-1}((-\infty, a])$ and h'. Suppose for a contradiction that $h'(t) \neq 0$. Then there exists a neighbourhood $U \ni t$ such that $h(s) \neq h(t)$ for every $s \in U \setminus \{t\}$. But as t is a Lebesgue point of $f^{-1}((-\infty, a])$, we have that $(f^{-1}((-\infty, a]) \cap U) \setminus \{t\} \neq \emptyset$. Hence, there exists a point in $s \in f^{-1}((-\infty, a])$ such that $h(s) \neq h(t)$,

which contradicts the definition of h. As those Lebesgue points have full measure, this proves the claim.

We similarly get that on $f^{-1}([b,\infty))$, it holds $h' \equiv 0$. Then

$$\int_{f^{-1}((a,b))} f' \,\mathrm{d}t = \int_0^1 h' \,\mathrm{d}t = h(1) - h(0). \tag{4.2.7}$$

The lemma follows as $h(0), h(1) \in [a,b]$. $\square$

With Proposition 4.1.3, we will now determine the metric speed of a curve through a point $y \in \mathbb{R}^d$ depending on the derivative.

First suppose that $y \in (0,1)^d$ and that y is a Lebesgue point of $\{g = 2\}$. In this argument, we will work with the definition of the metric d_g via (4.1.2). We will at first determine the shape and length of an almost distance minimising curve between two points that are close to each other. Fix an $\varepsilon > 0$. Then, by the Lebesgue differentiation theorem, there exists a $\delta > 0$ such that for each open interval $I \subset \mathbb{R}$ with $x_1(y) \in I$, and $\mathcal{L}^1(I) \leq \delta$, it holds

$$\frac{1}{\mathcal{L}^1(I)} \int_I |\theta - 2| \,\mathrm{d}\mathcal{L}^1 \leq \varepsilon. \tag{4.2.8}$$

Fix the null set $N_g^{(1)} \subset \mathbb{R}$ as the set of non-Lebesgue points of θ and define $N_g := N_g^{(1)} \times \mathbb{R}^{d-1}$. Note that then $O \cap N_g^{(1)} = \emptyset$.

Let $z \in (0,1)^d$ be such that $\mathsf{d}_{euc}(y,z) \leq \frac{1}{8}\mathrm{dist}_{euc}(y, \partial(0,1)^d)$. Then any 2-shortest curve connecting them, i.e., any curve $\gamma : [0,1] \to \mathbb{R}^d$ from y to z such that $\gamma \perp N_g$ and $L_g(\gamma) \leq 2\mathsf{d}_g(y,z)$, has to lie inside $(0,1)^d$. Let $\gamma : [0,1] \to (0,1)^d$ be a Lipschitz continuous curve transversal to N_g such that $\gamma(0) = y$ and $\gamma(1) = z$. We will now estimate $\mathsf{d}_g(y,z)$ depending on $z - y \in \mathbb{R}^d$. We assume that $x_1(z) \geq x_1(y)$ and $|x_1(z) - x_1(y)| < \delta$. The case $x_1(z) \leq x_1(y)$ is analogous.

Step 1: Reduction to curves with uniformly bounded x_1 coordinates.
Pick $a_y \leq x_1(y)$ and $a_z \geq x_1(z)$ such that $a_y, a_z \in O \cap (0,1)$, and

$$a_z - a_y \leq \delta, \quad x_1(y) - a_y \leq \varepsilon, \quad a_z - x_1(z) \leq \varepsilon, \quad a_z - a_y \leq 2|z-y|_{euc}. \tag{4.2.9}$$

Now define $\tilde{\gamma} : [0,1] \to \mathbb{R}^d$ via

$$\begin{aligned} x_1(\tilde{\gamma})(t) &= \max(a_y, \min(a_z, x_1(\gamma_t))), \\ x_i(\tilde{\gamma})(t) &= x_i(\gamma_t) \text{ for } i = 2, \dots, d. \end{aligned} \tag{4.2.10}$$

Then $\tilde{\gamma}$ is Lipschitz continuous and transversal to N_g. Indeed, on $(x_1 \circ \gamma)^{-1}((-\infty, a_y] \cup [a_z, \infty))$, we get that $\tilde{\gamma} \in \{a_y, a_z\} \times (0,1)^{d-1} \subset N_g^c \cap \{g \equiv \mathrm{Id}\}$ and on $(x_1 \circ \gamma)^{-1}((a_y, a_z))$, we have that $\gamma = \tilde{\gamma}$. Note that the methods from the proof of Lemma 4.2.2 yield that

$$|\dot{\tilde{\gamma}}_t|_{euc} \leq |\dot{\gamma}_t|_{euc} \text{ a.e. on } [0,1]. \tag{4.2.11}$$

Now, we can compute

$$\begin{aligned}
L_g(\gamma) &= \int_{[0,1]} \sqrt{\theta(x_1(\gamma_t))}|\dot{\gamma}_t|_{euc}\,\mathrm{d}t \\
&= \int_{(x_1\circ\gamma)^{-1}((a_y,a_z))} \sqrt{\theta(x_1(\gamma_t))}|\dot{\gamma}_t|_{euc}\,\mathrm{d}t + \int_{(x_1\circ\gamma)^{-1}((-\infty,a_y]\cup[a_z,\infty))} \sqrt{\theta(x_1(\gamma_t))}|\dot{\gamma}_t|_{euc}\,\mathrm{d}t \\
&\geq \int_{(x_1\circ\gamma)^{-1}((a_y,a_z))} \sqrt{\theta(x_1(\gamma_t))}|\dot{\gamma}_t|_{euc}\,\mathrm{d}t + \int_{(x_1\circ\gamma)^{-1}((-\infty,a_y]\cup[a_z,\infty))} |\dot{\tilde{\gamma}}_t|_{euc}\,\mathrm{d}t \\
&= \int_{(x_1\circ\gamma)^{-1}((a_y,a_z))} \sqrt{\theta(x_1(\tilde{\gamma}_t))}|\dot{\tilde{\gamma}}_t|_{euc}\,\mathrm{d}t + \int_{(x_1\circ\gamma)^{-1}((-\infty,a_y]\cup[a_z,\infty))} \sqrt{\theta(x_1(\tilde{\gamma}_t))}|\dot{\tilde{\gamma}}_t|_{euc}\,\mathrm{d}t \\
&= \int_{[0,1]} \sqrt{\theta(x_1(\tilde{\gamma}_t))}|\dot{\gamma}_t|_{euc}\,\mathrm{d}t = L_g(\tilde{\gamma}).
\end{aligned} \tag{4.2.12}$$

Hence, we found a curve $\tilde{\gamma}$ with the same endpoints as γ that is transversal to N_g with bounded x_1 coordinates that is at most as long as γ. From now on, we write $\gamma := \tilde{\gamma}$ with a slight abuse of notation.

Step 2. Reduction to the length of curves consisting of two straight lines.
Define

$$\begin{aligned}
v_{(1)} &:= \int_{(\theta\circ x_1\circ\gamma)^{-1}(1)} \dot{\gamma}_t\,\mathrm{d}t \in \mathbb{R}^d, \\
v_{(2)} &:= \int_{(\theta\circ x_1\circ\gamma)^{-1}(2)} \dot{\gamma}_t\,\mathrm{d}t \in \mathbb{R}^d.
\end{aligned}$$

Note that $v_{(1)} + v_{(2)} = z - y$ and by classical results about the Euclidean space it holds

$$\begin{aligned}
L_g(\gamma) &= \int_{[0,1]} \sqrt{\theta(x_1(\gamma_t))}|\dot{\gamma}_t|_{euc}\,\mathrm{d}t \\
&= \int_{(\theta\circ x_1\circ\gamma)^{-1}(2)} \sqrt{2}|\dot{\gamma}_t|_{euc}\,\mathrm{d}t + \int_{(\theta\circ x_1\circ\gamma)^{-1}(1)} |\dot{\gamma}_t|_{euc}\,\mathrm{d}t \\
&\geq \sqrt{2}|v_{(2)}|_{euc} + |v_{(1)}|_{euc}.
\end{aligned} \tag{4.2.13}$$

Write $O \cap (a_y, a_z) := \bigcup_{n\in\mathbb{N}} I_n$, where I_n are pairwise disjoint open intervals. By our choice of δ (see (4.2.8) and (4.2.9)), we get that $\mathcal{L}^1(O \cap (a_y, a_z)) \leq \varepsilon(a_z - a_y)$. Now, by Lemma 4.2.2, we get the following estimate on the first component $x_1(v_{(1)})$ of $v_{(1)} = \sum_{i=1}^d x_i(v_{(1)})e_i$.

$$|x_1(v_{(1)})| = \Big|\sum_n \int_{(x_1\circ\gamma)^{-1}(I_n)} (x_1 \dot{\circ} \gamma)(t)\mathrm{d}t\Big| \leq \sum_n \mathcal{L}^1(I_n) \leq \varepsilon|a_z - a_y| \leq 2\varepsilon|z - y|_{euc}.$$

Now define

$$w_{(1)} := v_{(1)} - x_1(v_{(1)})e_1 \in \{0\} \times \mathbb{R}^{d-1},\ w_{(2)} = v_{(2)} + x_1(v_{(1)})e_1 \in \mathbb{R}^d.$$

Then $w_{(1)} + w_{(2)} = z - y$ and

$$|w_{(1)} - v_{(1)}|_{euc}, |w_{(2)} - v_{(2)}|_{euc} \leq 2\varepsilon|z-y|_{euc}.$$

Hence

$$L_g(\gamma) \geq \sqrt{2}|w_{(2)}|_{euc} + |w_{(1)}|_{euc} - 6\varepsilon|z-y|_{euc}, \tag{4.2.14}$$

where $w_{(1)} + w_{(2)} = z - y$ and $w_{(1)} \in \operatorname{span}(e_2, \dots e_d)$.

Step 3. Optimisation of $w_{(1)}$ and $w_{(2)}$ under the above constraints.
By potentially applying a linear transform from $\mathrm{SO}(d-1)$ that rotates the last $d-1$ entries, we may assume that $z-y \in \operatorname{span}(e_1, e_2)$. We reduce the right hand side of (4.2.14), if we orthogonally project both $w_{(1)}$ and $w_{(2)}$ onto $\operatorname{span}(e_1, e_2)$, so we may assume that $w_{(1)}, w_{(2)} \in \operatorname{span}(e_1, e_2)$. Now, if $z - y = ae_1$, for some $a \in \mathbb{R}$ we get that $w_{(1)} = 0$ and hence

$$L_g(\gamma) \geq \sqrt{2}a - 6a\varepsilon. \tag{4.2.15}$$

Otherwise, we may write $z-y = a(be_1+e_2)$, where $a, b \in \mathbb{R}$, $a \neq 0$. Write $w_{(1)} = a(1-c)e_2$ and $w_{(2)} = a(be_1 + ce_2)$. Now, we want to choose the optimal parameter $c \in \mathbb{R}$ such that

$$\sqrt{2}|w_{(2)}|_{euc} + |w_{(1)}|_{euc},$$

that is (4.2.14) without the error term, becomes minimal under the constraint that $w_{(1)} \in \operatorname{span}(e_2)$. The error term in (4.2.14) is independent of c, hence this reduces to finding c such that

$$f_b(c) := |1-c| + \sqrt{2}\sqrt{b^2+c^2} \tag{4.2.16}$$

is minimised. From the definition of the function one can see that for $c \leq 0$ it holds $f_b(c) \geq f_b(0)$. Similarly, if $c \geq 1$, we get that $f_b(c) \geq f_b(1)$. Hence we only need to minimise for $c \in (0,1)$, in which f_b is continuously differentiable and check the boundary values. We have that

$$f_b(0) = 1 + \sqrt{2}|b|, f_b(1) = \sqrt{2(1+b^2)}.$$

Now the derivative for $c \in (0,1)$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}c} f_b(c) = -1 + \frac{\sqrt{2}c}{\sqrt{b^2+c^2}}.$$

The derivative can only vanish if $\sqrt{2}c_* = \sqrt{b^2 + c_*^2} \iff c_* = |b|$. Now, if $|b| \le 1$, we get that $c_* \in [0,1]$ and

$$f_b(c_*) = 1 - |b| + \sqrt{2(b^2 + b^2)} = 1 + |b|.$$

Then, we get that $f_b(0), f_b(1) \ge f_b(c_*)$. If $|b| \ge 1$, we only need to check the boundary values and get that $f_b(0) \ge f_b(1)$. Hence, for $|b| \ge 1$ the optimal value is $\sqrt{2(b^2+1)}$ and for $|b| \le 1$, the optimal value is $1 + |b|$.

Step 4. Description of the metric speed of a curve depending on its derivative u. Recall that we chose y to be a Lebesgue point of $\{g = 2\}$. Writing $z - y = u = (u_1, u')$ where $u_1 \in x_1(y) + (-\delta, \delta) \subset \mathbb{R}$ and $u' \in \mathbb{R}^{d-1}$, the previous three steps yield that

$$L_g(\gamma) \ge \begin{cases} |u_1| + |u'|_{euc} - 6\varepsilon|u|_{euc} & \text{if } |u_1| \le |u'|_{euc}, \\ \sqrt{2}|u|_{euc} - 6\varepsilon|u|_{euc} & \text{otherwise.} \end{cases}$$

More precisely, using that $\mathsf{d}_g \ge \boldsymbol{\delta}_{N_g}$, we get that for u as above, it holds

$$\mathsf{d}_g(y, y+u) \ge \begin{cases} |u_1| + |u'|_{euc} - 6\sigma(y,u_1)|u|_{euc} & \text{if } |u_1| \le |u'|_{euc}, \\ \sqrt{2}|u|_{euc} - 6\sigma(y,u_1)|u|_{euc} & \text{otherwise,} \end{cases} \tag{4.2.17}$$

where

$$\sigma(y,u_1) := \inf\{\varepsilon > 0, \left|\frac{1}{|I|}\int_I \theta \,\mathrm{d}x - 2\right| \le \varepsilon, \forall I \subset \mathbb{R} \text{ open}, x_1(y) \in I, \mathcal{L}^1(I) < |u_1|\}. \tag{4.2.18}$$

As $x_1(y)$ is chosen as a Lebesgue point of θ, we get that $\sigma(y,u_1) \to 0$ as $|u_1| \to 0$.

We will next prove an upper bound on $\mathsf{d}_g(y,z)$ by working with the regularisation, i.e., smooth metrics g_ϵ that converge to g almost everywhere and such that $\mathsf{d}_{g_\epsilon} \to \mathsf{d}_g$. We will construct curves $\hat{\gamma}$ that are given by the concatenation of four straight lines, two of which will be arbitrarily short. The remaining two lines will be optimal in the sense of steps 2 and 3. Fix $\epsilon > 0$. For any $\varsigma > 0$ and $g_\varsigma := \rho_\varsigma * g$, it holds $g_\varsigma \le 2\mathrm{Id}$. Hence, for all $\varsigma > 0$, we have that

$$\mathsf{d}_{g_\varsigma}(y,z) \le \sqrt{2}|u|_{euc}.$$

Moreover, we can find a rational $q \in \mathbb{Q} \cap (0,1)$ such that $|q - x_1(y)| \le \epsilon|u|_{euc}$. Fix a $0 < \alpha < \beta < 1$ such that $U := (\alpha,\beta)^{d-1}$ satisfies that $y, z \in (0,1) \times U$. Now for some $\varsigma_\epsilon > 0$ small enough, we have that for $\varsigma \in (0, \varsigma_\epsilon)$, it holds $g_\varsigma = \mathrm{Id}$ on $\{q\} \times U$. If $|u_1| < |u'|_{euc}$, write $b = \frac{u_1}{|u'|_{euc}}$. Then define the curves

$$\begin{aligned}
&\gamma_1 : [0,1] \to \mathbb{R}^d, t \mapsto y + t(q - x_1(y))e_1, \\
&\gamma_2 : [0,1] \to \mathbb{R}^d, t \mapsto y + (q - x_1(y))e_1 + t(1-|b|)u', \\
&\gamma_3 : [0,1] \to \mathbb{R}^d, t \mapsto y + (q - x_1(y))e_1 + (1-|b|)u' - t(q - x_1(y))e_1, \\
&\gamma_4 : [0,1] \to \mathbb{R}^d, t \mapsto y + (1-|b|)u' + t|b|u' + tu_1e_1.
\end{aligned}$$

We then get that

$$L_{g_\varsigma}(\gamma_1), L_{g_\varsigma}(\gamma_3) \leq \sqrt{2}|q - x_1(y)| \leq \sqrt{2}\epsilon|u|_{euc}.$$

Moreover as $g_\varsigma \equiv \mathrm{Id}$ on $\{q\} \times U \supset \gamma_2([0,1])$, we get that

$$L_{g_\varsigma}(\gamma_2) = (1 - |b|)|u'|_{euc}.$$

Finally, we get that

$$L_{g_\varsigma}(\gamma_4) \leq \sqrt{2}\big|bu' + u_1 e_1\big|_{euc} = 2|u_1|.$$

Hence, for all ς small enough, the concatenation $\hat{\gamma}$ of the four curves satisfies

$$\mathsf{d}_{g_\varsigma}(y,z) \leq L_{g_\varsigma}(\hat{\gamma}) \leq 3\epsilon|u|_{euc} + \frac{|u'|_{euc} - |u_1|}{|u'|_{euc}}|u'|_{euc} + 2|u_1| = |u_1| + |u'|_{euc} + 3\epsilon|u|_{euc}. \tag{4.2.19}$$

Hence, using $\mathsf{d}_g = \lim_{\varsigma\to 0} \mathsf{d}_{g_\varsigma}$ and the fact that ϵ was independent of $|u_1|$, sending $\epsilon \to 0$ yields that

$$\mathsf{d}_g(y, y+u) \leq \alpha(u) := \begin{cases} |u_1| + |u'|_{euc} & \text{if } |u_1| \leq |u'|_{euc}, \\ \sqrt{2}|u|_{euc} & \text{otherwise.} \end{cases} \tag{4.2.20}$$

Now considering the curve $\gamma^{y,u} : [0,1] \to \mathbb{R}^d, t \mapsto y + tu$, for $u \in \mathbb{R}^d$, (4.2.20) and (4.2.17) give that

$$0 \leq \lim_{h\to 0} \frac{\alpha(hu) - \mathsf{d}_g(\gamma_0^{y,u}, \gamma_h^{y,u})}{|h|} \leq \lim_{h\to 0} \frac{\sigma(y, hu_1)|hu|_{euc}}{|h|} = \lim_{h\to 0} \sigma(y, hu_1)|u|_{euc} = 0.$$

Hence

$$\lim_{h\to 0} \frac{\mathsf{d}_g(\gamma_0^{y,u}, \gamma_h^{y,u})}{|h|} = \lim_{h\to 0} \frac{\alpha(hu)}{|h|} = \alpha(u).$$

This completes the description of the metric speed of curves through points y that are Lebesgue points of $\{g = 2\}$ depending on the derivative u. This can be seen as the explicit expression of what is called the "derivative" of the distance function in [52].

For any point $y \in (0,1)^d$ such that $x_1(y) \in O$, we have that $g \equiv \mathrm{Id}$ in a neighbourhood of y. Let $\gamma : [0,1] \to M$ be an absolutely continuous curve through y and fix $t \in [0,1]$ such that $\gamma_t = y$. Suppose that γ is differentiable at time t. The metric speed $|\dot{\gamma}_t|$ of γ at time t depends only on the values of γ at an arbitrarily small neighbourhood $I \subset [0,1]$ of t. The continuity of γ allows to assume $\gamma(I) \subset O \times \mathbb{R}^{d-1}$. As g is constant, and hence smooth, in $O \times \mathbb{R}^{d-1}$, the results on smooth Riemannian metrics (see also [25, Proposition 4.10]) show that in this case $|\dot{\gamma}_t| = |\dot{\gamma}_t|_{g(y)} = |\dot{\gamma}_t|_{euc}$.

Similarly, for every $y \in ([0,1]^d)^c$, we have that $g \equiv 2\mathrm{Id}$ in a neighbourhood of y. Hence, for every absolutely continuous curve γ through y with t as above, we have that $|\dot\gamma_t| = |\dot\gamma_t|_{g(y)} = \sqrt{2}|\dot\gamma_t|_{euc}$.

Now note that for $\mathcal{L}^1$-almost all $x \in \mathbb{R}$, it holds that $x \in O$ or x is a Lebesgue point of $\{\theta = 2\}$. Write $N^{(2)} \subset \mathbb{R}$ for the null set of points satisfying neither of that and define the $\mathcal{L}^d$-null set $N_2 := N^{(2)} \times \mathbb{R}^{d-1} \cup (\partial(0,1)^d)$. In the spirit of Proposition 4.1.3, we have now for almost all points y, more precisely all points $y \in \mathbb{R}^d \setminus N_2$ characterised the metric speed of a curve through the point y depending on its derivative at y. In particular, we have shown that for almost every y, we have that if an absolutely continuous curve $\gamma : [0,1] \to \mathbb{R}^d$ passes through y (say at time $t \in [0,1]$) and is differentiable at t, then the metric speed $|\dot\gamma_t|$ exists and is given by $|\dot\gamma_t|_{euc}$ if $y \in (0,1)^d$ and $x_1(y) \in O$, $\alpha(\dot\gamma_t)$ if $y \in (0,1)^d$ and $x_1(y) \in \mathbb{R} \setminus (N^{(2)} \cup O)$, and $\sqrt{2}|\dot\gamma_t|_{euc}$ if $y \in ([0,1]^d)^c$.

Step 5. Example of functions that shows that the Cheeger energy does not satisfy the parallelogram identity.

This is the final step of the argument. Consider the smooth functions $f_1, f_2, f_3, f_4 : M = \mathbb{R}^d \to \mathbb{R}$ given by

$$f_1 : x \mapsto x_1, f_2 : x \mapsto x_2, f_3 := f_1 + f_2, \text{ and } f_4 := f_1 - f_2.$$

Now again pick a point $y \in (0,1)^d$ such that $x_1(y)$ is a Lebesgue point of $\{\theta = 2\}$. Pick a vector $0 \neq u \in \mathbb{R}^d$. Denote by u_2 the second component of u. Now, we may compute

$$\partial_u f_1 = u_1, \partial_u f_2 = u_2, \partial_u f_3 = u_1 + u_2, \partial_u f_4 = u_1 - u_2.$$

Then

$$\lim_{h \to 0} \frac{|f_i(\gamma_h^{y,u}) - f_i(\gamma_0^{y,u})|}{\mathsf{d}_g(\gamma_0^{y,u}, \gamma_h^{y,u})} = \frac{|\partial_u f_i|(\gamma_0^{y,u})}{\alpha(u)}, \text{ for } 1 \leq i \leq 4.$$

Now if $|u_1| \geq |u'|_{euc}$, we have that $\alpha(u) = \sqrt{2}|u|_{euc}$, hence

$$\frac{|\partial_u f_1|}{\alpha(u)} = \frac{|u_1|}{\sqrt{2}|u|_{euc}} \leq \frac{|u_1|}{\sqrt{2}|u_1|} = \frac{1}{\sqrt{2}}, \tag{4.2.21}$$

$$\frac{|\partial_u f_2|}{\alpha(u)} = \frac{|u_2|}{\sqrt{2}|u|_{euc}} \leq \frac{|u_2|}{\sqrt{2}|u_2|} = \frac{1}{\sqrt{2}}, \tag{4.2.22}$$

$$\frac{|\partial_u f_3|}{\alpha(u)} = \frac{|u_1 + u_2|}{\sqrt{2}|u|_{euc}} \leq \frac{|u_1| + |u_2|}{\sqrt{2(u_1^2 + u_2^2)}} \leq 1, \tag{4.2.23}$$

$$\frac{|\partial_u f_4|}{\alpha(u)} = \frac{|u_1 - u_2|}{\sqrt{2}|u|_{euc}} \leq \frac{|u_1| + |u_2|}{\sqrt{2(u_1^2 + u_2^2)}} \leq 1, \tag{4.2.24}$$

Note that (4.2.23) and (4.2.24) follow from the fact that the quadratic mean is greater than or equal to the arithmetic mean. Moreover, we note that the equality in (4.2.21) can be achieved by choosing $u = u_{(1)} := e_1$.

If $|u_1| < |u'|_{euc}$, we have that $\alpha(u) = |u_1| + |u'|_{euc}$, hence

$$\frac{|\partial_u f_1|}{\alpha(u)} = \frac{|u_1|}{|u_1| + |u'|_{euc}} \leq \frac{|u_1|}{2|u_1| + (|u'|_{euc} - |u_1|)} \leq \frac{1}{2}, \tag{4.2.25}$$

$$\frac{|\partial_u f_2|}{\alpha(u)} = \frac{|u_2|}{|u_1| + |u'|_{euc}} \leq \frac{|u_2|}{|u_2|} = 1, \tag{4.2.26}$$

$$\frac{|\partial_u f_3|}{\alpha(u)} = \frac{|u_1 + u_2|}{|u_1| + |u'|_{euc}} \leq \frac{|u_1| + |u_2|}{|u_1| + |u_2|} = 1, \tag{4.2.27}$$

$$\frac{|\partial_u f_4|}{\alpha(u)} = \frac{|u_1 - u_2|}{|u_1| + |u'|_{euc}} \leq \frac{|u_1| + |u_2|}{|u_1| + |u_2|} = 1, \tag{4.2.28}$$

Here, in (4.2.25), we used that $|u_1| < |u'|_{euc}$ which implies that $|u'|_{euc} - |u_1| > 0$. In (4.2.26), (4.2.27), and (4.2.28), one can see that $u = u_{(2)} := e_2$, $u = u_{(3)} := e_1 + e_2$, and $u = u_{(4)} := e_1 - e_2$ yield equality. We now define the functions $G_1, G_2, G_3, G_4 : \mathbb{R}^d \to [0, \infty]$ via

$$\begin{aligned} G_1 &:= \frac{1}{\sqrt{2}} + \frac{\sqrt{2} - 1}{\sqrt{2}} \mathbb{1}_{O \times (0,1)^{d-1}}, \\ G_2 &= \frac{1}{\sqrt{2}} + \frac{\sqrt{2} - 1}{\sqrt{2}} \mathbb{1}_{(0,1)^d}, \\ G_3 &= G_4 = 1 + (\sqrt{2} - 1) \mathbb{1}_{O \times (0,1)^{d-1}}. \end{aligned} \tag{4.2.29}$$

To summarise the above definitions more easily, note that for $i = 1, \ldots, 4$, we have that in $([0,1]^d)^c \cup (O \times (0,1)^{d-1})$ it holds $G_i = |\nabla_g f_i|_g$ and in $((0,1) \setminus O) \times (0,1)^{d-1}$ we have that G_i equals the respective upper bound established in (4.2.21)-(4.2.28).

Proposition 4.2.3. *For $1 \leq i \leq 4$, it holds that G_i is a weak upper gradient of f_i.*

Proof. Let $\gamma : [0,1] \to M$ be absolutely continuous and transversal to a $\mathcal{L}^d$-null set $N \supset N_g \cup N_2$. As $f_i \in C^1$, we may apply the chain rule. We have that for almost every $t \in [0,1]$, it holds that $\gamma_t \in ([0,1]^d)^c \cup (O \times (0,1)^{d-1})$ or $\gamma_t \in ((0,1) \setminus O) \times (0,1)^{d-1}$ and t is a differentiability point of γ. In the first case, we get that

$$\left| \frac{\mathrm{d}}{\mathrm{d}s} f_i \circ \gamma \Big|_{s=t} \right| = |\partial_{\dot{\gamma}_t} f_i(\gamma_t)| = |\langle \nabla_g f_i(\gamma_t), \dot{\gamma}_t \rangle_g| \leq |\nabla_g f_i|_g(\gamma_t) |\dot{\gamma}_t|_g = G_i(\gamma_t)|\dot{\gamma}_t|. \tag{4.2.30}$$

In the second case, we get that either $\dot{\gamma}_t = 0$, in which case $\frac{\mathrm{d}}{\mathrm{d}s} f_i \circ \gamma \big|_{s=t} = 0$ or

$$\left| \frac{\mathrm{d}}{\mathrm{d}s} f_i \circ \gamma \Big|_{s=t} \right| = |\partial_{\dot{\gamma}_t} f_i(\gamma_t)| = \frac{|\partial_{\dot{\gamma}_t} f_i(\gamma_t)|}{\alpha(\dot{\gamma}_t)} \alpha(\dot{\gamma}_t) \leq G_i(\gamma_t)|\dot{\gamma}_t|. \tag{4.2.31}$$

Hence, for any transversal curve, (4.2.30) and (4.2.31) together yield that

$$|f_i(\gamma_1) - f_i(\gamma_0)| \leq \int_0^1 |\partial_{\dot{\gamma}_t} f_i(\gamma_t)| \,\mathrm{d}t \leq \int_0^1 G_i(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t, \tag{4.2.32}$$

where in the last step we used (4.2.20) and (4.2.21)-(4.2.28). Now, we can conclude with Lemma 4.1.5. □

We next want to see that these are indeed the minimal weak upper gradients. For that we need the following result from [102, Lemma 4.23].

Lemma 4.2.4. *Let $k \in L^1_{\mathrm{loc}}(\mathbb{R}^d, \mathcal{L}^d)$, $x \in \mathbb{R}^d$, and $r > 0$. For $\delta \in (0,r) \cap \mathbb{Q}$, $v \in B_r^{euc}(0)$, we define*

$$F_{v,\delta,x} : t \to \frac{1}{\mathcal{L}^d(B_\delta(x))} \int_{B_\delta(x)} k(y + tv) \,\mathrm{d}\mathcal{L}^d(y), \quad \forall t,\ |t| < 1.$$

Then for $\mathcal{L}^d$-a.e. $x \in \mathbb{R}^d$ and all $\delta \in (0,r) \cap \mathbb{Q}$, $v \in B_r^{euc}(0)$, we have that $t = 0$ is a Lebesgue point of $F_{v,\delta,x}$.

Proposition 4.2.5. *For $1 \leq i \leq 4$, it holds that G_i is the minimal weak upper gradient of f_i.*

Proof. We will argue similarly to the proof of [102, Proposition 4.24] (here Proposition 3.3.12). Fix an $i \in \{1, 2, 3, 4\}$. We argue by contradiction. Suppose that G_i is not the minimal weak upper gradient of f_i. Then, there exists an $\varepsilon > 0$ such that the set $D_\varepsilon := \{x : |Df_i|_w(x) < G_i(x) - \varepsilon\}$ has positive $\mathcal{L}^d$-measure. Then we can choose a point $x \in D_\varepsilon \cap N_2^c$ such that x is a Lebesgue point of $|Df_i|$, G_i and such that $t = 0$ is a Lebesgue point of

$$\begin{aligned} F_{u_{(i)},\delta,x} &: t \to \frac{1}{\mathcal{L}^d(B_\delta(x))} \int_{B_\delta(x)} |Df_i|_w(y + tu_{(i)}) \,\mathrm{d}\mathcal{L}^d(y), \text{ and} \\ G_{u_{(i)},\delta,x} &: t \to \frac{1}{\mathcal{L}^d(B_\delta(x))} \int_{B_\delta(x)} G_i(y + tu_{(i)}) \,\mathrm{d}\mathcal{L}^d(y), \end{aligned}$$

for all $\delta \in (0,1) \cap \mathbb{Q}$. This is possible by Lemma 4.2.4. Then we may choose $\delta \in (0,1) \cap \mathbb{Q}$ small enough such that

$$\begin{aligned} &\frac{1}{\mathcal{L}^d(B_\delta(x))} \int_{B_\delta(x)} ||Df_i|_w(y) - |Df_i|_w(x)| \,\mathrm{d}\mathcal{L}^d(y) \leq \frac{\varepsilon}{8}, \text{ and} \\ &\frac{1}{\mathcal{L}^d(B_\delta(x))} \int_{B_\delta(x)} |G_i(y) - G_i(x)| \,\mathrm{d}\mathcal{L}^d(y) \leq \frac{\varepsilon}{8}. \end{aligned} \tag{4.2.33}$$

Moreover, we can choose $\tau > 0$ such that

$$\frac{1}{2\tau}\frac{1}{\mathcal{L}^d(B_\delta(x))}\int_{-\tau}^{\tau}\left|\int_{B_\delta(x)}|Df_i|_w(y+tu_{(i)})\,\mathrm{d}\mathcal{L}^d(y)-\int_{B_\delta(x)}|Df_i|_w(y)\,\mathrm{d}\mathcal{L}^d(y)\right|\mathrm{d}t \le \frac{\varepsilon}{8}, \text{ and}$$
$$\frac{1}{2\tau}\frac{1}{\mathcal{L}^d(B_\delta(x))}\int_{-\tau}^{\tau}\left|\int_{B_\delta(x)}G_i(y+tu_{(i)})\,\mathrm{d}\mathcal{L}^d(y)-\int_{B_\delta(x)}G_i(y)\,\mathrm{d}\mathcal{L}^d(y)\right|\mathrm{d}t \le \frac{\varepsilon}{8}. \tag{4.2.34}$$

Then (4.2.33) and (4.2.34) together with the fact that $|Df_i|_w(x) < G_i(x) - \varepsilon$, yields that

$$\frac{1}{2\tau}\frac{1}{\mathcal{L}^d(B_\delta(x))}\int_{-\tau}^{\tau}\int_{B_\delta(x)}G_i(y+tu_{(i)})-|Df_i|_w(y+tu_{(i)})\,\mathrm{d}\mathcal{L}^d(y)\mathrm{d}t \ge \varepsilon-\frac{4\varepsilon}{8}=\frac{\varepsilon}{2}. \tag{4.2.35}$$

We now define the curve $\gamma^{y,i} : [0,1] \to \mathbb{R}^d$ as $\gamma^{y,i} : t \mapsto y + 2\tau(t-\frac{1}{2})u_{(i)}$. Then, we define the test plan

$$\mathrm{d}\boldsymbol{\pi}(\gamma) := \begin{cases} \frac{1}{\mathcal{L}^d(B_\delta(x))}\mathrm{d}\mathcal{L}^d(y) & \text{if } \gamma = \gamma^{y,i}, \text{ for some } y \in B_\delta(x), \\ 0 & \text{otherwise.}\end{cases} \tag{4.2.36}$$

Now from the fact that d_g and d_{euc} are $\sqrt{2}$-equivalent, we get that for each $y \in \mathbb{R}^d$ and each $t \in [0,1]$ the metric speed $|\dot\gamma^{y,i}| \ge \frac{1}{\sqrt{2}}|u_{(i)}|_{euc}$.

$$\begin{aligned}&\int_0^1\int_{C^0([0,1],\mathbb{R}^d)}(G_i(\gamma_t)-|Df_i|_w(\gamma_t))|\dot\gamma_t|\,\mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t\\ &=\frac{1}{\mathcal{L}^d(B_\delta(x))}\int_{-\tau}^{\tau}\int_{B_\delta(x)}(G_i(y+tu_{(i)})-|Df_i|_w(y+tu_{(i)}))|\dot\gamma_t^{y,i}|\,\mathrm{d}\mathcal{L}^d(y)\mathrm{d}t \ge \frac{\tau\varepsilon}{\sqrt{2}}.\end{aligned} \tag{4.2.37}$$

Now we observe that for all points $y \in \mathbb{R}^d$, $t \in [0,1]$ such that $z := y + tu_{(i)} \in (O \times (0,1)^{d-1}) \cup ([0,1]^d)^c$, we have that $g \equiv \mathrm{Id}$ a neighbourhood of z and $\nabla_g f_i(z) = u_{(i)}$. Then by definition we have that $G_i(z)|\dot\gamma_t^{y,i}| = G_i|u_{(i)}|_{euc} = \langle\nabla_{euc}f_i, u_{(i)}\rangle_{euc} = \partial_{u_{(i)}}f_i$. Similarly, we have that for all points $y \in \mathbb{R}^d$, $t \in [0,1]$ such that $z := y + tu_{(i)} \in (N_2 \cup N_g)^c \setminus (O \times (0,1)^{d-1} \cup ([0,1]^d)^c)$ that $\partial_{u_{(i)}}f_i(z) = G_i(z)|\dot\gamma_t^{y,i}|$ (see (4.2.31) and the equality cases of (4.2.21), (4.2.26), (4.2.27), and (4.2.28)).

Hence, we get that for every y such that $\gamma^{y,i}$ is transversal to $N_2 \cup N_g$, it holds

$$f_i(\gamma_1^{y,i}) - f_i(\gamma_0^{y,i}) = \int_0^1 \partial_{u_{(i)}}f_i(\gamma_t^{y,i})\,\mathrm{d}t = \int_0^1 G_i(z)|\dot\gamma_t^{y,i}|\,\mathrm{d}t. \tag{4.2.38}$$

On the other hand, Lemma 4.1.5 together with (4.2.37) and (4.2.38) yields that

$$\begin{aligned}
&\int_0^1 \int_{C^0([0,1],\mathbb{R}^d)} |Df_i|_w(\gamma_t)|\dot{\gamma}_t|\, \mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t \\
&\leq \int_0^1 \int_{C^0([0,1],\mathbb{R}^d)} G_i(\gamma_t)|\dot{\gamma}_t|\, \mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t - \frac{\tau\varepsilon}{\sqrt{2}} \\
&= \int_0^1 \int_{C^0([0,1],\mathbb{R}^d)\cup\{\gamma\perp N_2\cup N_g\}} G_i(\gamma_t)|\dot{\gamma}_t|\, \mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t - \frac{\tau\varepsilon}{\sqrt{2}} \\
&< \int_{C^0([0,1],\mathbb{R}^d)\cup\{\gamma\perp N_2\cup N_g\}} |f_i(\gamma_1^{y,i}) - f_i(\gamma_0^{y,i})|\, \mathrm{d}\boldsymbol{\pi}(\gamma) = \int_{C^0([0,1],\mathbb{R}^d)} |f_i(\gamma_1^{y,i}) - f_i(\gamma_0^{y,i})|\, \mathrm{d}\boldsymbol{\pi}(\gamma).
\end{aligned}$$

This contradicts the definition of a test plan and hence proves the Lemma. □

Now we have got all the ingredients to prove the following:

Proposition 4.2.6. *The metric measure space $(\mathbb{R}^d, \mathsf{d}_g, \mathrm{vol}_g)$ is not infinitesimally Hilbertian.*

Proof. Pick a function $\phi \in C_c^\infty(\mathbb{R}^d)$ such that $\phi \equiv 1$ on $[-1,2]^d$. Consider the functions $\tilde{f}_i := \phi f_i$ for $1 \leq i \leq 4$. Then by the previous observations, we have that

$$\begin{aligned}
|D\tilde{f}_i|_w &= G_i \text{ in } (-1,2)^d \\
|D\tilde{f}_i|_w &= \frac{1}{\sqrt{2}}|\nabla_{euc}\tilde{f}_i|_{euc} \text{ in } ((-1,2)^d)^c.
\end{aligned}$$

More precisely, this gives

$$|D\tilde{f}_i|_w = |\nabla_g \tilde{f}_i|_g \text{ a.e. in } (O\times(0,1)^{d-1}) \cup ((0,1)^d)^c, \tag{4.2.39}$$

and

$$\begin{aligned}
|D\tilde{f}_1|_w &= \frac{1}{\sqrt{2}}, \text{ a.e. in } (0,1)^d \setminus O\times(0,1)^{d-1}, \\
|D\tilde{f}_2|_w &= |D\tilde{f}_3|_w = |D\tilde{f}_4|_w = 1 \text{ a.e. in } (0,1)^d \setminus (O\times(0,1)^{d-1}).
\end{aligned}$$

Then it immediately follows that

$$|D\tilde{f}_3|_w^2 + |D\tilde{f}_4|_w^2 = 2(|D\tilde{f}_1|_w^2 + |D\tilde{f}_2|_w^2) \text{ a.e. in } (O\times(0,1)^{d-1}) \cup ((0,1)^d)^c, \tag{4.2.40}$$

but

$$|D\tilde{f}_3|_w^2 + |D\tilde{f}_4|_w^2 - 2(|D\tilde{f}_1|_w^2 + |D\tilde{f}_2|_w^2) = -1 \text{ a.e. in } (0,1)^d \setminus (O\times(0,1)^{d-1}). \tag{4.2.41}$$

Moreover, we have that $\tilde{f}_3 = \tilde{f}_1 + \tilde{f}_2$ and $\tilde{f}_4 = \tilde{f}_1 - \tilde{f}_2$ and all four functions have finite Cheeger energy. But as $\mathcal{L}^d((0,1)^d \setminus O \times (0,1)^{d-1}) \geq 1 - 2\kappa > 0$ and the fact that vol_g and $\mathcal{L}^d$ are $2^{\frac{d}{2}}$-equivalent, we get that

$$\mathrm{Ch}(\tilde{f}_3) + \mathrm{Ch}(\tilde{f}_4) \neq 2(\mathrm{Ch}(\tilde{f}_1) + \mathrm{Ch}(\tilde{f}_2)), \tag{4.2.42}$$

which proves that the metric measure space $(\mathbb{R}^d, \mathsf{d}_g, \mathrm{vol}_g)$ is not infinitesimally Hilbertian.

□

4.2.3 An example of a Sobolev Riemannian metric, neither quasi-Riemannian nor infinitesimally Hilbertian

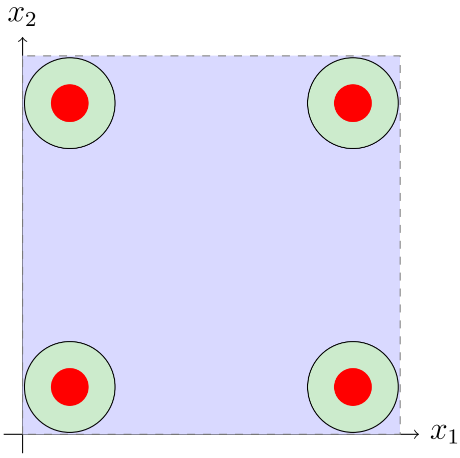
Fix a $d \geq 3$ and set $p := d - 1$. We will now refine the previous examples to construct a Riemannian metric g on $\mathbb{R}^d$ such that $g, g^{-1} \in L^\infty$, $g \in W^{1,p}_{\mathrm{loc}}$ such that (M, d_g, g) is not quasi-Riemannian and such that the metric measure space $(M, \mathsf{d}_g, \mathrm{vol}_g)$ is not infinitesimally Hilbertian.

Intuition in three dimensions

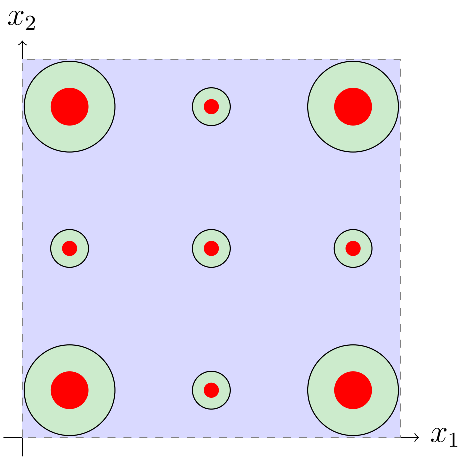
Before giving the general construction, we like to give some intuition in three dimensions, that explains how we can refine the previous example to a Sobolev Riemannian metric. The key to the previous example was to have (at least) one coordinate direction in which we could always travel cheaply, i.e., the dense set of lines orthogonal to the first coordinate axis on which the metric was small induced the non-Hilbertian behaviour.

In this section, we will construct Riemannian metrics with Sobolev regularity that rely on the same idea, here demonstrated in three dimensions. The metric g is going to be defined on $\mathbb{R}^3$ and is of the form $g(x_1, x_2, x_3) = \theta(x_1, x_2)\mathrm{Id}$, thus the metric depends only on the first two coordinates. We inductively construct $\theta : \mathbb{R}^2 \to [1, 2]$ as follows: As a starting point, we set $\theta_0 \equiv 2$. In the first iteration, we pick radii $r_1 > \rho_1 > 0$ and a finite set of points D_1 that are pairwise more than $2r_1$ away from each other.

We now define the smooth function θ_1 to be 1 inside the $B_{\rho_1}(D_1)$. Outside the $B_{r_1}(D_1)$, we want θ_1 to still equal 2, so $\theta_1^{-1}((1,2)) \subset B_{r_1} \setminus B_{\rho_1}(D_1)$.



In the second iteration, we pick another set $D_2 \subset \mathbb{R}^2 \setminus B_{r_1}(D_1)$ and radii $r_2 > \rho_2 > 0$ small enough such that $B_{r_1}(D_1) \cap B_{r_2}(D_2) = \emptyset$. We then obtain the smooth function $\theta_2 : \mathbb{R}^2 \to [1, 2]$ by modifying θ_1 inside $B_{r_2}(D_2)$ such that $\theta_2 \equiv 1$ on $B_{\rho_2}(D_2)$ (and $B_{\rho_1}(D_1)$) and $\theta_2^{-1}((1, 2)) \subset (B_{r_1} \setminus B_{\rho_1}(D_1)) \cup (B_{r_2} \setminus B_{\rho_2}(D_2))$.



Iterating such a process and choosing r_m, ρ_m, and D_m appropriately we can obtain a sequence of smooth functions θ_m that converge in $W^{1,2}_{\mathrm{loc}}$ to a function $\theta : \mathbb{R}^2 \to [1, 2]$ with the following properties:

- $\theta \equiv 2$ on a set $S \subset \mathbb{R}^2$ of positive Lebesgue measure,
- $\theta \equiv 1$ on an open set containing $D(= \bigcup_{m\in\mathbb{N}} D_m)$ that is dense in S.

Recalling that we define $g(x_1, x_2, x_3) := \theta(x_1, x_2)\mathrm{Id}$, we get that along the set of lines $D \times \mathbb{R}$, the metric g equals Id. These lines are dense in the set $S \times \mathbb{R}$, where $g \equiv 2\mathrm{Id}$. This will force quasi-Riemannianity and infinitesimal Hilbertianity to fail.

Note that as θ is a function on $\mathbb{R}^2$, $W^{1,2}_{\mathrm{loc}}$-regularity is the best regularity θ can have without being continuous and continuity would imply infinitesimal Hilbertianity (Section 3.3.2).

Construction

Define $\omega_{d-1} := \mathcal{L}^{d-1}(B_1^{\mathbb{R}^{d-1}}(0))$. We start with the following preparatory result:

Lemma 4.2.7. *For each $R > 0$ and $\varepsilon > 0$, there exists an $r > 0$ and a function $f \in C_c^{0,1}(B_R^{\mathbb{R}^{d-1}}(0), [0,1])$ with $f(B_r(0)) = \{1\}$ and such that*

$$\int_{\mathbb{R}^{d-1}} |\nabla_{euc} f|^{d-1}\,\mathrm{d}x < \varepsilon. \tag{4.2.43}$$

Proof. This proof uses the idea of the proof of [58, Theorem 4.16]. For each $\rho > 0$ consider the function $\eta_\rho \in C_c^{0,1}(B_\rho^{\mathbb{R}^{d-1}}(0), [0,1])$ defined by

$$\eta_\rho(y) := \begin{cases} 1 & \text{if } y \in B_{\rho/4}(0), \\ 1 - \frac{4(|y| - \frac{\rho}{4})}{\rho} & \text{if } y \in B_{\rho/2}(0) \setminus B_{\rho/4}(0) \\ 0 & \text{if } y \in B^c_{\rho/2}(0). \end{cases} \tag{4.2.44}$$

We can estimate that

$$\int_{\mathbb{R}^{d-1}} |\nabla_{euc} \eta_\rho|^{d-1}\,\mathrm{d}x \leq \int_{B_{\rho/2}(0)} \frac{4^{d-1}}{\rho^{d-1}} \leq 2^{d-1}\omega_{d-1}. \tag{4.2.45}$$

Define

$$S_j := \sum_{k=1}^{j} \frac{1}{k}.$$

Now for each $j \in \mathbb{N}$ define the function $f_j \in C_c^{0,1}(B_R(0), [0,1])$ as

$$f_j := \frac{1}{S_j} \sum_{k=1}^{j} \frac{\eta_{R/8^k}}{k}. \tag{4.2.46}$$

Then the construction of η_ρ implies that on $B_{R/8^{j+1}}(0)$ we have that $f_j \equiv 1$. Moreover, we have that $\operatorname{supp} \nabla_{euc}\eta_{R/8^k} = \overline{B_{R/(2\cdot 8^k)}(0)} \setminus B_{R/(4\cdot 8^k)}(0)$, hence for any $k \neq k' \in \mathbb{N}$, we get that $\operatorname{supp} \nabla_{euc}\eta_{R/8^k} \cap \operatorname{supp} \nabla_{euc}\eta_{R/8^{k'}} = \emptyset$. We may now estimate

$$\int_{\mathbb{R}^{d-1}} |\nabla_{euc} f_j|^{d-1}\,\mathrm{d}x = \frac{1}{S_j^{d-1}} \sum_{k=1}^{j} \frac{1}{k^{d-1}} \int_{\mathbb{R}^{d-1}} |\nabla_{euc}\eta_{R/8^k}|^{d-1}\,\mathrm{d}x \leq \frac{2^{d-1}\omega_{d-1}}{S_j^{d-1}} \sum_{k=1}^{j} \frac{1}{k^{d-1}}.$$

Now as $d-1 > 1$, we get that $\lim_{j\to\infty} \frac{2^{d-1}\omega_{d-1}}{S_j^{d-1}} \sum_{k=1}^{j} \frac{1}{k^{d-1}} = 0$. Hence, we can find a $j \in \mathbb{N}$ such that $\int_{\mathbb{R}^{d-1}} |Df_j|^{d-1}\,\mathrm{d}x < \varepsilon$. Then setting $r := \frac{R}{8^{j+1}}$ finishes the proof. $\square$

Pick a decreasing sequence $(R_m)_{m\in\mathbb{N}} \subset (0,1)$ such that $R_0 = \frac{1}{2}$ and for $m \geq 1$ it holds

$$R_m \leq \frac{1}{2^{m+3}}, \tag{4.2.47}$$

$$\sum_{m\in\mathbb{N}} 2^{(m+1)d+4} R_m < \frac{1}{4(1+\omega_{d-1})}. \tag{4.2.48}$$

We inductively define the finite sets $D_m \subset (0,1)^{d-1}$, open sets $O_m \subset (-1,2)^{d-1}$, a decreasing null sequence $(r_m)_{m\in\mathbb{N}} \subset (0,1)$ and functions $\theta_m \in C^\infty(\mathbb{R}^{d-1})$ as follows:

$$D_0 := O_0 := \emptyset,\ \theta_0 \equiv 2 \in C^\infty(\mathbb{R}^{d-1}), r_0 := \frac{1}{4}.$$

Now for $m \geq 1$, define

$$D_m := \Big(\frac{1}{2^{m+1}}\mathbb{Z}^{d-1} \cap (0,1)^{d-1}\Big) \setminus \overline{\big(\bigcup_{j\leq m-1} O_j\big)}, \tag{4.2.49}$$

$$r_m := \min\Big(R_m, \frac{1}{2}\mathrm{dist}_{euc}(D_m, \bigcup_{j\leq m-1} O_j), \frac{r_{m-1}}{2}\Big), \tag{4.2.50}$$

$$O_m := \bigcup_{x\in D_m} B_{r_m}(x). \tag{4.2.51}$$

Note that (4.2.50) together with (4.2.51) yield that

$$\overline{O_m} \cap \overline{O_j} = \emptyset \text{ for all } j < m. \tag{4.2.52}$$

Moreover, note that from $D_m \subset \frac{1}{2^{m+1}}\mathbb{Z}^{d-1}$, together with (4.2.47) and (4.2.50), we have that for $x, x' \in D_m$, $x \neq x'$, it holds

$$\overline{B_{r_m}(x)} \cap \overline{B_{r_m}(x')} = \emptyset. \tag{4.2.53}$$

Using Lemma 4.2.7, we can for each $m \in \mathbb{N}$ find a function $\eta_m \in C_c^{0,1}(\mathbb{R}^{d-1}, [0,1])$ such that there exists an $\rho_m \in (0, r_m/5)$ with $\eta_m|_{B_{\rho_m}(0)} \equiv 1$, $\eta_m|_{\mathbb{R}^{d-1}\setminus B_{r_m/5}(0)} \equiv 2$ and

$$2^{(m+1)(d+1)} \int_{\mathbb{R}^{d-1}} |D\eta_m|^{d-1}\,\mathrm{d}x \leq 2^{-m}. \tag{4.2.54}$$

Define the function $\psi_m \in C^\infty(\mathbb{R}^{d-1})$ via

$$\psi_m(y) := \begin{cases} 2 & \text{if } y \in O_m^c, \\ \eta_m\big(y-x\big) & \text{if } y \in B_{r_m}(x) \text{ for some } x \in D_m. \end{cases} \tag{4.2.55}$$

By (4.2.51) and (4.2.53), we get that ψ_m is well-defined. Now define

$$\theta_m := \min(\theta_{m-1}, \psi_m). \tag{4.2.56}$$

We observe that by (4.2.48), it holds

$$\begin{aligned} \mathcal{L}^{d-1}\big(\bigcup_{m\in\mathbb{N}} \overline{O_m}\big) &\leq \sum_{m=1}^{\infty} \#\big(\frac{1}{2^{m+1}}\mathbb{Z}^{d-1} \cap (0,1)^{d-1}\big)\mathcal{L}^{d-1}(B_{r_m}(0)) \\ &\leq \omega_{d-1}\sum_{m=1}^{\infty} 2^{(m+1)(d-1)+1} R_m^{d-1} \leq \frac{1}{4}. \end{aligned} \tag{4.2.57}$$

Define the set

$$S := (0,1)^{d-1} \setminus \bigcup_{m\in\mathbb{N}} \overline{O_m}. \tag{4.2.58}$$

Lemma 4.2.8. $\bigcup_{m\in\mathbb{N}} D_m$ *is dense in* S.

Proof. Assume this was not the case. Then there exists a $y \in S$ and an $r > 0$ such that $B_r(y) \cap \bigcup_{m\in\mathbb{N}} D_m = \emptyset$, where $B_r(y) = B_r^{\mathbb{R}^{d-1}}(y)$. By the monotonicity of the sequence R_m, there exists a $j \in \mathbb{N}$ such that

$$R_{j'} \leq \frac{r}{3}, \text{ for } j' \geq j. \tag{4.2.59}$$

Moreover, as by definition $\bigcup_{k\leq j} \overline{O_k} \not\ni y$ is compact, there exists an $r' \in (0, \frac{r}{3})$ such that

$$B_{r'}(y) \cap \bigcup_{k\leq j} \overline{O_k} = \emptyset. \tag{4.2.60}$$

Now as $(\bigcup_{l\in\mathbb{N}} \frac{1}{2^{l+1}}\mathbb{Z}^{d-1} \setminus \bigcup_{l\leq j} \frac{1}{2^{l+1}}\mathbb{Z}^{d-1}) \cap (0,1)^{d-1}$ is dense in $(0,1)^{d-1}$, can find $j < l \in \mathbb{N}$, $z \in \frac{1}{2^{l+1}}\mathbb{Z}^{d-1} \cap (0,1)^{d-1}$, such that $z \in B_{r'}(y)$. Then by our assumption, we have that $z \notin \bigcup_{m\in\mathbb{N}} D_m$, hence $z \notin D_l$. Considering the l-th step of the above construction and (4.2.49), we get that $z \notin D_l$ implies

$$z \in \bigcup_{k<l} \overline{O_k}.$$

But then (4.2.60) implies that

$$z \in \bigcup_{j<k<l} \overline{O_k}.$$

Hence there exists a $j < k < l$ such that $z \in \overline{O_k}$, which together with (4.2.51) implies that $z \in B_{r_k}(x)$ for some $x \in D_k$. But then (4.2.59) and (4.2.60) give that

$$|y - x|_{euc} \leq |z - y|_{euc} + |z - x|_{euc} \leq r_k + r' \leq R_j + r' < \frac{2r}{3}.$$

But then $x \in B_r(y) \cap D_k$, which gives a contradiction. □

Now, we will prove the following useful property:

Lemma 4.2.9. *For each point* $y \in \mathbb{R}^{d-1} \setminus \bigcup_{m\in\mathbb{N}} \overline{O_m}$*, we have that*

$$\lim_{r\to 0} \frac{\sum_{m\in\mathbb{N}} r_m \#\{x \in D_m : \overline{B_{r_m/5}(x)} \cap \overline{B_r(y)} \neq \emptyset\}}{r} = 0 \text{ and} \tag{4.2.61}$$

$$\lim_{r\to 0} \frac{\sum_{m\in\mathbb{N}} r_m^{d-1} \#\{x \in D_m : \overline{B_{r_m/5}(x)} \cap \overline{B_r(y)} \neq \emptyset\}}{r^{d-1}} = 0. \tag{4.2.62}$$

Proof. First, denote

$$\zeta_m(y, r) := \#\{x \in D_m : \overline{B_{r_m/5}(x)} \cap \overline{B_r(y)} \neq \emptyset\}. \tag{4.2.63}$$

Fix an $m \in \mathbb{N}$. As $y \notin \overline{O_m}$, we get that for each $x \in D_m$, it holds $|x-y|_{euc} > r_m$, hence

$$\mathrm{dist}_{euc}(y, \overline{B_{r_m/5}(x)}) > \frac{4r_m}{5}. \tag{4.2.64}$$

Thus, if $r \le \frac{4r_m}{5}$, it follows that $\zeta_m(y,r) = 0$. It now suffices to consider those m such that $r_m \le \frac{5}{4}r$. In that case, we get

$$\zeta_m(y,r) \le \#\{\frac{1}{2^{m+1}}\mathbb{Z}^{d-1} \cap \overline{B_{r+\frac{r_m}{5}}(y)} \ne \emptyset\} \le C(d) 2^{(d-1)(m+1)} \frac{5^{d-1}}{4^{d-1}} r^{d-1},$$

where $C(d)$ is a constant only depending on d. Then it follows that

$$\begin{aligned} \frac{\sum_{m\in\mathbb{N}} r_m^{d-1} \#\{x \in D_m : \overline{B_{r_m/5}(x)} \cap \overline{B_r(y)} \ne \emptyset\}}{r^{d-1}} &\le C(d) \sum_{m\colon 4r_m \le 5r} 2^{(d-1)(m+1)} r_m^{d-1} \\ &\le C(d) \sum_{m\colon 4r_m \le 5r} 2^{(d-1)(m+1)} r_m. \end{aligned} \tag{4.2.65}$$

Now (4.2.48) implies that the right-hand-side of (4.2.65) converges to 0 as $r \to 0$, which implies (4.2.62). Moreover,

$$\frac{\sum_{m\in\mathbb{N}} r_m \#\{x \in D_m : \overline{B_{r_m/5}(x)} \cap \overline{B_r(y)} \ne \emptyset\}}{r} \le r^{d-2} C(d) \sum_{m\colon 4r_m \le 5r} 2^{(d-1)(m+1)} r_m, \tag{4.2.66}$$

which using (4.2.48) once again yields (4.2.61). □

Proposition 4.2.10. *The functions θ_m converge to a function θ in $W^{1,d-1}_{\mathrm{loc}}(\mathbb{R}^{d-1})$ as $m \to \infty$.*

Proof. We first note that for all m, it holds $\theta_m \equiv 2$ outside $(-1,2)^{d-1}$, hence, it suffices to consider its behaviour inside $(-1,2)^{d-1}$. Note that (4.2.55) implies that $\psi_m^{-1}([1,2)) \subset O_m$. Then (4.2.52) together with (4.2.56) yields that for any $m' < m$ it holds

$$\theta_m = \theta_{m'} + \sum_{j=m'+1}^{m} (\psi_j - 2). \tag{4.2.67}$$

Then, as $(\psi_j - 2) \in C_c^\infty((-1,2)^{d-1})$, the Poincaré inequality together with (4.2.53) and 4.2.54 yields that

$$\begin{aligned} \|(\psi_j - 2)\|^{d-1}_{W^{1,d-1}(\mathbb{R}^{d-1})} &\le C \int_{(-1,2)^{d-1}} |\nabla_{euc}\psi_j|^{d-1} \,\mathrm{d}\mathcal{L}^{d-1} \\ &= C|D_j| \int_{B_{r_j}(0)} |\nabla_{euc}\eta_j|^{d-1}(x) \,\mathrm{d}\mathcal{L}^{d-1}(x) \\ &\le C 2^{(j+1)(d-1)} 2^{-j(d+1)} \le C 2^{-j}. \end{aligned} \tag{4.2.68}$$

Hence, for $m' \leq m$, we have that

$$\|\nabla_{euc}\theta_m - \nabla_{euc}\theta_{m'}\|^{d-1}_{L^{d-1}((-1,2)^{d-1})} \leq C \sum_{j=m'+1}^{m} 2^{-j}.$$

Note that the right hand side goes to 0 as $m' \to \infty$, thus, $(\nabla_{euc}\theta_m)_m$ is a null-sequence in $L^{d-1}(\mathbb{R}^{d-1})$. As for all m one has that $\theta_m - 2 \in C_c^\infty((-1,2)^{d-1}, \mathbb{R})$, the Poincaré inequality yields that θ_m converges to a function θ in $W^{1,d-1}_{\text{loc}}(\mathbb{R}^{d-1})$. ☐

Our inductive construction also implies that θ_m converges pointwise and is bounded in L^∞, hence, we can describe θ explicitly: For all $y \in \mathbb{R}^{d-1} \setminus \bigcup_{m\in\mathbb{N}} \overline{O_m}$, we have that $\theta_m(y) = 2$ for all $m \in \mathbb{N}$. For all $y \in \bigcup_{m\in\mathbb{N}} \overline{O_m}$, there exists a m^* such that $y \in \overline{O_{m^*}}$. Then for any $m \geq m^*$, (4.2.52) guarantees that $\theta_m(y) = \theta_{m^*}(y)$. Now as by the dominated convergence theorem, θ is the pointwise limit of the θ_m, we can conclude that

$$1 \leq \theta \leq 2 \text{ on } \mathbb{R}^{d-1}, \tag{4.2.69}$$

$$\theta \equiv 2 \text{ on } \mathbb{R}^{d-1} \setminus \bigcup_{m\in\mathbb{N}} \overline{O_m}, \tag{4.2.70}$$

$$\theta^{-1}([1,2)) \subset \bigcup_{m\in\mathbb{N}} \bigcup_{x\in D_m} B_{r_m/5}(x). \tag{4.2.71}$$

Moreover, we get that for each $m \in \mathbb{N}$, $x \in D_m$, it holds

$$\theta \equiv 1 \text{ on } B_{\rho_m/10}(x). \tag{4.2.72}$$

Proposition 4.2.11. *The set $N_\theta \subset \mathbb{R}^{d-1}$ of non-Lebesgue points of θ can be chosen as a subset of the $\mathcal{L}^{d-1}$-null set $\bigcup_{m\in\mathbb{N}} \partial O_m$.*

Proof. For each point $y \in \mathbb{R}^{d-1} \setminus \bigcup_{m\in\mathbb{N}} \overline{O_m}$, we can apply (4.2.62) and get that

$$\begin{aligned}\frac{1}{\mathcal{L}^{d-1}(B_r(y))} \int_{B_r(y)} |\theta(x) - 2| \mathrm{d}\mathcal{L}^{d-1} &\leq \frac{\mathcal{L}^{d-1}(B_r(y) \cap \bigcup_{m\in\mathbb{N}} \bigcup_{x\in D_m} B_{r_m/5}(x))}{\mathcal{L}^{d-1}(B_r(y))} \\ &\leq C \frac{\sum_{m\in\mathbb{N}} r_m^{d-1} \#\{x \in D_m : \overline{B_{r_m/5}(x)} \cap \overline{B_r(y)} \neq \emptyset\}}{r^{d-1}},\end{aligned}$$

where the last term converges to 0 as $r \to 0$. For each point $y \in \bigcup_{m\in\mathbb{N}} O_m$, we have that θ is continuous in a neighbourhood of y, hence y is a Lebesgue point. For each m, we have that ∂O_m is a $\mathcal{L}^{d-1}$-null set, hence so is their countable union. ☐

We are now ready to define the metric g on $M := \mathbb{R}^d$ via

$$g(x', x_d) = \theta(x')\mathrm{Id}, \ x' \in \mathbb{R}^{d-1}, x_d \in \mathbb{R}. \tag{4.2.73}$$

Again (M, g) satisfies Assumption 4.1.1 and $(M, \mathsf{d}_g, \mathrm{vol}_g)$ satisfies (2.2.1).

4.2.3.1 Analysis of quasi-Riemannianity

We now have enough information to prove Theorem 2. Note that by (4.2.57) and (4.2.58), we have that $S \subset (0,1)^{d-1}$ satisfies $\mathcal{L}^{d-1}(S) > 0$ and $\theta = 2$ on S (see Proposition 4.2.11). Then consider $E := S \times (0, \frac{1}{2}) \subset M$. Then $\mathrm{vol}_g(E) \geq \mathcal{L}^d(E) > 0$. For $y \in E$, define the curve $\gamma^y : [0,1] \to (0,1)^{d-1} \times (-2,2), t \mapsto y + te_d$ and define the test plan $\boldsymbol{\pi} \in \mathcal{P}(C([0,1],\mathbb{R}^d))$ via

$$\mathrm{d}\boldsymbol{\pi}(\gamma) := \begin{cases} \frac{1}{\mathcal{L}^d(E)} \mathrm{d}\mathcal{L}^d(y) & \text{if } \gamma = \gamma^y, \text{ for some } y \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Now the strategies from Subsection 4.2.1 together with Lemma 4.2.8 and Proposition 4.2.11 imply that for all $y \in E$, and almost every $t \in [0,1]$, it holds $|\dot{\gamma}_t^y| = 1 < \sqrt{2} = |\dot{\gamma}_t^y|_g$. Hence, $\boldsymbol{\pi}$ is supported on curves for which the metric speed is strictly smaller than the norm of the derivative, for almost every time.

Now we can consider the function $f \in C^1(M)$ given by $f : (x', x_d) \to x_d$. Then f has slope $|Df| = 1$ on $S \times \mathbb{R}$, but $|\nabla_g f|_g < 1$ on $S \times \mathbb{R}$. Finally, f and $\boldsymbol{\pi}$ together yield that $|Df|_w > |\nabla_g f|_g$ on a set of positive measure, which proves Theorem 2.

We will conclude this observation with the following remark.

Remark 4.2.12. *From the Sobolev embedding theorem, we get that for a d-dimensional manifold with a Riemannian metric $g \in W^{1,p}_{\mathrm{loc}}$ for some $p > d$, that g is automatically continuous. Then the results we saw in Subsection 3.3.2, yield that (M, d_g, g) satisfies condition (iii) from Definition 4.1.6 for C^1-functions. Theorem 2 shows that this is no longer given if $d \geq 3$ and $p \leq d-1$ and therewith works towards solving Problem 2 in [53], Chapter 3: spazi metrici quasi-riemanniani: the question what regularity assumption on g is necessary such that the d_g-slope of a Lipschitz function equals the g-norm of the gradient almost everywhere (see also (2.2), Theorem (2.18), and Theorem (6.1) in [51]).*

4.2.3.2 Analysis of infinitesimal Hilbertianity

We will now analyse the infinitesimal structure of the metric measure space $(M, \mathsf{d}_g, \mathrm{vol}_g)$ and prove Theorem 3. Pick a point $y = (y', y_d) \in M$ such that $y' \in S$. We will work in a neighbourhood around y and restrict to local coordinates. Pick a vector $u \in \mathbb{R}^d$. We want to understand the metric speed at time 0 of the curve $\gamma^{y,u} : [0, \xi] \to \mathbb{R}^d, t \mapsto y + tu$, $\xi > 0$. Define the null set

$$N_g := (\partial[0,1]^{d-1} \cup \bigcup_{m \in \mathbb{N}} \partial O_m) \times \mathbb{R} \subset M. \tag{4.2.74}$$

Fix an $\varepsilon > 0$ and find an $r_\varepsilon > 0$ such that for $r \in (0, r_\varepsilon)$, the expressions in (4.2.61) and (4.2.62) are smaller than or equal to ε. Now for some $0 < r < r_\varepsilon$ choose $z \in M$ such that $\mathsf{d}_{euc}(z, y) = \frac{r}{12}$, which using that d_g and d_{euc} are $\sqrt{2}$-equivalent implies that any $\frac{4}{3}$-shortest curve transversal to N_g (i.e. a curve $\gamma \perp N_g$ from y to z, such that $L_g(\gamma) \le \frac{4}{3}\mathsf{d}_g(y, z)$) from y to z lies inside $B_r^{euc}(y)$. Pick such a curve γ and assume it to be 1-d_{euc}-Lipschitz (which can be done if r is small enough, as d_g and d_{euc} are $\sqrt{2}$-equivalent).

Denote

$$\Omega_m := \bigcup_{x \in D_m} B^{\mathbb{R}^{d-1}}_{r_m/5}(x) \subset (-1, 2)^{d-1}. \tag{4.2.75}$$

We first note that by the definition of the metric g it holds that

$$L_g(\gamma) \ge \int_{\gamma^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R})} \sqrt{2}|\dot{\gamma}_t|_{euc}\,\mathrm{d}t + \int_{\gamma^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})} |\dot{\gamma}_t|_{euc}\,\mathrm{d}t. \tag{4.2.76}$$

This is in the spirit of the step 2 in Subsection 4.2.2, with the sole difference that we do not explicitly know $\gamma^{-1}(\{g \equiv \mathrm{Id}\})$, but we can use that $\gamma^{-1}(\{g \equiv \mathrm{Id}\}) \subset \gamma^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})$. Very roughly speaking, we will show that an efficient curve γ mainly moves in the d-th coordinate direction in $\gamma^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})$.

Now observe that

$$\gamma^{-1}\Big(\bigcup_{m\in\mathbb{N}} \overline{\Omega_m} \times \mathbb{R}\Big) = \bigcup_{m\in\mathbb{N}} \gamma^{-1}(\overline{\Omega_m} \times \mathbb{R}).$$

Hence, we can pick a $J \in \mathbb{N}$ such that

$$\mathcal{L}^1\Big(\gamma^{-1}\Big(\bigcup_{m\in\mathbb{N}} \overline{\Omega_m} \times \mathbb{R}\Big)\Big) \le \frac{r\varepsilon}{3} + \mathcal{L}^1\Big(\gamma^{-1}\Big(\bigcup_{m\le J} \overline{\Omega_m} \times \mathbb{R}\Big)\Big). \tag{4.2.77}$$

As γ is 1-d_{euc}-Lipschitz continuous, (4.2.77) implies that γ is very "short" in $\gamma^{-1}(\bigcup_{m>J} \overline{\Omega_m} \times \mathbb{R})$, so, again very roughly speaking, it suffices to study γ on $\gamma^{-1}(\bigcup_{m\le J} \overline{\Omega_m} \times \mathbb{R})$ to understand γ on $\gamma^{-1}(\bigcup_{m\in\mathbb{N}} \overline{\Omega_m} \times \mathbb{R})$.

For any $x \in \bigcup_{m\in\mathbb{N}} D_m$, write $m(x)$ as the natural number such that $x \in D_{m(x)}$. Now the set $\bigcup_{m\le J} D_m$ is finite so we may write it as $\{x_l\}_{l=1}^k \subset \mathbb{R}^{d-1}$ for some $k \in \mathbb{N}$. Write $m_l := m(x_l)$. In the following, we will essentially argue that the most efficient way to get from γ_0 to γ_1 is to pass the set $\overline{B^{\mathbb{R}^{d-1}}_{r_{m_l}/5}(x_l)} \times \mathbb{R}$ at most once for each $l \in \{1, \ldots, k\}$.

We will inductively define Lipschitz-continuous curves $\gamma^{(l)} : [0, 1] \to \mathbb{R}^d$ from y to z for $l = 0, \ldots, k$ as follows: Define $\gamma^{(0)} := \gamma$. Now for $l \ge 1$ define

$$t_l^{\min} := \min(t \in [0, 1] : \gamma_t^{(l-1)} \in \overline{B^{\mathbb{R}^{d-1}}_{r_{m_l}/5}(x_l)} \times \mathbb{R}),$$
$$t_l^{\max} := \max(t \in [0, 1] : \gamma_t^{(l-1)} \in \overline{B^{\mathbb{R}^{d-1}}_{r_{m_l}/5}(x_l)} \times \mathbb{R}).$$

Define

$$w_l := \int_{t_l^{\min}}^{t_l^{\max}} \dot\gamma_t^{(l-1)}\,\mathrm{d}t = \gamma_{t_l^{\max}} - \gamma_{t_l^{\min}} \in \mathbb{R}^d, \tag{4.2.78}$$

and

$$\gamma_t^{(l)} := \begin{cases} \gamma_t^{(l-1)} & \text{if } t \in [0, t_l^{\min}], \\ \gamma_{t_l^{\min}}^{(l-1)} + \frac{t - t_l^{\min}}{t_l^{\max} - t_l^{\min}} w_l & \text{if } t \in [t_l^{\min}, t_l^{\max}], \\ \gamma_t^{(l-1)} & \text{if } t \in [t_l^{\max}, 1]. \end{cases}$$

Claim 1. For each $l = 1, \dots, k$, $\gamma^{(l)}$ satisfies

$$\begin{aligned} &\int_{(\gamma^{(l-1)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R})} \sqrt{2}|\dot\gamma_t^{(l-1)}|_{euc}\,\mathrm{d}t + \int_{(\gamma^{(l-1)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})} |\dot\gamma_t^{(l-1)}|_{euc}\,\mathrm{d}t \\ &\geq \int_{(\gamma^{(l)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R})} \sqrt{2}|\dot\gamma_t^{(l)}|_{euc}\,\mathrm{d}t + \int_{(\gamma^{(l)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})} |\dot\gamma_t^{(l)}|_{euc}\,\mathrm{d}t. \end{aligned} \tag{4.2.79}$$

Proof of claim 1. This can be seen by noticing that

$$\begin{aligned} &(\gamma^{(l)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R}) = (\gamma^{(l-1)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R}) \setminus [t_l^{\min}, t_l^{\max}] \text{ and} \\ &(\gamma^{(l)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R}) = (\gamma^{(l-1)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R}) \cup [t_l^{\min}, t_l^{\max}]. \end{aligned} \tag{4.2.80}$$

Moreover, $\gamma^{(l-1)} = \gamma^{(l)}$ on $[t_l^{\min}, t_l^{\max}]^c$ and, as $\overline{B_{r_{m_l}/5}^{\mathbb{R}^{d-1}}(x_l)} \times \mathbb{R} \subset \mathbb{R}^d$ is convex, it holds that $\gamma^{(l)}([t_l^{\min}, t_l^{\max}]) \subset \overline{B_{r_{m_l}/5}^{\mathbb{R}^{d-1}}(x_l)} \times \mathbb{R}$. Thus, we have that

$$\begin{aligned} &\int_{[t_l^{\min}, t_l^{\max}]^c \cap (\gamma^{(l-1)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R})} \sqrt{2}|\dot\gamma_t^{(l-1)}|_{euc}\,\mathrm{d}t \\ &+ \int_{[t_l^{\min}, t_l^{\max}]^c \cap (\gamma^{(l-1)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})} |\dot\gamma_t^{(l-1)}|_{euc}\,\mathrm{d}t \\ &= \int_{[t_l^{\min}, t_l^{\max}]^c \cap (\gamma^{(l)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R})} \sqrt{2}|\dot\gamma_t^{(l)}|_{euc}\,\mathrm{d}t + \int_{[t_l^{\min}, t_l^{\max}]^c \cap (\gamma^{(l)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})} |\dot\gamma_t^{(l)}|_{euc}\,\mathrm{d}t, \end{aligned} \tag{4.2.81}$$

and

$$\begin{aligned} &\int_{[t_l^{\min}, t_l^{\max}] \cap (\gamma^{(l-1)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R})} \sqrt{2}|\dot\gamma_t^{(l-1)}|_{euc}\,\mathrm{d}t \\ &+ \int_{[t_l^{\min}, t_l^{\max}] \cap (\gamma^{(l-1)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})} |\dot\gamma_t^{(l-1)}|_{euc}\,\mathrm{d}t \\ &\geq \int_{[t_l^{\min}, t_l^{\max}]} |\dot\gamma_t^{(l-1)}|_{euc}\,\mathrm{d}t \geq |w_l|_{euc} = \int_{[t_l^{\min}, t_l^{\max}] \cap (\gamma^{(l)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})} |\dot\gamma_t^{(l)}|_{euc}\,\mathrm{d}t. \end{aligned} \tag{4.2.82}$$

The sum of (4.2.81) and (4.2.82) yields the claim. $\square$

Furthermore, we observe that the observations after (4.2.80) together with $\gamma^{(l)}([t_j^{\min}, t_l^{\max}]) \subset (\bigcup_{m\leq J} \overline{\Omega_m}) \times \mathbb{R} \subset (\bigcup_{m>J} \overline{\Omega_m})^c \times \mathbb{R}$ imply that

$$\begin{aligned}\mathcal{L}^1((\gamma^{(l)})^{-1}((\bigcup_{m>J} \overline{\Omega_m}) \times \mathbb{R})) &= \mathcal{L}^1((\gamma^{(l-1)})^{-1}((\bigcup_{m>J} \overline{\Omega_m}) \times \mathbb{R}) \setminus [t_l^{\min}, t_l^{\max}]) \\ &\leq \mathcal{L}^1((\gamma^{(l-1)})^{-1}((\bigcup_{m>J} \overline{\Omega_m}) \times \mathbb{R})), \end{aligned} \tag{4.2.83}$$

for all $l = 1, \dots, k$. For $l' \leq l$ denote by $I_{l'}^{(l)} := (\gamma^{(l)})^{-1}(\overline{B^{\mathbb{R}^{d-1}}_{r_{m_{l'}}/5}(x_{l'})} \times \mathbb{R})$.

Claim 2. For each $l = 1, \dots, k$, $l' \leq l$, we have that $I_{l'}^{(l)}$ is a closed interval.

Proof of claim 2. We will prove this by induction. For $l = 1$ the claim is true as $(\gamma^{(1)})^{-1}(\overline{B^{\mathbb{R}^{d-1}}_{r_{m_1}/5}(x_1)} \times \mathbb{R}) = [t_1^{\min}, t_1^{\max}]$. Now assume the claim holds true for some $l < k$. Now we have that $(\gamma^{(l+1)})^{-1}(\overline{B^{\mathbb{R}^{d-1}}_{r_{m_{l+1}}/5}(x_{l+1})} \times \mathbb{R}) = [t_{l+1}^{\min}, t_{l+1}^{\max}]$. By the construction of $\gamma^{(l+1)}$, (4.2.52), and (4.2.53), we have that $t_{l+1}^{\min}, t_{l+1}^{\max} \notin I_{l'}^{(l)}$. Hence, as $I_{l'}^{(l)}$ is a closed interval by the induction hypothesis, it holds that $[t_{l+1}^{\min}, t_{l+1}^{\max}] \supset I_{l'}^{(l)}$, or $[t_{l+1}^{\min}, t_{l+1}^{\max}] \cap I_{l'}^{(l)} = \emptyset$. In the first case, we get that $I_{l'}^{(l+1)} = \emptyset$ and in the second one it holds $I_{l'}^{(l+1)} = I_{l'}^{(l)}$. Both are closed intervals, which finishes the induction step and proves the claim. $\square$

Finally, we note that for all $l = 0, \dots, k$, $\gamma^{(l)}$ is 1-d_{euc}-Lipschitz. This is true by assumption for $l = 0$. For $l > 0$ this follows inductively, using that in $\mathbb{R}^d$, $w_l = \gamma^{(l-1)}_{t_l^{\max}} - \gamma^{(l-1)}_{t_l^{\min}}$, hence by the induction hypothesis, $|w_l|_{euc} \leq t_l^{\max} - t_l^{\min}$.

Now we can combine (4.2.76) with (4.2.79) to get that

$$L_g(\gamma) \geq \int_{(\gamma^{(k)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m})^c \times \mathbb{R})} \sqrt{2} |\dot{\gamma}_t^{(k)}|_{euc} \,\mathrm{d}t + \int_{(\gamma^{(k)})^{-1}((\bigcup_{m\in\mathbb{N}} \overline{\Omega_m}) \times \mathbb{R})} |\dot{\gamma}_t^{(k)}|_{euc} \,\mathrm{d}t.$$

Now (4.2.77) and (4.2.83), together with the fact that $\gamma^{(k)}$ is 1-d_{euc}-Lipschitz continuous yield that

$$L_g(\gamma) + r\varepsilon \geq \int_{(\gamma^{(k)})^{-1}((\bigcup_{m\leq J} \overline{\Omega_m})^c \times \mathbb{R})} \sqrt{2} |\dot{\gamma}_t^{(k)}|_{euc} \,\mathrm{d}t + \int_{(\gamma^{(k)})^{-1}((\bigcup_{m\leq J} \overline{\Omega_m}) \times \mathbb{R})} |\dot{\gamma}_t^{(k)}|_{euc} \,\mathrm{d}t. \tag{4.2.84}$$

As in the previous example, we define

$$\begin{aligned} v_{(1)} &:= \int_{(\gamma^{(k)})^{-1}((\bigcup_{m\leq J} \overline{\Omega_m}) \times \mathbb{R})} \dot{\gamma}^{(k)} \,\mathrm{d}t \in \mathbb{R}^d, \\ v_{(2)} &:= \int_{(\gamma^{(k)})^{-1}((\bigcup_{m\leq J} \overline{\Omega_m})^c \times \mathbb{R})} \dot{\gamma}^{(k)} \,\mathrm{d}t \in \mathbb{R}^d. \end{aligned}$$

Then,

$$v_{(1)} + v_{(2)} = z - y.$$

Applying classical facts on the Euclidean space to (4.2.84) yields that

$$L_g(\gamma) + r\varepsilon \geq \sqrt{2}|v_{(2)}|_{euc} + |v_{(1)}|_{euc}. \tag{4.2.85}$$

Let $P^{(d-1)} : \mathbb{R}^d \to \mathbb{R}^d$ be the orthogonal projection mapping $\mathbb{R}^{d-1} \times \mathbb{R} \ni (x', x_d) \mapsto (x', 0)$. Now by Claim 2 together with our choice of r_ε (see (4.2.61)), we get that

$$\begin{aligned}
|P^{(d-1)}v_{(1)}|_{euc} &= \Big|\sum_{l=1}^{k} \int_{(\gamma^{(k)})^{-1}(\overline{B^{\mathbb{R}^{d-1}}_{r_{m_l}/5}(x_l)} \times \mathbb{R})} P^{(d-1)}\dot{\gamma}^{(k)}_t \,\mathrm{d}t\Big|_{euc} \\
&\leq \sum_{l=1}^{k} \Big|\int_{(\gamma^{(k)})^{-1}(\overline{B^{\mathbb{R}^{d-1}}_{r_{m_l}/5}(x_l)} \times \mathbb{R})} P^{(d-1)}\dot{\gamma}^{(k)}_t \,\mathrm{d}t\Big|_{euc} \\
&= \sum_{l=1}^{k} |P^{(d-1)}\gamma^{(k)}(\max I^{(k)}_l) - P^{(d-1)}\gamma^{(k)}(\min I^{(k)}_l)| \\
&\leq \sum_{l=1}^{k} \operatorname{diam}(\overline{B^{\mathbb{R}^{d-1}}_{r_{m_l}/5}(x_l)}) \\
&\leq 2r\frac{\sum_{m\in\mathbb{N}} \frac{r_m}{5}|\{x \in D_m : \overline{B^{\mathbb{R}^{d-1}}_{r_m/5}(x)} \cap \overline{B^{\mathbb{R}^{d-1}}_{r}(y')} \neq \emptyset\}|}{r} \leq 2r\varepsilon,
\end{aligned} \tag{4.2.86}$$

where we wrote $y = (y', y_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Now define

$$w_{(1)} := v_{(1)} - P^{(d-1)}(v_{(1)}) = (v_{(1)})_d e_d \in \operatorname{span}(e_d) \subset \mathbb{R}^d,\ w_{(2)} = v_{(2)} + P^{(d-1)}(v_{(1)}) \in \mathbb{R}^d.$$

Then $w_{(1)} + w_{(2)} = z - y$ and

$$|w_{(1)} - v_{(1)}|_{euc}, |w_{(2)} - v_{(2)}|_{euc} \leq 2r\varepsilon.$$

Hence, with (4.2.85), we get that

$$L_g(\gamma) + 6r\varepsilon \geq \sqrt{2}|w_{(2)}|_{euc} + |w_{(1)}|_{euc}, \tag{4.2.87}$$

for vectors $w_{(1)}, w_{(2)}$ with $w_{(1)} + w_{(2)} = z - y$ and $w_{(1)} \in \operatorname{span}(e_d)$. We want to minimise the right hand side of (4.2.87) under the thereafter mentioned constraints. This will be done similarly as in the previous example (see step 3, from (4.2.14) onwards).

Write again $z - y = u = (u', u_d) \in \mathbb{R}^{d-1} \times \mathbb{R}$. By potentially applying a linear coordinate transform from $SO(d-1)$, we may assume that $u' \in \operatorname{span}(e_1)$. Then we reduce the right-hand-side of (4.2.87) by orthogonally projecting $w_{(1)}$ and $w_{(2)}$ onto $\operatorname{span}(e_1, e_d)$. Now if $u \in \operatorname{span}(e_1)$, we get that $w_{(1)} = 0$ and hence

$$L_g(\gamma) + 6r\varepsilon \geq \sqrt{2}|z - y|_{euc}.$$

Otherwise, we have that $u_d \neq 0$. Then u can be written as $a(be_1 + e_d)$. Then writing $w_{(1)} = a(1-c)e_d$ and $w_{(2)} = a(be_1+ce_d)$ reduces to minimising $f_b(c)$ as defined in (4.2.16). Together with $r = 12|u|_{euc}$, this yields

$$L_g(\gamma) \geq \begin{cases} |u_d| + |u'|_{euc} - 100\varepsilon|u|_{euc} & \text{if } |u_d| \geq |u'|_{euc}, \\ \sqrt{2}|u|_{euc} - 100\varepsilon|u|_{euc} & \text{otherwise.} \end{cases}$$

As before, together with $\mathsf{d}_g \geq \boldsymbol{\delta}_{N_g}$ (see (4.1.2)) this yields

$$\mathsf{d}_g(y, y+u) \geq \begin{cases} |u_d| + |u'|_{euc} - 100\sigma(y,u)|u|_{euc} & \text{if } |u_d| \geq |u'|_{euc}, \\ \sqrt{2}|u|_{euc} - 100\sigma(y,u)|u|_{euc} & \text{otherwise,} \end{cases} \tag{4.2.88}$$

where

$$\sigma(y,u) := \frac{\sum_{m\in\mathbb{N}} \frac{2r_m}{5} |\{x \in D_m : \overline{B^{\mathbb{R}^{d-1}}_{r_m/5}(x)} \cap \overline{B^{\mathbb{R}^{d-1}}_{|u|_{euc}}(y')} \neq \emptyset\}|}{|u|_{euc}} \to 0, \text{ as } |u|_{euc} \to 0,$$

as of Lemma 4.2.9.

We will next prove an upper bound on $\mathsf{d}_g(y,z)$ and work with the regularisation. Fix $\epsilon > 0$. For any $\varsigma > 0$ and $g_\varsigma := \rho_\varsigma * g$, it holds $g_\varsigma \leq 2\mathrm{Id}$. Hence,

$$\mathsf{d}_{g_\varsigma}(y,z) \leq \sqrt{2}|u|_{euc}.$$

By Lemma 4.2.8, we can find an $m \in \mathbb{N}$ and an $x \in D_m$ such that $|P^{(d-1)}(y) - (x,0)| \leq \epsilon|u|_{euc}$. Now for $\varsigma \in (0, \rho_m/20)$, it holds $g_\varsigma = 1$ on $\{x\} \times \mathbb{R}$. If $|u_d| > |u'|_{euc}$, write $b = \frac{|u'|_{euc}}{|u_d|}$ and note that the concatenation of the curves

$$\begin{aligned}
&\gamma_1 : [0,1] \to \mathbb{R}^d, t \mapsto y + t((x,0) - P^{(d-1)}y), \\
&\gamma_2 : [0,1] \to \mathbb{R}^d, t \mapsto y + ((x,0) - P^{(d-1)}y) + t(1-b)u_d e_d, \\
&\gamma_3 : [0,1] \to \mathbb{R}^d, t \mapsto y + ((x,0) - P^{(d-1)}y) + (1-b)u_d e_d - t((x,0) - P^{(d-1)}y), \text{ and} \\
&\gamma_4 : [0,1] \to \mathbb{R}^d, t \mapsto y + (1-b)u_d e_d + t((u',0) + bu_d e_d),
\end{aligned}$$

connects y with z and has g_ς length at most $|u'|_{euc} + |u_d|_{euc} + 3\epsilon$. As ϵ was arbitrary, we get that

$$\mathsf{d}_g(y, y+u) = \lim_{\varsigma\to 0} \mathsf{d}_{g_\varsigma}(y, y+u) \leq \alpha(u) := \begin{cases} |u_d| + |u'|_{euc} & \text{if } |u_d| \geq |u'|_{euc}, \\ \sqrt{2}|u|_{euc} & \text{otherwise.} \end{cases} \tag{4.2.89}$$

Together with Lemma 4.1.3, this gives that for any $y \in S \times \mathbb{R}$, the metric speed of a curve $\gamma : [0,1] \to M$, $t \in [0,1]$ such that $\dot{\gamma}_t =: u$ exists and $\gamma_t = y$ is given by $\alpha(u)$.

Now the argument works similarly as in the previous example. We note that for any $y \in (([0,1]^{d-1})^c \times \mathbb{R}) \cup (\bigcup_{m\in\mathbb{N}} O_m \times \mathbb{R})$ we have that g is continuous in a neighbourhood of y and the metric speed of a curve is given by the g-norm of its derivative, wherever it exists.

Proposition 4.2.13. *The metric measure space* $(\mathbb{R}^d, \mathsf{d}_g, \mathrm{vol}_g)$ *is not infinitesimally Hilbertian.*

We will only provide a limited amount of detail, as the idea of the proof is very similar to the one of Proposition 4.2.6.

Proof. Let $\tilde{\phi} \in C_c^\infty(\mathbb{R}^{d-1})$ such that $\tilde{\phi}((-2,2)^{d-1}) = \{1\}$ and $\tilde{\phi} \equiv 0$ on $((-3,3)^{d-1})^c$. Define $\phi \in C^\infty(M)$ locally via $\phi : \mathbb{R}^{d-1} \times \mathbb{R}, (x', x_d) \mapsto \tilde{\phi}(x')$ Now consider the functions

$$f_1 : x \mapsto \phi(x)x_1, f_2 : x \mapsto \phi(x)|x_d|, f_3 = f_1 + f_2, f_4 = f_1 - f_2.$$

Define the functions G_i for $i = 1, \ldots, 4$ vol_g-almost everywhere (more precisely on $M \backslash N_g$) via

$$G_1(z) := \begin{cases} \frac{1}{\sqrt{2}} & \text{if } z \in S \times \mathbb{R}, \\ |\nabla_g f_1|_g & \text{otherwise.} \end{cases}, G_2(z) := \begin{cases} 1 & \text{if } z \in S \times \mathbb{R}, \\ |\nabla_g f_2|_g & \text{otherwise.} \end{cases}$$
$$G_3(z) := \begin{cases} 1 & \text{if } z \in S \times \mathbb{R}, \\ |\nabla_g f_3|_g & \text{otherwise.} \end{cases}, G_4(z) := \begin{cases} 1 & \text{if } z \in S \times \mathbb{R}, \\ |\nabla_g f_4|_g & \text{otherwise.} \end{cases}$$

As in Proposition 4.2.5, one can show that G_i is the minimal weak upper gradient of f_i for $i = 1, \ldots, 4$. Now from the above definition, one can see that f_i, G_i are in $L^\infty_{\mathrm{loc}}(M)$. From the definition, we have that outside $S \times \mathbb{R}$, we have that $2G_1^2 + 2G_2^2 = G_3^2 + G_4^2$ almost everywhere. Inside $S \times \mathbb{R}$, we have that

$$2G_1^2 + 2G_2^2 - (G_3^2 + G_4^2) = 1 \neq 0. \tag{4.2.90}$$

As in the previous example, this is the main fact that will yield non-infinitesimal Hilbertianity. We only have to navigate around the problem that $G_i \notin L^2(\mathbb{R}^d, \mathrm{vol}_g)$. In a final step define a function $\eta \in C_c^\infty(\mathbb{R})$ such that $\eta(x) = \eta(-x)$ for all $x \in \mathbb{R}$ and such that $\eta \equiv 1$ on $(-1,1)$ and $\eta \equiv 0$ on $(-2,2)^c$. For $n \geq 1$ define η_n via $\eta_n \equiv 1$ on $(-n,n)$, $\eta_n(x) = \eta(|x| - n + 1)$ for $|x| \geq n$ and for $1 \leq i \leq 4$, define the functions $f_{i,n} : \mathbb{R}^d \to \mathbb{R}$ via

$$\begin{aligned} &f_{1,n} : (x_1, \ldots, x_d) = x \mapsto \eta_n(x_d)\phi(x)x_1, \\ &f_{2,n} : (x_1, \ldots, x_d) = x \mapsto \eta_n(x_d)\phi(x)(|x_d| - n + 1), \\ &f_{3,n} := f_{1,n} + f_{2,n}, \ f_{4,n} := f_{1,n} - f_{2,n}. \end{aligned}$$

Then $\operatorname{supp} f_{i,n} \subset [-3,3]^{d-1} \times [-n-1, n+1]$. Now, with similar methods as before, using the fact that the minimal weak upper gradient depends locally on the function, we get that the minimal weak upper gradient $|Df_{i,n}|_w$ of $f_{i,n}$ can be described via

$$|Df_{i,n}|_w(z) := \begin{cases} G_i(z) & \text{if } z \in S \times (-n,n), \\ |\nabla_g f_{i,n}|_g(z) & \text{if } z \in ((-3,3)^{d-1} \setminus S) \times (-n,n), \\ |Df_{i,1}|_w(z-(n-1)e_d) & \text{if } z \in (-3,3)^{d-1} \times [n,n+1], \\ |Df_{i,1}|_w(z+(n-1)e_d) & \text{if } z \in (-3,3)^{d-1} \times [-n-1,-n], \\ 0 & \text{otherwise.} \end{cases}$$

Now,

$$\begin{aligned}
&\int_M 2|Df_{1,n}|_w^2 + 2|Df_{2,n}|_w^2 - |Df_{3,n}|_w^2 - |Df_{4,n}|_w^2 \,\mathrm{dvol}_g \\
&= \int_{(-3,3)^{d-1}\times(-n,n)} 2|Df_{1,n}|_w^2 + 2|Df_{2,n}|_w^2 - |Df_{3,n}|_w^2 - |Df_{4,n}|_w^2 \,\mathrm{dvol}_g \\
&+ \int_{(-3,3)^{d-1}\times([-n-1,-n]\cup[n,n+1])} 2|Df_{1,n}|_w^2 + 2|Df_{2,n}|_w^2 - |Df_{3,n}|_w^2 - |Df_{4,n}|_w^2 \,\mathrm{dvol}_g \\
&= n\int_{S\times(-1,1)} 2G_1^2 + 2G_2^2 - G_3^2 - G_4^2 \,\mathrm{dvol}_g \\
&+ \int_{(-3,3)^{d-1}\times((-2,-1)\cup(1,2))} 2|Df_{1,1}|_w^2 + 2|Df_{2,1}|_w^2 - |Df_{3,1}|_w^2 - |Df_{4,1}|_w^2 \,\mathrm{dvol}_g \\
&= 2n\mathrm{vol}_g(S\times(0,1)) \\
&+ \int_{(-3,3)^{d-1}\times((-2,-1)\cup(1,2))} 2|Df_{1,1}|_w^2 + 2|Df_{2,1}|_w^2 - |Df_{3,1}|_w^2 - |Df_{4,1}|_w^2 \,\mathrm{dvol}_g,
\end{aligned}$$

where in the last two steps, we first used the fact that $2|\nabla_g f_{1,n}|_g^2 + 2|\nabla_g f_{2,n}|_g^2 - |\nabla_g f_{3,n}|_g^2 - |\nabla_g f_{4,n}|_g^2 = 0$ for all $n \in \mathbb{N}$ and then applied (4.2.90). We have that $|Df_{i,1}|_w^2 \in L^\infty$ and hence the last integral in the above equation is finite, so as $\mathrm{vol}_g(S \times (0,1)) > 0$, we get that

$$\lim_{n\to\infty} 2\mathrm{Ch}(f_{1,n}) + 2\mathrm{Ch}(f_{2,n}) - \mathrm{Ch}(f_{3,n}) - \mathrm{Ch}(f_{4,n}) = \infty. \tag{4.2.91}$$

In particular, there exists an $n \in \mathbb{N}$ such that $2\mathrm{Ch}(f_{1,n}) + 2\mathrm{Ch}(f_{2,n}) - \mathrm{Ch}(f_{3,n}) - \mathrm{Ch}(f_{4,n}) \neq 0$. □

This proves Theorem 3, taking into account that for $p' \in [1, d-1]$, one has $W^{1,p'}_{\mathrm{loc}} \subset W^{1,d-1}_{\mathrm{loc}}$.

Remark 4.2.14. *For $d \geq 4$, the above example shows that we can find a Riemannian metric g on $\mathbb{R}^d$ of the Geroch-Traschen class that induces a non-infinitesimally Hilbertian metric measure space.*

Remark 4.2.15. *For a manifold with a $W^{1,p}_{\mathrm{loc}}$-Riemannian metric such that $p > d$, the Sobolev embedding theorem together with the results from [102] immediately give us that $(M, \mathsf{d}_g, \mathrm{vol}_g)$ is infinitesimally Hilbertian. Theorem 3 shows that $g, g^{-1} \in L^\infty_{\mathrm{loc}} \cap W^{1,p}_{\mathrm{loc}}$ for*

$p \leq d-1$ is not sufficient for infinitesimal Hilbertianity. This works towards answering Question 3 leaving only the interval of $p \in (d-1, d]$, where it is unknown whether $g, g^{-1} \in L^\infty_{\mathrm{loc}} \cap W^{1,p}_{\mathrm{loc}}$ is sufficient to get infinitesimal Hilbertianity. Indeed, the Sobolev embedding theorem forces g to be continuous as soon as $g \in W^{1,p}_{\mathrm{loc}}$ for $p > d$. Note that using our construction, we cannot do better than $W^{1,d-1}_{\mathrm{loc}}$, as the metric g can be written as $g(x_1, \ldots, x_d) = \theta(x_1, \ldots, x_{d-1})\mathrm{Id}$, so $W^{1,p}_{\mathrm{loc}}$-regularity of θ for $p > d-1$ would force θ (as a function of $\mathbb{R}^{d-1}$) to be continuous and continuity implies infinitesimal Hilbertianity. In a work in progress, we are currently studying whether additional $W^{1,p}_{\mathrm{loc}}$-Sobolev regularity of the metric g for $p \in (d-1, d]$ (or in the case $d = 2$, $p \in [1, 2]$) implies infinitesimal Hilbertianity.

Remark 4.2.16. *One motivation behind the definition of infinitesimal Hilbertianity was to distinguish Finslerian structures from Riemannian ones, in particular in the context of synthetic lower Ricci curvature bounds. The distance in this example arises from a Riemannian metric of low regularity; however the corresponding metric measure space does not satisfy infinitesimal Hilbertianity. The directional dependence of the metric speed of curves established in* (4.2.88) *reveals the asymptotic shape of metric balls in the manifold and it is readily seen that the space $(M, \mathsf{d}_g, \mathrm{vol}_g)$ is not locally Minkowskian (see [109] and [94] for a discussion of infinitesimal Hilbertianity for locally Minkowskian spaces). Moreover, the construction immediately implies that this space does not satisfy neither distributional lower Ricci curvature bounds nor the $\mathsf{CD}(K, \infty)$-condition for any $K \in \mathbb{R}$.*

Chapter 5

Infinitesimal Minkowskianity for manifolds with continuous Lorentzian metrics

This chapter contains the results of a solo-authored paper draft [120]. We refer to Subsection 1.3.3 for an introduction to this chapter and to Section 2.3 for relevant preliminary definitions. The main result in this chapter is

Theorem 4. *Let M be a $(d+1)$-dimensional manifold and g be a continuous Lorentzian metric on M such that (M,g) is a causally simple spacetime. Then the metric measure spacetime $(M,\tau,\ell_g,\mathrm{vol}_g)$ is infinitesimally Minkowskian.*

Notation

Given $m \in \mathbb{N}$ and real numbers $\lambda_1,\dots,\lambda_m$, the expression $\mathrm{diag}(\lambda_1,\dots,\lambda_m)$ refers to the m-dimensional diagonal matrix with entries $\lambda_1,\dots,\lambda_m$. Given a matrix $A \in \mathbb{R}^{m\times m}$, $\|A\|_{op}$ denotes the operator norm of A induced by the Euclidean metric on $\mathbb{R}^m$.

Given a manifold M, a continuous and time orientable Lorentzian metric g on M and two vectors $v,w \in T_pM$ for some $p \in M$, we write $g(v,w) = \langle v,w\rangle_g = \sum_{i,j} g_{ij}v_iw_j$ for the g-inner product of v and w. Given a causal function f on M, we denote by $|\mathrm{d}f|$ the maximal weak subslope. We will denote by $\mathrm{d}f \in T^*M$ its differential, $\nabla_g f \in TM$ its gradient and by $|\nabla_g f|_g (= |\mathrm{d}f|_g) = \sqrt{-\sum_{i,j} g^{ij}\partial_i f \partial_j f}$ the g-norm of its gradient (= the g-norm of its differential), wherever they exist. For a causal curve $\gamma : [0,1] \to M$, we denote by $\dot\gamma_t \in T_{\gamma_t}M$ the derivative, by $|\dot\gamma_t|_g = \sqrt{-g(\dot\gamma_t,\dot\gamma_t)}$ the g-norm of the derivative, and by $|\dot\gamma_t|$ the causal speed of γ at time t, wherever these exist.
Given an open set $U \subset M$ and Lorentzian metrics g, g', we say that $g \prec g'$ (or $g' \succ g$) on U if for every $0 \neq v \in TU$ it holds that $g(v,v) \leq 0 \implies g'(v,v) < 0$.

5.1 Preliminary results from measure theory

The following measure theoretic results will be useful in Section 5.2. Let $m \in \mathbb{N}$.

Lemma 5.1.1. *Let $\Omega \subset \mathbb{R}^m$ be open and let μ be a locally finite Radon measure on Ω. For $\delta > 0$ define the set $\Omega_\delta := \{p \in \Omega : \overline{B_\delta(p)} \subset \Omega\}$. Then for every $\delta > 0$, the map $E_{\mu,\delta} : \Omega_\delta \to [0, \infty), p \mapsto \mu(\overline{B_\delta(p)})$ is $\mathcal{L}^m$-measurable and locally bounded.*

Proof. Fix a $\delta > 0$. Fix a compact sets $K_1 \subset K_2 \subset \Omega$ such that $\overline{B_\delta(K_1)} \subset K_2$. Now for each $p \in K_1$, it holds that

$$\mu(\overline{B_\delta(p)}) \leq \mu(K_2) < \infty.$$

This proves the local boundedness.

For $i = 0, \ldots, m-1$, define the sets P_i and the Radon measures μ_{i+1}, ν_i such that

- P_i is a countable union of i-dimensional spheres, where i-dimensional spheres are given by $\{q \in \mathbb{R}^m : |q - p| = r, q - p \in V\}$, where $p \in \mathbb{R}^m$, $r > 0$ and $V \subset \mathbb{R}^m$ is an i-dimensional linear subspace of $\mathbb{R}^m$. If $i = 0$, P_0 is going to be a countable set of points.
- $\nu_i = \mu_i \llcorner P_i$,
- $\mu_{i+1} = \mu_i - \nu_i$ is zero on spheres of dimension at most i.

Define $\mu_0 := \mu$. Now, define the set

$$P_0 := \{p \in K_2 : \mu(\{p\}) > 0\}.$$

This set has to be countable, as $P_0 = \bigcup_{n\in\mathbb{N}}\{p \in K_2 : \mu(\{p\}) > \frac{1}{n}\}$, and for each $n \in \mathbb{N}$, we have that $\#\{p \in K_2 : \mu(\{p\}) > \frac{1}{n}\} \leq n\mu(K_2) < \infty$. Now define the measures

$$\nu_0 := \mu \llcorner P_0,\ \ \mu_1 := \mu - \nu_0. \tag{5.1.1}$$

Now for $i \in \{1, \ldots, m-1\}$, define inductively the measures ν_i, μ_{i+1} as follows: Let

$$P_i := \{S : i\text{−dimensional sphere},\ S \subset K_2,\ \mu_i(S) > 0\}. \tag{5.1.2}$$

For every $n \in \mathbb{N}$, we have that

$$\#\{S : i\text{−dimensional sphere},\ S \subset K_2,\ \mu_i(S) > 1/n\} < n\mu_i(K_2) \leq n\mu(K_2) < \infty. \tag{5.1.3}$$

Indeed, suppose that this was not true. Then there exist two distinct i-dimensional spheres $S_\alpha, S_\beta \in P_i$ such that $\mu_i(S_\alpha \cap S_\beta) > 0$. Now, we have that for any two distinct

i-dimensional spheres, their intersection is the finite union of spheres of dimension at most $i-1$. By our assumption, the measure μ_i is zero on spheres of dimension at most $i-1$, which yields a contradiction and hence verifies (5.1.3). This proves that $P_i = \bigcup_{n\in\mathbb{N}}\{S : i\text{–dimensional sphere},\ S \subset K_2,\ \mu_i(S) > 1/n\}$ is countable. Now we define

$$\nu_i := \mu_i \llcorner P_i,\ \mu_{i+1} := \mu_i - \nu_i. \tag{5.1.4}$$

In total, this gives us a decomposition of the measure

$$\mu = \mu_m + \sum_{i=0}^{m-1} \nu_i, \tag{5.1.5}$$

where the ν_i are supported on countably many i-dimensional spheres (collected in P_i) and μ_m is zero on spheres of dimension smaller than m.

For each $i \leq m-1$, and any i-dimensional sphere S, we have that the set

$$D_{S,\delta} := \{p : S \subset \partial B_\delta(p)\}$$

is a sphere of dimension $m-i-1$. Any such sphere is a Borel $\mathcal{L}^m$-zero set. Then, the set

$$D := \bigcup_{i=0}^{m-1} \bigcup_{S\in P_i} D_{S,\delta} \tag{5.1.6}$$

is a Borel $\mathcal{L}^m$-zero set, which is a countable union of spheres of dimension at most $m-1$. For any point $p \in K_1 \setminus D$, we have that

$$\mu(\partial B_\delta(p)) = \mu_m(\partial B_\delta(p)) = 0.$$

Note that

$$\lim_{n\to\infty} \mu(B_{\delta+2^{-n}}(p) \setminus B_{\delta-2^{-n}}(p)) = \mu(\partial B_\delta(p)) = 0. \tag{5.1.7}$$

Fix an $\varepsilon > 0$ and $n \in \mathbb{N}$ such that $\mu(B_{\delta+2^{-n}}(p) \setminus B_{\delta-2^{-n}}(p)) \leq \varepsilon$. Then we have that for each $q \in B_{2^{-n-1}}(p)$, it holds that

$$\begin{aligned} |\mu(\overline{B_\delta(p)}) - \mu(\overline{B_\delta(q)})| &\leq \mu((\overline{B_\delta(p)} \setminus \overline{B_\delta(q)}) \cup (\overline{B_\delta(q)} \setminus \overline{B_\delta(p)})) \\ &\leq \mu(B_{\delta+2^{-n}}(p) \setminus B_{\delta-2^{-n}}(p)) \leq \varepsilon. \end{aligned} \tag{5.1.8}$$

Hence, $E_{\mu,\delta}$ is continuous at p. This holds for $p \in K_1 \setminus D$, so the map is continuous almost everywhere. Hence, $E_{\mu,\delta}$ is $\mathcal{L}^m$-measurable when restricted to K_1. As $K_1 \subset \Omega_\delta$ was arbitrary, this concludes the proof. $\square$

Lemma 5.1.2. *Let μ be a locally finite Radon measure on $\mathbb{R}^m$ such that $\mu \perp \mathcal{L}^m$. Then for $\mathcal{L}^m$-almost every point $p \in \mathbb{R}^m$, it holds that*

$$\lim_{\delta \to 0} \frac{\mu(B_\delta(p))}{\delta^m} = 0. \tag{5.1.9}$$

Proof. Suppose this was not the case. Then there exist an $\varepsilon > 0$ and a bounded set $A' \subset \mathbb{R}^m$ with $\mathcal{L}^m(A') > 0$ such that for each $p \in A'$ it holds

$$\limsup_{\delta \to 0} \frac{\mu(B_\delta(p))}{\delta^m} \geq 2\varepsilon > 0. \tag{5.1.10}$$

Note that by Lemma 5.1.1, we have that the set of all points satisfying (5.1.10) is Lebesgue measurable. As $\mu \perp \mathcal{L}^m$, there exists a decomposition of $\mathbb{R}^m$ into disjoint Borel sets L, S such that $\mathcal{L}^m(S) = 0 = \mu(L)$ and $L \cup S = \mathbb{R}^m$. Define $A := A' \setminus S$. Then $\mathcal{L}^m(A') = \mathcal{L}^m(A) > 0$. Pick an open set $\tilde{O} \supset A$. Then for each point $p \in A$, there exists a $\delta_p > 0$ such that $B_{\delta_p}(p) \subset \tilde{O}$ and such that

$$\frac{\mu(B_{\delta_p}(p))}{(\delta_p)^m} \geq \varepsilon. \tag{5.1.11}$$

Then

$$A \subset \bigcup_{p \in A} B_{\delta_p}(p) \subset \tilde{O}.$$

By the Besicovitch covering theorem, there exists a constant $c_m \in \mathbb{N}$ such that there exist c_m countable families $T_1, \ldots, T_{c_m}$ of points $p \in A$ such that

$$A \subset \bigcup_{i=1}^{c_m} \bigcup_{p \in T_i} B_{\delta_p}(p) =: O \subset \tilde{O}.$$

and such that for each i and $p, p' \in T_i$, it holds

$$B_{\delta_{p'}}(p') \cap B_{\delta_p}(p) = \emptyset. \tag{5.1.12}$$

By the pigeon hole principle, we get that there exists an $i \in \{1, \ldots, c_m\}$ such that

$$\mathcal{L}^m\Big(\bigcup_{p \in T_i} B_{\delta_p}(p)\Big) \geq \frac{1}{c_m} \mathcal{L}^m(A). \tag{5.1.13}$$

But now (5.1.12) together with (5.1.11) and (5.1.13) yields that

$$\mu\Big(\bigcup_{p \in T_i} B_{\delta_p}(p)\Big) \geq \frac{\varepsilon}{\mathcal{L}^m(B_1(0))} \sum_{p \in T_i} \mathcal{L}^m(B_{\delta_p}(p)) \geq \frac{\varepsilon}{c_m \mathcal{L}^m(B_1(0))} \mathcal{L}^m(A). \tag{5.1.14}$$

It follows that

$$\mu(\tilde{O}) \geq \mu(O) \geq \frac{\varepsilon}{c_m \mathcal{L}^m(B_1(0))} \mathcal{L}^m(A).$$

As $\tilde{O}$ is an arbitrary open set containing A, the outer regularity of Radon measures on $\mathbb{R}^m$ yields that

$$\mu(A) \geq \frac{\varepsilon}{c_m \mathcal{L}^m(B_1(0))} \mathcal{L}^m(A) > 0. \tag{5.1.15}$$

But $A \subset L$ and $\mu(L) = 0$ by assumption. This is a contradiction and finishes the proof. $\square$

Lemma 5.1.3. *Let $A \subset \mathbb{R}^m$ be Lebesgue measurable and $P, Q \subset A$ such that $\mathcal{L}^m(A \cap P^c) = 0$. Assume that for each $p \in P$ there exists a radius $r_p > 0$ such that $\mathcal{L}^m(Q^c \cap B_{r_p}(p)) = 0$. Then, $\mathcal{L}^m(A \cap Q^c) = 0$. In particular, Q is measurable.*

Proof. We may assume A to be bounded. Then $A \cap P \subset \bigcup_{p \in P} B_{r_p}(p)$. The Besicovitch covering theorem yields that there exists a countable set $(p_i)_{i \in \mathbb{N}} \subset P$ such that

$$A \cap P \subset \bigcup_{i \in \mathbb{N}} B_{r_{p_i}}(p_i).$$

Denote by $\mathcal{L}^{m*}$ the outer m-dimensional Lebesgue measure. Now we have that

$$\mathcal{L}^{m*}(A \cap Q^c) \leq \mathcal{L}^{m*}(A \cap P^c) + \sum_{i \in \mathbb{N}} \mathcal{L}^{m*}(B_{r_{p_i}}(p_i) \cap Q^c) = 0.$$

Now null sets are Lebesgue measurable, which proves the lemma. $\square$

The final lemma of this section is proved analogously to Lemma 5.2.18, so the proof is omitted.

Lemma 5.1.4. *Let $\Omega \subset \mathbb{R}^m$ be open, $f : \Omega \to \mathbb{R}$ be Lebesgue measurable and locally bounded, and $\Gamma \subset B_1(0)$ be a countable set. Then for $\mathcal{L}^m$-almost every point $p \in \Omega$ and all $\dot{\gamma} \in \Gamma$, we have that $t = 0$ is a Lebesgue point of the map*

$$L^f_{p,\dot{\gamma}} : (-\varepsilon, \varepsilon) \to \mathbb{R}, t \mapsto f(p + t\dot{\gamma}), \tag{5.1.16}$$

where ε is such that $B_\varepsilon(p) \subset \Omega$.

5.2 Application to manifolds with non-smooth Lorentzian metrics

We want to study how the theory on causal functions as it has been developed in [16] applies to the setting of a Lorentzian manifold with a metric of low regularity. From now on the setting will be a smooth $(d+1)$-manifold M, for $d \geq 1$, equipped with a continuous Lorentzian metric g, i.e., a continuous symmetric bilinear $(0,2)$-tensor of signature $(-,+,\dots,+)$. We say that a vector $v \in TM$ is *timelike* if $g(v,v) < 0$, *null* or *lightlike* if $g(v,v) = 0$ and $v \neq 0$, *spacelike* if $v = 0$ or $g(v,v) > 0$, and *causal* if it is timelike or null. We assume that g is *time orientable*, i.e., there exists a continuous timelike vector field X which tells us at each point, which cone is the future cone and which cone is the past cone. More precisely, we say that a causal vector $v \in TM$ is future directed, if $g(v,X) < 0$, otherwise we say that it is past directed. Time orientable Lorentzian manifolds are called *spacetimes*.

For timelike vectors $v \in TM$, we define $|v|_g := \sqrt{-g(v,v)}$. Recall that for $p \in M$ and $v, w \in T_pM$ causal, the reverse Cauchy-Schwartz inequality holds, i.e.,

$$|g(v,w)| \geq |v|_g |w|_g, \tag{5.2.1}$$

with equality if and only if $v = \alpha w$ for some $\alpha \in \mathbb{R}$.

A spacetime arising from a manifold with a continuous Lorentzian metric can naturally be seen as a metric measure spacetime as follows:

- Any complete Riemannian metric h on M induces a distance d_h on M such that (M, d_h) is a Polish metric space. Fix some such h and denote by τ_h the topology induced by d_h. Note that τ_h equals the topology induced by charts, so from now on, we will not always mention the topology explicitly.
- The metric g induces a volume measure locally given by $\mathrm{dvol}_g = \sqrt{-\det g}\, \mathrm{d}\mathcal{L}^{d+1}$.
- We say that an absolutely continuous curve $\gamma\colon [0,1] \to M$ is *classically causal* (c.c.) and future directed (past directed) if $g(\dot\gamma_t, \dot\gamma_t) \leq 0$ and $\dot\gamma_t$ is future directed (past directed) or zero for almost every $t \in [0,1]$.
- The time separation $\ell_g : M^2 \to \{-\infty\} \cup [0,\infty]$ is defined as

$$\ell_g(x,y) = \sup \Big\{ L_g(\gamma) := \int_0^1 \sqrt{-g(\dot\gamma_t, \dot\gamma_t)}\, \mathrm{d}t \Big|\ \gamma \in C^{0,1}([0,1], M), \\ \text{c.c., future dir., } \gamma_0 = x, \gamma_1 = y. \Big\}. \tag{5.2.2}$$

We then define the chronological future and past I^+, I^- of a point $p \in M$ via

$$I^+(p) = \{q \in M : p \ll q\},\ I^-(p) = \{q \in M : q \ll p\},$$

and the causal future and past J^+, J^- of a point $p \in M$ via

$$J^+(p) = \{q \in M : p \leq q\},\ J^-(p) = \{q \in M : q \leq p\},$$

where as in Section 2.3 the chronological relation $\ll := \ll_g$ is defined via $\ll_g := \ell_g^{-1}((0,\infty]) \subset M^2$ and the causal relation $\leq := \leq_g$ is defined via $\leq_g := \ell_g^{-1}([0,\infty]) \subset M^2$. From now on when speaking of classically causal curves, we always refer to future directed classically causal curves, even if not explicitly mentioned.

Recall that the spacetime (M, g) is said to be

- *strongly causal* if for every $p \in M$ and any neighbourhood $U \subset M$ of p, there exists a neighbourhood $V \subset U$ of p such that every classically causal and future directed curve with endpoints in V is contained in U.
- *causally simple* if there are no non-constant closed classically causal and future directed curves and if $\leq$ is closed in the manifold topology.
- *globally hyperbolic* if it is strongly causal and if for any two points $p, q \in M$ the set $J^+(p) \cap J^-(q)$ is compact.

The next result is known for smooth Lorentzian spacetimes (see for instance [98]) and the proof works similarly. Alternatively, Minguzzi's result on closed cone structures [97, Theorem 2.47] implies the following lemma.

Lemma 5.2.1. *If (M, g) is causally simple then (M, g) is strongly causal.*

Proof. Suppose for a contradiction that (M, g) is not strongly causal. Then there exists a $p \in M$ and an open set $U \subset M$ such that $p \in U$ and such that for every open set $V \in M$ with $p \in V$ there exist $q^-, q^+ \in V$ and a classically causal and future directed curve $\gamma : [0,1] \to M$ with $\gamma_0 = q^-$ and $\gamma_1 = q^+$ that leaves U. By possibly shrinking U we may assume that $U \ni p$ is relatively compact. For $n \in \mathbb{N}$, fix open sets V_n such that $p \in V_{n+1} \subset V_n \subset U$ and such that $\bigcap_{n\in\mathbb{N}} V_n = \{p\}$. Now, by our assumption, there exist $q_n^-, q_n^+ \in V_n$ and a causal curve $\gamma_n : [0,1] \to M$ from q_n^- to q_n^+ that leaves U. Denote

$$q_n^0 := \gamma\big(\min\{t \in [0,1] : \gamma_t \notin U\}\big) \in \partial U.$$

By potentially passing to a subsequence, the relative compactness of U and the choice of the V_n yields that

$$\lim_{n\to\infty} q_n^- = \lim_{n\to\infty} q_n^+ = p,$$

and that there exists a $q \in \partial U$ such that

$$\lim_{n\to\infty} q_n^0 = q.$$

Moreover, we have that $q_n^- \leq_g q_n^0 \leq_g q_n^+$, hence the fact that $\leq_g$ is closed yields that

$$p \leq_g q \leq_g p.$$

However, $p \notin \partial U$, hence $q \neq p$. Thus, there exists a non-constant closed classically causal curve, namely starting at p and passing through q. This contradicts the causal simplicity and hence proves the lemma. $\square$

Assumption 5.2.2. *From now on we will assume (M, g) to be causally simple.*

Remark 5.2.3. *The assumption of causal simplicity is of technical nature, as causal simplicity is necessary to enter the framework of a Polish metric measure spacetime as in [16]. Indeed, it is needed for $\{\ell_g \geq 0\}$ to be closed in the manifold topology. However, strong causality is implied by Lemma 5.2.1 and suffices for our proofs. By [122, Proposition 3.3], any globally hyperbolic continuous spacetime is causally simple and strongly causal.*

From now on, we will consider the metric measure spacetime $(M, \tau_h, \ell_g, \mathrm{vol}_g)$. When speaking of causal paths we refer to maps $\gamma : [0,1] \to M$ such that for each $s, t \in [0,1]$ with $s \leq t$, it holds $\gamma_s \leq_g \gamma_t$ (these are the future directed and causal paths as defined in [16]). Note that the notion of a causal path uses the definition of ℓ_g based on classically causal curves. When speaking of a causal curve, we refer to continuous causal paths.

Remark 5.2.4. *Notice that in the literature, it is sometimes employed to consider only those classically causal curves whose tangent vectors are non-zero almost everywhere (e.g., [122]). In the context of Lorentzian length spaces, causal curves are commonly defined to be non-constant [84, 35]. However, our definitions allow (classically) causal curves with constant segments, as this ensures that $\ell_g(x, x) = 0$ for $x \in M$ and better suits the definition of causal curves in [16].*

A priori, all we know is that the class of classically causal curves is contained in the class of causal curves. We quickly notice the following useful fact.

Lemma 5.2.5. *Let (M, g) be a spacetime and $V \subset U \subset M$ neighbourhoods such that every classically causal curve with endpoints in V is contained in U. Then, every causal path with endpoints in V is contained in U.*

Proof. Suppose there exists a causal path $\gamma\colon [0,1] \to M$ with $\gamma_0, \gamma_1 \in V$ and such that there is a $t \in (0,1)$ with $\gamma_t \notin U$. By the definition of causal paths, we have that $\gamma_0 \leq_g \gamma_t \leq_g \gamma_1$. Hence, there exist Lipschitz continuous classically causal curves $\lambda^1, \lambda^2\colon [0,1] \to M$ such that $\lambda^1_0 = \gamma_0$, $\lambda^1_1 = \lambda^2_0 = \gamma_t$, and $\lambda^2_1 = \gamma_1$. Then, the concatenation λ of λ^1 and λ^2 is a classically causal curve with endpoints in V that leaves U. This is a contradiction and proves the lemma. $\square$

This shows that the strong causality assumption also holds for causal paths, not only for classically causal curves. Similarly, the causal simplicity assumption holds for causal paths as well.

Lemma 5.2.6. *Let $p \in M$ and $0 \neq w \in T_pM$ be spacelike. Then there exist timelike vectors $u, v \in T_pM$ such that one of them is future directed and the other one past directed and such that $\langle u, w\rangle, \langle v, w\rangle > 0$.*

Proof. As this is a statement about the tangent space, we may assume to be working in $\mathbb{R}^{d+1} \cong T_pM$. Furthermore, we may assume the coordinates to be chosen in a way that $g(p) = \operatorname{diag}(-1, 1, \ldots, 1)$. Write $w = (w_0, \ldots, w_d) \in \mathbb{R}^{d+1}$. By applying a rotation, we may assume that $w_2 = \ldots = w_d = 0$. This reduces the problem to the case $d = 1$ and $g = \operatorname{diag}(-1, 1)$.
Now as w is spacelike, we get that $w_1^2 > w_0^2$. Define the vectors $\vartheta := (w_1, w_0), \vartheta' := -\vartheta = (-w_1, -w_0)$ and note that ϑ, ϑ' are both timelike but one is future directed and the other one past directed. Moreover, it holds that $g(w, \vartheta) = g(w, -\vartheta) = 0$. Fix an $\alpha > 0$ such that $\vartheta + [-\alpha, \alpha]e_1$ is timelike and hence in the same cone (future or past) as ϑ and such that similarly $\vartheta' + [-\alpha, \alpha]e_1$ is timelike and hence in the same cone as ϑ'. Now we distinguish cases:
Case 1. $w_1 > 0$.
Then $u = \vartheta + \alpha e_1$ and $v = \vartheta' + \alpha e_1$ yield the desired. Indeed,

$$\begin{aligned} g(w, u) &= g(w, \vartheta) + \alpha w_1 = \alpha w_1 > 0, \\ g(w, v) &= g(w, \vartheta') + \alpha w_1 = \alpha w_1 > 0. \end{aligned}$$

Case 2. $w_1 < 0$.
In this case, $u = \vartheta - \alpha e_1$ and $v = \vartheta' - \alpha e_1$ yield the desired. This proves the lemma. $\square$

Many of the following arguments will be local, so we will often work in small neighbourhoods $U \subset M$ that may be assumed to be contained in one coordinate patch. To keep notation short, we will oftentimes assume $U \subset \mathbb{R}^{d+1}$ without explicitly mentioning (pushforwards by) a chart $\psi : U \to \mathbb{R}^{d+1}$. A similar observation as the following one can be found in [16].

Proposition 5.2.7. *Let f be a causal function on M. Then for each point $p \in M$ at which f is differentiable, the gradient $\nabla_g f(p) \in T_pM$ is either zero or causal and past directed.*

Proof. Take any timelike and future directed vector $v \in T_pM$. Working in a small neighbourhood U around p, we may assume that $U \subset \mathbb{R}^{d+1}$ and identify the tangent spaces with $\mathbb{R}^{d+1}$. Then, by the continuity of g, there exists a neighbourhood V around p such that for each $q \in V$, $g(q)(v,v) < 0$, where v is seen as a vector in $\mathbb{R}^{d+1}$. Hence, there exists an $s > 0$ such that the curve $\gamma : [-s,s] \to M, t \mapsto p + tv$ is a causal and future directed curve. Note that γ is differentiable at $t = 0$. Then, the chain rule yields that

$$\lim_{h\to 0} \frac{f(\gamma_h) - f(\gamma_0)}{h} = \mathrm{d}f_p(v) = \langle \nabla_g f, v\rangle_g(p).$$

The causality of f and γ yields that

$$\frac{f(\gamma_h) - f(\gamma_0)}{h} \geq 0, \ \forall 0 \neq h \in [-s,s].$$

Hence, $\langle \nabla_g f, v\rangle_g(p) \geq 0$. As v was chosen arbitrarily, we get that for all timelike and future directed vectors $v \in T_pM$ it holds

$$\langle \nabla_g f, v\rangle_g(p) \geq 0.$$

Now assume $\nabla_g f(p)$ was non-zero and spacelike. Then by Lemma 5.2.6, there exists a past directed timelike vector $u \in T_pM$ such that $\langle \nabla_g f, u\rangle_g(p) > 0$. But then the vector $v := -u$ is timelike and future directed and satisfies $\langle \nabla_g f, v\rangle_g(p) < 0$, which is a contradiction. Similarly, considering $v = X(p)$, we get that $\nabla_g f(p)$ cannot be causal and future directed. □

Lemma 5.2.8. *Let $\gamma\colon [0,1] \to M$ be a causal path. Then for each $t \in [0,1]$, at which γ is differentiable, the derivative $\dot\gamma_t \in T_{\gamma_t}M$ is either 0 or causal and future directed.*

Proof. Fix a $t \in [0,1]$ such that γ is differentiable at t and denote $p := \gamma_t$ and $v := \dot\gamma_t$. We may pick local coordinates for an open set $U \ni p$ such that $p = 0 \in \mathbb{R}^{d+1}$ and such that $g(p) = \mathrm{diag}(-1,1,\ldots,1)$. For $\varepsilon > 0$, define the Lorentzian metric $m_\varepsilon := \mathrm{diag}(-1-\varepsilon,1,\ldots,1)$ on U. Then for all $\varepsilon > 0$, we have that there exists an open neighbourhood $U_\varepsilon \subset U$ such that $p \in U_\varepsilon$ and such that $m_\varepsilon \succ g$ on U_ε. Moreover, by the strong causality of (M,g), there exists a neighbourhood $p \in V_\varepsilon \subset U_\varepsilon$ such that every g-causal path with endpoints in V_ε is contained in U_ε.

Claim 1: v is causal or 0.

Proof of claim 1. Suppose for a contradiction that v is spacelike and nonzero. Then $g(p)(v,v) > 0$. As $\lim_{\varepsilon\to 0} m_\varepsilon = g(p)$, we have that there exists an $\varepsilon := \varepsilon_v > 0$ small enough such that $m_\varepsilon(v,v) > 0$. For $h > 0$ such that $\gamma_{t+h} \in V_\varepsilon$, define

$$v_h := \frac{\gamma_{t+h} - \gamma_t}{h} \in \mathbb{R}^{d+1}.$$

Now, we have that $v_h \to v$ as $h \to 0$, hence we can find a $h > 0$ such that $\gamma_{t+h} \in V_\varepsilon$ and

$$m_\varepsilon(v_h, v_h) \geq \frac{m_\varepsilon(v,v)}{2} > 0.$$

As γ is causal, we have that $\gamma_{t+h} = \gamma_t + hv_h \geq \gamma_t$, hence there exists a Lipschitz continuous, g-classically causal curve $\lambda : [0,1] \to M$ such that $\lambda_0 = \gamma_t$, $\lambda_1 = \gamma_{t+h}$. But then λ is contained in U_ε. Hence, λ is m_ε-classically causal. But as m_ε is constant on U_ε, this implies $m_\varepsilon(hv_h, hv_h) \leq 0$ which is a contradiction. This proves claim 1.

Claim 2: v is future directed or 0.

Proof of claim 2. We already know that v is causal, hence $g(p)(X(p), v) > 0$. Suppose for a contradiction that v is non-zero and past directed. Fix an $\varepsilon > 0$ such that $m_\varepsilon(X(p), v) > 0$ and consider $U_\varepsilon, V_\varepsilon$ as above. For $h > 0$ small enough consider again $v_h := \frac{\gamma_{t+h}-\gamma_t}{h} \in \mathbb{R}^{d+1}$. We can choose $h > 0$ such that $\gamma_{t+h} \in V_\varepsilon$ and $m_\varepsilon(X(p), v_h) > 0$. However, there exists a g-classically causal and future directed Lipschitz continuous curve $\lambda : [0,1] \to M$ such that $\lambda_0 = \gamma_t$, $\lambda_1 = \gamma_{t+h}$. But then λ is contained in U_ε. Hence, λ is m_ε-classically causal and future directed, so $m_\varepsilon(v_h, X(p)) < 0$. This is a contradiction. □

Remark 5.2.9. *Lemma 5.2.8 shows that in the setting of a manifold with a continuous Lorentzian metric the class of causal paths is a sensible extension of the class of classically causal curves, in the sense that the derivatives of these maps are causal or zero wherever they exist. This observation is in the flavour of [84, Proposition 5.9].*

Lemma 5.2.10. *Let g be continuous and strongly causal, $\gamma\colon [0,1] \to M$ a causal path. Then for each $t \in [0,1)$ where γ is differentiable, it holds that*

$$\lim_{h\searrow 0} \frac{\ell_g(\gamma_t, \gamma_{t+h})}{h} = |\dot\gamma_t|_g.$$

Proof. Pick a $t \in [0,1)$ such that γ is differentiable at t and denote $p := \gamma_t \in M$. We may restrict to a small neighbourhood U of p and using local coordinates, we may assume that $U \subset \mathbb{R}^{d+1}$ and $g(p) = \mathrm{diag}(-1,1,\dots,1)$. Furthermore, we may possibly shrink U such that it holds $g \prec m_2 := \operatorname{diag}(-2, 1, \dots, 1)$ on U and such that for every g-causal vector $v \in TU$ it holds $m_2(v, v) < g(v, v) \leq 0$.

We will first consider the case $\dot{\gamma}_t = 0$. Now, there exist $p^-, p^+ \in U$ such that $p \in U' := (J^-_{m_2}(p^+) \cap J^+_{m_2}(p^-))^\circ \subset U$ (see [99, Theorem 2.14]). Furthermore, by the strong causality of g, we can now pick a neighbourhood $V \subset U'$ of p such that every g-causal path with endpoints in V is contained in U'. Thus, for any $q \in V$, we have that any g-classically causal and future directed curve λ from p to q is then contained in U'. Hence it is also m_2-classically causal, future directed, and $L_g(\lambda) < L_{m_2}(\lambda)$, which yields that for $q \in V$, it holds

$$\ell_g(p, q) \leq \ell_{m_2}(p, q). \tag{5.2.3}$$

Moreover, note that for $q \in U'$, it holds

$$\ell_{m_2}(p, q) \leq 2\mathsf{d}_{euc}(p, q). \tag{5.2.4}$$

Now, as γ is continuous, we have that there exists a $h_0 > 0$ such that for $h \in (0, h_0)$, it holds $\gamma_{t+h} \in V$. Hence, considering only $h \in (0, h_0)$, we can apply (5.2.3) and (5.2.4) which when passing to the limit yields that

$$0 \leq \limsup_{h \searrow 0} \frac{\ell_g(\gamma_t, \gamma_{t+h})}{h} \leq \limsup_{h \searrow 0} \frac{2\mathsf{d}_{euc}(\gamma_t, \gamma_{t+h})}{h} = 2|\dot{\gamma}_t|_{euc} = 0. \tag{5.2.5}$$

This finishes the proof in the case $\dot{\gamma}_t = 0$.

From now on we assume that $\dot{\gamma}_t \neq 0$. Then Lemma 5.2.8 ensures that $\dot{\gamma}_t$ is causal and future directed. We will show that

$$\limsup_{h \searrow 0} \frac{\ell_g(\gamma_t, \gamma_{t+h})}{h} \leq |\dot{\gamma}_t|_g, \text{ and } \liminf_{h \searrow 0} \frac{\ell_g(\gamma_t, \gamma_{t+h})}{h} \geq |\dot{\gamma}_t|_g. \tag{5.2.6}$$

Working the neighbourhood U (as above) of p permits to assume $U \subset \mathbb{R}^{d+1}$ and allows us to assume that $p = 0$, $g(p) = \operatorname{diag}(-1, 1, \dots, 1)$, and that e_0 is future directed.

By potentially reparametrising γ, we may assume that $|\dot{\gamma}_t|_{euc} = 1$. Identifying the tangent space T_pM with $\mathbb{R}^{d+1}$ as well, we may think of $v := \dot{\gamma}_t$ as a vector in $\mathbb{R}^{d+1}$.

We start with the first inequality of (5.2.6) and prove it by contradiction. Suppose for a contradiction that there was an $\varepsilon > 0$ and a sequence $h_n \in (0, 1-t)$ with $\lim_{n\to\infty} h_n = 0$ and such that for all $n \in \mathbb{N}$, it holds

$$\frac{\ell_g(\gamma_t, \gamma_{t+h_n})}{h_n} > |\dot{\gamma}_t|_g + \varepsilon = |v|_{g(p)} + \varepsilon = |v|_{g(0)} + \varepsilon.$$

Define

$$\delta := \Big(\frac{\varepsilon}{12}\Big)^2. \tag{5.2.7}$$

Define the Lorentzian metric $m_\delta := \mathrm{diag}(-1-\delta, 1, \dots, 1)$ on U. Then we can find a neighbourhood $U_\delta \subset U$ of $p = 0$ such that for all $q \in U_\delta$ it holds

$$\|g(0) - g(q)\|_{op} \leq \delta,\ \|g(q) - m_\delta\|_{op} \leq 2\delta,\ m_\delta \succ g(q).$$

Moreover, we can shrink U_δ to even get that for each $q \in U_\delta$, $w \in T_qM$ g-causal, it holds

$$-g(q)(w,w) < -m_\delta(w,w). \tag{5.2.8}$$

Now define the set $C \subset \mathbb{R}^{d+1}$ as

$$C := \Bigg\{ u = (u_0, \dots, u_d) \in \mathbb{R}^{d+1}, u_0 \geq 0, (1+\delta)u_0^2 - \Big(\sum_{i=1}^d u_i^2\Big) \geq \Big(|v|_{g(0)} + \frac{\varepsilon}{3}\Big)^2 \Big(\sum_{i=0}^d u_i^2\Big) \Bigg\}.$$

In the spacetime $(\mathbb{R}^{d+1}, m_\delta)$, these are exactly those points u in the causal future of 0 that satisfy $\ell_{m_\delta}(0,u)^2 \geq (|v|_{g(0)} + \frac{\varepsilon}{3})^2 \mathsf{d}_{euc}(0,u)^2$. Applying [99, Theorem 2.14], we can find an open neighourhood $U'_\delta \subset U_\delta$ that is m_δ-globally hyperbolic, contains 0, and that is of the form $0 \in (I^+_{m_\delta}(q^-) \cap I^-_{m_\delta}(q^+))^\circ \subset U'_\delta$ for $q^-, q^+ \in U_\delta$. By the strong causality of the spacetime (M,g), we can find a neighbourhood $V_\delta \subset U'_\delta$ of 0 such that any g-causal path between two points in V_δ is contained in U'_δ.

Now, by the continuity of γ, there exists a $h_* > 0$ such that $\gamma([t, t+h_*]) \subset V_\delta$. For n large enough, we have that $h_n \leq h_*$, hence $p_n := \gamma_{t+h_n} \in V_\delta$. By our assumption, there exist Lipschitz continuous g-classically causal curves $\lambda_n \colon [0,1] \to M$ such that $\lambda_n(0) = p = 0$ and $\lambda_n(1) = p_n$ and

$$L_g(\lambda_n) \geq h_n\Big(\frac{\varepsilon}{2} + |v|_{g(0)}\Big). \tag{5.2.9}$$

Moreover, we have that $\lambda_n([0,1]) \subset U'_\delta$. By the definition of U_δ and U'_δ we have that (for n large enough) the λ_n are m_δ-classically causal. Then, using (5.2.8) and (5.2.9), we get that

$$\ell_{m_\delta}(0,p_n) = \ell_{m_\delta}(p,p_n) \geq L_{m_\delta}(\lambda_n) \geq L_g(\lambda_n) \geq h_n\Big(\frac{\varepsilon}{2} + |v|_{g(0)}\Big). \tag{5.2.10}$$

Moreover,

$$\mathsf{d}_{euc}(0,p_n) = \mathsf{d}_{euc}(p,p_n) = h_n|v|_{euc} + o(h_n) = h_n + o(h_n). \tag{5.2.11}$$

Now, if thinking of the $p_n = ((p_n)_0, \dots, (p_n)_d)$ as vectors in $\mathbb{R}^{d+1}$, and recalling that in the Minkowski space geodesics are direct straight connections, (5.2.10), and (5.2.11) we get that for n large enough, it holds

$$\begin{aligned}(1+\delta)(p_n)_0^2 - \Big(\sum_{i=1}^{d}(p_n)_i^2\Big) &= \ell_{m_\delta}^2(0,p_n) \geq h_n^2\Big(\frac{\varepsilon}{2} + |v|_{g(0)}\Big)^2 \\ &\geq \Big(\frac{\varepsilon}{3} + |v|_{g(0)}\Big)^2 \mathsf{d}_{euc}(0,p_n)^2 = \Big(\frac{\varepsilon}{3} + |v|_{g(0)}\Big)^2 \sum_{i=0}^{d}(p_n)_i^2.\end{aligned} \tag{5.2.12}$$

Hence, for n large enough, we have that $p_n \in C$. Note that

$$|m_\delta(v,v)| \leq |g(0)(v,v)| + 2\delta|v|_{euc}^2 = g(0)(v,v) + 2\delta. \tag{5.2.13}$$

Consider

$$p_h := \frac{\gamma_{t+h} - \gamma_t}{h} \in \mathbb{R}^{d+1} \cong T_pM.$$

As $\lim_{h\searrow 0} p_h = v$, (5.2.7) together with (5.2.13) yields that

$$\lim_{h\searrow 0} |p_h|_{m_\delta} = |v|_{m_\delta} \leq \sqrt{|v|_{g(0)}^2 + 2\delta|v|_{euc}^2} \leq |v|_{g(0)} + \frac{\varepsilon}{6}.$$

Then recalling that $\lim_{h\searrow 0} |p_h|_{euc} \to |v|_{euc} = 1$, we get that for h small enough, it holds

$$(1+\delta)(p_h)_0^2 - \Big(\sum_{i=1}^{d}(p_h)_i^2\Big) \leq \Big(|v|_{g(0)} + \frac{\varepsilon}{4}\Big)^2\Big(\sum_{i=0}^{d}(p_h)_i^2\Big).$$

As $h_n p_n = p_{h_n}$, this contradicts (5.2.12) and hence proves the first part of (5.2.6).

We will also prove the second part of (5.2.6) by contradiction. Assume there exists an $\varepsilon > 0$ and a sequence $h_n \in (0, 1-t)$ such that $\lim_{n\to\infty} h_n = 0$ and such that for all n it holds

$$0 \leq \frac{\ell_g(\gamma_t, \gamma_{t+h_n})}{h_n} < |\dot\gamma_t|_g - \varepsilon = |v|_{g(p)} - \varepsilon = |v|_{g(0)} - \varepsilon.$$

First note that this implies $|v|_{g(0)} \geq \varepsilon > 0$, i.e., v is timelike at $p = 0$. By the continuity of g, there exists an $r > 0$ such that for each $q \in B_r^{euc}(0)$, it holds

$$\|g(q) - g(0)\|_{op} \leq \frac{\varepsilon}{4}. \tag{5.2.14}$$

Define

$$v_n := \frac{\gamma_{t+h_n} - \gamma_t}{h_n},$$

and $\lambda_n\colon [0,1] \to M, s \mapsto sh_n v_n$. Now for n large enough, we have that $h_n \le r$, which shows that $\lambda_n([0,1]) \subset B^{euc}_r(0)$. Now note, $v_n \to v$ in $\mathbb{R}^{d+1}$ as $n \to \infty$, so for n large enough, λ_n is a g-classically causal and future directed curve. But then

$$\int_0^1 h_n |v_n|_{g(sh_nv_n)}\,\mathrm{d}s = L_g(\lambda_n) \le \ell_g(\gamma_t, \gamma_{t+h_n}) < h_n(|v|_{g(0)} - \varepsilon). \tag{5.2.15}$$

But by (5.2.14), we get that for n large enough and all $s \in [0,1]$ it holds

$$||v_n|_{g(sh_nv_n)} - |v|_{g(0)}| \le \frac{\varepsilon}{2}.$$

This contradicts (5.2.15) and therewith proves the second part of (5.2.6), which immediately yields the desired result. □

Lemma 5.2.11. *Let $\gamma \in \mathrm{LCC}([0,1], M)$. Then γ has at most countably many discontinuities.*

The proof of this lemma follows the same strategy as the proof of [16, Lemma 2.27], with the difference that we immediately assume left continuity of γ rather than, as they do in [16], working in a so-called forward metric spacetime.

Proof. Recall that we fixed a complete Riemannian metric h on M and denote by d_h the induced Polish distance on M. For some $c > 0$ we define the set

$$B := \{t \in [0,1] : \limsup_{s \to t} \mathsf{d}_h(\gamma_s, \gamma_t) > c\}.$$

If we can prove that B is countable for any $c > 0$, the result follows. Assume for a contradiction that there exists a $c > 0$ such that B is uncountable.

Claim: There exists a $T \in (0,1]$ such that for all $\varepsilon > 0$, it holds that $B \cap (T - \varepsilon, T]$ is uncountable.

Proof of the claim. We argue by contradiction. If the claim was false, then for every $T \in (0,1]$ there exists an $\varepsilon(T) > 0$ such that $B \cap (T - \varepsilon(T), T]$ is countable. Define

$$a := \inf\{T \in [0,1] : [T,1] \cap B \text{ is countable}\}.$$

We know that $a \le 1 - \varepsilon(1) < 1$. Assume $a > 0$. Then

$$(a,1] \cap B = \bigcup_{n \in \mathbb{N}} [a + n^{-1}, 1] \cap B \text{ is countable.}$$

But now, we know that there exists an $\varepsilon(a) > 0$ such that $B \cap (a - \varepsilon(a), a]$ is countable, hence $B \cap [a - \varepsilon(a)/2, 1]$ is countable, contradicting that a is infimal with this property. It follows that $a = 0$ and hence

$$B \cap [0,1] = \Big(\{0\} \cup \bigcup_{n \in \mathbb{N}} [n^{-1}, 1]\Big) \cap B \text{ is countable.}$$

This is a contradiction and proves the claim.

Now fix a T as in the claim. Using that γ is left continuous, we get that there exists an $\varepsilon > 0$ such that

$$\gamma((T-\varepsilon, T]) \subset B^{\mathsf{d}_h}_{c/3}(\gamma(T)). \tag{5.2.16}$$

But now, by assumption we have that $(T-\varepsilon, T] \cap B$ is uncountable, and hence nonempty, so there exists a $t \in (T-\varepsilon, T) \cap B$. By the definition of B, there exists an $s \in (T-\varepsilon, T)$ such that

$$\mathsf{d}_h(\gamma_s, \gamma_t) > c. \tag{5.2.17}$$

This contradicts (5.2.16) and hence proves that B is at most countable, finishing the proof of the lemma. $\square$

Lemma 5.2.12. *Let $\gamma\colon [0,1] \to M$ be a left continuous causal path. Then γ is differentiable at almost every $t \in [0,1]$.*

Proof. This proof uses the idea of [16, Theorem A.1]. First note that by Lemma 5.2.11, γ is continuous almost everywhere. Pick a point $s \in [0,1]$ such that γ is continuous at s. Denote $p := \gamma_s$ and pick a chart V around p such that $p = 0 \in V \subset \mathbb{R}^{d+1}$ and such that $g(p) = \mathrm{diag}(-1, 1, \dots, 1)$. We call the coordinates $(t, x_1, \dots, x_d)$ and suppose that ∂_t is future directed. Now there exists a neighbourhood $U \subset \mathbb{R}^{d+1}$ of p such that $g \prec \mathrm{diag}(-2, 1, \dots, 1) =: m_2$. Possibly shrinking U further, we may assume that U is m_2-globally hyperbolic and that it can be written as $I^+_{m_2}(p^-) \cap I^-_{m_2}(p^+)$, for some $p^-, p^+ \in V$. Using the strong causality, we may further restrict to a set W such that $p \in W \subset U$ and such that every g-causal path with endpoints in W is contained in U. Now there exists a linear transform into coordinates $(y_0, y_1, \dots, y_d)$ such that the future cone of m_2 is contained in the set

$$C := \Big\{ \sum_{i=0}^{d} \alpha_i \partial_{y_i} | \alpha_0, \dots, \alpha_d \geq 0 \Big\}.$$

As $m_2 \succ g$ in U, we get that the future cone of g is contained in C at every point $q \in U$. Now as γ is continuous at s, there exists an $\varepsilon > 0$ such that $\gamma([s-\varepsilon, s+\varepsilon]) \subset W$. Suppose that there exists an $i \in \{0, \dots, d\}$ and $\sigma \in [s, s+\varepsilon]$ such that $y_i(\gamma_s) > y_i(\gamma_\sigma)$. Then $w := \gamma_\sigma - \gamma_s \in \mathbb{R}^{d+1} \setminus C$. But by assumption, we have that $\gamma_s \leq_g \gamma_\sigma$, hence the definitions of W and U yield that $\gamma_s \leq_{m_2} \gamma_\sigma$. Using that m_2-geodesics are straight lines yields that w has to lie in the future cone of m_2, which yields a contradiction. Hence we have that for all i, it holds $y_i(\gamma_s) \leq y_i(\gamma_\sigma)$. Similarly, we get that for $\sigma \in [s-\varepsilon, s]$ and all i, it holds $y_i(\gamma_\sigma) \leq y_i(\gamma_s)$. Hence, we get that for every continuity point $s \in [0,1]$ of γ, there exist coordinates y and an $\varepsilon > 0$ such that $y \circ \gamma$ is increasing in every variable in $[s-\varepsilon, s+\varepsilon]$. Hence, for every continuity point $s \in [0,1]$, there exists an $\varepsilon > 0$ such that γ is differentiable almost everywhere in $[s-\varepsilon, s+\varepsilon]$.

Now Lemma 5.1.3 yields that γ is almost everywhere differentiable. $\square$

This leads us to the first important result of this chapter, stating that the causal speed can be identified as the causal length of the derivative. This can be seen as a Lorentzian version of [25, Corollary 3.8].

Corollary 5.2.13. *Let g be continuous and strongly causal. Let $\gamma \in \mathrm{LCC}([0,1], M)$. Then for almost every $t \in [0,1]$, the causal speed is given by*

$$|\dot{\gamma}_t| = |\dot{\gamma}_t|_g. \tag{5.2.18}$$

Proof. By Lemma 5.2.12, we have that γ is differentiable almost everywhere. Then by Lemma 5.2.10, we have that for almost every $t \in [0,1]$, it holds

$$\lim_{h \searrow 0} \frac{\ell_g(\gamma_t, \gamma_{t+h})}{h} = |\dot{\gamma}_t|_g. \tag{5.2.19}$$

To conclude, we use Definition 2.3.5 (see also [16, (2.28)]), which states that for almost every t, it holds

$$\lim_{h \searrow 0} \frac{\ell_g(\gamma_t, \gamma_{t+h})}{h} = |\dot{\gamma}_t|.$$

$\square$

Lemma 5.2.14. *Let M be a manifold and g be a continuous Lorentzian metric on M. Then* $\mathrm{Vis}(M) = M$*.*

Proof. Let $p \in M$. Fix an open neighbourhood $U \subset M$ of p and suppose from now on that $U \subset \mathbb{R}^{d+1}$. Pick a vector $v \in \mathbb{R}^{d+1}$ with $g(p)(v,v) < -\alpha < 0$ and such that v is future directed. We may assume that $|v|_{euc} = 1$. Then, there exists a neighbourhood $V \subset U$ of p such that v is future directed and $g(q)(v,v) < -\frac{\alpha}{2} < 0$ at every point $q \in V$. Let $\theta > 0$ such that $B_{4\theta}^{\mathbb{R}^{d+1}}(p) \subset V$. For $y \in B_\theta(p)$, define the causal curve $\gamma^y \colon [0,1] \to V, t \mapsto y + \theta(2t-1)v$. Then $\gamma_y([0,1]) \subset B_{4\theta}(p)$. Consider the test plan $\boldsymbol{\pi} \in \mathcal{P}(\mathrm{LCC}([0,1], M))$ given by

$$\mathrm{d}\boldsymbol{\pi}(\gamma) := \begin{cases} \frac{1}{\mathcal{L}^{d+1}(B_\theta(p))} \mathrm{d}\mathcal{L}^{d+1}(y) & \text{if } \gamma = \gamma^y \text{ for some } y \in B_\theta(p), \\ 0 & \text{otherwise.} \end{cases}$$

By Corollary 5.2.13, we get that

$$|\dot\gamma_t|\mathrm{d}\boldsymbol{\pi}(\gamma)\mathrm{d}t \geq \begin{cases} \frac{\sqrt{\alpha}\theta}{2\mathcal{L}^{d+1}(B_\theta(p))}\mathrm{d}\mathcal{L}^{d+1}(y)\mathrm{d}t & \text{if } \gamma = \gamma^y \text{ for some } y \in B_\theta(p), \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$\mathrm{d}(e_t)_\#|\dot\gamma_t|\boldsymbol{\pi} \geq \frac{\sqrt{\alpha}\theta}{2\mathcal{L}^{d+1}(B_\theta(p))}\mathbb{1}_{B_\theta(p+\theta(2t-1)v)}\mathrm{d}\mathcal{L}^{d+1},$$

hence, writing $(\mathsf{e}(t,\gamma))_\#|\dot\gamma_t|\boldsymbol{\pi} = \rho_{\boldsymbol{\pi}}\mathrm{d}\mathcal{L}^{d+1}$, we get that for every $q \in B_{\theta/8}(p)$, it holds

$$\rho_{\boldsymbol{\pi}}(q) \geq \frac{\sqrt{\alpha}\theta}{16\mathcal{L}^{d+1}(B_\theta(p))} > 0.$$

Hence, $p \in \mathrm{Vis}(M)$. As p was arbitrary, this proves the lemma. □

The next proposition has been proved for the smooth case in [98, Theorem 1.19] and [16, Theorem A.2]. We will closely follow their argument.

Lemma 5.2.15. *Let f be a causal function. Then for almost every $p \in \{|f| < \infty\}$, there exists a neighbourhood W of p such that the distributional derivative of f restricted to W is given by a co-vector valued, finite Radon measure $\mathcal{D}f$. In particular, f is differentiable almost everywhere on $\{|f| < \infty\}$.*

Proof. Lemma 2.3.11 and Lemma 5.2.14 yield that for almost every $p \in \mathrm{Vis}(M) \cap \{|f| < \infty\} = \{|f| < \infty\}$, there exists an open neighbourhood U of p such that f is finite on U. We may assume that $U \subset \mathbb{R}^{d+1}$ and that U is relatively compact. We may furthermore assume that $g(p) = \mathrm{diag}(-1, 1, \dots, 1)$. Now we can find an open neighbourhood $p \in V \subset U$ such that $m_{1/2} := \mathrm{diag}(-1, 2, \dots, 2) \prec g$ on V. There exists a basis $b_0, \dots, b_d$ of $\mathbb{R}^{d+1}$ such that the cone $C := \{\sum_{i=0}^d \alpha_i b_i | \alpha_0, \dots, \alpha_d \geq 0\}$ is contained in the causal future cone of $m_{1/2}$. Then, C is contained in the g-causal future cone of T_qM for all $q \in V$. Now there exist coordinates $y = (y_0, \dots, y_d)$ of $\mathbb{R}^{d+1}$ such that $\partial_{y_i} = b_i$ for all $0 \leq i \leq d$. In these coordinates, we have that for each $x, z \in V$ with $y_i(x) \leq y_i(z)$ for all $0 \leq i \leq d$, it holds that $x \leq_g z$. Hence the function f is monotone in each variable, which by [37, Theorem 4] yields $\mathcal{L}^{d+1}$-measurability (and hence vol_g-measurability). We may assume that $p = 0$ and that $W := (-a, a)^{d+1} \subset [-a, a]^{d+1} \subset V$ for some $a > 0$. Now for each $\beta_0, \dots, \beta_d \in [-a, a]$, and all $i \in \{0, \dots, d\}$, it holds

$$t \mapsto f(\beta_0, \dots, \beta_{i-1}, t, \beta_{i+1}, \dots, \beta_d), \quad t \in [-a, a],$$

is a monotone (and hence a bounded variation function in one variable) whose distributional differential is bounded above by $f(a, \dots, a) - f(-a, \dots, -a) < \infty$. By [4, Remark

3.104], this proves that f is locally a bounded variation function, i.e., its distributional derivative is a finite co-vector valued Radon measure $\mathcal{D}f$ on $[-a,a]$. Furthermore, [37, Theorem 14] states that f is $\mathcal{L}^{d+1}$-almost everywhere differentiable in $[-a,a]$.

$\square$

Before stating the next lemma, it is worth understanding the distributional derivative $\mathcal{D}f$ on $\{|f|<\infty\}$ a little better. Fix $p \in \{|f|<\infty\}$ as in Lemma 5.2.15 and choose an open neighbourhood W of p accordingly. Assume that $W \subset \mathbb{R}^{d+1}$. Fix a vector $v \in \mathbb{R}^{d+1}$ such that $g(p)(v,v)<0$ and shrink W to a neighbourhood $U \ni p$ such that v is future directed and timelike everywhere in U. Then, as $\mathcal{D}f$ is the distributional derivative of f, we have that for every $\varphi \in C_c^\infty(U)$, it holds

$$\int_U \varphi \, \mathrm{d}\mathcal{D}f(v) = \lim_{h \searrow 0} \int_U \varphi(y) \frac{f(y+hv)-f(y)}{h} \, \mathrm{d}\mathcal{L}^{d+1}(y).$$

This yields that $\mathcal{D}f(v)$ is the limit measure of the sequence $\frac{f(y+hv)-f(y)}{h}\mathrm{d}\mathcal{L}^{d+1}(y)$, as $h \searrow 0$, in the narrow topology.

Lemma 5.2.16. *Let p, v, U be as above. Given $q \in U$, $\delta, s > 0$ such that $B_{2\delta+2s}(q) \subset U$, it holds that*

$$\int_{-s}^{s} \int_{\overline{B_\delta(q)}} \mathrm{d}\mathcal{D}f(v)(y+tv)\mathrm{d}t \geq \int_{\overline{B_\delta(q)}} f(y+sv) - f(y-sv) \, \mathrm{d}\mathcal{L}^{d+1}(y).$$

Proof. By the upper semi-continuity on closed sets under the narrow convergence of measures, and using a telescoping argument, we get that

$$\begin{aligned}
&\int_{-s}^{s} \int_{\overline{B_\delta(q)}} \mathrm{d}\mathcal{D}f(v)(y+tv)\mathrm{d}t \\
&\geq \limsup_{n\to\infty} \int_{-s}^{s} \int_{\overline{B_\delta(q)}} \frac{n}{2s}\Big(f\Big(y+tv+\frac{2s}{n}v\Big) - f(y+tv)\Big) \, \mathrm{d}\mathcal{L}^{d+1}(y)\mathrm{d}t \\
&= \limsup_{n\to\infty} \int_{\overline{B_\delta(q)}} \int_{-s}^{s} \frac{n}{2s}\Big(f\Big(y+tv+\frac{2s}{n}v\Big) - f(y+tv)\Big) \, \mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y) \\
&= \limsup_{n\to\infty} \int_{\overline{B_\delta(q)}} \frac{n}{2s} \int_{[0,\frac{2s}{n})} f\Big(y+\Big(t+s+\frac{2s}{n}\Big)v\Big) - f(y+(t-s)v) \, \mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y) \\
&\geq \int_{\overline{B_\delta(q)}} f(y+sv) - f(y-sv) \, \mathrm{d}\mathcal{L}^{d+1}(y),
\end{aligned}$$

where the last inequality follows from the fact that f is increasing in v-direction. $\square$

Lemma 5.2.17. *Given a causal function f on (M,g) and a neighbourhood W as in Lemma 5.2.15. Suppose that $W \subset \mathbb{R}^{d+1}$. Denote by $\mathcal{D}f$ the distributional derivative of*

f in W and split it into its absolutely continuous part with respect to $\mathcal{L}^{d+1}$ denoted by $\mathcal{D}f^a = Df\mathcal{L}^{d+1}$ and its singular part $\mathcal{D}f^s$. Then for almost every $p \in W$, we have that if f is differentiable at p, it holds that $\mathrm{d}f(p) = Df(p)$.

Proof. By [4, Theorem 3.83], we have that for almost every p it holds that f is approximately differentiable at p and the approximate differential equals the density Df of the absolutely continuous part $\mathcal{D}f^a$ of the distributional derivative of f. Hence, this holds for almost every p, which satisfies that f is differentiable at p. But then the approximate differential and the differential are equal at such points, hence $Df = \mathrm{d}f$ almost everywhere. □

From now on, when writing $\mathrm{d}f$ we refer to the absolutely continuous part of the distributional derivative of f, which is a covector valued function that is almost everywhere defined and locally integrable in $\{|f| < \infty\}$. The following lemma has been proved in Riemannian signature in [102] (here Lemma 3.3.11).

Lemma 5.2.18. *Let $V \subset M$ be an open set that lies in one coordinate patch. Let $k \in L^1_{\mathrm{loc}}(V, \mathrm{vol}_g)$ and assume that $V \subset \mathbb{R}^{d+1}$. Let $x \in V$ and $\theta > 0$, such that $B^{euc}_{4\theta}(x) \subset V$. Let $\Gamma \subset B^{euc}_\theta(0)$ be a countable set. For $\delta \in (0,\theta) \cap \mathbb{Q}$, $\dot\gamma \in \Gamma$, we define*

$$F^k_{\dot\gamma,\delta,x} \colon t \mapsto \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(x)})} \int_{\overline{B_\delta(x)}} |\dot\gamma|_{g(y+t\dot\gamma)} k(y + t\dot\gamma)\, \mathrm{d}\mathcal{L}^{d+1}(y), \quad t \in (-1,1).$$

Then for $\mathcal{L}^{d+1}$-a.e. $x \in V$ and all $\delta \in (0,\theta) \cap \mathbb{Q}$, $\dot\gamma \in \Gamma$, we have that $t = 0$ is a Lebesgue point of $F^k_{\dot\gamma,\delta,x}$.

Here, $|\dot\gamma|_g$ should be interpreted as $\sqrt{|g(\dot\gamma,\dot\gamma)|}$, as we have not made any assumptions on whether $\dot\gamma$ is causal.

Proof. As g is continuous, we get that $k \in L^1_{\mathrm{loc}}(V, \mathcal{L}^{d+1})$. Fix $w \in V$, and choose $\theta > 0$ accordingly. Moreover, fix δ and $\dot\gamma$. For any $z \in B^{euc}_\theta(w)$, the Lebesgue differentiation theorem yields that $\mathcal{L}^1$-almost every $t \in (-1,1)$ is a Lebesgue point of $F^k_{\dot\gamma,\delta,z}$. In other words that means that for $\mathcal{L}^1$-almost every y on the line $\{z + t\dot\gamma, |t| < 1\}$, $t = 0$ is a Lebesgue point of $F^k_{\dot\gamma,\delta,y}$. The set of non-Lebesgue points on that line shall be denoted by $N_{\dot\gamma,\delta,z}$, and it holds that $\mathcal{L}^1(N_{\dot\gamma,\delta,z}) = 0$. Let $H_{\dot\gamma,w}$ be the d-dimensional hyperplane in $\mathbb{R}^{d+1}$ through w that is orthogonal to $\dot\gamma$, and intersected with $B^{euc}_\theta(w)$. Then, by Fubini's Theorem, it holds

$$\mathcal{L}^{d+1}\Big(\bigcup_{z \in H_{\dot\gamma,w}} N_{\dot\gamma,\delta,z}\Big) = \int_{H_{\dot\gamma,w}} \mathcal{L}^1(N_{\dot\gamma,\delta,z})\, \mathrm{d}\mathcal{L}^d(z) = 0.$$

Hence for $\mathcal{L}^{d+1}$-almost every $y \in B_\theta^{euc}(w)$, $t = 0$ is a Lebesgue point of $F^k_{\dot\gamma,\delta,y}$. Note that, by construction, $\dot\gamma$ and δ vary in a countable set. This shows the claim in an open neighbourhood around w. As w was arbitrary, the proof is complete by Lemma 5.1.3. $\square$

Under the same assumptions on $k, \theta, \delta, \Gamma$, Lemma 5.2.18 holds analogously for

$$\tilde F^k_{\dot\gamma,\delta,x} \colon t \mapsto \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(x)})} \int_{\overline{B_\delta(x)}} k(y + t\dot\gamma)\,\mathrm{d}\mathcal{L}^{d+1}(y), \quad t \in (-1,1). \tag{5.2.20}$$

The following proposition describes the maximal weak subslope of a causal function f. This is the last ingredient we need to prove infinitesimal Minkowskianity.

Proposition 5.2.19. *Let g be continuous and strongly causal and $f \colon M \to \overline{\mathbb{R}}$ be a causal function. Then, the maximal weak subslope $|\mathrm{d}f|$ is given by*

$$|\mathrm{d}f|(p) = G(p) := \begin{cases} |\nabla_g f|_g(p) & \text{if } |f(p)| < \infty \text{ and } f \text{ is differentiable at } p, \\ +\infty & \text{otherwise.} \end{cases} \tag{5.2.21}$$

Remark 5.2.20. *This proposition can be seen as a Lorentzian counterpart to [102, Proposition 4.24]. The proof essentially follows a similar strategy, with the additional difficulty that f is not necessarily C^1 and hence the distributional derivative might be measure valued. In the Riemannian case, proving the analogue statement for C^1-functions sufficed to identify the synthetic Sobolev space as the classical one (see [102, Section 4.4], Subsection 3.3.2), as the theory from [7, 8] allowed to pass from C^1 to all Sobolev functions.*

Before giving the proof, we summarise the strategy: We start with verifying that G is a weak subslope. Then, to prove maximality, we argue by contradiction and assume that $|\mathrm{d}f| > G$ on a set of positive measure:

- We pick a point p in $\{|\mathrm{d}f| > G\}$ at which f is differentiable and find a timelike vector $v \in T_pM$ such that $G(p)|v|_g = |\nabla_g f|_g(p)|v|_g \approx \langle \nabla_g f(p), v\rangle_g = (f \circ \gamma)'(0) = \lim_{h \searrow 0} \frac{f(p+hv) - f(p-hv)}{2h}$, where $\gamma \colon t \mapsto p + tv$. Recalling Definition 2.3.8, this suggests that $|\mathrm{d}f| \le G$ along the curve γ.
- To make this idea rigorous, we build a test plan of short parallel curves in v-direction around p and use Lemma 5.1.2 to localise away from the support of the singular part $\mathcal{D}f^s$ of the distributional derivative of $\mathcal{D}f$ of f.

Proof of Proposition 5.2.19. We start by proving that G is a weak subslope. Fix an arbitrary test plan $\boldsymbol{\pi} \in \mathcal{P}(\mathrm{LCC}([0,1], M))$. Moreover, let $N \subset M$ be a vol_g-null set such that f is differentiable or infinite on $M \setminus N$, which exists by Lemma 5.2.15. As in [119, Lemma 3.5], we have that for $\boldsymbol{\pi}$-almost every path γ, it holds that

$$\mathcal{L}^1(\gamma^{-1}(N)) = 0. \tag{5.2.22}$$

Consider a path $\gamma \in \mathrm{LCC}([0,1], M)$ such that (5.2.22) holds. We distinguish two cases.

Case 1. $f(\gamma([0,1])) \subset \mathbb{R}$.
Now, f is differentiable $\mathcal{H}^1$-almost everywhere on $\gamma([0,1])$ and the chain rule yields that for almost every $t \in [0,1]$, it holds

$$(f \circ \gamma)'(t) = \mathrm{d}f_{\gamma_t}(\dot{\gamma_t}) \geq |\nabla_g f|_g(\gamma_t)|\dot{\gamma}_t|_g, \tag{5.2.23}$$

where the last inequality follows from the reverse Cauchy-Schwartz inequality. Note, that we can apply it because $\nabla_g f$ is zero or causal and past directed by Lemma 5.2.7 and $\dot{\gamma}_t$ is zero or causal and future directed by Lemma 5.2.8. Now, as $f \circ \gamma \colon [0,1] \to \mathbb{R}$ is a monotone function, and $(f \circ \gamma)'(t)$ the absolutely continuous part of the distributional derivative of $f \circ \gamma$, this yields

$$f(\gamma_1) - f(\gamma_0) \geq \int_0^1 (f \circ \gamma)'(t)\,\mathrm{d}t \geq \int_0^1 |\nabla f|_g(\gamma_t)|\dot{\gamma}_t|_g\,\mathrm{d}t = \int_0^1 G(\gamma_t)|\dot{\gamma}_t|\,\mathrm{d}t. \tag{5.2.24}$$

Case 2. $f(\gamma([0,1])) \not\subset \mathbb{R}$.
Then, as f is causal, we get that $f(\gamma_0) = -\infty$ or $f(\gamma_1) = +\infty$. In both cases, we have that

$$f(\gamma_1) - f(\gamma_0) = +\infty,$$

hence G is a weak subslope.

Now, we want to verify that G is the maximal weak subslope. Suppose that was not the case. Then there exists an $\varepsilon > 0$ such that

$$\mathrm{vol}_g(\{|\mathrm{d}f| \geq G + \varepsilon\}) > 0.$$

Denote $S := \{|\mathrm{d}f| \geq G + \varepsilon\}$. Then, G is finite at vol_g-almost every point in S, hence, by the definition of G, f is finite at almost every point of S. We may assume that S lies inside one coordinate patch and will from now on work in $\mathbb{R}^{d+1}$. Define Γ to be a countable and dense set in $\partial B_1^{euc}(0) \subset \mathbb{R}^{d+1}$. There exists a $\zeta > 1$ such that the set

$$\mathcal{L}^{d+1}(\{|\nabla_g f|_g \leq \zeta\} \cap S) > 0. \tag{5.2.25}$$

By Lemma 5.1.4, Lemma 5.1.2, Lemma 5.2.18, and the Lebesgue differentiation theorem, we can choose a point $p \in S$ and an open neighbourhood W of p such that

- p is a $\mathcal{L}^{d+1}$-density point of $S \cap \{|\nabla_g f|_g \leq \zeta\}$ and a Lebesgue point of $G, \mathrm{d}f, \min(|\mathrm{d}f| - G, \varepsilon)$.
- f is of bounded variation and f is differentiable almost everywhere in W (see Lemma 5.2.15 and [4]).
- W is relatively compact and $W \subset \{|f| < \infty\}$ (see Lemma 2.3.11).
- there exists a $\theta > 0$ such that for all $\delta \in (0, \theta) \cap \mathbb{Q}$, $\dot{\gamma} \in \theta\Gamma$, it holds that $t = 0$ is a Lebesgue point of $F^k_{\dot{\gamma},\delta,p}$ and $\tilde{F}^k_{\dot{\gamma},\delta,p}$, as they are defined in Lemma 5.2.18 and (5.2.20) for $k = \mathrm{d}f, G, \min(|\mathrm{d}f| - G, \varepsilon)$. In the case of $\mathrm{d}f$ this shall be understood component wise.
- $\lim_{\delta \to 0} \frac{\mathcal{D}f^s(B_\delta(p))}{\delta^{d+1}} = 0$ (see Lemma 5.1.2).
- for every $\dot{\gamma} \in \Gamma$ and $\delta \in (0,1) \cap \mathbb{Q}$ small enough, $t = 0$ is a Lebesgue point of $L^{E_{\mathcal{D}f,\delta}}_{p,\dot{\gamma}}$ as defined in Lemma 5.1.4 and Lemma 5.1.1.

Define

$$\epsilon := \frac{\varepsilon}{200(1+\zeta)}. \tag{5.2.26}$$

Now, note that by Lemma 5.2.7, $\nabla_g f(p) \in T_pM$ is causal and past directed or 0. If $\nabla_g f(p) \neq 0$, there exists a timelike and future directed vector $\tilde{v} \in \mathbb{R}^{d+1} \cong T_pM$, such that $\frac{1}{2} \leq |\tilde{v}|_{euc} \leq 2$ and

$$\langle \nabla_g f, \tilde{v} \rangle_g(p) \leq (\epsilon + |\nabla_g f|_g)|\tilde{v}|_g(p). \tag{5.2.27}$$

Now, from our choice of Γ, we can approximate $\tilde{v}|\tilde{v}|^{-1}_{euc}$ to get a timelike and future directed vector $v \in \Gamma$ such that

$$|\nabla_g f|_g |v|_g(p) \leq \langle \nabla_g f, v \rangle_g(p) \leq (3\epsilon + |\nabla_g f|_g)|v|_g(p). \tag{5.2.28}$$

If $\nabla_g f(p) = 0$, then any timelike and future directed $v \in \Gamma$ will satisfy (5.2.28), so we may fix an arbitrary one.

Using a linear transformation $A \in SO(d+1)$ and a translation, we may assume that $v = e_0$ and $p = 0$. Then, by the continuity of g, we have that e_0 is timelike and future directed in a neighbourhood of $p = 0$, so by possibly shrinking $W \ni p = 0$, we get that e_0 is timelike and future directed in W and that there exists a $\lambda \in (0,1)$ such that for all $q \in W$, it holds

$$|e_0|_{g(q)} \geq \lambda. \tag{5.2.29}$$

Now, we can choose $\delta, s \in (0,1) \cap \mathbb{Q}$ such that $B_{2\delta+2s}(0) \subset W$ and such that for $k \in \{G, \mathrm{d}f, \min(\varepsilon, |\mathrm{d}f| - G)\}$, it holds

$$\frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} |k(y) - k(0)| \,\mathrm{d}\mathcal{L}^{d+1}(y) \le \lambda\epsilon, \tag{5.2.30}$$

$$\frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \Bigg| \frac{1}{2s} \int_{-s}^{s} \int_{\overline{B_\delta(0)}} k(y + te_0) |e_0|_{g(y+te_0)} \,\mathrm{d}\mathcal{L}^{d+1}(y)\mathrm{d}t \\ - \int_{\overline{B_\delta(0)}} k(y) |e_0|_{g(y)} \,\mathrm{d}\mathcal{L}^{d+1}(y) \Bigg| \le \lambda\epsilon, \tag{5.2.31}$$

as well as

$$\frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} |\langle \nabla_g f, e_0\rangle_g(y) - \langle \nabla_g f, e_0\rangle_g(0)| \,\mathrm{d}\mathcal{L}^{d+1}(y) \le \lambda\epsilon, \tag{5.2.32}$$

$$\frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \Bigg| \frac{1}{2s} \int_{-s}^{s} \int_{\overline{B_\delta(0)}} \langle \nabla_g f, e_0\rangle_g(y + te_0) \,\mathrm{d}\mathcal{L}^{d+1}(y)\mathrm{d}t \\ - \int_{\overline{B_\delta(0)}} \langle \nabla_g f, e_0\rangle_g(y) \,\mathrm{d}\mathcal{L}^{d+1}(y) \Bigg| \le \lambda\epsilon, \tag{5.2.33}$$

and

$$\frac{|\mathcal{D}f^s(\overline{B_\delta(0)})|}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \le \lambda\epsilon \text{ and } \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{-s}^{s} \int_{\overline{B_\delta(0+se_0)}} \mathrm{d}\mathcal{D}f^s(e_0) \,\mathrm{d}t \le 2s\lambda\epsilon. \tag{5.2.34}$$

For $y \in \overline{B_\delta(p)} = \overline{B_\delta(0)}$ define the causal curve

$$\gamma^y \colon [0,1] \to W, t \mapsto y + 2s\Big(t - \frac{1}{2}\Big)v = y + 2s\Big(t - \frac{1}{2}\Big)e_0. \tag{5.2.35}$$

Now, define the test plan $\boldsymbol{\pi}$ via

$$\mathrm{d}\boldsymbol{\pi}(\gamma) := \begin{cases} \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})}\mathrm{d}\mathcal{L}^{d+1}(y) & \text{if } \gamma = \gamma^y \text{ for some } y \in \overline{B_\delta(0)}, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.36}$$

Using Corollary 5.2.13, we get that

$$\int \int_0^1 G(\gamma_t)|\dot{\gamma}_t| \,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) \\ = \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} \int_0^1 |\nabla_g f|_g\Big(y + 2s\Big(t - \frac{1}{2}\Big)e_0\Big) 2s|e_0|_g \,\mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y) \\ = \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} \int_{-s}^{s} |\nabla_g f|_g(y + te_0)|e_0|_g \,\mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y). \tag{5.2.37}$$

Using (5.2.28) and (5.2.29), combined with (5.2.30) - (5.2.33) with $k = G$ we get that

$$\frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})}\Bigg| \int_{\overline{B_\delta(0)}} \int_{-s}^{s} |\nabla_g f|_g (y + te_0)|e_0|_g \,\mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y) \\ - \int_{\overline{B_\delta(0)}} \int_{-s}^{s} \langle \nabla_g f, e_0\rangle_g (y + te_0) \,\mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y)\Bigg| \leq 2s(4\lambda + 3|e_0|_{g(0)})\epsilon. \quad (5.2.38)$$

Using furthermore (5.2.30), (5.2.29), and (5.2.31) with $k \in \{\min(\varepsilon, |\mathrm{d}f| - G), G\}$ yields

$$\frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} \int_{-s}^{s} \big(|\mathrm{d}f| - |\nabla_g f|_g (y + te_0)\big)|e_0|_g \,\mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y) \geq 2s(|e_0|_{g(0)}\varepsilon - 4\lambda\epsilon). \tag{5.2.39}$$

Recall that our choice of W (see the discussion leading up to (5.2.29)) allows us to apply Lemma 5.2.16. Now, Lemma 5.2.17 together with Lemma 5.2.16 and (5.2.34) gives that

$$\begin{aligned} &\frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} \int_{-s}^{s} \langle \nabla_g f, e_0\rangle_g (y + te_0) \,\mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y) \\ &= \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{-s}^{s} \int_{\overline{B_\delta(0)}} \mathrm{d}\mathcal{D}f(e_0)(y + te_0)\mathrm{d}t - \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{-s}^{s} \int_{\overline{B_\delta(p)}} \mathrm{d}\mathcal{D}f^s(e_0)(y + te_0)\mathrm{d}t \\ &\geq \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} f(y + se_0) - f(y - se_0) \,\mathrm{d}\mathcal{L}^{d+1}(y) - 8s\lambda\epsilon. \qquad (5.2.40) \end{aligned}$$

Then, a similar computation as in (5.2.37) followed by the combination of (5.2.38), (5.2.39), and (5.2.40), yields that

$$\begin{aligned} &\int \int_0^1 |\mathrm{d}f|(\gamma_t)|\dot\gamma_t| \,\mathrm{d}t\mathrm{d}\boldsymbol{\pi}(\gamma) \qquad (5.2.41) \\ &= \frac{1}{\mathcal{L}^{d+1}(B_\delta(0))} \int_{\overline{B_\delta(0)}} \int_{-s}^{s} |\mathrm{d}f|(y + se_0)|e_0|_{g(y+se_0)} \,\mathrm{d}t\mathrm{d}\mathcal{L}^{d+1}(y) \\ &\geq \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} f(y + se_0) - f(y - se_0) \,\mathrm{d}\mathcal{L}^{d+1}(y) + 2s(|e_0|_{g(0)}\varepsilon - (8\lambda + 3|e_0|_{g(0)})\epsilon) \\ &> \frac{1}{\mathcal{L}^{d+1}(\overline{B_\delta(0)})} \int_{\overline{B_\delta(0)}} f(y + se_0) - f(y - se_0) \,\mathrm{d}\mathcal{L}^{d+1}(y) = \int f(\gamma_1) - f(\gamma_0) \,\mathrm{d}\boldsymbol{\pi}(\gamma), \qquad (5.2.42) \end{aligned}$$

where the last step follows from (5.2.26). Hence, $|\mathrm{d}f|$ is not a weak subslope. This is a contradiction and finishes the proof. □

This allows us to prove the main theorem:

Theorem 5.2.21 (Theorem 4)**.** *Let M be a $(d+1)$-dimensional manifold and g be a continuous Lorentzian metric on M such that (M, g) is a causally simple spacetime. Then the metric measure spacetime $(M, \ell_g, \mathrm{vol}_g)$ is infinitesimally Minkowskian.*

Proof. Take two causal functions f_1, f_2 on $(M, \ell_g, \mathrm{vol}_g)$. Note that

$$\{|f_1 + f_2| < \infty\} = \{|2f_1 + f_2| < \infty\} = \{|f_1| < \infty\} \cap \{|f_2| < \infty\}.$$

We aim to prove

$$2|\mathrm{d}(f_1 + f_2)|^2 + 2|\mathrm{d}f_1|^2 = |\mathrm{d}f_2|^2 + |\mathrm{d}(2f_1 + f_2)|^2 \ \mathrm{vol}_g - \text{a.e.} \tag{5.2.43}$$

Moreover note that $f_1 + f_2$ and $2f_1 + f_2$ are differentiable wherever both f_1 and f_2 are. Define

$$N := (\{|f_1| < \infty\} \cap \{|f_2| < \infty\}) \setminus \{f_1 \text{ and } f_2 \text{ are differentiable}\}. \tag{5.2.44}$$

Using that N is a vol_g-null set, Proposition 5.2.19 yields that in $\{|f_1| < \infty\} \cap \{|f_2| < \infty\}$, it holds

$$\begin{aligned} &2|\mathrm{d}(f_1 + f_2)|^2 + 2|\mathrm{d}f_1|^2 = 2|\nabla_g(f_1 + f_2)|_g^2 + 2|\nabla_g f_1|_g^2 \\ &= |\nabla_g f_2|_g^2 + |\nabla_g(2f_1 + f_2)|_g^2 = |\mathrm{d}f_2|^2 + |\mathrm{d}(2f_1 + f_2)|^2 \ \mathrm{vol}_g - \text{a.e.} \end{aligned} \tag{5.2.45}$$

Indeed, the middle equality follows from the fact that taking derivatives is linear and that g is a bilinear form. In any other case, both sides of (5.2.43) are infinite and hence equal. This finishes the proof. $\square$

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