Deconstructing Spacetime: Properties of Emergent Geometry in Matrix Theories



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Abstract

This thesis explores mechanisms for the emergence of spacetime from Matrix Theories. The overarching goal of this programme is to identify a microscopic quantum-mechanical description of a large number of degrees of freedom, which reduces to General Relativity in the large-distance limit.

We introduce and motivate the concept of emergent gravity in Chapter 1.

In Chapter 2, we discuss BFSS theory, a proposed non-perturbative description of M-Theory in the light-cone frame. We review the BFSS conjecture and describe the rich physical content of the theory. Particular attention is given to black hole configurations and how they are characterised in the theory.

Chapter 3 contains a proof of a soft graviton factorisation theorem in BFSS theory. The validity of such a theorem is equivalent to the conservation of charges associated with asymptotic symmetries, in the continuum M-Theory limit. Thus, proving a soft theorem constitutes an important consistency check of the conjecture. We take a direct approach by defining vertex-like operators, which are suitable to represent gravitons, and we show that their correlation functions satisfy the soft factorisation property.

In Chapter 4, we consider deformations of BFSS and IKKT theories, obtained through torus compactification with anti-periodic and mixed fermionic boundary conditions. We show how this procedure leads to a qualitatively different theory, with no supersymmetry and a mass term, but which retains a non-trivial fermionic sector. We leverage such a construction to explicitly realise a black hole configuration in terms of Matrix Theory degrees of freedom.

This thesis makes a step forward in the understanding of Quantum Gravity through the lens of Matrix Theory. The approach of deconstructing space and time into fundamental constituents is very powerful and offers a promising perspective on the nature of gravity, one where large-distance properties emerge in a particular limit of a more fundamental theory.

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Statement of Originality

This thesis is the work of the author, except where otherwise stated. It has not been submitted in whole or in part for any other qualification.
The thesis contains material based on the following publications:

- Davide Laurenzano and John Wheater, *"A soft theorem from vertex-like operators in BFSS theory,"* Journal of High Energy Physics **02** (2026) 045, `arXiv:2510.15488 [hep-th]`.

- Davide Laurenzano and John Wheater, *"Fuzzy Black Holes from Mass Generation in Matrix Compactification," Phys.Rev.D 114 (2026) 4, 046019*, `arXiv:2511.10430 [hep-th]`.

The text has been adapted and expanded for the purposes of this thesis. Except where specifically cited, the remaining material represents original work completed by the author.

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1 Introduction

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1.1 The problem of Quantum Gravity

The study of Quantum Gravity (QG) presents unprecedented challenges, pushing the limits of our understanding of the physical laws that govern space and time themselves. Two difficulties complicate the search for a consistent theory of Quantum Gravity.

The first arises from the unique nature of gravity compared to the other fundamental interactions. This distinction originates in Einstein's formulation of General Relativity in 1915, in which gravity is not described as a conventional force but rather as a manifestation of the curvature of spacetime induced by the presence of matter and energy. The fundamental object is the metric $g_{\mu\nu}$, which, for a given matter distribution encoded in the energy-momentum tensor $T_{\mu\nu}$, is obtained as a solution to Einstein's equations

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = \frac{8\pi G_N}{c^4} T_{\mu\nu} \,.$$

Here, the Ricci tensor $R_{\mu\nu}$ and the Ricci scalar R characterise the curvature of spacetime, and the dimensional parameters are the cosmological constant Λ, the gravitational

constant G_N and the speed of light c. The Ricci tensor and the Ricci scalar are defined as

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} \qquad R = g^{\mu\nu} R_{\mu\nu}$$

in terms of the Ricci tensor

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma}\,,$$

where the Christoffel symbols are defined as

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\sigma\nu} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right) .$$

The mostly minus signature for Lorentzian metrics is assumed in the following, including for the Minkowski metric $\eta_{\mu\nu}$, which raises and lowers indices in flat space, except where explicitly noted otherwise. The notion of a gravitational force is only emergent in the Newtonian limit, consisting of a weak gravitational field, slow motion of test particles and matter, and a quasi-static approximation for the metric and the sources. It is replaced more generally by the deformation of the spacetime manifold. A conventional quantisation of this formalism fails for the following reason. The Einstein equations above are derived from the Einstein–Hilbert action

$$S_{\rm EH} = \frac{1}{16\pi G_N} \int d^4x \, \sqrt{-g}\, R\,.$$

Since the mass dimension of the coupling is $[G_N] = -2$, the theory is non-renormalizable, and higher-loop diagrams generate ultraviolet divergences requiring counterterms with progressively more derivatives. These higher-derivative terms appear as

$$S_{\rm EFT} = \frac{1}{16\pi G_N} \int d^4x \, \sqrt{-g} \left[R + \frac{c_1}{M_p^2} R^2 + \frac{c_2}{M_p^2} R_{\mu\nu} R^{\mu\nu} + \frac{d_1}{M_p^4} R^3 + \frac{d_2}{M_p^4} R \nabla^2 R + \cdots \right],$$

and they are suppressed by the Planck mass M_p. Hence, the expansion is obtained as the Effective Field Theory (EFT), which can be derived from a more fundamental theory of quantum gravity which has M_p as its fundamental scale. Consequently, perturbative Einstein gravity does not supply an ultraviolet completion with finitely many parameters. Nonetheless, as a low-energy EFT, perturbative quantum gravity describes quantised metric field perturbations, which are spin-2 particles named gravitons, as first noted by M.Bronstein [1].

The second difficulty in formulating a theory of QG is that gravitational interaction is very weak compared to other forces. This can be expressed in terms of a Planck length, which sets the scale at which potential QG effects become sizeable. Its value is obtained from the fundamental constants c, G_N and the reduced Planck constant $\hbar$

$$\ell_p = \sqrt{\frac{\hbar G_N}{c^3}} \approx 1.616 \times 10^{-35} m$$

which is $O(10^{-20})$ times the size of a proton. This is far beyond current experimental reach, resulting in a lack of data which could guide theoretical predictions.

Nonetheless, because everything lives in spacetime and therefore gravitates, a quantum description of spacetime would likely shed new light on the nature of all fundamental interactions, and extend currently available theories such as the Standard Model of particle physics, potentially resolving part of its shortcomings. This is the case for string theory, arguably the most developed proposal for a theory of QG, which unifies all fundamental interactions in a quantum mechanical framework.

Many interesting approaches to QG have been proposed, each with its own advantages and flaws, which offer perspectives from different angles on the problem. However, the lack of experimental testability renders the various theories not yet falsifiable, which indeed represents a major issue. A fruitful direction for making progress in the search for QG is to find common ground for comparing different approaches and their predictions. One possibility for inferring some universal features is to focus on various formulations of QG in terms of large-N theories. This is the idea that continuous geometry emerges in a limit which involves a large number of fundamental degrees of freedom. Indeed, the large-N limit alone does not provide a sufficient condition: the mechanism of emergence will differ by case and will involve the specific details of every formulation.

My research explores this concept to investigate the mechanism of emergent space and time across different approaches. In this thesis, we will focus mostly on large-N Matrix Theories, which provide a concrete QG proposal motivated by string theory.

Primarily, we will work in the context of the Banks–Fischler–Shenker–Susskind (BFSS) conjecture [2]. This is the highly compelling idea that M-Theory, a proposed theory of QG in eleven dimensions, is non-perturbatively defined by large-N BFSS Matrix Theory. The latter is a quantum mechanical theory, with no intrinsic notion of space. One of the

main consequences is that geometry and gravity are effective, emergent notions, replaced at short distances by the dynamics of the fundamental degrees of freedom in the theory. Note that, within this proposal, space and time are treated on a different footing: time is a variable in the fundamental theory, whereas space is a derived concept.

Before exploring the theory in greater detail, we will provide a brief overview of the emergence of spacetime from large-N dynamics across different approaches to Quantum Gravity. Hopefully, this will help put this work into context, with the overarching goal of identifying common patterns between different proposals.

1.2 Emergent gravity from large-N theories

The idea that spacetime emerges from a more fundamental underlying theory is a common feature of many non-perturbative approaches to QG. This perspective is made concrete through the framework of holographic dualities, according to which a gravitational theory admits an alternative formulation in terms of a non-gravitational description. In this section, we review several proposals of large-N theories in which spacetime is understood to emerge through different mechanisms.

Matrix integrals. The integral of an exponential functional depending on many pure numbers, organised in large matrices, displays interesting features. This can be defined as

$$\mathcal{M}_N := \int [dM]\, e^{-N\mathrm{Tr}(V(M))}\,,$$

where $V(M)$ is a matrix-valued function of the $N \times N$ matrix M. The integral admits a Feynman diagram expansion in terms of ribbon graphs, meaning that every propagator appears as a pair of lines. Diagrams can be put into correspondence with polygonizations of two-dimensional Riemann surfaces. In particular, the expansion can be organised in powers of N, where the exponent corresponds to the Euler characteristic χ_h of surfaces

$$\mathcal{F}_N(\alpha) := \log \mathcal{M}_N = \sum_{h=0}^{\infty} e^{\chi_h \log N} f_h(\alpha)\,,$$

and α stands for the coupling constants in the theory. For some choices of the potential $V(M)$, this can be interpreted as the discretised version of the genus expansion in a two-dimensional world-sheet string theory, possibly coupled to matter. A continuum limit

is obtained by tuning the coupling constants to some critical values, while simultaneously taking N to be large, thus performing a double-scaled limit. See [3] for a review. The IKKT theory, introduced later, is an example of a matrix integral.

Matrix quantum mechanics. Matrix degrees of freedom can be promoted to quantum operators with explicit time dependence. In this case, the matrix integral becomes a quantum mechanical path integral. For a single Hermitian matrix model, if the potential has a $U(N)$ symmetry, the singlet sector reduces to the dynamics of the eigenvalues of the matrix, which are fermionic particles due to the Jacobian in the diagonalisation. Such theories can also be put in duality with world-sheet theories. For example, Liouville theory coupled to bosonic matter with central charge equal to one is dual to Matrix Quantum Mechanics [4]. BFSS theory is an example of a matrix quantum-mechanical theory, involving several non-commuting matrices.

Causal Dynamical Triangulations. A remarkable example of the discretisation of the gravitational path integral consists of triangulating Riemann surfaces. More specifically, the discretisation envisages a preferred time foliation and preserves causality, allowing for computations in Lorentzian signature [5]. This is a fully non-perturbative approach to QG. The two-dimensional case admits a dual description in terms of a dually weighted matrix integral, where local constraints on triangulations can be enforced at the level of the action [6]. Coupling of matter with the discretised geometry can also be considered [7–9]. The two-dimensional continuum geometry emerges in the large-N double scaling limit of the matrix integral. We will see how the idea of discretised geometries will also appear in Matrix Theories, where certain matrix configurations correspond to discretised membranes.

JT gravity. Jackiw–Teitelboim gravity is a two-dimensional theory of QG describing the universal low-energy dynamics of near-AdS_2 throats that arise in many near-extremal black holes. The standard example is the near-horizon region of a near-extremal Reissner–Nordström black hole, with related JT-like descriptions appearing in appropriate reductions of more general charged or rotating near-extremal black holes, including Kerr-type systems. A striking property of pure JT gravity is that its observables, such as the partition function and boundary-to-boundary correlators, are reproduced by a double-scaled random matrix integral [10], giving them the interpretation of ensemble averages.

This provides a useful point of comparison with BFSS matrix quantum mechanics: both are large-N type frameworks connected to black hole near-horizon physics.

Gauge/gravity duality. This is the principle that certain quantum theories of gravity, such as string theory or M-Theory on a given bulk spacetime, admit an equivalent description in terms of a lower-dimensional non-gravitational quantum theory, often a gauge theory. In the best-understood examples, the bulk geometry contains an asymptotically anti-de Sitter component, and the dual theory is defined on the corresponding boundary spacetime [11–13]. Most explicit realisations involve a large-N gauge theory, where the classical gravitational limit emerges at strong coupling. The canonical example is the duality between type IIB string theory on $AdS_5 \times S^5$ and four-dimensional $\mathcal{N} = 4$ super-Yang–Mills theory with gauge group $SU(N)$. Another example, discussed below, is BFSS matrix quantum mechanics, which in the appropriate large-N, strong-coupling, and finite-temperature regime is dual to the near-horizon black zero-brane background of type IIA string theory.

1.3 Black holes as a benchmark

If gravity were to be quantum at all, deviations from the classical behaviour would be observable at extremely small distances, comparable to the Planck length. As a consequence, there are very few physical systems where the quantum regime of gravity could be accessed. One of these is the singularity at the beginning of the Universe, as suggested by the Big Bang proposal. Near this singularity, General Relativity breaks down.

The other widely explored case is that of black holes. These are solutions of General Relativity which possess an event horizon, surrounding a region from which particles or radiation cannot escape. In physically relevant models of gravitational collapse, singularity theorems imply that such spacetimes exhibit singular behaviour, more precisely characterised by geodesic incompleteness: some geodesics cannot be extended beyond a finite value of their affine parameter [14, 15]. Generally, singularities signal a boundary in the regime of applicability of a mathematical formulation. In this case, the black hole singularity suggests that General Relativity fails to fully describe black holes. Hence, an extension of the gravitational theory, perhaps quantum, is needed to incorporate them.

Reversing the argument, black holes are windows into the quantum regime of gravity. Since their discovery, much progress has been made in understanding their properties from different perspectives. Beyond the singularity problem, which pertains to the Planckian regime, they present interesting properties also at large scales. Here, we focus on their thermodynamic properties. Black holes are known to be thermal and to have entropy. This is a famous result by Bekenstein and Hawking, who argued that the entropy reads (in natural units $\hbar = c = k_B = 1$)

$$S_{BH} = \frac{A}{4G_N} \,.$$

The generalised second law of thermodynamics states that the above black hole entropy plus the entropy outside the black hole is a non-decreasing quantity in every physical process.

The finite value for the black hole entropy constitutes a benchmark for every QG proposal. If a black hole behaves like an ordinary quantum system, one should be able to identify and count black hole microstates which contribute to the entropy. The assumption in the previous sentence is referred to as the central dogma [16]. This is supported by gauge/gravity duality, and we will provide a specific realisation in Chapter 4, where we will deconstruct a black hole state in terms of fundamental degrees of freedom of a specific quantum mechanical theory. For example, a highly compelling result in string theory is represented by the Strominger–Vafa counting of black hole microstates [17] in terms of bound states of BPS branes[1]. In this supersymmetric setting, the entropy can be related to the logarithm of the degeneracy of BPS states, which reproduces the Bekenstein–Hawking entropy.

For non-supersymmetric black holes, by contrast, there is in general no protected degeneracy that can be identified with the entropy. Rather, the entropy should be understood as characterising the density of quantum states at fixed energy and conserved charges,

$$S(E, Q) \sim \log \rho(E, Q),$$

where $\rho(E, Q)$ denotes the number of states in an appropriate microcanonical energy window around E and at specified macroscopic charges. This distinction also applies to

[1] BPS branes are reviewed in Chapter 2

extremal black holes when they are not supersymmetric. An example where this density of states perspective can be made particularly explicit is Jackiw–Teitelboim gravity, in which the quantum theory admits an exact expression for the spectral density.

A further question raised by black holes is whether their evaporation is compatible with unitary quantum evolution. This is the black hole information paradox. The problem can be formulated as follows: suppose that a black hole forms from the collapse of matter initially prepared in a pure quantum state, and assume that the complete quantum-gravitational evolution is unitary. Moving forward in time, the black hole behaves as a thermal object and emits approximately thermal Hawking radiation. In the semiclassical description, the radiation is entangled with degrees of freedom behind the horizon, and its von Neumann entropy increases as the black hole evaporates. If this description remains valid until the black hole has fully evaporated, the final state consists only of radiation in a mixed state with non-zero von Neumann entropy, even though the initial state was pure. Such an evolution from a pure state to a mixed state is incompatible with unitarity.

Resolving the paradox requires a microscopic understanding of black hole quantum mechanics, such that the evaporation process can be addressed within a unitary framework. This can be formalised by defining the fine-grained von Neumann entanglement entropy of the radiation. If the full evolution is unitary, and under further assumptions on the typicality of the state and the evaporation dynamics [18], this entropy should initially increase as the radiation becomes entangled with the remaining black hole. However, after the Page time (when the entanglement entropy equals the black hole entropy), it should decrease as information is transferred to the radiation, eventually returning to zero when the evaporation is complete. The expected behaviour is known as the Page curve. Significant progress towards deriving the Page curve from the gravitational path integral has been achieved [16, 19, 20], through the so-called island formula. Nonetheless, the microscopic interpretation of these results and their embedding in a complete theory of QG remain active areas of research.

1.4 Outline

In what follows, we will explore a specific realisation of large-N duality, namely BFSS theory. This thesis aims to address its relation to string theory and the mechanism of

emergent spacetime, both by reviewing the state of the art and by advancing the subject. We will sharpen the evidence in favour of the conjecture, make progress in understanding the physical implications of the theory, and tackle problems in black hole physics within the Matrix Theory framework. The material will be distributed across the following chapters:

- **Chapter 2** introduces BFSS theory, explains its connection to string theory, reviews the evidence for the BFSS conjecture, and discusses results relating to black hole solutions within the theory.

- **Chapter 3**, based on [21], proves a soft-factorisation theorem in BFSS theory. This result implies the existence of an infinite-dimensional asymptotic symmetry group containing the eleven-dimensional Poincaré group, thereby sharpening our understanding of the conjecture.

- **Chapter 4**, based on [22], studies deformations of IKKT and BFSS theories induced by torus compactification. When a mixed class of fermionic boundary conditions is included, the deformation becomes non-trivial. This makes it possible to construct black hole solutions explicitly in terms of microscopic degrees of freedom.

- **Chapter 5** summarises the main results of the dissertation and outlines possible directions for future research.

- In the **Appendices**, we provide supplementary material, which extends and clarifies the content of the main chapters.

2
Matrix Theory

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2.1 M-Theory

The idea of a quantum framework which unifies gravity with the other fundamental interactions finds a proposal in M-Theory. Although its full description has remained

elusive, we know that it comprises an eleven-dimensional spacetime. The known corners of the theory and the relative degrees of freedom form a rich physical setting. This includes the five consistent superstring theories, which are ten-dimensional, and are accessed by exploring specific regions of parameter space. More generally, M-Theory provides a unified framework, comprised of different descriptions, which are generally valid in distinguished patches of parameter space. The overlaps of validity give rise to dualities, such that the same physics can be formulated in two ways.

A prominent feature of M-Theory is the presence of extended objects. Clearly, a state where an extended object is inserted is bound to break the spacetime symmetry and supersymmetry of the original vacuum. However, the ones in M-Theory are of a particular kind: they preserve part of the supersymmetries and are named BPS, after Bogomol'nyi-Prasad-Sommerfield. As a consequence, they are non-perturbative, solitonic configurations, and the surviving symmetry guarantees that they remain part of the theory even when parameters and coupling constants are changed.

Among the BPS states, there is the fundamental $F1$ string, whose excitations yield the whole field content of string theory, which is comprised of a massless spectrum and massive excitations. Another example, highly relevant for the present dissertation, is represented by Dp-branes. These are $p+1$-dimensional objects, charged under $p+1$ form gauge fields, called Ramond-Ramond, which arise from the quantisation of closed strings as part of the massless spectrum.

On the other hand, when open strings are present, Dp-branes represent the locations where strings can end. This can be formalised by saying that, for every direction in the ten-dimensional spacetime of string theory, string endpoints can have Dirichlet or Neumann boundary conditions. The fundamental string is a $1+1$-dimensional object embedded in a $1+9$-dimensional space. Hence, the presence of a Dp-brane translates into the boundary conditions

$$\partial_\sigma x^\mu(t, \sigma = 0, \pi) = 0, \qquad \mu = 0, 1, \ldots, p$$

(Neumann boundary conditions) and

$$x^i(t, \sigma = 0, \pi) = 0, \qquad i = p+1, \ldots, 9$$

(Dirichlet boundary conditions) for a brane whose transverse coordinates sit at the origin. Here, σ is the coordinate along the string world-sheet, such that $0 \leq \sigma \leq \pi$ and the extremes correspond to the string endpoints.

The duality between the closed string and the open string perspectives on Dp-branes is at the heart of the gauge/gravity duality [11]. We will consider the explicit example with $D0$-branes in section 2.3.1.

2.1.1 Eleven-dimensional supergravity

A known corner of the M-Theory framework consists of its effective description at low energy, corresponding to $\mathcal{N} = 1$ supergravity in eleven dimensions. The $\mathcal{N} = 1$ label indicates that a single Majorana spinor supercharge generates supersymmetry transformations. In eleven dimensions, it has 32 real components. This represents the highest amount of supersymmetry that a theory of gravity admits, without introducing fields of spin higher than 2.

Indeed, among the degrees of freedom, we have the eleven-dimensional metric g_{MN}, $M, N = 0, \ldots, 10$. Supersymmetry then requires the existence of a fermionic superpartner, named gravitino ψ_M. For massless particles, the physical degrees of freedom are representations of the little group $SO(9)$. A convenient gauge fixing is to consider the traceless part. Then the counting of degrees of freedom becomes apparent:

$$\#g_{MN} = \frac{D^2 - D}{2} - 1 + D = 44 \qquad \text{for } D = 9$$

for the symmetric rank two tensor (metric) and

$$\#\psi_M = 128$$

for a vector of Majorana $SO(9)$ spinors. The condition that the number of bosonic and fermionic degrees of freedom match imposes the presence of a three-form bosonic field C_3, which provides the remaining 84 bosonic degrees of freedom. When this is included, the bosonic part of the action reads

$$S = \frac{1}{2\kappa_{11}^2} \int d^{11}x \sqrt{-g} \left(R - \frac{1}{2}|G_4|^2 \right) - \frac{1}{12\kappa_{11}^2} \int C_3 \wedge G_4 \wedge G_4 \,,$$

where $G_4 = dC_3$ and $\kappa_{11} = \sqrt{8\pi G_{11}}$ is the gravitational constant in eleven dimensions.

The supercharges, namely the generators of supersymmetry transformations, are packaged into a single Majorana spinor Q_α. The most general Lorentz-covariant extension of the super-Poincaré algebra is

$$\{Q_\alpha, Q_\beta\} = 2(C\Gamma^M)_{\alpha\beta}\, p_M + 2(C\Gamma^{MN})_{\alpha\beta} Z_{MN} + 2(C\Gamma^{M_1\cdots M_5})_{\alpha\beta} Z_{M_1\cdots M_5}, \qquad (2.1.1.1)$$

where C is the charge conjugation operator, Γ^M are gamma matrices and Γ^{MN}, $\Gamma^{M_1\ldots M_5}$ are anti-symmetric products thereof. The two central charges Z_{MN} and Z_{MNPQR} are associated with $M2$- and $M5$-membranes. At the level of fields, the infinitesimal supersymmetry transformation is

$$\delta_\epsilon \Phi = i[\bar{\epsilon} Q, \Phi],$$

with the commutator replaced by the appropriate graded commutator for fermionic fields. A bosonic configuration preserves the supersymmetry generated by ϵ when

$$\delta_\epsilon \Phi_{\mathrm{BPS}} = 0$$

for all fields in the configuration. Since the fermionic fields vanish on a purely bosonic background, the variations of bosonic fields vanish automatically. The non-trivial condition is therefore the vanishing of the gravitino variation,

$$\delta_\epsilon \psi_M = 0. \qquad (2.1.1.2)$$

A non-zero solution ϵ of this equation is called a *Killing spinor*.

For an extended brane configuration, equation (2.1.1.2) imposes an algebraic projection on ϵ. For a static planar $M2$-brane extended along the spatial directions x^1 and x^2, the relevant part of the supersymmetry algebra is

$$\{Q, Q^\dagger\} = 2\left(M\,\mathbb{I} + Z_{12}\Gamma^{012}\right), \qquad (2.1.1.3)$$

where M is the energy per unit area and Z_{12} is the membrane charge density. Here the $C\Gamma^0$ factor was reabsorbed when going from Q to $Q^\dagger$, using the Majorana condition. Since the left-hand side is a positive operator on the physical Hilbert space, its eigenvalues must be non-negative. Using

$$(\Gamma^{012})^2 = \mathbb{I},$$

the eigenvalues of (2.1.1.3) are proportional to $M \pm Z_{12}$. Positivity therefore implies the BPS bound

$$M \geq |Z_{12}|.$$

When the bound is saturated, one set of eigenvalues of the supercharge anticommutator vanishes. For an orientation such that $Z_{12} > 0$, the vanishing combination is selected by the projector

$$\Pi_- = \frac{1}{2}\left(\mathbf{1} - \Gamma^{012}\right).$$

Then (2.1.1.2) implies that the preserved supersymmetry parameters obey

$$\Pi_- \epsilon = 0. \tag{2.1.1.4}$$

For an anti-$M2$-brane, whose charge has the opposite sign, the projection condition is instead

$$\Gamma^{012}\epsilon = -\epsilon.$$

Equation (2.1.1.4) leaves 16 independent components of the original supersymmetry parameter. The $M2$-brane therefore preserves half of the supersymmetries of the vacuum and is said to be *half-BPS*. Equivalently, if the supercharges are decomposed according to

$$Q_\pm = \Pi_\pm Q,$$

then the supersymmetries preserved by the brane, say Q_- are those whose generators annihilate the BPS state,

$$Q_- \left|\mathrm{M2}\right\rangle = 0.$$

The remaining supercharges act non-trivially and generate the shortened BPS supermultiplet.

The same argument applies to an $M5$-brane extended along $x^1, \ldots, x^5$. Its BPS condition is $T_5 = |Z_{12345}|$, and the preserved supersymmetry parameters satisfy

$$\Gamma^{012345}\epsilon = \epsilon\,.$$

This condition again defines a rank-16 projector and hence preserves one half of the 32 supersymmetries. The BPS nature of these objects guarantees that they are present in the full M-Theory formulation.

The $M2$-brane couples to the three-form gauge potential electrically

$$S_{C_3} = T_2 \int_{M2} C_3$$

and the tension is related to the charge by the BPS constraint $T_2 = |Z_2|$. At the same time, the $M5$-brane couples to the three-form field magnetically or, equivalently, it couples to the dual six-form field electrically. This is obtained by dualising the field strength

$$G_7 = *G_4 = dC_6 - \frac{1}{2} C_3 \wedge G_4$$

and the magnetic coupling takes the form

$$S_{C_6} = T_5 \int_{M5} C_6 \,.$$

2.1.2 M-Theory and type IIA string theory

A further central aspect of M-Theory is the relation with a specific superstring theory, called type IIA. We will not review its construction in great detail, but we will introduce the useful aspects as we proceed.

We have discussed, in the previous section, the low-energy description of M-Theory as eleven-dimensional supergravity. At the same time, the low-energy limit of the type IIA string is a particular kind of supergravity in ten dimensions, named type IIA supergravity. This is characterised by the presence of odd-rank form gauge fields, which couple to Dp-branes with p-even. The two theories are connected by the following procedure. Starting from the eleven-dimensional theory, we consider one of the directions of the background geometry to take values on a circle. Namely, the topology is $\mathbb{R}^{1,9} \times S^1$. This is the simplest example of **compactification**. We can perform a Fourier decomposition of the dynamical fields on the compact direction. By restricting ourselves to the zero modes, we obtain the **dimensional reduction** of the theory. When applied to the supergraviton multiplet in eleven dimensions, this procedure reproduces all the massless degrees of freedom of type IIA supergravity.

Moreover, we can consider the non-zero modes, called Kaluza-Klein excitations. From the ten-dimensional perspective, they have a mass (using the mostly-minus flat metric)

$$M_{10}^2 = p_\mu p^\mu = p_{11}^2$$

where $\mu = 0, \ldots, 9$ and p_{11} is the momentum in the compact direction. This is quantised as a consequence of the compactification

$$x_{11} \sim x_{11} + 2\pi R_{11}\,.$$

Hence,

$$p_{11} = \frac{N}{R_{11}}, \qquad N \in \mathbb{Z}\,.$$

These excitations are half-BPS point-like objects. This follows from the fact that the supersymmetry algebra decomposes into

$$\{Q_\alpha, Q_\beta\} = 2(C\Gamma^\mu)_{\alpha\beta}\, p_\mu + 2(C\Gamma^{10})_{\alpha\beta}\, p_{11}\,,$$

hence $Z := p_{11}$ plays the role of central charge, and it is equal to the mass, thus saturating the BPS bound. This suggests their correspondence with $D0$-branes. The mass of BPS $D0$-branes in type IIA string theory is

$$M_{D0} = \frac{1}{\ell_s g_s}\,,$$

where ℓ_s is the string scale, controlling stringy corrections to the low-energy actions, and g_s is the string coupling, controlling quantum fluctuations to string amplitudes. Starting from a single $D0$-brane, the identification with the lowest Kaluza-Klein mode requires

$$R_{11} = \ell_s g_s\,.$$

Furthermore, Kaluza-Klein excitations with generic N are represented by threshold bound states of $D0$-branes, namely a bound state with zero binding energy. In fact, the energy cannot deviate from the value fixed by the BPS condition.

We see that the strong coupling limit $g_s \gg 1$ implies that the compactification radius becomes large. Hence, in this limit, we recover the eleven-dimensional picture. Beyond the low-energy, supergravity approximation, this suggests that the strong coupling limit of type IIA string theory should be viewed as a theory in eleven dimensions. In conclusion, **type IIA string theory at strong coupling is M-Theory**; particle states and extended objects are all uplifted to M-Theory configurations.

2.2 BFSS Matrix Theory

We have discussed, in Chapter 1, how large-N models play a crucial role in a variety of QG proposals, where gravity emerges from collective dynamics or from averages over ensembles. Then we have briefly introduced the framework of M-Theory, which consistently unifies gravity and quantum mechanics in a compelling setting. In the rest of the dissertation, we will be concerned with a large-N formulation of M-Theory. Altogether, this constitutes a proposal for a comprehensive description of M-Theory, and realises the emergence paradigm concretely.

In this section, we will introduce the various components of the theory, and we will explore its connection to M-Theory in different regions of its parameter space.

2.2.1 Field content and symmetries

Banks-Fischler-Shenker-Susskind theory is a matrix quantum mechanics, meaning that the degrees of freedom are organised in square matrices of size N and explicitly depend on a variable t, interpreted as time. The theory presents a $U(N)$ gauge redundancy, with gauge field $A(t)$ in the adjoint representation. The Lagrangian of the theory is

$$\mathcal{L}_{BFSS} = \frac{1}{g^2}\mathrm{Tr}\left[\frac{1}{2}\left(\mathcal{D}_t X^I(t)\right)^2 + \frac{1}{4}[X^I(t), X^J(t)]^2 + i\bar{\psi}(t)\mathcal{D}_t\psi(t) - \bar{\psi}(t)\Gamma^I[X_I(t), \psi(t)]\right] \quad (2.2.1.1)$$

where the trace is over the space of $N \times N$ matrices and guarantees gauge invariance. The dynamical degrees of freedom are bosonic $X^I(t),\ I = 1, \ldots, 9$ and fermionic $\psi_\alpha(t),\ \alpha = 1, \ldots, 16$ and they transform in the adjoint representation of $U(N)$

$$X^I(t) \to U X^I(t) U^{-1}, \qquad \psi_\alpha(t) \to U\psi_\alpha(t)U^{-1}\,.$$

The gauge boson $A(t)$, which appears in covariant derivatives

$$D_t\phi = \partial_t\phi - i[A(t), \phi]\,,$$

is non-dynamical, since the theory is one-dimensional and does not admit a kinetic term. It can be treated as a Lagrange multiplier, such that its equation of motion

imposes the Gauss law constraint

$$\hat{G}|\psi\rangle = 0, \qquad \hat{G} = 2i\sum_{I=1}^{9} : [X^I(t), \Pi^I(t)] : +2i\sum_{\alpha=1}^{16} : [\psi_\alpha(t), \bar{\psi}_\alpha(t)] : \tag{2.2.1.2}$$

on physical states $|\psi\rangle$; i.e. requiring them to be gauge invariant. Here

$$\Pi^I(t) := \frac{1}{g^2}\mathcal{D}_t X^I(t)$$

is the momentum conjugate to $X^I(t)$. As an $N \times N$ matrix, $\hat{G}$ is the generator of infinitesimal $U(N)$ gauge transformations. The commutator appearing in (2.2.1.2) refers to matrix indices, while the normal ordering indicates that canonical conjugate operators are always to the right when acting on the Hilbert space of physical states.

The theory is symmetric under a global $SO(9)$ group, and the degrees of freedom are explicitly characterised as representations of it. In particular, bosonic matrices $X^I(t)$ transform as $SO(9)$ vectors, while fermionic matrices $\psi_\alpha(t)$ are evaluated in the sixteen-dimensional real spinor representation of the group. The matrices Γ^I are the nine appropriate gamma matrices for this representation. Furthermore, the theory is maximally supersymmetric, which implies invariance under the action of two real fermionic supercharges of sixteen components each. The supercharges read

$$Q_\alpha = \mathrm{Tr}\left[\Pi_I(\Gamma^I\psi)_\alpha - \frac{i}{2g^2}[X^I, X^J](\Gamma_{IJ}\psi)_\alpha\right] \tag{2.2.1.3}$$

with $\Gamma_{IJ} = \frac{1}{2}[\Gamma_I, \Gamma_J]$, $\alpha = 1, \ldots, 16$, and

$$\tilde{Q}_\alpha = \frac{1}{g^2}\mathrm{Tr}\,[\psi_\alpha] \tag{2.2.1.4}$$

The first one generates a transformation with constant parameter ϵ given by

$$\begin{aligned}
\delta_\epsilon X^I &= i\epsilon^T\Gamma^I\psi \\
\delta_\epsilon A &= i\epsilon^T\psi \\
\delta_\epsilon\psi &= \left(D_t X^I\Gamma^I + \frac{i}{2}[X^I, X^J]\Gamma^{IJ}\right)\epsilon
\end{aligned}$$

whereas the second set of supercharges generates transformations with constant spinor parameter η which read

$$\begin{aligned}
\delta_\eta X^I &= 0 \\
\delta_\eta A &= 0 \\
\delta_\eta\psi &= \eta\,\mathbb{I}_N\,.
\end{aligned}$$

From the point of view of supersymmetry, the $SO(9)$ group is the R-symmetry, which acts on the supercharges non-trivially. We will elaborate more on the physical significance of the various symmetries in the following sections, after discussing the origin and the relevance of the theory.

2.2.2 Matrix Theory from dimensional reduction

Before establishing the connection between the model introduced in the previous subsection and M-Theory, it is instructive to discuss its origin and significance within perturbative string theory. Matrix Theory is obtained by dimensionally reducing maximally supersymmetric Yang–Mills theory with gauge group $U(N)$ and coupling g_{10d}

$$S = \frac{1}{g_{10d}^2} \int d^{10}x \, \mathrm{Tr} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{i}{2} \bar{\psi} \Gamma^\mu D_\mu \psi \right]$$

from ten dimensions to one. Here $F_{\mu\nu}$ is the field-strength tensor of the gauge field A^μ, ψ is the fermionic superpartner, and Γ^μ are gamma matrices in ten dimensions. The process of dimensional reduction on a compact manifold consists of dropping the spacetime dependence of the fields on the reduced coordinates

$$\partial_{x^I} A^\mu(t, x^I) = 0$$

such that the integration over these directions reproduces the volume of the manifold. Moreover, the spatial components of the gauge field become scalar fields

$$A^I(t, x^I) \to \Phi^I(t) = \frac{X^I(t)}{2\pi\alpha'}$$

and the Lagrangian becomes (2.2.1.1)[1], with coupling constant

$$g^2 = \frac{g_{10d}^2}{\mathcal{V}_9} \,,$$

where $\mathcal{V}_9$ is the volume of the reduced directions.

In the string theory context, ten-dimensional super Yang-Mills theory represents the low-energy effective description of the open string sector and its target space dynamics. This can be understood as follows. The spectrum of excitations in the open string sector

[1]For notational convenience, a field redefinition reabsorbing the $2\pi\alpha'$ factor in the definition of $X^I(t)$ is performed.

includes massless degrees of freedom, given by a $U(1)$ gauge field and the fermionic superpartner. Moreover, it includes a tower of massive states, with a mass scale set by

$$m_{heavy} = \frac{1}{\ell_s}$$

where ℓ_s is the string length. At low energies, namely $E \ll \frac{1}{\ell_s}$, the heavy modes are integrated out, and all that remains are the massless ones. To leading order, they are described by super Yang-Mills theory. Upon dimensional reduction from ten to $p+1$ dimensions, this represents the low-energy limit of open string theory, with the string endpoints constrained to lie on the world-volume of a stack of Dp-branes [23]. The scalar fields X^I, $I = p+1, \ldots, 9$, are naturally interpreted as the transverse coordinates of the branes in the target space. In this work, we will be primarily concerned with the case $p = 0$.

This construction can be extended to multiple Dp-branes, allowing open strings to end on different branes. For concreteness, consider two $D0$-branes located at $X^I = x_1^I$ and $X^I = x_2^I$. Open string states are then endowed with Chan–Paton labels $[i, j]$ [24], with $i, j = 1, 2$, indicating that the string begins on the i-th brane and ends on the j-th. Strings stretched between distinct branes acquire a mass proportional to the separation between the branes. More precisely,

$$m_{12} = \frac{|x_1^I - x_2^I|}{2\pi\ell_s^2}\,.$$

In the reduced theory, the degrees of freedom can be encoded in 2×2 background matrices which are commuting (hence, simultaneously diagonalisable)

$$X_{bg}^I = \begin{pmatrix} x_1^I & 0 \\ 0 & x_2^I \end{pmatrix}$$

where the diagonal entries provide the coordinates of the two $D0$-branes, together with a matrix of off-diagonal fluctuations Y_{12}^I which are massive. The resulting theory will have gauge group $U(1) \times U(1)$, corresponding to the charge of the string endpoints on the two branes, plus a massive W-boson. The limit where the separation between the two branes goes to zero is particularly interesting. It implies

$$r_{12} := |x_1^I - x_2^I| \to 0 \;\Rightarrow m_{12} \to 0$$

so that open strings stretching between different branes become massless, and the gauge symmetry is extended. This is equivalent to a reversed Higgs mechanism, where the gauge group becomes larger

$$U(1) \times U(1) \to U(2)$$

and typical configurations are given by non-commuting matrices

$$[X^I, X^J] \neq 0\,.$$

The procedure can be generalised in a straightforward manner to the case of N coincident $D0$-branes. The Chan–Paton factors $[i, j]$ now take values in $i, j = 1, \ldots, N$ and $N^2 - N$ fluctuations become massless when the branes are taken to be coincident, thus extending the gauge group

$$U(1)^N \to U(N)\,.$$

This is the explicit construction of BFSS theory (2.2.1.1), which highlights its physical interpretation: it represents the low-energy limit of open string theory on a collection of N coincident $D0$-branes. In this case, the location of the branes is given by

$$x^I = \frac{1}{N}\mathrm{Tr}[X^I]\,,$$

whereas off-diagonal matrix entries describe strings stretched between different branes. The fact that the theory describes a low-energy limit implies that $D0$-branes are not relativistic; hence they move with small velocities $v \ll 1$. Given the mass of a $D0$-brane

$$M_{D0} = \frac{1}{g_s \ell_s}\,,$$

the interpretation is valid in the perturbative regime, namely for weak string coupling $g_s \ll 1$, for the branes to be sufficiently heavy. The global $SO(9)$ symmetry group is interpreted, in this framework, as rotational invariance in the target space.

Before we proceed, it will be useful to note a simple fact. The overall centre of mass motion of the stack of $D0$-branes can be separated from the relative one. This is done by defining

$$x^I_{CM} : \frac{1}{N}\mathrm{Tr}[X^I]\,, \qquad \psi_{CM} := \frac{1}{N}\mathrm{Tr}[\psi]$$

and decomposing the matrix degrees of freedom into

$$X^I = x^I_{CM}\mathbb{I}_N + \tilde{X}^I$$
$$\psi = \psi_{CM}\mathbb{I}_N + \tilde{\psi}\,.$$

The centre of mass component transforms with respect to the $U(1)$ subgroup of the gauge symmetry. It is described by the Lagrangian

$$\mathcal{L} = \frac{N}{g^2}\left(\frac{1}{2}\dot{x}^I_{CM}\dot{x}^I_{CM} + i\bar{\psi}_{CM}\dot{\psi}_{CM}\right)$$

and decouples from the remaining degrees of freedom. The latter represent the relative motion between branes, and their Lagrangian $\mathcal{L}[\tilde{X}, \tilde{\psi}]$ takes the form (2.2.1.1), where now $\tilde{X}^I$ and $\tilde{\psi}$ are Hermitian, traceless matrices in the adjoint representation of $SU(N)$.

2.2.3 The BFSS conjecture

The idea that a one-dimensional gauge theory can describe the dynamics of non-perturbative objects in string theory is taken much further in the BFSS conjecture. This states that [2] **the Discrete Light-Cone Quantisation (DLCQ) of M-Theory is non-perturbatively described by BFSS Matrix Theory**. The full uncompactified eleven-dimensional theory is recovered in the limit $N \to \infty$. See [25–28] for reviews.

The DLCQ construction provides a simple relation between eleven- and ten-dimensional theories by compactifying M-Theory along a light-like direction. To see how this arises, consider flat eleven-dimensional Minkowski spacetime expressed in light-cone coordinates,

$$x^\mu = (x^0, x^I, x^{11}), \quad x^+ = \frac{x^0 + x^{11}}{\sqrt{2}}, \quad x^- = \frac{x^0 - x^{11}}{\sqrt{2}}$$

and compactify x^- on a circle of radius R, such that

$$x^- \sim x^- + 2\pi R\,.$$

As a direct consequence, the $+$ component of the conjugate momentum becomes quantised

$$P^+ = \frac{N}{R}, \quad N \in \mathbb{Z}_+\,.$$

The precise statement of the conjecture is that the DLCQ of M-Theory with N units of longitudinal momentum is non-perturbatively dual to Matrix Theory with gauge group

$U(N)$. For $N \to \infty$, we fix the longitudinal momentum to be finite, so that $R \sim N \to \infty$. This corresponds to the decompactification limit.

An interesting physical motivation for using DLCQ is that null compactifications generally turn field theories where particle production and annihilation occur, into theories where the particle number is fixed. One way to see this is to start from the wave equation

$$\left(-\partial_{x^0}^2 + \partial_{x^{11}}^2 + \partial_{x^I}^2\right) f(x^0, x^I, x^{11}) = 0\,,$$

which is typical of relativistic quantum field theories where the number of particles can vary. Changing to light-cone coordinates, we get

$$\left(\partial_{x^+}\partial_{x^-} + \partial_{x^I}^2\right) \tilde{f}(x^+, x^I, x^-) = 0$$

In DLCQ, we look for solutions having fixed momentum in the longitudinal direction

$$\tilde{f}(x^+, x^I, x^-) = e^{i\frac{N}{R}x^-} g(x^+, x^I)\,,$$

which results in

$$\left(i\frac{N}{R}\partial_{x^+} + \partial_{x^I}^2\right) g(x^+, x^I) = 0\,.$$

This equation is first order in light-cone time, and characterises non-relativistic quantum mechanical theories where particle production is absent. Indeed, this makes BFSS theory, in which the total number of $D0$ particles is fixed to be N, a suitable description of a DLCQ system.

We restore the units of ℓ_s. Consistency between the BFSS Lagrangian (2.2.1.1) and the low-energy limit of the open string sector of type IIA requires that the coupling satisfy

$$g^2 = \frac{g_s}{\ell_s^3}\,.$$

Moreover, the compactification of M-Theory into type IIA requires

$$R = g_s \ell_s, \qquad \ell_p^3 = g_s \ell_s^3$$

where $\ell_p = G_{11}^{\frac{1}{9}}$ is the Planck length in eleven dimensions. Altogether, this implies

$$g^2 = \frac{R^3}{\ell_p^6}\,.$$

This relation is useful to present the Lagrangian in Planck units (with $\ell_p = 1$), after a field rescaling, which gives

$$\mathcal{L}_{BFSS} = \mathrm{Tr}\left[\frac{1}{2R}\left(\mathcal{D}_t X^I(t)\right)^2 + \frac{R}{4}[X^I(t), X^J(t)]^2 + i\bar{\psi}(t)\mathcal{D}_t\psi(t) - R\bar{\psi}(t)\Gamma^I[X_I(t), \psi(t)]\right]. \quad (2.2.3.1)$$

The BFSS conjecture is remarkable in that it identifies the quantum mechanical theory governing the dynamics of non-relativistic $D0$-branes and open strings with a non-perturbative formulation of M-Theory in the discrete light-cone frame. A considerable body of evidence supports this proposal. Matrix Theory admits configurations corresponding to a wide range of M-Theory objects, including extended membranes, naturally incorporates multi-particle states, and reproduces the expected long-range interactions of eleven-dimensional light-front supergravity. We discuss these and other pieces of evidence in the following subsections.

2.2.4 Infinite Momentum Frame vs Discrete Light-Cone Quantisation

The DLCQ formulation that we have presented in the previous section is, in fact, a refinement of the BFSS conjecture due to Susskind [29]. The original statement relates BFSS theory with the Infinite Momentum Frame (IMF) sector of M-Theory. The IMF description involves the compactification of one space-like coordinate x^{11} on a circle of radius R_s, such that

$$\begin{pmatrix} x^{11} \\ t \end{pmatrix} \sim \begin{pmatrix} x^{11} - R_s \\ t \end{pmatrix}. \quad (2.2.4.1)$$

In the original formulation, BFSS theory corresponds to the sector of M-Theory with infinite momentum

$$p^{11} = \frac{N}{R_s}$$

with $N \to \infty$. On the other hand, the DLCQ version constitutes a stronger claim, where the duality holds even for finite N.

These two compactification procedures are related in a particular limit. First, we note that (2.2.4.1) can be obtained from the space-like compactification

$$\begin{pmatrix} x^{11} \\ t \end{pmatrix} \sim \begin{pmatrix} x^{11} - \sqrt{\frac{R^2}{2} + R_s^2} \\ t + \frac{R}{\sqrt{2}} \end{pmatrix}$$

through a boost with a parameter

$$\beta = \frac{1}{\sqrt{1 + \frac{2R_s^2}{R^2}}}.$$

This implies that the limit $R_s \to 0$ is equivalent to compactification along a light-like direction. More precisely, M-Theory in the discrete light-cone frame with N units of longitudinal momentum can be obtained by taking the $R_s \to 0$ limit of type IIA string theory with N $D0$-branes, after performing a large boost. To keep the eleven-dimensional Planck scale fixed in this limit, it is convenient to introduce an auxiliary string coupling $\tilde{g}_s$ and string length $\tilde{\ell}_s$, defined in terms of an auxiliary eleven-dimensional Planck length $\tilde{\ell}_{11}$ by

$$\tilde{g}_s = \left(\frac{R_s}{\tilde{\ell}_{11}}\right)^{3/2}, \qquad \tilde{\ell}_s = \sqrt{\frac{\tilde{\ell}_{11}^3}{R_s}},$$

where $\tilde{\ell}_{11}$ is related to the physical eleven-dimensional Planck length through

$$\frac{R_s}{\tilde{\ell}_{11}^2} = \frac{R}{\ell_{11}^2}.$$

The above limiting procedure allows for a compelling argument by Seiberg [30] and Sen [31] in favour of the duality. They noted that, starting from type IIA string theory with N $D0$-branes, described by a Born-Infeld action, the low-energy EFT expansion organises in powers of $O(\frac{R_s}{R})$, after changing units from the space-like to the light-like frame. The lowest-order term is given by the BFSS action, describing non-relativistic branes. Thus, schematically,

$$S_{D0} = S_{BFSS} + O\left(\frac{R_s}{R}\right)$$

so that in the limit of $R_s \to 0$ we have DLCQ quantisation of M-Theory (through the above limiting procedure) being described by BFSS theory. The physical intuition is

that, although the string scale $m_s = \frac{1}{\ell_s} \to 0$ goes to zero in this large boost limit, the light-cone energy of Matrix Theory excitations is even smaller. The latter reads

$$\Delta E = \frac{R_s}{R} P^-$$

and, as a consequence, the ratio gives

$$\frac{\Delta E}{m_s} = \frac{P^-}{R} \sqrt{R_s \ell_{11}^3}$$

which goes to zero for $R_s \to 0$. Hence, the relevant physics is captured by the low-energy limit of D-brane dynamics.

Naively, this would appear to be a proof of the conjecture. However, there are some subtleties. First, the argument assumes that a theory with the desired properties to represent M-Theory exists independently, such that its circle compactification reduces to type IIA string theory. Moreover, there is a potential order-of-limits ambiguity. The Seiberg–Sen argument takes the infinite-boost limit at fixed N, whereas recovering decompactified M-Theory requires a subsequent $N \to \infty$ limit. It is not established that these limits commute, or that the finite-N decoupling arguments are uniform in N. In particular, states whose masses or interaction strengths scale non-trivially with both the light-like limit parameter and N might decouple at every fixed N but remain relevant in a correlated large-N limit. Thus the finite-N DLCQ identification does not by itself prove that the large-N theory has precisely the spectrum and dynamics of decompactified M-Theory.

2.2.5 Objects in Matrix Theory

The type IIA picture introduced in Section 2.2.2 provides an explicit setting to construct semiclassical supergraviton states from matrix configurations. We have seen in Section 2.1.2 how, after compactification of the supergraviton multiplet from eleven to ten dimensions, massive threshold bound states of $D0$-branes are produced. From the Matrix Theory perspective, these can be represented by a semiclassical bosonic matrix configuration [32–35]

$$X^I(t) = \begin{pmatrix} x^I + v^I t & 0 & 0 & \cdots \\ 0 & x^I + v^I t & 0 & \cdots \\ 0 & 0 & \ddots & \\ \vdots & \vdots & & \end{pmatrix}$$

which solves the bosonic equations of motion

$$\ddot{X}^I = -R^2 \left[[X^I, X^J], X^J \right]$$

and corresponds to supergravitons with

$$p^+ = \frac{N}{R}, \qquad p^I = \frac{N}{R} v^I, \qquad p^- = \frac{N}{2R} v^2 \,.$$

Moreover, the supergraviton polarisation, namely the tensor index structure, is also naturally reproduced from the theory. This is accounted for by the fermionic Fock space, defined in terms of the operators

$$\psi_\alpha^\pm = \frac{\psi_\alpha \pm i\psi_{\alpha+8}}{\sqrt{2}}, \qquad \alpha = 1, \ldots, 8$$

where ψ^+ is a fermionic creation operator and ψ_α^- is an annihilation one. The dimensionality of this Fock space is precisely 256, which is suitable to represent all the supergraviton polarisations. The details of the Fock space representation are presented in [36].

Beyond the semiclassical approximation, bound states with zero energy have been studied in Matrix Theory for finite N. The existence has been proven by means of a Witten index calculation for $N = 2$ in [37, 38] and for more general N in [35, 39]. A further piece of evidence is provided by the fact that, as shown in [40], all ground states should be $SO(9)$ singlets. Hence, all ground states are bosonic and, since the Witten index is 1, there is a unique zero-energy state for $N = 2$.

Black hole configurations, with $E > 0$ are bound states of branes and are believed to be metastable, since they decay into Hawking radiation. We will largely discuss black holes in Matrix Theory in later sections of this chapter and in Chapter 4.

Furthermore, the theory allows for a continuum of scattering states with $E > 0$. These are asymptotic states where the wavefunction is strongly peaked around block-diagonal configurations, and each block is described by a ground-state wavefunction

$$\psi_0 \left(\vec{X}_i, \vec{x}_i \right) = \psi_0 \left(\vec{X}_i \right) e^{-i\vec{p}_i \cdot \vec{x}_i}$$

where $\vec{X}_i$ are $N_i \times N_i$ matrices and $x_i^I = \frac{1}{N_i} \mathrm{Tr} X_i^I$ is the centre of mass coordinate. On top of this wavefunction, the state is fully specified by a representative in the fermionic Fock space, which accounts for the supergraviton polarisation.

Membranes. In Section 2.1.1, we have seen how M-Theory describes extended BPS objects such as $M2$-membranes. These can also be reproduced in terms of Matrix Theory configurations. The representation involves non-commuting matrices

$$[X^I, X^J] \neq 0$$

and provides a discretisation of the membrane world-volume, where smooth geometry emerges in the large-N limit. Membrane charges appear explicitly in the supersymmetry algebra. By using the supersymmetry generators (2.2.1.3) and (2.2.1.4), it was shown in [41] that

$$\begin{aligned}
\{Q_\alpha, Q_\beta\} &= 2H\delta_{\alpha\beta} + 2Z^I \left(\Gamma_I\right)_{\alpha\beta} + 2Z^{IJKL} \left(\Gamma_{IJKL}\right)_{\alpha\beta} \\
\{Q_\alpha, \tilde{Q}_\beta\} &= 2P^I \left(\Gamma_I\right)_{\alpha\beta} + 2Z^{IJ} \left(\Gamma_{IJ}\right)_{\alpha\beta} \\
\{\tilde{Q}_\alpha, \tilde{Q}_\beta\} &= 2P^+ \delta_{\alpha\beta} \,.
\end{aligned} \tag{2.2.5.1}$$

Here

$$Z^I = i\mathrm{Tr}\left[\left\{P^J, \left[X^I, X^J\right]\right\} + \left[\left[X^I, \psi_\alpha\right], \psi_\alpha\right]\right]$$

is the charge of $M2$-membranes wrapped around the longitudinal dimension, which represents the string charge from a ten-dimensional perspective,

$$Z^{IJ} = -i\mathrm{Tr}\left[X^I, X^J\right]$$

is the unwrapped membrane charge, and

$$Z^{IJKL} = \mathrm{Tr}\left(X^{[I} X^J X^K X^{L]}\right)$$

is the wrapped 5-brane charge. The supersymmetry algebra agrees with the eleven-dimensional supergravity algebra (2.1.1.1) in the light-cone frame, except for the lack of the unwrapped $M5$-charge on the Matrix Theory side. Note that, for finite N, all these charges vanish identically (in fact, Z^I only vanishes on the space of gauge-invariant states, satisfying the Gauss law constraint (2.2.1.2)). In the $N \to \infty$ limit, they have a non-trivial value. Physically, this corresponds to the fact that extended object configurations in Matrix Theory have a vanishing net charge, and must therefore be compact.

In the following, we are going to provide some examples of extended configurations, obtained in terms of Matrix Theory fundamental degrees of freedom. The first example

is the **spherical membrane**. This is a two-sphere embedded in the eleven-dimensional target space. The embedding is realised by the matrix configuration

$$X^I = \frac{2r}{N} J^I, \quad I = 1, 2, 3\,,$$

where r sets the radial scale of the sphere and J^I are generators of $SU(2)$ in the spin $j = \frac{N-1}{2}$ representation. The sphere equation is given by the Casimir relation

$$X^I X_I = \frac{4r^2}{N^2} C_2(N) \mathbb{I} = r^2 \left(1 - \frac{1}{N^2}\right) \mathbb{I}\,,$$

where the $O(\frac{1}{N^2})$ correction to the sphere relation arises from the non-commutative nature of the configuration, and is indeed removed for $N \to \infty$. We can choose a basis in which $X^3 = \frac{2r}{N} J^3$ is diagonal. The eigenvalues are uniformly distributed in the range $[-r, r]$. This corresponds to a uniform distribution of $D0$-branes, arranged to form a spherical membrane. Note that this configuration does not give a static solution of the Matrix Theory equations of motion. Rather, it collapses due to the equation of motion

$$\ddot{r} = -a r^3\,,$$

where a is a constant. This fact motivates the work presented in Chapter 4, where we study deformations of BFSS theory which allow us to construct static membranes. As explained in sections 2.4 and 2.5.2, our approach will differ from the standard cases, where typically membranes are stabilised by introducing membrane fluxes. We will further interpret these as Schwarzschild black holes in Matrix Theory.

Genus one surfaces can also be realised in terms of matrix degrees of freedom. For example, the **torus membrane** can be constructed in terms of the matrices

$$U = \begin{pmatrix} 1 & & & & \\ & q & & & \\ & & q^2 & & \\ & & & \ddots & \\ & & & & q^{N-1} \end{pmatrix}, \quad V = \begin{pmatrix} 0 & 1 & & & \\ & 0 & 1 & & \\ & & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & & 0 \end{pmatrix}$$

where

$$q = e^{\frac{2\pi i}{N}}$$

and the non-commutativity relation is

$$UV = q^{-1} VU\,.$$

Then a two-torus embedded in $\mathbb{R}^4 \subset \mathbb{R}^9$ is realised as

$$X^1 = \frac{r}{2}\left(U + U^\dagger\right), \qquad X^2 = \frac{-ir}{2}\left(U - U^\dagger\right)$$
$$X^3 = \frac{s}{2}\left(V + V^\dagger\right), \qquad X^4 = \frac{-is}{2}\left(V - V^\dagger\right)$$

where r and s set the radii of the two circles

$$X_1^2 + X_2^2 = r^2, \qquad X_3^2 + X_4^2 = s^2$$

which are satisfied identically given that U and V are unitary matrices.

The non-compact membrane can be constructed by selecting two coordinates X_1, X_2 for the embedding of the non-commutative plane

$$X_1 = Q, \qquad X_2 = P, \qquad [Q, P] = \frac{i\epsilon^2}{2\pi}\mathbb{I}$$

in terms of a constant parameter ϵ. This can be thought of as a quantisation of the standard Poisson brackets

$$\{f(x_1, x_2), g(x_1, x_2)\} = \partial_1 f \partial_2 g - \partial_2 f \partial_1 g$$

which allows us to define generic operators describing fluctuations around the flat membrane.

The construction of wrapped and transverse $M5$-branes is considerably less well understood, except in a handful of special cases that require more sophisticated constructions. In particular, the transverse $M5$-brane presents a challenge because the corresponding central charge does not appear explicitly in the Matrix Theory supersymmetry algebra (2.2.5.1). The origin of the issue lies in the fact that, in light-front formalism, membranes must have Neumann boundary conditions in the light-like directions. Hence, $M5$-branes, associated with Dirichlet boundary conditions, cannot extend in those directions. They are believed to be present in the theory only for $N \to \infty$, or as solitonic objects, whose construction must rely on non-perturbative dualities [42]. In fact, a finite-size configuration with geometry $T^3 \times S^2$ was constructed in [43], but a general description of transverse $M5$-brane configurations remains elusive. See [44] for recent developments.

The situation is somewhat more favourable for the wrapped $M5$-brane, which corresponds to a $D4$-brane upon reduction to ten dimensions. In this case, the associated central charge is present in the BFSS supersymmetry algebra. Nevertheless, the known explicit solutions are still limited to a small class of highly symmetric configurations [42, 45, 46].

2.2.6 BFSS as a second quantised theory

One of the advantages of the Matrix Theory formulation of M-Theory is the fact that it naturally encodes multi-particle states. Let us focus on the bosonic part of the classical theory. The equations of motion, as derived from the Lagrangian (2.2.3.1), read

$$\ddot{X}^I = -R^2 \left[[X^I, X^J], X^J\right] .$$

This implies that, for the block diagonal matrix configuration

$$X^I = \begin{pmatrix} Y^I & 0 \\ 0 & Z^I \end{pmatrix} ,$$

where Y^I and Z^I are $N_1 \times N_1$ and $N_2 \times N_2$ matrices, respectively, with $N_1 + N_2 = N$; the equations of motion are imposed on the two blocks independently

$$\ddot{Y}^I = -R^2 \left[[Y^I, Y^J], Y^J\right] , \qquad \ddot{Z}^I = -R^2 \left[[Z^I, Z^J], Z^J\right] .$$

It is natural to interpret these configurations as classically independent, non-interacting objects. They may correspond to supergravitons, extended branes, or, more generally, any Matrix Theory configuration admitting a dual interpretation in the target space. This construction extends straightforwardly to states composed of an arbitrary number of constituents.

The existence of such multi-particle configurations is a direct consequence of the flat directions of the Matrix Theory potential. Flat directions obstruct the interpretation of the model as a first-quantised theory of the supermembrane. In fact, they result in a continuum spectrum for the Hamiltonian [47], which is not compatible with the interpretation of a discrete particle spectrum. On the other hand, they acquire a natural physical interpretation in the second-quantised picture, where they describe widely separated, non-interacting particles in the target space. In Section 2.2.8, we will see how this picture extends to the semiclassical analysis, including the fermionic degrees of freedom.

Within this interpretation, the continuous spectrum of the BFSS Hamiltonian finds a compelling explanation. Indeed, states with arbitrarily small energies can be constructed from multi-particle configurations whose constituents have small relative velocities.

2.2.7 Compactifications of Matrix Theory

In Section 2.1, we have commented on the fact that the landscape of M-Theory is tied together by dualities. In this section, we explore some of these dualities through the lens of Matrix Theory.

One of the main examples is T-duality. Let us consider ten-dimensional type IIA string theory, and suppose that one dimension of the target space is compact. More specifically, we consider x^{10} to take values on a circle S^1 with radius R. T-duality is the statement that this description is equivalent to type IIB string theory with one compact dimension of radius $\tilde{R} = \frac{\alpha'}{R}$. In other words, compactifications on a small circle can be mapped to compactifications on a large one, whose size is inversely proportional to the initial radius.

From the world-sheet perspective, T-duality swaps Dirichlet and Neumann boundary conditions. This implies that a system of unwrapped Dp-branes is mapped to a system of $D(p+1)$-branes wrapped along the compact direction. The argument can be repeated straightforwardly in each direction, for compactifications on a d-dimensional torus $\mathbb{T}^d$. In the context of the BFSS conjecture, we expect torus compactifications of a system of $D0$-branes to be T-dual to $(d+1)$-dimensional Super Yang-Mills theory compactified on a (dual) torus. This allows us to extend the BFSS conjecture to the DLCQ of M-Theory compactified on a d-dimensional torus [30]. Several non-trivial checks of this proposal have been carried out [48]. In Chapter 4, we will exploit this compactification procedure as the starting point for deriving a deformation of the BFSS model.

As an important caveat, compactifications of BFSS theory are well defined for $\mathbb{T}^d$ with $d \leq 5$ [30, 31]. For $d = 1, 2, 3$ T-duality with $(d+1)$-dimensional SYM theory follows smoothly. For $d = 4$, the SYM theory is non-renormalizable, and, therefore, it must be UV completed. The string coupling is strong, and the duality should be performed in the eleven-dimensional M-Theory framework, where the $D4$-branes are interpreted as wrapped $M5$-branes. The completion is known as the $5+1$-dimensional $(2,0)$ field theory. For $d = 5$, a similar analysis applies for $D5$-branes in the strong coupling limit of type IIB theory. Here, a different kind of duality, called S-duality, that maps strong coupling into weak coupling, can be leveraged to complete this to a theory on a stack of $NS5$-branes. This is called little string theory. For $d = 6$, the SYM theory is also non-renormalizable

and needs to be uplifted to eleven dimensions. However, the uplifted description does not decouple from bulk gravity, and it cannot be defined as a world-volume theory.

2.2.8 Perturbative matching to supergravity

A solid piece of evidence for the BFSS conjecture is the fact that long-range effective interactions between Matrix Theory objects have a direct correspondence with perturbative calculations in light-front linearised supergravity. We will review here the matching of the effective interaction potential between two gravitons at the lowest order.

We start from the DLCQ supergravity side. The effective supergravity action describing the interaction between two sources with energy-momentum tensors $\hat{T}_{\mu\nu}$ and $\tilde{T}_{\mu\nu}$, $\mu,\nu = 0,\ldots,10$ respectively reads [49]

$$S = -\frac{1}{4}\int d^{11}x \int d^{11}y\, \hat{T}_{\mu\nu}(x) D^{\mu\nu\rho\sigma}_{grav}(x-y)\tilde{T}_{\rho\sigma}(y)\,.$$

The graviton propagator in de Donder gauge is

$$D^{\mu\nu\rho\sigma}_{grav}(x-y) = 2\kappa^2\left(\eta^{\mu\rho}\eta^{\nu\sigma}+\eta^{\mu\sigma}\eta^{\nu\rho}-\frac{2}{9}\eta^{\mu\nu}\eta^{\rho\sigma}\right)\Box^{-1}(x-y)$$

in units where the gravitational coupling is $2\kappa^2 = (2\pi)^5 R^3$, and the scalar propagator in light-front coordinates reads

$$\Box^{-1}(x-y) = \frac{1}{2\pi R}\sum_n \int \frac{dk^- d^9k_\perp}{(2\pi)^{10}} \frac{e^{-i\frac{n}{R}x^- - ik^-x^+ + ik_\perp\cdot x_\perp}}{2\frac{n}{R}k^- - k_\perp^2}\,.$$

The comparison with the Matrix Theory potential is carried out for $n=0$. Physically, this describes the interaction between two sources mediated by the exchange of a graviton, which does not carry any momentum in the longitudinal direction. In this case, the scalar propagator simplifies to

$$\Box^{-1}(x-y) = -\frac{1}{2\pi R}\delta(x^+-y^+)\frac{15}{32\pi^4|x_\perp - y_\perp|^7}\,.$$

We expand the sources in terms of their multipole moments [50]

$$\begin{aligned}\int dx^-\, \hat{T}_{\mu\nu}(x) = \hat{\mathcal{T}}_{\mu\nu}(x^+)\delta(\vec{x}-\vec{x}_0) \\ - \hat{\mathcal{T}}^{(i)}_{\mu\nu}(x^+)\partial_i\delta(\vec{x}-\vec{x}_0) + \frac{1}{2}\hat{\mathcal{T}}^{(ij)}_{\mu\nu}(x^+)\partial_i\partial_j\delta(\vec{x}-\vec{x}_0) + \ldots \quad (2.2.8.1)\end{aligned}$$

where

$$\hat{\mathcal{T}}_{\mu\nu}^{(i_1\ldots i_n)}(x^+) = \int dx^- d^9x_\perp \hat{T}_{\mu\nu}(x^+, x^-, \vec{x}_0 + \vec{x})\, x^{i_1}\cdots x^{i_n}$$

are the multipole moments of the current. Calling $r = |\vec{x}_0 - \vec{y}_0|$ the distance between the sources, the leading monopole contribution is the zeroth-order term in (2.2.8.1), and the potential reads

$$V_{grav} = -\frac{15R^2}{4r^7}\left(\hat{\mathcal{T}}^{\mu\nu}\tilde{\mathcal{T}}_{\mu\nu} - \frac{1}{9}\hat{\mathcal{T}}^{\mu}_{\mu}\tilde{\mathcal{T}}^{\nu}_{\nu}\right), \tag{2.2.8.2}$$

in terms of the graviton stress tensor

$$\hat{\mathcal{T}}^{\mu\nu} = \frac{p_1^\mu p_1^\nu}{p_1^+}.$$

For ease of comparison with the Matrix Theory expression below, let us substitute the momentum

$$p_1^\mu = \frac{1}{\sqrt{2}R}(1 + v_1^2, 2v_1^I, 1 - v_1^2),$$

and similarly for the other graviton, resulting in

$$V_{grav} = -\frac{15}{16}\frac{v^4}{r^7}, \qquad v^I = v_1^I - v_2^I.$$

The result can be extended to higher orders by including more multipole moments in the expansion. Moreover, analogous potentials can be obtained for the interaction between $M2$-brane sources, mediated by the exchange of a three-form field C_3, and between $M5$-sources, mediated by the exchange of a dual six-form field C_6 (the gauge fields were introduced in Section 2.1.1).

The supergravity result can be obtained by a direct computation in Matrix Theory. This is strongly suggestive of the correspondence between BFSS quantum mechanics and the low-energy effective action of M-Theory. We will now review the derivation, which utilises the background field formalism. Unless otherwise specified, we will be working in string units ($\alpha' = 1$). Let us start with the explicit example of the two-graviton interaction. We split bosonic matrices into a background part, representing the interacting gravitons, and off-diagonal fluctuations, which account for interactions

$$X^I = B^I + Y^I.$$

Regarding fermions, as far as the leading-order interaction is concerned, we can restrict ourselves to the trivial fermionic background. Cases where the fermionic background is non-trivial have been addressed in the literature, see e.g. [50], and we will see an explicit example in Chapter 3. The decomposition is substituted back into the BFSS action, which is analysed at quadratic order in the fluctuations. Let us consider the explicit example of two gravitons in the $SU(2)$ theory. Assuming a large separation between particles, it is enough to consider a classical background. Then we have

$$B^1 = \frac{1}{2}\begin{pmatrix} vt & 0 \\ 0 & -vt \end{pmatrix}, \quad B^2 = \frac{1}{2}\begin{pmatrix} b & 0 \\ 0 & -b \end{pmatrix}, \quad B^I = 0, \quad I = 3, \dots, 9$$

for a two-graviton state with relative velocity v and impact parameter b. Then the distance between gravitons reads

$$r^2 = b^2 + (vt)^2 \,.$$

When plugging this background into the BFSS action, we find a system of harmonic oscillators comprising bosons, more precisely fluctuations Y^I, together with the gauge field A, ghost fields and fermions. The precise counting of the degrees of freedom is:

- **Bosons** 8 with frequency $\omega_b = r$, 1 with frequency $\omega_b = \sqrt{r^2 + 2v}$, 1 with frequency $\omega_b = \sqrt{r^2 - 2v}$;
- **Ghosts** 2 with frequency $\omega_g = r$;
- **Fermions** 8 with frequency $\omega_f = \sqrt{r^2 + v}$, 8 with frequency $\omega_f = \sqrt{r^2 - v}$.

The leading-order term in the long-range interaction potential can be calculated in a simple way by working in the quasi-static approximation [51]. This is the assumption that the time of the oscillations $t_{osc} \sim \frac{1}{\omega}$ is small compared to the time of interactions $t_{int} \sim \frac{r}{v}$, and the oscillators are frozen in the ground state throughout. In this approximation, the effective potential generated by the oscillators is given by the combination of the ground state energies

$$V_{eff} = \sum_b \omega_b - \sum_g \omega_g - \frac{1}{2}\sum_f \omega_f$$

which, substituting for the various frequencies, reads

$$V_{eff} = \sqrt{r^2 + 2v} + \sqrt{r^2 - 2v} + 6r - 4\sqrt{r^2 + v} - 4\sqrt{r^2 - v} \,.$$

This is expanded for small $\frac{v}{r^2}$, and the leading order term reads

$$V_{eff} = -\frac{15}{16}\frac{v^4}{r^7}\,. \tag{2.2.8.3}$$

This agrees with the supergravity result (2.2.8.2) for the leading-order potential between two gravitons with longitudinal momentum $p^+ = \frac{1}{R}$ in light-front eleven-dimensional supergravity. The result can be easily generalised to the case where the two gravitons have N_1 and N_2 units of longitudinal momentum. Restoring units, the potential reads

$$V = V_{eff} = -\frac{15}{16}\frac{N_1 N_2 v^4}{R^3 M_p^9 r^7}\,.$$

Such a result is particularly useful for the present dissertation, since it will provide the interaction term used to prove the soft theorem for the scattering of vertex-like operators in BFSS theory in Chapter 3. Note that the potential in equation (2.2.8.3) was obtained by means of a precise cancellation between bosonic degrees of freedom and fermionic ones. This is a consequence of the supersymmetry of the theory and provides a highly non-trivial check of the conjecture.

The more general formalism, which leads to the interaction potential (2.2.8.3), consists of using the explicit expression for the propagator of fluctuating degrees of freedom to evaluate the one-loop effective action for background fields. Once again, the effective Lagrangian can be arranged in a form which matches the supergravity expression (2.2.8.2). Explicit calculations show that the result depends on the graviton stress tensor [50]. Moreover, the supergravity multipole expansion can be reproduced on the Matrix Theory side, and allows us to compute higher-order terms in the large-distance expansion, whose expression will depend on higher moments of the stress tensor.

At the same time, analogous expressions can be obtained for the scattering between two $M2$-brane sources or two $M5$-branes, where the effective potential will depend on the respective integrated currents. All things considered, the leading-order potential between two matrix configurations, representing supergravity objects interacting in the target space, reads

$$V_{leading} = V_{gravity} + V_{electric} + V_{magnetic}$$

where the different terms are

$$V_{gravity} = -\frac{15R^2}{4r^7}\left(\hat{\mathcal{T}}^{\mu\nu}\tilde{\mathcal{T}}_{\mu\nu} - \frac{1}{9}\hat{\mathcal{T}}^{\mu}_{\mu}\tilde{\mathcal{T}}^{\nu}_{\nu}\right)$$
$$V_{electric} = -\frac{45R^2}{r^7}\hat{\mathcal{J}}^{\mu\nu\rho}\tilde{\mathcal{J}}_{\mu\nu\rho}$$
$$V_{magnetic} = -\frac{45R^2}{r^7}\hat{\mathcal{M}}^{+-\mu\nu\rho\sigma}\tilde{\mathcal{M}}_{+-\mu\nu\rho\sigma}$$

and all the components of the currents are expressed in terms of Matrix Theory degrees of freedom, and they are given explicitly in [50].

Beyond linearised supergravity, it becomes technically difficult to check the correspondence explicitly. Nonetheless, the leading-order three-graviton potential, which scales like

$$V_{3grav} \sim \frac{v^6}{r^{14}}$$

has been reproduced from a two-loop Matrix Theory computation for $N = 3$ [52, 53]. Indeed, more work is needed to establish the correspondence further.

2.3 Black holes in Matrix Theory

BFSS theory provides a compelling setting to study black holes as ordinary quantum systems. Although solving the full quantum mechanical problem has not been feasible at the technical level, the collective dynamics of the degrees of freedom at large N reproduces known black hole features. Hence, the theory provides a natural ground to address longstanding problems such as the information paradox and the meaning of spacetime singularities. We will now review various results for the two kinds of black holes that can be studied in Matrix Theory: the black zero-brane in type IIA supergravity and the Schwarzschild black hole in M-Theory.

In this section, we will mostly be concerned with the type IIA appearance of BFSS theory, discussed in Section 2.2.2, as opposed to the null reduction of M-Theory one. The duality with the black brane geometry will be similar in spirit to the AdS/CFT correspondence [11]. However, we will see in Section 2.3.3 how the duality with M-Theory in the Infinite Momentum Frame appears in a specific region of parameter space, and allows for a description of the Schwarzschild black hole solution in Matrix Theory.

2.3.1 Black brane in Matrix Theory

The $D0$-black hole is a solution of type IIA supergravity, where the relevant part of the action reads

$$I = \frac{1}{(2\pi)^7 \ell_s^8} \int d^{10}x \sqrt{g} \left[e^{-2\phi} \left(R + 4(\nabla\phi)^2 \right) - \frac{1}{4} F_{\mu\nu}^2 \right] ,$$

R being the Ricci scalar, ϕ a scalar field called the dilaton and $F_{\mu\nu}$ the field strength for the Ramond-Ramond gauge field. The solution has metric [54–57] (see [58] for a review)

$$ds^2 = -H^{-\frac{1}{2}}(r) f(r) dt^2 + H^{\frac{1}{2}}(r) \left(f^{-1}(r) dr^2 + r^2 d\Omega_8^2 \right) \tag{2.3.1.1}$$

effective string coupling

$$e^{\phi} = g_s H^{\frac{3}{4}}(r) ,$$

and gauge field

$$A_0 = \left(1 - H^{-1}(r) \right) \coth \alpha$$

which are specified in terms of two functions

$$H(r) = 1 + \frac{r_0^7 \sinh^2 \alpha}{r^7}, \qquad f(r) = 1 - \frac{r_0^7}{r^7} ,$$

where α parametrises deviations from extremality and r_0 represents the radius of the horizon. These are related to the mass M and the charge $Q = \frac{N}{g_s \ell_s}$ of the black hole via

$$M = \frac{(2\pi)^2}{7 d_0} \frac{r_0^7}{g_s^2 \ell_s^8} \left(7 \cosh^2 \alpha + 1 \right) , \qquad N = \frac{(2\pi)^2}{d_0} \frac{r_0^7}{g_s \ell_s^7} \left(\cosh \alpha \sinh \alpha \right) ,$$

where the constant is $d_0 = 240\pi^5$. The BPS bound reads

$$\frac{M}{N} \geq \frac{1}{g_s \ell_s} ,$$

which is saturated in the extremal limit, which consists of $\alpha \to \infty$ while keeping the charge fixed by simultaneously sending $r \to 0$. The black hole temperature can also be expressed in terms of α and r_0 as

$$T = \frac{7}{4\pi r_0 \cosh \alpha}$$

and vanishes in the extremal limit.

We will now explain the duality with Matrix Theory. The claim is that **the supergravity limit of type IIA string theory in the near-horizon region of the black zero-brane background is dual to Matrix Theory in the decoupling limit.** In fact, we will make the statement more precise by looking at the range of energies in which this applies exactly. The decoupling limit consists of

$$\alpha' \to 0, \qquad g_{YM}^2 = \frac{g_s}{\alpha'^{\frac{3}{2}}} = \text{fixed}\,,$$

where g_{YM}^2 is the Yang-Mills coupling of Matrix Theory [59] and $\alpha' = \ell_s^2$. We will focus on the $p \leq 3$ case. Then, in this limit, interactions between the world-line theory and the bulk, controlled by the ten-dimensional Newton constant G_{10}, decouple, since

$$G_{10} = 8\pi^6 g_s^2 \ell_s^8 \to 0\,.$$

The duality relates the weak coupling limit of one theory to the strong coupling limit of the other. Defining the effective dimensionless coupling in the Yang-Mills description at given energy U_0

$$g_{eff}^2 = g_{YM}^2 N U_0^{-3}\,,$$

it follows that the theory is weakly coupled for

$$U_0 \gg \left(g_{YM}^2 N\right)^{\frac{1}{3}}\,.$$

On the other hand, the supergravity description only applies for small effective string coupling and small curvature. The effective string coupling at the horizon can be written as

$$e^{\phi} = \frac{g_{eff}^{\frac{7}{2}}}{N}\,.$$

It controls the strength of loop corrections to the supergravity action, and the requirement that it is small implies

$$e^{\phi} \ll 1 \Longrightarrow U_0 \gg g_{YM}^{\frac{2}{3}} N^{\frac{1}{7}}\,.$$

The constraint on the curvature imposes [59]

$$\alpha' R = \frac{1}{g_{eff}} \ll 1 \Rightarrow U_0 \ll \left(g_{YM}^2 N\right)^{\frac{1}{3}}\,.$$

Hence, we trust the supergravity description for

$$g_{YM}^{\frac{2}{3}} N^{\frac{1}{7}} \ll U_0 \ll \left(g_{YM}^2 N\right)^{\frac{1}{3}} ,$$

where the Matrix Quantum Mechanics is strongly coupled. Various numerical tests of the correspondence, in this region of parameter space, have been performed [60–64]. The Yang-Mills coupling also sets the natural energy scale in the theory, through

$$E \sim \left(g_{YM}^2\right)^{\frac{1}{3}} = \frac{g_s^{\frac{1}{3}}}{\ell_s} .$$

We impose that the energy above extremality in the bulk picture, defined as $\Delta E := M - \frac{N}{g_s \ell_s}$, corresponds to the Yang-Mills energy. This implies that

$$U_0^7 := \left(\frac{r_0}{\alpha'}\right)^7 = a_p g_{YM}^4 \Delta E = \text{fixed} ,$$

where a_P summarises the numerical coefficients. On the Yang-Mills side, we can think of having a probe $D0$-brane at distance r_0. Then, U_0 represents the expectation value of the Higgs, coming from stretched open strings, as explained in Section 2.2.2, and sets the energy scale at which the theory is probed. In particular, this implies that in the decoupling limit

$$\frac{r_0}{\ell_s} \to 0 ,$$

namely, we are looking at sub-stringy distances in the Yang-Mills theory and at near-extremal configurations on the bulk side.

The duality originates from the fact that the supergravity solution and the Yang-Mills theory are the low-energy descriptions of closed and open sectors of type IIA string theory, respectively. In the decoupling limit, these are known to be equivalent. We have already seen in Section 2.2.2 that the low-energy limit of the open string sector on $D0$-branes is described by Matrix Theory. On the other hand, the energy of excited string states in the black brane background is

$$E = \frac{H^{-\frac{1}{4}}}{\ell_s} = (g_s N)^{\frac{1}{3}} \rho^{\frac{7}{4}} \frac{1}{\ell_s} ,$$

where ρ is the rescaled radial coordinate

$$\rho := \frac{r}{\alpha'} \frac{1}{(d_0 g_{YM}^2 N)^{\frac{1}{3}}} .$$

Hence, the excitations which survive at low energy live in the near-horizon region. To summarise, supergravity in the near-horizon region of the nearly extremal black zero-brane background is dual to Matrix Theory in the decoupling limit. Corrections in α' are suppressed in this limit.

2.3.2 Evaporation of the black zero-brane

The dual description of the black zero-brane in terms of Matrix Theory provides a compelling setting to address longstanding puzzles in black hole physics. In particular, a mechanism was proposed in [65] that partially explains the Hawking radiation process within the gauge theory's dynamics. BFSS theory was considered at temperature T in the type IIA region described above

$$g_{YM}^{\frac{2}{3}} N^{\frac{1}{7}} \ll T \ll \left(g_{YM}^2 N\right)^{\frac{1}{3}} .$$

In this high-temperature regime, the mass of open strings is parametrically smaller than the temperature [66]; the classical approximation holds, and the relevant configurations can be characterised in terms of the eigenvalue dynamics of the bosonic matrices. The dynamics has chaotic behaviour [67–69]. If we introduce a proper cutoff for matrix eigenvalues (i.e. putting the D0-branes in a box), which makes the phase space bounded, ergodicity should hold for generic initial conditions.

The black hole configuration is represented by a block matrix of size $N \times N$ where all the $D0$-branes and open strings are excited. This describes a bound state of N $D0$-particles and open strings connecting them. A useful quantity to consider is the eigenvalue distribution of the matrix

$$\mathcal{R} = \left(\sum_I X^I X_I\right)^{\frac{1}{2}}$$

representing the radial position of the particles. It was shown in [65] that such a distribution takes the form

$$\rho(r) \sim (\lambda T)^{-\frac{1}{4}} \left[\frac{r}{(\lambda T)^{\frac{1}{4}}}\right]^{-8(2N-3)} ,$$

where $\lambda = g_{YM}^2 N$ is the 't Hooft coupling. The above density suppresses branes travelling at large distances. Hence, the bosonic dynamics produce an effective confinement of branes in the black hole.

If we interpret the emission of individual branes as Hawking radiation, this suggests that the evaporation process is heavily suppressed. However, due to ergodicity, the system explores configurations where a particle wanders at distances $U \sim T$. When this occurs, the heavy string excitation resolves the full BFSS dynamics, which includes fermionic terms. At this point, the classical approximation breaks down. As we have shown in Section 2.2.8, flat directions open up and the brane develops a fast-decaying interaction potential. As a consequence, a quantum of radiation is emitted.

The proposed mechanism for evaporation predicts an effective threshold radius R_T where flat directions open up and branes are considered to decouple. Calling ℓ the average size of the bound state, the rate of the evaporation process can be estimated in terms of the difference in entropy between the pure black hole configuration and the state including a black hole and a quantum of radiation. The expression reads

$$\Gamma = (\lambda T)^{\frac{1}{4}} e^{-\Delta S} = (\lambda T)^{\frac{1}{4}} \frac{e^{S_>}}{e^{S_<}} = (\lambda T)^{\frac{1}{4}} \frac{\int_{R_T}^{\infty} x^{-8(2N-3)} dx}{\int_{\ell}^{R_T} x^{-8(2N-3)} dx} = \frac{(\lambda T)^{\frac{1}{4}}}{e^{(8(2N-3)-1)\log(R_T/\ell)} - 1}\,, \tag{2.3.2.1}$$

where the overall inverse-time scale is set by the Lyapunov exponent [65]

$$\lambda_L = (\lambda T)^{\frac{1}{4}}\,.$$

Given this rate, the entanglement entropy of radiation was computed in [70].

The state of the system at time t is taken to be, schematically, a superposition of states representing the black hole with $N - n$ branes, with n $D0$-branes having been emitted as radiation

$$|\psi(t)\rangle \sim \sum_{n=0}^{N-1} \mathcal{N}_n \sqrt{p_n(t)} |N-n, E, \vec{p}\rangle_{BH} |n, E', \vec{p}'\rangle_{rad}\,.$$

Here $p_n(t)$ are the probabilities that n branes have been emitted, which can be estimated using the expression for the rate (2.3.2.1), $\mathcal{N}_n$ are normalisations, and the states are characterised by energies and momenta. The density matrix constructed from this (pure) state is

$$\rho(t) = |\psi(t)\rangle \langle\psi(t)|$$

and the reduced density matrix can be obtained by tracing over the black hole region

$$\rho_{rad}(t) = \mathrm{Tr}_{BH}\rho(t)\,.$$

The von Neumann entanglement entropy

$$S(t)_{VN} = -\operatorname{Tr}(\rho_{rad}(t)\ln\rho_{rad}(t))$$

is computed explicitly, and the result is shown in Figure 2.1, where we fix the parameter $a = 16\log(\frac{R_T}{\rho})$.

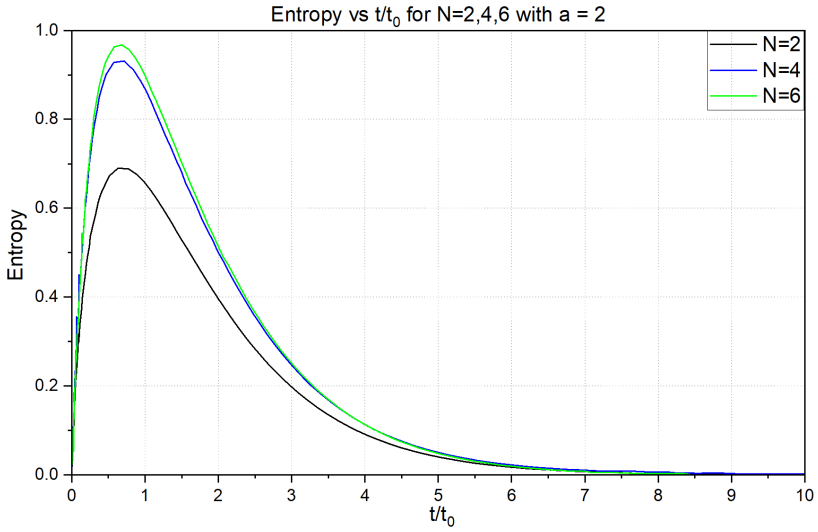


Figure 2.1: Entanglement entropy vs time for $a = 2$ and $N = 2, 4, 6$.

We see that the entropy follows a behaviour compatible with a Page curve: it increases up to a maximum value, before going back to zero at later times, signalling the recovery of information after evaporation. This example shows how the duality can be leveraged to resolve black hole evaporation in terms of a unitary process within the quantum mechanical dual theory.

2.3.3 M-Theory region and the Schwarzschild black hole

We will now continue to explore the vast landscape of BFSS theory by considering a different region of parameter space. At energies

$$U_0 \ll g_{YM}^{\frac{2}{3}} N^{\frac{1}{7}} \tag{2.3.3.1}$$

the effective string coupling becomes large. If we think of type IIA as M-Theory compactified on a circle of radius R_{11} (see Section 2.1.2), then locally, on a curved background

$$R_{11} = e^{\frac{2\phi}{3}} \ell_p \,.$$

In the regime (2.3.3.1), this radius becomes big, the eleventh dimension is macroscopic, and we expect to recover the M-Theory description. We must, therefore, uplift the ten-dimensional black brane solution. The natural way to achieve this is to consider a solution which is uniform in the eleventh dimension, compatible with dimensional reduction. This is a black string configuration, whose metric takes the form

$$ds^2_{string} = \ell_p^2 \left(-f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega_8^2 + dx_{11}^2\right) ,$$

where

$$f(r) = 1 - \frac{r_0^7}{r^7} .$$

To obtain the uplift of a black brane (2.3.1.1) with N units of Kaluza-Klein momentum, we need to perform a large boost:

$$\begin{pmatrix} t' \\ x'_{11} \end{pmatrix} = \begin{pmatrix} \cosh\alpha & \sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix} \begin{pmatrix} t \\ x_{11} \end{pmatrix} .$$

The resulting eleven-dimensional metric takes the form, in terms of the functions $H(r)$ and $f(r)$ defined in Section 2.3.1

$$ds^2_{11} = -H^{-1}(r)\, f(r)dt^2 + H(r)\left[dx_{11} + \left(H^{-1}(r) - 1\right)\coth\alpha dt\right)\Big]^2 + f^{-1}(r)dr^2 + r^2 d\Omega_8^2 \tag{2.3.3.2}$$

and the reduction on a circle reproduces the ten-dimensional metric, gauge field, and dilaton.

The black string is stable for a certain range of energies. In fact, if we lower the energy further, we encounter the **Gregory-Laflamme instability** [71, 72]. The origin of this effect lies in the comparison between the size of the compact direction and the dimension of the horizon. After the boost, one can show using (2.3.3.2) that the size of the compact dimension expands from $2\pi R_{11}$ at asymptotic infinity to $2\pi R_{11}\cosh\alpha$ at the horizon. The black string is stable as long as the configuration is uniformly smeared along the compact direction. However, when the size of this direction at the horizon exceeds the extent of the black hole, the geometry becomes that of a black hole localised in the compact direction. The instability arises when [73]

$$e^{\alpha} \sim \frac{r_0}{R_{11}} . \tag{2.3.3.3}$$

Given the entropy-to-radius relation

$$S_{string} = \ell_p^{-9} R_{11} r_0^8 \cosh\alpha$$

and the momentum-to-radius

$$P_{string} = M_{string} \sinh\alpha = \ell_p^{-9} R_{11} r_0^7 \sinh 2\alpha$$

we see that the condition (2.3.3.3) translates into

$$\frac{N}{R_{11}} = P_{string} \sim \frac{S_{string}}{R_{11}} \Rightarrow S \sim N\,.$$

For smaller entropies, the geometry becomes a Schwarzschild black hole. In terms of energy, the threshold value is

$$\left(\frac{r_0}{\ell_p}\right)^9 \sim N \Rightarrow U_0 \sim g_{YM}^{\frac{2}{3}} N^{\frac{1}{9}}\,.$$

Below the threshold, the Schwarzschild radius of the black hole reads

$$\left(\frac{r_s}{\ell_p}\right)^8 = \frac{(E\,N)^{\frac{1}{2}}}{g_{YM}^{\frac{1}{3}}} = \frac{(U_0^7\,N)^{\frac{1}{2}}}{g_{YM}^{\frac{7}{3}}}$$

and the gravity description can be trusted up until the point where the extent of the black hole becomes comparable with the eleven-dimensional Planck scale, namely

$$U_0 \gg g_{YM}^{\frac{2}{3}} N^{-\frac{1}{7}}\,.$$

To summarise, the Schwarzschild black hole regime of Matrix Theory is valid for

$$g_{YM}^{\frac{2}{3}} N^{-\frac{1}{7}} \ll U_0 \ll g_{YM}^{\frac{2}{3}} N^{\frac{1}{9}}\,.$$

A plot summarising the various regimes can be found in Figure 2.2.

2.4 Related Matrix Models: IKKT and BMN

BFSS theory is an example of a broader class of maximally supersymmetric matrix models motivated by string theory. Two closely related examples are the IKKT and BMN matrix models. These theories share with BFSS the idea that spacetime geometry is encoded in large-N matrix degrees of freedom, but they realise it in different ways.

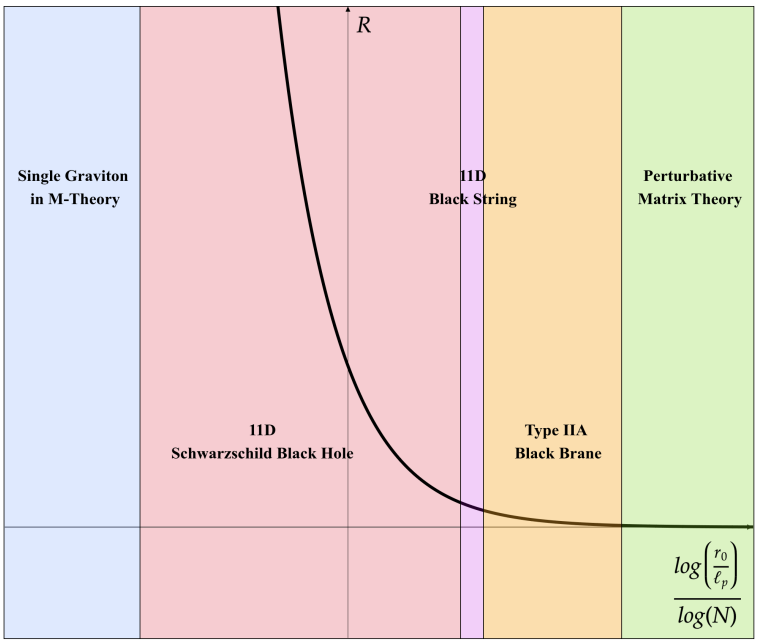


Figure 2.2: Gravitational dual of different regimes of Matrix Theory probed by a brane at distance r_0. From right to left, the effective string coupling increases and the eleventh dimension decompactifies. The transitions are, in order: **green** to **yellow** strongly to weakly curved gravity; **yellow** to **purple** gravitational solution uplifted to eleven dimensions; **purple** to **red** Gregory-Laflamme transition; **red** to **blue** small black hole to single graviton.

The Ishibashi–Kawai–Kitazawa–Tsuchiya (IKKT) model, also known as the type IIB matrix model [74], is obtained by reducing ten-dimensional maximally supersymmetric Yang-Mills theory all the way to zero dimensions. In contrast with BFSS, which is a matrix quantum mechanics theory with an explicit time direction, IKKT is defined purely as a matrix integral. Its bosonic degrees of freedom are ten Hermitian matrices A^I, with $I = 0, \ldots, 9$, together with fermionic matrices ψ_α, $\alpha = 1, \ldots, 32$ in the Majorana-Weyl spinor representation of the $SO(1,9)$ global symmetry group. Note that we are assuming a 32-component Dirac notation where a Weyl projection condition is enforced. Its action takes the form

$$S_{IKKT} = -\frac{1}{2g^2}\mathrm{Tr}\left[\frac{1}{2}[A^I, A^J]^2 + \bar{\psi}\Gamma^I[A_I, \psi]\right] .$$

The bosonic matrices A^I form a vector of $SO(1,9)$. Similar to the BFSS case 2.2.2, these encode information about spacetime coordinates in 1+9 dimensions. In particular, diagonal entries encode the location of N $D(-1)$ instantons, whereas off-diagonal degrees

of freedom describe open strings stretched between two branes. The fermionic matrices ψ act as superpartners of the bosons, and the theory is maximally supersymmetric.

The absence of a distinguished time coordinate makes the IKKT model a more covariant proposal than BFSS: all spacetime directions are treated on the same footing, and the interpretation of spacetime must emerge from the eigenvalue distribution and correlations of the matrices.

After Wick rotating to Euclidean signature, the partition function can be defined through the path integral

$$Z_{IKKT} = \int \mathcal{D}[A^I]\mathcal{D}[\psi_\alpha]\, e^{-S^E_{IKKT}} \,. \tag{2.4.0.1}$$

Naively, one might expect that this should diverge due to the flat directions in the potential. However, this has been proven not to be the case, and in fact the above integral converges [75, 76].

The theory is strictly tied to the BFSS model. It has been shown [66, 77] that BFSS theory at high temperature reduces to IKKT. Conversely, a compactification procedure, similar to the one we will introduce explicitly in Chapter 4, shows that IKKT compactified on a circle is dual to BFSS at finite temperature [78].

Emergent spacetime from IKKT theory has been largely explored. Manifolds with different geometries have been constructed in terms of matrix degrees of freedom [79–81], and Einstein gravity has been recovered [82–84]. Spontaneous symmetry breaking of the global symmetry has been studied analytically and numerically [85–88]. This highlighted the existence of a phase where four matrix dimensions become macroscopic. Such a phase has been interpreted in terms of an emergent cosmology from the theory [89–93]. Recent work on the IKKT model and physically motivated deformations thereof includes [94–98].

A second important example is the Berenstein–Maldacena–Nastase (BMN) [99] matrix model. This theory is a massive deformation of BFSS describing M-Theory on the maximally supersymmetric plane-wave background. The pp-wave background is the supergravity configuration with metric

$$\begin{aligned} ds^2 &= -2dx^+dx^- - H(x^I)(dx^+)^2 + (dx^I)^2 \\ H(x^I) &= \left(\frac{\mu}{3}\right)^2 (x^a)^2 + \left(\frac{\mu}{6}\right)^2 (x^\nu)^2 \end{aligned}$$

where $a = 1, 2, 3$ and $\nu = 4, \ldots, 9$, and three-form field-strength components

$$F_{+123} = \mu \,.$$

This configuration can be obtained from the maximally supersymmetric $AdS_4 \times S^7$ and $AdS_7 \times S^4$ geometries, representing the $M2$- and $M5$-configurations respectively, by means of a Penrose limit. The latter is a limit which zooms in on the neighbourhood of a null geodesic.

Given this gravitational solution, deformations of the dual theory are obtained by considering the coupling of higher moments of the stress tensor and membrane current to the metric and flux background [100, 101]. In particular, a mass term is obtained through the coupling between the stress tensor and the pp-wave metric. Moreover, a Myers-type cubic interaction and the corresponding supersymmetry-preserving fermionic term are generated from the coupling of the membrane current to the flux. The Lagrangian is

$$\mathcal{L}_{\text{BMN}} = \mathcal{L}_{BFSS} + \text{Tr}\left[-\frac{1}{2}\left(\frac{\mu}{3}\right)^2 (X^a)^2 - \frac{1}{2}\left(\frac{\mu}{6}\right)^2 (X^\nu)^2 + i\frac{\mu}{3}\epsilon_{abc}X^a X^b X^c + \frac{i\mu}{4}\bar{\psi}\Gamma^{123}\psi\right].$$

The BFSS model is recovered in the limit $\mu \to 0$. The physical effect of the BMN deformation is to lift the flat directions of BFSS and to admit stable fuzzy sphere vacua. Unlike ordinary BFSS, where fuzzy spherical membranes are not static solutions but instead oscillate under the BFSS equations of motion, the BMN model admits stable, spherical $M2$-membrane solutions as fuzzy spheres. This is a manifestation of the Myers effect [102], where the background flux polarises a collection of $D0$-branes into a membrane configuration. This is different from the approach we will adopt in Chapter 4, where we will explore an alternative avenue to obtain stable membranes in BFSS.

Thus, BFSS, IKKT, and BMN should be viewed as complementary realisations of the same underlying idea: a non-perturbative formulation of string or M-Theory in terms of large-N matrices.

2.5 Overview of the main chapters

Our goal in the remainder of the thesis will be to present our novel contributions to the field. The purpose of this section is to outline how they fit in the existing literature, parts of which we reviewed in this chapter, and expand it.

2.5.1 Soft theorems and the symmetries of the theory

We have seen in Section 2.2 that BFSS theory has a global $SO(9)$ symmetry, which can be interpreted as target space rotations. Moreover, the theory is a non-relativistic quantum mechanics, and hence admits Galilean boosts, which translate between reference frames moving at constant velocity. At the same time, in the large-N, M-Theory limit, we expect the theory to describe eleven-dimensional flat space, according to the BFSS conjecture. Hence, the S-Matrix should enjoy a full $ISO(1,10)$ Poincaré symmetry group.

The problem can be addressed by building on recent work [103, 104], which has demonstrated a connection between asymptotic symmetries and soft theorems in BFSS theory. This development has enabled a deeper comprehension of how the complete 11-dimensional Lorentz group manifests itself within Matrix Theory, alongside the recognition of an infinite-dimensional asymptotic symmetry group. These are diffeomorphisms of asymptotically flat spacetime generated by vector fields which become Killing vectors in the asymptotic region. The reasoning goes as follows: assuming a soft factorisation theorem to hold for scattering amplitudes of gravitons

$$\mathcal{A}(q_s, h_s; p_1, \theta_1; \ldots; p_n, \theta_n) = (S^{(-1)} + S^{(0)} + \ldots)\mathcal{A}(p_1, \theta_1; \ldots; p_n, \theta_n)\,,$$

where q_s is a soft momentum. It follows that

$$\langle \text{out}| \left(\mathcal{Q}_{out}[f] + \mathcal{Q}_{out}[Y^I]\right) - \left(\mathcal{Q}_{in}[f] + \mathcal{Q}_{in}[Y^I]\right) |\text{in}\rangle = O\left(\frac{1}{N}\right)\,,$$

where the explicit form of the universal factor and the associated conserved charges are given in [103]. This is the statement that infinitely many charges, associated with asymptotic symmetries, are conserved in all scattering processes. In particular, super-translations, which include and generalise eleven-dimensional translations, are parametrised by a function f, and super-rotations, which include and generalise the eleven-dimensional Lorentz group, are parametrised by a vector Y^I. Note that this conservation is only valid in the large-N limit. The conserved charges can be put in one-to-one correspondence with vector fields ξ^μ, which become Killing vectors in the asymptotic region and generate infinitesimal transformations. These vectors are also parametrised by f and Y^I.

Hence, proving a soft theorem in BFSS theory guarantees the presence of symmetries which are expected to emerge from the duality with M-Theory, but which have been elusive. In Chapter 3, we fill this gap by providing such a proof in the large-distance EFT setting.

2.5.2 Mass deformation and fuzzy black hole in Matrix Theory

In Chapter 1, we discussed how Quantum Gravity proposals must address the problem of the black hole and Big Bang singularities. Hence, it is important to understand how this question can be answered in Matrix Theories. This broadly motivates our work in Chapter 4.

In Section 2.3, we have seen that BFSS theory is suitable to describe black holes. However, an explicit solution of a Schwarzschild black hole, in terms of the theory's degrees of freedom, has remained elusive. Our idea is to adapt the construction of [105, 106], in the context of a bottom-up matrix quantum mechanics proposal. There, the black hole geometry is dual to a fuzzy sphere configuration, with non-trivial fermionic excitations. We have seen in Section 2.2.5 how, in ordinary BFSS theory, the fuzzy sphere is not a stable solution to the equations of motion. In order to fix this and adapt the construction of [105] to BFSS theory, we need to study a mechanism for generating a non-supersymmetric mass deformation along with a non-trivial fermionic sector. Chapter 4 illustrates a possible mechanism, consisting of torus compactification of the Matrix Theory target space with anti-periodic boundary conditions for fermions. The theory we will obtain has exactly the desired feature, and we will perform the black hole construction explicitly.

A further piece of motivation comes from a recent proposal for addressing the Big Bang singularity problem in IKKT theory, which we briefly reviewed in 2.4. Compelling evidence for the emergence of $1 + 3$ macroscopic dimensions has been found in the IKKT model. It was shown, numerically and analytically, that a time direction can be defined such that spontaneous symmetry breaking of rotational symmetry from $SO(10)$ to $SO(4) \times SO(6)$ occurs in the Euclidean model [85, 86]. Moreover, numerical studies of the Lorentzian theory have hinted at a similar mechanism [87]; the symmetry breaking pattern is $SO(1,9) \to SO(1,3) \times SO(6)$ and three space directions become macroscopic after a critical time, whereas the remaining six are of Planck size. Such evidence has motivated studies of matrix cosmology [89–92], in which it is proposed that early universe

cosmology and spacetime metric emerge from the Matrix Model. Moreover, the presence of fermionic degrees of freedom is an important feature of the theory; indeed, Monte Carlo results show that they are crucial for symmetry breaking to occur [85].

A key requirement for a $1 + 3$-dimensional spacetime to emerge is that a mass term is present in the theory [87]. More generally, cosmological solutions were found in various analyses of IKKT with a mass term [107–111]. The question arises as to the possible nature of such a mass term.

The problem of establishing a symmetry-breaking pattern which entails the emergence of spacetime in BFSS is made more complicated by the difficulties in simulating the model. However, some analytical results obtained using a Gaussian expansion method [112] suggest [113] that symmetry breaking does occur in BFSS. Despite this success, there is no conclusive evidence that the symmetry-breaking pattern is the one expected for the emergence of three large space directions, namely $SO(9) \to SO(3) \times SO(6)$. Hence, by analogy with IKKT, we are motivated to generalise the result for mass generation to BFSS theory, as a further step towards the achievement of such a symmetry-breaking pattern.

3 Soft Theorem

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3.1 Introduction

Chapter 2 introduced BFSS theory as a large-N, non-perturbative realisation of M-Theory. There, we presented several pieces of evidence in support of the BFSS conjecture. As explained in Section 2.5.1, recent results [103, 104] have demonstrated that a crucial confirmation of the BFSS conjecture, related to the presence of the expected symmetry group in the large-N limit, can be achieved by proving a soft graviton theorem in BFSS theory. In this chapter, we aim to provide such a proof.

Soft theorems [114] hold for a vast class of theories describing massless particles (including supergravity); they relate the scattering amplitude between states which include a soft particle with the amplitude between the corresponding states when such

a particle is removed. However, it was not until recently that it was realised that soft theorems are equivalent to Ward identities for the conservation of charges associated with asymptotic symmetries [115–124] (cf. [125, 126] for recent reviews of asymptotic symmetries in celestial holography). The statement of [103, 104] is that indeed this connection is also present in the BFSS theory. Soft theorems have been shown to be universally valid [127–132]; therefore, it would be natural to expect a soft theorem to hold in BFSS if the latter were to describe supergraviton dynamics.

A proof of such a theorem was given in [133, 134]. That proof was carried out by first showing that any scattering amplitude involving massless particles and satisfying a specific set of assumptions enjoys a soft factorisation. It was then argued that those assumptions should be satisfied by Matrix Theory scattering amplitudes, under the condition that amplitudes involving five or more non-soft particles are evaluated in a particular kinematic regime.

In this chapter, we approach the problem from a complementary perspective by proving a soft theorem for the effective theory describing supergraviton interactions in Matrix Theory. Specifically, we will construct composite operators which possess the right quantum numbers to represent gravitons in the target space. We will then show how their correlation functions enjoy a soft factorisation to the leading and subleading orders in the large-N limit.

This chapter is organised as follows: in Section 3.2 we analyse the statement of the soft theorem for BFSS theory; in Section 3.3 we outline the formalism in which the computation is carried out and the strategy we will adopt; Section 3.4 will then be devoted to the detailed proof of the leading soft theorem, while in Section 3.5 we will obtain the subleading soft factor; in Section 3.6 we discuss the result and provide an outlook on future developments of this work.

3.2 The Theorem

The soft graviton theorem provides a relationship between scattering amplitudes of the form

$$\mathcal{A}(q_s, h_s; p_1, \theta_1; \ldots; p_n, \theta_n) = (S^{(-1)} + S^{(0)} + \ldots)\mathcal{A}(p_1, \theta_1; \ldots; p_n, \theta_n)\,, \qquad (3.2.0.1)$$

where momenta are characterised by an overall scale $|p^\mu| \propto \omega$, and h_s, θ_i are graviton polarisations. The particle with momentum q_s and polarisation h_s is soft, namely $\omega_s \ll \omega_i, \forall i$, and the soft factors are such that

$$S^{(n)} \sim \left(\frac{\omega_s}{\omega_i}\right)^n .$$

In particular, the leading and subleading factors are

$$S^{(-1)} = \frac{\kappa}{2} h_{s\mu\nu} \sum_{j=1}^{n} \eta_s \eta_j \frac{p_j^\mu p_j^\nu}{q_s \cdot p_j}, \qquad S^{(0)} = i\frac{\kappa}{2} h_{s\mu\nu} \sum_{j=1}^{n} \eta_s \eta_j \frac{p_j^\mu J_j^{\nu\rho} q_{s,\rho}}{q_s \cdot p_j} , \tag{3.2.0.2}$$

where $\eta = -1$ if the corresponding particle is *incoming* and $\eta = 1$ if *outgoing*, $J_j^{\mu\nu} = L_j^{\mu\nu} + S_j^{\mu\nu}$ stands for the total angular momentum of the j-th particle and $\kappa = \sqrt{32\pi G_{11}}$. Soft terms are ubiquitous in theories with massless particles. We are concerned here with the case of 11-dimensional supergravity, so $\mu = 0, 1 \ldots, 10$.

To prove a result analogous to (3.2.0.1) for BFSS theory, we will use the large-distance effective theory describing interactions of $D0$-brane bound states. As we reviewed in Section 2.2.8, this effective interaction is obtained by integrating out quantum fluctuations Y around a classical background

$$X^I_{back} = X^I_1 \mathbb{I}_{N_1} \oplus X^I_2 \mathbb{I}_{N_2}, \quad X^I = X^I_{back} + Y^I.$$

Then, by defining r as the centre of mass distance between particles, the background matrix describes two $D0$-brane bound states composed of N_i branes and having transverse coordinates X^I_i. Quantum fluctuations give rise to massive excitations, both bosonic and fermionic, whose mass depends on r. By integrating out these modes at large distances, we obtain the effective theory for $D0$-brane bound state interactions, whose expression reads

$$\mathcal{L} = \frac{N_1}{2R}(\dot{X}^I_1)^2 + \frac{N_2}{2R}(\dot{X}^I_2)^2 - A\frac{N_1 N_2}{R^3 r^7}\left[(\dot{X}^I_1 - \dot{X}^I_2)^2\right]^2 + \mathcal{L}_{ang} + \mathcal{L}_{spin} , \tag{3.2.0.3}$$

where A is an order 1 constant (in units where the Planck length is set to 1). The first interaction term will be sufficient for the computation of $S^{(-1)}$, while the derivation of $S^{(0)}$ will require the explicit expression, which we will provide later, of the subleading contributions $\mathcal{L}_{ang}$ and $\mathcal{L}_{spin}$. The Lagrangian (3.2.0.3) can be naturally generalised to describe n such particles with pairwise interaction, leading to

$$\mathcal{L}_n = \sum_{i=1}^{n} \frac{N_i}{2R}(\dot{X}^I_i)^2 - \sum_{\substack{i\neq j \\ i,j=1}}^{n} \frac{A}{2}\frac{N_i N_j}{R^3 r^7}\left[(\dot{X}^I_i - \dot{X}^I_j)^2\right]^2 + \mathcal{L}_{ang} + \mathcal{L}_{spin}. \tag{3.2.0.4}$$

At large distances, where *in* and *out* particles are non-interacting, their non-relativistic momentum will be given by $p_i^I = \sqrt{2}\frac{N_i}{R}v_i^I$, where v_i^I is the velocity eigenvalue of $\dot{X}_i^I$.

We define asymptotic *in* and *out* states (corresponding to Matrix Theory time $\mp\infty$ respectively) as threshold bound states, each being characterised by the number of branes, the nine-dimensional velocity vector and the $SO(9)$ polarisation. An n particle *in* state is then written as

$$|N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n\rangle_{\text{in}}\,, \qquad N_1 + N_2 + \ldots + N_n = N\,,$$

and analogously for *out* states. Consequently, the scattering amplitude between n multi-particle states is given by the S-matrix element

$$\mathcal{A}(N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n) \\ = {}_{\text{out}}\langle N_1, v_1^I, \theta_1; \ldots; N_j, v_j^I, \theta_j | N_{j+1}, v_{j+1}^I, \theta_{j+1}; \ldots; N_n, v_n^I, \theta_n\rangle_{\text{in}}\,.$$

In this model, the parameter characterising the soft expansion is the number of branes in the bound state. In particular, a particle is soft (and its quantum numbers will be labelled by the subscript 's') if $\frac{N_s}{N_j} \ll 1\ \forall j$, so that the number of $D0$-branes in the bound state is dwarfed by that of any other state. Then, the soft theorem can be stated as

$$\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n) = \\ = (S^{(-1)} + S^{(0)} + \ldots)\mathcal{A}(N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n)\,, \quad (3.2.0.5)$$

where the expansion in soft factors is such that $S^{(k)} \sim \left(\frac{N_s}{N_j}\right)^k$. In the large-$N$ limit of Matrix Theory, the soft limit corresponds to the scaling $N_s \sim O(1)$, $N_j \sim N \to \infty$. Then the soft factors scale according to $S^{(k)} \sim \left(\frac{1}{N}\right)^k$.

The explicit forms of $S^{(-1)}$ and $S^{(0)}$, dubbed leading and subleading, respectively, in BFSS, have been computed in [103]. The expressions in (3.2.0.2) are translated to the $D0$-brane language by setting

$$p^\mu = \frac{N}{\sqrt{2}R}(1 + v^2, 2v^I, 1 - v^2)\,, \quad (3.2.0.6)$$

where v^I is the 9-dimensional transverse velocity and $v^2 = v^I v_I$. The polarisation tensor $h_{\mu\nu}$ is described by the symmetric traceless SO(9) tensor h_{IJ}, which accounts for the 44-dimensional polarisation basis of 11-dimensional gravity. We find

$$S^{(-1)} = -2\kappa\, h_{IJ} \sum_j \eta_s \eta_j \frac{N_j}{N_s} \frac{v_{sj}^I v_{sj}^J}{v_{sj}^2}\,,$$

and the subleading term is given later in (3.5.2.3). In [103] it was shown that $S^{(-1)}$ and $S^{(0)}$ are associated with the conservation of BFSS charges which generate asymptotic symmetry transformations. Our goal will be to demonstrate the validity of (3.2.0.5) within the effective field theory (3.2.0.4).

3.3 The formalism

In this section, we outline the formalism we will use to give a world-line proof for the soft theorem. Conceptually, the idea is to look at the scattering between Matrix Theory objects using the large-distance effective potential (3.2.0.4). The latter was computed with the motivation of showing that BFSS theory reproduces the interactions of linearised supergravity [50]. Here, we do not rely on the duality with eleven-dimensional supergravity to show a soft factorisation, but rather adopt a world-line formalism to carry out the computation in Matrix Theory, thus obtaining an independent check of the BFSS conjecture.

We consider the one-dimensional quantum theory of fields $X_i^I(t)$, $i = 1, \dots, n$, defined by a path integral with Lagrangian given by (3.2.0.4). We want to compute scattering amplitudes in a vertex operator formalism, which naturally accounts for the translation between world-line and target space quantities. The scattering amplitude involving n asymptotic particles takes the form

$$\begin{aligned}\mathcal{A}(N_1, v_1^I, \theta_1; \dots; N_n, v_n^I, \theta_n) &= \prod_{i=1}^{n} \int d\mu_{t_i} \, \langle \Omega | T[\mathcal{V}_1^*(t_1) \dots \mathcal{V}_n(t_n)] | \Omega \rangle \\ = \prod_{i=1}^{n} \int d\mu_{t_i} \, \langle \mathcal{V}_1^*(t_1) \dots \mathcal{V}_n(t_n) \rangle &= \prod_{i=1}^{n} \int d\mu_{t_i} \frac{\int \Pi_{k=1}^{n} \mathcal{D}X_k \, \mathcal{V}_k(t_k) e^{iS}}{\int \Pi_{k=1}^{n} \mathcal{D}X_k \, e^{iS}} \,, \end{aligned} \tag{3.3.0.1}$$

where $\langle\rangle$ denotes the expectation value with respect to the full action and T stands for time ordering. Moreover, we define

$$\int d\mu_{t_i} \; := \lim_{\alpha \to 0} \int_{-\infty}^{+\infty} dt_i \, e^{-\alpha |t_i|} \,,$$

namely, we integrate over operator insertion time, with an exponential regulator that kills divergences for large times. In the particular case of soft particle emission, the amplitude is

$$\begin{aligned}\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \dots; N_n, v_n^I, \theta_n) = \\ \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} \, \langle \Omega | T[\mathcal{V}_s^*(t_s) \mathcal{V}_1^*(t_1) \dots \mathcal{V}_n(t_n)] | \Omega \rangle \,. \end{aligned} \tag{3.3.0.2}$$

The operators appearing in the expressions (3.3.0.1) and (3.3.0.2) are vertex-like operators, and we propose that they represent asymptotic states in the target space. In particular, starred vertex operators create particles in the *out* state, while the non-starred create them in the *in* state. We describe their properties in Subsection 3.3.2. First, we will outline the formalism in which we will compute (3.3.0.2).

3.3.1 Correlation function in the effective theory

We evaluate (3.3.0.2) using perturbation theory in the interaction term of (3.2.0.3). Amplitudes such as (3.3.0.1) represent a correlation function in the eleven-dimensional target space. It follows that (3.3.0.2) corresponds to a soft emission or absorption, in which the soft $D0$-brane bound state is attached to one of the hard external legs; see figure 3.1. We therefore consider the series expansion corresponding to the interaction of

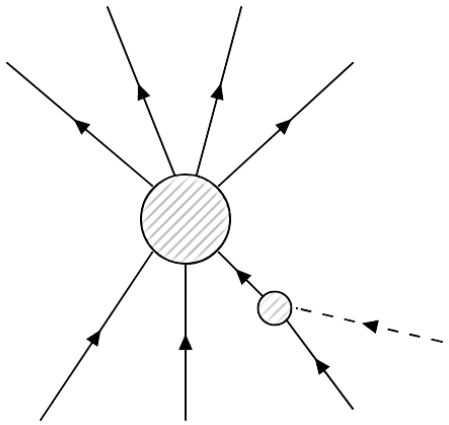


Figure 3.1: Soft absorption. The dashed line represents a soft $D0$-brane bound state, while the solid lines represent the hard ones. The big blob represents interactions between hard particles, whereas the small one stands for interactions between a soft and a hard particle.

the soft $D0$-brane bound state with one of the hard particles, to all orders in perturbation theory. As we will see in the next subsection, each vertex operator is a composite operator depending on $X_j(t)$ fields and creates a particle in the asymptotic initial or final state. Hence, the interaction time runs up to the earliest insertion time of the two particles involved in the case of *outgoing* particles, or from the latest insertion time in the case of *incoming* particles, or between the two in case of one *incoming* and one *outgoing* particle. In the following, we can pick the case in which both particles are in the *out* state for convenience, but the results can be easily extended to the other cases.

Defining

$$S_{sj} = \sum_{n=0}^{\infty} \int_{-\infty}^{\min(t_j,t_s)} du_1 \dots du_n \frac{i^n}{n!} \mathcal{L}_{sj}(u_1) \dots \mathcal{L}_{sj}(u_n)\,, \tag{3.3.1.1}$$

where $\mathcal{L}_{sj}$ is the interaction term in (3.2.0.3) with one of the interacting particles being soft, the correlation function (3.3.0.2) can be written as

$$\begin{aligned}&\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \dots; N_n, v_n^I, \theta_n) \\ &\qquad = \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} \sum_{j=1}^{n} \langle \mathcal{V}_1^*(t_1) \dots \overline{\mathcal{V}_s^*(t_s) S_{sj} \mathcal{V}_j^*(t_j)} \dots \mathcal{V}_n(t_n) \rangle_h \,. \end{aligned} \tag{3.3.1.2}$$

Here, the expectation value $\langle\rangle_h$ is defined with respect to an action containing only hard fields $X_i^I(t)$, $i = 1 \dots n$; and the overline indicates that the soft field $X_s^I(t)$ is contracted out of the string $\mathcal{V}_s^*(t_s) S_{sj}(u) \mathcal{V}_j^*(t_j)$. We will show how this computation yields a soft factorisation which reproduces (3.2.0.5). Note that the effective action (3.2.0.3) describes graviton interactions with no transfer of longitudinal momentum p^+. We will be able to use it to characterise soft emission/absorption because the longitudinal momentum transfer is negligible in the soft limit.

Before proceeding, it is helpful to introduce an auxiliary field. (For the moment, we focus on the leading interaction; in Section 3.5 we will extend the discussion to include the subleading contributions $\mathcal{L}_{ang}$ and $\mathcal{L}_{spin}$.) After a field redefinition, the Lagrangian has a canonically normalised kinetic term

$$\mathcal{L} = \frac{1}{2}(\dot{X}_s^I)^2 + \frac{1}{2}(\dot{X}_j^I)^2 - \frac{\lambda}{4}\left[\left(\sqrt{\frac{R}{N_s}}\dot{X}_s^I - \sqrt{\frac{R}{N_j}}\dot{X}_j^I\right)^2\right]^2,$$

where $\lambda = \frac{4AN_sN_j}{R^3 r^7}$. Then we introduce an auxiliary field $\sigma(t)$ and write the partition function as

$$Z = const \int \mathcal{D}X_s \mathcal{D}X_j \mathcal{D}\sigma \, e^{i\int dt \mathcal{L}'}\,,$$

where

$$\mathcal{L}' = \frac{1}{2}(\dot{X}_s^I)^2 + \frac{1}{2}(\dot{X}_j^I)^2 + \frac{\lambda}{4}\sigma^2 - \frac{\lambda}{2}\sigma\left(\sqrt{\frac{R}{N_s}}\dot{X}_s^I - \sqrt{\frac{R}{N_j}}\dot{X}_j^I\right)^2. \tag{3.3.1.3}$$

The new interaction term is more convenient to manipulate.

Standard graph combinatorial techniques show that the superficial degree of divergence of diagrams scales in the same way as powers of the coupling constant. This suggests

that the expansion parameter for the perturbative computation is $\frac{\lambda}{\epsilon}$, where ϵ is a short-distance cutoff which must be sent to zero at the end of the computation. To implement this cutoff, we introduce a kinetic term for σ

$$\mathcal{L}_{\sigma}^{kin} = \frac{\epsilon^2 \lambda}{4} \dot{\sigma}^2 \,. \tag{3.3.1.4}$$

Note that in DLCQ, the p^+ component of the momentum is kept fixed in the large-N limit. As a consequence, the light-like radius scales like $R \sim N$ and

$$\lambda = \frac{4 A N_s N_j}{R^3 r^7} \tag{3.3.1.5}$$

scales like $\sim \frac{1}{N^2}$ in the soft limit where $N_j \sim N$ and $N_s \sim O(1)$. In fact, in the homogeneous M-Theory region, cf. Chapter 2, we have $N^{\frac{1}{9}} < r < N^{\frac{1}{7}}$, and λ decays faster than $\frac{1}{N^2}$. Note that λ depends on time through the $\frac{1}{r^7}$ factor; for simplicity of notation, we will only write down this dependence explicitly when relevant. The Lagrangian of the theory now takes the form

$$\begin{aligned} \mathcal{L}_{ren} &= \mathcal{L}' + \mathcal{L}_{\sigma}^{kin} \\ &= \frac{1}{2}(\dot{X}_s^I)^2 + \frac{1}{2}(\dot{X}_j^I)^2 + \frac{\epsilon^2 \lambda}{4}\dot{\sigma}^2 + \frac{\lambda}{4}\sigma^2 - \frac{\lambda}{2}\sigma\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^I - \sqrt{\frac{R}{N_j}}\dot{X}_j^I\Big)^2 ; \end{aligned} \tag{3.3.1.6}$$

note that, after introducing the ϵ-regulator through the kinetic term (3.3.1.4), the theory is UV finite and does not require counterterms. This is a special feature of this one-dimensional quantum field theory. We provide a proof of this statement in Appendix A. Note that we can remove the cutoff after taking the $N \to \infty$ limit. Indeed, the scattering amplitude still takes the form (3.3.1.2), but now the interaction term in the Dyson series amounts to

$$\mathcal{L}_{sj} = -\frac{\lambda}{2}\sigma\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^I - \sqrt{\frac{R}{N_j}}\dot{X}_j^I\Big)^2 . \tag{3.3.1.7}$$

In the following, we will need expressions for the contractions between elementary operators, which can be derived from (3.3.1.6). Those will be used after we explicitly expand vertex-like operators appearing in (3.3.0.2) in terms of $X_j(t)$ fields. First, a contraction between position operators produces a two-point function

$$\overbrace{X_i^I(t_1) X_j^J}(t_2) = \langle T[X_i^I(t_1) X_j^J(t_2)] \rangle = -\frac{i}{2}\delta_{ij}\delta^{IJ}|t_1 - t_2| \,, \tag{3.3.1.8}$$

as obtained by considering a principal value prescription for the Fourier transform. Consequently, the contraction between a position and a velocity yields

$$\overbrace{X_i^I(t_1)\dot{X}_j^J}(t_2) = \langle T[X_i^I(t_1)\dot{X}_j^J(t_2)]\rangle = \frac{d}{dt_2}\langle T[X_i^I(t_1)X_j^J(t_2)]\rangle = \frac{i}{2}\delta_{ij}\delta^{IJ}\mathrm{sgn}(t_1-t_2)\,,$$

and between two velocities gives

$$\dot{X}_i^I\overbrace{(t_1)\dot{X}_j^J}(t_2) = i\delta_{ij}\delta^{IJ}\delta(t_1-t_2)\,. \tag{3.3.1.9}$$

Using these relations, we obtain

$$\overbrace{e^{ip\cdot X_i(t_1)}X_j^I}(t_2) = \frac{p^I}{2}|t_1-t_2|\,e^{ip\cdot X_i(t_1)}\delta_{ij}\,,$$

and

$$\overbrace{e^{ip\cdot X_i(t_1)}\dot{X}_j^I}(t_2) = -\frac{p^I}{2}\,\mathrm{sgn}(t_1-t_2)\,e^{ip\cdot X_i(t_1)}\delta_{ij}\,. \tag{3.3.1.10}$$

Finally, the Wick contraction of two σ fields is given by

$$\overbrace{\sigma(t)\sigma}(t') = \langle\sigma(t)\sigma(t')\rangle = \frac{i}{\epsilon\sqrt{\lambda(t)\lambda(t')}}e^{-\frac{|t-t'|}{\epsilon}}\,. \tag{3.3.1.11}$$

Having outlined the general framework for our computation, we can move on to consider the explicit form of vertex-like operators.

3.3.2 Vertex-like operators

The operators whose correlation functions are considered in the following take the form

$$\mathcal{V}_j(t_j) = \mathcal{N}_j\,p_j^+ : A_{IJ}\dot{X}_j^I(t_j)\dot{X}_j^J(t_j)e^{iX_j(t_j)\cdot p_j}\cos\left(\beta_j\int_{-\infty}^{+\infty}du\,\dot{X}_j(u)\cdot p_j\right):, \tag{3.3.2.1}$$

where $a\cdot b \equiv a_I b^I$ and $\mathcal{N}_j$ is a normalisation. The operator (3.3.2.1) is a vertex-like operator which should describe a massless 11-dimensional particle in the asymptotic region. The cosine term, which deviates from the usual vertex-operator form, is required to reabsorb a divergence in the two-point function, as we will show shortly. This is due to the one-dimensional nature of the theory, which prevents us from having a standard vertex operator structure. Nonetheless, our goal in the rest of the paper will be to show that operators in the form (3.3.2.1) reproduce interesting results for target space scattering amplitudes. The transverse momentum is

$$p_j^I = \sqrt{2}\frac{N_j}{R}v_j^I\,,$$

which is uplifted to a light-like 11-dimensional vector by the dictionary (3.2.0.6), while

$$p_j^+ = \frac{N_j}{R}\,,$$

and p_j^- is the energy in the target space, such that the on-shell condition

$$p_j^- = \frac{N_j}{R}\vec{v}_j^2$$

is satisfied. The polarisation A_{IJ} is an $SO(9)$ tensor, and is, therefore, a representation of the little group of 11-dimensional Lorentz symmetry, thus describing a massless particle. We will be interested in the case of a graviton, for which the tensor has rank 2. The construction can, however, be extended to higher rank representations of $SO(9)$.

The structure of (3.3.2.1) is determined by the requirement of asymptotic translation and rotation invariance. The normal ordering of terms in the vertex operator will be implied from now on, even when not written explicitly. As a consequence, we will never generate Wick contractions between elements of the same vertex operator.

The soft vertex operators correspond to a soft graviton insertion

$$\mathcal{V}_s(t_s) = \mathcal{N}_s\, q_s^+\, h_{IJ} : \dot{X}_s^I(t_s)\dot{X}_s^J(t_s)e^{iq_s\cdot X_s(t_s)} \cos\left(\beta_s \int_{-\infty}^{+\infty} du\, \dot{X}_s(u)\cdot q_s\right) : \,. \qquad (3.3.2.2)$$

Here q_s is a soft momentum (i.e. the overall scale is given by $\frac{N_s}{R} \ll \frac{N_j}{R}$ in the large-N limit) and h_{IJ}, which is symmetric and traceless, is the graviton polarisation. We perform a field redefinition such that the kinetic terms in (3.2.0.3) are canonically normalised. Hence, the vertex operator becomes

$$\mathcal{V}_s(t_s) = \mathcal{N}_s\, h_{IJ} : \dot{X}_s^I(t_s)\dot{X}_s^J(t_s)e^{i\sqrt{\frac{2N_s}{R}}v_s\cdot X_s(t_s)} \cos\left(\beta_s\sqrt{\frac{2N_s}{R}} \int_{-\infty}^{+\infty} du\, \dot{X}_s(u)\cdot v_s\right) : \,. \qquad (3.3.2.3)$$

One of the central claims of the present paper is that operators in the form (3.3.2.3) are suitable to create an asymptotic graviton state. In order to check the statement, we compute the two-point function of vertex operators and check it against the overlap of graviton states. Consider the correlator between two graviton vertex operators in the form (3.3.2.3), with the same number of branes

$$\mathcal{A}(N_k, v_k, h; N_k, \tilde{v}_k, \tilde{h}) = \int d\mu_{t_1} d\mu_{t_2} \langle T[\mathcal{V}_k^*(t_1)\tilde{\mathcal{V}}_k(t_2)]\rangle\,.$$

Defining

$$T = t_1 + t_2, \quad t_k = t_2 - t_1$$

we obtain

$$\mathcal{A}(N_k, v_k, h; N_k, \tilde{v}_k, \tilde{h}) = \int d\mu_T \, d\mu_{t_k} \langle T[\mathcal{V}_k^*(0)\mathcal{V}_k(t_k)]\rangle$$

which is the statement that the expectation value only depends on time differences. Analogously, we can make the overall zero mode of $X_k(t)$ fields explicit, which corresponds to a constant shift

$$X_k^I(t) = \tilde{X}_k^I(t) + \sqrt{\frac{N_k}{R}} x_0^I \,,$$

where the rescaling of the zero mode is a consequence of the field redefinition. The path integral over this mode then gives conservation of momentum in the target space

$$\int d^9 x_0^I \, e^{-i x_0^I (p_k^I - \tilde{p}_k^I)} = (2\pi)^9 \delta(\vec{p}_k - \tilde{\vec{p}}_k) \,.$$

Now, the integral over T can be thought of as an integral over a constant shift in time, and reads

$$\int d\mu_T = \int d\mu_T \, e^{-iT(p_k^- - \tilde{p}_k^-)} = 2\pi\delta(p_k^- - \tilde{p}_k^-) \,,$$

where the first equality is trivially true since we are working on-shell; hence the delta is really a $\delta(0)$, which is what we expect for the energy conservation condition for on-shell eleven-dimensional momenta in the form (3.2.0.6). Hence, the remaining part of the Green's function reads

$$\mathcal{A}(N_k, v_k, h; N_k, v_k, \tilde{h}) = \int d\mu_{t_k} \langle T[\mathcal{V}_k^*(0)\mathcal{V}_k(t_k)]\rangle \,.$$

To compute this correlation function, we must consider all possible Wick contractions. There are three options, depending on how the operators in the prefactor are contracted.

- **Option 1:** All prefactors are contracted together

$$\mathcal{A}_1(N_k, v_k, h; N_k, v_k, \tilde{h}) = 2|\mathcal{N}_k|^2 \int d\mu_{t_k} h_{IJ} \tilde{h}_{KL} \langle \dot{X}_k^I(0) \dot{X}_k^K(t_k)\rangle \langle \dot{X}_k^J(0) \dot{X}_k^L(t_k)\rangle$$
$$\langle \cos\left(\beta_k \sqrt{\frac{2N_k}{R}} \int_{-\infty}^{+\infty} du \, \dot{X}_k(u) \cdot v_k\right) \cos\left(\beta_k \sqrt{\frac{2N_k}{R}} \int_{-\infty}^{+\infty} du \, \dot{X}_k(u) \cdot v_k\right)\rangle$$
$$\langle e^{-i\sqrt{\frac{2N_k}{R}} v_k \cdot X_k(0)} e^{i\sqrt{\frac{2N_k}{R}} v_k \cdot X_k(t_k)}\rangle$$

- **Option 2:** One prefactor for each operator is contracted with the exponential

$$\mathcal{A}_2(N_k, v_k, h; N_k, v_k, \tilde{h}) = 4|\mathcal{N}_k|^2 \int d\mu_{t_k} h_{IJ}\tilde{h}_{KL}\langle \dot{X}_k^I(0)\dot{X}_k^K(t_k)\rangle$$
$$\langle \cos\left(\beta_k\sqrt{\frac{2N_k}{R}}\int_{-\infty}^{+\infty} du\, \dot{X}_k(u)\cdot v_k\right)\cos\left(\beta_k\sqrt{\frac{2N_k}{R}}\int_{-\infty}^{+\infty} du\, \dot{X}_k(u)\cdot v_k\right)\rangle$$
$$\langle e^{-i\sqrt{\frac{2N_k}{R}}v_k\cdot X_k(0)}\dot{X}_k^L(t_k)\rangle\langle \dot{X}_k^J(0)e^{i\sqrt{\frac{2N_k}{R}}v_k\cdot X_k(t_k)}\rangle$$

- **Option 3:** All prefactors are contracted with exponentials

$$\mathcal{A}_3(N_k, v_k, h; N_k, v_k, \tilde{h}) = |\mathcal{N}_k|^2 \int d\mu_{t_k} h_{IJ}\tilde{h}_{KL}$$
$$\langle \cos\left(\beta_k\sqrt{\frac{2N_k}{R}}\int_{-\infty}^{+\infty} du\, \dot{X}_k(u)\cdot v_k\right)\cos\left(\beta_k\sqrt{\frac{2N_k}{R}}\int_{-\infty}^{+\infty} du\, \dot{X}_k(u)\cdot v_k\right)\rangle$$
$$\langle e^{-i\sqrt{\frac{2N_k}{R}}v_k\cdot X_k(0)}\dot{X}_k^K(t_k)\dot{X}_k^L(t_k)\rangle\langle \dot{X}_k^I(0)\dot{X}_k^J(0)e^{i\sqrt{\frac{2N_k}{R}}v_k\cdot X_k(t_k)}\rangle\,.$$

After the prefactors are contracted, the final step is to contract the remaining exponentials, and the total amplitude is then the sum of the three contributions. The contraction between cosines reads

$$\langle \cos\left(\beta_k\sqrt{\frac{2N_k}{R}}\int_{-\infty}^{+\infty} du\, \dot{X}_k(u)\cdot v_k\right)\cos\left(\beta_k\sqrt{\frac{2N_k}{R}}\int_{-\infty}^{+\infty} dw\, \dot{X}_k(w)\cdot v_k\right)\rangle$$
$$= \cos\left(\frac{\beta_k^2}{\epsilon}\frac{N_k}{R}v_k^2\right)$$

where we regularised a divergent delta function $\delta(0)$ as the $\epsilon \to 0$ limit of $\frac{1}{2\epsilon}$, cf. equation (3.3.1.11). The first contribution is given by

$$\mathcal{A}_1(N_k, v_k, h; N_k, v_k, \tilde{h}) = -\frac{|\mathcal{N}_k|^2}{\epsilon}h_{IJ}\tilde{h}^{IJ}\cos\left(\frac{\beta_k^2}{\epsilon}\frac{N_k}{R}v_k^2\right). \tag{3.3.2.4}$$

As mentioned above, the role of the cosine factor in the vertex operator is to cancel the divergence that arises due to the one-dimensional, ultra-local nature of the theory. Hence, we choose

$$\beta_k = \sqrt{\frac{R}{v_k^2 N_k}\epsilon\left(\frac{\pi}{2} + c_k\epsilon\right)} \tag{3.3.2.5}$$

to cancel the divergence, where c_k is a constant, which we will specify shortly. It follows that

$$\mathcal{A}_1(N_k, v_k, h; N_k, v_k, \tilde{h}) = |\mathcal{N}_k|^2 c_k h_{IJ}\tilde{h}^{IJ}\,. \tag{3.3.2.6}$$

Note that, in principle, we should have listed options where either the prefactor or the exponential in vertex-like operators is contracted with the cosine factor. However, options where the exponent is contracted with the prefactor are in the form

$$\langle \int_{-\infty}^{+\infty} dudv\, \dot{X}^I(u)\dot{X}^J(v)X^K(t_k)X^L(t_k)\rangle = \int_{-\infty}^{+\infty} dudv\, \mathrm{sgn}(u-t_k)\mathrm{sgn}(v-t_k) = 0\,.$$

Moreover, when velocity operators are contracted with the cosine, e.g. as an alternative to contracting prefactors with the exponent in option 3, the result is

$$\langle \beta_k^2 \frac{2N_k}{R} v_k^I v_k^J \int_{-\infty}^{+\infty} dudv\, \dot{X}^I(u)\dot{X}^J(v)\dot{X}^K(t_k)\dot{X}^L(t_k)\rangle \propto \epsilon$$

and vanishes in the $\epsilon \to 0$ limit. Hence, the only non-vanishing contraction of the cosine terms is among themselves, thus removing the divergence and leaving the rest of the amplitude unaffected.

The remaining two options for Wick's contractions read

$$\mathcal{A}_2(N_k, v_k, h; N_k, v_k, \tilde{h}) = -i|\mathcal{N}_k|^2 c_k \epsilon \frac{2N_k}{R} h_{IJ} v_k^J \tilde{h}^{IL} v_{k,L} \tag{3.3.2.7}$$

and

$$\mathcal{A}_3(N_k, v_k, h; N_k, v_k, \tilde{h}) = -\frac{i}{2}|\mathcal{N}_k|^2 c_k \epsilon \frac{N_k}{R} \frac{h_{IJ} v_k^I v_k^J \tilde{h}_{KL} v_k^K v_k^L}{v_k^2}\,, \tag{3.3.2.8}$$

which vanish in the $\epsilon \to 0$ limit. It follows that only the first amplitude (3.3.2.6) contributes. Moreover, choosing

$$c_k = \frac{1}{|\mathcal{N}_k|^2} \tag{3.3.2.9}$$

fixes the overall normalisation, and the final result for the amplitude reads

$$\mathcal{A}(N_k, v_k, h; N_k, v_k, \tilde{h}) = \langle N_k, v_k, h | N_k, v_k, \tilde{h}\rangle = h_{IJ}\tilde{h}^{IJ}\,, \tag{3.3.2.10}$$

which reproduces the overlap between graviton states.

We have thus shown that the two-point function of vertex-like operators reproduces the correct graviton amplitude in the target space. Indeed, (3.3.2.10) implies that, if we attach an external line of a particle j to an amplitude in the form (3.3.0.1), we obtain, removing delta functions for momentum and energy conservation

$$\int d\mu_{t_{ext}} \int d\mu_{\tilde{t}_j} \langle \mathcal{V}_j(t_{ext})\tilde{\mathcal{V}}_{j,\alpha\beta}(\tilde{t}_j)\rangle \tilde{\mathcal{A}}^{\alpha\beta}(N_1, v_1^I, \theta_1; \ldots; N_j, v_j^I; \ldots; N_n, v_n^I, \theta_n)$$
$$= \theta_{\alpha\beta}\tilde{\mathcal{A}}^{\alpha\beta}(N_1, v_1^I, \theta_1; \ldots; N_j, v_j^I; \ldots; N_n, v_n^I, \theta_n)\,,$$

where $\tilde{\mathcal{A}}^{\alpha\beta}(N_1, v_1^I, \theta_1; \ldots; N_j, v_j^I; \ldots; N_n, v_n^I, \theta_n)$ is the tensor obtained by considering a generic scattering amplitude between vertex-like operators, with the external graviton polarisation removed. Similarly, the notation $\tilde{\mathcal{V}}_{j,\alpha\beta}(\tilde{t}_j)$ stands for a graviton operator which is not contracted to an external polarisation, such that the two-point function

$$\int d\mu_{t_{ext}} \int d\mu_{\tilde{t}_j} \langle \mathcal{V}_j(t_{ext}) \tilde{\mathcal{V}}_{j,\alpha\beta}(\tilde{t}_j) \rangle$$

simply reproduces the external graviton polarisation $\theta_{\alpha\beta}$.

We are now ready to use this construction of vertex-like operators to show the soft factorisation property at leading and subleading orders.

3.4 The leading soft term

In this section, we will show how to reproduce the leading term of the soft factorisation (3.2.0.5) from the formalism outlined in Section 3.3. We will carry out the computation in detail, thus providing a proof for the soft theorem for $D0$-brane bound states in BFSS at leading order in the large-N expansion.

We consider the amplitude (3.3.1.2), where we make vertex operators explicit. As explained in Section 3.3.2, vertex operators (3.3.2.3) have a cosine term which allows us to remove a divergence in the graviton two-point function. However, we saw that they play no role in Wick's contractions, since they give rise to vanishing terms when $\epsilon \to 0$. Hence, we will just set them equal to 1 in the following, for ease of notation.

The amplitude (3.3.1.2) reads

$$\begin{aligned}
&\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n) \\
&\qquad = \mathcal{N}_s^* \mathcal{N}_j^* \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} \langle \Omega | h_{IJ} T[\dot{X}_s^I(t_s) \dot{X}_s^J(t_s) e^{-i\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s)} \\
&\times \sum_{N=1}^{\infty} \left[\frac{i^N}{N!} \Big(\int du\, \mathcal{L}_{sj}(u) \Big)^N \right] A_{I_1 I_2} \dot{X}_j^{I_1}(t_j) \dot{X}_j^{I_2}(t_j) e^{-i\sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)} \prod_{k \neq j} \mathcal{V}_k(t_k)] | \Omega \rangle \,, \quad (3.4.0.1)
\end{aligned}$$

where, for simplicity, we assumed the hard particle to be also a graviton with polarisation $A_{I_1 I_2}$, and it is understood that in $\prod_{k \neq j} \mathcal{V}_k(t_k)$ operators corresponding to outgoing particles are complex conjugated. As explained in Section 3.3.2, the integral over zero modes of fields yields a delta function for total momentum conservation. We need to evaluate the various options for Wick's contractions involving the soft vertex operator, the

j-th hard vertex operators and interaction terms in (3.4.0.1). We can make a distinction between contractions among interaction terms in $\prod_i \mathcal{L}_{sj}(u_i)$, which we will name internal contractions; and contractions between operators in $\prod_i \mathcal{L}_{sj}(u_i)$ and terms in the external vertex operators, which we will name external contractions. It will be useful to first work out the result of internal contractions.

First we rewrite (3.4.0.1) as

$$\begin{aligned}
&\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n) \\
&\qquad = \mathcal{N}_s^* \mathcal{N}_j^* \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} \langle \Omega | h_{IJ} T[\dot{X}_s^I(t_s) \dot{X}_s^J(t_s) e^{-i\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s)} \\
&\qquad\quad \times \sum_{N=1}^{\infty} \left[\frac{i^N}{N!} \sum_{k=1}^{N} \sum_{n_1+\ldots+n_k=N} \frac{N!}{k! n_1! \ldots n_k!} \prod_{i=1}^{k} \Big(\int du\, \mathcal{L}_{sj}(u) \Big)^{n_i} \right] \\
&\qquad\qquad\qquad \times A_{I_1 I_2} \dot{X}_j^{I_1}(t_j) \dot{X}_j^{I_2}(t_j) e^{-i\sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)} \prod_{k \neq j} \mathcal{V}_k(t_k)] | \Omega \rangle \,.
\end{aligned}$$

Next, we rearrange the sums as

$$\sum_{N=1}^{\infty} \sum_{k=1}^{N} \sum_{n_1+\ldots+n_k=N} = \sum_{k=1}^{\infty} \sum_{N=k}^{\infty} \sum_{n_1+\ldots+n_k=N} = \sum_{k=1}^{\infty} \prod_{i=1}^{k} \sum_{n_i=1}^{\infty} ,$$

thus obtaining

$$\begin{aligned}
&\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n) \\
&\qquad = \mathcal{N}_s^* \mathcal{N}_j^* \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} \langle \Omega | h_{IJ} T[\dot{X}_s^I(t_s) \dot{X}_s^J(t_s) e^{-i\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s)} \\
&\qquad\quad \times \left[\sum_{k=1}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} \Big(i \int du\, \mathcal{L}_{sj}(u) \Big)^{n_i} \right] \\
&\qquad\qquad \times A_{I_1 I_2} \dot{X}_j^{I_1}(t_j) \dot{X}_j^{I_2}(t_j) e^{-i\sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)} \prod_{k \neq j} \mathcal{V}_k(t_k)] | \Omega \rangle \,. \quad (3.4.0.2)
\end{aligned}$$

Let us define a chain of length n as

$$C_n = \Big(-\frac{i}{2} \Big)^n \langle \prod_{i=1}^{n} \int du_i \, \big(\lambda(u_i) \sigma(u_i) \big) \Big(\sqrt{\frac{R}{N_s}} \dot{X}_s^I(u_i) - \sqrt{\frac{R}{N_j}} \dot{X}_j^I(u_i) \Big)^2 \rangle_\omega$$

where the angle brackets $\langle \rangle_w$ mean that all $\dot{X}_j$ operators except two are contracted among each other, in such a way that the contraction only involves operators evaluated at different times. Soft operators do not take part in internal contractions, since we are only considering the case in which the soft particle appears in external lines (soft emission or absorption). The two operators that remain uncontracted are called the

endpoints of the chain and will take part in external contractions. Performing the internal contraction of the $\dot{X}_j$ operators yields

$$C_n = -in!\frac{N_j}{R}\Big(-\frac{i}{2}\Big)^n\Big(\frac{2iR}{N_j}\Big)^n\int du\,\lambda^n(u)\sigma^n(u)\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^I(u)-\sqrt{\frac{R}{N_j}}\dot{X}_j^I(u)\Big)^2\,. \quad (3.4.0.3)$$

Now define

$$F[\sigma(u)] = \sum_{n=1}^{\infty}\Big(\frac{\sigma(u)\lambda(u)R}{N_j}\Big)^n = \frac{\sigma(u)\lambda(u)\frac{R}{N_j}}{1-\sigma(u)\lambda(u)\frac{R}{N_j}}\,, \quad (3.4.0.4)$$

so that

$$\sum_{n=1}^{\infty}\frac{1}{n!}C_n = -i\frac{N_j}{R}\int du\,F[\sigma(u)]\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^I(u)-\sqrt{\frac{R}{N_j}}\dot{X}_j^I(u)\Big)^2\,,$$

and for convenience, introduce

$$O_{res} := \left[\sum_{k=1}^{\infty}\frac{1}{k!}\prod_{i=1}^{k}\sum_{n_i=1}^{\infty}\frac{1}{n_i!}C_{n_i}\right]\,. \quad (3.4.0.5)$$

Then (3.4.0.2) becomes

$$\begin{aligned}\mathcal{A}(N_s,v_s^I,h_s;N_1,v_1^I,\theta_1;\ldots;N_n,v_n^I,\theta_n)\\ &= \mathcal{N}_s^*\mathcal{N}_j^*\int d\mu_{t_s}\prod_{i=1}^{n}\int d\mu_{t_i}\langle\Omega|h_{IJ}T[\dot{X}_s^I(t_s)\dot{X}_s^J(t_s)e^{-i\sqrt{\frac{2N_s}{R}}v_s\cdot X_s(t_s)}\\ &\quad\times O_{res}A_{I_1I_2}\dot{X}_j^{I_1}(t_j)\dot{X}_j^{I_2}(t_j)e^{-i\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)}\prod_{k\neq j}\mathcal{V}_k(t_k)]|\Omega\rangle\,. \quad (3.4.0.6)\end{aligned}$$

The next step is to work out external contractions involving the endpoints of the chains. In order to provide a non-zero contribution to leading (or first sub-leading) order in N, external contractions must satisfy:

A) Every remaining velocity operator in the chains, and all auxiliary fields σ, must be contracted, because otherwise the term will be killed by normal ordering;

B) The soft operators $\dot{X}_s(t_s)$ must be contracted out, each with a different chain. This is because, by (3.3.1.9), contracting the two soft operators in (3.4.0.6) with the *same* chain gives a term proportional to $h_{IJ}\delta^{IJ}$ which vanishes because the graviton polarisation is traceless; it follows that the $k=1$ term in (3.4.0.5) does not contribute. For the same reason the two hard velocity operators, $\dot{X}_j(t_j)$, cannot be contracted with the same chain;

C) At least one end of each chain must be contracted with an exponential operator. This is because terms in which $\dot{X}_s(t_s)$ is contracted with one end of a chain and $\dot{X}_j(t_j)$ with the other end lead to a factor $\delta(t_s - t_j)$ which kills the integral over t_s. As we will see in the explicit computation below, this implies that the amplitude is proportional to $\lambda(t_j) \sim \frac{1}{N^2}$, which is lower order with respect to subleading in N. Hence, we neglect these terms.

We can write down a Feynman rule for the external edges of a diagram corresponding to contractions with an external velocity:

$$\begin{aligned}\langle : \dot{X}_l^I(t) e^{-i\left(\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s) + \sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)\right)} : \sum_{n=1}^{\infty} \frac{1}{n!} C_n \rangle \\ = -\frac{N_j}{\sqrt{2}R} \sqrt{\frac{R}{N_l}} v_{sj}^I F[\sigma(t)] e^{-i\left(\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s) + \sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)\right)} \sum_{n=1}^{\infty} \frac{1}{n!} C_n \quad (3.4.0.7)\end{aligned}$$

where $l = s, j$ depending on whether the external velocity is soft or hard, and we defined $v_{sj}^I = (v_s^I - v_j^I)$. In particular, this implies an $F[\sigma]$ power counting for external contractions, and we will show in the following how this translates into a power counting in N.

There are three possibilities for the external contractions, and we specify for each of them the power of $F[\sigma]$ resulting from contractions with external velocities:

1. No operator from the hard vertex operator is contracted,

$$\begin{aligned}&\mathcal{A}^{(1)}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n) \\ &= \mathcal{N}_s^* \mathcal{N}_j^* \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} h_{IJ} \langle \Omega | T[\underline{\dot{X}_s^I(t_s) \dot{X}_s^J(t_s) e^{-i\left(\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s) + \sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)\right)}} \\ &\qquad \times \underline{O_{res}} A_{I_1 I_2} \dot{X}_j^{I_1}(t_j) \dot{X}_j^{I_2}(t_j) \prod_{k \neq j} \mathcal{V}_k(t_k)] | \Omega \rangle \,, \quad (3.4.0.8)\end{aligned}$$

   where the underline indicates Wick contractions. The power counting yields $F^2[\sigma(t_s)]$;

2. One of the operators from the hard vertex operator is contracted,

$$\begin{aligned}&\mathcal{A}^{(2)}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n) \\ &= \mathcal{N}_s^* \mathcal{N}_j^* \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} h_{IJ} \langle \Omega | T[\underline{\dot{X}_s^I(t_s) \dot{X}_s^J(t_s) e^{-i\left(\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s) + \sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)\right)}} \\ &\qquad \times \underline{O_{res}} A_{I_1 I_2} \underline{\dot{X}_j^{I_1}(t_j)} \dot{X}_j^{I_2}(t_j) \prod_{k \neq j} \mathcal{V}_k(t_k)] | \Omega \rangle \,. \quad (3.4.0.9)\end{aligned}$$

   The power counting yields $F^2[\sigma(t_s)] F[\sigma(t_j)]$;

3. Both operators from the hard vertex operator are contracted,

$$\begin{aligned}&\mathcal{A}^{(3)}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n)\\ &= \mathcal{N}_s^* \mathcal{N}_j^* \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} h_{IJ} \langle \Omega | T[\underline{\dot{X}_s^I(t_s) \dot{X}_s^J(t_s) e^{-i\left(\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s) + \sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)\right)}}\\ &\qquad \times \underline{O_{res}} A_{I_1 I_2} \underline{\dot{X}_j^{I_1}(t_j) \dot{X}_j^{I_2}(t_j)} \prod_{k \neq j} \mathcal{V}_k(t_k)] | \Omega \rangle \,. \quad (3.4.0.10)\end{aligned}$$

The power counting yields $F^2[\sigma(t_s)]F^2[\sigma(t_j)]$.

Note that we only highlighted the $F[\sigma]$ power counting stemming from contractions with external operators. There are $F[\sigma]$ factors coming from contractions of chains in O_{res} with the exponential operator. Those are the same for all three contributions and will be worked out explicitly in the following. We will compute the case (3.4.0.8) explicitly; the others are similar, and we will point out how the N power counting arises from the corresponding $F[\sigma]$ power counting as we go through the steps of the computation.

Firstly, the $k = 0$ term in (3.4.0.5) does not contribute because of rule C and the $k = 1$ term vanishes due to rule B. This implies that non-zero contributions are those for which the sum over k in (3.4.0.5) starts from two, namely

$$O_{res} = \left[\sum_{k=2}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} C_{n_i} \right] .$$

We select two chains to be contracted with the soft vertex operator. There are $\frac{k(k-1)}{2}$ equivalent options for this choice. We then contract them with external soft operators. After this operation, we obtain

$$\begin{aligned}&\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n)\\ &\qquad = \mathcal{N}_s^* \mathcal{N}_j^* \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} \langle \Omega | h_{IJ} T[v_{sj}^I v_{sj}^J e^{-i\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s)} \left(\frac{N_j}{R}\right)^2 \left(\frac{R}{N_s}\right)\\ &\times F^2[\sigma(t_s)] \left[\sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} C_{n_i} \right] A_{I_1 I_2} \dot{X}_j^{I_1}(t_j) \dot{X}_j^{I_2}(t_j) e^{-i\sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)} \prod_{k \neq j} \mathcal{V}_k(t_k)] | \Omega \rangle \,. \quad (3.4.0.11)\end{aligned}$$

Note that in the intermediate steps of the previous computation, factors of $\delta(u_{1,2} - t_s)$ appear due to (3.3.1.9). Then equation (3.3.1.1) tells us that, upon integration over $u_{1,2}$, this is non-vanishing only for $t_j \geq t_s$ (it is the other way around for two incoming particles, whereas for one *incoming* and one *outgoing* particle indeed the *outgoing* one comes later).

The Feynman rule (3.4.0.7) has been applied in order to obtain contractions of chains with external operators, and the factor $F^2[\sigma(t_s)]$ has been obtained, as anticipated. The analogous application of the Feynman rule (3.4.0.7) to the terms (3.4.0.9) and (3.4.0.10) yields the respective $F[\sigma]$ powers described above.

The next step consists of working out the σ contractions. A contraction of σ fields, in the $\epsilon \to 0$ limit, reads

$$\langle\sigma(u_1)\sigma(u_2)\rangle = \frac{i}{\epsilon\sqrt{\lambda(u_1)\lambda(u_2)}}e^{-\frac{|u_1-u_2|}{\epsilon}} \to \frac{2i}{\lambda(u_1)}\delta(u_1-u_2)\,. \tag{3.4.0.12}$$

in particular, the propagator between two σ's is proportional to $\frac{1}{\lambda}$. Moreover, we recall the definition (3.4.0.4) of $F[\sigma(t)]$ operators, which implies that every σ comes in the combination $\sigma(t)\lambda$. This, in turn, implies that for any pair of contracted σ's, we get a factor of $\lambda \sim O(\frac{1}{N^2})$. For example, we can compute

$$\langle F[\sigma(t_s)]\rangle = \frac{i}{\epsilon}\Big(\frac{R}{N_j}\Big)^2\lambda(t_s)\Big(1+O\big(\lambda(t_s)\big)\Big) = \frac{i}{\epsilon}\Big(\frac{R}{N_j}\Big)^2\lambda(t_s)\Big(1+O(\frac{1}{N^2})\Big) \tag{3.4.0.13}$$

and

$$\langle F[\sigma(t_s)]F[\sigma(t_s)]\rangle = \frac{i}{\epsilon}\Big(\frac{R}{N_j}\Big)^2\lambda(t_s)\Big(1+O(\frac{1}{N^2})\Big)\,. \tag{3.4.0.14}$$

We can work out the analogous contraction for $F[\sigma]$ powers in the terms (3.4.0.9) and (3.4.0.10). These read

$$\langle F[\sigma(t_s)]F[\sigma(t_s)]F[\sigma(t_j)]\rangle = -2\Big(\frac{R}{N_j}\Big)^4\frac{\lambda(t_s)}{\epsilon}\frac{\lambda(t_j)}{\epsilon}\Big(1+O(\frac{1}{N^2})\Big) \tag{3.4.0.15}$$

and

$$\langle F[\sigma(t_s)]F[\sigma(t_s)]F[\sigma(t_j)]F[\sigma(t_j)]\rangle = -2\Big(\frac{R}{N_j}\Big)^4\frac{\lambda(t_s)}{\epsilon}\frac{\lambda(t_j)}{\epsilon}\Big(1+O(\frac{1}{N^2})\Big)\,. \tag{3.4.0.16}$$

In particular, it is clear from (3.4.0.15) and (3.4.0.16) that those contributions are $O(\frac{1}{N^2})$ with respect to (3.4.0.14). This implies that the external contractions (3.4.0.9) and (3.4.0.10) are suppressed with respect to (3.4.0.8) and do not contribute to the soft theorem. Note that this is true even though the amplitudes they multiply are different. Those amplitudes involve only hard particles and their contractions are proportional to powers of $\frac{N_j}{R}$, which is of order N^0. Hence, correlators differing by $\dot{X}_j$ factors do not change the polynomial scaling with N of the prefactors. This remains true even when more terms are added to the EFT expansion (3.2.0.3).

We plug (3.4.0.14) back into (3.4.0.11), keeping only the leading order in the $O(\frac{1}{N^2})$ expansion, obtaining

$$\begin{aligned}
&\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \dots; N_n, v_n^I, \theta_n) \\
&\qquad = i\mathcal{N}_s^*\mathcal{N}_j^* \int d\mu_{t_s} \prod_{i=1}^{n} \int d\mu_{t_i} \langle\Omega| h_{IJ} T[v_{sj}^I v_{sj}^J \, e^{-i\sqrt{\frac{2N_s}{R}} v_s \cdot X_s(t_s)} \, \frac{\lambda(t_s)}{\epsilon} \Big(\frac{R}{N_s}\Big) \\
&\times \left[\sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} C_{n_i}\right] A_{I_1 I_2} \dot{X}_j^{I_1}(t_j) \dot{X}_j^{I_2}(t_j) e^{-i\sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t_j)} \prod_{k\neq j} \mathcal{V}_k(t_k)] |\Omega\rangle \,. \quad (3.4.0.17)
\end{aligned}$$

The next step is to contract the remaining chains with the exponential operator. This results in

$$\left[\sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \sum_{n_i=1}^{\infty} \frac{1}{n_i!} C_{n_i}\right] \longrightarrow \left[\sum_{k=0}^{\infty} \frac{1}{k!} \prod_{i=1}^{k} \Big(-i\frac{N_j}{2R} v_{sj}^2\Big) \int_{-\infty}^{t_s} du_i \, F[(\sigma_{u_i})]\right] .$$

Next, the remaining σ's must be contracted among themselves. In order to do this, we use (3.4.0.13) and only keep the dominant contribution in the $O(\frac{1}{N^2})$ expansion. The expectation value we need to compute takes the form $\langle e^{\kappa \int_{-\infty}^{t_s} du_i \, F[(\sigma_{u_i})]}\rangle$. We want to show that

$$\langle e^{\kappa \int_{-\infty}^{t_s} du_i \, F[(\sigma_{u_i})]}\rangle = e^{\kappa \int_{-\infty}^{t_s} du_i \, \langle F[(\sigma_{u_i})]\rangle} (1 + O(\epsilon)) \,. \quad (3.4.0.18)$$

In order to do so, we consider a term at order n in the expansion of $e^{\kappa \int_{-\infty}^{t_s} du_i \, \langle F[(\sigma_{u_i})]\rangle}$

$$\begin{aligned}
&e^{\kappa \int_{-\infty}^{t_s} du_i \, \langle F[(\sigma_{u_i})]\rangle} \supset \frac{\kappa^n}{n!} \int du_1 \dots du_n \langle F[\sigma(u_1)]\rangle \dots \langle F[\sigma(u_n)]\rangle \\
&= \frac{\kappa^n}{n!} \int du_1 \dots du_n \sum_{\substack{m_1,\dots,m_n=0 \\ m=m_1+\dots m_n}}^{\infty} c_{m_1} \dots c_{m_n} \Big(\frac{R}{N_j}\Big)^{2(n+m)} \Big(\frac{\lambda(u_1)}{\epsilon}\Big)^{1+m_1} \dots \Big(\frac{\lambda(u_n)}{\epsilon}\Big)^{1+m_n}
\end{aligned}$$

where we expanded F contractions to all orders with combinatorial coefficients c_{m_i}. Now we consider a term at order $n+1$ in

$$\begin{aligned}
&\langle e^{\kappa \int_{-\infty}^{t_s} du_i \, F[(\sigma_{u_i})]}\rangle - e^{\kappa \int_{-\infty}^{t_s} du_i \, \langle F[(\sigma_{u_i})]\rangle} \\
&\qquad \supset \frac{\kappa^{n+1}}{(n+1)!} \int du_1 \dots du_{n+1} \, (n+1) \langle F[\sigma(u_1)]\rangle \dots \langle F[\sigma(u_n)] F[\sigma(u_{n+1})]\rangle
\end{aligned}$$

where we evaluate a contraction between two F functionals evaluated at different times. The combinatorial factor is $n+1$ because time integrations are ordered, and we can only contract adjacent functionals together. The contraction evaluates to

$$\langle F[\sigma(u_n)] F[\sigma(u_{n+1})]\rangle = 2\Big(\frac{R}{N_j}\Big)^2 \delta(u_n - u_{n+1}) \lambda(u_n) \sum_{m_n=0}^{\infty} \tilde{c}_{m_n} \Big(\frac{R}{N_j}\Big)^{2m_n} \Big(\frac{\lambda(u_n)}{\epsilon}\Big)^{m_n}$$

so one time integration drops out with the delta function. We can group the two contributions, leading to

$$\begin{aligned}\langle e^{\kappa\int_{-\infty}^{t_s} du_i\, F[(\sigma_{u_i})]}\rangle \\ \supset \frac{\kappa^n}{n!}\int du_1\dots du_n \sum_{m_1,\dots,m_n=0}^{\infty} c_{m_1}\dots c_{m_{n-1}}(c_{m_n}-2\kappa\epsilon\tilde{c}_{m_n}) \\ \times\left(\frac{R}{N_j}\right)^{2(n+m)}\left(\frac{\lambda(u_1)}{\epsilon}\right)^{1+m_1}\dots\left(\frac{\lambda(u_n)}{\epsilon}\right)^{1+m_n}\end{aligned}$$

and we see that the term coming from contractions between different F is $O(\epsilon)$ compared to the one coming from $e^{\kappa\int_{-\infty}^{t_s} du_i\,\langle F[(\sigma_{u_i})]\rangle}$. Indeed, it is a general fact that, whenever we consider a contraction between two different F, we generate a delta function which eliminates a time integral, and we pay the price of a power of ϵ. The argument is generic in n and can be repeated to all orders in the expansion of the exponential, proving the statement (3.4.0.18).

As a result, the amplitude reads

$$\begin{aligned}\mathcal{A}(N_s,v_s^I,h_s;N_1,v_1^I,\theta_1;\dots;N_n,v_n^I,\theta_n) \\ = i\mathcal{N}_s^*\mathcal{N}_j^*\int d\mu_{t_s}\prod_{i=1}^{n}\int d\mu_{t_i}\langle\Omega|h_{IJ}T[v_{sj}^I v_{sj}^J\frac{\lambda(t_s)}{\epsilon}\Big(\frac{R}{N_s}\Big) \\ \times e^{\frac{R}{2N_j}v_{sj}^2\int_{-\infty}^{t_s}du_i\frac{\lambda(u_i)}{\epsilon}}A_{I_1I_2}\dot{X}_j^{I_1}(t_j)\dot{X}_j^{I_2}(t_j)e^{-i\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)}\prod_{k\neq j}\mathcal{V}_k(t_k)]|\Omega\rangle\,, \quad (3.4.0.19)\end{aligned}$$

where, because the soft exponent is not involved in any additional contraction, we set it to 1 by normal ordering. We can now carry out the integration over t_s. The integral takes the form

$$\int d\mu_{t_s}\frac{\lambda(t_s)}{\epsilon}\,e^{\frac{R}{2N_j}v_{sj}^2\int_{-\infty}^{t_s}du_i\frac{\lambda(u_i)}{\epsilon}} = \lim_{\alpha\to 0}\int_{-\infty}^{t_j}dt_s\,\frac{\lambda(t_s)}{\epsilon}\,e^{\frac{R}{2N_j}v_{sj}^2\int_{-\infty}^{t_s}du_i\frac{\lambda(u_i)}{\epsilon}}e^{-\alpha|t_s|}\,.$$

We evaluate the integral under the change of variable

$$k(t_s)=\int_{-\infty}^{t_s}du_i\,\frac{\lambda(u_i)}{\epsilon}-\frac{N_j}{R}\frac{2\alpha}{v_{sj}^2}|t_s|$$

and neglecting a term that goes to zero with α. We obtain

$$\int d\mu_{t_s}\frac{\lambda(t_s)}{\epsilon}\,e^{\frac{R}{2N_j}v_{sj}^2\int_{-\infty}^{t_s}du_i\frac{\lambda(u_i)}{\epsilon}} = \int_{-\infty}^{k(t_j)}dk\,e^{\frac{R}{2N_j}v_{sj}^2\,k} = \frac{N_j}{R}\frac{2}{v_{sj}^2}\Big(1+O(\frac{1}{N^2})\Big)\,.$$

By plugging back into equation (3.4.0.19) we obtain

$$\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n) = 2i\mathcal{N}_s^*\mathcal{N}_j^*\left(\frac{N_j}{N_s}\right)h_{IJ}\frac{v_{sj}^I v_{sj}^J}{v_{sj}^2}$$
$$\prod_{i=1}^{n}\int d\mu_{t_i}\langle\Omega|T[A_{I_1I_2}\dot{X}_j^{I_1}(t_j)\dot{X}_j^{I_2}(t_j)e^{-i\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)}\prod_{k\neq j}\mathcal{V}_k(t_k)]|\Omega\rangle\,. \quad (3.4.0.20)$$

We have freedom in choosing the normalisation constant of the soft vertex operator to match the target space gravitational coupling, which amounts to fixing

$$\mathcal{N}_s = -i\sqrt{32\pi G_{11}}\,,$$

and we obtain

$$\mathcal{A}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n)$$
$$= -2\sqrt{32\pi G_{11}}\,h_{IJ}\sum_j \eta_s\eta_j\frac{N_j}{N_s}\frac{v_{sj}^I v_{sj}^J}{v_{sj}^2}\mathcal{A}(N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n)\,, \quad (3.4.0.21)$$

which is the statement of the leading soft theorem, as anticipated. Note that we restored the sum over all hard particles. The sign factors η_s and η_j appear for the different cases of *outgoing* and *incoming* s and j particles (recall that $\eta_i = \pm 1$ if the particle is outgoing or incoming, respectively). We can, therefore, identify the leading soft factor in (3.2.0.5) as

$$S^{(-1)} = -2\sqrt{32\pi G_{11}}\,h_{IJ}\sum_j \eta_s\eta_j\frac{N_j}{N_s}\frac{v_{sj}^I v_{sj}^J}{v_{sj}^2}\,. \quad (3.4.0.22)$$

The remaining combinations for external contractions do not modify this conclusion. The computation of the terms (3.4.0.9) and (3.4.0.10) proceeds analogously, and we have pointed out how these terms provide an $O(\frac{1}{N^2})$ contribution with respect to the leading order, which is subleading in the soft limit and is, therefore, negligible.

3.5 Subleading term

In this section, we derive the subleading term of the soft theorem, which scales as N^0 in the large-N limit. The leading term of the soft theorem is encoded in the first interaction term of the effective field theory (3.2.0.3); the subleading term requires the inclusion of order $\frac{1}{r^8}$ terms in the effective interaction Lagrangian. It was shown in [50] that these consist of an angular momentum part $\mathcal{L}_{ang}$ and a spin part $\mathcal{L}_{spin}$. The subleading soft factor $S^{(0)}$ in (3.2.0.5) receives contributions from both. We will analyse the two cases separately.

3.5.1 Orbital angular momentum

The orbital angular momentum term in (3.2.0.3) (after canonically normalising the kinetic term) takes the form

$$\mathcal{L}_{ang} = -\frac{7\lambda(u)}{4r^2(u)} r^I(u)\Big(\sqrt{\frac{R}{N_s}}X_s^I(u) - \sqrt{\frac{R}{N_j}}X_j^I(u)\Big)\Big[\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^J(u) - \sqrt{\frac{R}{N_j}}\dot{X}_j^J(u)\Big)^2\Big]^2 . \tag{3.5.1.1}$$

In the auxiliary field formalism, the effective two-particle Lagrangian at order $O\left(\frac{1}{r^8}\right)$ is

$$\begin{aligned}\mathcal{L} = {} & \frac{1}{2}(\dot{X}_s^I)^2 + \frac{1}{2}(\dot{X}_j^I)^2 + \frac{\epsilon^2\lambda}{4}\dot{\sigma}^2 + \frac{\lambda}{4}\Big(1 + \frac{7}{r^2}r^I\Big(\sqrt{\frac{R}{N_s}}X_s^I - \sqrt{\frac{R}{N_j}}X_j^I\Big)\Big)\sigma^2 \\ & - \frac{\lambda}{2}\Big(1 + \frac{7}{r^2}r^I\Big(\sqrt{\frac{R}{N_s}}X_s^I - \sqrt{\frac{R}{N_j}}X_j^I\Big)\Big)\sigma\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^I - \sqrt{\frac{R}{N_j}}\dot{X}_j^I\Big)^2 + \mathcal{L}_{spin}. \end{aligned} \tag{3.5.1.2}$$

Note that the σ^2 term also receives an additional contribution. However, at this order in the EFT expansion, when it is reduced to order σ by performing Wick's contraction with interaction term insertions, this simply reproduces the interaction term in (3.5.1.2) and does not generate new terms. As a consequence, the angular momentum interaction term picks up a factor 1/2.

If we recall that $\lambda \propto \frac{1}{r^7}$, then we can write

$$\frac{r^I}{r^9} = -\frac{1}{7}\frac{\partial}{\partial\sqrt{\frac{R}{N_s}}x_s^I(u)}\Big(\frac{1}{r^7}\Big) = -\frac{1}{7}\sqrt{\frac{N_s}{R}}\frac{\partial}{\partial x_s^I(u)}\Big(\frac{1}{r^7}\Big), \tag{3.5.1.3}$$

where $x_s^I(u)$ is the particle coordinate. Integrating this derivative by parts, we can throw away the total derivative term, since the final amplitude will not depend on x_s at this order of the soft expansion. Then (3.5.1.3) can be written as

$$\frac{r^I}{r^9} = \frac{1}{7}\sqrt{\frac{N_s}{R}}\Big(\frac{1}{r^7}\Big)\frac{\partial}{\partial x_s^I(u)} = \frac{1}{7}i\sqrt{\frac{N_s}{R}}\Big(\frac{1}{r^7}\Big)\dot{X}_s^I(u)$$

since, in the asymptotic region, the conjugate momentum to the position operator is the velocity operator. Therefore, the angular momentum interaction term in (3.5.1.2) becomes

$$\begin{aligned}\mathcal{L}_{ang}^{\sigma} = -\frac{i}{4}\lambda(u)\dot{X}_s^I(u)\sqrt{\frac{N_s}{R}}\Big(\sqrt{\frac{R}{N_s}}X_s^I(u) - \sqrt{\frac{R}{N_j}}X_j^I(u)\Big) \\ \times\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^J(u) - \sqrt{\frac{R}{N_j}}\dot{X}_j^J(u)\Big)^2\sigma(u)\,. \end{aligned} \tag{3.5.1.4}$$

We only consider the case in which (3.5.1.4) is inserted once in the amplitude, since multiple insertions lead to a contribution at higher order in the large-N expansion. The

structure of internal contractions is, then, analogous to what we discussed for the leading term. The resulting interaction operator is

$$O_{res}^{ang} = \frac{i}{2}\int du\, \dot{X}_s^I(u)\sqrt{\frac{N_s}{R}}\Big(\sqrt{\frac{R}{N_s}}X_s^I(u) - \sqrt{\frac{R}{N_j}}X_j^I(u)\Big)\sum_{n=1}^{\infty}\frac{1}{n!}\tilde{C}_n(u)\, O_{res} \qquad (3.5.1.5)$$

where $\tilde{C}_n(u)$ is the integrand of (3.4.0.3). Any chain operator has two ends, as explained in Section 3.4. We say that the chain operator C_n is contracted with a velocity operator when one end is contracted with that velocity and the other end is contracted with the exponent. Moreover, we say that C_n is contracted with the exponent when both ends are. The bulk of the computation is analogous to the leading term derivation. We will not, therefore, go through all the details again and will limit ourselves to an analysis of the important features of the proof for the subleading theorem.

The possible options for external contractions are (we will be schematic to avoid long formulas):

- One of the soft operators is contracted with the explicit soft velocity in (3.5.1.5), no hard operator is contracted:

$$\begin{aligned}&\mathcal{A}_{sub}^{(1)}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n)\\ &\sim h_{IJ}\langle \tilde{C}_n(u)\, O_{res}\dot{X}_s^J(t_s)e^{-i\left(\sqrt{\frac{2N_s}{R}}v_s\cdot X_s(t_s)+\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)\right)}\rangle\langle \dot{X}_s^I(t_s)\dot{X}_s^K(u)\rangle; \qquad (3.5.1.6)\end{aligned}$$

  this will contribute to the subleading soft theorem;

- No soft operator is contracted with the explicit soft velocity in (3.5.1.5), no hard operator is contracted:

$$\begin{aligned}&\mathcal{A}_{sub}^{(2)}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n)\\ &\sim h_{IJ}\langle \dot{X}_s^I(t_s)\dot{X}_s^J(t_s)\tilde{C}_n(u)\, O_{res}\rangle\langle e^{-i\left(\sqrt{\frac{2N_s}{R}}v_s\cdot X_s(t_s)+\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)\right)}\dot{X}_s^K(u)\rangle; \qquad (3.5.1.7)\end{aligned}$$

  this term will also contribute to the subleading soft theorem;

- One of the soft operators is contracted with index K velocity, and one of the hard operators is contracted:

$$\begin{aligned}&\mathcal{A}_{sub}^{(3)}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \ldots; N_n, v_n^I, \theta_n)\\ &\sim h_{IJ}\langle \dot{X}_s^I(t_s)\dot{X}_s^K(u)\rangle\langle \tilde{C}_n(u)\dot{X}_j^{I_1}(t_j)\rangle\langle \dot{X}_s^J(t_s)e^{-i\left(\sqrt{\frac{2N_s}{R}}v_s\cdot X_s(t_s)+\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)\right)}O_{res}\rangle\,. \qquad (3.5.1.8)\end{aligned}$$

This term gives a contribution of order $O(\frac{1}{N^2})$ with respect to the subleading order, so it does not contribute to the soft theorem;

- No soft operator is contracted with index K velocity, one hard operator is contracted:

$$\begin{aligned}&\mathcal{A}^{(4)}_{sub}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \dots; N_n, v_n^I, \theta_n)\\ &\sim h_{IJ}\langle \dot{X}_s^I(t_s)\tilde{C}_n(u)\dot{X}_j^{I_1}(t_j)\rangle\langle \dot{X}_s^J(t_s)O_{res}\rangle\langle e^{-i\left(\sqrt{\frac{2N_s}{R}}v_s\cdot X_s(t_s)+\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)\right)}\dot{X}_s^K(u)\rangle;\end{aligned} \tag{3.5.1.9}$$

this term does not contribute for the same reason as the previous;

- Both hard operators are contracted:

$$\begin{aligned}&\mathcal{A}^{(5)}_{sub}(N_s, v_s^I, h_s; N_1, v_1^I, \theta_1; \dots; N_n, v_n^I, \theta_n)\\ &\qquad\sim h_{IJ}\langle \dot{X}_s^I(t_s)\dot{X}_s^J(t_s)\tilde{C}_n(u)\, O_{res}\dot{X}_j^{I_1}(t_j)\dot{X}_j^{I_2}(t_j)\rangle\\ &\qquad\qquad\langle e^{-i\left(\sqrt{\frac{2N_s}{R}}v_s\cdot X_s(t_s)+\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)\right)}\dot{X}_s^K(u)\rangle;\end{aligned} \tag{3.5.1.10}$$

this term is also $O(\frac{1}{N^2})$ with respect to the subleading order, so it does not contribute to the soft theorem.

We see that the only relevant contributions come from (3.5.1.6) and (3.5.1.7). Note that we did not explicitly list any option in which the factor $\left(\sqrt{\frac{R}{N_s}}X_s^I(u) - \sqrt{\frac{R}{N_j}}X_j^I(u)\right)$ is contracted. The reason is that the contractions involving the soft operator $X_s^I(u)$ give a term which vanishes due to δ functions coming from other contractions. Moreover, contractions with prefactors which involve a single chain yield potentially divergent terms, which are set to zero by dimensional regularisation, as explained in Appendix A. As for the hard operator $X_j^I(u)$, after external contractions, this will be evaluated at t_s due to the $\delta(u - t_s)$ factor that comes out of propagators. Then it can be integrated by parts, yielding a factor proportional to $X_j^I(t_j)$ plus a term which vanishes at this order in the large-N expansion. We also note that $X_j^I(t_j)$ can be extracted from the hard exponent as

$$X_j^I(t_j) = i\sqrt{\frac{R}{2N_j}}\frac{\partial}{\partial v_j^I}e^{-i\sqrt{\frac{2N_j}{R}}v_j\cdot X_j(t_j)}. \tag{3.5.1.11}$$

All in all, considering the result of Wick's contractions for the amplitude in (3.5.1.6), proceeding through steps that are analogous to those for the leading term, the resulting

S-Matrix element reads (restoring η_s, η_j factors)

$$\mathcal{A}^{(1)}_{sub}(N_s, v^I_s, h_{IJ}; N_1, v^I_1, \theta_1; \dots; N_n, v^I_n, \theta_n) \\ = \frac{1}{4}\sqrt{32\pi G_{11}}\, h_{IJ} \sum_j \eta_s \eta_j \Big(v^I_{sj} \partial_{v^J_j} + v^J_{sj} \partial_{v^I_j} \Big) \mathcal{A}(N_1, v^I_1, \theta_1; \dots; N_n, v^I_n, \theta_n)\,. \quad (3.5.1.12)$$

The same considerations apply to the contribution (3.5.1.7). The corresponding matrix element is

$$\mathcal{A}^{(2)}_{sub}(N_s, v^I_s, h_{IJ}; N_1, v^I_1, \theta_1; \dots; N_n, v^I_n, \theta_n) \\ = -\sqrt{32\pi G_{11}}\, h_{IJ} \sum_j \eta_s \eta_j \frac{v^I_{sj} v^J_{sj}}{v^2_{sj}} v^K_s \partial_{v^K_j} \mathcal{A}(N_1, v^I_1, \theta_1; \dots; N_n, v^I_n, \theta_n) \\ = -\sqrt{32\pi G_{11}}\, h_{IJ} \sum_j \eta_s \eta_j \Big(\frac{v^I_{sj} v^J_{sj}}{v^2_{sj}} v^K_{sj} \partial_{v^K_j} + 2\frac{N_j}{R} \frac{v^I_{sj} v^J_{sj}}{v^2_{sj}} \partial_{\frac{N_j}{R}} \Big) \mathcal{A}(N_1, v^I_1, \theta_1; \dots; N_n, v^I_n, \theta_n)\,, \quad (3.5.1.13)$$

where we used

$$v_j \cdot \partial_{v_j} e^{-i\sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t)} = 2\frac{N_j}{R} \partial_{\frac{N_j}{R}} e^{-i\sqrt{\frac{2N_j}{R}} v_j \cdot X_j(t)}$$

in the second equality. Equations (3.5.1.12) and (3.5.1.13) represent the orbital momentum contribution to the soft factorisation.

3.5.2 Spin

After canonically normalising the kinetic term, the spin contribution to (3.2.0.3) at order $\frac{1}{r^8}$ is [50]

$$\mathcal{L}_{\text{spin}} = -\frac{R}{N_j} \frac{7\lambda(u)}{2r^2(u)} r^I(u) \Big(\sqrt{\frac{R}{N_s}} \dot{X}^K_s(u) - \sqrt{\frac{R}{N_j}} \dot{X}^K_j(u) \Big)^2 \\ \times S^{IJ}_j \Big(\sqrt{\frac{R}{N_s}} \dot{X}^J_s(u) - \sqrt{\frac{R}{N_j}} \dot{X}^J_j(u) \Big)\,. \quad (3.5.2.1)$$

Here S^{IJ}_j stands for the BFSS spin generator acting on the j-th particle, defined as

$$S^{IJ}_j = \frac{1}{8R} Tr(\psi^\alpha \Gamma^{[IJ]}_{\alpha\beta} \psi^\beta)\,,$$

where ψ are BFSS fermion fields and $\Gamma^{[IJ]}$ are Lorentz generators built from gamma matrices. We will not rephrase (3.5.2.1) with an auxiliary field. As in the case of the

orbital angular momentum, we can rewrite (3.5.2.1) to obtain

$$\mathcal{L}_{\text{spin}} = -i\frac{R}{2N_j}\lambda(u)\sqrt{\frac{N_s}{R}}\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^K(u) - \sqrt{\frac{R}{N_j}}\dot{X}_j^K(u)\Big)^2 \times S_j^{IJ}\dot{X}_s^I(u)\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^J(u) - \sqrt{\frac{R}{N_j}}\dot{X}_j^J(u)\Big) \quad (3.5.2.2)$$

by using the argument around (3.5.1.3). As usual, internal contractions dress the interaction term. Since we are not working in the auxiliary field formalism for this term, there is still a self-interacting contribution which produces a non-vacuum-vacuum diagram. It follows that, for this term, we have two possible options for internal contractions:

- Self-interacting contribution: first we contract two of the velocities at time u among themselves, and then we construct chains. This evaluates to

$$O_{res}^s = -i\int du\,\sqrt{\frac{N_s}{R}}\Big(\frac{R}{N_j}\Big)\frac{\lambda(u)}{\epsilon}\Big(1+F[\sigma(u)]\Big) S_j^{IJ}\dot{X}_s^I(u)\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^J(u) - \sqrt{\frac{R}{N_j}}\dot{X}_j^J(u)\Big)O_{res}\,;$$

- Non-self-interacting contribution: two chains are attached to the new interaction operator. This gives

$$O_{res}^{ns} = -i\frac{N_j}{R}\sqrt{\frac{N_s}{R}}S_j^{IJ}\int du\,\lambda(u)\Big(1+F[\sigma(u)]\Big)^2\dot{X}_s^I(u) \times\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^J(u) - \sqrt{\frac{R}{N_j}}\dot{X}_j^J(u)\Big)\Big(\sqrt{\frac{R}{N_s}}\dot{X}_s^K(u) - \sqrt{\frac{R}{N_j}}\dot{X}_j^K(u)\Big)^2 O_{res}\,.$$

Each of these options is associated with a unique non-vanishing external contraction:

- **Self-interacting contraction:** the interaction term involves O_{res}^s. This gives a contribution $O(\frac{1}{N^2})$ with respect to the subleading term;
- **Non-self-interacting contraction:** the interaction term involves O_{res}^{ns}. This term contributes to the subleading soft factor.

The contribution of other contractions vanishes, analogously to the case of the orbital angular momentum. Out of those, only the second contributes to the subleading order

of the soft expansion. It follows that the spin contribution to the soft theorem can be written as (restoring η_s, η_j factors)

$$\mathcal{A}_{spin}(N_s, v_s, h_s; N_1, v_1, \theta_1, \dots, N_n, v_n, \theta_n) \\ = 2i\sqrt{32\pi G_{11}} \sum_j \eta_s \eta_j \frac{h_{IJ} v_{sj}^I}{v_{sj}^2} S_j^{JK} v_j^K \mathcal{A}(N_1, v_1, \theta_1, \dots, N_n, v_n, \theta_n) \,,$$

which represents the spin contribution to the subleading soft factor.

We can collect the orbital angular momentum term and the spin term together and summarise the subleading contribution to the soft theorem. This reads

$$S^{(0)} = \frac{1}{4}\sqrt{32\pi G_{11}} \sum_{j=1}^{n} \eta_s \eta_j h_{IJ} \Big(\Big(v_{sj}^I \partial_{v_j^J} + v_{sj}^J \partial_{v_j^I} \Big) \\ - 4 \frac{v_{sj}^I v_{sj}^J}{v_{sj}^2} v_{sj}^K \partial_{v_j^K} - 8 \frac{N_j}{R} \frac{v_{sj}^I v_{sj}^J}{v_{sj}^2} \partial_{\frac{N_j}{R}} + 8i \frac{v_{sj}^I}{v_{sj}^2} S_j^{JK} v_j^K \Big) \quad (3.5.2.3)$$

such that

$$\mathcal{A}(N_s, v_s, h_s; N_1, v_1, \theta_1, \dots, N_n, v_n, \theta_n) \\ = \Big(S^{(-1)} + S^{(0)} + \dots \Big) \mathcal{A}(N_1, v_1, \theta_1, \dots, N_n, v_n, \theta_n) \,,$$

which reproduces the result in [103], corresponding to the statement of the leading and subleading soft theorem for the BFSS theory.

As a final remark, we note that, in the EFT approach adopted in this paper, we have truncated all higher-order terms in the $O(\frac{1}{r})$ expansion from the Lagrangian (3.2.0.3), as they give a suppressed contribution in the large-distance regime of our computation. These terms include the supersymmetric completions of the $O(\frac{\dot{X}^4}{r^7})$ term with a higher number of fermionic fields (see [135]), and terms of order $O(\frac{\dot{X}^6}{r^{14}})$ or higher. Interestingly, at fixed r, the latter is the only term with the same power of N as the subleading order of the soft expansion. At large distances, and in particular in the uniform M-Theory regime $N^{\frac{1}{9}} < r < N^{\frac{1}{7}}$, it decays with N faster than the subleading term, and hence it does not spoil the validity of the theorem. All the others will give contributions which are also suppressed by powers of $O(\frac{1}{N})$ after considering their contractions with external vertex operators.

3.6 Conclusion

In this chapter, we derived a soft theorem for the BFSS theory by considering scattering amplitudes in the effective theory describing two-particle interactions between $D0$-brane bound states. In particular, the objects whose amplitudes enjoy a soft factorisation are vertex-like operators with the correct quantum numbers to describe supergravitons in the target space. We showed that the lowest order in the large-distance expansion, corresponding to $\sim \frac{1}{r^7}$, encodes the leading factor of the soft theorem, whereas terms at order $\sim \frac{1}{r^8}$ give rise to the subleading factor. The calculation requires the introduction of a term in the Lagrangian which is only relevant in the UV, after which the theory is super-renormalisable in dimensional regularisation. The soft correction to the correlation function is expanded in powers of N, and the leading $O(N)$ and subleading $O(N^0)$ soft factors are obtained. The result has precisely the form required by the soft theorem in BFSS. The expressions we obtain are, in fact, consistent with the ones predicted in [103] (up to a matching of the overall constant multiplying the amplitude), which are obtained by applying the BFSS duality dictionary to translate from the soft graviton theorem in 11-dimensional supergravity. In light of the connection between soft theorem and asymptotic symmetries, we conclude that, in the regime of validity of the effective field theory picture, our analysis confirms the realisation of the full 11-dimensional Lorentz symmetry, along with the presence of an infinite-dimensional asymptotic symmetry group.

Open questions and further directions of study are discussed in Chapter 5.

4 Matrix Compactification

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4.1 Introduction

In this chapter, we consider deformations of IKKT and BFSS theories which result in the generation of a non-supersymmetric mass term; see Section 2.5.2 for the physical motivation. An interesting mechanism was proposed in [136], where compactification of IKKT on a six-torus is considered. Upon choosing anti-periodic boundary conditions for fermions, it is shown that supersymmetry is broken completely; the effective theory of zero winding modes for a large radius of compactification is purely bosonic and

contains a mass term. This explains the origin of the mass deformation. We will review this mechanism in Section 4.2.

Then, we will generalise this result in Section 4.3 by considering mixed boundary conditions for fermions, where some degrees of freedom are given periodic boundary conditions, and others are given anti-periodic ones. We will obtain an effective theory that both retains some fermionic degrees of freedom and generates a mass term.

We will adapt this procedure to BFSS theory in 4.4. We will show that the result is qualitatively analogous to the IKKT case, and that anti-periodic boundary conditions for fermions still yield an effective theory for winding zero modes which presents a mass term and no fermions.

We will consider, in Section 4.5, a class of mixed boundary conditions in BFSS theory and show how they result in a non-supersymmetric mass-deformed theory, where half the fermionic degrees of freedom are retained, packaged in an $SO(3)$ Weyl spinor.

We will leverage such a theory in Section 4.6, where we will explicitly construct a black hole solution from a fuzzy two-sphere embedded in three matrix directions representing the background spacetime. Fuzzy sphere solutions of Matrix Theories have been considered in different settings. In the context of BMN theory, they have been proposed as the holographic dual to supergravity on the supersymmetric plane-wave background [99, 137]. Furthermore, in BFSS theory, they have been studied in relation to black hole physics [138–140].

Quantum excitations of the fermionic sector will account for the black hole entropy. This will scale like $S \sim N^{\beta}$, $\beta \leq 1$, N being the matrix size, in agreement with previous studies of black holes in Matrix Theory [73, 141, 142]. The saturated value $S \sim N$ signals the Gregory–Laflamme instability [71, 72] that leads to the black hole/black string transition. Hence, this work constitutes an explicit top-down construction of a Schwarzschild black hole from compactifications of Matrix Theory, where the Bekenstein-Hawking entropy is obtained as the degeneracy of the quantum state of fermionic degrees of freedom. We will comment on the mass-to-radius and the entropy-to-radius relations obtained in this way, and show how their consistency requires tuning the scaling of the initial torus radius with N.

Lastly, we comment on our results in Section 4.7.

4.2 Review of IKKT compactification

In this section, we review the results of [136] on the compactification of IKKT Matrix Theory on a six-torus $\mathbb{T}^6$. For ease of comparison, we will adhere closely to the formalism and notation of [136]. However, for clarity and completeness, we also give an explicit construction in Appendix B.1.

We introduced the IKKT matrix integral in Section 2.4. It will be useful to Wick rotate to Euclidean signature through the replacement $A^0 \to iA^0$, $\Gamma^j \to i\Gamma^j$, $j = 1, \dots, 9$, so that the Euclidean action reads

$$S^E_{IKKT} = -\frac{1}{2g^2}\mathrm{Tr}\left[\frac{1}{2}[A^I, A^J]^2 + i\bar{\psi}\Gamma^I[A_I, \psi]\right] ; \tag{4.2.0.1}$$

here g is the Yang-Mills coupling, A^I, $I = 0, \dots, 9$ are bosonic $N \times N$ matrices and ψ_α, $\alpha = 1, \dots, 32$ are fermionic $N \times N$ matrices. Note that, as pointed out in [94], in Euclidean signature, one cannot impose a reality condition on spinors consistently with supersymmetry. In this case, the notation $\bar{\psi}_\beta := (\psi^T C_{10})_\beta = \psi^\alpha C_{10,\alpha\beta}$, where C_{10} is the charge conjugation matrix in ten dimensions, should be regarded as a convenient definition. In fact, while in Lorentzian signature the fermionic degrees of freedom consist of one 32-component $SO(1,9)$ spinor satisfying a Majorana reality condition and a Weyl projection condition, in the Euclidean theory only the $SO(10)$ Weyl spinor ψ appears in the action and not its complex conjugate, and the path integration is restricted to Hermitian matrices. Of course, the overall number of degrees of freedom is the same in the two cases.

We analyse the case where the target space of the theory is compactified on a six-dimensional torus, where all directions have the same extent $2\pi L$. The construction is inspired by a technique which was first developed in the context of compactifications of the BFSS matrix model [143]. The same mechanism was later deployed in IKKT to compactify the theory on a thermal circle [144]. The main idea is to define the unitary operators U^a, $a = 4, \dots, 9$, which generate translations in the six target space directions we wish to compactify. If one further imposes that such operators commute with each other when they act on different directions, then translations can be generated independently.

The action of the U^a operators on fermionic matrices appearing in (4.2.0.1) can be either periodic or anti-periodic when circling around any of the torus directions.

For the moment, we consider the fully anti-periodic case, for which the action of translation operators reads

$$\begin{aligned} (U^a)^{-1} A^\mu U^a &= A^\mu \\ (U^a)^{-1} A^b U^a &= A^b + 2\pi L \delta^{ab} \\ (U^a)^{-1} \psi U^a &= -\psi \ . \end{aligned} \qquad \begin{aligned} &\mu = 0, \ldots, 3 \\ &a = 4, \ldots, 9 \end{aligned} \tag{4.2.0.2}$$

Since matrices A^I encode information about target space locations, the transformation (4.2.0.2) can be thought of as wrapping around the torus once. We can think of every compact target space direction as a real line where consecutive intervals of length $2\pi L$ are identified; then the transformation (4.2.0.2) translates one interval into the next. The Lagrangian of the theory obtained in this way will be invariant under the action of U^a. Note that this is true even for anti-periodic fermions, since fermionic fields appear in the Lagrangian in pairs.

This construction explicitly implements the method of mirror images introduced by Washington Taylor in [48]. The relations (4.2.0.2) are realised by viewing the Hilbert space of the matrices as a tensor product

$$X = Y \otimes Z \, , \tag{4.2.0.3}$$

where Y is the space of $N \times N$ matrices which is left invariant by the translation and Z is the space of $M \times M$ matrices acted upon by the translation operator (hence the original matrix size $\mathcal{N} = N \times M$). Introduce a pair of $M \times M$ matrices p^a and q^a, satisfying $[q^a, p^b] = i\delta^{ab}$. (Note that this only makes sense in the limit of $M \to \infty$, and we implicitly assume that we work in this limit.) Then, to implement the relations (4.2.0.2), the operators U^a are given by

$$U^a = \mathbb{I}_N \otimes e^{-2\pi i q^a} e^{-i p^a} \, , \tag{4.2.0.4}$$

and the matrix degrees of freedom by

$$\begin{aligned} A^\mu &= \sum_{n^b \in \mathbb{Z}^6} A^\mu(n^b) \otimes e^{i n^b p^b} \\ A^a &= \sum_{n^b \in \mathbb{Z}^6} A^a(n^b) \otimes e^{i n^b p^b} + 2\pi L \, \mathbb{I}_N \otimes q^a \\ \psi &= \sum_{r^b \in \mathbb{L}_6} \psi(r^b) \otimes e^{i r^b p^b} \, . \end{aligned} \tag{4.2.0.5}$$

Here $\mathbb{Z}^6$ is a six-dimensional hypercubic lattice with vertex positions labelled by integers, and

$$\mathbb{L}_6 = \left(\mathbb{Z} + \frac{1}{2}\right) \times \ldots \times \left(\mathbb{Z} + \frac{1}{2}\right) \tag{4.2.0.6}$$

is a six-dimensional hypercubic lattice with vertex positions labelled by half-integers.

Essentially, periodic/anti-periodic boundary conditions are imposed by requiring that winding numbers n^b, r^b take integer/half-integer values, respectively. In Appendix B.1, we provide an explicit formulation of this prescription, where the operators appearing in the decomposition of matrices and in the translation operator are realised as matrices in the $|q^a\rangle$ basis, both for periodic and anti-periodic fermions.

Physically, along each compact direction, $D(-1)$-instantons with coordinates r^a in a fundamental region $r^a \in [0, 2\pi L]$ are replicated an infinite number of times. Open strings connecting them can have winding number zero, when they stretch across two branes in the same fundamental region, or non-zero, in which case they connect two copies living in different regions. The matrices (4.2.0.5) can be represented in block form, where diagonal blocks describe fundamental regions, and off-diagonal blocks account for winding modes. See [136] for more details.

The consequence of the compactification procedure is a theory whose degrees of freedom wind around the torus directions. We name the resulting action the compact IKKT action. This is obtained by first defining the Fourier decomposition of fields in winding modes in terms of a six-dimensional momentum variable σ

$$A^I(\sigma) = \sum_{n^b \in \mathbb{Z}^6} A^I(n^b) e^{2\pi i L n^b \sigma^b}, \quad \psi(\sigma) = \sum_{r^b \in \mathbb{L}_6} \psi(r^b) e^{2\pi i L r^b \sigma^b}, \quad \sigma^b \in [0, L^{-1}]\,. \tag{4.2.0.7}$$

Inserting the decomposition (4.2.0.5) into the original IKKT action, tracing over the Z subspace, and making use of the property

$$\mathrm{Tr}[e^{i(n \pm m)^b p^b}] = M \delta_{n, \mp m}\,, \tag{4.2.0.8}$$

we find

$$S_C = \frac{1}{2g_{eff}^2} \int \frac{d^6\sigma}{(2\pi L)^{-6}} \mathrm{Tr}\Bigg[\frac{1}{2} F_{ab} F^{ab} + D_a A_\mu D^a A^\mu - \frac{1}{2}[A^\mu, A^\nu]^2 + \bar{\psi}\Gamma^a D_a \psi - i\bar{\psi}\Gamma^\mu [A_\mu, \psi]\Bigg]\,, \tag{4.2.0.9}$$

where the effective coupling is $g_{eff}^2 = \frac{g^2}{M}$. In the above expression, we have dropped the dependence of matrix fields on σ^b for simplicity of the notation, and we recall that all the fields are $N \times N$ matrices acting on the subset Y of the Hilbert space (4.2.0.3).

The goal of the compactification procedure is to provide a mechanism for mass generation. To do so, we need to consider the decompactification limit and, in particular, the effective theory of zero modes in this limit. The appearance of mass terms could lead to the breaking of the global $SO(10)$ symmetry. It will be useful to consider a decomposition of spinors which reflects the symmetry-breaking pattern we seek. Before turning our attention to the derivation of the effective theory, we review the spinor decomposition adopted in [136], as this will be central to our discussion of boundary conditions in the next section.

The decomposition of gamma matrices that we will adopt highlights a separation between $SO(4)$ (recall that we Wick rotated to Euclidean signature) and $SO(6)$ degrees of freedom. We consider the gamma matrices

$$\Gamma^\mu = \tilde{\Gamma}_7 \otimes \gamma^\mu, \quad \Gamma^a = \tilde{\Gamma}^a \otimes \mathbb{I}\,,$$

where $\tilde{\Gamma}^a$ are 8×8 gamma matrices of $SO(6)$ and $\tilde{\Gamma}_7$ is the corresponding chirality operator. The $SO(4)$ gamma matrices are further decomposed in the Weyl basis

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix},$$

where $\sigma^\mu = (\mathbb{I}, \sigma^i)$, $\bar{\sigma}^\mu = (\mathbb{I}, -\sigma^i)$ and σ^i are Pauli matrices. The 10-dimensional chiral operator and charge conjugation decompose as

$$\Gamma_{11} = \tilde{\Gamma}_7 \otimes \begin{pmatrix} \mathbb{I}_2 & 0 \\ 0 & -\mathbb{I}_2 \end{pmatrix}, \quad C_{10} = C_6 \otimes \begin{pmatrix} i\sigma_2 & 0 \\ 0 & -i\sigma_2 \end{pmatrix}, \tag{4.2.0.10}$$

where C_6 is the charge conjugation matrix in 6 dimensions. It follows that the 32-component spinor ψ can be written as

$$\psi = \begin{pmatrix} \psi_+^A \\ \psi_-^A \end{pmatrix}, \qquad A = 1, 2\,, \tag{4.2.0.11}$$

where $\psi_\pm^A$ are the two chiral components and each component is an 8-dimensional spinor acted upon by the $SO(6)$ Clifford algebra.

The spinor decomposition can be plugged into the compact action (4.2.0.9), yielding

$$S_C = \frac{1}{2g_{eff}^2}\int \frac{d^6\sigma}{(2\pi L)^{-6}}\mathrm{Tr}\left[\frac{1}{2}F_{ab}F^{ab} + D_a A_\mu D^a A^\mu - \frac{1}{2}[A^\mu, A^\nu]^2 \right.$$
$$+\bar{\psi}_+^A\tilde{\Gamma}^a\partial_a\psi_+^A + \bar{\psi}_-^A\tilde{\Gamma}^a\partial_a\psi_-^A - i\bar{\psi}_+^A\tilde{\Gamma}^a[A_a,\psi_+^A] - i\bar{\psi}_-^A\tilde{\Gamma}^a[A_a,\psi_-^A]$$
$$\left.+i\bar{\psi}_+^A(\sigma^\mu)^{AB}[A_\mu,\psi_-^B] - i\bar{\psi}_-^A(\bar{\sigma}^\mu)^{AB}[A_\mu,\psi_+^B]\right] , \quad (4.2.0.12)$$

along with the ghost theory associated with the Lorentz gauge condition $\partial_a A^a = 0$:

$$S_{gh} = \frac{1}{g_{eff}^2}\int \frac{d^6\sigma}{(2\pi L)^{-6}}\mathrm{Tr}\left[\partial^a\bar{c}D_a c\right] , \quad (4.2.0.13)$$

which arises from a gauge-fixing delta function in the measure of the path integral (2.4.0.1).

The idea is to look at the effective theory in the limit $L \gg g_{eff}^{1/2}$. By inserting the winding mode decomposition (4.2.0.7) into the compact action (4.2.0.12) and ghost theory (4.2.0.13), we find that non-zero winding modes on the torus acquire a mass proportional to the torus size $2\pi L$. This explicit mass term takes the form

$$S_m = \frac{1}{2g_{eff}^2}\sum_{n^a\in\mathbb{Z}^6}(2\pi L n^a)^2\mathrm{Tr}[A_I(-n^a)A^I(n^a)]$$
$$+\frac{1}{2g_{eff}^2}\sum_{r^a\in\mathbb{L}_6}(2\pi L i r^a)\mathrm{Tr}[\bar{\psi}_+^A(r^a)\tilde{\Gamma}^a\psi_+^A(r^a)] + \frac{1}{2g_{eff}^2}\sum_{r^a\in\mathbb{L}_6}(2\pi L i r^a)\mathrm{Tr}[\bar{\psi}_-^A(r^a)\tilde{\Gamma}^a\psi_-^A(r^a)] .$$

Note that the masses are proportional to the winding number. This implies, in particular, that the mass of open strings stretched between two well-separated branes, with coordinates x^J and y^J in the compact direction J and equal coordinates in the other directions, is $m \propto |y^J - x^J + 2\pi L n|$. Here n is the difference in winding number, and accounts for the stretching of the string when winding the compact dimension. It follows that, in the limit we are considering, such modes can be integrated out. The result of this operation is a Wilsonian effective theory for the zero modes. As explicitly shown in [136], boundary conditions for fermions play a crucial role, and the effective theory obtained depends on the choice. Indeed, anti-periodic conditions for fermions break supersymmetry explicitly. However, this does not produce any divergences in the one-loop effective action, since the divergent contributions arising from bosonic and fermionic non-zero modes running in the loop still cancel.

The outcome is an effective theory for zero modes, which, after rotating back to Lorentzian signature, takes the form

$$S_{eff}^0 = -\frac{1}{2g_{eff}^2}\mathrm{Tr}\left[\frac{1}{2}[A^I(0),A^J(0)]^2 - M_{IJ}A^I(0)A^J(0)\right] ,$$

where the mass matrix consists of two blocks

$$M_{IJ} = \begin{pmatrix} \eta_{\mu\nu} M_4^2 & 0 \\ 0 & \eta_{ab} M_6^2 \end{pmatrix} , \tag{4.2.0.14}$$

and breaks the $SO(1,9)$ symmetry to $SO(1,3) \times SO(6)$. The two terms M_4^2 and M_6^2 in (4.2.0.14) are given by

$$M_4^2 = 16(S_F - S_B)\frac{N}{L^2}, \qquad M_6^2 = \frac{32}{3}(S_F - S_B)\frac{N}{L^2} ,$$

where S_F and S_B are the infinite sums

$$S_B = \sum\nolimits'_{n^b \in \mathbb{Z}^6} \frac{1}{(2\pi n^b)^2}, \qquad S_F = \sum_{r^b \in \mathbb{L}_6} \frac{1}{(2\pi r^b)^2} ,$$

and the bosonic sum is performed over non-zero modes. These sums can be regularised, and the infinite part cancels in the difference, leaving a finite, positive number. Note that, because fermions do not have zero modes with this choice of boundary conditions, they completely decouple from the low-energy theory and supersymmetry is broken.

On the other hand, fully periodic boundary conditions for the fermions, in the form

$$(U^a)^{-1} \psi U^a = \psi ,$$

yield the low-energy action

$$S^0_{eff} = -\frac{1}{2g^2_{eff}} \mathrm{Tr} \left[\frac{1}{2} [A^I(0), A^J(0)]^2 + \bar{\psi}(0) \Gamma^I [A_I(0), \psi(0)] \right] ,$$

which is identical to the non-compact IKKT action with an effective coupling g_{eff}. In particular, supersymmetry is completely intact, and no mass term is generated.

In the following sections, we will extend these computations to the case of IKKT with alternative boundary conditions and then to BFSS theory.

4.3 IKKT with interchanging boundary conditions

In this section, we compute the Wilsonian effective action which results from integrating out non-zero modes from the compact IKKT action with mixed boundary conditions. We expect mixed boundary conditions to give rise to an effective theory which is an intermediate case between the fully supersymmetric/no mass case and the effective theory with a mass term and no fermions. Our goal is to obtain a theory which, as a consequence of compactification, features fermionic degrees of freedom, while still giving rise to a mass term. We consider two types of interchanging boundary conditions: the first swaps the two components with the same chirality, and the other exchanges the chiralities.

4.3.1 Component-reversing boundary conditions

We begin by considering the spinor decomposition (4.2.0.11). We impose that the Weyl spinor components are mapped into one another when wrapped around any direction on the torus, namely

$$(U^a)^{-1}\psi^1_\pm U^a = \psi^2_\pm, \qquad (U^a)^{-1}\psi^2_\pm U^a = \psi^1_\pm\,. \tag{4.3.1.1}$$

This is obtained by rewriting the spinor in the form

$$\psi^A_\pm = \psi^1_\pm \begin{pmatrix}1\\0\end{pmatrix} + \psi^2_\pm \begin{pmatrix}0\\1\end{pmatrix} := \frac{(\psi^\uparrow_\pm + \psi^\downarrow_\pm)}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix} + \frac{(\psi^\uparrow_\pm - \psi^\downarrow_\pm)}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix} = \frac{\psi^\uparrow_\pm}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix} + \frac{\psi^\downarrow_\pm}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

and using the Toeplitz decomposition

$$\begin{aligned}\psi^\uparrow_\pm &= \sum_{n^b\in\mathbb{Z}^6} \psi^\uparrow_\pm(n^b)\otimes e^{in^bp^b}\,,\\ \psi^\downarrow_\pm &= \sum_{r^b\in\mathbb{L}_6} \psi^\downarrow_\pm(r^b)\otimes e^{ir^bp^b}\,,\end{aligned} \tag{4.3.1.2}$$

where we recall that $\psi^\uparrow_\pm$ and $\psi^\downarrow_\pm$ are 8-component spinors of $SO(6)$. We also recall that $\mathbb{L}_6$ is the lattice (4.2.0.6) of half-integer winding modes. The decomposition (4.3.1.2) highlights how every chiral component of the spinor has a part with periodic and a part with anti-periodic boundary conditions

$$\psi^A_\pm = \Psi^\uparrow_\pm + \Psi^\downarrow_\pm, \qquad \begin{cases}\Psi^\uparrow_\pm = \dfrac{\psi^\uparrow_\pm}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\\[2ex] \Psi^\downarrow_\pm = \dfrac{\psi^\downarrow_\pm}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\end{cases}.$$

Note that these boundary conditions are equivalent to the transformation generated by the operator

$$T = \mathbb{I}_8 \otimes \begin{pmatrix}\sigma_1 & 0\\ 0 & \sigma_1\end{pmatrix}, \tag{4.3.1.3}$$

which acts on spinors when they are translated on a full circle around one of the torus directions. Following the spinor decomposition, the Weyl and Majorana [1] relations

$$\Gamma_{11}\psi = \psi, \qquad \bar\psi = \psi^T C_{10}$$

[1] As pointed out in the previous section, only in the Lorentzian theory can one impose the reality condition, while in Euclidean signature the symmetry group is $SO(10)$ which does not admit a Majorana-Weyl representation.

become

$$\tilde{\Gamma}_7\psi_\pm = \pm\psi_\pm, \qquad \psi_\pm = \pm i\sigma_2 C_6 \bar{\psi}_\pm^T ,$$

where we used (4.2.0.10). These are indeed preserved by the action of (4.3.1.3), which transforms the chirality and charge conjugation matrices as

$$T\,\Gamma_{11}T^{-1} = \Gamma_{11}, \qquad T\,C_{10}T^{-1} = -C_{10}\,.$$

The fermionic theory that we obtain, by inserting this decomposition with interchanging boundary conditions back into the IKKT action, reads

$$\begin{aligned} S_F = \frac{1}{2g_{eff}^2}\int \frac{d^6\sigma}{(2\pi L)^{-6}}\mathrm{Tr}\,\Big[&\bar{\Psi}_+^\uparrow\tilde{\Gamma}^a\partial_a\Psi_+^\uparrow + \bar{\Psi}_-^\uparrow\tilde{\Gamma}^a\partial_a\Psi_-^\uparrow + \bar{\Psi}_+^\downarrow\tilde{\Gamma}^a\partial_a\Psi_+^\downarrow + \bar{\Psi}_-^\downarrow\tilde{\Gamma}^a\partial_a\Psi_-^\downarrow \\ &-i\bar{\Psi}_+^\uparrow\tilde{\Gamma}^a[A_a,\Psi_+^\uparrow] - i\bar{\Psi}_-^\uparrow\tilde{\Gamma}^a[A_a,\Psi_-^\uparrow] - i\bar{\Psi}_+^\downarrow\tilde{\Gamma}^a[A_a,\Psi_+^\downarrow] - i\bar{\Psi}_-^\downarrow\tilde{\Gamma}^a[A_a,\Psi_-^\downarrow] \\ &+i\bar{\Psi}_+^\uparrow\sigma^\mu[A_\mu,\Psi_-^\uparrow] - i\bar{\Psi}_-^\uparrow\bar{\sigma}^\mu[A_\mu,\Psi_+^\uparrow] + i\bar{\Psi}_+^\downarrow\sigma^\mu[A_\mu,\Psi_-^\downarrow] - i\bar{\Psi}_-^\downarrow\bar{\sigma}^\mu[A_\mu,\Psi_+^\downarrow]\Big]\,. \end{aligned}$$

Here, the fields have been expanded in winding modes as

$$\begin{aligned} A^I(\sigma) &= \sum_{n^b\in\mathbb{Z}^6} A^I(n^b)e^{2\pi iLn^b\sigma^b} \\ \Psi_\pm^\uparrow(\sigma) &= \sum_{r^b\in\mathbb{Z}^6} \Psi_\pm^\uparrow(r^b)e^{2\pi iLr^b\sigma^b} \\ \Psi_\pm^\downarrow(\sigma) &= \sum_{r^b\in\mathbb{L}_6} \Psi_\pm^\downarrow(r^b)e^{2\pi iLr^b\sigma^b} \\ c(\sigma) &= \sum_{n^b\in\mathbb{Z}^6} c(n^b)e^{2\pi iLn^b\sigma^b}\,, \end{aligned} \tag{4.3.1.4}$$

where we also indicated the mode expansion of ghost fields (with periodic boundary conditions) for later convenience. Note that the two spinors, with opposite boundary conditions, are completely decoupled from one another. This follows from the different nature of their winding modes, which causes terms of the form $\bar{\Psi}_+^\downarrow\sigma^\mu[A_\mu,\Psi_-^\uparrow]$ to vanish due to the property (4.2.0.8). The vanishing of these terms causes a loss of fermionic degrees of freedom running in diagram loops, ultimately because the transformation (4.3.1.3) is not a symmetry of the compact action. We will show shortly that this results in generating a divergent mass term in the effective theory.

We proceed by rescaling the fields in the compact IKKT action, in order to make them dimensionless, using

$$A^I \to \lambda L A^I, \quad \psi \to \lambda L^{3/2}\psi, \quad c \to \lambda L c\,, \tag{4.3.1.5}$$

where λ is the dimensionless coupling constant $\lambda := \frac{g_{eff}}{L^2}$, which will serve as the perturbative expansion parameter. Substituting the mode expansion (4.3.1.4) into the compact action yields

$$S = S_0 + S_{kin} + S_{int}\,,$$

where S_0 is the zero-mode action, S_{kin} is the action for kinetic terms of non-zero modes and S_{int} are the interaction terms. In particular, the action for kinetic terms reads

$$\begin{aligned} S_{kin} = \frac{1}{2}\sum_{n^a\in\mathbb{Z}^6}(2\pi n^a)^2\mathrm{Tr}[A_I(-n^a)A^I(n^a)] + \frac{1}{2}\sum_{r^a\in\mathbb{Z}^6}(2\pi i r^a)\mathrm{Tr}[\bar{\psi}^{\uparrow}_{+}(r^a)\tilde{\Gamma}^a\psi^{\uparrow}_{+}(r^a)] \\ + \frac{1}{2}\sum_{r^a\in\mathbb{Z}^6}(2\pi i r^a)\mathrm{Tr}[\bar{\psi}^{\uparrow}_{-}(r^a)\tilde{\Gamma}^a\psi^{\uparrow}_{-}(r^a)] + \frac{1}{2}\sum_{s^a\in\mathbb{L}_6}(2\pi i s^a)\mathrm{Tr}[\bar{\psi}^{\downarrow}_{+}(s^a)\tilde{\Gamma}^a\psi^{\downarrow}_{+}(s^a)] \\ + \frac{1}{2}\sum_{s^a\in\mathbb{L}_6}(2\pi i s^a)\mathrm{Tr}[\bar{\psi}^{\downarrow}_{-}(s^a)\tilde{\Gamma}^a\psi^{\downarrow}_{-}(s^a)] + \sum_{n^a\in\mathbb{Z}^6}(2\pi n^a)^2\mathrm{Tr}[\bar{c}(n^a)c(n^a)]\,, \end{aligned} \quad (4.3.1.6)$$

which shows that non-zero modes are massive. We will use this fact to integrate them out at one-loop, in order to obtain the Wilsonian effective action for the zero modes,

$$\begin{aligned} S_0^{eff} &= -\ln\left(\prod\nolimits'_{n^a\in\mathbb{Z}^6}\prod\nolimits'_{r^b\in\mathbb{Z}^6}\prod_{s^c\in\mathbb{L}_6}\int[\mathcal{D}A(n^a)][\mathcal{D}\psi^{\uparrow}(r^b)][\mathcal{D}\psi^{\downarrow}(s^c)]e^{-S_C}\right) \\ &= S_0 - \ln\left\langle e^{-S_{int}}\right\rangle - \ln Z_{kin}\,, \end{aligned} \quad (4.3.1.7)$$

where $\ln Z_{kin}$ is a constant, independent of the zero modes, which we discard. The second term is the correction to the zero-mode effective action. The expectation value is computed in perturbation theory (we will evaluate it at one-loop) with respect to the kinetic action (4.3.1.6). The interaction can be decomposed as

$$S_{int} = \sum_{i=1}^{11} V_i\,, \quad (4.3.1.8)$$

where V_1, V_2 and V_{11} are purely bosonic or ghost terms, identical to the corresponding terms in the purely anti-periodic case [136], whereas V_i for $i = 3,\ldots,10$ include fermions and differ from [136] due to the choice of boundary conditions.

We first examine the bosonic terms, which are given by

$$\begin{aligned} V_1 &= -\frac{\lambda^2}{4}\sum_{n^a,m^a,k^a\in\mathbb{Z}^6}\mathrm{Tr}\left[[A^I(-n^a-m^a-k^a),A^J(n^a)][A_I(m^a),A_J(k^a)]\right]\,, \\ V_2 &= -\lambda\sum_{n^a,m^a\in\mathbb{Z}^6}\mathrm{Tr}\left[2\pi(n^a+m^a)A^I(-n^a-m^a)[A_a(n^a),A_I(m^a)]\right]\,. \end{aligned}$$

Note that at least one of the modes summed over is a zero mode. The interaction term with ghosts reads

$$V_{11} = -\lambda \sum_{m^a \in \mathbb{Z}^6} 2\pi m^a \mathrm{Tr}\left[\bar{c}(m^a)[A_a(0), c(m^a)]\right] .$$

There are also interaction terms which only involve non-zero modes, but those are integrated over in the Wilsonian effective theory calculation, and since they do not depend on zero modes, they contribute to the effective action as a constant which can be ignored.

The interaction term for fermions is composed of a periodic and an anti-periodic component. These are completely disconnected from each other. We define $V_3, \ldots, V_6$ to involve periodic spinors, given by

$$\begin{aligned}
V_3 &= -\frac{i}{2}\lambda \sum_{r^a \in \mathbb{Z}^6} \mathrm{Tr}\left[\bar{\psi}^{\uparrow}_{+}(r^a)\tilde{\Gamma}^a[A_a(0), \psi^{\uparrow}_{+}(r^a)]\right] , \\
V_4 &= -\frac{i}{2}\lambda \sum_{r^a \in \mathbb{Z}^6} \mathrm{Tr}\left[\bar{\psi}^{\uparrow}_{-}(r^a)\tilde{\Gamma}^a[A_a(0), \psi^{\uparrow}_{-}(r^a)]\right] , \\
V_5 &= \frac{i}{4}\lambda \sum_{r^a \in \mathbb{Z}^6} \mathrm{Tr}\left[\bar{\psi}^{\uparrow}_{+}(r^a)\sigma^{\mu}_{\uparrow}[A_\mu(0), \psi^{\uparrow}_{-}(r_a)]\right] , \\
V_6 &= -\frac{i}{4}\lambda \sum_{r^a \in \mathbb{Z}^6} \mathrm{Tr}\left[\bar{\psi}^{\uparrow}_{-}(r^a)\bar{\sigma}^{\mu}_{\uparrow}[A_\mu(0), \psi^{\uparrow}_{+}(r_a)]\right] .
\end{aligned}$$

The terms $V_7, \ldots, V_{10}$ are obtained by swapping $\psi^{\uparrow}_{\pm} \leftrightarrow \psi^{\downarrow}_{\pm}$, $\sigma^{\mu}_{\uparrow} \leftrightarrow \sigma^{\mu}_{\downarrow}$, and considering half-integers winding modes. Here

$$\begin{aligned}
\sigma^{\mu}_{\uparrow} &= \begin{pmatrix} 1 & 1 \end{pmatrix} \sigma^{\mu} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \sum_{i,j=1}^{2} \sigma^{\mu}_{ij} = 2 \begin{pmatrix} 1 \\ i \\ 0 \\ 0 \end{pmatrix} \\
\sigma^{\mu}_{\downarrow} &= \begin{pmatrix} 1 & -1 \end{pmatrix} \sigma^{\mu} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \sum_{ij=1}^{2} (-1)^{i+j} \sigma^{\mu}_{ij} = 2 \begin{pmatrix} 1 \\ -i \\ 0 \\ 0 \end{pmatrix} .
\end{aligned} \tag{4.3.1.9}$$

We can now compute the effective action for zero modes derived from the interaction terms above. Starting with bosons and ghosts, for which the computation proceeds analogously to [136], we have

$$S_0^{eff} \supset \langle V_1 \rangle - \frac{1}{2}\left\langle V_2^2 \right\rangle - \frac{1}{2}\left\langle V_{11}^2 \right\rangle ,$$

and expectation values are computed using the propagators

$$\begin{aligned}
\left\langle A^N(n^a) A^M(m^a) \right\rangle &= \frac{\delta^{MN}\delta_{n^a+m^a,0}}{(2\pi n^a)^2} , \\
\langle \bar{c}(n^a) c(m^a) \rangle &= \frac{\delta_{n^a,m^a}}{(2\pi n^a)^2} .
\end{aligned}$$

The result reads

$$\begin{aligned}
\langle V_1 \rangle &= 9\lambda^2 N S_{B_1} \mathrm{Tr}[A^I(0)A_I(0)] \\
\left\langle V_2^2 \right\rangle &= 2\lambda^2 N \left((17S_{B_2} + S_{B_1})\mathrm{Tr}[A^a(0)A_a(0)] + S_{B_1}\mathrm{Tr}[A^\mu(0)A_\mu(0)]\right) \\
\left\langle V_{11}^2 \right\rangle &= -2\lambda^2 N S_{B_2} \mathrm{Tr}[A^a(0)A_a(0)]\,,
\end{aligned}$$

where the factors

$$S_{B_1} = \sum\nolimits'_{n^a\in\mathbb{Z}^6} \frac{1}{(2\pi n^a)^2}, \qquad S_{B_2} = \sum\nolimits'_{n^a\in\mathbb{Z}^6} \frac{(2\pi n^1)^2}{(2\pi n^a)^4}\,, \tag{4.3.1.10}$$

come from propagators in the loop. Note that these sums are divergent and have to be regularised, as discussed in [136].

We now turn our attention to contributions to the effective action induced by integrating out fermions from the interaction terms $V_3, \ldots, V_{10}$. Fermionic propagators are inferred from (4.3.1.6) and read

$$\left\langle \bar\psi^u_{p,\alpha}(r^a)\psi^v_{q,\beta}(s^a)\right\rangle = -i\frac{2\pi r^a \tilde\Gamma^a_{\alpha\beta}\delta^{uv}\delta_{p,q}\delta_{r^a,s^a}}{(2\pi r^a)^2}\,,$$

where $u, v = \uparrow, \downarrow$, $p, q = \pm$ and $\alpha, \beta = 1, \ldots, 8$ are $SO(6)$ spinor indices. It follows that the contribution of fermionic interactions to the zero-mode effective action is

$$S_0^{eff} \supset -\frac{1}{2}\left\langle V_3^2\right\rangle - \frac{1}{2}\left\langle V_4^2\right\rangle - \langle V_5 V_6\rangle - \frac{1}{2}\left\langle V_7^2\right\rangle - \frac{1}{2}\left\langle V_8^2\right\rangle - \langle V_9 V_{10}\rangle\,. \tag{4.3.1.11}$$

Recalling that $\psi^\uparrow_\pm$ has periodic boundary conditions and $\psi^\downarrow_\pm$ has anti-periodic ones, a straightforward application of Wick's theorem yields

$$\begin{aligned}
\left\langle V_3^2\right\rangle &= -4\lambda^2 N(2S_{B_2} - S_{B_1})\mathrm{Tr}[A^a(0)A_a(0)] \\
\left\langle V_4^2\right\rangle &= -4\lambda^2 N(2S_{B_2} - S_{B_1})\mathrm{Tr}[A^a(0)A_a(0)] \\
\left\langle V_7^2\right\rangle &= -4\lambda^2 N(2S_{F_2} - S_{F_1})\mathrm{Tr}[A^a(0)A_a(0)] \\
\left\langle V_8^2\right\rangle &= -4\lambda^2 N(2S_{F_2} - S_{F_1})\mathrm{Tr}[A^a(0)A_a(0)]\,,
\end{aligned}$$

where S_{F_1} and S_{F_2} read

$$S_{F_1} = \sum_{s^a\in\mathbb{L}_6} \frac{1}{(2\pi s^a)^2}, \qquad S_{F_2} = \sum_{s^a\in\mathbb{L}_6} \frac{(2\pi s^1)^2}{(2\pi s^a)^4}\,.$$

These terms are also divergent but can be regularised with the same procedure as the bosonic ones, and the divergent part is the same [136]. In particular, the difference $S_{F_1} - S_{B_1}$ is a finite number which can be evaluated numerically.

Cross terms in (4.3.1.11) require a closer look. Let us analyse $\langle V_5 V_6 \rangle$. After contracting fermionic fields, we obtain

$$\langle V_5 V_6 \rangle = \lambda^2 N S_{B_1} \sigma^\mu_\uparrow \bar{\sigma}^\nu_\uparrow \mathrm{Tr}[A_\mu(0) A_\nu(0)] \,,$$

which we can rearrange using (4.3.1.9)

$$\sigma^\mu_\uparrow \bar{\sigma}^\nu_\uparrow \mathrm{Tr}[A_\mu(0) A_\nu(0)] = 4\mathrm{Tr}[A_0^2(0) + A_1^2(0)] \,,$$

and hence

$$\langle V_5 V_6 \rangle = 4\lambda^2 N S_{B_1} \mathrm{Tr}[A_0^2(0) + A_1^2(0)] \,. \tag{4.3.1.12}$$

Analogously, one can compute

$$\langle V_9 V_{10} \rangle = 4\lambda^2 N S_{F_1} \mathrm{Tr}[A_0^2(0) + A_1^2(0)] \,. \tag{4.3.1.13}$$

Note that (4.3.1.12) and (4.3.1.13) imply that fermions in the loop only contribute to the mass of two of the components of bosonic matrices in the $SO(4)$ sector, thus breaking the $SO(4)$ symmetry.

We can now summarise all contributions to the zero modes effective action (4.3.1.7) to order $O(\lambda^2)$ in perturbation theory. First, the action S_0 for zero modes at tree level receives contributions from both the bosonic part and the periodic components of fermions, and is given by

$$\begin{aligned} S_0 = -\frac{1}{2}\lambda^2 \mathrm{Tr} \Big[& \frac{1}{2}[A^I(0), A^J(0)]^2 + i\bar{\psi}^\uparrow_+(0)\tilde{\Gamma}^a[A_a(0), \psi^\uparrow_+(0)] + i\bar{\psi}^\uparrow_-(0)\tilde{\Gamma}^a[A_a(0), \psi^\uparrow_-(0)] \\ & - i\bar{\psi}^\uparrow_+(0)[A_0(0) + iA_1(0), \psi^\uparrow_-(0)] + i\bar{\psi}^\uparrow_-(0)[A_0(0) - iA_1(0), \psi^\uparrow_+(0)] \Big] \,. \end{aligned}$$

Including the above one-loop corrections, we obtain the full effective action

$$\begin{aligned} S_0^{eff} = S_0 - \lambda^2 N \Big(& 4(S_{F_1} - S_{B_1}) \mathrm{Tr}[A_0^2(0) + A_1^2(0)] \\ & -8 S_{B_1} \mathrm{Tr}[A_2^2(0) + A_3^2(0)] + \frac{8}{3}(S_{F_1} - S_{B_1}) \mathrm{Tr}[A_a^2(0)] \Big) \,. \end{aligned} \tag{4.3.1.14}$$

Note that the choice of interchanging boundary conditions (4.3.1.1) breaks the $SO(4)$ symmetry subgroup in two ways: through the interaction term, which couples fermionic zero modes to specific components of the bosonic field; and through the mass matrix, which is diagonal but takes different values for different components in the four-dimensional

subspace. Moreover, these boundary conditions effectively reduce the number of fermionic degrees of freedom which contribute to loop diagrams, due to the decoupling of $\psi^{\uparrow}$ and $\psi^{\downarrow}$ modes. As a consequence, cancellation of divergence at the one-loop level fails. In particular, we find a divergent mass

$$M_2^2 = M_3^2 = -16\lambda^2 N S_{B_1}$$

for two components of the bosonic field. This highlights a complication in generating a mass term alongside fermionic degrees of freedom from torus compactifications of IKKT with mixed boundary conditions. We will show in the next subsection that this issue also arises with alternative choices of boundary conditions. As we will see in Section 4.5, the same issue does not arise in BFSS theory, where we explicitly find a class of mixed boundary conditions which does not affect the cancellation at one-loop level and gives a finite result. This constitutes an interesting qualitative difference between the two models, which would be worth exploring further.

We can consider renormalising the divergence by introducing an *ad hoc* mass counterterm in the compact action on the torus (4.3.1.8). This is artificial from the point of view of the original IKKT model, where this term is forbidden by supersymmetry and global $SO(10)$. However, we explicitly showed how both these symmetries are broken in the compact action, so this term is allowed at the level of the EFT. We have freedom in choosing the renormalisation scheme. The choice we make is to cancel the divergence by introducing the counterterm action

$$S_{CT} = -\frac{\lambda^2}{2}\int \frac{d^6\sigma}{(2\pi L)^{-6}} 16 N S_{F_1} \mathrm{Tr}[A_2^2(0) + A_3^2(0)]\,,$$

such that

$$\begin{aligned} S_0^{eff} &= S_0 - \ln\left\langle e^{-S_{int}}\right\rangle - \ln Z_{kin} + S_{CT} \\ &= S_0 - \lambda^2 N \Bigg(4(S_{F_1} - S_{B_1})\mathrm{Tr}[A_0^2(0) + A_1^2(0)] \\ &\qquad + 8(S_{F_1} - S_{B_1})\mathrm{Tr}[A_2^2(0) + A_3^2(0)] + \frac{8}{3}(S_{F_1} - S_{B_1})\mathrm{Tr}[A_a^2(0)] \Bigg)\,. \end{aligned}$$

This choice is natural in the sense that it reinserts the contribution coming from fermions in the loop, which have been lost due to the mixed boundary conditions. Of course, other

choices are possible, and the finite value of the resulting mass is scheme-dependent. We then rescale fields back by inverting (4.3.1.5), and Wick rotate to restore Lorentzian signature, resulting in

$$S_0 = -\frac{1}{2g_{eff}^2}\mathrm{Tr}\left[\frac{1}{2}[A^I(0),A^J(0)]^2 + \bar{\psi}_+^{\uparrow}(0)\tilde{\Gamma}^a[A_a(0),\psi_+^{\uparrow}(0)] + \bar{\psi}_-^{\uparrow}(0)\tilde{\Gamma}^a[A_a(0),\psi_-^{\uparrow}(0)]\right.$$
$$\left. - \bar{\psi}_+^{\uparrow}(0)[A_0(0)-A_1(0),\psi_-^{\uparrow}(0)] + \bar{\psi}_-^{\uparrow}(0)[A_0(0)+A_1(0),\psi_+^{\uparrow}(0)]\right],$$

and

$$S_0^{eff} = S_0 + \frac{N}{L^2}\left(4(S_{F_1}-S_{B_1})\mathrm{Tr}[A_0^2(0)-A_1^2(0)]\right.$$
$$\left. -8(S_{F_1}-S_{B_1})\mathrm{Tr}[A_2^2(0)+A_3^2(0)] - \frac{8}{3}(S_{F_1}-S_{B_1})\mathrm{Tr}[A_a^2(0)]\right).$$

As expected, the theory we obtain shows mixed qualitative features compared to the purely periodic or anti-periodic cases: here, parts of the fermions are still present in the zero modes action, alongside a mass term. Both contributions break Lorentz symmetry explicitly.

Note that there are other options for boundary conditions, which also produce infinite mass terms in the effective action and require renormalisation. For example, we can give opposite boundary conditions to different components of the two-dimensional spinor or duplicate fermions and give each component different boundary conditions.

Another possibility would be to consider the theory to be compactified on $\mathbb{T}^4$. Then, upon considering the decompactification limit with mixed boundary conditions, we can obtain an effective theory with a broken $SO(6)$ subgroup and an unbroken $SO(4)$.

4.3.2 Chiral-reversing boundary conditions

A further type of boundary condition that can be considered is swapping chiral components of fermions when cycling around the torus. These are implemented by

$$(U^a)^{-1}\psi_+^A U^a = \psi_-^A, \qquad (U^a)^{-1}\psi_-^A U^a = \psi_+^A\,.$$

In other words, we decompose

$$\psi_+^A = \frac{1}{\sqrt{2}}(\psi_\uparrow^A + \psi_\downarrow^A), \qquad \psi_-^A = \frac{1}{\sqrt{2}}(\psi_\uparrow^A - \psi_\downarrow^A)\,, \tag{4.3.2.1}$$

where $\psi_\uparrow^A$ has periodic boundary conditions and $\psi_\downarrow^A$ is anti-periodic, and their Hilbert space decomposition (4.2.0.3) reads

$$\psi_\uparrow^A = \sum_{n^b \in \mathbb{Z}^6} \psi_\uparrow^A(n^b) \otimes e^{in^b p^b}$$
$$\psi_\downarrow^A = \sum_{r^b \in \mathbb{L}_6} \psi_\downarrow^A(r^b) \otimes e^{ir^b p^b} \ .$$

Substituting (4.3.2.1) into the compact action (4.2.0.9), we get the fermionic action

$$S_F^{CR} = \frac{1}{2g_{eff}^2} \int \frac{d^6\sigma}{(2\pi L)^{-6}} \mathrm{Tr} \left[\bar{\psi}_\uparrow^A \tilde{\Gamma}^a \partial_a \psi_\uparrow^A + \bar{\psi}_\downarrow^A \tilde{\Gamma}^a \partial_a \psi_\downarrow^A - i\bar{\psi}_\uparrow^A \tilde{\Gamma}^a [A_a, \psi_\uparrow^A] - i\bar{\psi}_\downarrow^A \tilde{\Gamma}^a [A_a, \psi_\downarrow^A] \right.$$
$$\left. + i\bar{\psi}_\uparrow^A (\sigma^j)_{AB} [A_j, \psi_\uparrow^B] - i\bar{\psi}_\downarrow^A (\sigma^j)_{AB} [A_j, \psi_\downarrow^B] \right] ,$$

where the index $j = 1, 2, 3$ runs over the three space components of $SO(4)$ and the superscript CR stands for *chiral-reversing* boundary conditions. The dependence on the momentum variable σ is obtained through the mode expansion

$$\psi_\uparrow^A(\sigma) = \sum_{n^b \in \mathbb{Z}^6} \psi_\uparrow^A(n^b) e^{2\pi i L n^b \sigma^b}$$
$$\psi_\downarrow^A(\sigma) = \sum_{r^b \in \mathbb{L}_6} \psi_\downarrow^A(r^b) e^{2\pi i L r^b \sigma^b} \ .$$

Note that the A_0 component of the bosonic vector is not coupled to spinors, due to the relation (4.2.0.8) which forbids terms of the form $\bar{\psi}_\downarrow^A (\sigma^0)_{AB} [A_0, \psi_\uparrow^B]$. The fermionic propagator in the space of winding modes is

$$\left\langle \bar{\psi}_{u,\alpha}^A(r^a) \psi_{v,\beta}^B(s^a) \right\rangle = -i \frac{2\pi r^a \tilde{\Gamma}_{\alpha\beta}^a \delta^{A,B} \delta_{u,v} \delta_{r^a, s^a}}{(2\pi r^a)^2} ,$$

where $u, v = \uparrow, \downarrow$. Integrating out the non-zero fermionic modes, we get in this case (after the usual field redefinition (4.3.1.5))

$$S_0^{eff} = S_0^{CR} - \lambda^2 N \left(-8 S_{B_1} \mathrm{Tr}[A_0^2(0)] + 4(S_{F_1} - S_{B_1}) \mathrm{Tr}[A_i^2(0)] + \frac{8}{3}(S_{F_1} - S_{B_1}) \mathrm{Tr}[A_a^2(0)] \right) , \tag{4.3.2.2}$$

and

$$S_0^{CR} = -\frac{1}{2} \lambda^2 \mathrm{Tr} \left[\frac{1}{2} [A^I(0), A^J(0)]^2 + i\bar{\psi}_\uparrow^A(0) \tilde{\Gamma}^a [A_a(0), \psi_\uparrow^A(0)] + i\bar{\psi}_\uparrow^A(0) (\sigma^i)_{AB} [A_i(0), \psi_\uparrow^B(0)] \right] .$$

Note that this effective theory for the zero modes is significantly different from (4.3.1.14). In particular, it features an explicit coupling of fermionic modes to all components of the bosonic vectors, except the zeroth one. The latter precisely corresponds to the bosonic

component, which acquires a divergent mass, due to the cancellation of some of the fermionic interactions which contribute at one loop. We can add a counterterm to the compact action in order to renormalise this divergence, and our choice of renormalisation scheme at one loop is

$$S_{CT} = -\frac{\lambda^2}{2}\int \frac{d^6\sigma}{(2\pi L)^{-6}} 16N\, S_{F_1}\mathrm{Tr}[A_0^2(0)]\,,$$

such that the renormalised effective action for zero modes reads

$$S_0^{eff,R} = S_0^{eff} + S_{CT} = S_0^{CR} - \lambda^2 N\Big(8(S_{F_1}-S_{B_1})\mathrm{Tr}[A_0^2(0)] \\ +4(S_{F_1}-S_{B_1})\mathrm{Tr}[A_i^2(0)] + \frac{8}{3}(S_{F_1}-S_{B_1})\mathrm{Tr}[A_a^2(0)]\Big)\,,$$

which is finite and preserves the $SO(3)$ symmetry subgroup of the $SO(4)$. We can rescale fields back and go to Lorentzian signature

$$S_0^{eff,R} = S_0^{eff} + S_{CT} = S_0^{CR} + \frac{N}{L^2}\Big(8(S_{F_1}-S_{B_1})\mathrm{Tr}[A_0^2(0)] \\ -4(S_{F_1}-S_{B_1})\mathrm{Tr}[A_i^2(0)] - \frac{8}{3}(S_{F_1}-S_{B_1})\mathrm{Tr}[A_a^2(0)]\Big)\,.$$

The conclusion is that mixed boundary conditions for fermions in IKKT theory dramatically change the zero-mode effective theory. They cause divergent mass terms to appear in the effective action. These can be renormalised in order to obtain a finite result. As a consequence, a subgroup of $SO(1,3)\times SO(6)$ is broken by the newly generated mass terms.

In the next two sections, we will turn our attention to BFSS theory. We will see that, in this case, the situation is different and, in particular, there is a class of mixed boundary conditions which do not further break $SO(3)\times SO(6)$.

4.4 Compactification of BFSS theory

In this section, we extend the mechanism of torus compactification to the BFSS Matrix Theory. We refer the reader to Chapter 2 for details of the theory. We will consider the cases of periodic and anti-periodic boundary conditions for fermions, and show that the result is equivalent to the corresponding cases in IKKT theory. In the next section, we will extend the discussion to include mixed boundary conditions.

As explained in Section 2.2.7, compactifications on a six-dimensional torus are ill-defined in BFSS theory. We therefore consider compactifications of the complementary

coordinates on a three-dimensional square torus $\mathbb{T}^3$, where each of the dimensions has size $2\pi L$. We expect that, in the decompactification limit, the results should be qualitatively equivalent, still producing different mass terms for the two sets of directions. We will show that this is indeed the case.

It will be convenient to work in Euclidean signature. To this end, we Wick rotate by means of

$$t \to it, \quad A \to iA, \quad \Gamma^I \to i\Gamma^I, \quad \psi \to i\psi\,,$$

and we obtain the Euclidean action

$$S^E_{BFSS} = \int dt \frac{1}{g^2} \mathrm{Tr}\left[\frac{1}{2}(\mathcal{D}_t X^I)^2 - \frac{1}{4}[X^I, X^J]^2 + \bar{\psi}\mathcal{D}_t\psi - i\bar{\psi}\Gamma^I[X_I, \psi]\right]. \tag{4.4.0.1}$$

Note that, in this signature, gamma matrices satisfy

$$\{\Gamma^I, \Gamma^J\} = -2\delta^{IJ}\,,$$

with a minus sign. The next step is to define a translation operator on the torus $\mathbb{T}^3$. This is done analogously to the method of Section 4.2, by defining a Hilbert space acted upon by matrices, which is decomposed into two components, as in equation (4.2.0.5). It follows that the degrees of freedom of the theory are decomposed as

$$\begin{aligned}
A(t) &= \sum_{n^b \in \mathbb{Z}^3} A(t, n^b) \otimes e^{in^b p^b}\,, \\
X^\mu(t) &= \sum_{n^b \in \mathbb{Z}^3} X^\mu(t, n^b) \otimes e^{in^b p^b}\,, \\
X^a(t) &= \sum_{n^b \in \mathbb{Z}^3} X^a(t, n^b) \otimes e^{in^b p^b} + 2\pi L \mathbb{I}_N \otimes q^a\,, \\
\psi(t) &= \sum_{r^b \in \mathbb{L}_3} \psi(t, r^b) \otimes e^{ir^b p^b}\,,
\end{aligned} \tag{4.4.0.2}$$

where we are considering the case of anti-periodic fermions explicitly, and we will comment on the periodic case later. The lattice $\mathbb{L}_3$ is the three-dimensional analogue of (4.2.0.6), namely

$$\mathbb{L}_3 = \left(\mathbb{Z} + \frac{1}{2}\right) \times \left(\mathbb{Z} + \frac{1}{2}\right) \times \left(\mathbb{Z} + \frac{1}{2}\right), \tag{4.4.0.3}$$

where lattice sites represent half-integer winding modes in each direction. Here, indices are $\mu = 4, \ldots, 9$ and $a = 1, 2, 3$, such that indices labelled by Latin letters $a, b, ..$ correspond

to compact directions (note that this is different from previous sections). Note that all the degrees of freedom now have an explicit time dependence. We substitute the decomposition (4.4.0.2) into the Euclidean action (4.4.0.1) and define the mode expansion of the fields

$$\begin{aligned}
A(t,\sigma) &= \sum_{n^b\in\mathbb{Z}^3} A(t,n^b)e^{2\pi i L n^b\sigma^b}\,,\\
X^\mu(t,\sigma) &= \sum_{n^b\in\mathbb{Z}^3} X^\mu(t,n^b)e^{2\pi i L n^b\sigma^b}\,,\\
X^a(t,\sigma) &= \sum_{n^b\in\mathbb{Z}^3} X^a(t,n^b)e^{2\pi i L n^b\sigma^b}\,,\\
\psi(t,\sigma) &= \sum_{r^b\in\mathbb{L}_3} \psi(t,r^b)e^{2\pi i L r^b\sigma^b}\,,\\
c(t,\sigma) &= \sum_{n^b\in\mathbb{Z}^3} c(t,n^b)e^{2\pi i L n^b\sigma^b}\,,
\end{aligned}$$

where $\sigma\in[0,L^{-1}]^3$ is the momentum mode along the torus. We obtain the compact BFSS action

$$\begin{aligned}
S^C_{BFSS} = \frac{1}{g^2_{eff}}\int\frac{d^3\sigma dt}{(2\pi L)^{-3}}\mathrm{Tr}\Big[&\frac{1}{4}F^{AB}F_{AB}+\frac{1}{2}\mathcal{D}_A X_\mu\mathcal{D}^A X^\mu\\
&-\frac{1}{4}[X^\mu,X^\nu]^2+\bar\psi\mathcal{D}_t\psi+\bar\psi\Gamma^a\mathcal{D}_a\psi-i\bar\psi\Gamma^\mu[X_\mu,\psi]\Big]\,, \quad (4.4.0.4)
\end{aligned}$$

where, for convenience, we have defined an index $A=0,1,2,3$, corresponding to the time and space coordinates of the fields. The effective gauge coupling is defined as $g^2_{eff}=\frac{g^2}{M}$, M being the dimension of the Hilbert subspace on which p^a and q^a are defined. We choose to work in Lorentz gauge $\partial_B X^B=0$, where $X^0(t,\sigma)=A(t,\sigma)$, implying that we must add a ghost term to the compact action (4.4.0.4)

$$S_{gh}=\frac{1}{g^2_{eff}}\int\frac{d^3\sigma\,dt}{(2\pi L)^{-3}}\mathrm{Tr}[\partial^A\bar c\mathcal{D}_A c]\,.$$

Furthermore, we Fourier transform the time variable in order to work in energy space, and perform the usual field rescaling, where we also rescale the time variable to make it dimensionless

$$X^\alpha\to\lambda L X^\alpha,\quad \psi\to\lambda L^{3/2}\psi,\quad c\to\lambda L c,\quad t\to\frac{t}{L},\quad \lambda^2=\frac{g^2_{eff}}{L^3}\,, \qquad (4.4.0.5)$$

where we introduced the index $\alpha=0,\dots,9$ and $X^0=A$. As a result, we ob-

tain the kinetic action

$$S_{kin} = \int dp \,\mathrm{Tr}\left[\sum_{n^a\in\mathbb{Z}^3} \frac{1}{2}(p^2 + n_a n^a) X_\alpha(p,n^a) X^\alpha(-p,-n^a) \right.$$
$$\left. + \sum_{s^a\in\mathbb{L}_3} i(p+\Gamma^a s_a)\bar{\psi}(p,s^a)\psi(p,s^a) + \sum_{n^a\in\mathbb{Z}^3}(p^2+n_a n^a)\bar{c}(p,n^a)c(p,n^a)\right], \quad (4.4.0.6)$$

and we observe that non-zero modes are massive. In particular, in the regime $L \gg g_{eff}^{2/3}$, the parameter λ is perturbative, and we can integrate these modes out at one-loop order. Analogously to the discussion around (4.3.1.7), we follow a Wilsonian approach to obtain an effective action for zero modes. The relevant interaction terms are those which couple zero modes to non-zero modes. The purely bosonic ones read

$$V_1 = -\frac{\lambda^2}{4} \tilde{\sum_{n^a,m^a,l^a\in\mathbb{Z}^3}} \int dp\, dq dr \,\mathrm{Tr}\left[[X^\alpha(p,n^a), X^\beta(q,m^a)]\right.$$
$$\left. [X_\alpha(r,l^a), X_\beta(-p-q-r,-n^a-m^a-l^a)]\right],$$

$$V_2 = -\lambda \tilde{\sum_{n^a,m^a\in\mathbb{Z}^3}} \int dp\, dq \mathrm{Tr}\,[(p+q)X_\alpha(-p-q,-n^a-m^a)[A(p,n^a), X^\alpha(q,m^a)]$$
$$+(n^a+m^a)X_\alpha(-p-q,-n^a-m^a)[X_a(p,n^a), X^\alpha(q,m^a)]],$$

where $\tilde{\sum}$ means that, in each term of the sum, one of the winding numbers is set to zero. Interactions involving fermions and ghost fields read

$$V_3 = -i\lambda \sum_{s^a\in\mathbb{L}_3} \int dp\, dq \,\mathrm{Tr}\Big[\bar{\psi}(p+q,s^a)[A(p,0),\psi(q,s^a)]$$
$$+ \bar{\psi}(p+q,s^a)\Gamma^a[X_a(p,0),\psi(q,s^a)]\Big],$$

$$V_4 = -i\lambda \sum_{s^a\in\mathbb{L}_3} \int dp\, dq \,\mathrm{Tr}\Big[\bar{\psi}(p+q,s^a)\Gamma^\mu[X_\mu(p,0),\psi(q,s^a)]\Big],$$

$$V_5 = \lambda \sum_{n^a\in\mathbb{Z}^3} \int dp\, dq \,\mathrm{Tr}\Big[(p+q)\bar{c}(p+q,n^a)[A(p,0),c(q,n^a)]$$
$$+ n^a\bar{c}(p+q,n^a)[X_a(p,0),c(q,n^a)]\Big].$$

We note that, in this case, it is convenient not to decompose spinors, for the sake of simplicity. In terms of the interaction potentials, the zero-mode effective action is given by

$$S_0^{eff} = S_0 - \ln\left\langle e^{-S_{int}}\right\rangle = S_0 + \langle V_1\rangle - \frac{1}{2}\sum_{i=2}^{5}\left\langle V_i^2\right\rangle$$

and expectation values are computed using Wick's theorem, where building blocks are two-point functions derived from the kinetic action (4.4.0.6)

$$\begin{aligned}
\langle X_\alpha(p,n^a)X_\beta(q,m^a)\rangle &= \frac{\delta_{\alpha\beta}\delta(p+q)\delta_{n^a+m^a,0}}{p^2+n^a n_a}\,,\\
\left\langle \bar\psi(p,s^a)\psi(q,r^a)\right\rangle &= -i\frac{\delta(p-q)\delta_{r^a,s^a}(p-\Gamma^a s_a)}{p^2+s^a s_a}\,,\\
\langle \bar c(n^a,p)c(m^a,q)\rangle &= \frac{\delta(p-q)\delta_{n^a,m^a}}{p^2+n^a n_a}\,.
\end{aligned}$$

As a result, the averages read

$$\begin{aligned}
&\langle V_1\rangle = 9\lambda^2 N S_{B_1}\int dp\mathrm{Tr}[X^\alpha(-p,0)X_\alpha(p,0)]\,,\\
&\left\langle V_2^2\right\rangle = 2\lambda^2 N\int dp\,[(17S_{B_3}+S_{B_1})\mathrm{Tr}[A(-p,0)A(p,0)]\\
&+(17S_{B_2}+S_{B_1})\mathrm{Tr}[X^a(-p,0)X_a(p,0)]+S_{B_1}\mathrm{Tr}[X^\mu(-p,0)X_\mu(p,0)]]\,,\\
&\left\langle V_3^2\right\rangle = -16\lambda^2 N\int dp\,[(2S_{F_3}-S_{F_1})\mathrm{Tr}[A(-p,0)A(p,0)]\\
&+(2S_{F_2}-S_{F_1})\mathrm{Tr}[X^a(-p,0)X_a(p,0)]]\,,\\
&\left\langle V_4^2\right\rangle = 16\lambda^2 N\int dp S_{F_1}\mathrm{Tr}[X^\mu(-p,0)X_\mu(p,0)]\,,\\
&\left\langle V_5^2\right\rangle = -2\lambda^2 N\int dp\,[S_{B_3}\mathrm{Tr}[A(-p,0)A(p,0)]+S_{B_2}\mathrm{Tr}[X^a(-p,0)X_a(p,0)]]\,,
\end{aligned}\tag{4.4.0.7}$$

where propagators in the loop give rise to factors

$$\begin{aligned}
S_{B_1} &= \sum\nolimits'_{n^a\in\mathbb{Z}^3}\int_{-\infty}^{\infty}dp\frac{1}{p^2+n^a n_a}\,, & S_{F_1} &= \sum_{r^a\in\mathbb{L}_3}\int_{-\infty}^{\infty}dp\frac{1}{p^2+r^a r_a}\,,\\
S_{B_2} &= \sum\nolimits'_{n^a\in\mathbb{Z}^3}\int_{-\infty}^{\infty}dp\frac{(n^1)^2}{(p^2+n^a n_a)^2}\,, & S_{F_2} &= \sum_{r^a\in\mathbb{L}_3}\int_{-\infty}^{\infty}dp\frac{(r^1)^2}{(p^2+r^a r_a)^2}\,,\\
S_{B_3} &= \sum\nolimits'_{n^a\in\mathbb{Z}^3}\int_{-\infty}^{\infty}dp\frac{p^2}{(p^2+n^a n_a)^2}\,, & S_{F_3} &= \sum_{r^a\in\mathbb{L}_3}\int_{-\infty}^{\infty}dp\frac{p^2}{(p^2+r^a r_a)^2}\,,
\end{aligned}\tag{4.4.0.8}$$

and primed sums run over non-zero modes only. These factors are infinite, but can be regularised, generalising the procedure of [136]. We will elaborate on this in Appendix B.2. We will also show that bosonic and fermionic terms have the same infinite part, which therefore cancels out when considering differences.

We collect all terms (4.4.0.7) together, invert the rescaling (4.4.0.5), and Wick rotate back to Lorentzian signature to obtain, for the zero modes effective action,

$$\begin{aligned}
S_0^{eff} = S_0 - \frac{1}{2}\int dt\,\Big[&M_A^2\mathrm{Tr}[A(t,0)A(t,0)]\\
&-M_{X^\mu}^2\mathrm{Tr}[X_\mu(t,0)X^\mu(t,0)]-M_{X^a}^2\mathrm{Tr}[X_a(t,0)X^a(t,0)]\Big]\,.
\end{aligned}$$

Here, the action S_0 reads

$$S_0 = \frac{1}{g_{eff}^2}\int dt \mathrm{Tr}\left[\frac{1}{2}\mathcal{D}_t X_I(t,0)\mathcal{D}_t X^I(t,0) + \frac{1}{4}[X^I(t,0), X^J(t,0)]^2 + i\partial_t\bar{c}(t,0)\mathcal{D}_t c(t,0)\right] ,$$

where $I, J = 1, \ldots, 9$ and the masses are

$$\begin{aligned} M_A^2 &= 16\frac{N}{L^2}\left(S_{F_1} - S_{B_1} + 2(S_{B_3} - S_{F_3})\right) , \\ M_{X^\mu}^2 &= 16\frac{N}{L^2}(S_{F_1} - S_{B_1}) , \\ M_{X^a}^2 &= 16\frac{N}{L^2}\left(S_{F_1} - S_{B_1} + 2(S_{B_2} - S_{F_2})\right) . \end{aligned}$$

By evaluating the p integrals in (4.4.0.8), we find that

$$S_{B_1} = 2S_{B_3}, \qquad S_{F_1} = 2S_{F_3} .$$

Moreover, these factors are related by the equations

$$S_{B_1} = S_{B_3} + 3S_{B_2}, \qquad S_{F_1} = S_{F_3} + 3S_{F_2} .$$

Altogether, the mass terms then simplify to

$$M_A^2 = 0, \quad M_{X^\mu}^2 = 16\frac{N}{L^2}(S_{F_1} - S_{B_1}), \quad M_{X^a}^2 = \frac{32}{3}\frac{N}{L^2}(S_{F_1} - S_{B_1}) , \tag{4.4.0.9}$$

where the difference $(S_{F_1} - S_{B_1})$ is finite after regularisation, as shown in Appendix B.2. The mass terms (4.4.0.9) show that, upon compactification on $\mathbb{T}^3$, with anti-periodic boundary conditions for fermions, the effective theory has a global symmetry $SO(3) \times SO(6)$, thus breaking the initial $SO(9)$. This extends the result of [136] to BFSS theory, proving that a mass term can be generated by compactification. It also hints at the possibility that the BFSS model enjoys symmetry breaking analogously to what is observed in numerical studies of the IKKT model. Furthermore, the gauge field remains massless in the effective theory. Note that this is not due to a supersymmetric cancellation, which does not occur in the case of anti-periodic boundary conditions, but to the fact that bosonic and fermionic terms cancel out separately due to gauge invariance.

The case of periodic boundary conditions for fermions is obtained straightforwardly by noting that, in that case, $S_{F_1} = S_{B_1}$. It follows that

$$M_A^2 = M_{X^\mu}^2 = M_{X^a}^2 = 0 ,$$

hence no mass term is generated. The zero-mode effective action now reads

$$S_0^{eff} = S_0^P = \int dt \frac{1}{g_{eff}^2} \mathrm{Tr} \left[\frac{1}{2} (\mathcal{D}_t X^I(t,0))^2 + \frac{1}{4} [X^I(t,0), X^J(t,0)]^2 \right.$$
$$\left. + i\bar{\psi}(t,0)\mathcal{D}_t\psi(t,0) - \bar{\psi}(t,0)\Gamma^I[X_I(t,0), \psi(t,0)] + i\partial_t\bar{c}(t,0)\mathcal{D}_t c(t,0) \right] ,$$

which is the same as the original BFSS action with an effective coupling constant.

4.5 BFSS with mixed boundary conditions

In this section, we consider a class of mixed boundary conditions for fermions which give rise to an effective theory with intermediate qualitative features. We will show that we can obtain a zero-mode effective theory where half the fermions are quenched, the remaining half being still present, and a mass term is generated.

In contrast to the IKKT case treated above, we will show that BFSS theory admits a class of mixed boundary conditions which preserves the divergence cancellation at one loop. As a consequence, the effective action is finite and does not require renormalisation. Most importantly, we will show that it features both a symmetry-breaking mass term and fermionic fields.

We consider the decomposition of sixteen-component fermionic degrees of freedom introduced in [145], which we review here. We choose the Γ^I matrices, $I = 1, \ldots, 9$, to be decomposed such that

$$\Gamma^a = \begin{pmatrix} -\sigma^a \otimes \mathbb{I}_4 & 0 \\ 0 & \sigma^a \otimes \mathbb{I}_4 \end{pmatrix}, \qquad \Gamma^\mu = \begin{pmatrix} 0 & \mathbb{I}_2 \otimes \gamma^\mu \\ -\mathbb{I}_2 \otimes \gamma^\mu & 0 \end{pmatrix} ,$$

where $\gamma_\mu = -\gamma_\mu^\dagger$ are anti-Hermitian 4×4 matrices satisfying the Clifford algebra relation

$$\{\gamma^\mu, \gamma^\nu\} = -2\delta^{\mu\nu}\mathbb{I}_4 .$$

The charge conjugation matrix is

$$C_9 = \begin{pmatrix} 0 & -i\sigma_2 \otimes \mathbb{I}_4 \\ i\sigma_2 \otimes \mathbb{I}_4 & 0 \end{pmatrix} ,$$

which allows for spinor decomposition in chiral components[2]

$$\psi = \begin{pmatrix} \lambda^{A\alpha} \\ i(\sigma^2)^{AB}\lambda^*_{B\alpha} \end{pmatrix}, \qquad A = 1, 2 , \quad \alpha = 1, \ldots, 4 .$$

[2]Note that we inverted the notation of [145] for spinor indices, in order to match with our conventions in previous sections.

It follows that the fermionic part of the compact BFSS Lagrangian (4.4.0.4) becomes

$$g^2\mathcal{L}_{ferm} = \mathrm{Tr}\left[2\lambda^\dagger_\alpha \mathcal{D}_t\lambda^\alpha - 2i\lambda^\dagger_\alpha\sigma^a\mathcal{D}_a\lambda^\alpha - i\lambda^\dagger_\alpha i\sigma^2(\gamma^\mu)_{\alpha\beta}[X_\mu,\lambda^*_\beta] - i\lambda^{T,\alpha}i\sigma^2(\gamma^\mu)^{\alpha\beta}[X_\mu,\lambda_\beta]\right], \tag{4.5.0.1}$$

where we only wrote down the α components of spinor indices explicitly.

One notable feature of the Lagrangian (4.5.0.1) is its invariance under

$$\lambda \to \lambda^*, \qquad \lambda^* \to \lambda\,,$$

which can be checked explicitly from the above expression (for the first two terms, transposition gives a minus sign due to the Grassmann nature of variables, and a further minus sign comes from integrating by parts). For this reason, we consider boundary conditions which implement this transformation when wrapping around a closed loop in any of the torus directions, namely

$$(U^a)^{-1}\lambda^\alpha U^a = \lambda^*_\alpha, \qquad (U^a)^{-1}\lambda^*_\alpha U^a = \lambda^\alpha\,.$$

This implies that we can decompose spinors into periodic and anti-periodic components

$$\lambda = \frac{1}{\sqrt{2}}(\lambda^\uparrow + i\lambda^\downarrow)\,, \qquad \lambda^* = \frac{1}{\sqrt{2}}(\lambda^\uparrow - i\lambda^\downarrow)\,,$$

with

$$\lambda^\uparrow(t) = \sum_{n^b\in\mathbb{Z}^3}\lambda^\uparrow(t,n^b)\otimes e^{in^bp^b}\,, \qquad \lambda^\downarrow(t) = \sum_{r^b\in\mathbb{L}_3}\lambda^\downarrow(t,r^b)\otimes e^{ir^bp^b}\,, \tag{4.5.0.2}$$

in terms of the operators p^a introduced in Section 4.2, where $\lambda^\uparrow$ and $\lambda^\downarrow$ are real. Here $\mathbb{L}_3$ is the three-dimensional lattice (4.4.0.3) of half integer modes.

When written in this way, all terms involving both components cancel out, and the Lagrangian is decomposed into two non-interacting pieces, one involving periodic spinors and the other anti-periodic

$$\mathcal{L}_{ferm} = \mathcal{L}^P + \mathcal{L}^{AP}\,. \tag{4.5.0.3}$$

Here

$$S^P = \int\frac{dtd^3\sigma}{(2\pi L)^{-3}}\mathcal{L}^P = \frac{1}{g^2_{eff}}\int\frac{dtd^3\sigma}{(2\pi L)^{-3}}\mathrm{Tr}\left[\lambda^{\uparrow,T}\mathcal{D}_t\lambda^\uparrow - i\lambda^{\uparrow,T}\sigma^a\mathcal{D}_a\lambda^\uparrow - i\lambda^{\uparrow,T}i\sigma^2\gamma^\mu[X_\mu,\lambda^\uparrow]\right],$$

and

$$S^{AP} = \int \frac{dtd^3\sigma}{(2\pi L)^{-3}}\mathcal{L}^{AP} = \frac{1}{g_{eff}^2}\int \frac{dtd^3\sigma}{(2\pi L)^{-3}}\mathrm{Tr}\left[\lambda^{\downarrow,T}\mathcal{D}_t\lambda^{\downarrow} - i\lambda^{\downarrow,T}\sigma^a\mathcal{D}_a\lambda^{\downarrow} + i\lambda^{\downarrow,T}i\sigma^2\gamma^\mu[X_\mu,\lambda^{\downarrow}]\right],$$

where the σ dependence comes from the usual mode expansion

$$\begin{aligned}
A(t,\sigma) &= \sum_{n^b\in\mathbb{Z}^3} A(t,n^b)e^{2\pi iLn^b\sigma^b},\\
X^\mu(t,\sigma) &= \sum_{n^b\in\mathbb{Z}^3} X^\mu(t,n^b)e^{2\pi iLn^b\sigma^b},\\
X^a(t,\sigma) &= \sum_{n^b\in\mathbb{Z}^3} X^a(t,n^b)e^{2\pi iLn^b\sigma^b},\\
\lambda^{\uparrow}(t,\sigma) &= \sum_{r^b\in\mathbb{Z}^3} \lambda^{\uparrow}(t,r^b)e^{2\pi iLr^b\sigma^b},\\
\lambda^{\downarrow}(t,\sigma) &= \sum_{r^b\in\mathbb{L}_3} \lambda^{\downarrow}(t,r^b)e^{2\pi iLr^b\sigma^b}.
\end{aligned}$$

We plug this back into (4.5.0.3), Fourier transform with respect to time, and perform the usual rescaling

$$X^\alpha \to kLX^\alpha, \quad \lambda^b \to kL^{3/2}\lambda^b, \quad c \to kLc, \quad t \to \frac{t}{L}, \quad k^2 = \frac{g_{eff}^2}{L^3},$$

with $\alpha = 0,\dots,9$, $b = \uparrow,\downarrow$. It is then straightforward to obtain fermionic propagators from the kinetic terms, which take the form

$$\begin{aligned}
\left\langle \lambda^{\uparrow}_{A,\alpha}(p,r^a)\lambda^{\uparrow}_{B,\beta}(q,s^a)\right\rangle &= -i\frac{\delta(p-q)\delta_{r^a,s^a}\delta_{\alpha\beta}(\delta_{AB}\,p + i\sigma^a_{AB}r_a)}{p^2 + r_a r^a},\\
\left\langle \lambda^{\downarrow}_{A,\alpha}(p,r^a)\lambda^{\downarrow}_{B,\beta}(q,s^a)\right\rangle &= -i\frac{\delta(p-q)\delta_{r^a,s^a}\delta_{\alpha\beta}(\delta_{AB}\,p + i\sigma^a_{AB}r_a)}{p^2 + r_a r^a}.
\end{aligned}$$

We integrate these out to compute expectation values of non-zero modes. The discussion for bosons is unchanged from Section 4.4. On the other hand, integrating out the fermions yields important physical consequences. The effective theory reads, in Lorentzian signature,

$$\begin{aligned}
S_0^{eff} &= S_0 - \ln\left\langle e^{-S_{int}}\right\rangle\\
&= S_0 + \frac{1}{2}\int dt\left[M^2_{X^\mu}\mathrm{Tr}[X_\mu(t,0)X^\mu(t,0)] + M^2_{X^a}\mathrm{Tr}[X_a(t,0)X^a(t,0)]\right], \quad (4.5.0.4)
\end{aligned}$$

where

$$M^2_{X^\mu} = 8\frac{N}{L^2}(S_{F_1} - S_{B_1}), \qquad M^2_{X^a} = \frac{16}{3}\frac{N}{L^2}(S_{F_1} - S_{B_1}), \qquad (4.5.0.5)$$

and

$$S_0 = \frac{1}{g_{eff}^2}\int dt \mathrm{Tr}\left[\frac{1}{2}\mathcal{D}_t X_I(t,0)\mathcal{D}_t X^I(t,0) + \frac{1}{4}[X^I(t,0), X^J(t,0)]^2 \right.$$
$$+i\lambda^{\uparrow,T}(t,0)\mathcal{D}_t\lambda^{\uparrow}(t,0) - \lambda^{\uparrow,T}(t,0)\sigma^a[X_a(t,0),\lambda^{\uparrow}(t,0)]$$
$$\left. -i\lambda^{\uparrow,T}(t,0)i\sigma^2\gamma^\mu[X_\mu(t,0),\lambda^{\uparrow}(t,0)] + i\partial_t\bar{c}(t,0)\mathcal{D}_t c(t,0)\right] .$$

We see that periodic fermions cancel with half of the bosonic degrees of freedom in loops and have zero modes, which appear in the effective theory. The anti-periodic fermions do not have zero modes but simply cancel the divergent part of bosonic propagators, thus generating a mass in the effective theory. As a result, we have obtained an effective field theory in which fermions appear explicitly. Bosons have symmetry-breaking masses which are exactly half the masses of the anti-periodic case.

In the next section, we will leverage this construction to study black hole solutions in the theory we have obtained.

4.6 Black hole in the zero-mode theory

In this section, we consider possible solutions of the theory (4.5.0.4) with the correct features to represent a black hole. In particular, we show that the construction of [105] can be adapted to our effective theory. There, the approach is bottom-up, and the matrix theory deployed is constructed ad hoc to reproduce black hole solutions. Here, the compactification procedure of BFSS with mixed boundary conditions provides a top-down approach.

4.6.1 Fuzzy sphere background

The key idea is to consider classical solutions to the bosonic part of (4.5.0.4) as the background geometry. We then treat fermionic excitations as quantum fluctuations on top of the geometry. We focus on solutions such that

$$A = 0, \qquad c = 0, \qquad X^\mu = 0\,,$$

which is made possible by the symmetry-breaking pattern $SO(9) \to SO(3) \times SO(6)$ we found in Section 4.5. Furthermore, we consider static solutions with

$$\partial_t X^I = 0\,.$$

From a target space perspective, this amounts to saying that, for those solutions, six spatial coordinates have no extent[3], whereas the remaining three transverse directions are macroscopic. Note that the zero-mode effective action obtained as a consequence of torus compactification of the original Matrix Theory lives in a non-compact target space.

We can write the relevant part of the Hamiltonian obtained from (4.5.0.4) as

$$H = R\,k^4\,\mathrm{Tr}\left[\frac{1}{2}P^iP^i - \frac{1}{4}[X^i,X^j]^2 - X^iX^i + \lambda^{\uparrow,T}\sigma^i[X_i,\lambda^{\uparrow}]\right], \tag{4.6.1.1}$$

where we have changed units to highlight the dependence on the light-like compactification radius R in M-Theory (see [2]) and rescaled the fields according to their scaling dimension

$$X^i \to kX^i, \quad \lambda^{\uparrow}(t,0) \to k^{\frac{3}{2}}\lambda^{\uparrow}(t,0), \quad P^i \to k^2P^i, \quad k^2 = \frac{\sqrt{R}\,M_{X^i}^2}{2}. \tag{4.6.1.2}$$

The classical equations of motion for the bosonic sector, for a static solution, read

$$\left[X^i,[X^i,X^j]\right] = 2X^j, \qquad P^i = \partial_t X^i = 0, \tag{4.6.1.3}$$

where $i = 1,2,3$. These equations are solved by the $s = \frac{N-1}{2}$ representation of $SU(2)$, which defines a fuzzy sphere through

$$[X^i,X^j] = i\epsilon^{ijk}X^k, \tag{4.6.1.4}$$

along with the Casimir relation

$$\sum_i X^{i\,2} = \frac{N^2-1}{4}\mathbb{I}_N.$$

Fuzzy spheres have been considered in Matrix Theories in relation to the Myers dielectric effect [102]. For example, in BMN theory [99], they arise due to the presence of a membrane flux, which polarises the $D0$ branes into an extended configuration. This requires working in the pp-wave background described in Section 2.4, where the fuzzy sphere describes an emergent $M2$ brane in the large-N limit. In the case considered here, the fuzzy sphere is stabilised by the mass deformation obtained through the compactification procedure, and in particular it lives in the flat space background of BFSS theory.

Let us now consider the fermionic term

$$\mathcal{H}_{ferm} = R\,k^4\mathrm{Tr}\left[\lambda^{\uparrow,\dagger}_{\alpha,a}(t,0)\sigma^j_{ab}[X_j(t,0),\lambda^{\uparrow,\alpha}_b(t,0)]\right], \tag{4.6.1.5}$$

[3]This is true for the semiclassical background, whereas quantum fluctuations are Planck-sized.

where $\lambda^{\uparrow,\dagger} = \lambda^{\uparrow,T}$ because it is a real spinor, and from now on we drop the $\uparrow$ superscript for convenience. We recall from Section 4.5 that spinors carry indices $a = 1, 2$ and $\alpha = 1, \ldots, 4$, which we have made explicit for clarity. On the fuzzy sphere background, this can be arranged as a collection of fermion harmonic oscillators, given by

$$\mathcal{H}_{ferm} = R\,k^4 \sum_{n=1}^{N-1} \left(\sum_{r=1}^{2n+2} \sum_{\alpha=1}^{4} \lambda_{+,n} \xi^{r,n,\alpha\dagger} \xi_\alpha^{r,n} + \sum_{r=1}^{2n} \sum_{\alpha=1}^{4} |\lambda_{-,n}| \chi^{r,n,\alpha\dagger} \chi_\alpha^{r,n} - 8n(n+1) \right) . \tag{4.6.1.6}$$

Here $\lambda_{+,n}$, $\lambda_{-,n}$ are the eigenvalues of the operator

$$\mathrm{X} := \sigma^j [X_j, \cdot] \,,$$

given by

$$\lambda_{\pm,n} = -\frac{1}{2} \pm \left(n + \frac{1}{2} \right) ,$$

and $\xi_\alpha^{r,n}$, $\chi_\alpha^{r,n}$ are fermionic annihilation operators, obtained by projecting the matrices $\lambda^\uparrow$ on the corresponding eigenvectors. We prove this result in Subsection 4.6.3.

We recognise (4.6.1.6) as the Hamiltonian of a set of fermionic oscillators with increasing frequencies, given by $\omega_n^\pm = R\,k^4 |\lambda_{\pm,n}|$. Since we are working at zero temperature, a state in this system is constructed by progressively filling up the Fermi sea, starting from the lowest energy shell and occupying shells of increasing energies as we increase the number of excited oscillators. A state of given energy is thus constructed by filling up some energy shells completely and partially filling up the highest contributing energy shell. The degeneracy Ω of such a state is then given by the number of inequivalent ways to partially fill this shell. In principle, we should also include bosonic quantum fluctuations. We will analyse them in Section 4.6.4 and show that they do not contribute to our construction.

4.6.2 Black hole solution

We want to interpret the geometry described above, along with a non-trivial fermionic state, as a black hole in the target space. As explained in [73, 141, 142] and reviewed in Section 2.3.3, any black hole solution in Matrix Theory needs to fit into the longitudinal direction. Our interpretation of the above fuzzy sphere geometry is that it represents the $x^- = 0$ transverse slice of the black hole, which is centred on $x^- = 0$ in the fundamental region along the compact light-like direction. Furthermore, we impose that the solution

is spherically symmetric in the target space, including the longitudinal direction. This requires a boost, which expands the longitudinal dimension at the horizon to fit the black hole, such that the momentum in the longitudinal direction is non-zero. This momentum is quantised in terms of the light-like radius R

$$P^+ = \frac{N}{R}\,.$$

The net effect is a solution living in five total dimensions, one of which is longitudinal and has size R before the boost. As explained above, the symmetry-breaking pattern $SO(9) \to SO(3) \times SO(6)$ allows for solutions which only extend in three space directions. We consider the other space directions to have a Planck-sized extent. In the target space description, they are integrated over and produce a volume factor $V_{6d} = \ell_p^6$, ℓ_p being the Planck length in eleven dimensions. As a consequence, starting from the eleven-dimensional Newton constant $G_{11} = \ell_p^9$, we get the five-dimensional one through

$$G_5 = \frac{G_{11}}{V_{6d}} = \ell_p^3\,.$$

In the boosted frame, where the system has a non-trivial P^+, the Hamiltonian gives the light-cone energy. This is related to the invariant mass through

$$H = E_{LC} = \frac{R}{2N} M_{5D}^2\,,$$

where M_{5D} is the mass of the Schwarzschild black hole in five dimensions[4], which is related to the Schwarzschild radius, r_0, by

$$M_{5D} = r_0^2\,.$$

The bosonic part of the Hamiltonian yields, when evaluated on the fuzzy sphere background,

$$H_{bos} = -\frac{R\,k^2}{2} N R_f^2\,,$$

where $R_f = \frac{k\,N}{2}$ is the physical fuzzy sphere radius in the large-N limit. This must be added to the fermionic energy.

In order to have a black hole solution, the fuzzy sphere must have a size smaller than the Schwarzschild radius. We show in Appendix B.3 that this is achieved by

[4]Recall that we are working in units such that the Planck length is one.

choosing a specific class of fermionic states. This is obtained by filling the shell of energy $E = |\lambda_{-,N-2}| = \lambda_{+,N-1}$ comprising both branches, in a way that excites $j = y(16N - 16)$ oscillators, $0 < y < 1$. As shown in Appendix B.3, the fermionic energy can be computed, and the total light-cone energy, at leading order in N, reads

$$E_{LC} = \frac{61R\,k^2}{6} N R_f^2\,,$$

such that the condition that the fuzzy sphere sits behind its Schwarzschild radius,

$$r_0 \geq R_f\,,$$

is satisfied.

It is worth noting that the fermionic contribution to the energy is of the same parametric order as the classical bosonic contribution. The highly excited fermionic sector is expected to backreact non-trivially on the emergent geometry. A fully self-consistent treatment of this backreaction may modify the simple picture of the configuration as a thin fuzzy shell and could potentially lead to an interpretation in terms of an extended interior lying behind the Schwarzschild radius. Determining the geometry associated with this backreaction would be an interesting direction for future work.

Since, given the fuzzy sphere background (4.6.1.4), all the degeneracy is captured by the fermionic sector of the theory, we can compute the entropy from such degeneracy. We are free to choose the fraction y of excited fermions in the highest shell. Let us consider the half-filled case $y = \frac{1}{2}$. Then

$$S = \log\Omega = \log\binom{16N-16}{y(16N-16)} \approx \gamma(N-1) + O(\log N), \qquad \gamma \approx 11.1\,,$$

scales linearly with N in the large-N limit. The coefficient γ comes from Stirling's approximation

$$\log\binom{16N-16}{y(16N-16)} \approx 16(N-1)H(y) - \frac{1}{2}\log\Big(32\pi(N-1)y(1-y)\Big)$$

with

$$H(y) = \left[-y\log y - (1-y)\log(1-y)\right]\,.$$

The appropriate entropy-radius relation for this configuration, namely

$$S \propto r_0^3\,,$$

is only satisfied if $r_0 \sim N^{\frac{1}{3}}$, which requires $k \propto N^{-\frac{2}{3}}$. Note that the scaling of k follows from the mass term, which is generated through compactification, see equation (4.6.1.2). This, in turn, is related to the torus dimension L (eq. (4.5.0.5)). It follows that the compactification limit and the large-N limit should be taken in a double-scaled fashion, such that

$$L \propto N^{\frac{7}{6}}, \qquad N \to \infty \,. \tag{4.6.2.1}$$

Note that, under this condition, we recover the expected logarithmic form of the entropy correction $S_{log} \propto \log r_0^3$.

The scaling of $S \propto N$ is the one expected for the transition between a black hole localised in the longitudinal direction and a black string stretched along it [73]. Furthermore, if the scaling is $S \propto N^\beta$, $\beta \uparrow 1$, we are approaching the transition, corresponding to the Gregory–Laflamme instability from the black hole state to the black string. In our model, we can construct solutions with lower entropy by choosing a larger fraction of fermionic oscillators to be excited in the largest shell of the Fermi sea $y > \frac{1}{2}$. This, in turn, causes the Schwarzschild radius to increase compared to the fuzzy sphere radius.

We can, in particular, realise a black hole configuration with entropy $S \propto N^\alpha$, $0 < \alpha < 1$ by exciting a fraction y of oscillators in the leading shell, such that y tends to 1 parametrically with N. In this case, the degeneracy given by Stirling's formula for large N is

$$\Omega \sim \frac{e^{16NH(y)}}{\sqrt{2\pi 16Ny(1-y)}}, \quad H(y) = -y\log y - (1-y)\log(1-y), \quad 0 < y < 1 \,.$$

Choosing

$$y = 1 - \frac{cN^{\alpha-1}}{\log N}, \quad 0 < \alpha < 1 \,,$$

c being an arbitrary constant of order 1, we obtain

$$S = \log\Omega = 16NH(y) - \frac{1}{2}\log\left(32\pi Ny(1-y)\right), \quad H(y) \sim (1-\alpha)cN^{\alpha-1} \,,$$

at leading order in N. Again, applying the entropy-radius relation and the relation between the Schwarzschild radius and the generated mass, we obtain the corresponding scaling of k as

$$k \propto N^{\frac{\alpha}{3}-1} \Rightarrow L \propto N^{\frac{3}{2}-\frac{\alpha}{3}} \,.$$

Our approach, therefore, shows that it is possible to construct a black hole solution as a fuzzy sphere background plus fermions in a state that generates the black hole entropy. We saw that, by taking the decompactification limit in the original theory in a specific way, see equation (4.6.2.1), we can approach the limiting case of the black hole/black string transition, represented by the entropy scaling like $S \sim N$. Moreover, we recover the known relations between physical quantities characterising a black hole.

The fuzzy sphere solution we found in this section is still valid even for different values of L and, therefore, of the mass parameter M_X, which do not satisfy the Bekenstein-Hawking formula for the entropy. These solutions cannot consistently represent black holes, which can be seen by looking at the boost along the compact dimension. We can show that the condition on the entropy to be smaller than the Bekenstein-Hawking formula puts a constraint on the boost parameter. Consider the case where the entropy is maximum $S \sim N$, we have

$$P^+ = \frac{N}{R} = \frac{S}{R}\,.$$

Such momentum is obtained through a longitudinal boost, in terms of which it reads [73]

$$P^+ = r_0^2\, e^{\rho}\,,$$

where ρ is the boost parameter. If the entropy does not satisfy the Bekenstein-Hawking formula, we have

$$S < r_0^3 \Longrightarrow e^{\rho} < \frac{r_0}{R}\,,$$

for a large boost. This implies that the longitudinal direction after the boost is smaller than the Schwarzschild radius, and the localised black hole interpretation breaks down.

More generally, in the case of $S \propto N^{\alpha}$, configurations that give $S < r_0^3$ correspond to solutions which are not consistent with the black hole interpretation. Therefore, we can only interpret our fuzzy sphere plus fermions construction as a black hole if the torus radius scales with N appropriately and the Bekenstein-Hawking formula is satisfied.

4.6.3 Fermionic theory on the fuzzy sphere background

In this section, we prove that the fermionic Hamiltonian consists of a collection of harmonic oscillators when evaluated on the fuzzy sphere background. We consider the operator

$$\mathrm{X} := \sigma_i[X^i, \cdot], \qquad \mathrm{X} : \mathbb{C}^2 \otimes \mathrm{End}_0(\mathcal{V}_s) \to \mathbb{C}^2 \otimes \mathrm{End}_0(\mathcal{V}_s)\,,$$

where the commutator $[X^i, \cdot]$ acts on the space of $N \times N$ matrices as traceless endomorphisms of the spin $s = \frac{N-1}{2}$ representation $\mathcal{V}_s$. The $\mathbb{C}^2$ factor is the space of two-dimensional complex spinors acted on by σ^i. Altogether, X acts on a $2(N^2-1)$ -dimensional complex space. We call this an adjoint-like representation, constructed from the spin s representation of $SU(2)$.

Let us now consider the square operator

$$\mathrm{X}^2 = \sigma_i\sigma_j \left[X^i, [X^j, \cdot]\right]\,,$$

which, as a consequence of (4.6.1.4), satisfies

$$\mathrm{X}^2 + \mathrm{X} - \left[X^i, [X^i, \cdot]\right] = 0\,, \tag{4.6.3.1}$$

where the operator

$$\Delta := \left[X^i, [X^i, \cdot]\right]$$

is the Casimir operator of the adjoint-like representation and defines a generalisation of the Laplacian to the non-commutative fuzzy sphere geometry. Indeed, this acts as the identity on the $\mathbb{C}^2$ component. We can decompose

$$\mathrm{End}(\mathcal{V}_s) \cong \mathcal{V}_s \otimes \mathcal{V}_s^* \cong \mathcal{V}_s \otimes \mathcal{V}_s\,,$$

where the second isomorphism comes from the fact that $SU(2)$ representations are self-dual. It follows that we can carry out a Clebsch–Gordan decomposition

$$\mathrm{End}_0(\mathcal{V}_s) \cong \bigoplus_{k=1}^{2s} \mathcal{V}_k\,, \tag{4.6.3.2}$$

such that the Casimir operator acts on each factor as

$$\Delta|_{\mathcal{V}_k} = k(k+1)\mathbb{I}_{2k+1}\,, \tag{4.6.3.3}$$

where the dimension of each Clebsch–Gordan factor is $dim(\mathcal{V}_k) = 2k + 1$.

Given an eigenvector u_λ of X, equation (4.6.3.1) reads

$$\lambda^2 u_\lambda + \lambda u_\lambda - \Delta u_\lambda = 0\,.$$

We decompose u_λ consistently with (4.6.3.2)

$$u_\lambda = v_1 + \ldots + v_{2s}\,,$$

and use (4.6.3.3) (note that v_0 is not present because of the tracelessness condition). As a consequence of linear independence, we get

$$\Big(\lambda^2 + \lambda - k(k+1)\Big)\, v_k = 0, \quad k = 1, \ldots, 2s\,.$$

We see that, for each k, u_λ can be decomposed into $v_{k\pm} \in \mathcal{V}_{k\pm\frac{1}{2}}$, lying respectively in spin $k \pm \frac{1}{2}$ representations of $SU(2)$, and having eigenvalues[5]

$$\lambda_{\pm,k} = -\frac{1}{2} \pm \left(k + \frac{1}{2}\right)\,.$$

It follows that the spectral decomposition of $\mathrm{X}|_{\mathcal{V}_n}$ in terms of eigenvalues and eigenvectors is given by

$$\mathrm{X}|_{\mathcal{V}_n} = \sum_{r=1}^{2n+2} \lambda_{+,n} U^{r,n\dagger}_{a\alpha\gamma} U^{r,n}_{b\beta\sigma} + \sum_{r=1}^{2n} \lambda_{-,n} V^{r,n\dagger}_{a\alpha\gamma} V^{r,n}_{b\beta\sigma}\,,$$

where $U^{r,n}$ and $V^{r,n}$ are eigenvectors corresponding to positive and negative eigenvalues respectively, obtained by representing $\mathcal{V}_n$ in the space of $N \times N$ matrices, such that $a, b = 1, 2$ are spinor indices and $\alpha, \beta, \gamma, \sigma = 1, \ldots, N$ are matrix indices. From (4.6.3.2), the full operator is then

$$\mathrm{X} = \sum_{n=1}^{N-1} \mathrm{X}|_{\mathcal{V}_n} = \sum_{n=1}^{N-1} \left(\sum_{r=1}^{2n+2} \lambda_{+,n} U^{r,n\dagger}_{a\alpha\gamma} U^{r,n}_{b\beta\sigma} + \sum_{r=1}^{2n} \lambda_{-,n} V^{r,n\dagger}_{a\alpha\gamma} V^{r,n}_{b\beta\sigma} \right)\,.$$

[5]Note that the total dimension of this space is indeed

$$\sum_{k=1}^{N-1} 4k + 2 = 2(N^2 - 1)\,,$$

and that

$$\mathrm{Tr}(\mathrm{X}|_{\mathcal{V}_k}) = (2k+2)k + 2k(-k-1) = 0\,,$$

as is required by the traceless structure of X.

Having obtained a spectral decomposition for the operator, we substitute it back into the fermionic Lagrangian (4.6.1.5). This allows us to define fermionic oscillators

$$\xi_\alpha^{r,n} := U_{b\beta\sigma}^{r,n}\lambda_{b\beta\sigma,\alpha}^{\uparrow}, \qquad \chi_\alpha^{r,n\dagger} := V_{b\beta\sigma}^{r,n}\lambda_{b\beta\sigma,\alpha}^{\uparrow}\,,$$

where the indices take respective values $b = 1, 2$ for $SO(3)$ spinor indices, $\beta, \sigma = 1, \ldots, N$ for matrix $SU(N)$ indices, and $\alpha = 1, \ldots, 4$ for $SO(6)$ spinor indices. These oscillators satisfy canonical anti-commutation relations

$$\{\xi_\alpha^{r,n}, \xi^{q,m,\gamma\dagger}\} = \delta^{rq}\delta^{n,m}\delta_\alpha^\gamma, \qquad \{\chi_\alpha^{r,n}, \chi^{q,m,\gamma\dagger}\} = \delta^{rq}\delta^{nm}\delta_\alpha^\gamma.$$

As a consequence, the fermionic Hamiltonian reads

$$\mathcal{H}_{ferm} = R\,k^4 \sum_{n=1}^{N-1}\left(\sum_{r=1}^{2n+2}\sum_{\alpha=1}^{4}\lambda_{+,n}\xi^{r,n,\alpha\dagger}\xi_\alpha^{r,n} + \sum_{r=1}^{2n}\sum_{\alpha=1}^{4}|\lambda_{-,n}|\chi^{r,n,\alpha\dagger}\chi_\alpha^{r,n} - 8n(n+1)\right),$$

which proves the claim.

4.6.4 Bosonic fluctuations

In this section, we show that bosonic fluctuations do not affect our black hole construction. First, we analyse semiclassical fluctuations of the background geometry. We will show that the solution is stable under those perturbations. Then, we consider quantum bosonic fluctuations and show that they do not contribute significantly to the black hole solution, in the large-N limit.

Classical bosonic fluctuations

The analysis of semiclassical fluctuations of the fuzzy sphere can be adapted from [105]. The idea is to consider $X^i + \delta X^i$, where X^i define the fuzzy sphere geometry (4.6.1.4) and δX^i are perturbations. Then, at quadratic order in the perturbations, the Lagrangian for fluctuations can be read off from (4.6.1.1)

$$\mathcal{L}_{\delta X^i} = R\,k^4\,\mathrm{Tr}\left[\frac{1}{2}\delta\dot{X}^i\delta\dot{X}^i + \frac{1}{4}\left([X^i,\delta X^j] - [X^j,\delta X^i]\right)^2 + \frac{1}{2}[X^i,X^j][\delta X^i,\delta X^j] + \delta X^i\delta X^i\right].$$

We can define the derivative operator

$$L^i := [X^i, \cdot]\,,$$

which, following the argument of Section 4.6.3, is an angular momentum operator. It therefore has an eigenbasis $|\ell, m_z\rangle$ with

$$L^2|\ell, m_z\rangle = \ell(\ell+1)|\ell, m_z\rangle, \quad L_3|\ell, m_z\rangle = m_z|\ell, m_z\rangle\,, \quad 0 \leq \ell \leq N-1, \quad -\ell \leq m_z \leq \ell\,.$$

The angular momentum ℓ is truncated due to the fuzziness of the sphere. Moreover, we can define $(\varepsilon_i)_{jk} := -i\epsilon_{ijk}$, which is an angular momentum with $\ell_\varepsilon = 1$, since

$$[\varepsilon_i, \varepsilon_j] = i\epsilon_{ijk}\,\varepsilon_k, \qquad \varepsilon^i\varepsilon^i = 2\,.$$

As shown in [105], the Lagrangian in terms of these operators reads

$$\mathcal{L}_{\delta X^i} = R\,k^4\,\mathrm{Tr}\left[\frac{1}{2}\delta\dot{X}^i\delta\dot{X}^i - \frac{1}{2}\delta X^i N_{ij}\delta X^j\right], \tag{4.6.4.1}$$

where

$$N_{ij} = \Big((\varepsilon\cdot L)^2 - (\varepsilon\cdot L) - 2\Big)\,.$$

It follows from the equation of motion

$$\delta\ddot{X}_i + N_{ij}\delta X_j = 0$$

that the problem of the stability of the various fluctuation modes becomes the analysis of the eigenvalues of N_{ij}. In particular, positive eigenvalues correspond to stable modes, whereas negative eigenvalues signal unstable perturbations. Note that

$$\varepsilon\cdot L = \frac{1}{2}\left(J(J+1) - \ell(\ell+1) - 2\right)\,,$$

where J corresponds to the total angular momentum $J^i = L^i + \varepsilon^i$. Then the eigenvalue problem

$$N_{ij}\phi^j = \lambda\phi^i$$

yields eigenvalues

$$\lambda = \begin{cases} -2 & \text{for } \ell = 0, J = 1 \text{ and } \ell = 1, J = 2 \\ 0 & \text{for } \ell \geq 1, J = \ell \text{ and } \ell = 2, J = 3 \\ \ell(\ell+3) & \text{for } \ell \geq 1, J = \ell - 1 \\ (\ell+1)(\ell-2) & \text{for } \ell \geq 3, J = \ell + 1 \end{cases} \tag{4.6.4.2}$$

and eigenfunctions given by the Clebsch–Gordan composition of the polarisation vector ε^p and spherical harmonics $\hat{Y}^\ell_{m_z}$

$$\phi^i = \sum_{m_z=-\ell}^{\ell}\sum_{p=0,\pm1} C^{J,M_Z}_{\ell,m_z;1,p}\varepsilon^p_i\hat{Y}^\ell_{m_z}\,.$$

Modes with $\ell \geq 1, J = \ell - 1$ and $\ell \geq 3, J = \ell + 1$ have positive eigenvalues, and therefore represent stable fluctuations. Two modes have negative eigenvalues; the mode $\ell = 0, J = 1$, which corresponds to an overall $U(1)$ shift, is absent because the X^i are traceless. The mode $\ell = 1, J = 2$ describes fluctuations with

$$\delta X^i = S^i_j X^j$$

called squashing modes, where S^i_j is a symmetric traceless tensor and X^i are the fuzzy sphere background matrices. We show in Appendix B.4 that the squashing modes are rendered stable by the corresponding fluctuations in the fermionic sector. In particular, we compute the potential, quadratic in the fluctuation, generated by the perturbation in the fermionic sector, in the state given by the Fermi sea with partially filled highest shell. We show that the sum of the bosonic and fermionic contributions gives a positive potential, corresponding to a stable perturbation.

Regarding zero modes, those with $J = \ell$ represent gauge transformations, which are not physical perturbations. This can be seen by first counting the number of such modes

$$\#_{zero} = \sum_{\ell=1}^{N-1} (2\ell + 1) = N^2 - 1\,,$$

which matches the number of independent generators. Moreover, an infinitesimal gauge transformation takes the form

$$\delta X^i = i[X^i, \Lambda] = iL^i\Lambda\,,$$

which is an eigenfunction of $\epsilon \cdot L$ with eigenvalue -1. We can check this explicitly

$$(\epsilon \cdot L)_{kl} L^l \Lambda = -i\epsilon_{jkl} L^j L^l = \frac{1}{2}\epsilon_{jkl}\epsilon_{jlm} L^m = -L^k\,,$$

which proves the statement. The remaining zero modes, with $\ell = 2, J = 3$, represent marginal fluctuations. However, the quartic potential can be evaluated, and it provides a positive contribution, which makes the perturbations marginally stable.

A similar analysis can be done for fluctuations in the directions orthogonal to the fuzzy sphere, to check that they also remain stable. The background geometry in this case is given by $X^\mu = 0$, $\mu = 4, \ldots, 9$, and we are interested in perturbations δX^μ. The perturbation Lagrangian at quadratic order takes a much simpler form

$$\mathcal{L}_{\delta X^\mu} = Rk^4 \mathrm{Tr}\left[\frac{1}{2}\delta\dot{X}^\mu\delta\dot{X}^\mu + \frac{1}{2}[X^i, \delta X^\mu][X^i, \delta X^\mu] + \frac{3}{2}\delta X^\mu \delta X^\mu\right]\,.$$

The potential can be further simplified by noting that

$$\mathrm{Tr}\left[[X^i,\delta X^\mu][X^i,\delta X^\mu]\right] = -\mathrm{Tr}\left[\delta X^\mu\Big[X^i,[X^i,\delta X^\mu]\Big]\right] = -\mathrm{Tr}\left[\delta X^\mu L^2\delta X^\mu\right],$$

which results in

$$\mathcal{L}_{\delta X^\mu} = Rk^4\mathrm{Tr}\left[\frac{1}{2}\delta\dot{X}^\mu\delta\dot{X}^\mu - \frac{1}{2}\delta X^\mu(L^2-3)\delta X^\mu\right]. \tag{4.6.4.3}$$

As a consequence, the equations of motion have the form

$$\delta\ddot{X}^\mu + \Big(L^2-3\Big)\,\delta X^\mu = 0\,,$$

and the problem of the stability of fluctuations reduces to the positivity of the eigenvalues of the operator (L^2-3). The eigenvalue problem

$$\Big(L^2-3\Big)\,\phi = \mu\,\phi \tag{4.6.4.4}$$

is solved by the spherical harmonics with eigenvalues determined by quantum numbers $0 \le \ell \le N-1$, $-\ell \le m_z \le \ell$, such that

$$\phi = \hat{Y}^\ell_{m_z}, \quad \mu = \ell(\ell+1)-3\,. \tag{4.6.4.5}$$

This implies that almost all the eigenvalues are positive, and therefore correspond to stable perturbations. The negative eigenvalues are: the $\ell=0$ mode, which represents a $U(1)$ shift, which is forbidden for traceless matrices; the $\ell=1$ modes, which, similarly to the case of $\ell=1, J=2$ fluctuations above, correspond to squashing modes

$$\delta X^{i+3} = A^i_j X^j\,,$$

where A^i_j is the 3×3, anti-symmetric matrix parametrising the fluctuation. Here we have chosen a parametrisation where the fluctuation involves three directions. The instability is fixed by including perturbations in the fermionic sector, analogously to the case above for perturbations along the fuzzy sphere directions, as shown in Appendix B.4.

Note that the stability of fluctuations strongly depends on the ratio between the two types of masses (4.5.0.5) associated with bosonic matrices, which is a consequence of the top-down derivation from BFSS through torus compactification. In fact, the -3 contribution appearing in (4.6.4.4) depends on the ratio directly. More negative values would lead to unstable fluctuations. The fuzzy sphere background is, therefore, stable against classical bosonic fluctuations.

Quantum bosonic fluctuations

The quadratic potentials in (4.6.4.1) and (4.6.4.3) for bosonic fluctuations can be quantised as simple harmonic oscillators. Since those terms represent fluctuations on top of the fuzzy sphere background, we subtract the zero-point energy from the quantised Hamiltonian, such that the energy of the bosonic ground state is that of the fuzzy sphere. We decompose the fluctuations δX^i, δX^μ into eigenmodes of the respective quadratic potential, and quantise every eigenmode independently. The quantum Hamiltonian for the fluctuations reads

$$H_{\delta X} = Rk^4 \sum_{\ell=1}^{N-1} \sum_{J=\ell-1,\ell+1} \sum_{m_z=-J}^{J} \sqrt{\lambda_{\ell,J}}\, a^\dagger_{\ell,J,m_z} a_{\ell,J,m_z} + Rk^4 \sum_{\ell=1}^{N-1} \sum_{m_z=-\ell}^{\ell} \sqrt{\mu_\ell}\, b^\dagger_{\ell,m_z} b_{\ell,m_z}\,.$$

Here, $\lambda_{\ell,J}$ and μ_ℓ are the eigenvalues of the classical bosonic modes (4.6.4.2), (4.6.4.5) and a_{ℓ,J,m_z}, b_{ℓ,m_z} are the annihilation operators for each mode. In the above expression, the modes stabilised by the fermionic sector, as explained in the previous section, are understood to have real frequencies. Matrix Theory is at zero temperature, which implies that bosonic fluctuations are frozen in the ground state and do not contribute to the energy and entropy of the black hole solution we constructed. We conclude that bosonic fluctuations do not affect our solution, which is composed of a fuzzy sphere background with fermionic excitations which partially fill up the Fermi sea, and bosonic harmonics in the overall ground state. It would be interesting to study the effect of thermal fluctuations in the case where $T \neq 0$. We leave this to future work.

4.7 Conclusion

In this chapter, we analysed torus compactifications of IKKT and BFSS theories as a mechanism for the generation of a mass term. Building on recent results, we imposed a novel set of boundary conditions on fermionic degrees of freedom in IKKT theory. These amount to acting with a non-trivial operator on the spinors when winding around any of the torus dimensions. In the case of IKKT, we analysed two kinds of mixed boundary conditions. In both cases, the action on the spinor is not a symmetry of the compact theory. As a consequence, some fermionic degrees of freedom decouple and do not contribute to loop diagrams, thus failing to cancel bosonic divergences. Hence, the

theory requires some *ad hoc* renormalisation. In the decompactification limit, non-zero modes become heavy and can be integrated out, thus obtaining an effective theory for zero modes. Such a theory breaks the original $SO(1,9)$ symmetry to a subgroup of $SO(1,3) \times SO(6)$, where the $SO(1,3)$ itself is broken to a subgroup.

We then analysed BFSS theory, firstly with anti-periodic boundary conditions, and showed that the resulting effective theory generates a mass term and all fermions are quenched. Then, we considered mixed boundary conditions. In this case, we were able to act on spinors with a symmetry operator when winding around the torus. As a consequence, divergences cancel automatically, and no renormalisation is required. The result is an effective theory with broken $SO(9) \to SO(3) \times SO(6)$, a mass term, and half the fermionic degrees of freedom compared to the initial theory.

In the last part of the chapter, we took this top-down construction as a starting point for explicitly characterising black hole solutions using Matrix Theory degrees of freedom. We found a configuration where a black hole spacetime is described by a fuzzy two-sphere, while fermionic excitations and their degeneracies give rise to the black hole entropy. Moreover, we showed how classical bosonic perturbations are stable and do not disrupt the fuzzy sphere configuration. Their quantum counterparts are frozen in the ground state and do not contribute to the solution.

This construction provides a direct realisation of a black hole in Matrix Theory, with features that are consistent with previous studies of Schwarzschild black holes in the M-Theory region of BFSS theory. In particular, we reproduce the $S \leq c\,N$ scaling of the entropy, $c = O(1)$ being a constant. The solution is made compatible with the entropy-to-radius relation, including logarithmic corrections, by requiring the torus dimension L to go to infinity as a power of the matrix size N, in the decompactification limit.

It would be interesting to investigate the saturated value of the entropy further, in order to assess whether the black hole/black string transition can be captured by this construction. In particular, a possible future direction would be the explicit construction of black string solutions from matrix degrees of freedom.

Fuzzy sphere solutions have been considered in IKKT in connection with the emergence of gravity from the matrix model, see e.g. [94, 97, 146]. As an interesting future direction, one could try to replicate the analysis we carried out in the context of BFSS theory to the

zero-mode effective theory obtained through torus compactification with mixed fermionic boundary conditions of IKKT. The main difference to the polarised IKKT model [94] is that the mass deformation derived through compactification breaks supersymmetry of the compact action. Further directions of investigation are discussed in Chapter 5.

5 Conclusions

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This thesis has investigated two different aspects of emergent spacetime from Matrix Theories: the symmetry structure and its black hole solutions. Central to the investigation is the use of dualities, which provide powerful tools for translating the rich physical structure of these theories into concrete results in Quantum Gravity. The main message is that the quantum dynamics of a large number of degrees of freedom can provide a concrete window into M-Theory within the framework of the BFSS conjecture.

Soft Theorems and the symmetries of BFSS theory. In the first part of the thesis, we built on the recent result in Ref. [103] that the soft supergraviton theorem in BFSS theory is equivalent to the conservation of charges associated to asymptotic symmetries. These constitute an infinite-dimensional symmetry group of the S-Matrix, which includes and extends the eleven-dimensional Poincaré symmetry. Hence, proving a soft theorem in BFSS is equivalent to resolving the longstanding problem of how the full Poincaré group manifests itself in the theory. We worked in the large-distance effective theory, which describes the interaction between two gravitons, and constructed vertex-like operators with the right quantum numbers to represent gravitons in the target space. We then considered the correlation functions where a soft operator is inserted and showed

that this factorises into a universal term times the amplitude without the soft particle. The universal term has the expected form, thus proving the validity of a soft theorem in this setting. The calculation highlights the way asymptotic symmetries are encoded in the degrees of freedom of the theory: the leading soft theorem, corresponding to the conservation of super-translation charges, is obtained from the leading order graviton interaction, depending on the gravity stress tensor; the subleading factorisation, which is equivalent to the presence of super-rotation symmetries, is derived from graviton interactions generated by the first moment of the stress tensor.

Compactification of the target space and black holes. The second part of this thesis was focused on non-supersymmetric deformations of IKKT and BFSS theories, which include a mass term. In particular, we adapted the mechanism proposed in [136] for IKKT, which consists of a torus compactification with anti-periodic boundary conditions in the fermionic sector. First, we reproduced this construction in the context of BFSS Matrix Theory and showed that a mass term is generated consistently. Then, we proposed a new class of mixed boundary conditions for fermions, which resulted in a theory with intermediate features with respect to the fully periodic and the fully anti-periodic settings. In particular, in the BFSS case, the presence of a mass term, along with a non-trivial fermionic sector, allowed for the construction of a Schwarzschild black hole solution. This is composed of a fuzzy two-sphere with fermionic excitations in a particular state of the Fermi sea. We saw how this construction satisfies the Gregory-Laflamme instability bound. The result is an explicit construction of a Schwarzschild black hole in terms of the degrees of freedom of BFSS theory.

5.1 State of the art in the BFSS conjecture

Throughout this work, we have examined the emergence of spacetime through the lens of the BFSS conjecture. After reviewing the existing evidence for this proposal, we presented several new results that provide further support for and insight into this framework. We summarise, in this section, the current understanding of the duality between BFSS theory and M-Theory, highlighting the ways in which our contributions extend the existing literature.

- **Perturbative evidence**: Correspondence between interaction potentials in the light-front formulation of supergravity and the equivalent objects in Matrix Theory. A one-loop calculation has been carried out for two-body interactions with no longitudinal momentum transfer of supergravitons and membranes, including up to four fermion terms in the background and higher multipole moments. The two-graviton potential has been calculated up to two loops. The calculations reproduce classical supergravity interaction plus some additional terms, interpreted as quantum corrections [36, 49, 50, 147–153] (see [154, 155] for recent developments in perturbative computations). Moreover, non-renormalisation theorems exclude the presence of further v^4 and v^6 terms beyond two loops [156, 157], v being the relative velocity between gravitons. Beyond the linearised approximation, the leading-order three-graviton potential has been reproduced from a two-loop calculation for $N = 3$ [52, 53].

- **Supergravitons and multi-particle states**: Semiclassical configurations which represent supergravitons exist in Matrix Theory. Multi-particle states are allowed due to the presence of flat directions, both classically and quantum mechanically (via supersymmetry). Quantum mechanically, the currently known states are bound states with zero binding energy, whose wavefunction factorises when Higgsing occurs. Moreover, there are metastable states and scattering states with $E > 0$. In Chapter 3, we introduced vertex-like operators in the large-distance effective theory, chosen to carry the quantum numbers of supergraviton scattering states. This suggests a natural route toward an S-Matrix description of these states.

- **Black holes in Matrix Theory**: Different kinds of black holes are obtained in the theory in different regions of parameter space. In the holographic region, dual to type IIA string theory, the bound state of $D0$-branes and strings is dual to the black zero-brane. Several numerical tests have confirmed the duality [60–64]. The evaporation process for the black zero-brane has been studied [65, 70]. In the M-Theory region, Schwarzschild black holes are known to be accounted for by the theory. We provided, in Chapter 4, an explicit realisation in the deformed model obtained from BFSS through torus compactification with mixed boundary conditions. This advances our understanding of how black holes are encoded in the $D0$-brane dual description.

- **Extended objects**: $M2$-membranes have been successfully constructed in BFSS theory in terms of matrix configurations, and their charges have been computed. This applies to both membranes that are extended in the transverse directions and to those that wrap the longitudinal direction, corresponding to strings in ten dimensions. More results are needed for $M5$-branes. In this case, the only branes that have been explicitly constructed have specific, highly symmetric geometries. Moreover, the five-brane charge corresponding to transverse branes does not appear in the supersymmetry algebra, constituting a further source of difficulty. Progress on the construction of $M5$-branes has been achieved in BMN theory [158, 159]. The result we obtained on mass-deformed BFSS theory in Chapter 4 opens the possibility of carrying out a similar analysis in the flat space background.

In summary, the BFSS conjecture provides a compelling framework for exploring the nature of Quantum Gravity, and a substantial body of evidence has accumulated in support of its validity. At the same time, many important questions remain unanswered, leaving a range of exciting directions for future research.

5.2 Outlook

Directly related to the novel work presented in this thesis, several directions appear promising and can be pursued:

1. **Soft Theorems in full BFSS theory.** Deriving soft theorems directly from scattering amplitudes in BFSS Matrix Theory, beyond the long-distance effective field theory limit, offers an interesting line of inquiry. Achieving this would require formulating a scattering matrix framework in which matrix configurations corresponding to multi-particle asymptotic states can be encoded and systematically related to asymptotic scattering data. Such a derivation would provide deeper insight into how spacetime symmetries are encoded in the BFSS model and, consequently, into the mechanisms underlying the emergence of Einstein gravity.

2. **Soft Theorems for the other massless particles.** It would be interesting to extend the analysis of soft theorems in BFSS theory to the rest of the supergraviton multiplet, namely to the gravitino and three-form field. The connection to asymptotic

symmetries in this case would shed light on the symmetry content of the theory for $N \to \infty$.

3. **Soft emission and Hawking radiation.** In the black hole configuration, the emission of a $D0$-brane corresponds to Hawking radiation. This can be interpreted from an S-Matrix point of view, which provides a suggestive analogy between Hawking radiation and soft emission. It would be interesting to further explore this correspondence.

4. **Non-perturbative effects from Matrix Theories.** One could apply the compactification procedure of IKKT and BFSS to different background manifolds. A promising line of inquiry is to characterise possible deformations depending on the compact manifold and the fermionic boundary conditions. Moreover, instantonic configurations on compact backgrounds could be theoretically computed directly in Matrix Theories. This should amount to computing non-perturbative corrections to string theory effective actions.

5. **Finite mass generation in IKKT.** Toroidal compactifications of the IKKT matrix model with mixed boundary conditions generically give rise to divergent mass terms. An interesting direction for future research is to investigate whether these divergences can be eliminated or consistently regularised. Resolving this issue would pave the way for extending the analysis of fuzzy black holes to the IKKT model.

6. **Explicit black hole construction in the type IIA region.** The analysis of the mass deformation and the explicit construction of black hole solutions could be extended to the high-temperature, holographic, type IIA supergravity regime of the matrix model. Such an investigation could provide further insight into the phase structure of the theory and the transitions between its different phases, while also establishing a closer connection with previous studies of the black hole evaporation mechanism in BFSS.

More broadly, I believe that one of the main questions to address in the field is how to obtain general relativity, its quantum corrections and other interactions in the large-N limit of BFSS theory. For example, we have analysed a way of constructing Schwarzschild black holes in terms of Matrix Theory configurations. The question is how to translate

this statement into a metric satisfying Einstein's equations in the continuum limit. More generally, a key open challenge is the construction of a precise dictionary relating matrix configurations to continuous geometries in the large-N limit. Progress in this direction has already been made through studies of matrix entanglement, in which geometric information encoded in target-space entanglement is extracted from the entanglement structure of the corresponding Matrix Theory degrees of freedom [160–162]. Developing such a dictionary would represent a significant step towards understanding the emergence of spacetime from the microscopic matrix dynamics.

It is my hope that this thesis has conveyed how the study of large-N Matrix Theories provides a compelling perspective on Quantum Gravity and the emergence of spacetime. While many fundamental questions remain open, I believe that continued progress in the understanding of large-N theories, and Matrix Theories in particular, will play an important role in addressing these challenges and further advancing our understanding of Quantum Gravity.

Appendices

A

Supplementary material for Chapter 3

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A.1 UV finiteness of the effective theory

Here we show that the theory defined by the Lagrangian (3.3.1.6) does not feature UV divergences, as long as we keep the regulator ϵ non-zero.

Although we worked in position space in the main text, it will be convenient to express the arguments of the present appendix in momentum space. We will be concerned with diagrams involving propagators between two velocity operators $\dot{X}$ and between two σ fields. The momentum space version of (3.3.1.9) and (3.3.1.11) reads

$$\langle\sqrt{\lambda}\sigma(p)\sqrt{\lambda}\sigma(0)\rangle = \frac{2i}{\epsilon^2 p^2 + 1} \tag{A.1.0.1}$$

and

$$\langle\dot{X}(p)\dot{X}(0)\rangle = i \tag{A.1.0.2}$$

respectively. It follows that the degree of divergence ω of a given diagram is given by

$$\omega = L - 2n_\sigma \tag{A.1.0.3}$$

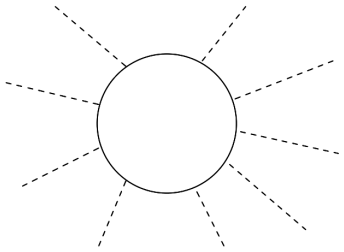


Figure A.1: Linearly divergent diagram with $l_{\dot{X}} = 0$ and $n_\sigma = 0$. This vanishes in dimensional regularisation.

where L is the number of loops and n_σ is the number of internal σ lines. This is also equal to

$$\omega = -\frac{v}{2} + \frac{l_\sigma}{2} - \frac{l_{\dot{X}}}{2} + 1\,, \tag{A.1.0.4}$$

where v is the number of vertices, l_σ is the number of external σ lines and $l_{\dot{X}}$ is the number of external $\dot{X}$ lines. Let us analyse the degree of divergence case by case:

- No external σ legs and $l_{\dot{X}} \geq 2$

$$l_\sigma = 0, \quad l_{\dot{X}} \geq 2 \quad \Rightarrow \quad \omega < 0 \tag{A.1.0.5}$$

  These diagrams are convergent.

- No external $\dot{X}$ legs, $l_{\dot{X}} = 0$, in this case

$$L = 1 + n_\sigma \quad \Rightarrow \quad \omega = 1 - n_\sigma\,. \tag{A.1.0.6}$$

  then for $n_\sigma = 0$ the diagram is linearly divergent. However, the divergence is proportional to $\int dp$, which vanishes in dimensional regularisation. The corresponding Feynman diagram is shown in Figure A.1. For $n_\sigma = 1$, $\omega = 0$, and we have an overlapping log divergence which takes the form $\int dr dp \, \frac{1}{\epsilon^2 p^2 + 1}$, which again vanishes due to $\int dr$ in dimensional regularisation. The corresponding Feynman diagram is shown in Figure A.2.

  Indeed, if we go to higher values of n_σ, ω becomes negative and the corresponding diagrams are convergent;

- Two external $\dot{X}$ legs, $l_{\dot{X}} = 2$, in this case

$$L = n_\sigma \quad \Rightarrow \quad \omega = -n_\sigma\,. \tag{A.1.0.7}$$

  Then the only possible divergence arises when $n_\sigma = 0$, then $\omega = 0$. However, there is no Feynman diagram corresponding to this situation.

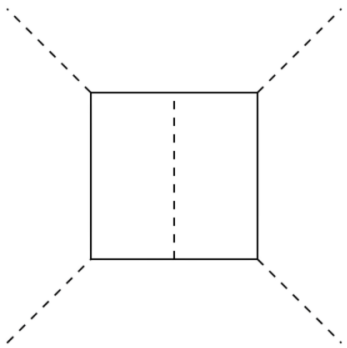


Figure A.2: Overlapping log divergent diagram with $l_{\dot{X}} = 0$ and $n_\sigma = 1$. This vanishes in dimensional regularisation.

- $l_{\dot{X}} \geq 4$, then $\omega < 0$ and the corresponding diagrams are convergent.

To summarise, due to the fact that the theory is one-dimensional, the structure of divergences is very simple, and not only is the theory super-renormalizable, but all possible divergences cancel out.

B

Supplementary material for Chapter 4

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B.1 Alternative block decomposition

We reformulate the block decomposition of matrix degrees of freedom introduced in Section 4.2 and used throughout Chapter 4. In particular, we will adopt an explicit matrix decomposition which highlights the block matrix structure of bosonic terms and realises the mode decomposition for anti-periodic fermions.

We have seen how compactifying Matrix Theories on a torus amounts to replicating the fundamental region an infinite number of times. This is implemented by splitting the Hilbert space of matrix degrees of freedom into two components

$$X = Y \otimes Z \,,$$

where Y represents the degrees of freedom in the fundamental region and Z contains information about transitions between different regions. Consider the decom-

position of the matrix

$$A^\mu = \sum_{n^b \in \mathbb{Z}^6} A^\mu(n^b) \otimes e^{in^b p^b} .$$

We want to provide an explicit matrix representation for the operator $e^{in^b p^b}$. Let us take the one-dimensional example for simplicity of notation. The higher-dimensional case follows immediately, since every torus direction is parametrised independently. The natural choice is the following:

$$\left(e^{inp}\right)_{kj} := C^n_{kj} = \delta_{k,j+n} ,$$

which represents a matrix with entries 1 on the diagonal at distance n from the central one and 0 elsewhere. Moreover, it satisfies

$$C^n C^m = C^{n+m} .$$

Such matrices give an explicit representation of operators e^{inp} in the $|q\rangle$ basis of the Hilbert space Z. The decomposition in winding modes now reads

$$A^\mu = \sum_{n \in \mathbb{Z}} A^\mu(n) \otimes C^n .$$

Similarly, we define a matrix Q which labels the fundamental regions

$$Q_{kj} = j\, \delta_{kj}$$

and exponentiates to

$$\left(e^{-2\pi i Q}\right)_{kj} = e^{-i2\pi j} \delta_{kj} = \delta_{kj} ,$$

where the last equality simply follows from the fact that matrix indices are integers. Now, the matrix corresponding to the compact direction is decomposed as

$$A^9 = \sum_{n \in \mathbb{Z}} A^9(n) \otimes C^n + 2\pi L \mathbb{I}_N Q .$$

The commutator with C^n is

$$[Q, C^n] = n C^n . \tag{B.1.0.1}$$

It follows that the translation operator can be written as

$$U = \mathbb{I}_N \otimes e^{-2\pi i Q} C^1 = \mathbb{I}_N \otimes C^1 ,$$

then the action on the matrices has the desired form

$$(U)^{-1}A^{\mu}U = A^{\mu}$$
$$(U)^{-1}A^{9}U = A^{9} + 2\pi L\mathbb{I}_N \otimes \mathbb{I}_M \,.$$

Moreover, when fermions are periodic, their decomposition is straightforward

$$\psi = \sum_{n\in\mathbb{Z}} \psi(n) \otimes C^n \,.$$

We now extend the construction to accommodate anti-periodic fermions. This requires doubling the Hilbert space, such that

$$Z = \mathcal{H}_{\mathbb{Z}} \oplus \mathcal{H}_{\mathbb{Z}+\frac{1}{2}} \,.$$

Namely, we double the winding modes lattice to account for both integer and half-integer modes, which take values in the respective lattices. Anti-periodic fermions are given by the block matrix

$$\psi = \begin{pmatrix} 0 & \psi(-\frac{1}{2}) & 0 & \psi(-\frac{3}{2}) & 0 & \cdots \\ \psi(\frac{1}{2}) & 0 & \psi(-\frac{1}{2}) & 0 & \psi(-\frac{3}{2}) & \cdots \\ 0 & \psi(\frac{1}{2}) & 0 & \psi(-\frac{1}{2}) & 0 & \cdots \\ \psi(\frac{3}{2}) & 0 & \psi(\frac{1}{2}) & 0 & \psi(-\frac{1}{2}) & \cdots \\ 0 & \psi(\frac{3}{2}) & 0 & \psi(\frac{1}{2}) & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the zeros sit on the integer lattice, where the bosons live. On the doubled Hilbert space, we define

$$C^n_{kj} = \delta_{k,j+2n} \,,$$

where the indices k, j run over the duplicated lattice and n can be either integer or half-integer. In particular, the operator

$$C^{1/2}_{kj} = \delta_{k,j+1}$$

acts as a shift between the two lattices. The appropriate modification of the Q operator is

$$Q_{kj} = \frac{j}{2}\delta_{kj} \,,$$

and the translation operator on the torus becomes

$$U = \mathbb{I}_N \otimes \left(e^{-2\pi i Q}C^1\right)_{kj} = \mathbb{I}_N \otimes (-1)^k C^1_{kj} \,.$$

As a consequence, anti-periodic fermions are decomposed as

$$\psi = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi(r)\otimes C^r\,,$$

and the commutation relation (B.1.0.1) (which is also valid for half-integer n) implies

$$(U)^{-1}\psi U = -\psi\,,$$

as required.

This construction generalises naturally to the case of mixed boundary conditions. The fermions are decomposed into a periodic and an anti-periodic component, the details of the decomposition depending on the specific choice of boundary conditions (we have considered different examples in the main text). Schematically

$$\psi = \psi^{\uparrow} + \psi^{\downarrow}\,,$$

and our construction implies that the periodic component $\psi^{\uparrow}$ is decomposed using integer modes, thus living on the periodic lattice, while $\psi^{\downarrow}$ is anti-periodic and lives on the shifted lattice.

With these decompositions, the interaction of an uncompactified boson with a fermion takes the form

$$\mathrm{Tr}\,\bar{\psi}\Gamma^I[A_I,\psi] = \sum_{n\in\mathbb{Z};r,p\in\mathbb{Z}+\frac{1}{2}|n+r+p=0} \mathrm{Tr}_Y\,\psi(r)[A(n),\psi(p)] \qquad \text{(B.1.0.2)}$$

where the trace on the right-hand side is taken over the N-dimensional subspace (4.2.0.3). We see that this term has a single $U(N)$ symmetry, just as the theory with periodic fermions has.

B.2 Loop propagator regularisation for BFSS

In this section, we extend the regularisation method of [136] for loop propagators to the case of coefficients arising from loops in the Wilsonian effective theory analysis of compact BFSS theory.

We begin by considering the bosonic sum

$$S_{B_1} = \sum\nolimits'_{n^a\in\mathbb{Z}^3} \int_{-\infty}^{+\infty} dp \frac{1}{p^2+n^a n_a}\,,$$

and we introduce a regulator α such that

$$S_{B_1} = \lim_{\alpha\to 0} {\sum}'_{n^a\in\mathbb{Z}^3} \int_{-\infty}^{+\infty} dp \frac{e^{-\alpha^2 n^a n_a - \alpha^2 p^2}}{p^2 + n^a n_a} \,.$$

We rewrite this as

$$S_{B_1} = \lim_{\alpha\to 0} {\sum}'_{n^a\in\mathbb{Z}^3} \int_{-\infty}^{+\infty} dp \int_0^{+\infty} dt\, e^{-(t+\alpha^2)n^a n_a} e^{-(t+\alpha^2)p^2} \,,$$

which can be rewritten in terms of the θ-function

$$\theta(t) = \sum_{n=-\infty}^{+\infty} e^{-\pi t n^2} \,,$$

such that

$$S_{B_1} = \lim_{\alpha\to 0} \sqrt{\pi} \int_{-\infty}^{\infty} dp \int_{\alpha^2/\pi}^{+\infty} dt e^{-tp^2} (\theta^3(t) - 1) \,.$$

Focusing on the divergent part of the integral, we have

$$\int_{\alpha^2/\pi}^{1} dt\, e^{-tp^2} (\theta^3(t) - 1) = \int_{\alpha^2/\pi}^{1} dt \frac{e^{-tp^2}}{t^{3/2}} (\theta^3(1/t) - 1) + F(\alpha; p) \,,$$

where

$$F(\alpha; p) = \int_{\alpha^2/\pi}^{1} dt\, e^{-tp^2} \left(\frac{1}{t^{3/2}} - 1 \right)$$

represents the divergent component of the integral. Here we made use of the property of the θ-function

$$\theta(t) = \frac{1}{t^{1/2}} \theta(1/t) \,.$$

By rearranging the various terms, we obtain

$$S_{B_1} = \lim_{\alpha\to 0} \sqrt{\pi} \int_{-\infty}^{+\infty} dp \int_1^{+\infty} dt \left[\frac{e^{-p^2/t}}{\sqrt{t}} + e^{-p^2 t} \right] (\theta^3(t) - 1) + \sqrt{\pi} \int_{-\infty}^{+\infty} dp F(\alpha; p) \,,$$

where the first integral is convergent, and we regularised the divergence to be in the second term.

We now consider the fermionic (in the case of anti-periodic boundary conditions) sum

$$S_{F_1} = \sum_{r^a\in\mathbb{L}_3} \int_{-\infty}^{+\infty} dp \frac{1}{p^2 + r^a r_a} \,,$$

which we regulate as

$$S_{F_1} = \sum_{r^a\in\mathbb{Z}^3} \int_{-\infty}^{+\infty} dp \frac{e^{-\alpha^2((r^a+u^a)^2+p^2)}}{p^2 + (r^a + u^a)^2}, \quad u^a = \frac{1}{2}(1, \ldots, 1) \,.$$

We can rewrite this as

$$S_{F_1} = \int_{-\infty}^{+\infty} dp \int_0^{+\infty} dt \sum_{r^a \in \mathbb{Z}^3} e^{-(\alpha^2+t)(p^2+(r^a+u^a)^2)} = \sqrt{\pi} \int_{-\infty}^{+\infty} dp \int_{\alpha^2/\pi}^{+\infty} dt\, \theta^3(t|u) e^{-tp^2} \,,$$

where the generalized θ-function takes the form

$$\theta(t|u) = \sum_{n=-\infty}^{+\infty} e^{-\pi t(n+u)^2} \,.$$

Again, we look at the divergent part, which reads

$$\int_{\alpha^2/\pi}^{1} dt\, e^{-tp^2} \theta^3(t|\frac{1}{2}) = \int_{\alpha^2/\pi}^{1} dt\, \frac{e^{-tp^2}}{t^{3/2}} (e^{-\frac{3}{4}\pi t} \theta^3(\frac{1}{t}|\frac{it}{2}) - 1) + \tilde{F}(\alpha; p) \,,$$

where

$$\tilde{F}(\alpha; p) = \int_{\alpha^2/\pi}^{1} dt\, \frac{e^{-tp^2}}{t^{3/2}}$$

and we used the property

$$\theta(t|a) = \frac{e^{-\frac{\pi}{4}t}}{t^{1/2}} \theta(\frac{1}{t}|\frac{it}{2}) \,.$$

It follows that

$$S_{F_1} = \sqrt{\pi} \int_{-\infty}^{+\infty} dp \int_1^{+\infty} dt \left(\frac{e^{-\frac{p^2}{t}}}{\sqrt{t}} \left(e^{-\frac{3\pi}{4t}} \theta^3(t|\frac{i}{2t}) - 1 \right) + e^{-p^2 t} \theta^3(t|\frac{1}{2}) \right) + \sqrt{\pi} \int_{-\infty}^{+\infty} dp\, \tilde{F}(\alpha; p) \,,$$

where the second term is the divergent one.

As a consequence, when we consider the difference $(S_{F_1} - S_{B_1})$, the divergent parts cancel out, giving

$$\int_{-\infty}^{+\infty} dp \left(\tilde{F}(\alpha; p) - F(\alpha; p) \right) = \int_{\alpha^2/\pi}^{1} dt \int_{-\infty}^{+\infty} dp\, e^{-tp^2} = 2\sqrt{\pi} \,,$$

such that the difference reads

$$S_{F_1} - S_{B_1} = 2\pi - \sqrt{\pi} \int_{-\infty}^{+\infty} dp \int_1^{+\infty} dt \left[\left(\frac{e^{-\frac{p^2}{t}}}{\sqrt{t}} + e^{-p^2 t} \right) (\theta^3(t) - 1) \right.$$
$$\left. - \left(\frac{e^{-\frac{p^2}{t}}}{\sqrt{t}} (e^{-\frac{3\pi}{4t}} \theta^3(t|\frac{i}{2t}) - 1) + e^{-p^2 t} \theta^3(t|\frac{1}{2}) \right) \right] \,,$$

which is a finite number and can be evaluated numerically.

B.3 Filling up the Fermi sea

In this section, we focus on the fermionic Hamiltonian (4.6.1.6), which represents a collection of fermionic oscillators. We show how the Fermi sea should be filled up in order to interpret the fuzzy sphere solution as a black hole.

We begin by noting that $|\lambda_{-,n-1}| = \lambda_{+,n}$, so that for $\lambda_{+,1} = 1 \leq E_n \leq |\lambda_{-,N-2}| = N-1$ the shell with energy E_n (ignoring the prefactor) is composed of the $4(2n+2)$ oscillators $\xi_\alpha^{j,n}$ and the $4(2n-2)$ oscillators $\chi_\alpha^{j,n-1}$, giving total dimension $\#(E_n) = 16n$. It follows that the shell with the largest dimension is the one with energy $E_{N-1} = |\lambda_{+,N-1}|$, having dimension $\#(E_{N-1}) = 4(4N-4)$. Moreover, the state with the largest degeneracy is the one for which such a shell is half-filled, which yields

$$\Omega = \binom{16N-16}{8N-8} \propto \frac{2^{16(N-1)}}{\sqrt{N}}\,.$$

At leading order in the large-N expansion, we can compute the fermionic energy contribution from the shell with energy E_{N-1}. This follows straightforwardly from (4.6.1.6). In particular, we are free to excite $j = 4(1+x)(2N-2)$ oscillators, $-1 \leq x \leq 1$ being the fraction of excited oscillators. The energy reads

$$H_{ferm} = \frac{8R\,k^4}{3}N^3 + O(N^2) = \frac{32R\,k^2}{3}N\,R_f^2 + O(N^2)$$

where $R_f = \frac{k\,N}{2}$ is the physical fuzzy sphere radius in the large-N limit. At the same time, the bosonic part of the Hamiltonian yields, when evaluated on the fuzzy sphere background

$$H_{bos} = -\frac{R\,k^2}{2}NR_f^2$$

So the total energy reads

$$H = \frac{R\,k^2}{6}(64-3)NR_f^2\,.$$

As explained in Section 4.6.2, the Hamiltonian gives the lightcone energy, and it is related to the invariant mass through

$$H = E_{LC} = \frac{R}{2N}M^2\,.$$

The mass-radius relation

$$M_{5D} = r_0^2$$

then implies

$$r_0 = \left(\frac{244}{3}\right)^{\frac{1}{4}} R_f \,.$$

In order for the fuzzy sphere geometry to represent a black hole, its radius must be smaller than the Schwarzschild radius. We see that this condition is satisfied when the Fermi sea is filled in this way. The fraction x of excited oscillators in the E_N determines the entropy. In the main text, we will express the results in terms of the fraction y defined as

$$y = \frac{1+x}{2}, \qquad 0 \leq y \leq 1.$$

B.4 Stability of the squashing modes

In this section, we prove that the squashing modes of bosonic fluctuations are stabilised by the fermionic sector. We have seen in Section 4.6.4 that quadratic bosonic fluctuations with $\ell = 1, J = 2$ are unstable. These correspond to the squashing modes of the fuzzy sphere

$$\delta X^i = S^i_j X^j \,,$$

where S^i_j is a symmetric traceless tensor and X^j are the bosonic matrices for the fuzzy sphere background ($SU(2)$ generators). The quadratic potential reads

$$\Delta V_{bos} = Rk^4 \mathrm{Tr}\left[\frac{1}{2}\delta X^i N_{ij} \delta X^j\right]$$

and, given that those are eigenmodes of N_{ij} with eigenvalue -2 we get

$$\Delta V_{bos} = -Rk^4\, S^i_j S^i_k \mathrm{Tr}\left[X^j X^k\right] \,.$$

Moreover, using

$$\mathrm{Tr}\left[X^j X^k\right] = \frac{N(N^2-1)}{12}\delta^{jk}$$

we obtain

$$\Delta V_{bos} = -Rk^4 \frac{N^3}{12} \mathrm{Tr}[S^2]$$

at leading order in N. The negative sign corresponds to an instability. We will now see how this is compensated by including quadratic perturbations, due to the same

modes, from the fermionic sector. The perturbation to the fermionic operator, due to the squashing modes, reads

$$\delta O_f = Rk^4 \,\mathrm{Tr}\left[\lambda^{\uparrow,\dagger}\sigma^i[\delta X^i, \lambda^{\uparrow}]\right] = Rk^4 \,\mathrm{Tr}\left[\lambda^{\uparrow,\dagger}\sigma^i S^i_j L^j \lambda^{\uparrow}\right], \quad L^j = [X^j, \cdot]\,.$$

We will compute the effective potential generated by the fluctuations using second-order perturbation theory on the state of the system, which we recall is the Fermi sea with partially filled highest shell. To this end, it is useful to decompose the fermions as

$$\lambda^{\uparrow}_{b\beta\sigma,\alpha} = U^{r,n,\dagger}_{b\beta\sigma}\xi^{r,n}_{\alpha} + V^{r,n,\dagger}_{b\beta\sigma}\chi^{r,n,\alpha\dagger}\,, \tag{B.4.0.1}$$

such that

$$\delta O_f = Rk^4\left(P^{r,q}_{nm}\xi^{r,n,\alpha\dagger}\xi^{q,m}_{\alpha} - Q^{r,q}_{nm}\chi^{r,n,\alpha\dagger}\chi^{q,m}_{\alpha} + T^{r,q}_{nm}\xi^{r,n,\alpha\dagger}\chi^{q,m,\alpha\dagger} + T^{r,q\,\dagger}_{nm}\chi^{q,m}_{\alpha}\xi^{r,n}_{\alpha}\right)$$

with

$$P^{r,q}_{nm} := U^{r,n}AU^{q,m,\dagger}, \quad Q^{r,q}_{nm} := V^{r,n}AV^{q,m,\dagger}, \quad T^{r,q}_{nm} := U^{r,n}AV^{q,m,\dagger}, \quad A := \sigma^i S^i_j L^j\,,$$

and, in the last expression, we suppressed matrix indices for simplicity of notation.

We will compute the Hamiltonian at second order in perturbation theory. First, we will show that the first-order term vanishes. This reads

$$\Delta V_{fo} = \frac{1}{\Omega}\sum_{S\in\mathcal{H}_0}\langle S|\delta O_f|S\rangle\,,$$

where $\mathcal{H}_0$ is the Hilbert space of the state of the system, composed of the Fermi sea with partially filled highest shell, and Ω is the degeneracy. Hence, $|S\rangle$ are states in this Hilbert space. The only terms which will contribute are those that preserve the particle number. Hence, we get non-trivial contributions from

$$\langle S|\xi^{r,n,\alpha\dagger}\xi^{q,m}_{\alpha}|S\rangle = 4\delta^{rq}\delta^{nm}\,,$$

and similarly for χ oscillators. It follows that

$$\Delta V_{fo} = 4Rk^4\sum_n\sum_r\left(P^{r,r}_{n,n} - Q^{r,r}_{n,n}\right)$$

which can be rewritten as

$$\Delta V_{fo} = 4Rk^4\sum_n \mathrm{Tr}\left[P_{+,n}A - P_{-,n}A\right]\,,$$

where

$$P_{+,n} = \frac{\mathrm{X}_n + (n+1)}{2n+1}, \qquad P_{-,n} = \frac{n - \mathrm{X}_n}{2n+1}$$

are the projectors in the + and − sectors, constructed from the operator $\mathrm{X}_n = (\sigma^i L_i)|_n$. We can use

$$\begin{aligned} \mathrm{Tr}\,(A) &= 0 \\ \mathrm{Tr}\left(\sigma^i \sigma^j\right) &= 2\delta^{ij} \\ \mathrm{Tr}\left(L^i L^j\right) &= \frac{1}{3} n(n+1)(2n+1)\delta^{ij} \end{aligned}$$

to show that

$$\Delta V_{fo} \propto \mathrm{Tr}\,(S) = 0\,.$$

We now proceed to compute the effective Hamiltonian generated at quadratic order by the perturbation. This is determined by

$$\Delta V_{ferm} = \frac{1}{\Omega} \sum_{S \in \mathcal{H}_0} \sum_{R \neq S} \langle S|\delta O_f|R\rangle \frac{1}{E_S - E_R} \langle R|\delta O_f|S\rangle\,,$$

and we are summing over all possible states that can be connected by the action of δO_f. The denominator gives the energy difference between the two states. Let us examine the action of δO_f on states $|S\rangle$. The leading contribution in the large-N expansion is obtained by acting with creation and annihilation operators on all the levels of the Fermi sea. When the action only involves the top, half-filled level, the corresponding contribution will be subleading in N. The operator $\xi^{r,n,\alpha\dagger}\xi^{q,m}_{\alpha}$ removes and adds one oscillator. This requires $r = q$ and $n = m$. As a consequence, this would impose $R = S$, which does not appear in the sum. Hence, this term never contributes, at leading order in N. The same applies to $\chi^{r,n,\alpha\dagger}\chi^{q,m}_{\alpha}$.

The action of $\xi^{r,n,\alpha\dagger}\chi^{q,m,\alpha\dagger}$ creates oscillators on states which are already filled, and therefore provides a vanishing contribution. The only non-zero term comes from removing two fermions from the sea by the action of $\chi^{q,n}_{\alpha}\xi^{r,m}_{\alpha}$. Moreover, P, Q, T terms are constructed from the interpolation of the operator A between different eigenstates. But A can only interpolate between states with the same orbital angular momentum n. This imposes $n = m$ on all P, Q, T factors. It follows that the only non-zero term

is $T^{r,q\,\dagger}_{n,n}\chi^{q,n}_{\alpha}\xi^{r,n}_{\alpha}$. This selects a state $|R\rangle$ where two oscillators have been removed. The corresponding energy difference is

$$E_S - E_R = Rk^4(\lambda_{+,n} - \lambda_{-,n}) = Rk^4(2n+1)\,.$$

Putting it all together, we get

$$\Delta V_{ferm} = 4Rk^4 \sum_n \frac{|T^{r,q}_{n,n}|^2}{2n+1}\,,$$

where the factor of 4 from summing over $\alpha = 1, \ldots, 4$. Hence, computing the fermionic correction reduces to the evaluation of the factor

$$|T^{r,q}_{n,n}|^2 = \mathrm{Tr}\left[P_{+,n}AP_{-,n}A\right]\,.$$

Expanding the expression, we get

$$|T^{r,q}_{n,n}|^2 = \frac{1}{(2n+1)^2}[n(n+1)\mathrm{Tr}\left(A^2\right) - \mathrm{Tr}\left(\mathrm{X}A^2\right) - \mathrm{Tr}\left(\mathrm{X}A\mathrm{X}A\right)]\,.$$

The traces can be evaluated using standard $SU(2)$ identities and read

$$\begin{aligned}
\mathrm{Tr}\left(A^2\right) &= \frac{2}{3}n(n+1)(2n+1)\mathrm{Tr}\left[S^2\right] \\
\mathrm{Tr}\left(\mathrm{X}A^2\right) &= \frac{1}{3}n(n+1)(2n+1)\mathrm{Tr}\left[S^2\right] \\
\mathrm{Tr}\left(\mathrm{X}A\mathrm{X}A\right) &= -\frac{2}{15}n(n+1)(2n+1)(n-1)(n+2)\mathrm{Tr}\left[S^2\right]\,.
\end{aligned}$$

Putting everything together

$$|T^{r,q}_{n,n}|^2 = \frac{n(n+1)(2n-1)(2n+3)}{5(2n+1)}\mathrm{Tr}\left[S^2\right]\,.$$

It follows that

$$\sum_n \frac{|T^{r,q}_{n,n}|^2}{(2n+1)} = \frac{N^3}{15}\mathrm{Tr}\left[S^2\right] + O(N^2)\,,$$

and, as a consequence

$$\Delta V_{ferm} = Rk^4\frac{4N^3}{15}\mathrm{Tr}\left[S^2\right] + O(N^2)\,.$$

Considering both bosonic and fermionic contributions, the quadratic perturbation potential for the squashing modes reads

$$\begin{aligned}
\Delta V_{squash} &= \Delta V_{bos} + \Delta V_{ferm} \\
&= Rk^4N^3\left(-\frac{1}{12} + \frac{4}{15}\right)\mathrm{Tr}\left[S^2\right] + O(N^2) = Rk^4N^3\frac{11}{60}\mathrm{Tr}\left[S^2\right] + O(N^2)
\end{aligned}$$

which is positive. Hence, the perturbation is stable.

A similar analysis can be carried out for the $\ell = 1$ modes in the directions orthogonal to the fuzzy sphere. These modes can be written in terms of fuzzy sphere matrices as

$$\delta X^{i+3} = A^i_j X^j \,,$$

where A^i_j is the 3×3, anti-symmetric matrix parametrising the fluctuation. The analogous analysis to the previous case can be carried out using the fermionic term

$$\delta O'_f = i\lambda^{\uparrow,T}(t,0) i\sigma^2 \gamma^{i+3} A^i_j [X_j, \lambda^{\uparrow}(t,0)]$$

obtained by perturbing the corresponding term in the full action (4.5.0.4). This fixes the instability of the $\ell = 1$ mode in the orthogonal directions.

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