Geometry of Actions, Expanders and Warped Cones



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Dedicata ai miei genitori

Rosa & Mariano

per essere sempre stati un punto di riferimento

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Abstract

In this thesis we introduce a notion of *graphs approximating actions* of finitely generated groups on metric and measure spaces. We systematically investigate expansion properties of said graphs and we prove that, under mild conditions, a sequence of graphs approximating a fixed action ρ forms a family of *expanders* if and only if ρ is *expanding in measure*.

We subsequently prove that the notion of expansion in measure for measure-preserving actions is equivalent to the well studied notion of *spectral gap*. This enables us to rely on a number of known results to construct numerous new families of expander (and superexpander) graphs.

Proceeding in our investigation, we show that the graphs approximating an action $\Gamma \curvearrowright X$ are uniformly quasi-isometric to the level sets of the associated *warped cone* $\mathcal{O}_\Gamma(X)$. Warped cones are 'cone like' structures that were introduced by J. Roe to construct examples of metric spaces with interesting coarse geometry (it has recently been shown that they violate the coarse Baum-Connes conjecture). The existence of such a relation between approximating graphs and warped cones has twofold advantages: on the one hand it implies that warped cones arising from actions that are expanding in measure coarsely contain families of expanders, on the other hand it provides a geometric model for the approximating graphs allowing us to study the geometry of the expander thus obtained.

The rest of the work is devoted to the study of the coarse geometry of warped cones (and approximating graphs). We do so in order to prove rigidity results which allow us to prove that our construction is flexible enough to produce a number of non coarsely equivalent new families of expanders. As a by-product, we also show that some of these expanders enjoy some rather peculiar geometric properties, *e.g.* we can construct expanders that are *coarsely simply connected*.

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Chapter 1

Introduction

In order to better explain the circumstances that brought me to develop this thesis the way it is, I decided to introduce the results I obtained and the related concepts in chronological order.[1] Possibly, this is not the most efficient way to present these contents, but I think that it is worth following this approach for the sake of drawing a clearer picture of the framework where they developed.

Baum–Connes conjecture(s)

Given a locally compact topological group G, P. Baum and A. Connes defined an assembly map from the K-theory of a (sort of) classifying space of G to the K-theory of the reduced C^*-algebra of G. They then conjectured that this assembly map is an isomorphism [BC00, BCH94]. This conjecture was extended in various ways, for example by considering K-theories with non-trivial coefficients (*Baum–Connes conjecture with coefficients*), by constructing an analogue of the assembly map for coarse geometries of metric spaces instead of classifying spaces of groups (*coarse Baum–Connes conjecture*) or by considering different C^*-algebras (*e.g.* considering $C^*_{\max}$ one gets the *maximal Baum–Connes conjecture*).

This suite of conjectures has been a huge driving force for the last couple of decades. In fact, the existence of such an isomorphism between those seemingly unrelated objects would yield far reaching consequences. To start with, this would provide a way for computing K-theories of C^*-algebras (a notoriously hard thing to do); but it would also bear vastly important consequences on the study of manifolds. Indeed, (some instances of) the Baum–Connes conjecture can be seen as an analytic

[1] Unfortunately, I am only familiar with a part of the whole story. For this reason, my exposition is bound to provide only a partial account of the developments in this subject(s).

analogue of the Borel conjecture, and the injectivity of the assembly map for a discrete group G implies that G satisfies the Novikov conjecture on higher signatures.

Currently, the Baum–Connes conjecture(s) have been proved for large classes of groups, but some counterexamples have also been found [HLS02]. In either way, some of these advances were based on the (non)existence of coarse embeddings into Banach spaces.

Coarse embeddings in Banach spaces

A standard technique for studying complicated spaces is to try to embed them into some larger and better behaved spaces. This technique turned out to be useful in the study of the Baum–Connes conjecture(s) as well, where the role of the larger space is played by Banach spaces.

The study of different notions of embeddings of metric spaces into Banach spaces has been carried out since the late 50's. It was known that every separable metric space is homeomorphic to a subset of $L^2(0,1)$, but in [Enf70] Enflo provided an example of a separable metric space which does not embed into a Hilbert space with a *uniform homeomorphism*. Apparently unaware of Enflo's result, Gromov later asked [Gro93, p218] whether every separable metric space embeds into a Hilbert space *coarsely* (see Subsection 2.6.3 for a definition).

It was proved by G. Yu in [Yu00] that the coarse Baum–Connes conjecture holds true for every metric space which coarsely embed into a Hilbert space. In particular, this allowed him to deduce that the assembly map is injective for finitely generated groups that admit such an embedding. This breakthrough gave new relevance to the study of coarse embeddings into Banach spaces. In fact, at the time of Yu's result there were no known examples of separable metric spaces which could not be coarsely embedded into Hilbert spaces.

Building on Enflo's ideas, Dranishnikov, Gong, Lafforgue and Yu constructed in [DGLY02] a separable metric space that does not coarsely embed into an Hilbert space. Their example consisted of a family of locally finite graphs with growing degrees and therefore it did not have *bounded geometry* (see Subsection 2.1.3 for a definition).

In order to find possible counterexamples to the injectivity of the assembly map on finitely generated groups (and hence counterexamples to the Novikov conjecture), it was necessary to find finitely generated groups whose Cayley graph could not coarsely embed into Hilbert spaces. A natural approach for doing so was to look for groups whose Cayley graph contained a coarsely embedded copy of a space which was known to be impossible to coarsely embed in a Hilbert space. For this purpose the graphs in

[DGLY02] were of no use, as their unbounded geometry prevented them from being coarsely embedded into any finite degree Cayley graph. Still, it was pointed out by Gromov that the reason why those graphs could not coarsely embed into Hilbert spaces had to do with their expansion properties. He thus linked the coarse Baum–Connes conjecture with the major research topic of expander graphs.

Families of expanders

A *family of expanders* is a sequence of finite graphs that have vertex sets of increasing cardinality, are sparse (*i.e.* have uniformly bounded degrees), but are at the same time highly connected (*i.e.* their *Cheeger constant* is bounded from below), see Subection 2.7.2 for the precise definition. Such notion was first defined in the 70s, and since then it motivated a great deal of research. Indeed, the existence of families of expander graphs is a fundamental tool to solve some concrete problems in computer science and in the study of computational complexity.

The existence of families of expanders was first proved by Pinsker [Pin73] by probabilistic means. Indeed, he managed to show that random sequences of graphs with bounded degrees are expanders with high probability. Still, to be able to use the expander graphs in concrete problems it was necessary to have explicit examples, and it turned out that defining explicit families of expanders was quite a challenge. This is because computing bounds for the Cheeger constant of a graph is an extremely difficult task in general.

The first concrete examples of expander graphs were built by Margulis [Mar73] using the machinery of Kazhdan's property (T). This construction made it clear that in order to construct more examples it was necessary to use a combination of techniques from different branches of mathematics. Indeed, the modern approaches to the theory of expander graphs use deep results of combinatorics, algebra and analysis. The necessity for such a diverse set of tools boosted the study of families of expanders not only for their concrete applications but also for their intrinsic interest. The pay-off was a number of important implications in other unexpected subjects as well (*e.g.* geometry and topology). See the surveys [HLW06] and [Lub12] for a great introduction to expander graphs and their applications.

Counterexamples to the Baum–Connes conjecture

The application of the theory of expander graphs that is more relevant for our discussion is given by the role they play in disproving the (coarse) Baum–Connes conjecture.

In [Mat97], Matoušek proved that a family of expander graphs cannot be coarsely embedded into a Hilbert space (this was also realised by Gromov in [Gro03]). This fact led Higson to produce a first counterexample to the coarse Baum-Connes conjecture [Hig99].

As opposed to the spaces built in [DGLY02], expander graphs do have bounded geometry and it it could hence be possible to coarsely embed them into the Cayley graph of a finitely generated group. This was indeed done by Gromov in [Gro03] (see also [AD08]). There he used probabilistic methods to build finitely generated groups that contain a 'weakly' embedded family of expanders and therefore do not coarsely embed into Hilbert spaces (it was later proved by Osajda that there exist groups containing isometrically embedded families of expanders [Osa14]). These groups were then used in [HLS02] to produce counterexamples to (some versions of) the Baum-Connes conjecture.

Without entering into details, we will just report that the counterexamples to the coarse Baum-Connes conjecture built in [Hig99, HLS02] are based on the existence, for a coarse disjoint union of expanders X, of non-compact ghost projections for the Roe algebra $C^*(X)$.

Warped cones

The idea of the warped cone construction was introduced by J. Roe in [Roe95] (see also [Roe96]) as a way of producing examples of metric spaces satisfying interesting coarse geometric properties and, in particular, to produce new counterexamples to the coarse Baum–Connes conjecture. While passing relatively unnoticed in the years following their introduction, warped cones have lately attracted a great deal of interest and lively research; starting with [DN17], many papers followed in quick succession: [Saw15, NS17, Vig16, Saw17a, SW17, WW17, dLV17, Saw17b, FNvL17] and finally [Vig17a]. We will now take some time to explore this construction as explained in [Roe05].

Given an action by diffeomorphisms of a finitely generated group $\Gamma = \langle S \rangle$ on a compact Riemannian manifold (M, ϱ), the *warped cone* associated with it is the infinite metric space $\big(\mathcal{O}_\Gamma(M), \delta_\Gamma\big)$ obtained from the infinite Riemannian cone $\big(M \times [1, \infty), t^2\varrho + dt^2\big)$ by warping the metric. That is, the metric δ_Γ is obtained from the cone metric $t^2\varrho + dt^2$ by imposing the condition that for every element s in the generating set S the distance between any two points of the form (x, t) and $(s \cdot x, t)$ be at most 1. Note that the warped metric δ_Γ depends on the choice of generating set.

Still, the coarse geometry of $\mathcal{O}_\Gamma(M)$ does not depend on this choice (see Section 8.1 for more on this).

One of the key features of this construction is that it can be used to produce metric spaces with very diverse coarse geometries, and all these spaces are going to have bounded geometry. In particular, among other things Roe was able to prove that if Γ is a finitely generated subgroup of a compact Lie group G such that the warped cone arising from the action by left multiplication $\Gamma \curvearrowright G$ coarsely embeds into a Hilbert space, then Γ must have the *Haagerup property*.[2] We remark here that Roe subsequently asked whether the converse was true:

Question 1.0.1 (Roe)**.** *Is it true that if a finitely generated subgroup Γ of a Lie group G has the Haagerup property then the warped cone $\mathcal{O}_\Gamma(G)$ coarsely embeds into a Hilbert space?*

We will be able to give a negative answer to the above shortly (Corollary 1.0.6).

Note that from Roe's result it follows that it is possible to construct warped cones that do not coarsely embed into Hilbert spaces (it is enough to choose a subgroup $\Gamma < G$ with property (T)). He was also hoping to prove that under some circumstances warped cones could be used to produce new counterexamples to the coarse Baum–Connes conjecture —remember that by this time the only known counterexamples to that conjecture are those built using families of expanders.

Strong evidence that it should be possible to construct such counterexamples was given by Druţu and Nowak in [DN17]. In fact, they managed to prove that if the action $\Gamma \curvearrowright M$ has a *spectral gap* (see Subection 2.4.4) then the Roe algebra $C^*(\mathcal{O}_\Gamma(M))$ has non-compact ghost projections (this was one of the key steps in [DGLY02]). Very recently, Sawicki announced that such warped cones do indeed violate the coarse Baum–Connes conjecture [Saw17c].

This was what was known at the time I started investigating the geometry of warped cones. Since all the known counterexamples to the coarse Baum–Connes conjecture were based on the presence of expanders, it was natural to try to clarify the relation between expanders and warped cones. In this thesis I make this relation explicit, and, for the most part of it, I build on these ideas to investigate the geometry of actions, expanders and warped cones.

[2]The Haagerup property is an analytic version of amenability (which we shall not define). All amenable groups and free groups have the Haagerup property. See *e.g.* [DK17b] for more on this.

Approximating graphs

Let again $\Gamma = \langle S \rangle$ be a finitely generated group acting by diffeomorphisms on a compact Riemannian manifold[3] M (see Section 2.3 for some basics of Riemannian geometry). Fixing a parameter $r > 0$, we now wish to "approximate" this action up to error $\approx r$ with a finite graph. The idea for doing so is to choose a finite partition $\mathcal{P}$ of M into many (regular) regions $R \subset M$ that have comparable sizes (*i.e.* diameter, volume and eccentricity: see Section 2.1 for precise definitions) and so that the diameter of these regions is approximately r. We then define an *approximating graph at scale* r as the graph $\mathcal{G}_r(\Gamma \curvearrowright M)$ whose vertex set is the set of regions in the partition $\mathcal{P}$, and so that two regions $R, R' \in \mathcal{P}$ are linked by an edge in $\mathcal{G}_r(\Gamma \curvearrowright M)$ if there exists a generator $s \in S$ so that the intersection $s(R) \cap R'$ is *not* trivial (formal definitions are in Subsection 5.1.1).

Note that the the graph $\mathcal{G}_r(\Gamma \curvearrowright M)$ is not uniquely defined, as it depends both on the choice of the partition $\mathcal{P}$ and on the generating set S. Still, different choices produce coarsely equivalent graphs (uniformly in the parameter r). It makes therefore sense to study the coarse geometry of the approximating graphs $\mathcal{G}_r(\Gamma \curvearrowright M)$ as $r > 0$ varies.

Consider now the warped cone $\mathcal{O}_\Gamma(M)$ associated with the same action, and for every $t \geq 1$ let $\mathcal{O}^t_\Gamma(M)$ denote the *level set at height* t. That is, $\mathcal{O}^t_\Gamma(M)$ is the subset $M \times \{t\} \subset \mathcal{O}_\Gamma(M)$ equipped with the restriction of the warped metric δ_Γ. One of the key observations from which this work originated is the following result.

Proposition A (Proposition 8.2.6)**.** *As t varies in $[1, \infty)$, the metric space $\mathcal{O}^t_\Gamma(M)$ and the finite graph $\mathcal{G}_{t^{-1}}\big(\Gamma \curvearrowright M\big)$ are naturally uniformly coarsely equivalent.*

The above result clarifies the interplay between approximating graphs and warped cones. This is in turn fruitful in two directions: on the one hand it allows one to study warped cones by looking at finite approximations of the action; and on the other hand it allows to study the "geometry" of the approximations of the action $\Gamma \curvearrowright M$ (and hence the action itself) by studying the geometry of the associated warped cone.

Actions expanding in measure

The second key notion that we introduce in this thesis is that of *measure expanding action* (Definition 5.2.1). This mimics the behaviour of expander graphs in the setting

[3]All the results stated in the rest of the introduction are actually proved in a much greater generality in this thesis. Here we restricted to smooth actions on manifolds to free our exposition from the technical details.

of actions on probability spaces. In the specific case of actions of groups on Riemannian manifolds (equipped with their Riemannian volume), we say that the action $\Gamma \curvearrowright M$ is *α-expanding* for some $\alpha > 0$ if we have that for every measurable subset $A \subset M$ with $\mathrm{Vol}(A) \leq \frac{1}{2}\mathrm{Vol}(M)$ the measure of the union of the images of A under the generators of Γ is at least α-times larger than the measure of A:

$$\mathrm{Vol}\left(\bigcup_{s\in S} s(A)\right) \geq (1+\alpha)\,\mathrm{Vol}(A).$$

The action is *expanding in measure* if it is α-expanding for some $\alpha > 0$.

We can then prove the following characterisation:

Theorem B (Theorems 5.2.12 and 7.1.3)**.** *For any sequence of parameters $r_k \to 0$, the sequence of approximating graphs $\mathcal{G}_{r_k}(\Gamma \curvearrowright M)$ is a family of expanders if and only if the action $\Gamma \curvearrowright M$ is expanding in measure.*

The idea of the proof of the above theorem is that the condition on measure-expansion is equivalent to producing a lower bound on the Cheeger constants of the approximating graphs. Then one only has to show that graphs approximating an action by diffeomorphisms on a compact manifold have uniformly bounded degree.

The freedom on the choice of the sequence r_k and the partition implies that we can build expanders with vertex sets of arbitrary cardinality:

Corollary 1.0.2 (Proposition 7.1.8)**.** *For every unbounded increasing sequence of cardinalities $n_k \in \mathbb{N}$ there exists a family of expanders where the k-th graph has precisely n_k vertices.*

Combining Proposition A and Theorem B we obtain the following complete characterisation of expansion in warped cones.

Corollary 1.0.3 (Theorem 8.2.7)**.** *For any unbounded sequence $t_k \to \infty$, the level sets $\mathcal{O}_\Gamma^{t_k}(M)$ are coarsely equivalent to a sequence of expanders if and only if the action $\Gamma \curvearrowright M$ is expanding in measure.*

I should now point out that, for some of the applications, we are also going to consider approximating graphs that are quite different in spirit from those used so far (see *e.g.* Section 7.2). Indeed, up to this point it was implicit that we were fixing an action on a compact space and then using the approximating graphs to obtain finer and finer approximations of that action. Still, if one was simply looking for sequences of graphs they could also allow for the action to vary and consider sequences

of approximating graphs of the form $\mathcal{G}(\Gamma_i \curvearrowright M_i)$ with $i \in \mathbb{N}$. In this case (the proof of) Theorem B still implies that if the actions are expanding in measure *uniformly* then this sequence is a family of expanders (the converse implication does not need to hold).

Application I: a large number of examples of expanders

Theorem B can be used to produce a great number of families of expanders, as suffices to prove that an action is expanding in measure to be able to use it to construct explicit expanders.

The easiest way to construct examples of measure expanding actions is by considering the case of measure preserving actions *i.e.* actions $\Gamma \curvearrowright M$ such that for every $\gamma \in \Gamma$ the map $\gamma \colon M \to M$ preserves the Riemannian volume. Indeed, in this setting we have the following characterisation:

Theorem C (Proposition 6.1.2)**.** *A measure preserving action $\Gamma \curvearrowright M$ is expanding in measure if and only if it has a spectral gap.*

This criterion has various consequences. To begin with, recall that Druţu and Nowak proved that warped cones admit non-compact ghost projections when the action has spectral gap, *i.e.* precisely when Theorem B implies that the action is expanding in measure. We can hence apply Proposition A and Theorem B to deduce that the counterexamples to the coarse Baum–Connes conjecture constructed using warped cones [Saw17c] always contain families of expanders. Note however that, a priori, the presence of a coarsely embedded family of expanders in a metric space does not necessarily imply that the metric space violates the coarse Baum–Connes conjecture.

Corollary 1.0.4. *The level sets of the warped cones which are candidate counterexamples to the coarse Baum–Connes conjecture are coarsely equivalent to a family of expanders.*

On a different note, Theorem C allows us to use well developed analytical and representation theoretical tools to produce many examples of expanding actions. For example, one can immediately deduce that every ergodic action of a group with Kazhdan property (T) will be expanding in measure.

More generally, a number of natural measure preserving actions are known to have spectral gaps (see *e.g.* [BG07, BdS14, CG11, Sha00, Bek03, GJS99]). To mention an example that will return over and over in this thesis, it is proved in [BG07] that if

$a, b \in \mathrm{SO}(3, \overline{\mathbb{Q}})$ are two matrices with algebraic coefficients that generate a non-abelian free group, then their action by rotations on $\mathbb{S}^2$ has spectral gap. Whence we get:

Corollary 1.0.5. *The level sets of the warped cones $\mathcal{O}_{\langle a,b\rangle}\big(\mathrm{SO}(3,\mathbb{R})\big)$ and $\mathcal{O}_{\langle a,b\rangle}\big(\mathbb{S}^2\big)$ are coarsely equivalent to families of expander graphs.*

Note that the free group does have the Haagerup property. Still, we already explained that expanders cannot be coarsely embedded into Hilbert spaces. This implies a strongly negative answer to Question 1.0.1 (this answer was previously and independently obtained also by Nowak and Sawicki in [NS17]).

Corollary 1.0.6. *There exist subgroups of compact Lie groups $\Gamma < G$ that have the Haagerup property and so that the warped cones $\mathcal{O}_\Gamma(G)$ do not coarsely embed into Hilbert spaces.*

We would also like to point out that there is an old conjecture by Gamburd, Jakobson and Sarnak [GJS99] stating that if G is a compact simple Lie group, then for a generic k-tuple $(g_1, \dots, g_k) \in G^k$ the action by left multiplication of the generated group $\langle g_1, \dots, g_n\rangle$ on G has a spectral gap. By Corollary 1.0.3, this conjecture can be equivalently restated as follows:

Conjecture 1.0.7 (Gamburd–Jakobson–Sarnak)**.** *For a generic k-tuple $(g_1, \dots, g_k)$ in a compact simple Lie group G, the level sets of the warped cone $\mathcal{O}_{\langle g_1,\dots,g_n\rangle}(G)$ form a family of expanders.*

When leaving the world of measure preserving actions, one would expect that it should be even easier to produce examples of measure expanding actions. Still, in that case most of the analytic tools that we could use are not available any more and, interestingly enough, at the moment I hardly know of any provably expanding action which is not measure preserving (the only exceptions that I am aware of are some actions on Poisson boundaries and the work of Bourgain and Yehudayoff [BY13]).

Application II: a more general approach to expanders

One of the advantages of the approach to families of expanders via approximating graphs, is that it somehow unifies various strategies for constructing expanders.

The idea that when a graph comes naturally from an action on a measure space it should be possible estimate its Cheeger constant by studying expanding properties of the action has already been in the air for a while. In fact, specific instances of it have been used more or less implicitly in many works on expanders (*e.g.* [Mar73,

GG81, Sha97, BY13]). These techniques clearly fall under the umbrella of graphs approximating actions.

Another classical and extremely successful way of producing expanders is by considering Cayley graphs of finite groups (*e.g.* a sequence of Cayley graphs of finite quotients of an infinite finitely generated group Γ). In some sense, this can also be seen as a special case of approximating graphs. Indeed, the Cayley graph $\mathrm{Cay}(\Gamma, S)$ can be seen as the graph approximating the action by (right) multiplication of Γ on itself, where the finite partition of Γ is simply the discrete one *i.e.* every region is a single element of Γ. Then, proving that a sequence of finite Cayley graphs $\mathrm{Cay}(\Gamma_i, S_i)$ forms a family of expanders is equivalent to proving that the Γ_i-actions are uniformly expanding in measure (here we are using the uniform measure on each Γ_i).

In particular, if the groups Γ_i are given by finite quotients $\Gamma_i = \Gamma/N_i$ and S is a generating set of Γ, then the graphs $\mathrm{Cay}(\Gamma/N_i, S)$ form a family of expanders if and only if the actions $\Gamma/N_i \curvearrowleft \Gamma$ are uniformly expanding in measure—this can also be rephrased by saying that the action of Γ on the profinite group obtained as the completion Γ with respect to the groups N_i (equipped with an appropriate measure) must be expanding. In this setting, the approach with approximating graphs reduces to the well-known fact that such a sequence is a family of expanders if and only if the sequence of subgroups N_i has property (τ). See Subsection 2.8.3 and Remark 6.1.7 for more on this.

This was to say that approximating graphs cover most of the known constructions of expanders (the major construction that does not seem fall into this family is that of zig-zag products [RVW02]). It should still be emphasized that approximating graphs also produce a multitude of completely new families of expanders, many of which have rather interesting and unexpected geometric properties coming from the link between approximating graphs and warped cones (continue reading for more on this).

Application III: superexpanders[4]

We already discussed at length that expanders do not coarsely embed into a Hilbert space. Still, it is a deep open problem whether there exist expanders that can be coarsely embedded into some superreflexive Banach space (a Banach space is *superreflexive* if and only if it is isomorphic to a uniformly convex Banach space, see Section 2.4 for more on Banach spaces). This question was first asked by V. Lafforgue and, besides being a natural question to ask, it has meaningful implications both

[4]The results here discussed are joint work with Tim de Laat.

in view of the Baum–Connes conjecture (one could try to extend Yu's proof of the conjecture to spaces that coarsely embed into more general Banach spaces) and in theoretical computer science.

In [Pis10], Pisier introduced the class of uniformly curved Banach spaces and showed that expanders are not coarsely embeddable into such spaces. This was a great step forward, as there are no known examples of superreflexive Banach spaces that are not uniformly curved. However, the aforementioned problem remains open.

We shall say that a family of expanders is a superexpander (Definition 2.7.11) if it does not coarsely embed into any uniformly convex Banach space. We remark that, despite the results of Pisier, only a small number of families of expanders are known to be superexpanders. The first examples of superexpanders were obtained by Lafforgue as Cayley graphs of quotients of a group with *Lafforgue's strong Banach property (T)* [Laf08]. The main other source of superexpanders, obtained by means of a combinatorial construction, was provided by Mendel and Naor in [MN14].

It turns out that approximating graphs are a good tool to produce numerous more examples of superexpanders. Indeed, it is relatively simple to prove that if an action $\Gamma \curvearrowright M$ has a strong Banach-valued spectral gap (see Subsection 2.4.5) then the approximating graphs $\mathcal{G}_r(\Gamma \curvearrowright M)$ cannot coarsely embed into uniformly convex Banach spaces (Proposition 5.4.3). We can hence use such actions to obtain superexpanders.[5]

The easiest way to produce actions with such strong Banach-valued spectral gap is to consider ergodic actions of groups with Lafforgue strong Banach property (T) (this technique was also suggested in [NS17]). For the sake of concreteness, one can prove (see Section 6.3) that for every $d \geq 5$ the group $\Gamma_d \coloneqq \mathrm{SO}(d, \mathbb{Z}[\frac{1}{5}])$ is an infinite group with strong Lafforgue property (T). Whence we obtain the following:

Theorem D (de Laat–V; Corollary 7.1.6)**.** *For every $d \geq 5$ and every infinitesimal sequence $r_k \to 0$, both sequences of graphs $\mathcal{G}_{r_k}\big(\Gamma_d \curvearrowright \mathrm{SO}(d, \mathbb{R})\big)$ and $\mathcal{G}_{r_k}\big(\Gamma_d \curvearrowright \mathbb{S}^{d-1}\big)$ give superexpanders.*

Geometric invariants of actions

Besides its interest in relation with the Baum–Connes conjecture, the warped cone construction is also intriguing in view of its relation with the study of dynamics of actions. In fact, the coarse geometry of warped cones encodes information about both the base space M and the action of the group Γ. This seems to suggest that it is

[5] This was also shown in [Saw17a].

worthwhile to further investigate how much information on the action it is possible to recover studying warped cones. Especially because we already know that it is possible to find geometric characterisations of some properties of the action, an example of this behaviour being precisely the fact that the action is expanding if and only if the level sets are coarsely equivalent to expanders. See also [Saw17b, FNvL17] for more instances of this interplay between geometric and dynamic properties.

It is worth remarking that there have already been a number of attempts to meaningfully codify information about actions into some geometric or topological object (one example being the notion topological entropy). Finding such a construction could lead to unexpected bridges between geometry and dynamical systems, from which there would surely be much to gain.

It is still too soon to know whether warped cones could be such link because not much research went into them yet. In particular, from what I could see there have not yet been meaningful instances of dynamical problems solved by studying of the geometry of warped cones. Besides, it might even be the case that it is not quite the coarse geometry of the warped cone that should play the main role, but some other weaker (or stronger) sort of geometric information.

At any rate, it is surely well worth to study the coarse geometry of warped cones. Indeed, even if there was no substantial return in terms of dynamical information it would still clarify the structure of these important counterexamples to Baum–Connes and the expanders obtained from them. This is what we are going to do in the rest of this thesis.

The coarse geometry of warped cones I: local structure[6]

Let $\Gamma \curvearrowright M$ be an action by isometries on a compact Riemannian manifold—we remark here that the results of this section and application are the only ones for which it is actually essential for M to be a manifold and not a wilder geometric object. We now wish to study the local geometry of the warped cone $\mathcal{O}_\Gamma(M)$ around a point $(x,t) \in M \times [1,\infty)$ as t gets larger.

More precisely, we fix a radius $r > 0$, a point $x \in M$ and we study the geometry of the ball of radius r centred on the point (x,t) as t goes to infinity. If the action of Γ is free at x (*i.e.* if $\gamma \cdot x = x$ for some $\gamma \in \Gamma$ then γ is the identity element) it is easy to show that the ball of radius r centred at (x,t) is actually isometric to the ball of radius r centred at the point $((x,t),e)$ in the product $\mathcal{O}(M) \times \mathrm{Cay}(\Gamma, S)$, where

[6]The material in this section and the subsequent application is also joint work with de Laat.

$\mathcal{O}(M)$ denotes the (unwarped) infinite Riemannian cone and the product is equipped with the ℓ^1-norm. Moreover, it is a basic exercise of Riemannian geometry to check that, as t increases, the ball of radius r centred at (x,t) in $\mathcal{O}(M)$ becomes more and more 'similar' to a ball of radius r in the Euclidean space $\mathbb{E}^{d+1}$ (*i.e.* $\mathbb{R}^{d+1}$ equipped with the Euclidean metric), where $d = \dim(M)$.

Summing up we see that, as soon as one considers level sets 'sufficiently high up', the local geometry of the warped cone $\mathcal{O}_\Gamma(M)$ close to points where the action of Γ is free will just look like the local geometry of $\mathbb{E}^{d+1} \times \mathrm{Cay}(\Gamma, S)$. At this point we can adapt an argument of Khukhro and Valette [KV17] to prove the following rigidity result.

Theorem E (de Laat–V; Theorem 8.3.3 and Appendix B)**.** *Let $\Gamma \curvearrowright M$ and $\Lambda \curvearrowright N$ be essentially free actions by isometries on Riemannian manifolds. If the associated warped cones are coarsely equivalent then $\Gamma \times \mathbb{Z}^{\dim(M)+1}$ is quasi-isometric to $\Lambda \times \mathbb{Z}^{\dim(N)+1}$.*

Theorem E gives us the first tools to show that some warped cones cannot be coarsely equivalent (a similar result was also obtained by Sawicki [Saw17b]). In particular this result applies very well to actions of groups that are relatively rigid in the sense that they are not quasi-isometric to other groups with 'Euclidean factors of the wrong dimension'.

Application IV: genuinely different expanders

Here and in the sequel, we say that two sequences of metric spaces $(X_k)_{k\in\mathbb{N}}$ and $(Y_k)_{k\in\mathbb{N}}$ are coarsely equivalent if there exists coarse equivalences $X_k \sim Y_k$ with constants uniform in k. Note that using this definition it is easy to produce examples of sequences of expanders that are not coarsely equivalent, for example by carefully selecting appropriate subsequences from a family of expanders (see Appendix A). Still, such examples are in some sense trivial; *e.g.* because they could very well have coarsely equivalent subsequences. Also in view of this fact, we say that two sequences of metric spaces are *coarsely disjoint* (Definition 8.2.3) if they do not admit coarsely equivalent subsequences.

It follows from (the proof of) Theorem E that if $\Gamma \curvearrowright M$ and $\Lambda \curvearrowright N$ are two actions by isometries and $\Gamma \times \mathbb{Z}^{\dim(M)}$ is not quasi-isometric to $\Lambda \times \mathbb{Z}^{\dim(N)}$ then for any two unbounded sequences t_k and s_k the families of level sets $\mathcal{O}^{t_k}_\Gamma(M)$ and $\mathcal{O}^{s_k}_\Lambda(N)$ are coarsely disjoint. Rigidity results for high rank lattices [KL97] imply the following:

Corollary 1.0.8. *The superexpanders arising from graphs approximating the actions $\Gamma_d \curvearrowright \mathrm{SO}(d,\mathbb{R})$ or $\Gamma_d \curvearrowright \mathbb{S}^{d-1}$ as in Theorem D are coarsely disjoint as $d \geq 5$ varies.*

Moreover, the understanding of the local geometry given by Theorem E can be further specialised to prove that the above superexpanders are coarsely disjoint from every family of superexpanders obtained as cayley graphs of quotients of groups with Lafforgue's strong Banach property (T).

The coarse geometry of warped cones II: coarse fundamental group

A very different approach to coarse rigidity for warped cones is inspired by [DK17a] and makes use of discrete fundamental groups. The *discrete fundamental group at scale* θ of a metric space (X, d) was defined by Barcelo, Capraro and White [BCW14] as the group $\pi_{1,\theta}(X)$ which is the analogue of the fundamental group of X where continuous loops are replaced by closed θ-paths (*i.e.* finite sequences of points with $d(x_i, x_{i+1}) \leq \theta$) which are considered up to θ-homotopies (see Section 3.1 for a detailed discussion).

From our perspective, the usefulness of the discrete fundamental groups is that the study of the groups $\pi_{1,\theta}(X)$ for (families of) metric spaces can provide some strong coarse invariants. Indeed, even if it is not true in general that $\pi_{1,\theta}(X)$ is invariant under coarse equivalences, it is easy to show that a coarse equivalence $X \to Y$ induces a homomorphism of $\pi_{1,\theta}(X)$ to $\pi_{1,\theta'}(Y)$ where the parameter θ' is explicitly bounded in term of θ and the constants of the coarse equivalence. This information can sometimes be enough to prove that such a coarse equivalence cannot exist.

It turns out that it is possible to explicitly compute the discrete fundamental groups of the level sets of warped cones of actions of free groups (the case of more general groups then follows, as any action can be seen as a quotient of a non-faithful action of a free group). In particular, we prove the following:

Theorem F (Theorem 8.4.1)**.** *For every $\theta \geq 1$ there exists a t_0 large enough so that for every $t \geq t_0$ we have*

$$\pi_{1,\theta}\big(\mathcal{O}^t_{F_S}(M)\big) \cong \big(\pi_1(M) \rtimes F_S\big)/\langle\langle K_\theta \rangle\rangle$$

where K_θ can be described explicitly and depends on the set of elements $w \in F_S$ for which the homeomorphism $w\colon M \to M$ has fixed points.

As sample consequences of Theorem F we report the following.

Corollary 1.0.9 (Corollary 8.4.4)**.** *If $d \geq 3$ is odd and $F_S \curvearrowright \mathbb{S}^{d-1}$ is any action by rotations, then*

$$\pi_{1,\theta}\Big(\mathcal{O}^t_{F_S}\big(\mathbb{S}^{d-1}\big)\Big) = \{0\}$$

for every t and $\theta \geq 1$.

Corollary 1.0.10 (Corollary 8.4.8)**.** *If Γ is finitely presented, $\Gamma \curvearrowright M$ is a free action and $\pi_1(M) = \{0\}$, then*

$$\pi_{1,\theta}\big(\mathcal{O}^t_\Gamma(M)\big) \cong \Gamma$$

for every θ and t large enough.

Note that from Theorem F it follows that, once θ is fixed, the discrete fundamental group of the level set $\mathcal{O}^t_{F_S}(M)$ does not depend on t (as soon as it is large enough). In many instances, it is also the case that (the limit for large t) of the group $\pi_{1,\theta}\big(\mathcal{O}^t_{F_S}(M)\big)$ does not even depend on $\theta \gg 1$. When this happens, we say that the warped cone has *stable discrete fundamental group* and we denote such limit by $\pi_{1,\infty}\big(F_S \curvearrowright M\big)$ (in general, we can define this group as the direct limit of the discrete fundamental groups). It is the case that if a finitely generated group Γ acts freely on M, then the warped cone has stable discrete fundamental group if and only if Γ is finitely presented.

This discrete fundamental group 'at scale infinity' can be used as a 'global' coarse invariant.[7] In fact we have:

Theorem G (Theorem 8.5.9)**.** *If (the level sets of) two warped cones $\mathcal{O}_\Gamma(M)$ and $\mathcal{O}_\Lambda(N)$ are coarsely equivalent and $\mathcal{O}_\Gamma(M)$ has stable discrete fundamental group, then $\mathcal{O}_\Lambda(N)$ has stable discrete fundamental group as well and*

$$\pi_{1,\infty}\big(\Gamma \curvearrowright M\big) \cong \pi_{1,\infty}\big(\Lambda \curvearrowright N\big).$$

Application V: even more different expanders with unexpected properties

Since approximating graphs are uniformly coarsely equivalent to level sets of warped cones, their discrete fundamental groups will be (roughly) the same as those of such level sets. This immediately implies the existence of coarsely simply connected expanders and superexpanders:

[7] Very recently, the discrete fundamental group of warped cones was also independently investigated by Fisher, Nguyen and van Limbeek [FNvL17]. They restricted their attention to free minimal actions by isometries on homogeneous spaces, and, in that restricted settings, they managed to use these invariants to prove some surprisingly powerful rigidity results (up to finite quotients, coarsely equivalent warped cones must come from actions that are conjugate). This allowed them to explicitly produce continua of coarsely disjoint warped cones (and hence expanders and superexpanders).

Corollary 1.0.11. *Let $d \geq 3$ be an odd number. For every expanding action by rotations $F_S \curvearrowright \mathbb{S}^{d-1}$ and infinitesimal sequence $r_k \to 0$, the sequence graphs $\mathcal{G}_{r_k}\big(F_S \curvearrowright \mathbb{S}^{d-1}\big)$ is a family of coarsely simply connected expander graphs.*

In particular, if $d \geq 5$ is odd, the superexpanders $\mathcal{G}_{r_k}\big(\Gamma_d \curvearrowright \mathbb{S}^{d-1}\big)$ are coarsely simply connected.

Moreover we also get the following peculiar behaviour:

Corollary 1.0.12. *If a warped cone $\mathcal{O}_\Gamma(M)$ has stable discrete fundamental group, then any sequence of approximating graphs $\mathcal{G}_{r_k}\big(\Gamma \curvearrowright M\big)$ will have constant θ-discrete fundamental group for every k and θ large enough.*

Finally, the fact that the discrete fundamental groups of $\mathcal{O}^t_{F_S}(M)$ do not depend on t should be contrasted with the theorem of Delabie–Khukhro [DK17a] stating that for every fixed θ and for any sufficiently small finite index subgroup $\Lambda_k \lhd \Lambda$ we have $\pi_{1,\theta}\big(\mathrm{Cay}(\Lambda/\Lambda_k)\big) \cong \Lambda_k$. This provides strong evidence that (level sets of) warped cones are generally coarsely disjoint from sequences of Cayley graphs of finite quotients. In particular, we can prove the following:

Theorem H (Theorem 8.6.1)**.** *If $\Lambda_k \lhd_f \Lambda$ is a nested residual sequence of finite index normal subgroups and the sequence of Cayley graphs $\mathrm{Cay}\big(\Lambda/\Lambda_k, S\big)$ is coarsely equivalent to a sequence of approximating graphs $\mathcal{G}_{r_k}\big(\Gamma \curvearrowright M\big)$, then $\mathcal{O}_\Gamma(M)$ has stable discrete fundamental group if and only if Λ is finitely presented. Moreover, when this is the case $\Lambda_k \cong \pi_{1,\infty}\big(\Gamma \curvearrowright M\big)$ for every $k \in \mathbb{N}$ large enough.*

This result allows us to show that for various classes of groups and warped cones no sequence of approximating graphs can be coarsely equivalent to a sequence of Cayley graphs of finite quotients and vice versa. In particular, we obtain the following:

Theorem 1.0.13 (Corollary 8.6.8)**.** *The superexpanders $\mathcal{G}_{t_k}\big(\Gamma_d \curvearrowright \mathrm{SO}(d,\mathbb{R})\big)$ are not coarsely equivalent to any sequence of Cayley graphs of finite (nested, residual) quotients of a group Λ.*

Chapter 2

Preliminaries

This chapter is meant to be a reference chapter for the precise conventions and standard results that will be used throughout the thesis. As such, it is not meant to be read from the beginning to the end, but it will be referred to as needed in the exposition of the original material.

2.1 Topologies, metrics and measures

Throughout the thesis, all the topological spaces that we will have to consider will be metrisable, if not directly metric spaces. We will usually denote by X a metric space, and we will sometimes write it as (X, d) when we want to stress that d is the metric on it.

Given $x \in X$ and $r \geq 0$, we will denote by $B(x, r) \coloneqq \{y \in X \mid d(x, y) < r\}$ the *open ball* of radius r centred at x and by $\overline{B}(x, r) \coloneqq \{y \in X \mid d(x, y) \leq r\}$ be the *closed ball*. Note that the closed ball needs not to be the closure of an open ball, in general. Recall that a metric space is *proper* if every closed ball is compact.

For a subset $Y \subseteq X$ and a radius $r \geq 0$ we denote by $N_r(Y)$ and $\overline{N}_r(Y)$ its open and closed[1] neighbourhoods of radius r:

$$N_r(Y) = \left\{x \in X \;\middle|\; \exists y \in Y,\ d(x, y) < r\right\} = \bigcup_{y \in Y} B(y, r)$$

$$\overline{N}_r(Y) = \left\{x \in X \;\middle|\; \exists y \in Y,\ d(x, y) \leq r\right\} = \bigcup_{y \in Y} \overline{B}(y, r).$$

We say that $Y \subset X$ is *discrete* if for every $y \in Y$ there exists an open neighbourhood $U \subseteq X$ such that $Y \cap U = \{y\}$. The *diameter* of a set $Y \subseteq X$ is $\operatorname{diam}(Y) \coloneqq \sup\{d(y, y') \mid y, y' \in Y\}$.

[1]The closed neighbourhood of a non compact set needs not be closed. We still call it closed neigbourhood because "neighbourhood with a bar" sounds silly.

2.1.1 General topology (for metric spaces)

A *path* on X is a continuous map $\gamma\colon [0,1] \to X$. We will usually denote by γ^* the *reverse path* $\gamma^*(t) \coloneqq \gamma(1-t)$. We will often implicitly confound paths with their images. The space X is *path-connected* if every two points in X can be joined by a path. We will usually only deal with path connected spaces.

A *free homotopy* between two paths $\gamma, \gamma'\colon [0,1] \to X$ is a continuous function $H\colon [0,1]\times[0,1] \to X$ whose restrictions to $[0,1]\times\{0\}$ and $[0,1]\times\{1\}$ coincide with γ and γ'. A *homotopy* is a free homotopy that keeps the endpoints fixed, *i.e.* such that the restrictions to $\{0\}\times[0,1]$ and $\{1\}\times[0,1]$ are constants. If X is path-connected, every pair of paths are freely homotopic, but they need not be homotopic.

A *closed path* or *loop* in X is a path $\gamma\colon [0,1] \to X$ with $\gamma(0) = \gamma(1)$ (equivalently, it is a continuous mapping of the circle $\gamma\colon \mathbb{S}^1 \to X$). Homotopies of loops coincide with homotopies of paths, while free homotopy of loops differ from free homotopies of paths in that they are not allowed to break the loops into open paths (*i.e.* $H(0,t) = H(1,t)$ for every $t \in [0,1]$). We say that a loop is *null-homotopic* if it is homotopic to a constant path or, equivalently, if it is freely homotopic to a constant path.

If γ and γ' are two paths so that $\gamma(1) = \gamma'(0)$, we denote by $\gamma\gamma'$ the path obtained concatenating them. The *fundamental group* based at $x \in X$ is the group $\pi_1(X,x)$ of closed paths based at x up to homotopy, equipped with the concatenation operation. If X is path connected, the isomorphism class of its fundamental group does not depend on the choice of x and we will sometimes simply denote it by $\pi_1(X)$.

The following material is mostly taken from [Sak13, Section 5.14].

Definition 2.1.1. The space X is *locally path connected* (shortened as l.p.c.) if for every point $x \in X$ and neighbourhood U of x there exists a neighbourhood $x \in V \subseteq U$ such that every two points in V are joined by a path in U. We say that X is *uniformly locally path connected* (u.l.p.c.) if for every $\epsilon > 0$ there exists a $\delta > 0$ such that for every $x \in X$ any two points in $B(x,\delta)$ are connected by a path γ with image completely contained in $B(x,\epsilon)$.

Remark 2.1.2. It is easy to show that if X is l.p.c. then its path-connected components are both open and closed (a *path-connected component* is a maximal path-connected subset). In particular, if X is also connected then it is path-connected.

Some authors use different definitions for l.p.c. spaces (*e.g.* by insisting that the neighbourhood V be open). These notions are equivalent because of the following:

Lemma 2.1.3. *A space X is l.p.c. if and only if every point in X admits a neighbourhood basis of path-connected open neighbourhoods.*

Proof. The 'if' part of the statement is clear. For the 'only if' part, note that if X is l.p.c. then every open subset of X is itself l.p.c. In particular, for every point $x \in X$ and open neighbourhood U of x, the space U will be l.p.c. It follows by Remark 2.1.2 that the connected component of U containing x will be a path-connected open neighbourhood of x in U (and hence X). □

Definition 2.1.4. The space X is *semi-locally simply connected* (s.l.s.c.) if for every $x \in X$ there exists a $\epsilon(x) > 0$ small enough so that every loop with image contained in $B\big(x, \epsilon(x)\big)$ is null-homotopic in X (but is not necessarily null-homotopic in the ball). We say that X is *uniformly semi-locally simply connected* (u.s.l.s.c.) if it is semi-locally simply connected and one can choose $\epsilon(x)$ to be constant (*i.e.* uniform over $x \in X$).

Remark 2.1.5. We recall that semi-local simple connectedness is one of the weakest conditions that allows one to prove that a space X admits a universal cover $\widetilde{X}$.

Lemma 2.1.6. *If X is compact and l.p.c (respectively, s.l.s.c) then it is u.l.p.c (respectively, u.s.l.s.c).*

Proof. Let X be compact and l.p.c. and fix $\epsilon > 0$. Then, for every $x \in X$ let $\delta(x) > 0$ be the largest radius such that any two points in $B(x, \delta(x))$ can be connected by a path in $B(x, \epsilon)$. If X was not u.l.p.c., there would be a sequence of points x_n such that $\delta(x_n) \to 0$. By compactness, we can assume that the sequence x_n converges to a point $\bar{x}$. There exists a $\bar{\delta} < \frac{\epsilon}{2}$ such that points in $B(\bar{x}, \bar{\delta})$ are connected by a path in $B\big(\bar{x}, \frac{\epsilon}{2}\big)$. But now we have that for every $y \in B\big(\bar{x}, \frac{\bar{\delta}}{2}\big)$ the radius $\delta(y)$ must be at least $\frac{\bar{\delta}}{2}$, which leads to a contradiction.

The same argument proves the analogous statement for s.l.s.c. □

At one point we will need the fundamental group of an infinite product of topological spaces. Curiously enough, all my standard references for general topology always restrict their attention to products of finitely many spaces. I feel hence obliged to give a sketch of proof for the following:

Lemma 2.1.7. *Let $(X_\alpha, x_\alpha)_{\alpha \in I}$ be an arbitrary collection of pointed connected topological spaces. Then*

$$\pi_1\Big(\prod_{\alpha \in I} X_\alpha, (x_\alpha)_{\alpha \in I}\Big) \cong \prod_{\alpha \in I} \pi_1(X_\alpha, x_\alpha).$$

Sketch of proof. By the universal property of topological products, there is a bijection between continuous maps $f\colon Y \to \prod_{\alpha\in I} X_\alpha$ and products of continuous maps $f_\alpha\colon Y \to X_\alpha$ for $\alpha \in I$. In particular, this yields a bijection

$$\pi_1\Big(\prod_{\alpha\in I} X_\alpha, (x_\alpha)_{\alpha\in I}\Big) \longleftrightarrow \prod_{\alpha\in I} \pi_1(X_\alpha, x_\alpha)$$

Because loops are given by products of loops and homotopies are given by products of homotopies. It is then enough to check that this bijection is a homomorphism, which is easily done. □

2.1.2 Lengths of paths and discrete paths

A continuous path $\gamma\colon [0,1] \to X$ is *rectifiable* if the supremum

$$|\gamma| \coloneqq \sup\Big\{\sum_{i=1}^{n} d\big(\gamma(i-1), \gamma(i)\big)\Big\}$$

is finite, where the supremum is taken over any finite sequence of times $0 = t_0 < \cdots < t_n = 1$ and $n \in \mathbb{N}$. When this is the case, we say that $|\gamma|$ is the *length* of γ (if γ is not rectifiable then it is convenient to set $|\gamma| = +\infty$).

Note that the length of a path joining two points $x, y \in X$ is at least $d(x,y)$. A metric space is a *path-metric space* if for every pair of points $x, y \in X$ the distance $d(x,y)$ is equal to the infimum of the lengths of paths connecting x to y. If a path between x and y realises their distance $|\gamma| = d(x,y)$, it is said to be a *geodesic*. A metric space is *geodesic* if every two points are joined by a geodesic (in particular, a geodesic metric space is a path metric space).

Definition 2.1.8. We say that (X,d) has *homotopy rectifiable paths* if every continuous path in X is homotopic to a path of finite length.

Remark 2.1.9. If every two points in a metric space X are connected by a rectifiable path, then one can defined an *induced path-metric* by defining the distance between two points to be the infimum of the lengths of the paths connecting them. This might cause some confusion when considering subsets of metric spaces, as the induced metric and the induced path-metric will generally be different.

Let now $\theta > 0$ be a fixed a parameter. A *discrete path* at *scale* θ (or *θ-path*) is a θ-Lipschitz map $Z\colon [n] \to X$ where $[n]$ is the subset $\{0, 1, 2, \ldots, n\} \subset \mathbb{R}$ with the subset metric. Equivalently, Z can be seen as an ordered sequence of points $(z_0, \ldots, z_n)$ in X with $d(z_i, z_{i+1}) \leq \theta$; we will use both notations in the sequel.

The space X is said to be *θ-connected* if any two points of X are connected via a θ-path. A θ-connected metric space is *θ-geodesic* if any two points $z, z' \in X$ are connected via a θ-path $(z_0, \dots, z_n)$ such that $d(z, z') = d(z_0, z_1) + \cdots + d(z_{n-1}, z_n)$.

2.1.3 Separated subsets and bounded geometry

Given $\epsilon > 0$, we say that a subset Y of a metric space (X, d) is *ϵ-separated* if $d(y, y') \geq \epsilon$ for every two $y, y' \in Y$ with $y \neq y'$. Note that if Y is ϵ-separated, then the balls $B(y, \frac{\epsilon}{2})$ with $y \in Y$ are all disjoint in X.

We say that Y is *r-dense* if the union of all the balls $B(y, r)$ with $y \in Y$ covers the whole of X. We say that a subset $Y \subseteq X$ is a *(r, ϵ)-net* if it is r-dense and ϵ-separated. When letting $r = \epsilon$, we will simply call (r, r)-nets *r-nets*.

Note that an r-separated set is maximal (with respect to the ordering given by the inclusion) if and only if it is also r-dense. In particular, it follows from Zorn's Lemma that in every metric space there exist r-nets.

Definition 2.1.10. A metric space has *bounded geometry* if for every $\epsilon > 0$ there is a function $f_\epsilon \colon \mathbb{R}_+ \to \mathbb{N}$ such that every ball of radius r can be covered with $f_\epsilon(r)$ balls of radius ϵ.

Remark 2.1.11. Note that a metric space X has bounded geometry if and only if every ϵ-net $Y \subset X$ in it admits a function $f_Y \colon \mathbb{R}_+ \to \mathbb{N}$ such that every (closed) ball of radius r in Y has at most $f_Y(r)$ points.

2.1.4 Eccentricity and quasi-symmetries

A subset of a metric space is *bounded* if it has finite diameter. We define the *eccentricity* of a bounded subset A of a metric space X as

$$\xi(A) \coloneqq \inf \left\{ \frac{R}{r} \;\middle|\; \exists x \in A \text{ such that } B(x, r) \subseteq A \subseteq B(x, R) \right\}.$$

Note that the eccentricity of a set is equal to the eccentricity of its closure.

Let $\eta \colon [0, +\infty) \to [0, +\infty)$ be a homeomorphism (*i.e.* a strictly increasing unbounded continuous function sending 0 to 0). A homeomorphism $f \colon X \to X$ is *η-quasi-symmetric* if it satisfies

$$\frac{d\big(f(x), f(y)\big)}{d\big(f(x), f(z)\big)} \leq \eta\left(\frac{d(x, y)}{d(x, z)}\right)$$

for every choice of points $z \neq x \neq y$ in X. We say that f is *quasi-symmetric* if it is η-quasi-symmetric for some $\eta \colon [0, \infty) \to [0, \infty)$.

The intuitive meaning of quasi-symmetric homeomorphisms is that they are homeomorphisms that send circles to ellipses of uniformly bounded eccentricity. In fact, note that if f is η-quasi-symmetric and $A \subset X$ is bounded, then the eccentricity of the image $f(A)$ is at most $\eta(\xi(A))$.

Bi-Lipschitz homeomorphisms are clearly quasi-symmetric. In particular, diffeomorphisms of a compact manifold are quasi-symmetric (Corollary 2.3.3).

2.1.5 Measures and measurable maps

We say that a subset A of a measure space (X, ν) is *ν-null* if it is measurable and $\nu(A) = 0$. It is *ν-conull* if $X \smallsetminus A$ is ν-null.

Given two measures ν and ν' defined on the same σ-algebra of X, ν' is *absolutely continuous* with respect to ν (denoted $\nu' \ll \nu$) if every ν-null subset of X is also ν'-null. The measures ν and ν' are *equivalent* if $\nu' \ll \nu$ and $\nu \ll \nu'$. If two measures are equivalent we also say that they represent the same *measure class*.

If ν and ν' are σ-finite measures on X and $\nu' \ll \nu$, then there exists a measurable function $f\colon X \to [0, \infty)$ such that

$$\nu'(A) = \int_A f(x) d\nu(x).$$

Such function is unique up to ν-null sets and is called *Radon-Nikodym* derivative. It is usually denoted by $\frac{d\nu'}{d\nu}$. Note that ν and ν' are equivalent if and only if the Radon-Nikodym derivative is strictly positive in a ν-conull set.

A map between measure spaces $f\colon (X, \nu) \to (Y, \mu)$ is *measurable* if the preimage of a measurable set is measurable. A measurable map defines a *push forward measure* $f_*\nu$ on Y letting $f_*\nu(A) \coloneqq \nu\big(f^{-1}(A)\big)$ for every measurable set $A \subseteq Y$.

A measurable map $f\colon (X, \nu) \to (X, \nu)$ is *measure class preserving* if $f_*(\nu)$ is equivalent to ν. It is *measure preserving* if $f_*\nu = \nu$.

2.1.6 Borel measures

We will often consider topological (metric) spaces that are also equipped with a measure ν. The measure ν is *Borel* if it is defined in the Borel σ-algebra. If not state otherwise, the measure ν will always be a Borel measure. If we want to stress the fact that X is a metric and measure space we denote it by (X, d, ν).

A measure ν on a Hausdorff topological space is a *Radon measure* if it is *locally finite* (every point has a neighbourhood of finite measure) and *inner regular* (the

measure of every measurable set $A \subseteq X$ is equal to the supremum of the measures of its compact subsets $K \subseteq A$). Most of the measures we will consider are Radon.

A Borel measure is *regular* if it is a Radon measure and it is also *outer regular* (the measure of every measurable set $A \subseteq X$ is equal to the infimum of the measures of the open sets containing it).

Definition 2.1.12. A metric and measure space (X, d, ν) is *doubling* if there exists a constant D such that

$$\nu\big(\overline{B}(x, 2r)\big) \leq D\nu\big(\overline{B}(x, r)\big)$$

for every $x \in X$ and every radius $r > 0$. The smallest such constant D is the *doubling constant* of X.

2.2 Groups and actions

Throughout this manuscript, Γ and Λ will always denote discrete countable groups. Moreover, we will usually only deal with finitely generated groups. When Γ is a finitely generated group, we will usually denote by S a finite generating set $\Gamma = \langle S \rangle$. In the sequel, the group Γ will usually by tacitly assumed to come together with a fixed generating set S.

We denote by S^{-1} the set of inverses of elements in S, and we will consistently use the notation $S^{\pm}$ and $S_e^{\pm}$ to denote the sets $S \cup S^{-1}$ and $S \cup S^{-1} \cup \{e\}$, where $e \in \Gamma$ is the identity element.

For any set S, we denote by F_S the *free group generated by* S. That is, F_S is a free group of rank $|S|$ and it can be seen as the group of finite words with letters in the alphabet $S^{\pm}$. The set S is a natural generating set of F_S.

2.2.1 Topological groups

When we do not require the group to be countable (*e.g.* when dealing with Lie groups), we will usually denote it by G. Often, G will be a *topological group* (*i.e.* G is a topological space and both the multiplication map $(g, h) \mapsto gh$ and the inverse map $g \mapsto g^{-1}$ are continuous). A finitely generated group Γ can be considered as a topological group with the discrete topology.

If a topological group is locally compact and Hausdorff, then it admits a unique *Haar measure* up to rescaling. That is, there is a non-zero measure m defined on the Borel σ-algebra of G such that for every measurable subset $A \subseteq G$

- $m(gA) = m(A)$ for every $g \in G$ (left invariant);

- $m(K)$ is finite for every compact $K \subseteq G$;
- $m(A) = \sup\{m(K) \mid K \subseteq A \text{ compact}\}$ (inner regular);
- $m(A) = \inf\{m(U) \mid A \subseteq U \text{ open}\}$ (outer regular).

Any measure having these properties must be equal to a multiple of m.

If a topological space X is locally compact and *second countable* (*i.e.* there exists a countable basis for the topology), then it admits an *exhaustion by compact sets*. That is, there exists a sequence of compact sets $K_n \subseteq X$ such that $K_n \subseteq \mathring{K}_{n+1}$ and $X = \bigcup_{n\in\mathbb{N}} K_n$. It follows that if G is a locally compact second countable Hausdorff topological group then its Haar measure is σ-finite and hence one can apply fundamental results such as Fubini's Theorem.

2.2.2 Actions on spaces

We will denote a left action ρ of a group G on a set X by $\rho\colon G \curvearrowright X$, or simply $G \curvearrowright X$ if we do not need to specify a name for it.[2] Similarly, we denote right actions by $X \curvearrowleft G : \rho$ or $X \curvearrowleft G$. If we refer to an action without specifying whether it is left or right we usually mean that it is a left action.

We will usually denote the image of a point $x \in X$ under the action of an element $g \in G$ by $g \cdot x$. Depending on the context, it will be convenient to use different notation, such as $g(x)$ or $\rho(g)x$ (the latter is especially useful when one wishes to keep the action ρ in the notation to prevent confusion with other actions).

Given an action $G \curvearrowright X$ and subsets $H \subseteq G$ and $A \subseteq X$, we denote by $H \cdot A$ the union

$$H \cdot A \coloneqq \bigcup_{g\in H} g(A) \subseteq X.$$

If G is a topological group and X is a topological space, we will implicitly assume that an action of G on X is a *continuous action* (*i.e.* it is given by a continuous map $G \times X \to X$). An action of Γ on a topological space X is *by homeomorphisms* if $g\colon X \to X$ is a homeomorphism for every $g \in G$.

If (X, d) is a metric space, an action $G \curvearrowright X$ is *by isometries* (resp. *bi-Lipschitz maps, quasi-symmetric maps*) if $g\colon X \to X$ is an isometry (resp. bi-Lipschitz, quasi-symmetric) for every $g \in G$. Note that in the case of bi-Lipschitz actions we do not require that all the maps share the same bi-Lipschitz constants, and similarly for

[2] For linear actions on Banach spaces (*i.e.* a representation) we will generally use the symbol π instead of ρ.

actions by quasi-symmetric map. Note also that if $g\colon X \to X$ is Lipschitz for every $g \in G$ then it is also bi-Lipschitz.

If (X, ν) is a measure space, an action $G \curvearrowright X$ is *by measurable maps* if the map $g\colon X \to X$ is measurable for every $g \in G$. If G is a topological group, an action is *measurable* if it is defined by a measurable map $G \times X \to X$, where G is equipped with the Borel σ-algebra. Note that if Γ is a discrete group then an action is measurable if and only if it is by measurable maps.

An action by measurable maps on a measure space (X, ν) is *measure preserving* (resp. *measure class preserving*) if $g_*\nu = \nu$ (resp. $g_*\nu$ is equivalent to ν) for every $g \in G$. If an action on (X, μ) is measure preserving (resp. measure class preserving) then μ is said to be *invariant* (resp. *quasi-invariant*).

Given an action $G \curvearrowright X$, for every $g \in G$ we denote its *fixed points set* by $\mathrm{Fix}(g) = \{x \in X \mid g(x) = x\}$. For a subset $H \subseteq G$ we let $\mathrm{Fix}(H) = \bigcap_{g\in G} \mathrm{Fix}(g)$. An action $G \curvearrowright X$ is *free* if $\mathrm{Fix}(g) = \emptyset$ for every $g \in G \smallsetminus \{e\}$. If X is a measure space, the action is *essentially free* if $\mathrm{Fix}(g)$ has measure 0 for every $g \in G \smallsetminus \{e\}$. An action is *faithful* if $\mathrm{Fix}(g) \neq X$ for every $g \in G \smallsetminus \{e\}$.

Given a subset $A \subseteq X$, its *stabiliser* is the subgroup

$$\mathrm{Stab}_G(A) = \{g \in G \mid g(A) = A\},$$

and when G is clear form the context we simply denote it by $\mathrm{Stab}(A)$. A subset $A \subset X$ is *G-invariant* if $\mathrm{Stab}(A) = G$. If the action is by measurable maps, a measurable set A is *essentially G-invariant* if $g(A) = A$ up to measure zero sets (*i.e.* the symmetric difference $g(A) \triangle A$ is a ν-null set) for every $g \in G$. Similarly, a measurable function $f\colon X \to \mathbb{R}$ is *G-invariant* if $f(x) = f(g \cdot x)$ everywhere and it is *essentially G-invariant* if $f(x) = f(g \cdot x)$ ν-almost everywhere.

Definition 2.2.1. An action by measurable maps on a measure space $G \curvearrowright (X, \nu)$ is *ergodic* if every G-invariant measurable set $A \subseteq X$ is either ν-null or ν-conull.

It is easy to show that if Γ is a countable group then an action $\Gamma \curvearrowright X$ is ergodic if and only if every essentially Γ-invariant set is either null or conull. This is also true in greater generality:

Theorem 2.2.2. *Let G be a locally compact second countable Hausdorff topological group and let $G \curvearrowright (X, \nu)$ be a measurable action. Then the following are equivalent:*

(i) $G \curvearrowright X$ is ergodic;

(ii) every essentially G-invariant subset is either ν-null or ν-conull;

(iii) every G-invariant function is constant almost everywhere;

(iv) every essentially G-invariant function is constant almost everywhere.

See [BM00, Theorem 1.3] for a proof.[3]

Given two actions $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ and a function $\varphi\colon \Gamma_1 \to \Gamma_2$, we say that a map $F\colon X_1 \to X_2$ is *φ-equivariant* if $F(g \cdot x) = \varphi(g) \cdot F(x)$ for every $x \in X_1$, $g \in \Gamma_1$. Morphisms in the category of actions on sets are given by φ-equivariant maps where φ is a group homomorphism.

2.2.3 Cayley graphs and word lengths

Given a finitely generated group Γ and a finite generating set S, the *(left) Cayley graph* is the (undirected, simplicial) graph[4] $\mathrm{Cay}(\Gamma, S)$ having one vertex for every element of Γ and such that if h and g are two distinct elements in Γ then $\{g, h\}$ is an edge in $\mathrm{Cay}(\Gamma, S)$ if and only if $h = gs$ for some $s \in S^{\pm}$.

The natural path-metric on $\mathrm{Cay}(\Gamma, S)$ coincide with the *(left) word metric* on Γ. That is, given an element $g \in \Gamma$ its *word length* $|g|$ is the length of the shortest word representing it

$$|g| = \min\{n \mid g = s_1 \cdots s_n \text{ with } s_1, \dots, s_n \in S^{\pm}\}.$$

The *(left) word distance* between two elements $g, h \in \Gamma$ is defined as $|h^{-1}g|$.

The action by left multiplication of Γ on itself induces an action by isometries on its Cayley graph. On the contrary the (right) action by right multiplication does not induce isometries on the Cayley graph (it does not send edges to edges), but it has *bounded displacement*. That is, given $h \in \Gamma$ then $d(g \cdot h, g) = |h|$ for every $g \in \Gamma$ and, in particular, it is bounded.

Similarly, we define the *right Cayley graph* as the graph $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Gamma, S)$ with Γ as vertex set and with an edge $\{g, h\}$ if and only if $h = sg$ for some $s \in S^{\pm}$. This time it is the action by right multiplication to induce an action by isometries on $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Gamma, S)$, while the action by left multiplication has bounded displacement. The path metric on $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Gamma, S)$ coincides with the *right word distance*, *i.e.* the distance defined by $(g, h) \to |gh^{-1}|$.

[3] In [BM00] the action is also assumed to be measure class preserving, but this hypothesis is not necessary for the proof.

[4] See Section 2.7 for our conventions on graphs.

Remark 2.2.3. The map sending g to g^{-1} for every $g \in G$ defines a natural isomorphism between $\mathrm{Cay}(\Gamma, S)$ and $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Gamma, S)$. Still, it is necessary to pay attention to whether the Cayley graph being used is the left or right one because it is important to know which one between the left and right multiplication induces an action by isometries.

2.2.4 Warped metrics

Let S be a finite set of homeomorphisms of a metric space (X, d).

Definition 2.2.4. The *warped metric* induced on X by S is the largest metric δ_S such that

- $\delta_S(x, y) \leq d(x, y)$ for every $x, y \in X$;
- $\delta_S(x, s(x)) \leq 1$ for every $s \in S$.

Lemma 2.2.5. *The warped metric δ_S is well-defined and for every pair of points x and y in X we have*

$$\delta_S(x, y) = \inf\Big\{n + \sum_{i=0}^{n} d(x_i, y_i)\Big\} \tag{2.1}$$

where the infimum is taken over $n \in \mathbb{N}$ and $(n+1)$-tuples $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ such that $x = x_0$, $y = y_n$ and for every $i = 1, \ldots, n$ there exists a $s_i \in S^{\pm}$ such that $x_i = s_i(y_{i-1})$.

Moreover, if X is proper then the infimum is actually a minimum.

Proof. It is clear that the expression on the RHS of (2.1) defines a metric on X that satisfies the requirements of Definition 2.2.4. It follows that δ_S exists and we have

$$\delta_S(x, y) \geq \inf\Big\{n + \sum_{i=0}^{n} d(x_i, y_i)\Big\}.$$

On the other hand, for every pair of $(n+1)$-tuples $x_0, \ldots, x_n$ and $y_0, \ldots, y_n$ with $x = x_0$, $y = y_n$ and $x_i = s_i(y_{i-1})$ we have

$$\begin{aligned}\delta_S(x, y) &\leq \delta_S(x_0, y_0) + \delta_S(y_0, x_1) + \delta_S(x_1, y_1) + \cdots + \delta_S(x_n, y_n) \\ &\leq d(x_0, y_0) + 1 + d(x_1, y_1) + \cdots + d(x_n, y_n) = n + \sum_{i=0}^{n} d(x_i, y_i),\end{aligned}$$

which proves the other inequality.

Let now X be a proper metric space, x, y be any two points in X and let $x_0^{(k)}, \ldots, x_{n(k)}^{(k)}$ and $y_0^{(k)}, \ldots, y_{n(k)}^{(k)}$ be sequences of tuples converging to the infimum

in (2.1). Since $n(k)$ is bounded, we can assume it constant $n(k) \equiv n$. Since (X, d) is proper and $y_0^{(k)}$ is contained in the closed ball $\overline{B}_{(X,d)}\big(x_0, \delta_S(x, y)\big)$, there exists a subsequence that converges to a point $\bar{y}_0$. Also, since S is finite we can assume that $s_1^{(k)}$ does not depend on k, and since the action is by homeomorphisms we deduce that the sequence $x_1^{(k)} = s_1 \cdot y_0^{(k)}$ converges to the point $\bar{x}_1 \coloneqq s_1(\bar{y}_0)$. Iterating this process, we produce a subsequence such that the $(n+1)$-tuples converge to two $(n+1)$-tuples $\bar{x}_0, \dots, \bar{x}_n$ and $\bar{y}_0, \dots, \bar{y}_n$ that realise the infimum in (2.1). □

The choice of the set of homeomorphisms S naturally induces an action of the free group F_S on X. Vice versa, given an action of a finitely generated group Γ and a finite generating set S of Γ, one can consider the induced warped metric δ_S. When we want to stress that a warped metric δ_S is coming from a group action, we will often denote it by δ_Γ. This is a slight abuse of notation as the metric depends on the choice of the generating set. This is one of the reasons why we use the tacit convention that finitely generated groups are thought of as equipped with a fixed finite generating set.

Remark 2.2.6. For any choice of a point $x \in X$, the orbit map $\Gamma \to X$ defined by $g \mapsto g(x)$ gives rise to a 1-Lipschitz map of the right Cayley graph to the warped space $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Gamma, S) \to (X, \delta_\Gamma)$. Indeed, two vertices $g, h \in \Gamma$ are joined by an edge if and only if $h = s^{\pm} g$ with $s \in S$, hence $\delta_\Gamma(g(x), h(x)) = \delta_\Gamma\big(g(x), s^{\pm}(g(x)\big) \leq 1$.

When dealing with actions by isometries, the warped distance has a more explicit formula.

Lemma 2.2.7. *If $\Gamma = \langle S \rangle$ is acting by isometries on (X, d), then*

$$\delta_S(x, y) = \inf_{g \in \Gamma} \big(d(x, g(y)) + |g|\big)$$

where $|g|$ is the word length of g in Γ.

Proof. For any $g \in \Gamma$, let $|g| = n$ and let $s_1 \cdots s_n$ be a word in $S^{\pm}$ such that $g = s_1 \cdots s_n$. Then letting

$$\begin{aligned} x_0 = x \qquad & y_0 = s_1 \cdots s_n(y) \\ x_1 = & y_1 = s_2 \cdots s_n(y) \\ x_2 = & y_2 = s_3 \cdots s_n(y) \\ & \cdots \\ x_n = & y_n = y, \end{aligned}$$

yields $(n+1)$-tuples as in Lemma 2.2.5. Thus we have $\delta_S(x, y) \leq \inf_{g \in \Gamma} \big(d(g(x), y) + |g|\big)$.

On the other hand, given $(n+1)$-tuples $x_0, \dots, x_n$ and $y_0, \dots, y_n$ and elements $s_1, \dots, s_n$ as in Lemma 2.2.5, let

$$\begin{aligned} z_0 &= y_0 \\ z_1 &= s_1^{-1}(y_1) \\ &\cdots \\ z_n &= s_1^{-1} \cdots s_n^{-1}(y_n). \end{aligned}$$

Then, since the action is by isometries, we have:

$$\begin{aligned} \sum_{i=0}^{n} d(x_i, y_i) &= d(x_0, y_0) + d(x_1, y_1) + \cdots + d(x_n, y_n) \\ &= d(x_0, z_0) + d\big(s_1(y_0), s_1(z_1)\big) + \cdots + d\big(s_n(y_{n-1}), s_n \cdots s_1(z_n)\big) \\ &= d(x_0, z_0) + d(z_0, z_1) + \cdots + d(z_{n-1}, z_n) \geq d(x, z_n) \end{aligned}$$

and, as $z_n = g(y)$ for $g = s_1^{-1} \cdots s_n^{-1}$, the converse inequality follows. □

2.2.5 Limits and co-limits of groups

In this subsection G will simply denote an abstract (*i.e.* non topological) group. A *directed poset* is a set I with a partial order $\leq$ such that for every pair of elements $i, j \in I$ there exists an element $k \in I$ such that $i \leq k$ and $j \leq k$.

Definition 2.2.8. A *direct system of groups* over a directed poset $(I, \leq)$ is the data of a family of groups $(G_i)_{i \in I}$ and homomorphisms $f_{ij} \colon G_i \to G_j$ for every ordered pair $i \leq j$ in I, such that

- $f_{ii} = \mathrm{id}_{G_i}$ for every $i \in I$,
- $f_{ik} = f_{jk} \circ f_{ij}$ for every ordered triple $i \leq j \leq k$ in I.

The *direct limit* of such a direct system is the group

$$\varinjlim G/G_i := \coprod_{i \in I} G_i \Big/ \sim$$

obtained quotienting the disjoint union of the groups G_i by the relation $\sim$ defined by imposing that $x_i \in G_i$ is equivalent to $x_j \in G_j$ if and only if there exists an index $k \in I$ larger than i and j such that $f_{ik}(x_i) = f_{jk}(x_j)$.

It follows from the definition that for every $i \in I$, the inclusion of G_i in the union $\coprod_{i\in I} G_i$ induces a homomorphism $\varphi_i \colon G_i \to \varinjlim G_i$. Moreover, if we are given a family of group homomorphisms $\psi_i \colon G_i \to H$ such that $\psi_i = \psi_j \circ f_{ij}$ for every $i \leq j$, then there exists a unique homomorphism $\psi \colon \varinjlim G_i \to H$ such that $\psi_i = \psi \circ \varphi_i$ for every $i \in I$ (*i.e.* the direct limit is a *co-limit* in the category of groups).

Lemma 2.2.9. *Given a group G, a directed poset $(I, \leq)$ and a family of normal subgroups $G_i \lhd G$ such that $G_i \subseteq G_j$ for every $i \leq j$, then the quotients G/G_i form a direct system of groups and the direct limit is given by the quotient*

$$\varinjlim G_i = G \Big/ \Big(\bigcup_{i\in I} G_i \Big).$$

Sketch of proof. It is clear that the groups G/G_i together with the quotient maps form a direct system. Let $G_\infty \coloneqq \bigcup_{i\in I} G_i$; the projections $p_i \colon G/G_i \to G/G_\infty$ are coherent with the direct system and therefore—by the universal property of the direct limit—we obtain homomorphism $p \colon \varinjlim G/G_i \to G/G_\infty$ that commutes with the maps p_i. Since p_i is surjective, so is p.

As the direct limit is a quotient of $\coprod G/G_i$, an element in $\ker(p)$ is of the form $[xG_i]$ for some $i \in I$ and $x \in G$. Moreover, xG_i is in the kernel of p_i and therefore $x \in G_\infty$. It follows that $x \in G_j$ for some $j \in J$ and hence $[xG_i] = [G_j] = e$ in $\varinjlim G/G_i$. □

Definition 2.2.10. An *inverse system of groups* over a directed poset $(I, \leq)$ is the data of a family of groups $(G_i)_{i\in I}$ and homomorphisms $f_{ij} \colon G_j \to G_i$ for every ordered pair $i \leq j$ in I, such that

- $f_{ii} = \mathrm{id}_{G_i}$ for every $i \in I$,
- $f_{ik} = f_{ij} \circ f_{jk}$ for every ordered triple $i \leq j \leq k$ in I.

The *inverse limit* (or *projective limit*) of such a direct system is the subgroup of coherent elements in the direct product $\prod_{i\in I} G_i$:

$$\varprojlim G_i \coloneqq \big\{ (x_i)_{i\in I} \;\big|\; x_i = f_{ij}(x_j) \text{ for every } i \leq j \big\} \subseteq \prod_{i\in I} G_i.$$

For every $i \in I$, the projection $\prod_{i\in I} G_i \to G_i$ induces a group homomorphism $\varphi_i \colon \varprojlim G_i \to G_i$. Moreover, if we are given a family of group homomorphisms $\psi_i \colon H \to G_i$ such that $\psi_i = f_{ij} \circ \psi_j$ for every $i \leq j$, then there exists a unique homomorphism $\psi \colon H \to \varprojlim G_i$ such that $\psi_i = \varphi_i \circ \psi$ for every $i \in I$ (*i.e.* the inverse limit is a *limit* in the category of groups).

Remark 2.2.11. In the sequel we will have to consider a direct system of groups G_θ where the parameter θ ranges in the interval $(0, +\infty)$ equipped with the usual ordering $\leq$. Taking the direct limit $\varinjlim G_\theta$ is the natural limit process to consider as θ grows to infinity. On the contrary, if we want to define the limit of the system as θ goes to 0 we have to use the inverse limit. Indeed, the correct poset to consider to define the limit for $\theta \to 0$ would be $(0, \infty)$ equipped with the opposite ordering $\geq$, and in this case the system of groups is not direct but inverse. Neatly enough, the notation helps us as the limit for $\theta \to 0$ then becomes $\varprojlim G_\theta$.

2.3 Riemannian geometry

In this section M will always be a connected *smooth manifold*, *i.e.* a second countable topological space locally homeomorphic to $\mathbb{R}^n$ and equipped with a differentiable structure coming from a C^∞-atlas. The results we need are covered in most introductory books on Riemannian geometry (*e.g.* [GHL12]).

2.3.1 Basic facts and definitions

Let M be a smooth manifold and for every $x \in M$ let T_xM denote its *tangent space* at x and let TM be the *tangent bundle* of M. A *Riemannian metric tensor* ϱ on M is choice of inner products $\varrho_x \colon T_xM \times T_xM \to \mathbb{R}$ for every $x \in M$ that depends smoothly on x. A *Riemannian manifold* (M, ϱ) is a smooth manifold equipped with a Riemannian metric tensor.

For every smooth path $\gamma \colon [0,1] \to M$, applying the differential of γ to the (positive) unit vector defines a map

$$\begin{aligned} \dot{\gamma} \colon [0,1] &\to TM \\ t &\longmapsto d\gamma_t(1) \end{aligned}$$

such that $\dot{\gamma}(t) \in T_{\gamma(t)}M$ for every $t \in [0,1]$. The *speed* of a smooth γ is the function $t \mapsto \|\dot{\gamma}(t)\|_\varrho \coloneqq \sqrt{\varrho_{\gamma(t)}\big(\dot{\gamma}(t), \dot{\gamma}(t)\big)}$. The *length* γ is the integral of its speed

$$\|\gamma\| \coloneqq \int_0^1 \|\dot{\gamma}(t)\|_\varrho dt.$$

The distance between two points in M is defined to be the infimum of the lengths of (piecewise) smooth curves connecting them. This defines a *Riemannian metric* d on M. We will often denote Riemannian manifolds by (M, d), where d is the metric (not the Riemannian tensor).

Remark 2.3.1. Every smooth path $\gamma\colon [0,1] \to M$ can be reparametrised so that it has constant speed. Every smooth path also admits an *arc length* reparametrisation, *i.e.* it can be made into a path $\gamma\colon [0, \|\gamma\|] \to M$ of constant speed 1.

We will denote by $\mathbb{E}^n$, $\mathbb{H}^n$ and $\mathbb{S}^n$ the n-dimensional Euclidean space,[5] hyperbolic space and sphere respectively. Given a Riemannian manifold (M, ϱ) and any positive constant t we denote by tM the Riemannian manifold $(M, t^2\varrho)$ obtained rescaling the Riemannian metric tensor.[6] From the metric space point of view, this is equivalent to rescaling the distance function by t.

If M and N are differentiable manifolds with $\dim(N) \leq \dim(M)$, ϱ is a Riemannian metric tensor on M and $F\colon N \to M$ is a *local embedding* (*i.e.* a smooth map such that the differential d_xF is not singular at any point $x \in N$), then N can be endowed with the *pull-back* Riemannian metric tensor $F^*\varrho$ defined by $F^*\varrho_x(v,w) \coloneqq \varrho_{F(x)}\big(dF(v)_x, dF(w)_x\big)$.

Let (M, ϱ) and (M', ϱ') be Riemannian manifolds with $\dim(M) \leq \dim(M')$. If $F\colon M \to M'$ is a local embedding then it preserves the lengths of curves if and only if $\varrho = F^*\varrho'$. More in general, if we are given Riemannian metric tensors, the norm of the differential of a smooth map $F\colon (M, \varrho) \to (M', \varrho')$ is well-defined

$$\|dF_x\| = \sup\big\{\|dF_x(v)\|_{\varrho'} \;\big|\; v \in T_xM,\ \|v\|_{\varrho} = 1\big\}$$

and we have:

Lemma 2.3.2. *Let (M, ϱ) and (M', ϱ') be Riemannian manifolds. If a local embedding $F\colon M \to M'$ satisfies $\|dF_x\| \leq L$ for every $x \in M$, then F is an L-Lipschitz map.*[7]

Sketch of proof. It is enough to note that for every smooth curve $\gamma\colon [0,1] \to M$ we have

$$\|F \circ \gamma\| = \int_0^1 \|d(F\circ\gamma)_t\|_{\varrho'} dt = \int_0^1 \big\|dF_{\gamma(t)}(\dot\gamma(t))\big\|_{\varrho'} dt \leq \int_0^1 \big\|dF_{\gamma(t)}\big\| \|\dot\gamma\|_{\varrho} dt.$$

□

Corollary 2.3.3. *If $F\colon (M, \varrho) \to (M', \varrho')$ is a diffeomorphism such that both $\|dF\|$ and $\|dF^{-1}\|$ are bounded by L, then it is L-bi-Lipschitz. In particular, if M is compact then any self-diffeomorphism $F\colon M \to M$ is a bi-Lipschitz map.*

[5] The Euclidean space $\mathbb{E}^n$ is just the vector space $\mathbb{R}^n$ equipped with the standard Euclidean metric. We prefer to use the notation $\mathbb{E}^n$ to stress that we think of it as a metric space (as opposed to a vector space).

[6] To rescale the metric by t one needs to rescale the metric tensor by t^2.

[7] The converse is also true, but we will not need it.

A Riemannian manifold (M, ϱ) is naturally equipped with a Borel measure given by the *Riemannian volume*. If $A \subset M$ is a Borel set contained in a chart Ω of M, $x_1, \dots, x_n$ are the *local coordinates* of Ω and ϱ takes the form of $\sum_{i,j=1}^{n} \varrho_{ij}(x) dx^i dx^j$ when written in local coordinates (*i.e.* we have $\varrho_x\big(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\big) = \varrho_{ij}(x)$), then the measure of A is defined by

$$\mathrm{Vol}(A) \coloneqq \int_A \sqrt{\det(\varrho_{ij})} dx_1 \cdots dx_n$$

where the integral is taken with respect to the Lebesgue measure of the chart.[8]

Since the expression $\sqrt{\det(\varrho_{ij})}$ is smooth and always positive, the Riemannian volume inherits a number of properties from the Lebesgue measure. For example, it is a regular measure (Subsection 2.1.6). Moreover, the Sard Theorem implies that if $F\colon M \to N$ is a smooth map, and $X \subseteq M$ is the set of points where $\mathrm{rk}(dF) < \dim(N)$, then $\mathrm{Vol}(F(X)) = 0$. In particular, a submanifold $M \subset N$ of dimension strictly smaller $\dim(N)$ has volume 0 in N.

Every Riemannian manifold (M, ϱ) has a well-defined notion of curvature. More precisely, the *sectional curvature* is a function $\kappa_\varrho(x, E)$ that associates a real number to any 2-dimensional subspace E of any tangent space T_xM.

Theorem 2.3.4 ([GHL12, Theorem 3.82])**.** *For every $n \in \mathbb{N}$ and $\kappa \in \mathbb{R}$ there exists a unique complete simply connected Riemannian manifold M^n_κ of dimension n and constant curvature $\kappa_\varrho(x, E) \equiv \kappa$ up to isometry. Specifically, we have*

- $M^n_\kappa = \frac{1}{\sqrt{-\kappa}}\mathbb{H}^n$ *if* $\kappa < 0$*;*
- $M^n_\kappa = \mathbb{E}^n$ *if* $\kappa = 0$*;*
- $M^n_\kappa = \frac{1}{\sqrt{\kappa}}\mathbb{S}^n$ *if* $\kappa > 0$*.*

We say that a manifold has *pinched curvature* if there exists a constant $\kappa \geq 0$ such that $-\kappa \leq \kappa_\varrho(x, E) \leq \kappa$ for every $x \in M$ and $E \subseteq T_xM$. The sectional curvature is a continuous function, and therefore every compact Riemannian manifold has pinched curvature.

[8] We are being somewhat sloppy on this definition because most of the results we need regarding the Riemannian volume can be taken as black boxes. Full details can be found in [GHL12, Section 3.H.2]

2.3.2 The exponential map

For the proofs of the results stated in this section we refer to [GHL12, Section 2.C].

A *Riemannian geodesic* is a path of constant speed that is a *local geodesic*[9] (*i.e.* for every $0 < s < t < 1$ close enough the restriction of the path to the interval $[s, t]$ is a geodesic in (M, d)). Note that a Riemannian geodesic needs not be a metric geodesic (in the sense of Subsection 2.1.2), while a metric geodesic is a Riemannian geodesic if and only if it has constant speed.

Theorem 2.3.5. *A Riemannian geodesic must be smooth. Moreover, for every $x \in M$ and $v \in T_xM$ there exists a $\epsilon > 0$ such that there is a unique Riemannian geodesic $\gamma\colon (-\epsilon, \epsilon) \to M$ such that $\gamma(0) = p$ and $\dot\gamma(0) = v$. In particular, γ has constant speed $\|v\|_\varrho$.*

Corollary 2.3.6. *Let $\gamma\colon [a, b] \to M$ and $\gamma'\colon [a', b'] \to M$ be two Riemannian geodesics having equal constant speed. If there exists a time $t \in [a, b] \cap [a', b']$ such that $\gamma(t) = \gamma'(t)$ and $\dot\gamma(t) = \dot\gamma'(t)$, then γ and γ' coincide on $[a, b] \cap [a', b']$.*

Corollary 2.3.7. *For every $x \in M$ and $v \in T_xM$ there exists a unique maximal geodesic $\gamma\colon (a, b) \to M$ with $-\infty \le a < 0 < b \le +\infty$ such that $\gamma(0) = x$ and $\dot\gamma(0) = v$.*

Let $U \subseteq TM$ be the of points $(x, v) \in TM$ such that the (unique) geodesic $\gamma_{x,v}$ with $\gamma_{x,v}(0) = x$ and $\dot\gamma_{x,v}(0) = v$ is defined on the whole interval $[0, 1]$. The *exponential map* is the function $\exp\colon TM \to M$ sending (x, v) to $\gamma_{x,v}(1)$. The restriction of exp to a single tangent space T_xM is denoted by $\exp_x\colon U \cap T_xM \to M$.

Theorem 2.3.8. *The set U is open in TM and* exp *is smooth. Moreover, for every $x \in M$ the differential $d(\exp_x)_0\colon T_xM \to T_xM$ naturally coincides with the identity.*

For every $x \in M$ the inner product ϱ_x makes T_xM a metric space. Note that up to renormalisation (equivalently, up to changing coordinate system on M) T_xM is thus isometric to the Euclidean space $\mathbb{E}^n$.

Corollary 2.3.9. *For every $x \in M$ and $\epsilon > 0$ there exists a $\delta > 0$ small enough so that $\exp_x$ is defined on the ball $B(0, \delta) \subset T_xM$ and the restriction $\exp_x|_{B(0,\delta)}$ gives a $(1 + \epsilon)$-bi-Lipschitz equivalence between $B(0, \delta)$ and its image.*

Moreover, if M is compact the parameter δ can be chosen independently of $x \in M$.

[9] Riemannian geodesics are usually defined by differentiable means. In this case, the fact that they are local geodesics is a theorem.

Sketch of proof. Since U is open, the existence of the small ball where $\exp_x$ is defined is obvious. Since $d(\exp_x)_0 = \mathrm{id}$, the pull-back $\exp_x^* \varrho$ coincides with ϱ_x at $0 \in T_xM$. By continuity of the differential, it follows that $\|d(\exp_x)\|$ (resp. $\|d(\exp_x)^{-1}\|$) is uniformly close to 1 in a neighbourhood of $0 \in T_xM$ (resp. $x \in M$). The fact that $\exp_x$ is $(1+\epsilon)$-bi-Lipschitz then follows from Lemma 2.3.2. The 'moreover' part of the statement is a standard compactness argument. □

The following is a classical result:

Theorem 2.3.10 (Hopf-Rinow)**.** *Let (M, ϱ) be a Riemannian manifold. The following are equivalent:*

(i) (M, d) is a complete metric space;

(ii) there exists a point $x \in M$ so that $\exp_x$ is defined on the whole of T_xM,

(iii) for every point $x \in M$ the exponential $\exp_x$ is defined on the whole of T_xM and it is surjective onto M.

Lemma 2.3.11. *Let M be a complete Riemannian manifolds and let x_1, x_2 be two distinct points in it. Then the set of points $y \in M$ such that $d(y, x_1) = d(y, x_2)$ has measure 0 in M.*

Sketch of proof. The distance function $f_x(y) \coloneqq d(y, x)$ is differentiable almost everywhere.[10] For a point $y \in M$ let $w \in T_xM$ be a smallest vector such that $y = \exp_x(w)$. Then the function f_x evaluated at y equals $\|w\|$ and its *gradient* $\nabla f_x(y)$ coincides with the image under the differential $d(\exp_x)_w$ of the (normalised) vector $\frac{w}{\|w\|}$.

The function $f_{x_1} - f_{x_2}$ is differentiable almost everywhere with gradient $\nabla f_{x_1} - \nabla f_{x_2}$. If its gradient vanishes at a point $y = \exp_{x_1}(w_1) = \exp_{x_2}(w_2)$ it follows that $d(\exp_{x_1})_{w_1}[\frac{w_1}{\|w_1\|}] = d(\exp_{x_2})_{w_2}[\frac{w_2}{\|w_2\|}]$ and hence $x_1 = x_2$ because they coincide with the point obtained following for time $\|w_1\| = \|w_2\|$ the unique geodesic leaving x with derivative $-\nabla f_{x_1} = -\nabla f_{x_2}$.

It follows that, when defined, the gradient $\nabla(f_{x_1} - f_{x_2})$ is never trivial and hence the set $\{y \in M \mid f_{x_1}(y) = f_{x_2}(y)\}$ has measure 0. □

[10] The only points where f_x is not smooth are x itself and the *cut-locus* of x, which is known to have measure 0 (see *e.g.* [GHL12, Lemma 3.96]).

2.3.3 Volume of balls in manifolds with pinched curvature

The *injectivity radius at a point* x of a complete Riemannian manifold (M, ϱ) is

$$\operatorname{inj}(x, M) \coloneqq \sup\{r > 0 \mid \exp_x \text{ is injective on } B(0, r) \subset T_x M\};$$

the *injectivity radius* of M is $\operatorname{inj}(M) \coloneqq \inf\{\operatorname{inj}(x, M) \mid x \in M\}$. The injectivity radius at any point is strictly positive by Theorem 2.3.8. The injectivity radius $\operatorname{inj}(x, M)$ is a continuous function of x. Therefore, if M is compact then $\operatorname{inj}(M)$ is strictly positive.[11]

Proposition 2.3.12. *Let $\kappa \in \mathbb{R}$ and $r \geq 0$, let $\operatorname{Vol}_n^{(\kappa)}(r)$ be the volume of the ball of radius r in the (unique) complete simply-connected Riemannian manifold of constant sectional curvature κ. Then the following expressions hold:*

$$\operatorname{Vol}_n^{(\kappa)}(r) = w_{n-1}|\kappa|^{-\frac{n-1}{2}} \int_0^r \sinh^{n-1}\big(s\sqrt{-\kappa}\big)ds \qquad \text{if } \kappa < 0$$

$$\operatorname{Vol}_n^{(\kappa)}(r) = w_{n-1} \int_0^r s^{n-1} ds \qquad \text{if } \kappa = 0$$

$$\operatorname{Vol}_n^{(\kappa)}(r) = w_{n-1}\kappa^{-\frac{n-1}{2}} \int_0^r \sin^{n-1}\big(s\sqrt{\kappa}\big)ds \qquad \text{if } \kappa > 0 \text{ and } r \leq \frac{\pi}{\sqrt{\kappa}}$$

where w_{n-1} is the volume of the sphere $\mathbb{S}^{n-1}$.

Theorem 2.3.13 (Bishop-Gunther-Gromov [GHL12, Theorem 3.101])**.** *Let (M, ϱ) be a Riemannian manifold of dimension n, $x \in M$ any point and let $r \leq \operatorname{inj}(x, M)$. For every $\kappa \in \mathbb{R}$, the following hold:*

- *if every sectional curvature of M is at least κ then $\operatorname{Vol}(B_M(x, r)) \leq \operatorname{Vol}_n^{(\kappa)}(r)$;*
- *if every sectional curvature of M is at most κ then $\operatorname{Vol}(B_M(x, r)) \geq \operatorname{Vol}_n^{(\kappa)}(r)$.*

For a fixed Riemannian manifold (M, d), let $v_M(r)$ and $V_M(r)$ denote the infimum and supremum Riemannian volume of a ball of radius r in M:

$$v_M(r) \coloneqq \inf\{\operatorname{Vol}\big(B(x, r)\big) \mid x \in M\} \qquad V_M(r) \coloneqq \sup\{\operatorname{Vol}\big(B(x, r)\big) \mid x \in M\}.$$

Lemma 2.3.14. *Let (M, ϱ) be a complete n-dimensional Riemannian manifold with pinched sectional curvature $-\kappa \leq \kappa_\varrho \leq \kappa$ and positive injectivity radius. Let $C \coloneqq \min\big\{\operatorname{inj}(M), \frac{\pi}{\sqrt{\kappa}}\big\}$. Then for every choice of radii $r \leq R \leq C$ there exists a constant $Q = Q(n, \kappa, C, R/r)$ depending only on n, κ, C and the ratio R/r such that*

$$\frac{V_M(R)}{v_M(r)} \leq Q.$$

[11] Complete non-compact manifolds need not have positive injectivity radius, not even under the assumption of constant curvature.

Proof. By Theorem 2.3.13, for every $x \in M$ and $0 \leq t \leq C$ we have

$$\mathrm{Vol}_n^{(\kappa)}(t) \leq \mathrm{Vol}(B_M(x,t)) \leq \mathrm{Vol}_n^{(-\kappa)}(t)$$

Thus we obtain

$$\frac{V_M(R)}{v_M(r)} \leq \frac{\mathrm{Vol}_n^{(-\kappa)}(R)}{\mathrm{Vol}_n^{(\kappa)}(r)},$$

and applying Proposition 2.3.12 we deduce

$$\frac{V_M(R)}{v_M(r)} \leq \frac{\int_0^R \sinh^{n-1}\big(s\sqrt{\kappa}\big)ds}{\int_0^r \sin^{n-1}\big(s\sqrt{\kappa}\big)ds}.$$

Let $\beta = R/r$ and define the function

$$f_\beta(t) \coloneqq \frac{\int_0^{\beta t} \sinh^{n-1}\big(s\sqrt{\kappa}\big)ds}{\int_0^t \sin^{n-1}\big(s\sqrt{\kappa}\big)ds} = \frac{\int_0^t \beta \sinh^{n-1}\big(\beta s\sqrt{\kappa}\big)ds}{\int_0^t \sin^{n-1}\big(s\sqrt{\kappa}\big)ds}.$$

The function $f_\beta(t)$ is differentiable at every $0 < t \leq C$, and

$$\lim_{t\to 0} f_\beta(t) = \lim_{t\to 0} \frac{\beta \sinh^{n-1}\left(\beta t\sqrt{\kappa}\right)}{\sin^{n-1}\left(t\sqrt{\kappa}\right)} = \beta^n.$$

Taking W to be the maximum value of the continuous function f_β on the compact set $[0, C/\beta]$ gives the desired bound. □

Corollary 2.3.15. *Any compact Riemannian manifold (M, ϱ) is a doubling metric measure space (Definition 2.1.12).*

Proof. Since the diameter of M is finite, it is enough to prove the doubling condition for balls of arbitrarily small radius. Since $\mathrm{inj}(M) > 0$ and M has pinched curvature, for small balls we can apply Lemma 2.3.14 and obtain $Q(n, \kappa, C, 2)$ as a doubling constant. □

2.4 Functional analysis

In this manuscript we will use complex Hilbert and Banach spaces.[12] We will generally denote a Banach space by E and a Hilbert space by $\mathcal{H}$. Two norms on a Banach space are *equivalent* if they induce the same topology on the underlying vector space (equivalently, both of them can be bounded in term of the other). Recall that a Banach space is *separable* if it admits a countable dense subset.

[12] We decided to use scalars in $\mathbb{C}$ because the majority of the works that we refer to use complex numbers. Most of what follows is true for real valued Banach spaces as well.

We will denote by $S(E)$ the *unit sphere* in the Banach space E (*i.e.* the set of unit vectors) and we identify the unit sphere of $\mathbb{C}$ with the unit circle $\mathbb{S}^1$. For any (measurable) subset A of a (measure) space X, we denote by $\mathbb{1}_A \colon X \to \mathbb{R}$ its *indicator function*, *i.e.* the (measurable) function assigning the value 1 to points in A and 0 to points in the complement.

2.4.1 Duals and operator topologies

The dual of a Banach space E is denoted by E^*. Every Banach space is naturally embedded in its bidual $E \subseteq (E^*)^*$. A Banach space is *reflexive* if this containment is an equality.

Given a vector $v \in E$ and functional $w \in E^*$ we denote the evaluation $w(v)$ by $\langle v, w\rangle$. If $\mathcal{H}$ is a Hilbert space, this notation is coherent with the canonical identification $\mathcal{H} \cong \mathcal{H}^*$ sending a vector w to the functional $w^* \coloneqq \langle \,\cdot\, , w\rangle$. That is, $\langle v, w^*\rangle$ coincides with the inner product $\langle v, w\rangle$ for every $v, w \in \mathcal{H}$.

Given two Banach spaces E_1, E_2, we denote by $\mathcal{L}(E_1, E_2)$ the space of continuous linear operators from E_1 to E_2. The *norm* of a linear operator between $T \colon E_1 \to E_2$ is defined as

$$\|T\|_{\mathcal{L}(E_1,E_e)} \coloneqq \sup_{v\in S(E_1)} \|T(v)\|_{E_2} = \sup_{v\in E_1\smallsetminus\{0\}} \frac{\|T(v)\|_{E_2}}{\|v\|_{E_1}};$$

when the spaces are clear from the context, we will simply denote it by $\|T\|$. This norm defines the *norm topology* on $\mathcal{L}(E_1, E_2)$. The *weak operator topology* on $\mathcal{L}(E_1, E_2)$ is defined as the weakest topology such that the function

$$T \mapsto \langle T(v), w\rangle$$

is continuous for every $v \in E_1$ and $w \in E_2^*$.

The *strong operator topology* on $\mathcal{L}(E_1, E_2)$ is defined as the weakest topology such that the function $T \mapsto T(v)$ is continuous for every $v \in E_1$. There is a (strict) inclusion of topologies on $\mathcal{L}(E_1, E_2)$:

$$\text{(Weak Op. Top.)} \subset \text{(Strong Op. Top.)} \subset \text{(Norm Topology)}.$$

We denote by $\mathrm{GL}(E) \subset \mathcal{L}(E, E)$ the set of continuous invertible linear endomorphism with continuous inverse, and by $\mathrm{U}(E) \subset \mathrm{GL}(E)$ the set of automorphism that preserve the norm of E. The spaces $\mathrm{GL}(E)$ and $\mathrm{U}(E)$ are equipped with the subset topologies coming from $\mathcal{L}(E, E)$.

2.4.2 L^p-spaces

Given a probability space (X, ν) and a parameter $1 \leq p \leq \infty$ we will denote by $L^p(X, \nu)$ the Banach space of complex valued functions of X with finite L^p-norm

$$\|f\|_p = \Big(\int_X |f(x)|^p d\nu(x)\Big)^{\frac{1}{p}}.$$

Remark 2.4.1. At some point we will have to consider the L^p-space of real valued functions. To avoid confusion we will denote it by $L^p(X, \nu; \mathbb{R})$ or simply $L^p(X; \mathbb{R})$.

Given a function $f\colon X \to \mathbb{C}$, we denote by $\arg(f)\colon X \to \mathbb{S}^1 = S(\mathbb{C})$ the *argument* of f, that is defined by

$$\arg(f)(x) \coloneqq \frac{f(x)}{|f(x)|}$$

when $f(x) \neq 0$ and arbitrarily (*e.g.* constantly equal to 1) when $f(x) = 0$.

Theorem 2.4.2 (Mazur)**.** *For every $1 \leq p, q < \infty$ the Mazur map $M_{p,q}\colon L^p(X) \to L^q(X)$ defined by*

$$M_{p,q}(f) \coloneqq \arg(f)|f|^{\frac{p}{q}}$$

restricts to a uniformly continuous homeomorphism of the unit spheres $S(L^p(X)) \to S(L^q(X))$.

See [BL98, Theorem 9.1] for a proof. We will also use the following:

Theorem 2.4.3 (Banach, Lamperti [Lam58])**.** *For $1 \leq p < \infty$ with $p \neq 2$, any unitary operator $U\colon L^p(X, \nu) \to L^p(X, \nu)$ must be of the form*

$$Uf(x) = f(\varphi(x))h(x)\Big(\frac{d\varphi_*\nu}{d\nu}(x)\Big)^{\frac{1}{p}};$$

where $\varphi\colon X \to X$ is some measure-class preserving transformation, $\frac{d\varphi_*\nu}{d\nu}$ is the Radon-Nikodym derivative and $|h(x)| = 1$ almost everywhere.

2.4.3 Uniformly convex Banach spaces

Given a Banach space E, its *convexity modulus* is the function $\delta_E\colon [0, 2] \to [0, 1]$ given by

$$\delta_E(\epsilon) = \inf\left\{1 - \frac{\|x + y\|}{2} \;\middle|\; \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \epsilon\right\}.$$

The Banach space E is *uniformly convex* if $\delta_E(\epsilon) > 0$ for every $\epsilon > 0$.

We say that a Banach space E is *superreflexive* if it admits an equivalent norm that makes it into a uniformly convex Banach space.[13] A superreflexive Banach space is also reflexive.

A group $G < \mathrm{GL}(V)$ of continuous linear transformations of a Banach space E is *uniformly equicontinuous* if there is a uniform upper bound on the operator norms $\|g\|_{\mathrm{GL}(E)}$ for $g \in G$. The next two results are proved (for real Banach spaces) in [BFGM07, Subsection 2.b]

Fact 2.4.4. *If E is a superreflexive Banach space and G is a group of equicontinuous linear transformations of E, then there exists an equivalent norm on E that is uniformly convex and G-invariant.*

A linear action of a group G on a vector space is called a *representation* of G. We will denote representations by either $\pi\colon G \to \mathrm{GL}(V)$ or by $\pi\colon G \curvearrowright V$ (as opposed to our conventional use of the symbol ρ for actions).

Fact 2.4.5. *Let G be a topological group and $\pi\colon G \to \mathrm{GL}(E)$ a representation on a superreflexive Banach space E such that $\pi(G)$ is equicontinuous. Then the following are equivalent:*

(i) *π is continuous w.r.t the weak operator topology;*

(ii) *π is continuous w.r.t the strong operator topology;*

(iii) *the map $G \times E \to E$ is continuous (*i.e. *π is a continuous action as defined in* Subsection 2.2.2*).*

2.4.4 Spectral gaps for (probability) measure preserving actions

Let G be a locally compact Hausdorff second countable group and $\pi\colon G \to \mathrm{GL}(E)$ a (weakly continuous) representation on a Banach space. Following [BFGM07] we give the following:

Definition 2.4.6. The representation π has *almost invariant vectors* if there exists a sequence $v_n \in E \smallsetminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{\operatorname{diam}\big(\pi(K)v_n\big)}{\|v_n\|} = 0$$

[13] This is not the classic definition of superreflexive spaces. The fact that these definitions coincide is a theorem of Pisier.

for every compact set $K \subseteq G$ (equivalently, there exists a sequence of unit vectors $v_n \in S(E)$ with $\operatorname{diam}\big(\pi(K)v_n\big) \xrightarrow{n\to\infty} 0$ for every compact K).

We take the following from [BFGM07, Remark 4.3]:

Lemma 2.4.7. *Given a unitary representation of a group on an L^p space $\pi_p\colon G \to \mathrm{U}(L^p(X,\nu))$, let $\pi_q\colon G \curvearrowright L^q(X,\nu)$ be the action defined by $\pi_q(g) \coloneqq M_{p,q} \circ \pi_p(g) \circ M_{q,p}$. Then π_q is a an action by unitary maps. Moreover, π_q has almost invariant vectors if and only if so does π_p.*

Sketch of proof. For every $g \in G$, the map $\pi_q(g)\colon L^q(X,\nu) \to L^q(X,\nu)$ is a well-defined function. The fact that it is linear and unitary follows from the Banach-Lamperti Theorem (Theorem 2.4.3) through a straightforward computation.

With regard to the almost invariant vectors, let v_n be a sequence of almost invariant vectors for π_p with $\|v_n\|_p = 1$ for every $n \in \mathbb{N}$. Since the Mazur maps $M_{p,q}$ are uniformly continuous (Theorem 2.4.2) there exists an $\epsilon > 0$ such that $\|M_{p,q}(v_n)\|_q \geq \epsilon$ for every $n \in \mathbb{N}$. Still, $\operatorname{diam}\big(\pi_q(K)M_{p,q}(v_n)\big) \to 0$ for every compact set $K \subseteq G$ and therefore the vectors $\|M_{p,q}(v_n)\|_q$ are almost invariant. □

For every $1 \leq p < \infty$, a measure preserving action $\rho\colon G \curvearrowright (X,\nu)$ induces an action by linear isometries $\pi^p_\rho\colon G \curvearrowright L^p(X)$ by precomposition:

$$\pi^p_\rho(g)f(x) \coloneqq f(g^{-1} \cdot x)$$

for every $g \in G$, $f \in L^p(X)$. When p or ρ are clear from the context, we simply denote this representation by π_ρ or π. Note that a priori π_ρ needs not be continuous (this will not be an issue when the acting groups is a countable group Γ equipped with the discrete topology).

Remark 2.4.8. If μ is σ-finite and $L^2(X)$ is a separable Hilbert space, then any measurable measure-preserving[14] action $\rho\colon G \curvearrowright (X,\mu)$ gives rise to a (unitary) representation π_ρ continuous with respect to the strong operator topology [BdlHV08, Proposition A.6.1].

[14] If μ is σ-finite and $G \curvearrowright X$ is a measure class preserving action, then one obtains an action by linear isometries $\pi^p_\rho\colon G \curvearrowright L^p(X)$ by letting

$$\pi^p_\rho(g)f(x) \coloneqq \Big(\frac{dg^{-1}_*\mu}{d\mu}(x)\Big)^{\frac{1}{p}} f(g^{-1} \cdot x)$$

where $\frac{dg^{-1}_*\mu}{d\mu}$ is the Radon–Nikodym derivative (Subsection 2.1.5). The above is also known as *Koopman representation* induced by $G \curvearrowright X$. When $p = 2$ and $L^2(X)$ is separable, [BdlHV08, Proposition A.6.1] still applies and hence the Koopman representation is continuous (this might be true in greater generality, but I did not have the time to investigate further).

If ν is a probability measure, the constant functions belong to $L^p(X)$ and they are invariant vectors for the representation $\pi_\rho\colon G \curvearrowright L^p(X)$. The canonical complement of the subspace of constant functions in L^p is given by the subspace of functions with zero average

$$L_0^p(X) \coloneqq \left\{ f \in L^p(X) \middle| \int_X f(x) d\nu(x) = 0 \right\},$$

and the representation π_ρ restricts to a representation $G \curvearrowright L_0^p(X)$ that we will (with an abuse of notation) denote again by π_ρ.

Let (X, ν) be a probability space, $\Gamma = \langle S \rangle$ a finitely generated group, $\rho\colon \Gamma \curvearrowright X$ a measure preserving action and π_ρ the induced continuous representation. Notice that the action $\pi_\rho\colon \Gamma \curvearrowright L_0^p(X)$ has almost invariant vectors if and only if there exists a sequence $f_n \in L_0^p(X) \setminus \{0\}$ such that

$$\lim_{n\to\infty} \frac{\|s \cdot f_n - f_n\|_p}{\|f_n\|_p} = 0$$

for every $s \in S$.

Definition 2.4.9. A probability measure preserving action $\rho\colon \Gamma \curvearrowright (X, \nu)$ of a finitely generated group $\Gamma = \langle S \rangle$ has a *spectral gap* in L_0^p if there exists a $\delta > 0$ such that the induced representation $\pi_\rho^p\colon \Gamma \curvearrowright L_0^p$ satisfies

$$\sum_{s \in S^\pm} \|s \cdot f - f\|_p \geq \delta \|f\|_p. \tag{2.2}$$

for every $f \in L_0^p(X)$. We also say that a family of measure preserving transformations $\rho_i\colon \Gamma \curvearrowright (X_i, \nu_i)$, $i \in I$ has *uniform spectral gap* if they all have spectral gap with a constant δ independent from $i \in I$.

Remark 2.4.10. In (2.2) we take the sum over $S^\pm$ because this is the most common convention. Taking the sum over S would yield the same result (with a different constant) because S is finite and $\|s^{-1} \cdot f - f\|_p = \|s \cdot f - f\|_p$. This could not be the case if the action was not assumed to be measure preserving.

Remark 2.4.11. A measure preserving action of a finitely generated group has spectral gap if and only if the induced unitary representation is isolated from the trivial representation (*i.e.* it admits a Kazhdan pair). See Remark 2.5.21.

The action $\rho\colon \Gamma \curvearrowright (X, \nu)$ has a spectral gap in L_0^p if and only if π_ρ does not have almost invariant vectors in L_0^p. It hence follows from Lemma 2.4.7 that the existence of a spectral gap for ρ does not depend on the choice of $1 \leq p < \infty$.

2.4.5 Banach valued spectral gap

We refer to [DU77, Chapter II] for details. A Banach valued function on a measure space $f\colon X \to E$ is *measurable* if there exists a sequence of simple functions $f_n\colon X \to E$ (*i.e.* finite sums of $v\mathbb{1}_A(x)$ with $A \subseteq X$ and $v \in E$) such that $\|f - f_n\|_E \to 0$ almost everywhere in X.

A measurable function is *integrable* if $\int_X \|f(x)\|_E dx$ is finite. The *Bochner integral* of an integrable function is defined by

$$\int_E f(x)dx = \lim_{n\to\infty} \int_X f_n(x)dx \in E$$

where the f_n are simple functions converging to f and their integral is the obviously weighted sum of vectors in E. Such integral is well-defined.

A Banach valued function is *p-integrable* if $\int_X \|f(x)\|_E^p dx < +\infty$. Note that if (X, ν) is a probability space and $p \geq 1$, then every p-integrable function is integrable and hence admits a Bochner integral.

Let (X, ν) be a probability space and E a Banach space. The *Bochner space* $B^p = L^p(X, \nu; E)$ is the Banach space of p-integrable functions $f\colon X \to E$ equipped with the norm

$$\|f\|_{B^p} = \left(\int_X \|f(x)\|_E^p \, d\nu(x) \right)^{\frac{1}{p}}.$$

Remark 2.4.12. Note that the Bochner space $L^p(X; \mathbb{C})$ is trivially the same as the usual L^p-space $L^p(X)$. Moreover, the notation is also coherent with the real case $L^p(X; \mathbb{R})$ (even though the latter is not a complex Banach space).

Remark 2.4.13. An L^2-Bochner spaces over a Hilbert space has a naturally defined inner products that makes it into a Hilbert space.

We let $L_0^2(X, \nu; E)$ be the subspace of $B \coloneqq B^2$ of functions with zero mean (*i.e.* 0 Bochner integral). Any measure preserving action $\rho\colon \Gamma \curvearrowright (X, \nu)$ induces a unitary action on B and on $L_0^2(X, \nu; E)$ by $\gamma \cdot f(x) = f(\gamma^{-1} \cdot x)$.

Definition 2.4.14. We say that ρ has *E-spectral gap* if there exists a constant $\epsilon > 0$ so that for every $f \in L_0^2(X, \nu; E)$, we have

$$\sum_{s \in S^{\pm}} \|f - s \cdot f\|_B \geq \epsilon \|f\|_B$$

where S is a finite generating set of Γ.

2.4.6 The Rademacher type

Let $(\Omega, \mathbb{P})$ be a *standard probability space* (*e.g.* the interval $[0,1]$ with the Lebesgue measure on the Borel σ-algebra) and let $X_n \colon \Omega \to \mathbb{C}$ with $n \in \mathbb{N}$ be a family of independent complex Gaussian $\mathcal{N}(0,1)$ random variables (a *complex* $\mathcal{N}(0,1)$ *random variable* is a random variable such that the real and imaginary part are independent standard Gaussian random variables).

A Banach space E is said to have *type* $p \geq 1$ if there exists a constant $T > 0$ such that for every $n \in \mathbb{N}$ and every choice of n vectors $v_1, \dots, v_n \in E$, we have

$$\left\| \sum_{i=1}^{n} X_i v_i \right\|_{L^2(\Omega;E)} \leq T \left(\sum_i \|v_i\|_E^p \right)^{1/p}.$$

Every Banach space has type 1; therefore E is said to have *non-trivial type* if it has type p for some $p > 1$.

Fact 2.4.15. *Every uniformly convex Banach space has non-trivial type.*

Remark 2.4.16. The converse is false. See [Mau03] for more details on type.

2.5 Unitary representations

A *unitary representation* of G is a continuous[15] homomorphism $\pi \colon G \to \mathrm{U}(\mathcal{H})$ of the group G into the group of unitary operators of a Hilbert space $\mathcal{H}$. We will generally denote a representation simply by π, but we will use the notation $(\pi, \mathcal{H})$ if we want to stress that $\mathcal{H}$ is the Hilbert space that is being acted on. If we want to stress that the acting group is G, we call π a *G-representation*.

2.5.1 Basic facts and definitions

If $\mathcal{H}' < \mathcal{H}$ is a π-invariant closed subspace, the restriction of $\pi|_{\mathcal{H}'}$ is a *subrepresentation* of π. A representation π is *irreducible* if it has no invariant closed subspaces (besides $\{0\}$ and $\mathcal{H}$ itself). A representation $(\pi', \mathcal{H}')$ is *contained* in $(\pi, \mathcal{H})$ (denoted by $\pi' \subseteq \pi$) if it is isomorphic to a subrepresentation of π; *i.e.* there exists an linear isometric embedding $T \colon \mathcal{H}' \hookrightarrow \mathcal{H}$ such that $T \circ \pi'(g) = \pi(g) \circ T$ for every $g \in G$ (*i.e.* a G-equivariant linear isometric embedding). Note in particular that $T(\mathcal{H}')$ must be a π-invariant vector subspace of $\mathcal{H}$ and that it is closed as it is an isometric image of a Hilbert space.

[15] By Fact 2.4.5 the specific choice of topology does not matter.

Every group G admits a *trivial representation i.e.* G acts as the identity on the one-dimensional vector space $\mathbb{C}$. We denote the trivial representation by I_G. Note that the trivial representation is irreducible, because $\mathbb{C}$ has no non-trivial vector subspaces.

Remark 2.5.1. The unitary representation induced by a measure preserving action of a group does *not* contain the trivial representation if and only if the action is ergodic 2.2.1.

Let $(\pi_i, \mathcal{H}_i)_{i\in I}$ be a family of unitary representations of the group G. The *Hilbert direct sum* $\bigoplus_{i\in I}\mathcal{H}_i$ is the Hilbert space given by the set of elements $v = (v_i)_{i\in I} \in \prod_{i\in I}\mathcal{H}_i$ such that $\sum_{i\in I}\|v_i\|^2_{\mathcal{H}_i} < \infty$ equipped with the inner product $\langle v, w\rangle \coloneqq \sum_{i\in I}\langle v_i, w_i\rangle_{\mathcal{H}_i}$. The *direct sum* of the representations $(\pi_i)_{i\in I}$ is the unitary representation

$$\bigoplus_{i\in I}\pi_i \colon G \to \mathrm{U}\Big(\bigoplus_{i\in I}\mathcal{H}_i\Big)$$

acting as π_i on the i-th coordinate. If the set I is finite, the direct sum coincides with the standard (finite) direct sum of representations. If I is countable, the Hilbert sum coincides with the *ℓ^2-sum*.

Given any natural number $n \geq 1$, we denote by $n\pi$ the direct sum of n copies of π. It is customary to denote by $\infty\pi$ the direct sum of a countably many copies of π. If $(\pi, \mathcal{H})$ is a representation and $\mathcal{H}$ is finite dimensional, we let $\dim(\pi) \coloneqq \dim(\mathcal{H})$.

Proposition 2.5.2 ([BdlHV08, Proposition A.1.8])**.** *Let $(\pi_i)_{i\in I}$ be a family of representations of G and π an irreducible G-representation. Then $\pi \subseteq \bigoplus_{i\in I}\pi_i$ if and only if $\pi \subseteq \pi_i$ for some $i \in I$.*

Let G be a locally compact second countable Hausdorff topological group. Then G admits a left-invariant Haar measure m (Subsection 2.2.1).

Definition 2.5.3. The *(left) regular representation* of G is the unitary representation $\lambda_G \colon G \to \mathrm{U}\big(L^2(G, m)\big)$ induced from the action of G on itself by (left) multiplication (the representation is induced by precomposition as in Subsection 2.4.4).

The following is a classical result.

Theorem 2.5.4 (Peter-Weyl)**.** *Let G be a compact group. Then every irreducible representation is finite dimensional and every representation is isomorphic to a direct sum or irreducible representations.*

Moreover, the regular representation of the compact group G is isomorphic to the direct sum $\bigoplus_{\pi\in\widehat{G}}(\dim(\pi))\pi$, and it hence contains all the irreducible representations ($\widehat{G}$ is the set of irreducible representations, see Remark 2.5.14).

See [Rob83, Chapter 5] for a proof.[16]

Remark 2.5.5. For general topological groups one loses the fact that any representation is direct sum of irreducible representations. What remains true is that every representation is direct sum of *cyclic* representations [BdlHV08, Proposition C.4.9]. It is also possible to prove that every representation is a *direct integral* of irreducible representations [BdlHV08, Section F.5].

2.5.2 Weak containments and equivalences

Let $(\pi, \mathcal{H})$ be a unitary representation. The *diagonal matrix coefficients* of π are the functions $G \to \mathbb{C}$ of the form $g \mapsto \langle \pi(g)v, v\rangle$ for some fixed $v \in \mathcal{H}$.

Remark 2.5.6. The diagonal matrix coefficient of π are also called *positive functions associated with* π. This is because diagonal matrix *are* positive functions (in the sense of *kernels* defined on G) [BdlHV08, Proposition C.4.3]. Vice versa, every positive function of G is a diagonal matrix element for some unitary representation π of G [BdlHV08, Proposition C.4.9].

Definition 2.5.7. Let $(\pi, \mathcal{H})$ and $(\pi', \mathcal{H}')$ be unitary representations of G, we say that π is *weakly contained in* π' (denoted by $\pi \prec \pi'$) if every diagonal matrix coefficient of π can be approximated uniformly on compact sets by finite sums of matrix coefficients of π'. That is, for every $v \in \mathcal{H}$, $K \subseteq G$ compact and $\epsilon > 0$ there exist $w_1, \dots, w_n \in \mathcal{H}'$ such that

$$\left| \langle \pi(g)v, v\rangle - \sum_{i=1}^{n} \langle \pi'(g)w_i, w_i\rangle \right| < \epsilon$$

for every $g \in K$.

It is clear that the weak containment defines a transitive relation, and one define an equivalence relation saying that two unitary representations π, π' are *weakly equivalent* (denoted $\pi' \sim \pi$) if $\pi \prec \pi'$ and $\pi' \prec \pi$.

Lemma 2.5.8. *For every unitary representation π of G and $n \in \mathbb{N}$ we have $n\pi \sim \pi$. More generally, for every set I the sum $\bigoplus_{i\in I} \pi$ of $|I|$ copies of π is weakly equivalent to π.*

[16] This theorem is actually a collection of various results. The statement we gave is a rewording of [BdlHV08, Theorem A.5.2].

Sketch of proof. We have to show that $n\pi \prec \pi$. A diagonal matrix coefficient of $n\pi$ takes the form $\langle n\pi(g)\overline{v}, \overline{v}\rangle$ where $\overline{v} = (v_1, \dots, v_n)$ is a vector in $\mathcal{H} \oplus \cdots \oplus \mathcal{H}$, and the claim follows because

$$\langle n\pi(g)\overline{v}, \overline{v}\rangle = \langle \pi(g)v_1, v_1\rangle + \cdots + \langle \pi(g)v_n, v_n\rangle.$$

To deal with infinite sums, let $\overline{v} = (v_i)_{i\in I}$. It is enough to notice that for every $\epsilon > 0$ there exists a finite subset $J \subset I$ such that $\sum_{i\in I\smallsetminus J}\|v_i\|^2 < \epsilon$, whence we deduce that the diagonal matrix coefficient associated with $\overline{v}$ is approximated up to ϵ by the sum of the diagonal matrix coefficients $\langle \pi(g)v_j, v_j\rangle$ for $j \in J$. □

Corollary 2.5.9. *Let G be a compact group. Then every unitary representation π of G is weakly contained in the regular representation λ_G. Moreover, if π has no invariant vectors, then it is weakly contained in the restriction of λ_G to the subspace of zero-integral functions $L^2_0(G)$.*

Proof. By the Peter-Weyl Theorem (Theorem 2.5.4) we know that π is isomorphic to a direct sum of irreducible representations, and that every irreducible representation is contained in λ_G. It follows that π can be embedded in $|I|\lambda_G$ for some cardinal $|I|$. On the other hand, $|I|\lambda_G \prec \lambda_G$ (Lemma 2.5.8), thus we obtain $\pi \subseteq |I|\lambda_G \prec \lambda_G$.

For the 'moreover' statement, it is enough to note that if π has no invariant vectors then its decomposition as sum of irreducible representations does not contain I_G as a factor. Since $\lambda_G = \lambda_G|_{L^2_0(G)} \oplus I_G$, it follows that $\pi \subset |I|\big(\lambda_G|_{L^2_0(G)}\big) \prec \lambda_G|_{L^2_0(G)}$. □

A diagonal matrix coefficient $\phi_v(g) \coloneqq \langle \pi(g)v, v\rangle$ is *normalised* if $1 = \phi_v(e) = \|v\|^2$ (*i.e.* it is the diagonal matrix coefficient associated with a unit vector).

Lemma 2.5.10 ([BdlHV08, Remark F.1.2])**.** *If π and π' are unitary representation of G such that $\pi \prec \pi'$ and ϕ_v is a normalised diagonal matrix coefficient of π, then ϕ_v can be approximated uniformly on compact sets by convex sums of normalised matrix coefficients of π'.*

Sketch of proof. Let ϕ be a diagonal matrix coefficient of π and $\psi_1, \dots, \psi_n$ be diagonal matrix coefficients of π' such that $|\phi(g) - \sum_{i=1}^n \psi_i(g)| < \epsilon$ for every $g \in K$. We can assume that $e \in K$. Then $|1 - \sum_{i=1}^n \psi_i(e)| = |\phi(e) - \sum_{i=1}^n \psi_i(e)| < \epsilon$.

Let $\lambda \coloneqq \sum_{j=1}^n \psi_j(e)$ (so that $\lambda \approx 1$). We obtain an approximation by convex combination of normalised coefficients:

$$\begin{aligned}\left|\phi - \sum_{i=1}^n \frac{\psi_i(e)}{\psi_1(e) + \cdots + \psi_n(e)}\frac{\psi_i}{\psi_i(e)}\right| &= \left|\frac{\lambda\phi - \sum_{i=1}^n \psi_i}{\lambda}\right| \\ &\leq \frac{1}{|\lambda|}\Big(|\phi - \textstyle\sum_{i=1}^n \psi| + |(1-\lambda)\phi|\Big)\end{aligned}$$

and this can be made arbitrarily small because $|\phi|$ is bounded as continuous on a compact set. □

Proposition 2.5.11 ([BdlHV08, Proposition F.1.4]). *Let π and π' be unitary representations of a locally compact Hausdorff second countable group. If π is irreducible and $\pi \prec \pi'$ then every normalised diagonal matrix coefficient of π can be approximated uniformly on compact sets by normalised diagonal matrix coefficients of π' (as opposed to* convex sums *of normalised coefficients).*

The following is not relevant to our exposition, we include it because we could not resist the temptation:

Proposition 2.5.12 (Hulanicki–Reiter [BdlHV08, Theorem G.3.2]). *Let G be a locally compact second countable Hausdorff topological group. Then the following are equivalent:*

(i) G is amenable;

(ii) $I_G \prec \lambda_G$;

(iii) $\pi \prec \lambda_G$ for every unitary representation π of G.

2.5.3 The Fell topology

Given any set $\mathcal{S}$ of unitary representations (the class of unitary representations is not a set, thus one needs to restrict to a set to define a topology), let $\mathcal{R}$ be the set of equivalence classes of representations in $\mathcal{S}$ (up to weak equivalence).

For a given representation $(\pi, \mathcal{H}) \in \mathcal{S}$ and any choice of diagonal matrix coefficients $\phi_1, \dots, \phi_n$, compact set $K \subseteq G$ and constant $\epsilon > 0$, let $W(\pi, \phi_1, \dots, \phi_n, K, \epsilon) \subseteq \mathcal{S}$ be the set of representations π' that admit functions $\psi_1, \dots, \psi_n$ that are finite sums of diagonal matrix coefficients and such that $|\phi_i(g) - \psi_i(g)| < \epsilon$ for every $g \in K$ and $i = 1, \dots, n$.

The sets $W(\pi, \phi_1, \dots, \phi_n, K, \epsilon)$ are closed under the weak equivalence of unitary representation. They hence descend to $\mathcal{R}$ and they form the basis of a topology on $\mathcal{R}$ which is known as the *Fell topology.* [17]

In this setting, (the proof of) Proposition 2.5.11 gives the following:

Proposition 2.5.13 ([BdlHV08, Proposition F.2.4][18]). *Let G be a locally compact Hausdorff second countable group and $\mathcal{S}$ a set of unitary representations. If $\pi \in \mathcal{S}$ is an irreducible representation, then a basis of neighbourhoods of π for the Fell topology is given by the sets $\widetilde{W}(\pi, \phi_1, \dots, \phi_n, K, \epsilon)$, where ϕ_i are normalised diagonal matrix*

[17] Equivalently, the Fell topology can be defined by saying that a *net* of (equivalence classes of) unitary representations $(\pi_i)_{i \in I}$ converges to π if and only if $\pi \prec \bigoplus_{j \in J} \pi_j$ for every subnet $J \subseteq I$.

[18] The authors of [BdlHV08] state this result without mentioning that the coefficients can be assumed to be normalised. Our statement follows from theirs together with the proof of Lemma 2.5.10.

coefficients, $K \subseteq G$ is compact, $\epsilon > 0$ and $\widetilde{W}(\pi, \phi_1, \dots, \phi_n, K, \epsilon)$ is the set of representations in $\mathcal{F}$ that admit normalised matrix coefficients ψ_i such that $|\phi_i(g) - \psi_i(g)| < \epsilon$ for every $g \in K$.

Remark 2.5.14. The family of (isomorphism classes of) irreducible unitary representations of a topological group G is a set [BdlHV08, Remark C.4.13], which can hence be endowed with the Fell topology. We shall denote this set by $\widehat{G}$.

2.5.4 Operator algebras and weak containments

Let G be a locally compact Hausdorff second countable topological group, let m be its Haar measure and $L^1(G)$ the space of integrable complex-valued functions of G. For every unitary representation $(\pi, \mathcal{H})$ of G one can define a continuous linear map [19] (denoted again by π) from $L^1(G)$ to the space of linear endomorphisms $\mathcal{L}(\mathcal{H}, \mathcal{H})$ letting

$$\pi(f) \coloneqq \int_G f(x)\pi(x)dm(x)$$

for every $f \in L^1(G)$ (here the integral is meant in the Bochner sense, see Subsection 2.4.5). Alternatively $\pi(f)$ is the unique linear operator such that

$$\langle \pi(f)v, w\rangle = \int_G f(x)\langle \pi(x)v, w\rangle dm(x)$$

for every $v, w \in \mathcal{H}$.

Theorem 2.5.15 ([Dix82, Section 18])**.** *Let π and π' be unitary representations of G. Then $\pi \prec \pi'$ if and only if $\|\pi(f)\| \leq \|\pi'(f)\|$ for every $f \in L^1(G)$.* [20]

More generally, Let $\mathcal{P}(G)$ be the space of probability measures on G. For every probability measure $\mu \in \mathcal{P}(G)$ the *μ-convolution operator* $\pi(\mu) \in \mathcal{L}(\mathcal{H}, \mathcal{H})$ is defined by $\int_G \pi(x)d\mu(x)$. That is, it is the unique operator such that

$$\langle \pi(\mu)v, w\rangle = \int_G \langle \pi(x)v, w\rangle d\mu$$

[19] The mapping $L^1(G) \to \mathcal{L}(\mathcal{H}, \mathcal{H})$ is actually a *$*$-representation* (a continuous homomorphism of $*$-algebras) and is *non-degenerate* (for every $v \in \mathcal{H} \setminus \{0\}$ there exists an $f \in L^1(G)$ such that $\pi(f)v \neq 0$). It is a theorem that the converse is also true: every non-degenerate $*$-representation of $L^1(G)$ arises from this construction. Here $L^1(G)$ is the *convolutive algebra* (the product of functions is their convolution) and the involution is given by $f^*(x) \coloneqq \overline{f(x^{-1})}$. The involution on $\mathcal{L}(\mathcal{H}, \mathcal{H})$ is given by taking the adjoint operator.

[20] In [Dix82, Section 18] it is also shown that $\pi \prec \pi'$ if and only if $C^* \ker(\pi') \subseteq C^* \ker(\pi)$ (where $C^* \ker$ is the kernel of the $*$-representation). From Theorem 2.5.15 and Proposition 2.5.12 it hence follows that G is amenable if and only if the reduced and maximal C^*-algebras of G coincide.

for every $v, w \in \mathcal{H}$. The definition of $\pi(\mu)$ agrees with that of $\pi(f)$ when f is an integrable function of norm 1 and $\mu(x) = f(x)m(x)$. Since the set $\mathcal{P} \cap L^1(G)$ is dense in $\mathcal{P}(G)$, we deduce the following from Theorem 2.5.15.

Corollary 2.5.16. *Let π and π' be unitary representations of a locally compact Hausdorff second countable group G. Then $\pi \prec \pi'$ if and only if $\|\pi(\mu)\| \leq \|\pi'(\mu)\|$ for every $\mu \in \mathcal{P}(G)$.*

Note that Corollary 2.5.9 implies the following:

Corollary 2.5.17. *Let G be a compact group, $\mu \in \mathcal{P}(G)$ a probability measure and π a unitary representation without non-trivial fixed vectors. Then $\|\pi(\mu)\| \leq \|\lambda_G|_{L^2_0(G)}(\mu)\|$ where $\lambda_G|_{L^2_0(G)}$ is the restriction of the regular representation to the space of zero-average functions.*

2.5.5 Kazhdan sets and pairs

We use the notation from [Sha00].

Definition 2.5.18. Let $K \subseteq G$ be a compact subset and $\epsilon > 0$ a constant. If $\mathcal{H}$ is a Hilbert space and $\pi \colon G \to U(\mathcal{H})$ is a unitary representation, a (K, ϵ)-*invariant vector* is a vector $v \in \mathcal{H}$ such that $\|\pi(g)v - v\| < \epsilon\|v\|$ for every $g \in K$.

Given a family $\mathcal{F}$ of unitary representations of G, we say that a pair (K, ϵ) is a *Kazhdan pair* for $\mathcal{F}$ if every representation $\pi \in \mathcal{F}$ does not have non-zero (K, ϵ)-invariant vectors. When this is the case, K (resp. ϵ) is a *Kazhdan set* (resp. a *Kazhdan constant*) for $\mathcal{F}$.

Lemma 2.5.19. *Let π be a unitary representation of a locally compact Hausdorff second countable group G. The following are equivalent:*

(i) *π admits a Kazhdan pair;*

(ii) *π does not admit almost invariant vectors (Definition 2.4.6);*

(iii) *π does not weakly contain the trivial representation;*

(iv) *$\|\pi(f)\| < \|f\|_1$ for some function $f \in L^1(G)$;*

(v) *$\|\pi(\mu)\| < 1$ for some probability measure $\mu \in \mathcal{P}(G)$.*

Sketch of proof. *(i)⇒(ii)* If π admits a sequence $v_n \in \mathcal{H}$ of almost invariant vectors then for every compact K and $\epsilon > 0$, v_n is going to be a (K, ϵ)-invariant vector for n large enough because $K \cup \{e\}$ is compact in G and hence $\operatorname{diam}(\pi(K) \cdot v_n \cup \{v_n\})$ must be small for large n.

(ii)⇒(i) Let K_n be an exhaustion by compact sets of G (Subsection 2.2.1) and let v_n be a $(K_n, \frac{1}{n})$-invariant vector. Then the sequence v_n is a sequence of almost invariant vectors because every compact set $K \subseteq G$ is contained in a K_n for n large enough.

(i)⇔(iii) By Lemma 2.5.10, we know that $I_G \prec \pi$ if and only every normalised diagonal matrix coefficient of I_G can be approximated on compact sets by normalised diagonal matrix coefficients of π; *i.e.* for every $K \subseteq G$ compact and $\epsilon > 0$ there exists $w \in S(\mathcal{H})$ such that $|1 - \langle \pi(g)w, w\rangle| < \epsilon$ for every $g \in K$.

On the other hand (K, ϵ') is *not* a Kazhdan pair for π if and only if there exists a $w \in S(\mathcal{H})$ such that

$$(\epsilon')^2 > \|\pi(g)w - w\|^2 = \|\pi(g)w\|^2 + \|w\|^2 - 2\operatorname{Re}(\langle \pi(g)w, w\rangle) = 2\big(1 - \operatorname{Re}(\langle \pi(g)w, w\rangle)\big)$$

for every $g \in K$. The claim follows because $1 - \operatorname{Re}\big(\langle \pi(g)v, v\rangle\big) \leq |1 - \langle \pi(g)v, v\rangle|$ and the fact that, since $|\langle \pi(g)v, v\rangle| \leq 1$, it is a simple exercise of Euclidean geometry to show that

$$1 - \operatorname{Re}\big(\langle \pi(g)v, v\rangle\big) \geq \frac{|1 - \langle \pi(g)v, v\rangle|^2}{2}.$$

(iii)⇔(iv)⇔(v) Follow from Theorem 2.5.15 and Corollary 2.5.16. □

More in general, the following holds:

Proposition 2.5.20. *Let G be a locally compact Hausdorff second countable group and let $\mathcal{F}$ be a set of unitary representations of G. The following are equivalent:*

(i) $\mathcal{F}$ admits a Kazhdan pair;

(ii) $\mathcal{F}$ does not contain the trivial representation I_G, and I_G is isolated in the Fell topology on $\mathcal{F} \cup \{I_G\}$.

Sketch of proof. By Proposition 2.5.13, I_G is isolated in $\mathcal{F} \cup \{I_G\}$ if and only if there is a compact $K \subseteq G$ and $\epsilon > 0$ such that $\widetilde{W}(I_G, 1, K, \epsilon) \cap \mathcal{F} = \emptyset$. That is, for every $\pi \in \mathcal{F}$ and $v \in S(\mathcal{H})$ we have $|1 - \langle \pi(g)v, v\rangle| \geq \epsilon$ for some $g \in K$.

The statement follows thanks to the identity:

$$\|\pi(g)v - v\|^2 = 2\|v\|^2 - 2\operatorname{Re}\big(\langle \pi(g)v, v\rangle\big).$$

□

Remark 2.5.21. According to Definition 2.4.9, a probability measure preserving action $\rho\colon \Gamma \curvearrowright (X,\nu)$ of a finitely generated group $\Gamma = \langle S\rangle$ has a spectral gap (in L^2_0) if there exists a $\delta > 0$ such that the induced representation $\pi\colon \Gamma \curvearrowright L^2_0$ satisfies $\sum_{s\in S^\pm}\|\pi(s)f - f\|_2 \geq \delta\|f\|_2$ for every $f \in L^2_0(X)$. If S is a Kazhdan set of ρ with Kazhdan constant ϵ, this inequality clearly holds with $\delta = \epsilon$. Vice versa, when the inequality holds then S must be a Kazhdan set with Kazhdan constant at least as large as $\frac{\delta}{|S^\pm|}$.

Let now G be a topological group and $S \subset G$ a finite subset. The probability measure μ_S on G is defined as the average of the *delta functions* of the points of S

$$\mu_S \coloneqq \frac{1}{S}\sum_{s\in S}\delta_s,$$

where δ_s is the measure assigning measure 1 to the point $\{s\}$ and 0 to its complement.

Proposition 2.5.22. *Let $\mathcal{F}$ be a set of unitary representations of topological group G and $S \subset G$ be finite. The following are equivalent:*

(i) S is a Kazhdan set for $\mathcal{F}$;

(ii) $S^\pm_e$ is a Kazhdan set for $\mathcal{F}$;

(iii) there exists $\epsilon > 0$ such that $\left\|\pi(\mu_{S^\pm_e})\right\| < 1-\epsilon$ for every $\pi \in \mathcal{F}$.

Sketch of proof. The first two conditions are clearly equivalent. To prove *(ii)⇔(iii)*, note that the operator $\pi(\mu_{S^\pm_e})$ is *self-adjoint* and thus we have

$$\left\|\pi(\mu_{S^\pm_e})\right\| = \sup_{v\in S(\mathcal{H})}\left|\langle\pi(\mu_{S^\pm_e})v, v\rangle\right|$$

(see [Rud91, Chapter 12]). Moreover, since $\pi(\mu_{S^\pm_e})$ is self-adjoint the inner product $\langle\pi(\mu_{S^\pm_e})v, v\rangle$ is real. The fact that $e \in S^\pm_e$ assures us that $\langle\pi(S^\pm_e)v, v\rangle$ with $\|v\| = 1$ is bounded away from -1 because we have

$$\langle\pi(\mu_{S^\pm_e}), v\rangle = \frac{1}{|S^\pm_e|}\|v\|^2 + \frac{1}{|S^\pm_e|}\sum_{s\in S^\pm_e\smallsetminus\{e\}}\langle\pi(s)v, v\rangle \geq \frac{1}{|S^\pm_e|} - \frac{|S^\pm_e|-1}{|S^\pm_e|} = -1 + \frac{2}{|S^\pm_e|}.$$

It follows that $\|\pi(\mu_{S^\pm_e})\| \leq 1-\epsilon$ for some $\epsilon < \frac{2}{|S^\pm_e|}$ if and only if

$$\epsilon > \left|1 - \langle\pi(\mu_{S^\pm_e})v, v\rangle\right| = \left|\frac{1}{|S^\pm_e|}\sum_{s\in S^\pm_e}\big(1 - \langle\pi(s)v, v\rangle\big)\right|$$

for every $v \in S(\mathcal{H})$, and this happens if and only if there is a $\epsilon' > 0$ such that for every $v \in S(\mathcal{H})$ we have $1 - \langle \pi(s)v, v\rangle > \epsilon'$ for some $s \in S_e^{\pm}$. From this we conclude that $S_e^{\pm}$ is a Kazhdan set using the argument in the proof of Proposition 2.5.20. [21] □

Remark 2.5.23. One can also show that if a family $\mathcal{F}$ of unitary representations of G admits a Kazhdan pair (K, ϵ) and K' is any compact generating set for the group G, then then also K' is a Kazhdan set for $\mathcal{F}$ (for an appropriate Kazhdan constant ϵ'). See [BdlHV08, Remark 1.1.2].

2.5.6 Kazhdan property (T)

Let G be a topological group

Definition 2.5.24. The group G has *Kazhdan property (T)* if the family $\mathcal{U}_0$ of all continuous unitary representations without non-trivial invariant vectors[22] admits a Kazhdan pair (K, ϵ). Such pair (K, ϵ) (resp. set, constant) is a *Kazhdan pair* (resp. *set, constant*) *of the group* G. In the remainder, when we say that a set K is a Kazhdan set without specifying any family of representations we mean that K is a Kazhdan set of the group.

Remark 2.5.25. If a compact subset $K \subseteq G$ is a Kazhdan set for every unitary G-representation π with no non-trivial invariant vectors, then it is a Kazhdan set of G. That is, if $\epsilon_\pi > 0$ is the largest constant such that (K, ϵ_π) is a Kazhdan pair for π, then there must exist a $\epsilon > 0$ such that $\epsilon_\pi \geq \epsilon$ for every π. Indeed if this was not the case one would get a contradiction by considering the direct sum of a sequence of representations $\pi_n \in \mathcal{U}_0$ with $\epsilon_{\pi_n} \to 0$. [23]

Theorem 2.5.26. *Let G be topological group. The following are equivalent:*

(i) G has Kazhdan property (T);

(ii) if a unitary representation π weakly contains I_G then it has invariant vectors (π contains I_G as a subrepresentation);

[21] Using more spectral theory, Proposition 2.5.22 could have been proved as follows: since $\pi(\mu_{S_e^{\pm}})$ is self adjoint, it has real spectrum and its norm is equal to the spectral radius. Moreover, since it has no residual spectrum the point spectrum realises the spectral radius. Thus it has norm one if and only if there exist eigenvectors (or approximate eigenvectors) with eigenvalue 1, *i.e.* almost invariant vectors.

[22] Technically, $\mathcal{U}_0$ is not a set. The formal definition should be that there exists a pair (K, ϵ) that is a Kazhdan pair for every set of unitary representations without non-trivial invariant vectors.

[23] This arguments extends to any family of unitary representations $\mathcal{F}$ that is closed under countable direct sum. It fails in the general setting.

(iii) the trivial representation is isolated in the Fell topology from any set of unitary representations without non-trivial fixed points;

Moreover, if G is locally compact Hausdorff second countable all of the above are equivalent to:

(iv) no unitary representation $\pi \in \mathcal{U}_0$ admits a sequence of almost invariant vectors;

(v) for every $\pi \in \mathcal{U}_0$ there exists a $f \in L^1(G)$ such that $\|\pi(f)\| < \|f\|_1$;

(vi) for every $\pi \in \mathcal{U}_0$ there exists a $\mu \in \mathcal{P}(G)$ such that $\|\pi(\mu)\| < 1$;

(vii) the trivial representation is isolated in the Fell topology in the set of irreducible unitary representation $\widehat{G}$ (see Remark 2.5.14).

Sketch of proof. *(i)⇒(ii)* Let K be a Kazhdan set of G. If π does not have non-trivial invariant vectors, then K is Kazhdan set for π. The statement follows from the proof of *(i)⇒(iii)* in Lemma 2.5.19.

(ii)⇒(i) If G does not have Kazhdan property T, for every pair (K, ϵ) with $K \subseteq G$ compact and $\epsilon > 0$ there exists a unitary representation $\pi_{(K,\epsilon)}$ in $\mathcal{U}_0$ that has (K, ϵ)-invariant vectors. Let I be the set of such pairs (K, ϵ). Then the direct sum $\bigoplus_{(K,\epsilon)\in I} \pi_{(K,\epsilon)}$ is a unitary representation without non-trivial fixed vectors that does not admit a Kazhdan pair.

(ii)⇒(iii) If $\{\pi_i \mid i \in I\}$ is a set of representations with no non-trivial fixed vectors that is not isolated from I_G, then the direct sum $\bigoplus_{i\in I} \pi_i$ is a representation without non-trivial invariant fixed vectors that weakly contains I_G.

(iii)⇒(ii) If π has no non-trivial fixed vectors and $I_G \prec \pi$ then $\{\pi\}$ is not isolated from I_G.

Assume now that G is locally compact Hausdorff second countable.

(ii) ⇔(iv) ⇔(v) ⇔(vi) Follow from Lemma 2.5.19.

(i)⇒(vii) Is a special case of *(i)⇒(ii)*.

(vii)⇒(i) See [BdlHV08, Theorem 1.2.5] for a proof. □

Lemma 2.5.27. *A finite set S in a compact group G is a Kazhdan set if and only if it is a Kazhdan set for the restriction of the regular representation λ_G to the invariant subspace $L^2_0(G)$ (equivalently, the action of $\Gamma = \langle S \rangle < G$ by left multiplication on G has a spectral gap).*

Proof. By Proposition 2.5.22, to prove that S is a Kazhdan set it is enough to check that there is an $\epsilon > 0$ such that for every unitary representation π without non-trivial invariant vectors we have $\left\|\pi(\mu_{S_e^{\pm}})\right\| < 1-\epsilon$. This readily proved combining Corollary 2.5.17 and Proposition 2.5.22. □

A fundamental feature of Kazhdan property (T) is the following:

Theorem 2.5.28 (Kazhdan [Kaz67][24])**.** *Let G be a locally compact Hausdorff second countable group and H a closed subgroup such that G/H admits a finite G-invariant regular Borel measure. Then G has property (T) if and only if H does. In particular, this holds if $H = \Gamma$ is a lattice in G.*

2.5.7 Strong Banach property (T)

Strong Banach property (T) was introduced by Lafforgue [Laf08, Laf09]. It is a reinforcement of property (T) in that it implies that the trivial representation is not only isolated from other unitary representations, but also from representations by operators on a Banach space with slowly growing norm. We use the version of strong property (T) relative to classes of Banach spaces, which is implicit in [Laf09], and which appeared explicitly in [DLS16]. The latter article also gives a characterization of strong property (T), which is what we take as its definition here.

A *continuous length function* of a topological group G is a continuous function $\ell\colon G \to [0,\infty)$ such that $\ell(g) = \ell(g^{-1})$ and $\ell(gh) \leq \ell(g) + \ell(h)$ for every $g, h \in G$.

Definition 2.5.29. Let $\mathcal{E}$ be a class of Banach spaces. A locally compact group G has *strong property* (T) with respect to $\mathcal{E}$, denoted by ($\mathrm{T}_{\mathcal{E}}^{\mathrm{strong}}$), if for every length function ℓ on G there exists a sequence of compactly supported symmetric Borel measures μ_n on G such that for every Banach space E in $\mathcal{E}$ there exists a constant $t > 0$ such that the following holds: for every representation $\pi\colon G \to B(E)$ continuous in the strong operator topology and satisfying $\|\pi(g)\|_{B(E)} \leq Le^{t\ell(g)}$ with $L \in \mathbb{R}_+$, the sequence $\pi(\mu_n)$ converges in the norm topology on $B(E)$ to a projection onto the $\pi(G)$-invariant vectors in E.

The *strong Banach property (T) of Lafforgue* corresponds to taking $\mathcal{E}$ to be the class of Banach spaces with nontrivial type. We need strong Banach property (T) because of the following:

[24] We took the statement from [BdlHV08, Theorem 1.7.1]. We refer to said book for a proof.

Proposition 2.5.30. *If Γ is a finitely generated group with strong Banach property (T) of Lafforgue, then every ergodic measure preserving action $\rho\colon \Gamma \curvearrowright (X, \nu)$ has E-spectral gap for every Banach space E with non-trivial type. Moreover, this spectral gap is uniform in the class of Banach spaces with nontrivial type, i.e. the constant ϵ in the definition of spectral gap in Section 2.4.5 does not depend on E nor on the action.*

This result is well known to experts. For the specific case of classes of uniformly convex Banach spaces and isometric representations on these spaces (which is the setting we will be using it), the result also follows from [DN17, Theorem 4.6].

2.6 Coarse geometry

D Let (X, d_X) and (Y, d_Y) be metric spaces. Given a constant $C > 0$, we say that two maps $f, g\colon (X, d_X) \to (Y, d_Y)$ are *C-close* if $d_Y(f(x), g(x)) < C$ for every $x \in X$. Recall that a subset Z of (X, d_X) is *C-dense* in X if for every $x \in X$ there exists a $z \in Z$ such that $d_X(x, z) < C$.

2.6.1 Quasi-isometries

We use the following definition of quasi-isometry:

Definition 2.6.1. Let $f\colon (X, d_X) \to (Y, d_Y)$ be a (possibly discontinuous) map between metric spaces and let $A, L \geq 0$ be constants. We say that the map f is

- (L, A)-*coarsely Lipschitz* if

$$d_Y(f(x), f(x')) \leq L d_X(x, x') + A$$

  for every $x, x' \in X$;

- a (L, A)-*quasi-isometric embedding* if

$$L^{-1} d_X(x, x') - A \leq d_Y(f(x), f(x')) \leq L d_X(x, x') + A$$

  for every $x, x' \in X$;

- a (L, A)-*quasi-isometry* if it is (L, A)-coarsely Lipschitz and there exists an (L, A)-coarsely Lipschitz quasi-inverse $\bar{f}\colon (Y, d_Y) \to (X, d_X)$ so that $f \circ \bar{f}$ and $\bar{f} \circ f$ are A-close to id_Y and id_X respectively.

Two spaces (X, d_X) and (Y, d_Y) are *quasi-isometric* if there is a quasi-isometry between them (this is clearly an equivalence relation).

We say that two sequences of metric spaces $(X_n, d_{X_n})_{n\in\mathbb{N}}$ and $(Y_n, d_{Y_n})_{n\in\mathbb{N}}$ are *uniformly quasi-isometric* if there exist quasi isometries $f_n\colon X_n \to Y_n$ sharing the same constants L, A.

Remark 2.6.2. In the definition of (L, A)-coarsely Lipschitz maps the constant L can be smaller than 1. On the contrary, for a map to be an (L, A)-quasi-isometry or an (L, A)-quasi-isometric embedding of a space with infinite diameter it follows from the definition that the constant L must be at least 1.

In literature a slightly different definition of quasi-isometry is often used (coarse surjective quasi-isometric embedding). We decided to use Definition 2.6.1 because it makes the notation tidier when dealing with discrete fundamental groups (Chapters 3 and 4). The difference between the two definitions is very much inessential and it only affects the values of the constants A and L as accounted for by the next lemma.

Lemma 2.6.3. *If $f\colon (X, d_X) \to (Y, d_Y)$ is a (L, A)-quasi-isometry then it is a (L, A)-quasi-isometric embedding and $f(X)$ is A-dense in Y (*i.e. *f is* coarsely surjective*). Vice versa, if $f\colon (X, d_X) \to (Y, d_Y)$ is a (L, A)-quasi-isometric embedding and $f(X)$ is C-dense then f is a (L, D)-quasi-isometry where*

$$D = \max\big\{A + 2C, C, L(A + C)\big\}.$$

Sketch of proof. Let $\bar{f}$ be a quasi-inverse of f. Then $f(X)$ is A-dense because $f \circ \bar{f}$ is A-close to id_Y. Moreover, f is an (L, A)-quasi-isometric embedding because we have

$$L d_Y\big(f(x), f(x')\big) + A \geq d_X\big(\bar{f}(f(x)), \bar{f}(f(x'))\big) > d(x, x') - 2A$$

and we already noted that $L \geq 1$.

For the other implication, for every y in Y let $z(y) \in Y$ be a point in $f(X)$ that is at distance less than C from y and define $\bar{f}(y)$ to be a pre-image of $z(y)$ under f (just choose one). It is then clear that $f \circ \bar{f}$ is C-close to id_Y. It is also easy to verify that $\bar{f}$ is a $(L, A + 2C)$-coarsely Lipschitz and that $\bar{f} \circ f$ is $L(A + C)$-close to id_X. □

2.6.2 Nets and Vietoris-Rips graphs

Recall from Subsection 2.1.3 that an (r, ϵ)-net is a subset that is r-dense and ϵ-separated.

Remark 2.6.4. If $Y \subseteq X$ is a (r, ϵ)-net and $f\colon X \to Z$ is a (L, A)-quasi-isometry, then $f(Y)$ is a $(Lr + 2A, \epsilon/L - A)$-net in Z (the $(\epsilon/L - A)$-separatedness is only meaningful when $\epsilon > LA$).

Metric spaces can be 'discretised' using Vietoris-Rips graphs:

Definition 2.6.5. For a fixed constant $C > 0$ the *Vietoris-Rips graph* at scale C of a metric space X is the graph[25] $\mathcal{VR}(C, X)$ having one vertex for every point $x \in X$ and an edge $\{x, y\}$ if and only if $d(x, y) < C$.

The Vietoris-Rips graph of a continuous space X will have infinite degree. Still, the Vietoris-Rips graph of a net of X tends to be well-behaved and provide a 'good coarse model' for X.

Lemma 2.6.6. *Let (X, d_X) be a geodesic metric space and $Y \subseteq X$ be r-dense. Then for every $C \geq 3r$ the Vietoris-Rips graph $\mathcal{VR}(C, Y)$ is connected and the inclusion $\mathcal{VR}(C, Y) \hookrightarrow X$ is a (L, A)-quasi-isometry where*

$$L = \max\left\{\frac{1}{r}, C\right\} \qquad \text{and} \qquad A = \max\{1, r\}.$$

Sketch of proof. For every $x \in X$ let $f(x) \in Y$ be a point such that and $d_X(x, f(x)) < r$. For any pair of points $x, x' \in X$ with $d_X(x, x') < r$ we have $d_Y(f(x), f(x')) < 3r$, and hence $f(x)$ and $f(x')$ are joined by an edge in $\mathcal{VR}(C, Y)$.

Since X is geodesic, for every pair of points $x, x' \in X$ there exists a sequence $x = x_0, \dots, x_n = x'$ with $d_X(x_{i-i}, x_i) < r$ for every $i = 1, \dots, n$ and where $n = \lceil d_X(x, x')/r \rceil + 1$. It follows that the sequence $f(x_0), \dots, f(x_n)$ is a path in $\mathcal{VR}(C, Y)$. This proves both that $\mathcal{VR}(C, Y)$ is connected and that the map $f\colon (X, d_X) \to \mathcal{VR}(C, Y)$ is $(1/r, 1)$-coarsely Lipschitz.

Moreover, if $\{y, y'\}$ is an edge in $\mathcal{VR}(C, Y)$ then $d_X(y, y') < C$. As $\mathcal{VR}(C, Y)$ is connected, the inclusion map is C-Lipschitz. This concludes the proof because the compositions of these maps are respectively r-close to id_X and equal to id_Y. □

Proposition 2.6.7. *A metric space X is quasi-isometric to a geodesic metric space if and only if for every r-dense subset $Y \subset X$ there is a constant $C \gg r$ such that $\mathcal{VR}(C, Y)$ is connected and its inclusion in X is a quasi-isometry.*

Sketch of proof. Since a connected graph is geodesic (in the sense of Remark 2.7.1), when X is quasi-isometric to a connected Vietoris-Rips graph $\mathcal{VR}(C, Y)$ there is nothing to prove.

[25] See Section 2.7 for our conventions on graphs.

Vice versa, let $f\colon X \to Z$ be a (L, A)-quasi-isometry into a geodesic metric space. Note that for every $C \geq 3r$ the inclusion $\mathcal{VR}(C, Y) \hookrightarrow X$ is C-Lipschitz and has r-dense image (as in the proof of Lemma 2.6.6). By Lemma 2.6.3, it is hence enough to show that there are constants L' and A' such that

$$d_X(y, y') \geq \frac{1}{L'} d_{\mathcal{VR}(C,Y)}\big(y, y'\big) - A'$$

for every $y, y' \in Y$.

The set $f(Y)$ is $(Lr + 2A)$-dense in Z. Thus, for every $C' \geq 3(Lr + 2A)$ the inclusion $\mathcal{VR}(C', f(Y)) \hookrightarrow Z$ is a quasi-isometry by Lemma 2.6.6. If $\{f(y), f(y')\}$ is an edge in $\mathcal{VR}(C', f(Y))$ and $C \gg r$ is such that $C \geq L(C' + A)$, then $\{y, y'\}$ must be an edge in $\mathcal{VR}(C, Y)$ as well. For such a C, it follows that

$$d_{\mathcal{VR}(C,Y)}\big(y, y'\big) \leq d_{\mathcal{VR}(C', f(Y))}\big(f(y), f(y')\big) \asymp d_Z\big(f(y), f(y')\big).$$

Composing the above inequality with that coming from the existence of the quasi-inverse $\bar{f}\colon Z \to X$, we deduce that the constants L' and A' that we seek for do indeed exist (to produce optimal constants one would have to merge this proof with the proof of Lemma 2.6.6). □

Note that if a space X has bounded geometry (Definition 2.1.10) and $Y \subseteq X$ is ϵ-separated, then for every $C > 0$ the Vietoris-Rips graph $\mathcal{VR}(C, Y)$ has bounded degree (it has degree bounded by $f_Y(C)$, in the notation of Remark 2.1.11).

Proposition 2.6.8. *Let X be quasi-isometric to a geodesic metric space. Then X is quasi-isometric to a space with bounded geometry if and only if one (and hence every) Vietoris-Rips graph $\mathcal{VR}(C, Y)$ has bounded degree, where $Y \subseteq X$ is an (r, ϵ)-net with $C \gg r \geq \epsilon \gg 1$ large enough.*

Sketch of proof. One direction is clear because a graph has bounded geometry if and only if it has bounded degree, and by Proposition 2.6.7 we know that X is quasi-isometric to $\mathcal{VR}(C, Y)$ as soon as Y is a (r, ϵ)-net set and C is large enough.

For the converse implication, Let $f\colon X \to Z$ be an (L, A)-quasi-isometry where Z has bounded geometry and let $Y \subseteq X$ be ϵ-separated for some large ϵ. Enlarging it if necessary, we can assume that Y is a ϵ-net, and hence $f(Y)$ is $(L\epsilon + A, \epsilon/L - A)$-net in Z. To conclude it is hence enough to note that when ϵ is large enough the map f induces an inclusion of graphs $\mathcal{VR}(C, Y) \hookrightarrow \mathcal{VR}\big(LC + A, f(Y)\big)$ and the latter has bounded degree. □

2.6.3 Coarse embeddings and equivalences

Definition 2.6.9. A map $f\colon (X, d_X) \to (Y, d_Y)$ between metric spaces is called a *coarse embedding* if there are two increasing and unbounded *control functions* $\rho_-, \rho_+\colon [0,\infty) \to [0,\infty)$, such that

$$\rho_-\big(d_X(x_1,x_2)\big) \le d_Y\big(f(x_1),f(x_2)\big) \le \rho_+\big(d_X(x_1,x_2)\big)$$

for every $x_1, x_2 \in X$.

A coarse embedding $f\colon (X, d_X) \to (Y, d_Y)$ is a *coarse equivalence* if it has a coarse inverse; *i.e.* there exists a coarse embedding $\bar{f}\colon (Y, d_Y) \to (X, d_X)$ so that both $f \circ \bar{f}$ and $\bar{f} \circ f$ are at bounded distance from the identity functions.

Remark 2.6.10. As in Lemma 2.6.3, a coarse embedding is a coarse equivalence if and only if it is coarsely surjective.

It is clear that quasi-isometric spaces are coarsely equivalent, and it turns out that for (coarsely) geodesic spaces the converse is also true:

Lemma 2.6.11. *If X is a geodesic metric spaces then every coarse embedding $f\colon X \to Y$ is coarsely Lipschitz.*

If (X, d_X) and (Y, d_Y) are quasi-isometric to geodesic metric spaces and X is coarsely equivalent to Y, then X and Y are quasi-isometric.

Proof. Let x and y be any two points in a geodesic space X and let $x = x_0, \dots, x_n = y$ be a sequence of points with $d_X(x_i, x_{i-1}) \le 1$ and $n = \lceil d_X(x,y) \rceil$. Then we have

$$d_Y\big(f(x),f(y)\big) \le \sum_{i=1}^{n} d_Y\big(f(x_i),f(x_{i-1})\big) \le \lceil d_X(x,y) \rceil \rho_+(1).$$

Let now X and Y be quasi-isometric to geodesic metric spaces X' and Y'. Then X' and Y' are coarsely equivalent. Since the coarse embedding $X' \to Y'$ and its coarse inverse must be coarsely Lipschitz then they are quasi-isometries. □

Remark 2.6.12. Note that it is not true that a coarse embedding must be a quasi-isometric embedding (not even for geodesic spaces).

Since all the spaces we are going to work with are quasi-isometric to geodesic metric spaces, we will often use 'quasi-isometric' and 'coarsely equivalent' as synonyms for metric spaces. To take advantage of this redundancy of terminology, we use the following:

Definition 2.6.13. Two sequences of metric spaces $(X_n, d_{X_n})_{n\in\mathbb{N}}$ and $(Y_n, d_{Y_n})_{n\in\mathbb{N}}$ are *coarsely equivalent* if there is a sequence of coarse equivalences $f_n\colon (X_n, d_{X_n}) \to (Y_n, d_{Y_n})$ with the same control functions ρ_- and ρ_+.

Remark 2.6.14. Sequences of geodesic metric spaces are coarsely equivalent if and only if they are uniformly quasi-isometric. That is, in our definition of coarse equivalence we dropped the term 'uniformly' from the notation. This is not a standard convention, but it makes sense in our context because when we study the coarse geometry of a sequence of spaces we actually only care about its uniform aspects.

2.7 Graphs and expanders

Unless otherwise stated, we will only consider simplicial non-oriented graphs *i.e.* a graph $\mathcal{G}$ is the datum of a set of vertices $\mathrm{V}(\mathcal{G})$ and a set of unoriented edges $\mathrm{E}(\mathcal{G}) \subseteq \big\{\{v, w\} \;\big|\; v, w \in \mathrm{V}(\mathcal{G}),\ v \neq w\big\}$. We will usually denote graphs with calligraphic letters.

A *map of graphs*, or a *graph morphism* $f\colon \mathcal{G} \to \mathcal{G}'$ is a map between the vertex sets $f\colon \mathrm{V}(\mathcal{G}) \to \mathrm{V}(\mathcal{G}')$ that sends edges to edges.

Given a vertex $v \in \mathrm{V}(\mathcal{G})$, its *neighbours* are the vertices $w \in \mathrm{V}(\mathcal{G})$ such that $\{v, w\}$ is an edge of $\mathcal{G}$, and we denote the set of neighbours by ∂v. The *degree of* v is $\deg(v) \coloneqq |\partial v|$ (this could be infinite). The *degree of* $\mathcal{G}$ is the supremum $\deg(\mathcal{G}) \coloneqq \sup_{v\in\mathrm{V}(\mathcal{G})} \deg(v)$. Most of the graphs that we are going to use will have bounded degrees, but at times we will also have deal with graphs with unbounded degree.

2.7.1 Graphs as metric spaces

A *path of length* n on a graph $\mathcal{G}$ is a sequence of vertices $v_0, \dots, v_n$ where consecutive vertices are joined by an edge. We say that $\mathcal{G}$ is *connected* if every two points are joined by a path in $\mathcal{G}$.

If $\mathcal{G}$ is a connected graph, we obtain a *path metric* on the set of vertices $\mathrm{V}(\mathcal{G})$ by imposing that the distance between two vertices be the minimal length of a path joining them. We will often consider connected graphs as metric spaces by considering their vertex sets equipped with the path metrics.

In particular, when we write $v \in \mathcal{G}$ we mean that v is a vertex in $\mathcal{G}$, and when we talk about maps between a graph and another metric space X we actually mean maps

between X and the vertex set of the graph. Note that a map between graphs seen as metric spaces needs not be a graph morphism.

The set of neighbours of a vertex $v \in \mathcal{G}$ is equal to the set of points of $\mathcal{G}$ at distance 1 from v. In particular, a map between connected graphs (seen as metric spaces) is a a graph morphism if and only if it is 1-Lipschitz, and it is an isomorphism if and only if it is an isometry.

Remark 2.7.1. Every graph can be represented geometrically by gluing a copy of the interval $[0, 1]$ for every edge. Clearly, the graph is connected if and only if the topological space thus obtained is connected, and when this is the case the topological space can be given a natural path metric whose restriction to the vertex set coincides with the graph path metric. For this reason we think of graphs as geodesic metric spaces (even though, strictly speaking, they are only quasi-isometric to geodesic metric spaces).

The *girth* of a graph $\mathcal{G}$ is the length of the shortest non-trivial loop in $\mathcal{G}$ (equivalently, the length of the shortest simple closed loop).

2.7.2 Expanders

For every graph $\mathcal{G}$ and every set of vertices $W \subset \mathrm{V}(\mathcal{G})$ we will denote by ∂W the *external vertex boundary*

$$\partial W \coloneqq \big\{ v \in \mathrm{V}(\mathcal{G}) \smallsetminus W \;\big|\; \exists w \in W \text{ s.t. } \{v, w\} \text{ is an edge} \big\}.$$

Note that ∂W is equal to the set of vertices at distance 1 from W. Equivalently, it can be expressed in terms of its neighbourhood of radius 1:

$$\partial W = N_1(W) \smallsetminus W.$$

We will denote by ∂_{int} the *internal vertex boundary*

$$\partial_{\mathrm{int}} W \coloneqq \big\{ w \in W \;\big|\; \exists v \in \mathrm{V}(\mathcal{G}) \smallsetminus W \text{ s.t. } \{v, w\} \text{ is an edge} \big\}.$$

Note that the internal vertex boundary of W coincides with the external vertex boundary of its complement $\mathrm{V}(\mathcal{G}) \smallsetminus W$. In particular, we have

$$\partial_{\mathrm{int}} W = N_1\big(\mathrm{V}(\mathcal{G}) \smallsetminus W\big) \cap W.$$

We define the *Cheeger constant* of the graph $\mathcal{G}$ as the infimum

$$h(\mathcal{G}) \coloneqq \inf \left\{ \frac{|\partial W|}{|W|} \;\middle|\; W \subset \mathrm{V}(\mathcal{G}) \text{ finite}, |W| \leq \frac{1}{2}\big|\mathrm{V}(\mathcal{G})\big| \right\}$$

(if the graph is infinite the condition on the cardinality of the set W is vacuous).

Definition 2.7.2. A sequence of finite graphs $(\mathcal{G}_n)_{n\in\mathbb{N}}$ is a *family of expanders* if $|\mathrm{V}(\mathcal{G}_n)| \to \infty$ and there are two constants $C, \epsilon > 0$ so that every graph $\mathcal{G}_n$ has degree bounded above by C and Cheeger constant $h(\mathcal{G}_n)$ at least ϵ.

Remark 2.7.3. We will sometimes be informal and simply write 'expander' instead of 'family of expanders'. Even when we do so, we still think of it as a family of spaces and our usual conventions applies (*e.g.* if we say that an expander does not coarsely embed in a Banach space we mean that the corresponding family of graphs does not uniformly coarse embed in that Banach space).

Remark 2.7.4. In the literature the Cheeger constant is usually defined using the edge-boundary, we chose to use this less standard definition because it is more convenient for our purposes. The two notions are coarsely equivalent when dealing with graphs with bounded degree.

For families of graphs with bounded degree the property of being expanders is an invariant of coarse equivalence:

Lemma 2.7.5. *Let $(\mathcal{X}_n)_{n\in\mathbb{N}}$ and $(\mathcal{Y}_n)_{n\in\mathbb{N}}$ be two coarsely equivalent sequences of (connected) graphs with uniformly bounded degree. Then $\mathcal{X}_n$ is a family of expanders if and only if so is $\mathcal{Y}_n$.*

Proof. As both $\mathcal{X}_n$ and $\mathcal{Y}_n$ are assumed to have uniformly bounded degree, it is enough to show that if the graphs $\mathcal{Y}_n$ do not have a uniform bound on their Cheeger constants then neither do the $\mathcal{X}_n$'s.

Let $f_n\colon \mathcal{X}_n \to \mathcal{Y}_n$ be a sequence of coarse embeddings with A-dense image and control uniform functions ρ_- and ρ_+. Assume that there is a sequence of subsets $F_n \subset \mathcal{Y}_n$ such that $|F_n| \leq |\mathcal{Y}_n|/2$ and $|\partial F_n|/|F_n| \to 0$, we claim that the sequence of their preimages $T_n \coloneqq f_n^{-1}(F_n)$ behaves similarly in $\mathcal{X}_n$.

First of all we want to show that the sets T_n are large, so let us consider the subsets of 'very internal' vertices of F_n:

$$I_n = \{w \in F_n \mid B(w, A) \subseteq F_n\}.$$

Since f_n is coarsely surjective, we have that $\forall w \in I_n$ there exists $v \in \mathcal{X}_n$ such that $d(f(v), w) < A$ and hence $v \in T_n$. Note that if $w, w' \in F_n$ have distance $d(w, w') \geq 2A$ and $v, v' \in T_n$ are such that $d(f(v), w) < A$ and $d(f(v')w') < A$ then $v \neq v'$. We deduce that the image of T_n contains at least $|I_n|/D^{2A}$ vertices, where D is the bound on the degree of the graphs.

Since we have

$$I_n = F_n \smallsetminus \Big(\bigcup \{ B(w, A) \mid w \in \partial F_n \} \Big),$$

we deduce:

$$|T_n| \geq \frac{|I_n|}{D^{2A}} \geq \frac{|F_n| - D^A |\partial F_n|}{D^{2A}} \geq \lambda_n |F_n|$$

for some sequence of values λ_n that approach $1/D^{2A}$ as n increases.

We now wish to bound $|\partial T_n|$ linearly with $|\partial F_n|$. The bounding function ρ_- is unbounded, thus there exists a $r > 0$ so that $\rho_-(r) \geq 1$. Therefore, if two vertices v, v' in $\mathcal{X}_n$ have distance $d(v, v') \geq r$ then they have different images $f_n(v) \neq f_n(v')$. We deduce that the image of ∂T_n contains at least $|\partial T_n|/D^r$ vertices. Note that a vertex v is in ∂T_n if and only if it lies outside T_n and it neighbours with a vertex v' in T_n. In particular, $f(v') \in F_n$ must be within distance $A + \rho_+(1)$ of ∂F_n and hence $f_n(y)$ is at most at distance $\kappa \coloneqq 2\rho_+(1) + A$ from ∂F_n. Therefore we have

$$\frac{|\partial T_n|}{D^r} \leq |\partial F_n| D^\kappa,$$

thus we get:

$$\frac{|\partial T_n|}{|T_n|} \leq \frac{D^{\kappa + r}}{\lambda_n} \frac{|\partial F_n|}{|F_n|}$$

and right hand side tends to zero.

The only issue now is that T_n might have more than $|\mathcal{X}_n|/2$ elements. If that is the case we only need to take its complement. Indeed, let O_n be the set of vertices that are 'very external' from F_n

$$O_n = \{ w \notin F_n \mid B(w, A) \subseteq \mathcal{Y}_n \smallsetminus F_n \}.$$

As $|\mathcal{X}_n \smallsetminus T_n| \geq |f_n^{-1}(O_n)|$, by the same argument of above we have

$$|\mathcal{X}_n \smallsetminus T_n| \geq \frac{|O_n|}{D^{2A}} \geq \frac{|\mathcal{Y}_n \smallsetminus F_n| - D^A |\partial F_n|}{D^{2A}} \geq \lambda_n |\mathcal{Y}_n \smallsetminus F_n| \geq \lambda_n |F_n|$$

and we can conclude because

$$|\partial(\mathcal{X}_n \smallsetminus T_n)| \leq D |\partial T_n|.$$

□

With a slight abuse of notation, we give the following:

Definition 2.7.6. A sequence of metric spaces $\big(X_n, d_{X_n}\big)_{n \in \mathbb{N}}$ *forms a family of metric expanders* if it is coarsely equivalent to a family of expander graphs $\mathcal{X}_n$.

Remark 2.7.7. Note that Lemma 2.7.5 implies that a family of graphs with uniformly bounded degrees form a family of metric expanders if and only if it is a genuine family of expander graphs (Definition 2.7.2).

Remark 2.7.8. Let $(X_n, d_{X_n})_{n\in\mathbb{N}}$ be a sequence of metric spaces that are uniformly quasi-isometric to a sequence of geodesic metric spaces. It follows from Lemma 2.7.5, Proposition 2.6.7 and Proposition 2.6.8, that $(X_n, d_{X_n})_{n\in\mathbb{N}}$ forms a sequence of metric expanders if and only if the sequence of Vietoris-Rips graphs $\mathcal{VR}(C, Y_n)$ is a family of expanders, where $Y_n \subseteq X_n$ is an (r, ϵ)-net for some fixed $C \gg r \geq \epsilon \gg 1$ large enough.

2.7.3 Superexpanders

One of the many remarkable features of expander graphs is that it is not possible to embed them into any Hilbert space without greatly distorting the metric. More precisely, the following is a well known[26] (see [Mat97] and also [Oza04, Appendix A]):

Theorem 2.7.9. *A family of expanders does not coarsely embed into any L^p space for $1 \leq p < \infty$.*

Remark 2.7.10. It immediately follows from Theorem 2.7.9 that metric expanders do not coarsely embed into any L^p space.

It is sometimes possible to prove stronger non-embeddability results for expanders. We will use the following:

Definition 2.7.11. A *family of superexpanders* is a family of expanders that cannot be coarsely embedded into any uniformly convex Banach space.

Remark 2.7.12. In Definition 2.7.11 it is necessary to restrict one's attention to some subclass of Banach spaces, because every metric space X can be isometrically embedded in the Banach space of continuous functions on X equipped with the supremum norm. The class of uniformly convex Banach spaces is a good candidate because it retains aspects of Euclidean geometry while being still a very large class (it is also a natural choice to make in the context of the Baum-Connes conjecture).

Remark 2.7.13. Often, superexpanders are defined as expanders that satisfy some strong Banach-valued spectral gap. It is always true that the existence of such a spectral gap implies that there cannot exist embeddings into uniformly convex Banach spaces (and hence this definition is stronger than the one we gave). Still, the two conditions are probably not equivalent in general.

[26] In [Mat97] it is proved that expanders do not coarsely embed into any ℓ^p-space. From this it is possible to deduce that they do not coarsely embed into any L^p-space *e.g.* using Ostrovskii's results about coarse embeddings of locally finite metric spaces into Banach spaces.

The following is a major open question (attributed to V. Lafforgue):

Question 2.7.14. *Is every family of expanders a superexpander?*

It is interesting to note that, while there are no known examples of expanders that are not superexpanders, it is usually quite difficult to construct new explicit families of superexpanders.

2.8 Box spaces

The precise notations and conventions we use in this sections are not quite standard, but are convenient for our purposes.

2.8.1 Schreier coset graphs

Given a subgroup $\Lambda' < \Lambda$ of a finitely generated group $\Lambda = \langle S \rangle$, let $\Lambda/\Lambda' = \{h\Lambda' \subset \Lambda \mid h \in \Lambda\}$ be the set of *left cosets* and $\Lambda'\backslash\Lambda = \{\Lambda' h \subset \Lambda \mid h \in \Lambda\}$ the set of *right cosets* of Λ' in Λ. The *(left) Schreier coset graph* (or simply *(left) Schreier graph*) is the graph $\mathrm{Schr}(\Lambda'\backslash\Lambda, S)$ whose vertex set is the set of right cosets $\Lambda'\backslash\Lambda$ and where two vertices $\{\Lambda' h_1, \Lambda' h_2\}$ form an edge if and only if $\Lambda' h_1 = \Lambda' h_2 s$ for some $s \in S^{\pm}$.

The left Schreier graph with respect to the trivial group $\{e\}$ coincides with the left Cayley graph $\mathrm{Cay}(\Lambda, S)$. If $N \lhd \Lambda$ is a normal subgroup, then right cosets are also left cosets and the Schreier graph $\mathrm{Schr}(N\backslash\Lambda, S)$ coincides with the Cayley graph of the quotient $\mathrm{Cay}(\Lambda/N, \overline{S})$, where $\overline{S}$ is the image of the generating set S under the quotient map.

Similarly, the *right Schreier graph* $\overline{\mathrm{Schr}}^{\mathrm{r}}(\Lambda/\Lambda', S)$ is the graph whose vertices are the left cosets Λ/Λ' and such that $\{h_1\Lambda', h_2\Lambda'\}$ is an edge if and only if $h_1\Lambda' = sh_2\Lambda'$ for some $s \in S^{\pm}$. The right Schreier graph with respect to the trivial group $\{e\}$ coincides with the right Cayley graph $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Lambda, S)$. If $N \lhd \Lambda$ is a normal subgroup, the right Schreier graph $\overline{\mathrm{Schr}}^{\mathrm{r}}(\Lambda/N, S)$ coincides with the right Cayley graph of the quotient $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Gamma/N, \overline{S})$.

The group Λ has a natural left action on the set of left cosets, but this does *not* induce an action by graph morphisms on the right Schreier graph. Instead, it defines an action by maps with bounded displacement. The same happens for the right action on the set of right cosets and the left Schreier graph.

Note that the inverse map $h\Lambda' \mapsto \Lambda' h^{-1}$ gives an isomorphism of graphs between $\overline{\mathrm{Schr}}^{\mathrm{r}}(\Lambda/\Lambda', S)$ and $\mathrm{Schr}(\Lambda'\backslash\Lambda, S)$.

2.8.2 Finite index subgroups and box spaces

Let Λ be a (finitely generated) discrete group. If Λ' is a finite index subgroup, we denote it by $\Lambda' <_f \Lambda$. The index of a subgroup is denoted by $[\Lambda : \Lambda']$.

A sequence of finite index subgroups $\Lambda_n <_f \Lambda$ with $n \in \mathbb{N}$ is *residual* if $\bigcap_{n\in\mathbb{N}} \Lambda_n = \{e\}$. Such a sequence is a *filtration* if it is nested: $\Lambda_{n+1} <_f \Lambda_n$ for every $n \in \mathbb{N}$. A filtration is *normal* if it is composed by groups that are normal in Λ ($\Lambda_n \lhd_f \Lambda$ for every $n \in \mathbb{N}$). The group Λ is *residually finite* if it admits a residual sequence (or, equivalently, a residual filtration or a residual normal filtration).

Definition 2.8.1. Given a finitely generated group $\Lambda = \langle S \rangle$ and sequence $(\Lambda_k)_{k\in\mathbb{N}}$ of finite index subgroups of increasing index, the *(left) box space* $\square_{(\Lambda_k)}\Lambda$ of a finitely generated group Λ is the family of left Schreier graphs $\big(\mathrm{Schr}(\Lambda_k/\Lambda, S)\big)_{k\in\mathbb{N}}$.

A (left) box space is *residual* if it comes from a residual sequence of subgroups; it is *nested* if it comes from a filtration; and it is *normal* if it comes from sequence of normal subgroups (in which case the Schreier graphs are Cayley graphs of the quotients).

The *(residual, nested, normal) right box spaces* are defined analogously.

Box spaces are usually thought of as sequences of metric spaces. In particular, in this setting it does not matter whether we had taken the left or right box spaces as they are isometric. Note also that, when the generating set is not specified, box spaces are only defined up to coarse equivalence. Our usual conventions for coarse equivalences of families of metric spaces apply to box spaces as well.

Remark 2.8.2. Quite often in literature only residual nested normal box spaces are considered (and they are hence just called box spaces). Moreover, box spaces are often made into a single metric space (as opposed to a family of spaces) by giving the disjoint union of Cayley graphs $\mathrm{Cay}(\Lambda/\Lambda_k)$ a metric by imposing that the distance between two components be larger than the sum of their diameters. One can hence study the coarse geometry of a box space treating it as a single object.

As long as one is coherent, it does not matter what viewpoint is taken. Indeed, it is shown in [KV17] that if two box spaces $\square_{(\Lambda_k)}\Lambda$ and $\square_{(\Gamma_k)}\Gamma$ are coarsely equivalent (when seen as metric objects) then, possibly discarding a finite number of indices, one has that $\mathrm{Cay}(\Lambda/\Lambda_k)$ is uniformly quasi-isometric to $\mathrm{Cay}(\Gamma/\Gamma_k)$ (and therefore they are coarsely equivalent as sequences of metric spaces). The converse also holds because the distance between different components is bound to go to infinity.

2.8.3 Box spaces and expanders

A sequence $(\Lambda_k)_{k\in\mathbb{N}}$ of finite index proper subgroups of a group Λ is said to have *property* (τ) if the (left) actions on the sets of (left) cosets have uniform spectral gap (Subsection 2.4.4). Here the coset sets are equipped with the (renormalised) counting measure. It follows from Proposition 2.5.20 and Remark 2.5.21 that the sequence $(\Lambda_k)_{k\in\mathbb{N}}$ has property (τ) if and only if the sequence of representations $\Lambda \to \mathrm{U}\big(L^2_0(\Lambda/\Lambda_k)\big)$ are isolated from the trivial representations (*i.e.* admits a Kazhdan set). If $(N_k)_{k\in\mathbb{N}}$ is a sequence of normal proper finite index subgroups, this is also equivalent to saying that the trivial representation of Λ is isolated in the family of irreducible unitary representations of Λ factoring through Λ/Λ_k for some $k \in \mathbb{N}$. In particular, note that any sequence of subgroups in a group with property (T) has property (τ).

The following theorem was the first source of explicit expander graphs (Definition 2.7.2):

Theorem 2.8.3 (Margulis)**.** *A box space $\square_{(\Lambda_k)}\Lambda$ is a family of expanders if and only if the sequence $(\Lambda_k)_{k\in\mathbb{N}}$ has property (τ).*

Remark 2.8.4. We will obtain a proof of Theorem 2.8.3 as a corollary of a more general construction (Remark 6.1.7). In some sense, one could say that the bulk of this thesis is a generalisation of this theorem.

Similarly, it is the case that when the actions $\Lambda \curvearrowright \Lambda/\Lambda_k$ have uniform strong Banach valued spectral gaps, then the box space $\square_{(\Lambda_k)}\Lambda$ is a superexpander. In particular this happens any time the group Λ has the strong property (T) of Lafforgue.

In fact, V. Lafforgue was the first to produce examples of superexpanders (as a consequence of his work on strong property (T) and the Baum-Connes conjecture). He showed that if Λ is a cocompact lattice in an almost simple higher rank algebraic group over a non-Archimedean local field, and $(\Lambda_k)_{k\in\mathbb{N}}$ is a normal residual filtration of Λ with trivial intersection, then the box space $\square_{(\Lambda_k)}\Lambda$ is a superexpander [Laf08] (precisely because in this case Λ has Lafforgue's strong Banach property (T) by [Laf08, Laf09, Lia14]).

Definition 2.8.5. A *Lafforgue expander* is a normal nested residual box space of a lattice in an almost simple higher rank algebraic group over a non-Archimedean local field.

Chapter 3

Discrete fundamental groups for metric spaces

Most of the material of this chapter (with the exception of the coarse-geometric statements) is included in [Vig17b].

3.1 Definition and coarse geometry

3.1.1 Discrete fundamental groups

Let X be a metric space and fix a parameter $\theta > 0$. Recall from Subsection 2.1.2 that a discrete path at scale θ (or θ-*path*) is a θ-Lipschitz map $Z\colon [n] \to X$ (equivalently, it is an ordered sequence of points $(z_0, \dots, z_n)$ in X with $d(z_i, z_{i+1}) \leq \theta$). We will continue to use the notation Z^* to denote the reverse θ-path (*i.e.* the θ-path obtained by reversing the ordering of the tuple of points defining Z).

We say that a θ-path $Z'\colon [m] \to X$ is a *lazy version* (or *lazification*) of Z if it is obtained from it by repeating some values, *i.e.* if $m > n$ and $Z' = Z \circ f$ where $f\colon [m] \to [n]$ is a surjective monotone map.

Given two θ-paths Z_1 and Z_2 of the same length n, a *free θ-grid homotopy* between them is a θ-Lipschitz map $H\colon [n] \times [m] \to X$ such that $H(\cdot, 0) = Z_1$ and $H(\cdot, m) = Z_2$ (here the product $[n] \times [m]$ is equipped the ℓ^1-metric). A *θ-grid homotopy* is a free θ-grid homotopy so that $H(0, t) = Z_1(0) = Z_2(0)$ and $H(n, t) = Z_1(n) = Z_2(n)$ for every $t \in [m]$.

Definition 3.1.1. Two θ-paths Z_1 and Z_2 are *θ-homotopic* (denoted by $Z_1 \sim_\theta Z_2$) if they are equivalent under the equivalence relation induced by lazifications and θ-grid homotopies. Equivalently, $Z_1 \sim_\theta Z_2$ if and only if there exist lazy versions Z_1' and Z_2' of Z_1 and Z_2 which are homotopic via a θ-grid homotopy.

If the endpoint of a θ-path coincides with the starting point of a second θ-path, the two θ-paths can be concatenated. Note that for every θ-path Z, the concatenation ZZ^* is θ-homotopic to the constant path $Z(0)$. Since the operation of concatenation is clearly compatible with θ-homotopies, we can make the following definition:

Definition 3.1.2 ([BCW14])**.** Let x_0 be a base point in a metric space X. The *discrete fundamental group* at *scale* θ (or *θ-fundamental group*) is the group $\pi_{1,\theta}\big(X, x_0\big)$ consisting of θ-homotopy classes of closed θ-paths with endpoints x_0, equipped with the operation of concatenation.

Remark 3.1.3. Just as for the usual fundamental group, if X is a θ-connected metric space then the isomorphism class of the discrete fundamental group does not depend on the choice of the base point. Moreover, if x_0 and x_0' are points at distance at most θ, then the map sending a θ-path $(x_0, x_1, \dots, x_0)$ to the θ-path $(x_0', x_0, x_1, \dots, x_0, x_0')$ induces a canonical isomorphism $I_{(x_0,x_0')}\colon \pi_{1,\theta}(X, x_0) \xrightarrow{\cong} \pi_{1,\theta}(X, x_0')$.

3.1.2 Coarse geometry on discrete paths

One can also think of the θ-fundamental group as the quotient of the fundamental group where short cycles are ignored[1] (see Remark 3.2.2). In particular, it is no surprise that one can try to use discrete fundamental groups to construct invariants of quasi-isometries.

If $f\colon (X, d_X) \to (Y, d_Y)$ is a (L, A)-coarse Lipschitz map and $\theta' \geq L\theta + A$, then f induces a homomorphism $f_*\colon \pi_{1,\theta}(X, x_0) \to \pi_{1,\theta'}(Y, f(x_0))$ by composition: $f_*([Z]) \coloneqq [f \circ Z]$. We collect some basic results about the interplay between quasi-isometries and discrete fundamental groups in the following lemma (some of these statements are also proved in [BCW14]).

Lemma 3.1.4. *Let (X, d_X) and (Y, d_Y) be 1-geodesic metric spaces; $x_0 \in X$ a fixed base point; $f\colon (X, d_X) \to (Y, d_Y)$ an (L, A)-coarse Lipschitz map; $\theta, \theta' > 1$ constants with $\theta' \geq L\theta + A$ and $f_*\colon \pi_{1,\theta}(X, x_0) \to \pi_{1,\theta'}(Y, f(x_0))$ the induced homomorphism. Then the following hold.*

(i) If $g\colon (X, d_X) \to (Y, d_Y)$ is (L, A)-coarse Lipschitz and θ'-close to f then we have $g_* = I_{(f(x_0),g(x_0))} \circ f_*$, where $I_{(f(x_0),g(x_0))}$ is the canonical isomorphism of Remark 3.1.3.

[1] This is the reason why discrete fundamental groups are also known as *coarse fundamental groups* in the literature

(ii) If f is an (L,A)-quasi-isometry, and $\theta \geq L + A$ then f_* is surjective.

(iii) If $\theta \leq \theta'$ then $(\mathrm{id}_X)_*\colon \pi_{1,\theta}(X,x_0) \to \pi_{1,\theta'}(X,x_0)$ is surjective.

(iv) If f is an (L,A)-quasi isometry, $(\mathrm{id}_X)_*\colon \pi_{1,\theta}(X,x_0) \to \pi_{1,L\theta'+A}(X,x_0)$ is an isomorphism and $\theta \geq L + A$, then f_* is an isomorphism.

Proof. (i): Given a closed θ-path Z with endpoints x_0, the path $I_{(g(x_0),g(x_0))} \circ g(Z)$ is a lazification of $g(Z)$. It is now enough to notice that the paths $I_{(f(x_0),g(x_0))} \circ f(Z)$ and $I_{(g(x_0),g(x_0))} \circ g(Z)$ are θ'-grid homotopic because the maps f and g are θ'-close.

(ii): Since (Y,d_Y) is 1-geodesic, any θ'-path Z' in Y is θ'-homotopic to a 1-path. Indeed, one can concatenate the 1-paths realising the distance between consecutive points $Z'(i), Z'(i+1)$ obtaining this way a 1-path that is θ'-homotopic to Z'.

Given an element $[Z'] \in \pi_{1,\theta'}(Y, f(x_0))$, we can assume that Z' is a 1-path. Let $\bar{f}$ be a quasi-inverse of f, then $\bar{f}(Z')$ is a $(L+A)$-path and hence a θ-path. It is then easy to see that $I_{(\bar{f}(f(x_0)),x_0)}(\bar{f}(Z'))$ is a closed θ-path based at x_0 whose image under f is θ'-homotopic to Z'.

(iii): This is a special case of (ii).

(iv): The quasi-inverse $\bar{f}$ induces a homomorphism $\bar{f}_*$ from $\pi_{1,\theta'}(Y, f(x_0))$ to $\pi_{1,L\theta'+A}(X, \bar{f} \circ f(x_0))$. As $\bar{f} \circ f$ is A-close to id_X and therefore θ-close, by (i) we have that $(\bar{f} \circ f)_*$ coincides with $I_{(x_0,\bar{f}\circ f(x_0))} \circ (\mathrm{id}_X)_*$ and it is hence an isomorphism. The claim follows because f_* is surjective by (ii). □

Remark 3.1.5. Analogous statements hold for coarse equivalences and coarse embeddings (this is relevant only for people wishing to study the discrete fundamental groups of spaces that are not quasi-isometric to geodesic metric spaces).

3.2 The interplay between continuous and discrete

3.2.1 Discretisations of continuous paths

Given a continuous path $\gamma\colon [0,1] \to X$ and a sequence of times $0 = t_0 \leq t_1 \leq \cdots \leq t_n = 1$, we say that the discrete path $\big(\gamma(t_0), \ldots, \gamma(t_n)\big)$ is a *θ-discretisation* of γ if $\gamma(t_{i-1}, t_i) \subseteq \overline{B}\big(\gamma(t_{i-1}), \theta\big)$ for every $i = 1, \ldots, n$. We will denote this θ-path by $\widehat{\gamma}^{\theta}_{(t_0,\ldots,t_n)}$. We will systematically use the following simple lemma:

Lemma 3.2.1. *Every continuous path admits a θ-discretisation. If two paths α and β are (freely) homotopic then any two discretisations $\widehat{\alpha}^{\theta}_{t_0,\ldots,t_n}$ and $\widehat{\beta}^{\theta}_{t'_0,\ldots,t'_m}$ are (freely) θ-homotopic.*

Proof. Fix the parameter $\theta > 0$. For any continuous path $\gamma\colon [0,1] \to X$ consider the open cover of $[0,1]$ given by the preimages $\gamma^{-1}\big(B(x, \frac{\theta}{2})\big)$ with $x \in X$. By the Lebesgue's Number Lemma, there exists $N \in \mathbb{N}$ large enough so that for every $t \in [0,1]$ the segment $[t - 1/N, t + 1/N]$ is fully contained in one of those open set. It follows that letting $t_i \coloneqq i/N$ for $i = 0, \dots, N$ yields a θ-discretisation of γ.

It remains to prove that continuous homotopies induce discrete homotopies. Let $H\colon [0,1]^2 \to X$ be an homotopy between two paths α and β. As above, we can use Lebesgue's Number Lemma to deduce that there exists $N \in \mathbb{N}$ large enough so that for every $(t,s) \in [0,1]^2$ the image under H of the ball $B\big((t,s), 1/N\big)$ has diameter smaller than θ. Consider the map $\widehat{H}^\theta\colon [N]^2 \to X$ defined by

$$\widehat{H}^\theta(i,j) \coloneqq H\left(\frac{i}{N}, \frac{j}{N}\right).$$

Then the maps $\widehat{H}^\theta(\cdot, 0)\colon [N] \to X$ and $\widehat{H}^\theta(\cdot, N)\colon [N] \to X$ are θ-discretisations of α and β respectively, and $\widehat{H}^\theta$ is a free θ-grid homotopy between them.

To conclude the proof of the lemma it is hence enough to show that any two θ-discretisations of the same path are θ-homotopic. This is readily done. Let $\widehat{\gamma}^\theta_{t_0,\dots,t_n}$ and $\widehat{\gamma}^\theta_{t'_0,\dots,t'_m}$ be two θ-discretisations of a path γ. We can assume that the inequalities $t_0 < \dots < t_n$ are strict and—up to choosing a common refinement for the discretisations—we can also assume that $n \leq m$ and that for every t_i there exists a $j \geq i$ so that $t_i = t'_j$. We now define a surjective function $f\colon [m] \to [n]$ letting $f(j) \coloneqq \max\{i \mid t_i \leq t'_j\}$. Then the θ-path $\big(\gamma(t_{f(0)}), \dots, \gamma(t_{f(m)})\big)$ is a lazification of $\big(\gamma(t_0), \dots, \gamma(t_n)\big)$ and it is θ-homotopic to $(\gamma(t'_0), \dots, \gamma(t'_m))$ via a θ-grid homotopy consisting of a single step.

The same proof clearly implies the statement for free homotopies as well. □

From Lemma 3.2.1 it follows that there is a well-defined θ-discretisation map $\widehat{\cdot}^\theta\colon \pi_1(X, x_0) \to \pi_{1,\theta}\big(X, x_0\big)$. Since the concatenation of the discretisations of two paths is a discretisation of the concatenation of the paths, this map is a homomorphism.

For convenience, we will generally drop the specific times t_i from the notation and simply write $\widehat{\gamma}^\theta$ to denote a θ-discretisation of γ.

Remark 3.2.2. Note that when the image of a continuous path has diameter smaller than θ then its θ-discretisation will be trivially θ-homotopic to a constant θ-path. Moreover, every closed θ-path of length at most 4 is θ-homotopic to a constant path because $(z_0, z_1, z_2, z_3, z_4 = z_0) \sim_\theta (z_0, z_1, z_1, z_0, z_0) \sim_\theta (z_0, z_0, z_0, z_0, z_0)$.

In some sense, the above is the only way in which loops become null-homotopic in $\pi_{1,\theta}$. For instance, it is proved in [BKLW01] that the 1-fundamental group of a graph is isomorphic to the fundamental group of the graph quotiented by the normal group

generated by all the loops of length at most 4 in the graph. This can be generalised to all geodesic metric spaces (Corollary 4.3.6), but we decided to postpone this discussion to Subsection 4.2.2, where we will prove it in more generality for geodesic warped metric spaces.

3.2.2 Relations with the fundamental group at small scale

This subsection and the next section are an excursus on the relation between the fundamental group of a metric space and the 'limit as θ tends to 0' of its discrete fundamental groups $\pi_{1,\theta}(X, x_0)$. Albeit not being relevant to the rest of this thesis (where we focus on large scale properties), we developed these contents in in an attempt to answer some questions raised in [BCW14].

We already noted that, given a pair $\theta' \leq \theta$, we have a natural homomorphism $\pi_{1,\theta'}(X, x_0) \to \pi_{1,\theta}(X, x_0)$. In particular, it makes sense to consider the inverse limit $\varprojlim \pi_{1,\theta}(X, x_0)$ of this system of groups (see Remark 2.2.11). Note that, for $\theta' < \theta$, any θ'-discretisation $\widehat{\gamma}^{\theta'}_{t_0,\dots,t_n}$ of a continuous path γ is also a θ-discretisation. Therefore Lemma 3.2.1 also implies that the following diagram commutes:

$$\begin{array}{ccc} & \pi_1(X, x_0) & \\ {\scriptstyle \widehat{\cdot}^{\theta'}} \swarrow & & \searrow {\scriptstyle \widehat{\cdot}^{\theta}} \\ \pi_{1,\theta'}(X, x_0) & \longrightarrow & \pi_{1,\theta}(X, x_0). \end{array}$$

and hence the discretisation maps induce a group homomorphism of the inverse limit $\widehat{\cdot}\colon \pi_1(X, x_0) \to \varprojlim \pi_{1,\theta}(X, x_0)$. We will denote the image of (the homotopy class of) a continuous path γ by $\widehat{\gamma}$.

N.B. This use of the symbols $\widehat{\cdot}$ and $\widehat{\gamma}$ is specific of this subsection and the subsequent section. From Chapter 4 onward they will assume a different meaning. We are confident that no confusion will arise, because the context is going to be quite clearly different.

It is natural to ask when the discretisation map $\widehat{\cdot}\colon \pi_1(X, x_0) \to \varprojlim \pi_{1,\theta}(X, x_0)$ is an isomorphism (see also [BCW14, Question 3]). In the next theorem we prove that this is the case for a large class of well-behaved spaces. The conditions needed for the proof of Theorem 3.2.3 are (in some sense) sharp; see Section 3.3. Recall that u.l.p.c. and u.s.l.s.c are short for 'uniformly locally path connected' and 'uniformly semi-locally path connected' respectively (Definitions 2.1.1 and 2.1.4).

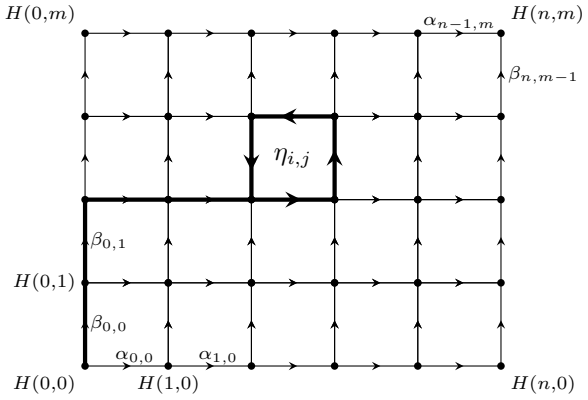


Figure 3.1: Constructing a continuous homotopy

Theorem 3.2.3. *If a connected metric space X is u.l.p.c. and u.s.l.s.c. then the discretisation map $\widehat{\cdot}\colon \pi_1(X,x_0) \to \varprojlim \pi_{1,\theta}(X,x_0)$ is an isomorphism.*

Proof. Injectivity: since X is u.s.l.s.c., there exists $\epsilon > 0$ so that every loop contained in a ball of radius at most ϵ is null-homotopic in X. Moreover, since X is u.l.p.c. there exists a $0 < \delta \leq \epsilon/2$ so that any two points in $B(x,\delta)$ are joined by a path in $B(x,\epsilon/2)$. Fix a parameter $0 < \theta < \delta$ and let γ be a continuous closed path whose θ-discretisation $\widehat{\gamma}^\theta$ is trivial in $\pi_{1,\theta}(X,x_0)$; we claim that γ is null-homotopic in X.

We can assume that $\widehat{\gamma}^\theta\colon [n] \to X$ is homotopic to the constant path via a θ-grid homotopy $H\colon [n]\times[m] \to X$. By construction, for every $0 \leq i \leq n-1$ and $0 \leq j \leq m$ there exist continuous paths $\alpha_{i,j}$ joining the point $H(i,j)$ with $H(i+1,j)$ and having its image completely contained in the ball of radius $\epsilon/2$ centred at $H(i,j)$. Similarly, for every $0 \leq i \leq n$ and $0 \leq j \leq m-1$ there exist analogous paths $\beta_{i,j}$ going from $H(i,j)$ to $H(i,j+1)$. Here we assume that the paths $\alpha_{i,0}$ are the subpaths of γ that are given by the choice of the discretisation $\widehat{\gamma}^\theta$, while the paths $\alpha_{i,m}$, $\beta_{0,j}$ and $\beta_{n,j}$ are constant.

We can concatenate these paths to obtain a path $\xi_{i,j}$ joining x_0 to $H(i,j)$ letting $\xi_{i,j} \coloneqq \big[(\beta_{0,0})\cdots(\beta_{0,j-1})\big]\big[(\alpha_{0,j})\cdots(\alpha_{i-1,j})\big]$. We now define some closed loops as follows (see Figure 3.1):

$$\eta_{i,j} \coloneqq (\xi_{i,j})(\alpha_{i,j})(\beta_{i+1,j})(\alpha_{i,j+1}^{-1})(\beta_{i,j}^{-1})(\xi_{i,j}^{-1}).$$

By construction, the closed path $(\alpha_{i,j})(\beta_{i+1,j})(\alpha_{i,j+1}^{-1})(\beta_{i,j}^{-1})$ is contained in the ball of radius ϵ centred at $H(i,j)$. Therefore, the loops $\eta_{i,j}$ are null-homotopic and it is easy to see that the path γ is homotopic to the product of the $\eta_{i,j}$ (concatenating them in the appropriate order), and therefore $[\gamma]$ is the trivial element in $\pi_1(X,x_0)$. It follows that the map $\widehat{\cdot}^\theta$ is injective and, a fortiori, $\widehat{\cdot}$ is injective as well.

Surjectivity: a generic element of $\varprojlim \pi_{1,\theta}(X, x_0)$ can be represented by a family $(Z_\theta)_{\theta>0}$—where Z_θ is a closed θ-path based at x_0—so that for every $\theta' < \theta$ we have $Z_{\theta'} \sim_\theta Z_\theta$. Again, since X is u.l.p.c., for every $\theta > 0$ there exists a $0 < \delta(\theta) \leq \theta$ such that any two points in $B(x, \delta(\theta))$ are joined by a path in $B(x, \theta)$ (in what follows we assume $\delta(\theta)$ to be a decreasing function of θ). It follows that the points of the $\delta(\theta)$-path $Z_{\delta(\theta)}$ can be joined with some small continuous paths and, concatenating these paths, we obtain a continuous loop γ_θ and we see that $Z_{\delta(\theta)}$ is a θ-discretisation of γ_θ.

From the proof of injectivity it follows that there exists a $\bar{\theta} > 0$ small enough so that $\widehat{\cdot}^{\bar{\theta}}$ is injective. Note that for every $0 < \theta < \bar{\theta}$ we have

$$\widehat{\gamma_\theta}^{\,\theta} \sim_\theta Z_{\delta(\theta)} \sim_{\delta(\bar{\theta})} Z_{\delta(\bar{\theta})} \sim_{\bar{\theta}} \widehat{\gamma_{\bar{\theta}}}^{\,\bar{\theta}}.$$

It follows that $\widehat{\gamma_\theta}^{\,\theta}$ and $\widehat{\gamma_{\bar{\theta}}}^{\,\bar{\theta}}$ are $\bar{\theta}$-homotopic and hence γ_θ and $\gamma_{\bar{\theta}}$ are homotopic. Therefore

$$\widehat{\gamma_{\bar{\theta}}}^{\,\theta} \sim_\theta \widehat{\gamma_\theta}^{\,\theta} \sim_\theta Z_{\delta(\theta)} \sim_\theta Z_\theta$$

for every $0 < \theta < \bar{\theta}$, and hence $\widehat{\gamma_{\bar{\theta}}} = ([Z_\theta])_{\theta>0} \in \varprojlim \pi_{1,\theta}(X, x_0)$. □

Applying Lemma 2.1.6, we obtain the following:

Corollary 3.2.4. *If a connected space X is l.p.c., s.l.s.c. and compact then the discretisation $\widehat{\cdot}\colon \pi_1(X, x_0) \to \varprojlim \pi_{1,\theta}(X, x_0)$ is an isomorphism.*

Note that from the proof of Theorem 3.2.3 it follows that if X is u.l.p.c. and u.s.l.s.c., then the projections $\varprojlim \pi_{1,\theta}(X, x_0) \to \pi_{1,\theta}(X, x_0)$ are injective for every θ small enough. However, the following example shows that such projections need not be surjective:

Example 3.2.5. Consider in $\mathbb{R}^3 = \mathbb{R} \times \mathbb{C}$ the cylindrical shell whose bases are disks of radius 1 centred at $(0, 0, 0)$ and $(1, 0, 0)$. Let $X_0 \subset \mathbb{R}^3$ be the space obtained by adding to the cylinder the segments joining $\left(\frac{1}{2}, e^{\frac{k\pi}{4}i}\right)$ with $\left(\frac{1}{2}, \frac{1}{2}e^{\frac{k\pi}{4}i}\right)$ for $k = 1, \ldots, 8$. Let $X_1 \subset \mathbb{R}^3$ be a copy of X_0, but translated by 1 on the x coordinate and with the y and z coordinates rescaled by a half. Similarly, X_2 is a translated and rescaled copy of X_1 and so on. Finally, let $X \coloneqq \bigcup_{n\in\mathbb{N}} X_n$ be the 'telescope space' obtained taking the union all such cylinders (with the ambient metric of $\mathbb{R}^3$). See Figure 3.2.

The space X is u.s.l.s.c. (it is simply connected). To see that it is also u.l.p.c, note that if it was not for the segments pointing toward the centre of the cylindrical shells one could just let $\delta = \epsilon$ to obtain the local connectedness. It is hence enough to prove

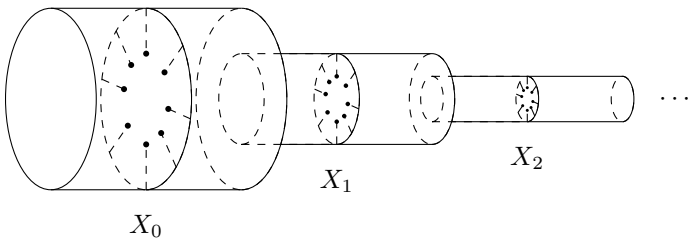


Figure 3.2: Example of space with non surjective projections

that the disjoint union of the middle sections $Y_n \coloneqq X_n \cap \{(n+1/2, y, z) \mid y, z \in \mathbb{R}\}$ is u.l.p.c.

For a fixed $\epsilon > 0$ let $\bar{n} \in \mathbb{N}$ be the largest number such that $\epsilon < 2^{-\bar{n}}$. For every $n > \bar{n}$ the (path-connected) space Y_n is contained in the ball of radius ϵ centred at any of its points. It is hence enough to deal with the spaces Y_n with $n \leq \bar{n}$. Here, letting $\delta = 2^{-\bar{n}} \left\| \frac{1}{2} - \frac{1}{2} e^{\frac{\pi}{4} i} \right\|$ will do the job because the balls of radius δ in Y_n are path connected.

We can hence use Theorem 3.2.3 to deduce that $\varprojlim \pi_{1,\theta}(X) = \{0\}$. Still, we claim that for every $n \in \mathbb{N}$ the discrete path Z_n in $X_n \subset X$ given by the points $\left(n + \frac{1}{2}, \frac{1}{2^{n+1}} e^{\frac{k\pi}{4} i}\right)$ represents a non trivial θ-path in $\pi_{1,\theta}(X)$ for $\theta = 2^{-n} \left\| \frac{1}{2} - \frac{1}{2} e^{\frac{\pi}{4} i} \right\|$.

Indeed, we claim that all the points $\left(n + \frac{1}{2}, \frac{1}{2^{n+1}} e^{\frac{k\pi}{4} i}\right)$ must belong to the image of any θ-path θ-homotopic to Z_n (and it is therefore a non-constant θ-path). Note that if $Z = (z_1, \ldots, z_m)$ is a θ-path such that for some $j < m$ we have $z_j = \frac{1}{2^{n+1}} e^{\frac{k\pi}{4} i}$ and $z_{j+1} = \frac{1}{2^{n+1}} e^{\frac{(k+1)\pi}{4} i}$, and $Z' = (z'_1, \ldots, z'_m)$ is a θ-path that is θ-close to Z then we must have $z'_j = z_j$ and $z'_{j+1} = z_{j+1}$ because the only pair of θ-close points that are also θ-close to z_j and z_{j+1} are z_j and z_{j+1} themselves. The claim follows because the paths in a θ-grid homotopy are a sequence of θ-close paths.

One may think that it should be true in general that when the limit homomorphism $\hat{\cdot}\colon \pi_1(X, x_0) \to \varprojlim \pi_{1,\theta}(X, x_0)$ is an isomorphism then the maps $\hat{\cdot}^\theta$ should be injective for θ small enough. The next example shows that this is not the case.

Example 3.2.6. Let $\frac{1}{2^n} \cdot \mathbb{S}^1$ be the circle of radius $1/2^n$ in $\mathbb{R}^2$ equipped with the metric d_n induced from $\mathbb{R}^2$ and let $X \coloneqq \prod_{n\in\mathbb{N}} \frac{1}{2^n} \cdot \mathbb{S}^1$ equipped with the ℓ_1-metric, *i.e.* $d\big((x_n)_{n\in\mathbb{N}}, (y_n)_{n\in\mathbb{N}}\big) = \sum_{n\in\mathbb{N}} d_n(x_n, y_n)$. Note that this is indeed a metric on the set $\prod_{n\in\mathbb{N}} \mathbb{S}^1$ because the series $\sum_{n\in\mathbb{N}} 2^{-n}$ converges. We claim that, with the topology coming from this metric, X is homeomorphic to the $(\mathbb{S}^1)^{\mathbb{N}}$ with the product topology.

To prove that a topology τ' is at least as fine as a topology τ defined on the same set X, it is enough to show that for every $x \in X$ and $U \in \tau$ containing x, there exists a $U' \in \tau'$ such that $x \in U' \subseteq U$. In our case, the projections $p_n\colon X \to \frac{1}{2^n} \cdot \mathbb{S}^1$ are

1-Lipschitz and therefore the topology coming from the metric is at least as fine as the product topology. To prove that the two topologies coincide it is hence enough to show that the product topology is at least as fine as the metric one. For every $\epsilon > 0$, there is a $\bar{n}$ large enough so that $\sum_{n \geq \bar{n}} \operatorname{diam}\left(\frac{1}{2^n} \cdot \mathbb{S}^1\right) \leq \frac{\epsilon}{2}$. It follows that for every point $\overline{x} \coloneqq (x_n)_{n \in \mathbb{N}} \in X$ the ball $B(\overline{x}, \epsilon)$ contains the set

$$U \coloneqq p_0^{-1}\Big(B\Big(x_0, \frac{\epsilon}{2\bar{n}}\Big)\Big) \cap \cdots \cap p_{\bar{n}-1}^{-1}\Big(B\Big(x_{\bar{n}-1}, \frac{\epsilon}{2\bar{n}}\Big)\Big),$$

which is open in the product topology.

Since X is homeomorphic to an infinite product of $\mathbb{S}^1$, its fundamental group is the infinite direct product $\prod_{n \in \mathbb{N}} \mathbb{Z}$ (see Lemma 2.1.7). On the contrary, for every fixed $\theta > 0$, every θ-path in X is θ-close to a path in a finite dimensional section $1 \cdot \mathbb{S}^1 \times \cdots \times \frac{1}{2^{\bar{n}}} \cdot \mathbb{S}^1 \subset X$ and from this it is simple to deduce that $\pi_{1,\theta}(X, x_0)$ is an abelian group of finite rank. In particular, the θ-discretisation $\widehat{\cdot}^{\theta} \colon \pi_1(X, x_0) \to \pi_{1,\theta}(X, x_0)$ is not injective.

On the other hand, we will now show that $\widehat{\cdot} \colon \pi_1(X, x_0) \to \varprojlim \pi_{1,\theta}(X, x_0)$ is an isomorphism. Let $f_n = p_0 \times \cdots \times p_n$ be the projection of X onto the product $X_n \coloneqq (1 \cdot \mathbb{S}^1) \times \cdots \times (2^{-n} \cdot \mathbb{S}^1)$. Since f_n is 1-Lipschitz, it induces homomorphisms $(f_n)_* \colon \pi_{1,\theta}(X, x_0) \to \pi_{1,\theta}(X_n, f_n(x_0))$ for every $\theta > 0$, and by the universal property of inverse limits these maps lift to a homomorphism $(f_n)_* \colon \varprojlim \pi_{1,\theta}(X, x_0) \to \varprojlim \pi_{1,\theta}(X_n, f_n(x_0))$. Moreover, $(f_n)_*$ commutes with the discretisation procedure, thus we obtain a commutative diagram:

$$\begin{array}{ccc} \pi_1(X, x_0) & \xrightarrow{\quad\widehat{\cdot}\quad} & \varprojlim \pi_{1,\theta}(X, x_0) \\ \big\downarrow{\scriptstyle (f_n)_*} & & \big\downarrow{\scriptstyle (f_n)_*} \\ \mathbb{Z}^{n+1} \cong \pi_1(X_n, f_n(x_0)) & \xrightarrow[\cong]{\widehat{\cdot}} & \varprojlim \pi_{1,\theta}(X_n, f_n(x_0)), \end{array}$$

where the bottom map is an isomorphism by Theorem 3.2.3. Note that the maps $(f_n)_* \colon \varprojlim \pi_{1,\theta}(X, x_0) \to \mathbb{Z}^{n+1}$ are coherent with the natural projections $\mathbb{Z}^m \to \mathbb{Z}^n$ for every $n \leq m$ and hence they give rise to a map to the projective limit (over n):

$$f_* \colon \varprojlim_{\theta > 0} \pi_{1,\theta}(X, x_0) \longrightarrow \varprojlim_{n \in \mathbb{N}} \mathbb{Z}^n \cong \prod_{n \in \mathbb{N}} \mathbb{Z}.$$

With some diagram chasing, it is easy to verify that the composition $\pi_1(X, x_0) \xrightarrow{\widehat{\cdot}} \varprojlim \pi_{1,\theta}(X, x_0) \xrightarrow{f_*} \prod_{n \in \mathbb{N}} \mathbb{Z}$ coincides with the natural isomorphism $\pi_1(X, x_0) \cong \prod_{n \in \mathbb{N}} \mathbb{Z}$. In particular, it follows that $\widehat{\cdot}$ is injective. To show that it is also surjective it is enough to check that f_* is injective. This is readily done: if $([Z_\theta])_{\theta > 0}$ is a non trivial element

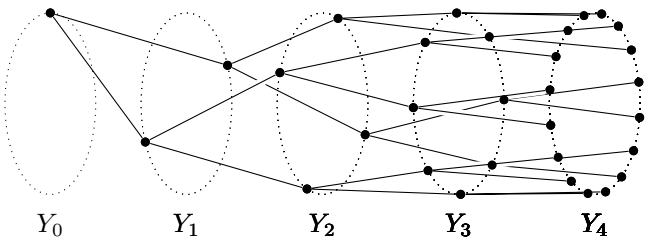


Figure 3.3: Failing surjectivity: non compact example

in $\varprojlim \pi_{1,\theta}(X, x_0)$, there is a $\theta > 0$ such that $[Z_\theta]$ is not trivial in $\pi_{1,\theta}(X, x_0)$. For n large enough the discrete path Z_θ is θ-close to a path contained in X_n. Since f_n is a 1-Lipschitz retraction, it follows that $(f_n)_*([Z_\theta])$ cannot be trivial in $\pi_{1,\theta}(X_n, f_n(x_n))$ and therefore $([Z_\theta])_{\theta>0}$ is mapped to a non trivial element of $\prod_{n\in\mathbb{N}} \mathbb{Z}$.

3.3 Some counterexamples

It this section we show that the hypotheses of Theorem 3.2.3 are (quite) sharp. More precisely, we show that each of the hypotheses of Corollary 3.2.4 (compactness, local path connectedness and semi-local simple connectedness) is necessary, in the sense that if any of these three hypotheses is dropped then both injectivity and surjectivity of the discretisation map $\hat{\cdot}$ may fail.

All the following examples are constructed as subsets of $\mathbb{R}^n$ for some $n = 2, 3$ and they are equipped with the restriction of the Euclidean metric.

3.3.1 Counterexamples to surjectivity

Here we provide examples examples showing that $\hat{\cdot}$ needs not be surjective if some hypotheses are dropped.

Example 3.3.1 (not compact). In $\mathbb{R}^3 = \mathbb{C} \times \mathbb{R}$, for every $n \in \mathbb{N}$ let S_n be the unit circle centred at $(0, 0, n)$ lying on the plane orthogonal to the axis $(0, 0, 1)$, and let $Y_n \subset S_n$ be the subset of points at angle $2\pi\frac{k+1/2}{2^n}$ for some $k \in \mathbb{N}$. Consider now the space $X \subset \mathbb{R}^3$ obtained as the union of the segments joining a point in Y_n with the two closest points in Y_{n+1} (see Figure 3.3). The space X is l.p.c. and simply connected (it is in fact contractible, because it is homeomorphic to a tree).

Still, $\varprojlim \pi_{1,\theta}(X)$ is not trivial. Indeed, note that each set Y_n can be seen as a θ_n-path for $\theta_n = \|1 - e^{\frac{2\pi}{2^n}i}\|$. Moreover, for every $m > n$ the θ_m-path Y_m is (freely) θ_n-homotopic to Y_n and hence the sequence $([Y_n])_{n\in\mathbb{N}}$ should represent a well-defined conjugacy class in the inverse limit. To obtain an actual element of the group (as

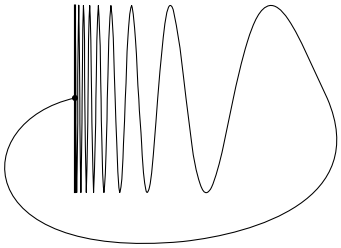


Figure 3.4: Failing surjectivity: non locally path connected example

opposed to elements defined only up to conjugacy) one has to fix a base point x_0 and a coherent sequence of closed path based at x_0 that are freely θ_n-homotopic to the θ_n paths Y_n. A natural choice is to let $x_0 = (e^{\pi i}, 0) = Y_0$, let β_n be a continuous path going from x_0 the point $(e^{\frac{\pi}{2^n}i}, n) \in Y_n$ and consider the θ_n-path $Z_n \coloneqq \widehat{\beta}^{\theta_n} Y_n (\widehat{\beta}^{\theta_n})^*$ (recall that the star denotes the reverse path). Then $Z_n \sim_{\theta_n} Z_m$ for every $m \geq n$ and hence the sequence $([Z_n])_{n\in\mathbb{N}}$ determines an element $[Z]$ in $\varprojlim \pi_{1,\theta}(X, x_0)$ (here we are using the fact that the inverse limit can be computed by using any cofinal sequence in the index set).

To show that $[Z]$ is not trivial, it is enough to show that one of its components is not trivial. With a similar argument of Example 3.2.6, one can show that every θ_3-path θ_3-homotopic to Z_3 must contain the whole of the set Y_3 in its image and hence cannot be trivial (the analogous statement is true for every Z_n with $n \geq 3$). The only subtlety in applying said argument is that one cannot consider every single jump between points in Y_3, but only jumps between pairs of adjacent points of Y_3 that are connected to different points of Y_2.

Example 3.3.2 (not l.p.c.). In $\mathbb{R}^2$, let X be the union of the graph of the function $\sin(1/x)$ for $x \in (0, 1]$ with the segment I joining $(0, -1)$ to $(0, 1)$ and a path joining one end of the graph to I (see Figure 3.4). This space is compact and simply connected, but $\varprojlim \pi_{1,\theta}(X) = \mathbb{Z}$. Indeed, by 'collapsing the singularity' one can produce an obvious 1-Lipschitz surjection $f\colon X \to \mathbb{S}^1$ and this in turns induces a homomorphism $f_*\colon \varprojlim \pi_{1,\theta}(X) \longrightarrow \varprojlim \pi_{1,\theta}(\mathbb{S}^1) \cong \pi_1(\mathbb{S}^1)$. It is easy to show that f_* is surjective (this is all we need to do to show that the discretisation map $\curlywedge\colon \pi_1(X) \to \varprojlim \pi_{1,\theta}(X)$ is not surjective). Proving that f_* is also injective is slightly more tedious, but it can be done by showing that θ-close θ-paths in $\mathbb{S}^1$ are θ-homotopic to images under f_* of θ-homotopic θ-paths of X.

Example 3.3.3 (not s.l.s.c.). Let X be the Hawaiian earring, *i.e.* the union of the circles of radius $1/n$ centred at $(1/n, 0)$ in $\mathbb{R}^2$ (Figure 3.5a). This space is the most common example of a non locally simply connected space and—despite still being a



(a) Hawaiian earring (b) Points at distance θ in Y_r

Figure 3.5: Failing surjectivity: non semi-locally simply connected example

rather mysterious object—its fundamental group has been quite thoroughly studied. It is known that $\pi_1(X)$ injects in the projective limit of the fundamental group of the largest cycles $\pi_1(X) \hookrightarrow \varprojlim_n \big(\mathbb{Z}^{*n}\big)$ but its image (which can be completely described) is not the whole group. Specifically, let $F_n = \mathbb{Z}^{*n}$ be freely generated by the set $\{a_1, \dots, a_n\}$, labelled so that the projection $F_{n+1} \to F_n$ is the map sending a_{n+1} to the identity. Let w_n be a reduced word in F_n and let $(w_n)_{n\in\mathbb{N}}$ represent an element of $\varprojlim_n \big(F_n\big)$. Then $(w_n)_{n\in\mathbb{N}}$ is in the image of $\pi_1(X)$ if and only if for every $i \in \mathbb{N}$ the number of times that the letter a_i appears in the words w_n is bounded uniformly on n (and hence eventually constant). For example, one can produce an element of $\varprojlim_n \big(F_n\big)$ that is not in the image of $\pi_1(X)$ by considering products of commutators: $w_n \coloneqq [a_1, a_2][a_1, a_3] \cdots [a_1, a_n]$ (see [MM86, Eda92]).

We will now show that for every $n \in \mathbb{N}$ there is an appropriate value of θ so that $\pi_{1,\theta}(X) \cong F_n$, and that the discretisation $\widehat{\cdot}$ coincides with the injection $\pi_1(X) \hookrightarrow \varprojlim_n \big(\mathbb{Z}^{*n}\big)$ mentioned above. From this it will follow that the discretisation homomorphism is not surjective.

Let $X_n \subset X$ be the subset consisting of the n largest circles. We claim that there is a value θ_n such that the discretisation at scale θ_n induces an isomorphism $F_n \cong \pi_1(X_n) \xrightarrow{\widehat{\cdot}^{\theta_n}} \pi_{1,\theta_n}(X)$—here we are using that the discretisation of a path in X_n is also a discretisation in X.

For $r < 1$, let Y_r be the subset of $\mathbb{R}^2$ obtained by removing the open ball $B\big((1,0),1\big)$ and adding back the closed ball $\overline{B}\big((r,0),r\big)$. One can show that the discretisation map $\widehat{\cdot}^{\theta}\colon \pi_1(Y_r) \to \pi_{1,\theta}(Y_r)$ is always surjective and that it is injective if and only if there are two points on either side of $(2,0)$ that are at distance at most θ from one another and from the ball $\overline{B}((r,0),r)$ (see Figure 3.5b). In particular, there is a threshold value $\theta(r)$ such that $\pi_{1,\theta(r)}(Y_r) = \{0\}$ and $\widehat{\cdot}^{\theta}$ is an isomorphism for every $\theta < \theta(r)$. The function $r \mapsto \theta(r)$ is strictly decreasing.

Let $\theta'(r) \coloneqq \theta(r) - \epsilon(r)$ where $\epsilon(r)$ is a positive decreasing function that takes small enough values (it will be clear in the sequel how small they need to be). We claim that letting $\theta_n \coloneqq \frac{1}{n}\theta'\big(\frac{n}{n+1}\big)$ makes $\widehat{\cdot}^{\theta_n}\colon \pi_1(X_n) \to \pi_{1,\theta_n}(X)$ injective. Performing an homothety of factor n on $\mathbb{R}^2$, we obtain that the rescaled image of X_n is contained in $\mathbb{R}^2 \smallsetminus B((1,0),1)$ while the rescaled image of X is contained in $Y_{\frac{n}{n+1}}$. Let γ be a non trivial continuous closed path in X_n. If the discretisation $\widehat{\gamma}^{\theta_n}$ was θ_n-homotopic to a constant path in X, after rescaling we would obtain that $\widehat{\gamma}^{\theta_n}$ is sent to a $\theta'\big(\frac{n}{n+1}\big)$-discretisation of the rescaling of γ that is $\theta'\big(\frac{n}{n+1}\big)$-homotopic to a constant path in $Y_{\frac{n}{n+1}}$, a contradiction because $\theta'\big(\frac{n}{n+1}\big) < \theta\big(\frac{n}{n+1}\big)$.

For the surjectivity, note that $\widehat{\cdot}^{\theta_n}\colon \pi_1(X_n) \longrightarrow \pi_{1,\theta_n}(X_n)$ is surjective (this is true for every θ) and it is hence enough to show that the inclusion $X_n \subseteq X$ induces a surjection of θ_n-discrete fundamental groups. Note that every θ-path in X is θ-homotopic to a concatenation of θ-paths that are contained in a single circle of X (a 'jump' between two circles is θ-homotopic to the concatenation of two θ-paths along those circles joining its endpoints with the origin). Note now that any θ_n-path in $X_{n+1} \smallsetminus X_n$ is θ_n-homotopic to a path in $X_{n+2} \smallsetminus X_{n+1}$ because rescaling by $n+1$ we have that

$$(n+1)\theta_n = \frac{n+1}{n}\Big[\theta\Big(\frac{n}{n+1}\Big) - \epsilon\Big(\frac{n}{n+1}\Big)\Big] > \theta\Big(\frac{n+1}{n+2}\Big)$$

and the latter is precisely the quantity needed to be able to homotope a discrete path from the rescaling of the n^{th} circle to the rescaling of the $(n+1)^{\text{th}}$ circle. By induction, it follows that every θ_n-path in in $X \smallsetminus X_n$ can be θ_n-homotoped to the constant path.

We thus proved that $\pi_{1,\theta_n}(X) \cong F_n$. Moreover, it follows from the construction that the projection $F_{n+1} \to F_n$ is given by collapsing the $(n+1)^{\text{th}}$ generator. Since $\theta_n \to 0$ as n goes to infinity, we deduce that $\varprojlim \pi_{1,\theta}(X) \cong \varprojlim_n F_n$.

Judging from the above example, one might hope to be able to characterise the image of $\pi_1(X)$ into $\varprojlim \pi_{1,\theta}(X)$ as the 'sequences of words (in some generating set of $\pi_{1,\theta}(X)$) whose projections in $\pi_{1,\theta}(X)$ are eventually constant for every fixed θ'. The next example shows that it is unlikely to find such a characterisation.

Example 3.3.4 (not s.l.s.c. *bis*). Let $X_0 \subset \mathbb{R}^2$ be the (empty) square with corners $(1,1),(-1,1),(-1,-1),(1,-1)$ and let X_1 be obtained from the square X_0 by adding the central cross (*i.e.* the two segments joining $(-1,0)$ to $(1,0)$ and $(0,-1)$ to $(0,1)$). Now, let X_{n+1} be obtained from X_n by adding the central crosses to all the left-most squares of X_n and let $X = \bigcup_n X_n$ be a "Hawaiian window".

Consider now the infinite path $\gamma\colon [0,\infty) \to X$ starting from the far right and zig-zagging from the top to the bottom while moving to the left—as the graph of

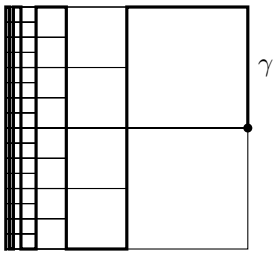


Figure 3.6: Hawaiian window

$\sin(1/x)$ would do—(see Figure 3.6). For every θ, Let Z_θ be the finite θ-path obtained by following the discretisation $\widehat{\gamma}^\theta$ until it gets at distance at most θ from the edge at the far left, and then walking back to the origin by reaching the bottom edge and following it to the right. Now, $([Z_\theta])_{\theta>0}$ is a coherent family of discrete paths and it therefore determines an element $[Z] \in \varprojlim \pi_{1,\theta}(X)$. Still, such $[Z]$ cannot be described as the discretisation of any continuous path, because any such path would have to follow γ in its endless travel and therefore it could not be the continuous image of a closed interval. Note however that $[Z]$ does seem to have 'constant projections onto $\pi_{1,\theta}(X)$'.

3.3.2 Counterexamples to injectivity

As in Subsection 3.3.1, we show how injectivity of $\widehat{\cdot}$ may fail if one hypothesis is dropped.

Example 3.3.5 (not compact). It is sufficient to note that $\mathbb{R}^2 \setminus \{0\}$ has trivial θ-discrete fundamental group for every $\theta > 0$. If one also desires to have a complete metric space it is enough to consider a manifold with cusps.

Example 3.3.6 (not l.p.c.). In $\mathbb{R}^3$, consider the graph of $\sin(\pi/x)$ as (x, y) range in $[0, 1] \times [-1, 1]$. Let $X \subset \mathbb{R}^3$ be the union of this graph with the limit square A and two squares B_1, B_2 running sideways the graph as in Figure 3.7. The space X is compact and it is semi-locally simply connected (for any point in $A \setminus (B_1 \sqcup B_2)$ there is a small neighbourhood that consists of a countable union of simply connected path-components. All the other points have a contractible neighbourhood).

Note that X has trivial θ-discrete fundamental group for every θ. Indeed, every θ-path is trivially θ-homotopic to a θ-path consisting of points that are at distance at least $\theta/2$ from the limit square A. It is easy to show that every θ-path in the graph of $\sin(\pi/x)$ with x ranging in $[\theta/2, 1]$ is θ-homotopic to the discretisation of a continuous path, and from this we can deduce that $\pi_{1,\theta}(X)$ is trivial because $X \setminus N_{\theta/2}(A)$ is simply connected.

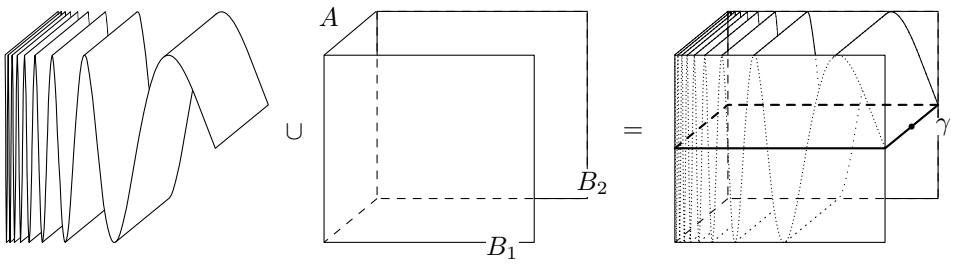


Figure 3.7: Non locally path connected example

Still, we claim that the path γ cutting through the three squares and closing up straight through the graph of $\sin(\pi/x)$ is not homotopic to a constant path and hence $\pi_1(X) \neq \{0\}$. Indeed, if there was such an homotopy we would obtain a continuous map $F\colon \mathbb{D} \to X$ with $F|_{\partial\mathbb{D}} = \gamma$ (here $\mathbb{D}$ is the unit disk in $\mathbb{R}^2$). Since $F^{-1}(B_1)$ and $F^{-1}(B_2)$ are compact, they have positive distance in $\mathbb{D}$. It follows that any point in $\partial(F^{-1}(A)) \setminus F^{-1}(B_1 \sqcup B_2)$ has a neighbourhood disjoint from $F^{-1}(B_1 \cup B_2)$ and we can hence find a path α lying outside $F^{-1}(B_1 \cup B_2)$ and having one endpoint in $F^{-1}(A)$ and the other in its complement. This yields a contradiction, as $F \circ \alpha$ would be a path reaching the limit square A by running along (a portion of) the graph of $\sin(\pi/x)$.

Example 3.3.7 (not s.l.s.c.). The basic idea is similar to that of Example 3.3.6, but it is a little more technical. Instead of adding to the graph of $\sin(\pi/x)$ three squares, we add a whole triangular prism built over the triangle with vertices $(0,-1),(0,1),(1,1)$. Moreover, we also add on the vertical plane passing through $(0,-1),(1,-1)$ the subset of $\mathbb{R}^2$ obtained by filling in the space contained between the graph of $\sin(\pi/x)$ and the x-axis as x ranges in $(0,1]$ (see Figure 3.8). Let B denote the prisms, C the subset of the plane before mentioned and A the segment joining $(0,-1,-1)$ and $(0,-1,1)$.

The space X thus obtained is compact and l.p.c. (every point in A has a basis of path-connected neighbourhoods, while every other point has contractible neighbourhoods). As in Example 3.3.6, we can show that $\pi_{1,\theta}(X)$ is trivial, it is hence enough to show that X is not simply connected. We claim that the same path γ as in Example 3.3.6 will show that the discretisation map is not injective.

Assume by contradiction that γ is null-homotopic. Then there is a map $F\colon \mathbb{D} \to X$ that coincides with γ on $\partial\mathbb{D}$. We begin by noting that the image of F must contain the whole of $X \setminus B$. Indeed, for every point $w = (x,y,z) \in X \setminus B$ we must have $x > 0$ and we can hence fix a constant $0 < \epsilon < x$. The space $Y_\epsilon \coloneqq X \cup \big([0,\epsilon] \times [-1,1] \times [-1,1]\big)$ is homotopy equivalent to a disk, while the space $Y_\epsilon \setminus \{w\}$ is homotopy equivalent to a holed disk (unless w belongs to the boundary of the disc, in which case it is contained

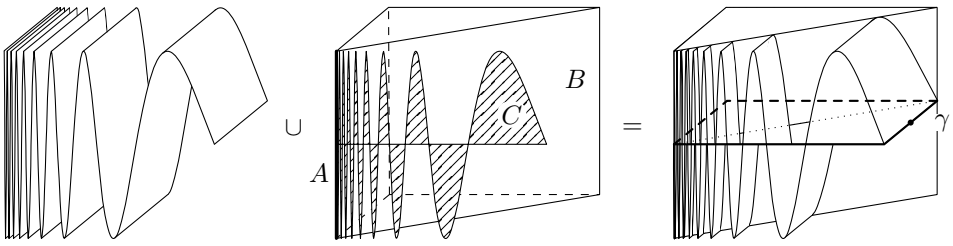


Figure 3.8: Non semi-locally simply path connected example

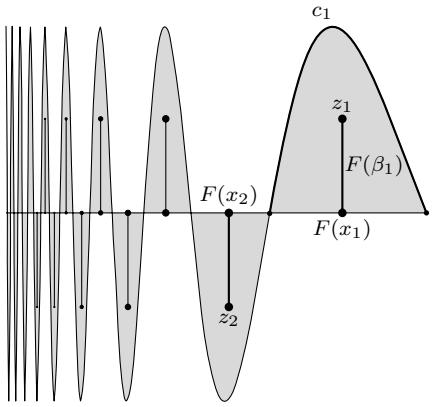


Figure 3.9: Detail of the set C with relevant paths and points

in $\mathrm{Im}(F|_{\partial\mathbb{D}})$) and hence γ is not null-homotopic in $Y_\epsilon \setminus \{w\}$. It follows that the image of F cannot be contained in $Y_\epsilon \setminus \{w\}$ and hence $w \in \mathrm{Im}(F)$.

Let $x_n \in \partial\mathbb{D}$ be the point sent to the midpoint between $\left(\frac{1}{n}, -1, 0\right)$ and $\left(\frac{1}{n+1}, -1, 0\right)$. Let also $c_n \subset C_n$ denote the graph of $\sin(\pi/x)$ as x ranges in $\left[\frac{1}{n+1}, \frac{1}{n}\right]$. Note that c_n disconnects X, and let Ω_n be the component of $X \setminus c_n$ containing $F(x_n)$. Note also that $F^{-1}(c_n)$ is a compact that disconnects $\mathbb{D}$ and let $U_n \subset \mathbb{D}$ be the component of $\mathbb{D} \setminus F^{-1}(c_n)$ containing x_n. We claim that $F(U_n)$ coincides with Ω_n.

Let $p_n \colon X \to X$ be a continuous map contracting Ω_n to the curve c_n and restricting to the identity on $X \setminus \Omega_n$. Let $F' \colon \mathbb{D} \to X$ be the map coinciding with F on U_n and defined by the composition $p_n \circ F$ on the complement $\mathbb{D} \setminus U_n$. This map is continuous as the two definitions agree on ∂U_n. Since F' coincides with F on $\partial\mathbb{D}$, it should define a null-homotopy of γ in X and its image should hence contain the whole of Ω_n. On the other hand, we know by construction that $F'^{-1}(\Omega_n)$ is contained in U_n, and thus the claim is proved as $F(U_n) = F'(U_n) = \Omega_n$.

Let now $z_n \in X$ be the point on the vertical line through x_n lying at distance $1/2$ away from x_n. Since the connected set U_n is open, it is locally path connected and hence it is also path connected. Since $z_n \in F(U_n)$, we can hence choose a point $y_n \in F^{-1}(z_n) \cap U_n$ and a path β_n in U_n joining x_n to y_n (Figure 3.9).

Note that, by construction, any path $F(\beta_n)$ with n odd stays at distance at least $1/2$ from all the points z_m with m even (and vice versa). Since $\mathbb{D}$ is compact, the map F must be uniformly continuous, and hence there exists an $\epsilon > 0$ such that the path β_n with n odd stays at distance at least ϵ from all the points y_m with m even (and vice versa).

Since $\mathbb{D}$ is compact, both the sequence $(y_{2n})_{n\in\mathbb{N}}$ and $(y_{2n+1})_{n\in\mathbb{N}}$ will admit converging subsequences. It follows that there exist intertwined indices $2n < 2m+1 < 2n' < 2m'+1$ such that both $d_{\mathbb{D}}\big(y_{2n}, y_{2n'}\big) < \frac{\epsilon}{2}$ and $d_{\mathbb{D}}\big(y_{2m+1}, y_{2m'+1}\big) < \frac{\epsilon}{2}$. We can hence join up those pairs of points with segments of length at most $\epsilon/2$. Concatenating these small paths with the paths $\beta_{2n}, \beta^*_{2n'}$ and $\beta_{2m+1}, \beta^*_{2m'+1}$ we obtain two disjoint paths in $\mathbb{D}$ linking intertwined pairs of points of $\partial\mathbb{D}$, which yields the desired contradiction.

Chapter 4

Discrete fundamental groups for warped spaces

Recall from Subsection 2.2.4 that the warped metric δ_S associated with a finite set S of homeomorphisms of a metric space (X, d) is the largest metric on X such that $\delta_S(x, y) \leq d(x, y)$ and $\delta_S(x, s \cdot x) \leq 1$ for every $x, y \in X$ and $s \in S$. We call the space (X, δ_S) *warped space*. In this chapter we will always assume that X is path-connected.

The aim of this chapter is to compute the θ-discrete fundamental group of (geodesic) warped spaces in terms of the fundamental group of X and the set of homeomorphisms S. In some instances, if $\Gamma \curvearrowright X$ is an action of a finitely generated group $\Gamma = \langle S \rangle$, we will also be able to express the θ-discrete fundamental group in term of X and (the action of) Γ, *i.e.* the description does not depend on the specific choice of the generating set S.

Remark 4.0.1. Everything that follows will still make sense when S is the empty set, and it therefore implies the analogous statements for geodesic metric spaces (as opposed to warped metric spaces).

4.1 Jumping-fundamental group

In this section we introduce the jumping-fundamental group of a warped space as an analogue of the fundamental group for metric spaces and we compute it in term of $\pi_1(X)$ and S.

Remark 4.1.1. One could avoid introducing the jumping-fundamental group by considering the fundamental group of the space obtained by gluing together the mapping tori

$$X_s \coloneqq \frac{X \times [0, 1]}{(x, 1) \sim (s(x), 0)}$$

along their bases $X \times \{0\}$ (this topological space can be used to produce yet another model for warped spaces). We preferred not to do so because we wanted to keep clear the distinction between topological paths and 'paths coming from the action'.

4.1.1 Definitions and notation

We begin by giving the following:

Definition 4.1.2. A *jumping-path* in X is a finite sequence of continuous paths $\gamma_0, \dots, \gamma_n \colon [0,1] \to X$ and elements $\vec{s}_1, \dots, \vec{s}_n$ with $\vec{s}_i = s_i^{\pm 1}$ for some $s_i \in S$, such that $\gamma_i(0) = \vec{s}_i \cdot \gamma_{i-1}(1)$ for every $i = 1, \dots, n$. Such jumping-path will be denoted by

$$\vec{\gamma} \coloneqq \gamma_0 \overset{\vec{s}_1}{\rightsquigarrow} \gamma_1 \overset{\vec{s}_2}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \gamma_n.$$

The *jumping pattern* of $\vec{\gamma}$ is the ordered sequence $(\vec{s}_1, \dots, \vec{s}_n)$.

We define the *length* of the jumping-path $\vec{\gamma}$ as

$$\|\vec{\gamma}\| \coloneqq n + \sum_{i=0}^{n} \|\gamma_i\|,$$

where $\|\gamma_i\|$ is the length of the path γ_i in (X, d). Note that if (X, d) is a geodesic metric space, by choosing the paths γ_i to be geodesics we deduce from Lemma 2.2.5 that the warped distance can then be expressed as

$$\delta_S(x, y) = \inf \big\{ \|\vec{\gamma}\| \;\big|\; \vec{\gamma} \text{ jumping-path between } x \text{ and } y \big\}.$$

Definition 4.1.3. A jumping-path $\vec{\gamma}$ between two points $x, y \in X$ is *geodesic* if it realises their distance $\|\vec{\gamma}\| = \delta_S(x, y)$. The warped metric space (X, δ_S) is *jumping-geodesic* if (X, d) is a geodesic metric space and every two points in (X, δ_S) are joined by a geodesic jumping-path.

Remark 4.1.4. It follows from the second part of Lemma 2.2.5 that if (X, d) is a proper geodesic metric space, then the warped space (X, δ_S) is jumping-geodesic. Note also that if (X, δ_S) is jumping-geodesic then it is a 1-geodesic metric space (defined in Subsection 2.1.2).

For notational convenience, if one of the internal paths γ_i of a jumping-path is constant, we will omit it from the notation and simply write $\overset{s_i}{\rightsquigarrow} \overset{s_{i+1}}{\rightsquigarrow}$. If the constant path is the initial (or terminal) one, we will keep its value in the notation: $x_0 \overset{s_1}{\rightsquigarrow} \cdots$ (or $\cdots \overset{s_n}{\rightsquigarrow} x_n$). We will denote the concatenation of jumping-paths simply by $\vec{\gamma}_1 \vec{\gamma}_2$.

If a jumping-path $\vec{\gamma} = \vec{\gamma}_1 \overset{\vec{s}}{\rightsquigarrow} \overset{\vec{s}^{-1}}{\rightsquigarrow} \vec{\gamma}_2$ has two consecutive jumps such that one is the opposite of the other, we say that the jumping-path $\vec{\gamma}' = \vec{\gamma}_1\vec{\gamma}_2$ obtained skipping those jumps is a *contraction* of $\vec{\gamma}$. Vice versa, adding a consecutive pair of opposite jumps is an *extension*.

Definition 4.1.5. Two jumping-paths with the same jumping pattern $\gamma_0 \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \gamma_n$ and $\gamma_0' \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \gamma_n'$ are *spatially-homotopic* if there exist free homotopies H_i between γ_i and γ_i' so that for every $t \in [0,1]$ and $i = 1, \dots, n$

- $H_0(0,t) = \gamma_0(0) = \gamma_0'(0)$,
- $H_n(1,t) = \gamma_n(1) = \gamma_n'(1)$,
- $H_i(0,t) = \vec{s}_i \cdot H_{i-1}(1,t)$.

Two jumping-paths are *homotopic* if you can transform one to the other with a finite number of space-homotopies, contractions and extensions.

The operation of concatenation between jumping-paths is compatible with homotopies. Given a continuous path γ, we denote by γ^* the path obtained walking along γ in the opposite direction. Similarly, if $\vec{\gamma} = \gamma_0 \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \gamma_n$ is a jumping-path, we define $\vec{\gamma}^*$ as

$$\vec{\gamma}^* \coloneqq \gamma_n^* \overset{\vec{s}_n^{-1}}{\rightsquigarrow} \cdots \overset{\vec{s}_0^{-1}}{\rightsquigarrow} \gamma_0^*.$$

Note that the concatenation $\vec{\gamma}\vec{\gamma}^*$ is homotopic to the constant jumping-path; we can therefore give the following:

Definition 4.1.6. The *jumping-fundamental group* is the group $J_S\Pi_1(X, x_0)$ of homotopy classes of closed jumping-paths based at a fixed point $x_0 \in X$, equipped with the operation of concatenation.

4.1.2 Constructing a homomorphism of the free group

The fundamental observation allowing us to compute jumping-fundamental groups is the following:

Lemma 4.1.7. *The jumping-paths $\gamma \overset{\vec{s}_1}{\rightsquigarrow} x_1$ and $x_0 \overset{\vec{s}_1}{\rightsquigarrow} \vec{s}_1(\gamma)$ are homotopic.*

Proof. The functions

$$H_0(r,t) \coloneqq \gamma(r(1-t)) \quad \text{and} \quad H_1(r,t) \coloneqq \vec{s}_1\big(\gamma(1-t+rt)\big)$$

define a space-homotopy between them. □

Corollary 4.1.8. *Every jumping-path is homotopic to a jumping-path where all the jumps are performed last:* $\gamma \overset{\vec{s}_1}{\rightsquigarrow} \overset{\vec{s}_2}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} x_0$.

Let $x_0 \in X$ be a fixed base point. For every $s \in S$ choose a continuous path α_s joining x_0 to $s \cdot x_0$. Then $\alpha_s \overset{s^{-1}}{\rightsquigarrow} x_0$ is a closed jumping-path and we can hence define a homomorphism ψ_S of the free group by extension:

$$\begin{aligned} \psi_S \colon F_S &\longrightarrow J_S\Pi_1(X, x_0) \\ s &\longmapsto \left[\alpha_s \overset{s^{-1}}{\rightsquigarrow} x_0\right]. \end{aligned}$$

Recall that in $J_S\Pi_1(X, x_0)$ we have

$$\left[\alpha_s \overset{s^{-1}}{\rightsquigarrow} x_0\right]^{-1} = \left[\left(\alpha_s \overset{s^{-1}}{\rightsquigarrow} x_0\right)^*\right] = \left[x_0 \overset{s}{\rightsquigarrow} \alpha_s^*\right].$$

Let $\alpha_{s^{-1}}$ be the path $s^{-1}(\alpha_s^*)$. Then by Lemma 4.1.7 we have

$$\left[x_0 \overset{s}{\rightsquigarrow} \alpha_s^*\right] = \left[s^{-1}(\alpha_s^*) \overset{s}{\rightsquigarrow} x_0\right] = \left[\alpha_{s^{-1}} \overset{s}{\rightsquigarrow} x_0\right]$$

and therefore $\psi_S(s^{-1}) = \left[\alpha_{s^{-1}} \overset{s}{\rightsquigarrow} x_0\right]$. We can hence continue to use the notation $\vec{s}$ to denote an element in $S^{\pm}$ and we have $\psi_S(\vec{s}) = \left[\alpha_{\vec{s}} \overset{\vec{s}^{-1}}{\rightsquigarrow} x_0\right]$.

Convention. Given a word $w = \vec{s}_1 \cdots \vec{s}_n$ of elements of $S^{\pm}$, we use the shorthand $\overset{w}{\rightsquigarrow}$ to denote the concatenation $\overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow}$. We denote by w_{rev} the reverse word $w_{\mathrm{rev}} \coloneqq \vec{s}_n \cdots \vec{s}_1$, so that $\overset{w_{\mathrm{rev}}}{\rightsquigarrow}$ is short for $\overset{\vec{s}_n}{\rightsquigarrow} \cdots \overset{\vec{s}_1}{\rightsquigarrow}$.

If $w = \vec{s}_1 \cdots \vec{s}_n$ is a word of elements of $S^{\pm}$, we can define the path α_w in X as the concatenation

$$\alpha_w \coloneqq \big(\alpha_{\vec{s}_1}\big)\big(\vec{s}_1(\alpha_{\vec{s}_2})\big) \cdots \big(\vec{s}_1 \circ \cdots \circ \vec{s}_{n-1}(\alpha_{\vec{s}_n})\big).$$

Note that we have $\alpha_{w_1 w_2} = \alpha_{w_1} w_1(\alpha_{w_2})$ and $\alpha_{w^{-1}} = w^{-1}(\alpha_w^*)$.

Lemma 4.1.7 now implies that $\psi_S(w)$ is the homotopy class $\left[\alpha_w \overset{w_{\mathrm{rev}}^{-1}}{\rightsquigarrow} x_0\right]$.

Remark 4.1.9. Here w is denoting both a word with letters in $S^{\pm}$ and the corresponding element in the free group F_S. This ambiguity does not cause any trouble because paths associated with equivalent words are homotopic (via contractions and extensions).

4.1.3 Computing the jumping-fundamental group

The choice of the paths α_s also induces a homomorphism $\phi_S \colon F_S \to \mathrm{Aut}\big(\pi_1(X, x_0)\big)$ where $\phi_S(s)$ is the automorphism given by

$$\phi_S(s)[\gamma] \coloneqq \left[(\alpha_s)s(\gamma)(\alpha_s^*)\right]$$

for every $[\gamma] \in \pi_1(X, x_0)$. Just as before, it is easy to check that for every word $w = \vec{s}_1 \cdots \vec{s}_n$ we have

$$\phi_S(w)[\gamma] = \big[\alpha_w w(\gamma) \alpha_w^*\big].$$

Theorem 4.1.10. *The choice of paths α_s induces an isomorphism of groups*

$$\Phi_S \colon \pi_1(X, x_0) \rtimes_{\phi_S} F_S \longrightarrow J_S\Pi_1(X, x_0),$$

where the map Φ_S sends $([\gamma], w)$ to $\big[\gamma\alpha_w \overset{w_{\text{rev}}^{-1}}{\rightsquigarrow} x_0\big]$.

Proof. The fundamental group $\pi_1(X, x_0)$ is a natural subgroup of $J_S\Pi_1(X, x_0)$ and we already noted that $\psi_S \colon F_S \to J_S\Pi_1(X, x_0)$ is a homomorphism. From Lemma 4.1.7 it follows that

$$\begin{aligned}\Big[\psi_S(s)\gamma\psi_S(s^{-1})\Big] &= \Big[\alpha_s \overset{s^{-1}}{\rightsquigarrow} \gamma\alpha_{s^{-1}} \overset{s}{\rightsquigarrow} x_0\Big] \\ &= \Big[\alpha_s s(\gamma) s(\alpha_{s^{-1}}) \overset{s}{\rightsquigarrow} \overset{s^{-1}}{\rightsquigarrow} x_0\Big] \\ &= \phi_S(s)[\gamma].\end{aligned}$$

Since S is a generating set, we deduce that $\big[\psi_S(w)\gamma\psi_S(w^{-1})\big] = \phi_S(w)[\gamma]$ for every $w \in F_S$ and hence ψ_S and the inclusion homomorphism induce a homomorphism of groups $\Phi_S \colon \pi_1(X, x_0) \rtimes_{\phi_S} F_S \to J_S\Pi_1(X, x_0)$. The expression for Φ_S follows from the previous discussion, as $\Phi_S([\gamma], w) = \Phi_S([\gamma], e)\Phi_S(e, w) = [\gamma]\psi_S(w)$.

We can promptly show that the map Φ_S is injective: by our definition of homotopy of jumping-paths, if a jumping-path $\vec{\boldsymbol{\gamma}}$ is homotopic to a constant path then its jumping pattern must eventually reduce to the trivial word via cancellation of consecutive opposite jumps. Therefore, if $\Phi_S([\gamma], w) = e$, the word w must represent the trivial element in F_S; and it is easy to see that $\Phi_S([\gamma], e) = e$ if and only if $[\gamma]$ is also trivial.

Surjectivity also follows from Lemma 4.1.7. Indeed, by Corollary 4.1.8 we know that every jumping-path is homotopic to a jumping-path where all the jumps are performed last $\vec{\boldsymbol{\gamma}} = \gamma \overset{\vec{s}_1}{\rightsquigarrow} \overset{\vec{s}_2}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} x_0$. Then, letting $w = \vec{s}_1^{-1} \cdots \vec{s}_n^{-1}$ we have

$$\big[\vec{\boldsymbol{\gamma}}\big] = \big[\gamma\alpha_w^*\alpha_w \overset{w_{\text{rev}}^{-1}}{\rightsquigarrow} x_0\big] = \Phi_S\big([\gamma\alpha_w^*], w\big).$$

□

Remark 4.1.11. It appears to us that jumping-jundamental groups and warped spaces might provide a sort of analogue (for semi-direct products of groups) of fundamental groups and classifying spaces (for general groups). We did not further develop this idea, though.

4.2 Discretisations and jumping-fundamental group

For the rest of the chapter we will let θ be a fixed parameter greater than or equal to 1 and that (X, d) has homotopy rectifiable paths (Definition 2.1.8).

Since the parameter θ is fixed, when no confusion can arise we will lighten the notation and simply write $\widehat{\gamma}$ instead of $\widehat{\gamma}^\theta$ to denote the θ-discretisation of a continuous path γ in X (see Subsection 3.2.1).

N.B. From now on the $\widehat{\cdot}$ and $\widehat{\gamma}$ will simply be short for $\widehat{\cdot}^\theta$ and $\widehat{\gamma}^\theta$. They are *not* the limit maps of Sections 3.2 and 3.3.

4.2.1 Discretisation of jumping-paths

Since we fixed $\theta \geq 1$, any jump $x \overset{\vec{s}}{\rightsquigarrow} \vec{s} \cdot x$ can be seen as a θ-path of length one in (X, δ_S). We can therefore define the discretisation of a jumping-path $\vec{\boldsymbol{\gamma}} = \gamma_0 \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \gamma_n$ as the concatenation of the discretisations of its continuous pieces and jumps:

$$\widehat{\boldsymbol{\gamma}} \coloneqq \widehat{\gamma}_0 \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \widehat{\gamma}_n.$$

Note that here the discretisation of the continuous pieces is done using the metric d and not the warped distance δ_S, as we find the geometry of the former more intuitive.

It is still true that the θ-homotopy class of a θ-discretisation only depends on the homotopy class of the jumping-path:

Lemma 4.2.1. *If two jumping-paths $\vec{\boldsymbol{\gamma}} = \gamma_0 \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \gamma_n$ and $\vec{\boldsymbol{\gamma}}' = \gamma'_0 \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_m}{\rightsquigarrow} \gamma'_m$ are homotopic then their discretisations $\widehat{\boldsymbol{\gamma}}$ and $\widehat{\boldsymbol{\gamma}}'$ are θ-homotopic.*

Proof. It is clear that contractions and expansions of jumping-paths produce θ-homotopic jumping-paths. It is therefore enough to prove the statement when $m = n$ and $\vec{\boldsymbol{\gamma}}$ and $\vec{\boldsymbol{\gamma}}'$ are spatially-homotopic.

Let $H_i \colon [0,1]^2 \to X$ be the homotopy between γ_i and γ'_i. By compactness, there exists an $N \in \mathbb{N}$ large enough so that the maps $\widehat{H}_i \colon [N]^2 \to X$ defined by

$$\widehat{H}_i(p, q) \coloneqq H_i\left(\frac{p}{N}, \frac{q}{N}\right)$$

are free θ-grid homotopy between the extremal θ-paths $\widehat{H}_i(\,\cdot\,, 0) \colon [N] \to X$ and $\widehat{H}_i(\cdot, N) \colon [N] \to X$, and moreover the θ-paths $\widehat{H}_i(\cdot, 0)$ and $\widehat{H}_i(\cdot, N)$ are θ-discretisations of γ_i and γ'_i respectively (this is the same argument of Lemma 3.2.1).

Note that the θ-paths $\widehat{H}_i(\,\cdot\,, 0)$ are concatenated via the jumps $\vec{s}_i$ (which can be seen as a step of a θ-path) and the same goes for the θ-paths $\widehat{H}_i(\,\cdot\,, N)$ and the

θ-grid homotopies $\widehat{H}_i$ as well. Therefore, by concatenation we obtain two θ-paths $\widehat{H}_0(\cdot,0) \overset{\vec{s}_1}{\rightsquigarrow} \widehat{H}_1(\cdot,0) \rightsquigarrow \cdots \rightsquigarrow \widehat{H}_N(\cdot,0)$ and $\widehat{H}_0(\cdot,N) \overset{\vec{s}_1}{\rightsquigarrow} \widehat{H}_1(\cdot,N) \rightsquigarrow \cdots \rightsquigarrow \widehat{H}_N(\cdot,N)$ which are θ-homotopic θ-discretisations of $\vec{\gamma}$ and $\vec{\gamma}'$. Then the lemma easily follow from the fact that different discretisations of the same continuous path are θ-homotopic. □

From Lemma 4.2.1 it follows that the discretisation procedure induces a well-defined discretisation map $\widehat{\cdot}\colon J_S\Pi_1(X,x_0) \to \pi_{1,\theta}\big((X,\delta_S),x_0\big)$. As it is clear that the discretisation of a concatenation of jumping-paths is θ-homotopic to the concatenation of their discretisations, the discretisation map $\widehat{\cdot}$ is an homomorphism.

4.2.2 Kernel of the discretisation homomorphism

Let $\mathcal{T}_\theta$ be the set of closed jumping-paths which are composition of four—non necessarily closed—jumping-paths of length at most θ:

$$\mathcal{T}_\theta \coloneqq \big\{\vec{\gamma} \;\big|\; \vec{\gamma} \text{ closed, } \vec{\gamma} = \vec{\gamma}_0\vec{\gamma}_1\vec{\gamma}_2\vec{\gamma}_3 \text{ with } \|\vec{\gamma}_i\| \leq \theta\big\}$$

and let $\mathrm{FT}_\theta \subseteq J_S\Pi_1(X,x_0)$ be the set of homotopy classes of the jumping-paths that are freely homotopic to jumping-paths in $\mathcal{T}_\theta$:

$$\mathrm{FT}_\theta \coloneqq \big\{\big[\vec{\gamma}\big] \;\big|\; \vec{\gamma} \sim_f \vec{\gamma}' \text{ for some } \vec{\gamma}' \in \mathcal{T}_\theta\big\},$$

where a *free homotopy* (denoted $\sim_f$) is a homotopy of closed jumping-paths where the space-homotopies are not required to keep the endpoints fixed as long as $H_0(0,t) = H_n(1,t)$ for every $t \in [0,1]$. Moreover, cyclic contractions and extensions are allowed as well (*i.e.* the jumping-path $x_0 \overset{\vec{s}}{\rightsquigarrow} \vec{\gamma} \overset{\vec{s}^{-1}}{\rightsquigarrow} x_0$ id freely homotopic to $\vec{\gamma}$).

Note that the set FT_θ is invariant under conjugation in $J_S\Pi_1(X,x_0)$.

Theorem 4.2.2. *Assume that (X,d) has homotopy rectifiable paths and that the warped space (X,δ_S) is jumping-geodesic. Then the discretisation homomorphism $\widehat{\cdot}$ is surjective and its kernel is generated by FT_θ. Therefore we have*

$$\pi_{1,\theta}\big((X,\delta_S),x_0\big) \cong J_S\Pi_1(X,x_0)/\langle \mathrm{FT}_\theta\rangle.$$

Proof. Let $z\colon [n] \to (X,\delta_S)$ be any θ-path. As (X,δ_S) is jumping-geodesic, we can choose for every $i = 1,\dots,n$ a geodesic jumping-path $\vec{\zeta}_i$ between z_{i-1} and z_i. We denote the composition of these paths by $z_{geo} \coloneqq \vec{\zeta}_1 \cdots \vec{\zeta}_n$ (this is a sort of 'geodesification' of the discrete path). Note that $z \sim_\theta \widehat{z}_{geo}$ via the map sending the last point of $\widehat{\zeta}_i$ to z_i and all the others to z_{i-1}. This proves the surjectivity of $\widehat{\cdot}$.

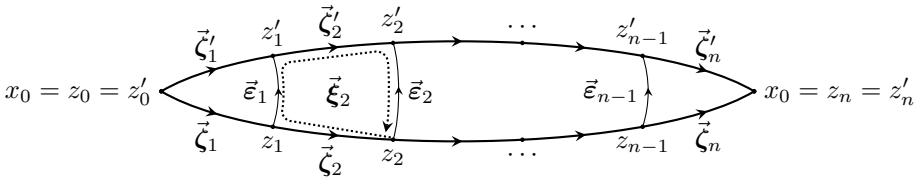


Figure 4.1: Geodesifications filling in a discrete homotopy.

We now have to study the kernel. Let $\vec{\gamma} = \vec{\gamma}_0\vec{\gamma}_1\vec{\gamma}_2\vec{\gamma}_3$ be a jumping-path in $\mathcal{T}_\theta$ and for each $0 \leq i \leq 3$ let $z_i = \vec{\gamma}_i(0) \in X$ be the starting point of $\vec{\gamma}_i$. Let also $z_4 \coloneqq z_0$; then the sequence $(z_0, \ldots, z_4)$ is a closed θ-path and it is easy to see that it is θ-homotopic to a constant path as

$$(z_0, z_1, z_2, z_3, z_4 = z_0) \sim_\theta (z_0, z_1, z_1, z_0, z_0) \sim_\theta (z_0, z_0, z_0, z_0, z_0)$$

are all θ-grid homotopies. As above, we also have $\widehat{\gamma}_i \sim_\theta (z_i, z_{i+1})$ and therefore the discretised path $\widehat{\gamma}$ is itself θ-homotopic to the constant path.

If a jumping-path $\vec{\gamma}$ is freely homotopic to $\vec{\gamma}' \in \mathcal{T}_\theta$, tracing the movement of the base point under the free homotopy we obtain a jumping-path $\vec{\boldsymbol{\beta}}$ joining x_0 to $\vec{\gamma}'(0)$ and we see that $\vec{\gamma}$ is genuinely homotopic to $\vec{\boldsymbol{\beta}}\vec{\gamma}'\vec{\boldsymbol{\beta}}^*$. We deduce that

$$\widehat{\gamma} \sim_\theta \widehat{\boldsymbol{\beta}}\widehat{\gamma}'\widehat{\boldsymbol{\beta}}^* \sim_\theta \widehat{\boldsymbol{\beta}}\widehat{\boldsymbol{\beta}}^* \sim_\theta x_0$$

and therefore $\mathrm{FT}_\theta \subseteq \ker(\widehat{\Phi}_S)$.

For the inverse inclusion, we begin by showing that if two θ-paths are θ-homotopic via a 1-step θ-grid homotopy, then their 'geodesifications' differ by a product of paths in FT_θ. Specifically, let $z, z' \colon [n] \to (X, \delta_S)$ be closed θ-paths with base point x_0 and so that $\delta_S(z_i, z'_i) \leq \theta$ for every $i \in [n]$. As above, let $z_{geo} \coloneqq \vec{\zeta}_1 \cdots \vec{\zeta}_n$ and $z'_{geo} \coloneqq \vec{\zeta}'_1 \cdots \vec{\zeta}'_n$ be concatenations of geodesic jumping-paths. Choose geodesic jumping-paths $\vec{\boldsymbol{\varepsilon}}_i$ joining z_i to z'_i and let $\vec{\boldsymbol{\xi}}_i \coloneqq \vec{\zeta}_i^*\vec{\boldsymbol{\varepsilon}}_{i-1}\vec{\zeta}'_i\vec{\boldsymbol{\varepsilon}}_i^*$; then the jumping-path z'_{geo} is homotopic to the composition $\vec{\zeta}_1\vec{\boldsymbol{\xi}}_1 \cdots \vec{\zeta}_n\vec{\boldsymbol{\xi}}_n$ (see Figure 4.1).

Let now $\vec{\boldsymbol{\beta}}_i$ be a jumping-path joining x_0 to z_i; then in $J_S\Pi_1(X, x_0)$ we have

$$[z_{geo}] = \left[\vec{\zeta}_1 \cdots \vec{\zeta}_n\right] = \left[\vec{\zeta}_1\vec{\boldsymbol{\beta}}_1^*\right]\left[\vec{\boldsymbol{\beta}}_1\vec{\zeta}_2\vec{\boldsymbol{\beta}}_2^*\right] \cdots \left[\vec{\boldsymbol{\beta}}_{n-1}\vec{\zeta}_n\right]$$

and

$$[z'_{geo}] = \left[\vec{\zeta}'_1 \cdots \vec{\zeta}'_n\right] = \left[\vec{\zeta}_1\vec{\boldsymbol{\beta}}_1^*\right]\left[\vec{\boldsymbol{\beta}}_1\vec{\boldsymbol{\xi}}_1\vec{\boldsymbol{\beta}}_1^*\right]\left[\vec{\boldsymbol{\beta}}_1\vec{\zeta}_2\vec{\boldsymbol{\beta}}_2^*\right] \cdots \left[\vec{\boldsymbol{\beta}}_{n-1}\vec{\zeta}_n\right]\left[\vec{\boldsymbol{\xi}}_n\right].$$

Note that $\vec{\boldsymbol{\xi}}_i \in \mathcal{T}_\theta$, and therefore $\vec{\boldsymbol{\beta}}_i\vec{\boldsymbol{\xi}}_i\vec{\boldsymbol{\beta}}_i^* \in \mathrm{FT}_\theta$. As FT_θ is invariant under conjugation, it follows that $[z_{geo}] \equiv [z'_{geo}] \pmod{\mathrm{FT}_\theta}$.

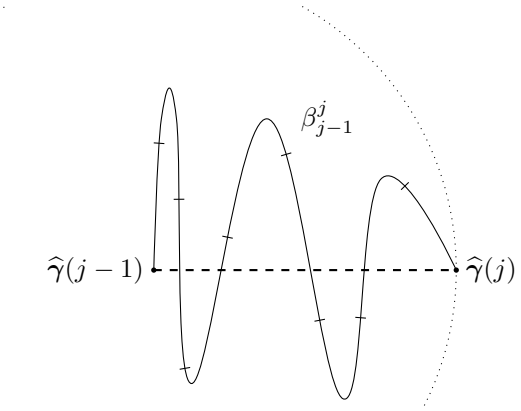


Figure 4.2: Subdividing a piece of a continuous paths.

Let $\vec{\gamma}$ be any closed jumping-path and $\widehat{\gamma}\colon [n] \to X$ its discretisation; we claim that $\left[\vec{\gamma}\right] \equiv \left[\widehat{\gamma}_{geo}\right]$ (mod FT_θ)—note that we do not claim that $\vec{\gamma}$ and $\widehat{\gamma}$ be homotopic. By hypothesis we can assume that $\vec{\gamma}$ is composed of rectifiable paths. Let β^j_{j-1} be the sub-path of $\vec{\gamma}$ going from $\widehat{\gamma}(j-1)$ to $\widehat{\gamma}(j)$—it could either be a continuous path or a single jump. We will now argue as above to conclude that β^j_{j-1} and the geodesic between $\widehat{\gamma}(j-1)$ and $\widehat{\gamma}(j)$ differ by a product of loops in $\mathcal{T}_\theta$. This is clearly doable if β^j_{j-1} is a single jump; while if β^j_{j-1} is a continuous path it is can be subdivided in finitely many pieces of length at most θ (because it is rectifiable) and the argument can be applied when seeing β^j_{j-1} as the concatenation of these smaller pieces (see Figure 4.2).

Assume now that $\vec{\gamma}$ and $\vec{\gamma}'$ are jumping-paths with θ-homotopic discretisations; then there exists a θ-grid homotopy $\widehat{H}\colon [n] \times [m] \to X$ between lazified versions of $\widehat{\gamma}$ and $\widehat{\gamma}'$. Let $\widehat{\gamma}^{(k)} \coloneqq \widehat{H}(\cdot, k)$, note that $\widehat{\gamma}_{geo} = \widehat{\gamma}^{(0)}_{geo}$ and $\widehat{\gamma}'_{geo} = \widehat{\gamma}^{(m)}_{geo}$ as lazifying does not modify the actual paths. By the above discussion, for every $k = 1, \dots, m$ we have $\left[\widehat{\gamma}^{(k-1)}_{geo}\right] \equiv \left[\widehat{\gamma}^{(k)}_{geo}\right]$ (mod FT_θ) and, as we know that $\left[\vec{\gamma}\right] \equiv \left[\widehat{\gamma}_{geo}\right]$ and $\left[\vec{\gamma}'\right] \equiv \left[\widehat{\gamma}'_{geo}\right]$ (mod FT_θ), this concludes the proof of the theorem. □

Remark 4.2.3. This is a sort of continuous version of [BKLW01, Theorem 2.7]. It is also clear from the proof that some version of this theorem can be proved with weaker hypotheses (*e.g.* spaces with path metrics). We decided not to do so to avoid unnecessary complications.

4.3 Explicit computations

Recall that we are assuming that (X, d) to has homotopy rectifiable paths, (X, δ_S) is jumping-geodesic and $\theta \geq 1$. Combining Theorem 4.2.2 with Theorem 4.1.10 we

obtain a (non-canonical, as it depends on the choice of paths α_s) surjection

$$\widehat{\Phi}_S \colon \pi_1(X, x_0) \rtimes_{\phi_S} F_S \twoheadrightarrow \pi_{1,\theta}\big((X, \delta_S), x_0\big).$$

We now wish to study the kernel of $\widehat{\Phi}_S$ more explicitly.

4.3.1 General remarks

Let $\mathrm{Sh}_{\mathrm{path}} \coloneqq \big\{[\gamma] \in \pi_1(X, x_0) \;\big|\; \gamma \sim_f \gamma',\ \|\gamma'\| \leq 4\theta\big\}$ be the set of continuous loops based at x_0 that are freely homotopic to a continuous loops of length at most 4θ in (X, d). It is then clear that $\mathrm{Sh}_{\mathrm{path}} \times \{e\}$ is in the kernel of $\widehat{\Phi}_S$. Moreover, as it is a subset of the normal factor of the semidirect product, taking the quotient by $\langle\!\langle \mathrm{Sh}_{\mathrm{path}} \times \{e\} \rangle\!\rangle$ preserves the structure of semidirect product. Therefore $\widehat{\Phi}_S$ factors as

$$\begin{array}{ccc} \pi_1(X, x_0) \rtimes_{\phi_S} F_S & \xrightarrow{\widehat{\Phi}_S} & \pi_{1,\theta}\big((X, \delta_S), x_0\big) \\ \downarrow & \nearrow & \\ \pi_1(X, x_0)/\langle\!\langle \mathrm{Sh}_{\mathrm{path}} \rangle\!\rangle \rtimes_{\phi_S} F_S & & \end{array}$$

where the normal closure $\langle\!\langle \mathrm{Sh}_{\mathrm{path}} \rangle\!\rangle$ is taken in the whole semidirect product and therefore we have

$$\langle\!\langle \mathrm{Sh}_{\mathrm{path}} \rangle\!\rangle = \big\langle \{\phi_S(w)[\gamma] \mid w \in F_S,\ [\gamma] \in \mathrm{Sh}_{\mathrm{path}}\} \big\rangle.$$

Equivalently, $\langle\!\langle \mathrm{Sh}_{\mathrm{path}} \rangle\!\rangle$ is the subgroup generated by the set of continuous paths which are freely homotopic to the image under some $w \in F_S$ of a short closed path.

Let $\mathrm{Sh}_{\mathrm{jump}} \coloneqq \big\{w \in F_S \;\big|\; \exists y \in X,\ d(y, w \cdot y) + |w| \leq 4\theta\big\}$ be the set of elements $w \in F_S$ that move some point $y \in (X, d)$ by less than 4θ minus the length of the reduced word of w.

Lemma 4.3.1. *Let (X, d) be a geodesic metric space and let θ be a natural number. Then for every $w \in \mathrm{Sh}_{\mathrm{jump}}$ there exists a $[\gamma] \in \pi_1(X, x_0)$ so that $([\gamma], w) \in \ker\big(\widehat{\Phi}_S\big)$.*

Proof. Let y be a point so that $d(y, w \cdot y) + |w| \leq 4\theta$, β be a continuous path from x_0 to y, and let ε be a continuous geodesic path from y to $w \cdot y$. Note that, as θ is an integer, the closed jumping-path $\varepsilon \overset{w_{\mathrm{rev}}^{-1}}{\rightsquigarrow} y$ can be decomposed into four sub-paths of length at most θ, and it is therefore in $\mathcal{T}_\theta$.

By Lemma 4.1.7 we have

$$\big[\alpha_w \overset{w_{\mathrm{rev}}^{-1}}{\rightsquigarrow} x_0\big] = \big[\alpha_w w(\beta) \overset{w_{\mathrm{rev}}^{-1}}{\rightsquigarrow} \beta^*\big] = \big[\alpha_w w(\beta)\varepsilon^*\beta^*\big]\big[\beta\varepsilon \overset{w_{\mathrm{rev}}^{-1}}{\rightsquigarrow} \beta^*\big],$$

and since $\big[\beta\varepsilon \overset{w_{\mathrm{rev}}^{-1}}{\rightsquigarrow} \beta^*\big]$ is in FT_θ we can conclude the proof by letting $\gamma \coloneqq \beta\varepsilon w(\beta^*)\alpha_w^*$ (see Figure 4.3). □

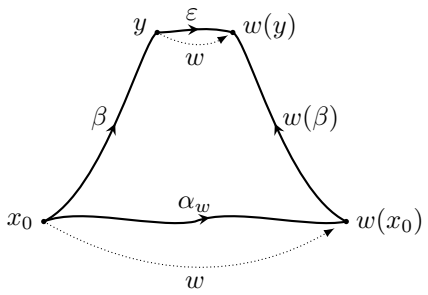


Figure 4.3: Constructing a continuous path that trivializes a short jump w.

Remark 4.3.2. Note that in Lemma 4.3.1 it is important that θ be integer, as it is necessary to be able to split a closed jumping-path length at most 4θ into four jumping-paths of length at most θ.

Corollary 4.3.3. *If (X,d) is geodesic, $\theta \in \mathbb{N}$ and $s \in \mathrm{Sh}_{\mathrm{jump}}$ for every $s \in S$, then $\pi_{1,\theta}\big((X,\delta_S),x_0\big)$ is isomorphic to a quotient of $\pi_1(X,x_0)$.*

The converse of Lemma 4.3.1 is not true in general. Still, it does hold if the action is by isometries.

Lemma 4.3.4. *If the action $F_S \curvearrowright (X,d)$ is by isometries, then there exists a γ so that $([\gamma],w) \in \ker\big(\widehat{\Phi}_S\big)$ only if $w \in \langle\langle \mathrm{Sh}_{\mathrm{jump}} \rangle\rangle$.*

Proof. Let $\vec{\boldsymbol{\gamma}} = \gamma_1 \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \gamma_n$ be a jumping-path and let $w = \vec{s}_n^{-1} \cdots \vec{s}_1^{-1}$. As the action is by isometries, the jumping-path $\gamma' \overset{w_{\mathrm{rev}}^{-1}}{\rightsquigarrow} x_0$ homotopic to $\vec{\boldsymbol{\gamma}}$ as in Corollary 4.1.8 has the same length of $\vec{\boldsymbol{\gamma}}$. It follows that if $\vec{\boldsymbol{\gamma}}$ is (freely homotopic to) a path in $\mathcal{T}_\theta$ then w is (conjugate to) an element in $\mathrm{Sh}_{\mathrm{jump}}$. □

Corollary 4.3.5. *If F_S acts by isometries and $\mathrm{Sh}_{\mathrm{jump}} = \{e\}$, then*

$$\pi_{1,\theta}\big((X,\delta_S),x_0\big) \cong \pi_1(X,x_0)/\langle\langle \mathrm{Sh}_{\mathrm{path}} \rangle\rangle \rtimes_{\phi_S} F_S.$$

Proof. Every jumping-path in $\mathcal{T}_\theta$ must be continuous as it cannot have any jump. It follows that the kernel of $\widehat{\Phi}_S$ is equal to $\langle\langle \mathrm{Sh}_{\mathrm{path}} \rangle\rangle$. □

By considering the action of the trivial group, we obtain as a special case the following:

Corollary 4.3.6. *Let (X,d) be a geodesic metric space. For every $\theta > 0$[1] we have*

$$\pi_{1,\theta}\big((X,d),x_0\big) \cong \pi_1(X,x_0)/\langle\langle \mathrm{Sh}_{\mathrm{path}} \rangle\rangle.$$

[1]We do not need θ to be integer, as there are no jumps to be considered.

4.3.2 Recovering the group contribution

Let $\Gamma = \langle S \rangle$ be a finitely generated group and let $\Gamma \curvearrowright (X, d)$ be an action. This induces a warped metric δ_S on X and we can apply the theory developed in the previous section to express $\pi_{1,\theta}\big((X\delta_S), x_0\big)$ in term of $\pi_1(X, x_0)$ and F_S. Still, the description of $\pi_{1,\theta}$ that we obtain this way is not completely satisfactory. Indeed, one would expect the discrete fundamental group to encode information about the acting group Γ, and this is not apparent in what was done so far.[2] The next result tries to uncover the dependence of $\pi_{1,\theta}$ on Γ, but in order to do so we first need to prove the following lemma:

Lemma 4.3.7. *Let $G \rtimes_\phi H$ be a semidirect product and $N \lhd H$ a normal subgroup. Assume that there exists a function $f\colon N \to G$ such that*

- *$\phi(n)(g) = g^{f(n)}$ for every $n \in N$ and $g \in G$ (where $g^{f(n)} \coloneqq f(n)nf(n)^{-1}$ denotes the conjugation by $f(n)$);*
- *$\phi(h)(f(n)) = f(n^h)$ for every $h \in H$ (f is ϕ-equivariant).*

Define Q to be the quotient

$$Q \coloneqq \frac{\big(G \rtimes_\phi H\big)}{\langle\!\langle \big(f(n)^{-1}, n\big) \mid n \in N \rangle\!\rangle};$$

then there is a natural short exact sequence

$$1 \longrightarrow \langle\!\langle f(N) \rangle\!\rangle_Q \longrightarrow Q \xrightarrow{\bar{p}} \frac{G}{\langle\!\langle f(N) \cup [f(N), G] \rangle\!\rangle} \rtimes_{\bar{\phi}} \frac{H}{N} \longrightarrow 1$$

where $[f(N), G]$ denotes the set $\big\{g^{f(n)}g^{-1} \mid n \in N,\ g \in G\big\}$ and $\langle\!\langle f(N) \rangle\!\rangle_Q$ is the normal subgroup generated by the image of $f(N)$ in Q.

Proof. We first need to show that the semidirect product on the right hand side is well-defined. For every $g \in G$, $n \in N$ and $h \in H$ we have

$$\phi(h)\big(f(n)\big) = f\big(n^h\big)$$

and

$$\phi(h)\big(g^{f(n)}g^{-1}\big) = \big(\phi(h)(g)\big)^{\phi(h)(f(n))}\phi(h)(g)^{-1} = \big(\phi(h)(g)\big)^{f(n^h)}\phi(h)(g)^{-1}.$$

[2] This is especially upsetting in Chapter 8 where we consider warped systems. Indeed, the discrete fundamental group should depend on the coarse geometry of the warped systems, and this depends only on Γ and not on the choice of generating set S.

Since N is a normal subgroup of H, we deduce that $\langle\!\langle f(N) \cup [f(N), G] \rangle\!\rangle$ is preserved by ϕ and we therefore obtain an H-action on the quotient $G/\langle\!\langle f(N) \cup [f(N), G] \rangle\!\rangle$. Moreover, N acts trivially on $G/\langle\!\langle f(N) \cup [f(N), G] \rangle\!\rangle$ as $\big(\phi(n)(g)\big)g^{-1}$ is in $[f(N), G]$ by definition. It follows that ϕ naturally induces the required homomorphism

$$\bar{\phi}\colon H/N \to \mathrm{Aut}\Big(\frac{G}{\langle\!\langle f(N) \cup [f(N), G] \rangle\!\rangle}\Big)$$

and we obtain a natural surjection

$$p\colon \big(G \rtimes_\phi H\big) \longrightarrow \frac{G}{\langle\!\langle f(N) \cup [f(N), G] \rangle\!\rangle} \rtimes_{\bar{\phi}} \frac{H}{N}$$

sending (g, h) to $(\bar{g}, \bar{h})$. It only remains to study the kernel of such surjection.

The elements $\big(f(n)^{-1}, n\big)$ trivially belongs to the kernel of p for every $n \in N$ and therefore p factors through the quotient Q. We obtain this way a surjective homomorphism $\bar{p}$ of Q onto the semidirect product. It only remains to show that the kernel of $\bar{p}$ is exactly $\langle\!\langle f(N) \rangle\!\rangle_Q$. Note that one containment is obvious, as $\big(f(n), e\big)$ is in $\ker(p)$ for every $n \in N$.

Vice versa, if (g, h) belongs to $\ker(p)$ then $h = n$ for some $n \in N$, so that $(g, h) \equiv \big(gf(n), e\big) \mod \langle\!\langle \big(f(n)^{-1}, n\big) \mid n \in N \rangle\!\rangle$. We can hence restrict to the study of elements of $\ker(p)$ of the form (g', e). Given such an element, g' can be expressed as a product of conjugates of $f(n)^{\pm 1}$ or $\big(g^{f(n)}g^{-1}\big)^{\pm 1}$ with $n \in N$ and $g \in G$. It is hence enough to observe that (the equivalence class of) a conjugate $f(n)^a$ is in $\langle\!\langle f(N) \rangle\!\rangle_Q$ by definition and that

$$\big(g^{f(n)}g^{-1}\big)^a = \Big(f(n)\big(f(n)^{-1}\big)^g\Big)^a$$

is in $\langle\!\langle f(N) \rangle\!\rangle_Q$ as well. □

Let now Γ be a finitely generated group acting on (X, d) and choose a presentation $\Gamma = \langle S \mid R \rangle$ (R could be infinite). Let $R_\theta \subset F_S$ be the subset of $\langle\!\langle R \rangle\!\rangle_{F_S}$ of words of length at most 4θ. Note that R_θ does not need to be a subset of R (R_θ depends only on the choice of S, not R). We will denote by Γ_θ the finitely presented group $\langle S \mid R_\theta \rangle$.

Note that if $r \in \langle\!\langle R \rangle\!\rangle$ is a relation of Γ, then the continuous path α_r is closed and it hence defines an element in $\pi_1(X, x_0)$. We can now prove the following.

Proposition 4.3.8. *Let $\Gamma = \langle S \mid R \rangle$ be a finitely generated group acting on (X, x_0) by homeomorphisms, fix $\theta \in \mathbb{N}$ and let G_θ be the quotient*

$$G_\theta := \frac{\pi_1(X, x_0)}{\langle\!\langle \big\{ [\alpha_r] \;\big|\; r \in \langle\!\langle R_\theta \rangle\!\rangle \big\} \cup \big\{ [\alpha_r \gamma \alpha_r^* \gamma^{-1}] \;\big|\; [\gamma] \in \pi_1(X, x_0),\ r \in \langle\!\langle R_\theta \rangle\!\rangle \big\} \rangle\!\rangle}.$$

Then the quotient $\pi_{1,\theta}\big((X, \delta_S), x_0\big)/\langle\!\langle \widehat{\Phi}_S([\alpha_r^*], e) \mid r \in R_\theta \rangle\!\rangle$ is isomorphic to a quotient of $G_\theta \rtimes \Gamma_\theta$.

Proof. This will be an application of Lemma 4.3.7, where $\pi_1(X, x_0)$ and F_S play the role of G and H respectively, and N corresponds to $\langle\langle R_\theta \rangle\rangle$. Note that the function sending $r \in \langle\langle R_\theta \rangle\rangle$ to $[\alpha_r] \in \pi_1(X, x_0)$ satisfies

$$\phi_S(r)[\gamma] = [\alpha_r r(\gamma) \alpha_r^*] = [\alpha_r][\gamma][\alpha_r]^{-1} = [\gamma]^{[\alpha_r]}$$

and it is also ϕ_S-equivariant, as we have

$$\begin{aligned} \phi_S(w)[\alpha_r] &= \big[\alpha_w w(\alpha_r) \alpha_w^*\big] \\ &= \big[\alpha_w w\big(\alpha_r w^{-1}(\alpha_w^*)\big)\big] \\ &= \big[\alpha_w w\big(\alpha_r r(\alpha_{w^{-1}})\big)\big] = \big[\alpha_{wrw^{-1}}\big] \end{aligned}$$

(here we used essentially that r acts as the identity on X). We can hence apply Lemma 4.3.7 to obtain a short exact sequence

$$1 \to \ \langle\langle [\alpha_r] \mid r \in R_\theta \rangle\rangle_Q \to \ \frac{\pi_1(X, x_0) \rtimes_{\phi_S} F_S}{\langle\langle \big\{ ([\alpha_r^*], r) \ \big| \ r \in R_\theta \big\} \rangle\rangle} \ \xrightarrow{\bar{p}} \ \big(G_\theta \rtimes_{\bar{\phi}} \Gamma_\theta\big) \to 1.$$

Note that in the expression above we used the fact that $[\alpha_r]^{-1} = [\alpha_r^*]$, $\alpha_{r_1 r_2} = \alpha_{r_1} \alpha_{r_2}$ and that $\alpha_{wrw^{-1}} = \alpha_w w(\alpha_r) \alpha_w^* = \phi_S(w)(\alpha_r)$ to deduce that

$$\langle\langle [\alpha_r] \mid r \in \langle\langle R_\theta \rangle\rangle \rangle\rangle_Q = \langle\langle [\alpha_r] \mid r \in R_\theta \rangle\rangle_Q$$

and

$$\langle\langle \big\{ ([\alpha_r]^{-1}, r) \ \big| \ r \in \langle\langle R_\theta \rangle\rangle \big\} \rangle\rangle = \langle\langle \big\{ ([\alpha_r^*], r) \ \big| \ r \in R_\theta \big\} \rangle\rangle.$$

However, in general it is not possible to restrict to $r \in R_\theta$ in the definition of G_θ, as $\alpha_{wrw^{-1}}$ is conjugate to α_r only in $\pi_1(X, x_0) \rtimes F_S$, and not in $\pi_1(X, x_0)$.

The homomorphism $\widehat{\Phi}_S$ is a surjection of $\pi_1(X, x_0) \rtimes F_S$ onto $\pi_{1,\theta}\big((X, \delta_S), x_0\big)$. Given $r \in R_\theta$, we have

$$\widehat{\Phi}_S\big(([\alpha_r^*], r)\big) = \big[\alpha_r^* \alpha_r \overset{r_{\mathrm{rev}}^{-1}}{\rightsquigarrow} x_0\big] = \big[x_0 \overset{r_{\mathrm{rev}}^{-1}}{\rightsquigarrow} x_0\big]$$

and the latter is trivial as it is a closed θ-path of length at most 4θ. This implies that $\widehat{\Phi}_S$ factors through the quotient by $\langle\langle \big\{ ([\alpha_r^*], r) \ \big| \ r \in R_\theta \big\} \rangle\rangle$.

To conclude the proof it is enough to note that $\widehat{\Phi}_S$ also factors through a surjection

$$\big(G_\theta \rtimes_{\bar{\phi}} \Gamma_\theta\big) \to \ \frac{\pi_{1,\theta}\big((X, \delta_S), x_0\big)}{\langle\langle \widehat{\Phi}_S([\alpha_r^*], e) \mid r \in R_\theta \rangle\rangle}.$$

□

Corollary 4.3.9. *If the paths α_r are null-homotopic in X for every $r \in R$, then $\pi_{1,\theta}\big((X, \delta_S), x_0\big)$ is isomorphic to a quotient of $\pi_1(X, x_0) \rtimes \Gamma_\theta$.*

4.3.3 The special case of box spaces

As already noted in Corollary 4.3.6, Theorem 4.2.2 can be applied also to metric geodesic metric spaces without an action (so that jumping-paths are merely continuous paths). In particular, it will prove convenient for computing the discrete fundamental group of a box space.

The next result is basically a rewriting of [DK17a, Lemma 3.4]. We include a sketch of a proof for the convenience of the reader, as the statement we give here is slightly more precise and general than what is proved by Delabie–Khukhro.

Let now $\Lambda = \langle T \rangle$ be finitely generated group, let $R_\theta \subset F_T$ be the of words of length at most 4θ that are trivial under the surjection $F_T \to \Lambda$ and let Λ_θ be the finitely presented group $\Lambda_\theta \coloneqq \langle T \mid R_\theta \rangle$. Note that Λ is naturally a quotient of Λ_θ.

Theorem 4.3.10 (Delabie–Khukhro)**.** *Given $\theta \in \mathbb{N}$ and a subgroup $H < \Lambda$ such that the word length $|g|$ is at least 4θ for every $g \in H \smallsetminus \{e\}$, the θ-discrete fundamental group of the (left) Schreier graph* $\mathrm{Schr}(H\backslash\Lambda, T)$ *is given by*

$$\pi_{1,\theta}\big(\mathrm{Schr}(H\backslash\Lambda, T)\big) \cong \frac{H_\theta}{\langle \{ h \in H_\theta \mid \exists g \in \Lambda_\theta \text{ s.t. } |ghg^{-1}| \leq 4\theta \} \rangle}$$

where H_θ is the preimage of H in Λ_θ. In particular, if H is normal in G and $|h| > 4\theta$ for every $h \in H \smallsetminus \{e\}$ then then $\pi_{1,\theta}\big(\mathrm{Schr}(H\backslash\Lambda, T)\big) \cong H_\theta$.

Sketch of proof. The Schreier graph $\mathrm{Schr}(H\backslash\Lambda, T)$ is isometric to the Schreier graph $\mathrm{Schr}(H_\theta\backslash\Lambda_\theta, T)$. This can be made into a geodesic metric space by gluing in an interval $[0, 1]$ for every edge of the graph.[3] The set of homotopy classes of closed paths in $\mathrm{Schr}(H_\theta\backslash\Lambda_\theta, T)$ and based at $H_\theta \in H_\theta\backslash\Lambda_\theta$ is in bijective correspondence with the set of (reduced) words in F_T that represent elements in $p^{-1}(H_\theta) \subseteq F_T$ (here p denotes the surjection $F_T \to \Lambda_\theta$).

We can apply Theorem 4.2.2 to the space $\mathrm{Schr}(H_\theta\backslash\Lambda_\theta, T)$ to deduce that

$$\pi_{1,\theta}\big(\mathrm{Schr}(H_\theta\backslash\Lambda_\theta, T)\big) \cong \frac{\pi_1\big(\mathrm{Schr}(H_\theta\backslash\Lambda_\theta, T)\big)}{\langle \mathrm{FT}_\theta \rangle} \cong \frac{p^{-1}(H_\theta)}{\langle \mathrm{FT}_\theta \rangle}.$$

By definition, R_θ is contained in FT_θ. Since $p^{-1}(H_\theta)/\langle\langle R_\theta \rangle\rangle = H_\theta \subseteq \Lambda_\theta$, we have

$$\pi_{1,\theta}\big(\mathrm{Schr}(H_\theta\backslash\Lambda_\theta, T)\big) \cong \frac{H_\theta}{\langle \mathrm{FT}_\theta \cap H_\theta \rangle}.$$

[3] The θ-fundamental group of the graph seen as a discrete set of vertices coincides with the θ-fundamental group of the geodesic metric space obtained gluing in the intervals because we are assuming that θ is integer. Otherwise there could be θ-paths that are non-trivial in the 'discrete graph' while they are trivial in the 'geodesic graph' (the issue is always that we need to be able to reduce to discrete paths of length four, which might not be possible in the discrete setting if $\theta \notin \mathbb{N}$).

To conclude, it is now enough to note that free homotopies (among reduced words) correspond to conjugations and hence FT_θ corresponds to the set of elements that can be conjugated to words of length at most 4θ. □

Corollary 4.3.11. *If Λ is finitely presented and $(H_k)_{k\in\mathbb{N}}$ is a residual nested normal filtration of Λ, for every $\theta \gg 0$ there exists an $n \in \mathbb{N}$ large enough so that $\pi_{1,\theta}\big(\mathrm{Schr}(\Lambda/H_k, T)\big) \cong H_k$ for every $k \geq n$.*

Proof. Let $\Lambda = \langle T \mid R\rangle$ be a finite presentation. Then there exists $\theta \gg 0$ such that $\langle\langle R_\theta \rangle\rangle = \langle\langle R \rangle\rangle$. Moreover, if H_k is a residual nested normal filtration then all the non-trivial elements of H_k will have word length at least 4θ for every k large enough. The statement now follows from the 'in particular' statement of Theorem 4.3.10. □

Chapter 5

Approximating graphs

In this chapter we introduce a procedure to approximate actions on spaces with families of graphs. We then provide criteria allowing us to establish when said graphs satisfy linear isoperimetric inequalities, have bounded degrees or cannot be coarsely embedded into Banach spaces.

5.1 Approximating actions with graphs

Let Γ be a finitely generated group with a fixed finite generating set S. Recall that we denote by $S_e^{\pm}$ the set $S \cup S^{-1} \cup \{e\}$.

5.1.1 Definitions of approximating graphs

The 'right' definition of graph approximating an action depends on the structure of the space that is being acted on.

Let (X, ν) be a measure space. A *measurable partition* of (X, ν) is a countable (possibly finite) family of disjoint measurable subsets (regions) $\mathcal{P} = \{R_i \mid i \in I\}$ such that

$$\nu\left(X \smallsetminus \coprod_{i \in I} R_i\right) = 0.$$

Let also $\Gamma \curvearrowright X$ be a measurable action (equivalently, an action by measurable maps. See Subsection 2.2.2).

Definition 5.1.1. Given a measurable action $\rho\colon \Gamma \curvearrowright X$ and a measurable partition $\mathcal{P} = \{R_i \mid i \in I\}$ of (X, ν), their *measure approximating graph* is the graph $\mathcal{G}\big(\rho\colon \Gamma \curvearrowright X\ ; \mathcal{P}\big)$ whose set of vertices is the set of regions $\mathrm{V}\big(\mathcal{G}\big(\rho\colon \Gamma \curvearrowright X\ ; \mathcal{P}\big)\big) = \mathcal{P}$ and such that the pair $\{R_i, R_j\}$ is an edge if and only if there exists an element $s \in S_e^{\pm}$

with

$$\nu\big((s \cdot R_i) \cap R_j\big) \neq 0.$$

We can give a similar definition for actions on topological spaces, but in this setting we choose a slightly different set of edges to retain more local information.

Definition 5.1.2. Given a continuous action $\rho\colon \Gamma \curvearrowright X$ and a partition $\mathcal{P} = \{R_i \mid i \in I\}$ of X, their *topology approximating graph* is the graph $\widetilde{\mathcal{G}}\big(\rho\colon \Gamma \curvearrowright X\ ; \mathcal{P}\big)$ whose set of vertices is the set of regions $\mathrm{V}\big(\widetilde{\mathcal{G}}\big(\rho\colon \Gamma \curvearrowright X\ ; \mathcal{P}\big)\big) = \mathcal{P}$ and such that the pair $\{R_i, R_j\}$ is an edge if and only if there exists an element $s \in S_e^{\pm}$ with

$$s \cdot \overline{R_i} \cap \overline{R_j} \neq \emptyset.$$

Remark 5.1.3. Note that, since we are considering closures of regions and the set $S_e^{\pm}$ contains the identity element, two adjacent regions $R, R' \in \mathcal{P}$ always form an edge in $\widetilde{\mathcal{G}}(\rho\colon \Gamma \curvearrowright X; \mathcal{P})$. This is the reason why we say that the topology approximating graphs retain local information about X.

When no confusion can arise, we will lighten the notation for the measure approximating graphs by simply writing $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{P})$, $\mathcal{G}(\Gamma \curvearrowright X)$ or $\mathcal{G}(\mathcal{P})$ (and similarly for the topology approximating graph). The choice of which notation to use will depend on what the parameter of interest is.

5.1.2 Comparisons among approximating graphs

Let X be a topological space and assume that ν is a measure on its Borel σ-algebra. Then a continuous action $\Gamma \curvearrowright X$ is also measurable and, for every measurable partition $\mathcal{P}$, the identity map on the vertex set induces a natural inclusion of graphs $\mathcal{G}(\mathcal{P}) \hookrightarrow \widetilde{\mathcal{G}}(\mathcal{P})$. One could expect that the above inclusion should be a coarse equivalence, but the next example shows that this is not the case:

Example 5.1.4. Let $S = \{a, b\} \subset \mathrm{SO}(3, \mathbb{R})$ be a pair of rotations that generate a non-abelian free subgroup of $\mathrm{SO}(3, \mathbb{R})$, then they induce an essentially free action of the free group $F_2 \curvearrowright \mathbb{S}^2$. Let $x \in \mathbb{S}^2$ be an element with trivial stabiliser. For every $n \in \mathbb{N}$ there is a $\epsilon > 0$ such that all the balls $B(w \cdot x, \epsilon)$ with $|w| \leq n$ have disjoint closures. Complete this set to a partition $\mathcal{P}_n$ of $\mathbb{S}^2$.

In the measure approximating graph, the ball of radius n around the vertex corresponding to $B(x, \epsilon)$ is in bijective correspondence with the ball of radius n in the free group F_2. In particular, if $R \in \mathcal{P}$ is a region such that $\overline{R} \cap \overline{B}(x, \epsilon) \neq \emptyset$, we have

that the distance between the corresponding vertices in $\mathcal{G}(\mathcal{P}_n)$ is at least $n+1$, while they are adjacent in $\widetilde{\mathcal{G}}(\mathcal{P}_n)$.

This implies that the inclusions $\mathcal{G}(\mathcal{P}_n) \hookrightarrow \widetilde{\mathcal{G}}(\mathcal{P}_n)$ are not uniform quasi-isometries.

Remark 5.1.5. The example above makes it clear what we mean when we say that the topology approximating graph encodes more local information than the measure approximating graph.

Remark 5.1.6. Using discrete fundamental groups it is possible to show that the families of graphs $\big(\mathcal{G}(\mathcal{P}_n)\big)_{n\in\mathbb{N}}$ and $\big(\widetilde{\mathcal{G}}(\mathcal{P}_n)\big)_{n\in\mathbb{N}}$ as in Example 5.1.4 are not coarsely equivalent (meaning that there are no uniform quasi-isometries between them, even if considering other maps besides the natural inclusions).

Example 5.1.4 also allows us to point at another typical feature of approximating graphs. That is, while it is possible (but highly atypical) for a measure approximating graph to have high girth, the girth of a topological approximating graph associated with an action on a (locally) connected space X is almost always 3 (which is the lowest possible girth for a graph with no multiple edges). Indeed, let R_1 and R_2 be adjacent regions in a partition $\mathcal{P}$ of X. Let $R' \in \mathcal{P}$ be a region intersecting $s(\overline{R}_1 \cap \overline{R}_2)$. Then the vertices corresponding to R_1, R_2 and R' are all joined by edges in $\widetilde{\mathcal{G}}(\mathcal{P})$. Note that in this argument we are assuming that R' is different from R_1 and R_2, which we can assume in most cases.

Having said that, we wish to remark that in most cases measure approximating and topology approximating graphs appear to be equivalent. In fact, one should be able to prove some statement implying that for 'generic' partitions the two graphs coincide up to a bounded error. Nevertheless, in the sequel we will specify when a statement is regarding approximating graphs in the measurable or topological settings (most of the times the results will hold for both).

In virtue of our assumption that groups come with a fixed generating set, we have so far systematically ignored the fact that the definition of approximating graphs depends on the choice of a generating set for the group acting. With the next easy lemma we wish to amend for this by showing that (in most cases) the coarse geometry of said graphs does not depend on this choice.

Lemma 5.1.7. *Let S and T be two finite generating sets for the group Γ. If an action $\Gamma \curvearrowright (X, \nu)$ is measure-class preserving then the measure approximating graphs $\mathcal{G}^S(\mathcal{P})$ and $\mathcal{G}^T(\mathcal{P})$ constructed with respect to these generating sets are uniformly quasi-isometric (*i.e. *the quasi-isometry constants only depend on S and T).*

The analogous statement holds for continuous actions on topological spaces and topology approximating graphs.

Proof. By considering a generating set containing both, we can assume that $S \subseteq T$. We then have a natural inclusion $\mathcal{G}^S(\mathcal{P}) \hookrightarrow \mathcal{G}^T(\mathcal{P})$. Since T is finite, there exists an n such that every element $t \in T$ can be expressed as a product of at most n elements of S. It follows that if $\{R, R'\}$ is an edge of $\mathcal{G}^T(\mathcal{P})$ then the vertices R and R' are joined by a path of length at most n in $\mathcal{G}^S(\mathcal{P})$. Indeed, let $t = s_1 \cdots s_n$ and let $\nu(t(R) \cap R') > 0$. Since the partition is countable and the action preserves measure zero sets, there exists a region $R_1 \in \mathcal{P}$ such that $R_1 \cap s_1^{-1}\big(t(R) \cap R'\big)$ has positive measure, and hence $\{R_1, R'\}$ is an edge in $\mathcal{G}^S(\mathcal{P})$. By construction, $s_2 \cdots s_n(R) \cap R_1$ has positive measure. We can hence conclude that R and R_1 are joined by a path of length $n-1$ by induction.

The proof in the topological setting is analogous. □

5.1.3 A minimal regularity condition

For the graph $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{P})$ to give any interesting information on the dynamical system, we need to require some tameness conditions on the action itself and on the partition considered. The most important among such requirements is some kind of control on the ratios of the measures of the regions of the partition $\mathcal{P}$.

Definition 5.1.8. A partition $\mathcal{P}$ of a measure space has *bounded measure ratios* if the measure of every region $R \in \mathcal{P}$ is finite and there exists a constant $Q \geq 1$ such that for every couple of regions R_i, R_j in $\mathcal{P}$ one has

$$\frac{1}{Q} \leq \frac{\nu(R_i)}{\nu(R_j)} \leq Q.$$

All the statements that we will prove in the sequel need the partitions to have bounded measure ratios. For most of the statements we will also need extra hypotheses, but these will vary from case to case.

Remark 5.1.9. As a general rule, the ideal situation to work in is that of actions by diffeomorphisms (or even isometries) on compact connected Riemannian manifolds. Most of the proofs that follow can be made substantially easier in this rather restricted settings.

5.2 Measure expansion and Cheeger constants

The major player of this section is the following notion of expansion for actions on measure spaces.

Definition 5.2.1. A measurable action $\rho\colon \Gamma \curvearrowright (X, \nu)$ is *expanding in measure* if there exists a constant $\alpha > 0$ such that $\nu(S_e^{\pm} \cdot A) \geq \big(1 + \alpha\big)\nu(A)$ for every measurable set $A \subset X$ with finite measure and $\nu(A) \leq \nu(X)/2$ (the latter condition is vacuous when X has infinite measure). When this is the case, we say that the action is *α-expanding*.

Note that the set A is always contained in $S_e^{\pm} \cdot A$ because $e \in S_e^{\pm}$. Moreover, since S is finite, an action is expanding in measure if and only if there exists a constant $\alpha' > 0$ such that for every subset A with finite measure and $\nu(A) \leq \nu(X)/2$ there exists an element $s \in S^{\pm}$ such that $\nu\big(s(A) \triangle A\big) \geq \alpha' \nu(A)$ (this reformulation is more reminiscent of a definition of Kazhdan property (T)).

Remark 5.2.2. We designed this notion of expansion to be the right condition for the subsequent results of this chapter. Still, it turned out that this notion is not quite new in the literature. In the terminology of [GMP16], the action ρ is expanding in measure if X is a *domain of expansion* for it. In [BY13] such a ρ is said to be a *continuous expander* (if the action is also differentiable).

5.2.1 Expansion of the action implies expansion of the graphs

The fundamental observation at the base of our work is the following lemma:

Lemma 5.2.3. *Let $\rho\colon \Gamma \curvearrowright (X, \nu)$ be an α-expanding action. For every partition $\mathcal{P}$ with measure ratios bounded by a constant Q, the measure approximating graph $\mathcal{G}(\mathcal{P})$ has Cheeger constant bounded away from zero*

$$h\big(\mathcal{G}(\mathcal{P})\big) \geq \epsilon > 0$$

and the constant $\epsilon = \epsilon(\alpha, Q)$ depends only on the expansion parameter α and the bound on measure ratios Q.

Proof. Let W be any finite set of vertices of $\mathcal{G}(\mathcal{P})$ with $|W| \leq |\mathcal{G}(\mathcal{P})|/2$ and consider the measurable set

$$A \coloneqq \coprod_{R_i \in W} R_i.$$

Up to measure zero sets we have

$$S_e^{\pm} \cdot A \subseteq \bigcup \Big\{ R_i \;\Big|\; \nu\big((S_e^{\pm} \cdot A) \cap R_i\big) > 0 \Big\} = \coprod_{W \cup \partial W} R_i =: B.$$

If $\nu(A) \leq \nu(X)/2$ then

$$\nu(B) \geq (1+\alpha)\nu(A)$$

and since $\mathcal{P}$ has bounded measure ratios we conclude

$$Q\big|\partial W\big|\left(\inf_{R\in\mathcal{P}}\nu(R)\right) \geq \nu(B \smallsetminus A) \geq \alpha\nu(A) \geq \alpha\big|W\big|\left(\inf_{R\in\mathcal{P}}\nu(R)\right);$$

whence

$$\frac{|\partial W|}{|W|} \geq \frac{\alpha}{Q}.$$

On the other hand, if $\nu(A) > \nu(X)/2$ let $C \coloneqq X \smallsetminus (S_e^{\pm}\cdot A)$. Since $S_e^{\pm}$ is symmetric, the set $(S_e^{\pm}\cdot C)\smallsetminus C$ is contained in $(S_e^{\pm}\cdot A)\smallsetminus A$. Then we have:

$$\nu\big((S_e^{\pm}\cdot A)\smallsetminus A\big) \geq \alpha\nu(C) = \alpha\Big(\nu(X) - \nu(A) - \nu\big((S_e^{\pm}\cdot A)\smallsetminus A\big)\Big)$$

whence

$$\nu(B\smallsetminus A) \geq \nu\big((S_e^{\pm}\cdot A)\smallsetminus A\big) \geq \frac{\alpha}{1+\alpha}\big(\nu(X)-\nu(A)\big).$$

Since $|W| \leq |\mathcal{P}|/2$, by the bound on measure ratios we get

$$\nu\big(X\smallsetminus A\big) \geq \frac{1}{Q}\nu(A).$$

Using the same argument as above and combining the inequalities so obtained we conclude that

$$\frac{|\partial W|}{|W|} \geq \min\left\{\frac{\alpha}{Q}, \frac{\alpha}{(1+\alpha)Q^2}\right\}$$

as desired. □

Corollary 5.2.4. *Let (X,ν) be a probability space and $\Gamma \curvearrowright (X,\nu)$ an action expanding in measure. Assume we are given a sequence of finite measurable partitions $\mathcal{P}_n$ with $|\mathcal{P}_n| \to \infty$ and measure ratios uniformly bounded by the same constant Q. Then the sequence of measure approximating graphs $\mathcal{G}(\mathcal{P}_n)$ share a uniform lower bound on their Cheeger constants. In particular, they form a family of expanders if and only if they have uniformly bounded degree.*

Remark 5.2.5. If X is also a topological space, since the measure approximating graph is a subgraph of the topology approximating graph having the same set of vertices, it follows that the Cheeger constant of $\widetilde{\mathcal{G}}(\mathcal{P})$ is bounded from below by the Cheeger constant of $\mathcal{G}(\mathcal{P})$. In particular, Lemma 5.2.3 and Corollary 5.2.4 hold for topology approximating graphs as well.

5.2.2 A converse implication

In certain situations, considering 'finer and finer' measurable partitions on the same dynamical system $\Gamma \curvearrowright (X, \nu)$ can yield a converse to Lemma 5.2.3. To make sense of the intuition behind choosing 'finer' partitions, it is helpful to restrict to spaces with more structure. A natural setting for doing so is that of metric spaces equipped with Radon measures on their Borel σ-algebras (see Subsection 2.1.6).

Lemma 5.2.6. *Let X be a locally compact metric space. Then, for every compact subset $K \subseteq X$ there exists a radius $r > 0$ small enough so that the closed neighbourhood $\overline{N}_r(K)$ is still compact.*

Proof. By local compactness, every point $x \in K$ has a compact neighbourhood and hence there exists a $r_x > 0$ such that the closed ball $\overline{B}(x, r_x)$ is compact. The collection of open balls $B(x, r_x)$ is an open cover of K, thus there exist finitely many $x_1, \ldots, x_n$ such that $B(x_i, r_{x_i})$ cover K.

Let $r > 0$ be small enough so that the neighbourhood $N_r(K)$ is contained in the union $\bigcup_{i=1}^n B(x_i, r_{x_i})$. Then we have

$$\overline{N}_r(K) \subseteq \bigcup_{i=1}^n \overline{B}(x_i, r_{x_i})$$

and therefore $\overline{N}_r(K)$ is compact. □

Proposition 5.2.7. *Let (X, d, ν) be a locally compact metric space with a Radon measure thereof and let $\mathcal{P}_n$ be a sequence of measurable partitions of X with uniformly bounded measure ratios. Assume that for every compact set $K \subseteq X$ there is a decreasing sequence of positive numbers $(r_{K,n})_{n\in\mathbb{N}}$ such that $r_{K,n} \to 0$ and $\nu\big(\overline{E}_{K,n}\big) \to 0$, where*

$$E_{K,n} \coloneqq \bigcup \big\{ R \;\big|\; R \in \mathcal{P}_n,\ \overline{R} \cap K \neq \emptyset,\ \operatorname{diam}(R) > r_{K,n} \big\}$$

and $\overline{E}_{K,n}$ is its closure.

Then, for any continuous action $\Gamma \curvearrowright X$ the existence of a uniform positive Cheeger constant $\epsilon > 0$ for the topology approximating graphs $\widetilde{\mathcal{G}}(\mathcal{P}_n)$ implies that the action is expanding in measure.

Proof. The idea of the proof is that the isoperimetric information on the approximating graphs translates well to subsets of X that can be described as union of regions in $\mathcal{P}_n$. To use this information we thus need to show that every subset of X can be suitably approximated as a union of regions.

Since ν is a Radon measure, it is enough to prove that there is a positive constant $\alpha > 0$ so that for every compact set $K \subset X$ with finite measure and $\nu(K) \leq \nu(X)/2$ we have $\nu(S_e^{\pm} \cdot K) \geq (1+\alpha)\nu(K)$.

Fix such a compact set K and an appropriate sequence $r_{K,n} \to 0$ and note that by hypothesis we have:

$$\nu(K) = \lim_{n\to\infty} \big[\nu\big(K \smallsetminus \overline{E}_{K,n}\big) + \nu\big(K \cap \overline{E}_{K,n}\big)\big] = \lim_{n\to\infty} \nu\big(K \smallsetminus \overline{E}_{K,n}\big).$$

For any set $A \subseteq X$ we will denote by $\mathrm{V}_n(A) \subseteq \mathcal{P}_n$ the set of cells in $\mathcal{P}_n$ whose closure intersects A

$$\mathrm{V}_n(A) \coloneqq \big\{R \in \mathcal{P}_n \;\big|\; \overline{R} \cap A \neq \emptyset\big\}$$

and denote by $\mathcal{N}_n[A]$ their union

$$\mathcal{N}_n[A] \coloneqq \bigcup_{R\in \mathrm{V}_n(A)} R \quad \subseteq X$$

and by $\overline{\mathcal{N}}_n[A]$ the union of their closures[1]

$$\overline{\mathcal{N}}_n[A] \coloneqq \bigcup_{R\in \mathrm{V}_n(A)} \overline{R} \quad \subseteq X.$$

By construction, for every $A \subseteq K$ and for every region $R \in \mathcal{P}_n$ contained in $E_{K,n}$ we have that R is not in $\mathrm{V}_n(A \smallsetminus \overline{E}_{K,n})$. Since $E_{K,n}$ is a union of regions, we deduce that $\mathcal{N}_n[A \smallsetminus \overline{E}_{K,n}] \subseteq \mathcal{N}_n[A] \smallsetminus E_{K,n}$. Moreover, since the diameter of a region R is equal to the diameter of its closure and $A \subseteq K$, if R is a region contained in $\mathcal{N}_n[A] \smallsetminus E_{K,n}$ then R is contained in the closed neighbourhood $\overline{N}_{r_{K,n}}(A)$. We thus obtain:

$$\overline{\mathcal{N}}_n\big[K \smallsetminus \overline{E}_{K,n}\big] \subseteq \overline{\mathcal{N}_n[K] \smallsetminus E_{K,n}} \subseteq \overline{\overline{N}_{r_{K,n}}(K) \smallsetminus E_{K,n}}. \tag{5.1}$$

As $A \subseteq \mathcal{N}_n[A]$ for every $A \subseteq X$, we have:

$$\nu\big(K \smallsetminus \overline{E}_{K,n}\big) \leq \nu\big(\mathcal{N}_n\big[K \smallsetminus \overline{E}_{K,n})\big]\big) \leq \nu\Big(\overline{\mathcal{N}_n\big[K \smallsetminus \overline{E}_{K,n})\big]}\Big) \leq \nu\big(\overline{N}_{r_{K,n}}(K)\big),$$

and since both the first and the last expression tend to $\nu(K)$ we deduce that there exist the limits

$$\lim_{n\to\infty} \nu\big(\mathcal{N}_n\big[K \smallsetminus \overline{E}_{K,n}\big]\big) = \lim_{n\to\infty} \nu\Big(\overline{\mathcal{N}_n\big[K \smallsetminus \overline{E}_{K,n}\big]}\Big) = \nu(K). \tag{5.2}$$

[1]Note that if $\mathrm{V}_n(A)$ is infinite $\overline{\mathcal{N}}_n[A]$ does not need to be closed. This is not going to be important though.

Equation (5.2) allows us to express $\nu(K)$ as a limit of measures of finite unions of regions of $\mathcal{P}_n$. We now need to find a similar estimate for $S_e^{\pm} \cdot K$. For n large enough the neighbourhood $\overline{N}_{r_{K,n}}\big(K\big)$ is compact by Lemma 5.2.6, hence for every $s \in S_e^{\pm}$ the restriction of s to said compact neighbourhood is a uniformly continuous map. It follows that there is an infinitesimal decreasing sequence $r'_n \to 0$ such that

$$s \cdot \overline{N}_{r_{K,n}}(K) \subseteq \overline{N}_{r'_n}\big(\overline{N}_{r_{K,n}}\big(s \cdot K\big)\big)$$

for every $s \in S_e^{\pm}$ and hence

$$S_e^{\pm} \cdot \overline{N}_{r_{K,n}}(K) \subseteq \overline{N}_{r'_n}\Big(\overline{N}_{r_{K,n}}\big(S_e^{\pm} \cdot K\big)\Big) \subseteq \overline{N}_{r'_n + r_{K,n}}\big(S_e^{\pm} \cdot K\big). \tag{5.3}$$

Again by Lemma 5.2.6, we can fix a n_0 large enough so that the set

$$C \coloneqq \overline{N}_{r'_{n_0} + r_{K,n_0}}\big(S_e^{\pm} \cdot K\big)$$

is compact. In particular, we obtain a new infinitesimal sequence $r_{C,n}$ from the hypothesis and, as for (5.1), we have

$$\mathcal{N}_n[A] \smallsetminus E_{C,n} \subseteq \overline{N}_{r_{C,n}}(A) \smallsetminus E_{C,n} \subseteq \overline{N}_{r_{C,n}}(A) \tag{5.4}$$

for every $A \subseteq C$.

For $n \geq n_0$, applying (5.1), (5.3) and (5.4), we obtain a chain of containments:

$$\begin{aligned}
\mathcal{N}_n\big[S_e^{\pm} \cdot \overline{\mathcal{N}}_n\big[K \smallsetminus \overline{E}_{K,n}\big]\big] &\subseteq \mathcal{N}_n\Big[S_e^{\pm} \cdot \Big(\overline{\overline{N}_{r_{K,n}}(K) \smallsetminus E_{K,n}}\Big)\Big] \\
&\subseteq \mathcal{N}_n\big[S_e^{\pm} \cdot \overline{N}_{r_{K,n}}(K)\big] \\
&\subseteq \mathcal{N}_n\big[\overline{N}_{r'_n + r_{K,n}}\big(S_e^{\pm} \cdot K\big)\big] \\
&\subseteq \Big(\mathcal{N}_n\big[\overline{N}_{r'_n + r_{K,n}}\big(S_e^{\pm} \cdot K\big)\big] \smallsetminus E_{C,n}\Big) \cup E_{C,n} \\
&\subseteq \overline{N}_{r_{C,n} + r'_n + r_{K,n}}\big(S_e^{\pm} \cdot K\big) \cup E_{C,n}.
\end{aligned}$$

As the measure of the last term converges to $\nu\big(S_e^{\pm} \cdot K\big)$, we deduce

$$\nu\big(S_e^{\pm} \cdot K\big) \geq \limsup_{n \to \infty} \nu\Big(\mathcal{N}_n\big[S_e^{\pm} \cdot \overline{\mathcal{N}}_n\big[K \smallsetminus \overline{E}_{K,n}\big]\big]\Big)$$

and together with (5.2) this yields:

$$\frac{\nu(S_e^{\pm} \cdot K)}{\nu(K)} \geq \limsup_{n \to \infty} \frac{\nu\Big(\mathcal{N}_n\big[S_e^{\pm} \cdot \overline{\mathcal{N}}_n\big[K \smallsetminus \overline{E}_{K,n}\big]\big]\Big)}{\nu\Big(\mathcal{N}_n\big[K \smallsetminus \overline{E}_{K,n}\big]\Big)}. \tag{5.5}$$

By hypothesis, the partitions $\mathcal{P}_n$ have uniformly bounded measure ratios. That is, there exists a constant Q such that for any n and any $R, R' \in \mathcal{P}_n$ we have $\nu(R) \leq Q\nu(R')$. It follows that for every pair of sets $A, B \subseteq X$ we have an estimate

$$\frac{\nu\big(\mathcal{N}_n[A]\big)}{\nu\big(\mathcal{N}_n[B]\big)} \geq \frac{|\mathrm{V}_n(A)|}{|\mathrm{V}_n(B)|} Q^{-1}.$$

Now, the key point of the proof is that for any set $A \subseteq X$ we have

$$\mathrm{V}_n\Big(S_e^{\pm} \cdot \overline{\mathcal{N}}_n[A]\Big) = \mathrm{V}_n(A) \sqcup \partial\, \mathrm{V}_n(A).$$

Thus we get

$$\frac{\nu\big(\mathcal{N}_n\big[S_e^{\pm} \cdot \overline{\mathcal{N}}_n[A]\big]\big)}{\nu\big(\mathcal{N}_n[A]\big)} = 1 + \frac{\nu\Big(\mathcal{N}_n[S_e^{\pm} \cdot \overline{\mathcal{N}}_n[A]] \smallsetminus \mathcal{N}_n[A]\Big)}{\nu\big(\mathcal{N}_n[A]\big)} \geq 1 + \frac{\big|\partial\, \mathrm{V}_n(A)\big|}{|\mathrm{V}_n(A)|} Q^{-1}$$

and if we apply this inequality to the sets $A_n \coloneqq K \smallsetminus \overline{E}_{K,n}$, inequality (5.5) becomes

$$\frac{\nu(S_e^{\pm} \cdot K)}{\nu(K)} \geq 1 + \limsup_{n\to\infty} \frac{\big|\partial\, \mathrm{V}_n(A_n)\big|}{|\mathrm{V}_n(A_n)|} Q^{-1}.$$

It is hence enough to find a uniform bound $\alpha > 0$ such that

$$\limsup_{n\to\infty} \frac{\big|\partial\, \mathrm{V}_n(A_n)\big|}{|\mathrm{V}_n(A_n)|} \geq \alpha.$$

If $|\mathrm{V}_n(A_n)|$ is less than or equal to $|\mathrm{V}_n(X)|/2$ then

$$\frac{\big|\partial\, \mathrm{V}_n(A_n)\big|}{|\mathrm{V}_n(A_n)|} \geq \epsilon$$

by definition of Cheeger constant. If this is not the case, we need to use an argument similar to that of Lemma 5.2.3. That is, denote by W_n the complement set $W_n \coloneqq \mathrm{V}_n(X) \smallsetminus \mathrm{V}_n(A_n)$ and notice that

$$\partial\, \mathrm{V}_n(A_n) = \partial_{\mathrm{int}}(W_n) \supseteq \partial\big(W_n \smallsetminus \partial_{\mathrm{int}}(W_n)\big),$$

where $\partial_{\mathrm{int}}(W_n)$ denotes the interior vertex boundary, *i.e* the set of vertices of W_n which are endpoints of edges with one endpoint in the complement $\mathrm{V}_n(X) \smallsetminus W_n$. It follows that

$$|\partial\, \mathrm{V}_n(A_n)| \geq \epsilon\big|W_n \smallsetminus \partial_{\mathrm{int}}(W_n)\big| = \epsilon\big(|W_n| - |\partial\, \mathrm{V}_n(A_n)|\big).$$

Since $\lim_{n\to\infty} \nu\big(\mathcal{N}_n(A_n)\big) = \nu(K) \leq \frac{1}{2}\nu(X)$, we can once more use the bound on measure ratios to get that $|\mathrm{V}_n(A_n)| \leq Q|W_n|$ for n large enough. Thus we have

$$\begin{aligned}\limsup_{n\to\infty} \frac{|\partial \mathrm{V}_n(A_n)|}{|\mathrm{V}_n(A_n)|} &\geq \limsup_{n\to\infty} \frac{1}{|\mathrm{V}_n(A_n)|} \min\left\{\epsilon|\mathrm{V}_n(A_n)| \,,\, \frac{\epsilon}{1+\epsilon}|W_n|\right\} \\ &\geq \limsup_{n\to\infty} \min\left\{\epsilon \,,\, \frac{\epsilon}{Q(1+\epsilon)}\right\}\end{aligned}$$

as desired. □

Remark 5.2.8. Proposition 5.2.7 implies a fortiori that a uniform lower bound on the Cheeger constant on measure approximating graphs forces the action to be expanding in measure.

Definition 5.2.9. Given a partition $\mathcal{P}$ of a metric space, we define its *mesh* as the supremum of the diameters of its regions:

$$\mathrm{mesh}(\mathcal{P}) \coloneqq \sup\{\mathrm{diam}(R) \mid R \in \mathcal{P}\}.$$

Note that if $\mathcal{P}_n$ is a sequence of partitions with $\mathrm{mesh}(\mathcal{P}_n) \to 0$, then all the sets $E_{K,n}$ as in Proposition 5.2.7 can be assumed to be empty simply by letting $r_{K,n} \coloneqq \mathrm{mesh}(\mathcal{P}_n)$ for every compact set $K \subseteq X$. In particular, Proposition 5.2.7 immediately implies the following:

Corollary 5.2.10. *Given a continuous action $\Gamma \curvearrowright X$ on a locally compact metric space equipped with a Radon measure and a sequence of measurable partitions $\mathcal{P}_n$ with uniformly bounded measure ratios and $\mathrm{mesh}(\mathcal{P}_n) \to 0$; if the topology approximating graphs $\widetilde{\mathcal{G}}(\mathcal{P}_n)$ have Cheeger constant uniformly bounded away from 0 then the action is expanding in measure.*

Remark 5.2.11. It is easier to prove Corollary 5.2.10 directly rather than proving Proposition 5.2.7, but we decided to provide a more general statement that could be applied *e.g.* to metric spaces with cusps as well.

5.2.3 An 'if and only if' conclusion

We can combine Proposition 5.2.7 with Lemma 5.2.3 to deduce an if and only if characterisation of expansion in measure in terms of lower bounds of Cheeger constants.

Theorem 5.2.12. *Let $\Gamma \curvearrowright (X, d, \nu)$ be a continuous action on a metric space with a Radon measure and let $\mathcal{P}_n$ a family of finite measurable partitions of X with uniformly*

bounded measure ratios and such that for every compact set $K \subseteq X$ there is a decreasing sequence $r_{K,n} \to 0$ such that $r_{K,n} \to 0$ and $\nu\big(\overline{E}_{K,n}\big) \to 0$, where

$$E_{K,n} \coloneqq \bigcup \big\{ R \;\big|\; R \in \mathcal{P}_n,\ \overline{R} \cap K \neq \emptyset,\ \operatorname{diam}(R) > r_{K,n} \big\}$$

and $\overline{E}_{K,n}$ is its closure. Then, the following are equivalent:

(i) the action is expanding in measure;

(ii) all the measure approximating graphs $\mathcal{G}(\mathcal{P}_n)$ share a common lower bound on their Cheeger constant;

(iii) all the topology approximating graphs $\widetilde{\mathcal{G}}(\mathcal{P}_n)$ share a common lower bound on their Cheeger constant.

Moreover, if $\alpha > 0$ is the constant of expansion in measure, Q is the bound on the measure ratios, $\epsilon = \inf_{n\in\mathbb{N}} h\big(\mathcal{G}(\mathcal{P}_n)\big)$ and $\widetilde{\epsilon} = \inf_{n\in\mathbb{N}} h\big(\widetilde{\mathcal{G}}(\mathcal{P}_n)\big)$, then we have the following estimates:

$$\begin{aligned} \widetilde{\epsilon} &\geq \epsilon \\ \epsilon &\geq \min\left\{ \frac{\alpha}{Q}, \frac{\alpha}{(1+\alpha)Q^2} \right\} \\ \alpha &\geq \min\left\{ \frac{\widetilde{\epsilon}}{Q}, \frac{\widetilde{\epsilon}}{(1+\widetilde{\epsilon})Q^2} \right\}. \end{aligned}$$

Remark 5.2.13. We wish to remark that if it is already known that the approximating graphs share a uniform bound on their degrees, then one can modify the proofs of Lemma 5.2.3 and Proposition 5.2.7 in order to extend them to measurable actions of semigroups and groupoids. Indeed, the key point where we used the existence of inverses is when we find a lower bound on the size of the boundary of a set consisting of more than a half of the total number of vertices. There, we look at the complement and use the inverses in order to relate the sizes of the (external) boundaries of a set and its complement. This sort of bound can be proved more easily if one already knows that the degrees are bounded.

5.3 Bounds on degrees

In this section we prove that some fairly mild condition on the geometry of a metric space X, the partition $\mathcal{P}$ and the action $\Gamma \curvearrowright X$ are enough to imply that the topology (and hence measure) approximating graph $\widetilde{\mathcal{G}}(\Gamma \curvearrowright X; \mathcal{P})$ has bounded degree. In

particular, this will allow us to use the results of Section 5.2 to build families of expanders.

Recall that the doubling constant of a doubling metric measure space (X, d, ν) is the smallset D such that $\nu\big(B_x(2r)\big) \leq D\nu\big(B_x(r)\big)$ for every $x \in X$ and $r > 0$. Recall also that the eccentricity $\xi(A)$ of a bounded set $A \subset X$ is the infimum of the ratios $\frac{R}{r}$ such that $B_x(r) \subseteq A \subseteq B_x(R)$ for some $x \in A$; and that a homeomorphism is η-quasi-symmetric if

$$\frac{d\big(f(x), f(y)\big)}{d\big(f(x), f(z)\big)} \leq \eta\left(\frac{d(x,y)}{d(x,z)}\right)$$

for every choice of points $z \neq x \neq y$ in X (Subsection 2.1.4).

Definition 5.3.1. A measurable map $f\colon (X, \nu) \to (X, \nu)$ has *measure distortion* bounded by $\Theta \geq 1$ if

$$\frac{1}{\Theta}\nu(A) \leq \nu\big(f(A)\big) \leq \Theta\nu(A)$$

for every measurable set $A \subseteq X$.

An action $\rho\colon G \curvearrowright (X, d)$ has bounded measure distortion if $\rho(g)$ has bounded measure distortion for every $g \in G$. We do not require the bounds to be uniform.

We can now prove the following:

Proposition 5.3.2. *Let (X, d, ν) be a doubling measure space and $\mathcal{P}$ be a measurable partition with measure ratios bounded by $Q \geq 1$ and such that all the regions $R \in \mathcal{P}$ are bounded and have uniformly bounded eccentricity $\xi(R) \leq \xi$ for some constant $\xi > 0$.*

Given an action $\rho\colon \Gamma \curvearrowright (X, d, \nu)$ by quasi-symmetric maps with bounded measure distortion, let $\eta\colon [0, \infty) \to [0, \infty)$ and $\Theta \geq 1$ be such that the homeomorphism $\rho(s)$ is η-quasi-symmetric and has measure distortion bounded by Θ for every generator $s \in S_e^{\pm}$.

Then the topology approximating graph $\widetilde{\mathcal{G}}(\Gamma \curvearrowright X; \mathcal{P})$ has bounded degree and this bound depends only on η, Θ, Q, ξ and the doubling constant D.

Proof. Fix any $s \in S_e^{\pm}$ and, for any region $R \in \mathcal{P}$, let

$$J \coloneqq \big\{R_i \in \mathcal{P} \;\big|\; \overline{R}_i \cap s(\overline{R}) \neq \emptyset\big\}.$$

It is enough to prove that there is a uniform bound on $|J|$.

Since s is η-quasi-symmetric and $\xi(R) = \xi(\overline{R})$, the image $s(\overline{R})$ has eccentricity at most $\eta(\xi)$. Thus, for every $\delta > 0$ there are $x \in X$ and $0 < r_1 \leq r_2$ with $r_2 \leq (\eta(\xi)+\delta)r_1$ such that

$$B_x(r_1) \subseteq s(R) \subseteq s(\overline{R}) \subseteq B_x(r_2).$$

Since δ is arbitrarily small we will ignore it in the sequel.

We can then bound the measure $\nu\big(B_x(r_2)\big)$ due to the doubling condition on X:

$$\nu\big(B_x(r_2)\big) \leq D^{\lceil \log_2(r_2/r_1)\rceil}\nu\big(B_x(r_1)\big) \leq D^{\lceil \log_2(\eta(\xi))\rceil}\nu\big(s(R)\big)$$

where $\lceil t\rceil$ denotes the smallest integer $k \geq t$. Letting $L_1 = D^{\lceil \log_2(\eta(\xi))\rceil}$ we get

$$\nu\big(B_x(r_2)\big) \leq L_1\Theta\nu(R)$$

by bounded measure distortion.

Choose a finite subset $J' \subseteq J$ (a priori, J could still be infinite at this point). For an arbitrary $\delta > 0$, choose $r \geq 0$ such that $\overline{R}_i \subseteq B_x(r_2 + r + \delta)$ for every region $R_i \in J'$. Again, since δ is small we will ignore it in the sequel. By definition, there exists a region R_j (here $j \in J'$) with closure intersecting $s(\overline{R})$ non trivially and having diameter $\operatorname{diam}(R_j) \geq r$. Thus, there exists $y \in X$ with $B_y(r/2\xi) \subseteq R_j$.

Notice that

$$\nu\left(\coprod_{i\in J'} R_i\right) \leq \nu\big(B_y(2(r+r_2))\big) \leq D^{\left\lceil \log_2\left(\frac{2(r+r_2)2\xi}{r}\right)\right\rceil}\nu\big(B_y(r/2\xi)\big).$$

Letting $L_2(t) \coloneqq D^{\lceil \log_2(4\xi(1+t))\rceil}$ yields

$$\frac{|J'|}{Q}\nu(R_j) \leq \nu\big(B_y(2r+r_2)\big) \leq L_2(r_2/r)\nu\big(B_y(r/2\xi)\big) \leq L_2(r_2/r)\nu(R_j), \tag{5.6}$$

where the RHS is set as $+\infty$ if $r = 0$. As the RHS is a decreasing function of r, and r increases as $J' \subset J$ does, we deduce that there exists a number $r_3 \geq 0$ such that $\overline{R}_i \subseteq B_x(r_2 + r_3)$ for every region $R_i \in J$. Inequality (5.6) then implies

$$\frac{|J|}{Q} \leq L_2\Big(\frac{r_2}{r_3}\Big).$$

On the other hand we have

$$\nu\left(\coprod_{i\in J} R_i\right) \leq \nu\big(B_x(r_3 + r_2)\big) \leq D^{\left\lceil \log_2\left(\frac{(r_3+r_2)}{r_2}\right)\right\rceil}\nu\big(B_x(r_2)\big),$$

thus letting $L_3(t) = D^{\lceil \log_2(t+1)\rceil}$ we get

$$\frac{|J|}{Q}\nu(R) \leq L_3(r_3/r_2)\nu\big(B_x(r_2)\big) \leq L_3(r_3/r_2)L_1\Theta\nu(R).$$

Thus we conclude

$$\begin{aligned} |J| &\leq \min\Big\{QL_2\big(r_2/r_3\big), Q\Theta L_3\big(r_3/r_2\big)L_1\Big\} \\ &\leq \sup_{t>0}\left(\min\left\{QL_2\big(t\big), Q\Theta L_3\Big(\frac{1}{t}\Big)L_1\right\}\right) \end{aligned}$$

and the latter is bounded. □

Corollary 5.3.3. *If an action $\Gamma \curvearrowright X$ and a sequence of partitions $\mathcal{P}_n$ satisfy the hypotheses of Theorem 5.2.12 and Proposition 5.3.2, then the measure (or topology) approximating graphs $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{P}_n)$ form a family of expanders if and only if the action is expanding in measure.*

Proposition 5.3.2 is fairly general in that it deals with a large class of maps and measure spaces. Still, at times its hypotheses might be cumbersome to work with. It is especially so when one already knows that the action and the partition already satisfy much stronger hypotheses that make some of the requirements of Proposition 5.3.2 redundant. For example, it is easy to prove the following:

Remark 5.3.4. Let (X, d) be a metric space with a partition $\mathcal{P}$ so that there exists a function $\zeta\colon [0, \infty) \to \mathbb{N}$ such that for every point $x \in X$ and every radius r the ball $B_x(r)$ intersects at most $\zeta(r)$ regions of $\mathcal{P}$.

Then, if $f\colon X \to X$ is a L-Lipschitz map and $Y \subset X$ is any subset, then $f(Y)$ intersect at most $\zeta\big(L \operatorname{diam}(Y)/2\big)$ regions of $\mathcal{P}$. In particular, in this case a (uniform) bound on the diameter of the regions in $\mathcal{P}$ yields a (uniform) bound on their degree as vertices of the approximating graph.

5.4 Obstructions to coarse embeddings

In this section we study the (non) existence of uniform coarse embeddings of approximating graphs into Banach spaces.[2] In the context of measure preserving actions with spectral gap (Subsections 2.4.4 and 2.4.5), it will be easy to prove that such coarse embeddings cannot exist. In the sequel this will allow us to produce explicit families of superexpanders. Similar results have been independently obtained in [NS17] and [Saw17a].

N.B. The results of this section hold only for measure preserving actions on probability spaces.

Let $\Gamma \curvearrowright (X, \nu)$ be a measure preserving action on a probability space, $\mathcal{P}$ a finite measurable partition of X and E a Banach space. Staying true to our conventions, a function $\hat{f}$ from the (measure) approximating graph $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{P})$ to the Banach space E is actually a function defined on the vertex set—namely $\mathcal{P}$. In particular, $\hat{f}$ naturally induces a function $f\colon X \to E$ assigning to a point $x \in X$ the value of the region $R_x \in \mathcal{P}$ containing it (this is defined almost everywhere).

[2] This study is motivated from the fact that families of expanders cannot be coarsely embedded into any L^p-space (Theorem 2.7.9).

Note that the function f thus defined is a simple function on a probability space and hence belongs to the Bochner space $L^2(X, \nu; E)$ (Subsection 2.4.5) *i.e.* the norm $\|f\|_B = \int_X \|f\|_E d\nu$ is finite. Recall also that the measure-preserving action $\gamma \curvearrowright X$ induces a unitary action on $L^2(X, \nu; E)$ by pre-composition.

The proof of the next two lemmas are adapted to measure approximating graphs, but the same arguments work for topology approximating graphs as well.

Lemma 5.4.1. *If $\hat{f}$ is a coarse embedding with control functions ρ_- and ρ_+, then for every $s \in S$ we have $\|f - s \cdot f\|_B \leq \rho_+(1)$.*

Proof. Note that for almost every $x \in X$ if we let $R_x, R_{s\cdot x} \in \mathcal{P}$ be the regions containing x and $s \cdot x$, respectively, then $d(R_x, R_{s\cdot x}) \leq 1$. Therefore, we have

$$\begin{aligned}\left\|f - s^{-1} \cdot f\right\|_B^2 &= \int_X \|f(x) - f(s \cdot x)\|_E^2 \, d\nu(x) \\ &= \int_X \left\|\hat{f}\left(R_x\right) - \hat{f}\left(R_{s\cdot x}\right)\right\|_E^2 \, d\nu(x) \\ &\leq \int_X \left(\rho_+(1)\right)^2 \, d\nu = \left(\rho_+(1)\right)^2.\end{aligned}$$

□

Lemma 5.4.2. *If $\mathcal{P}$ has Q-bounded measure ratios, the degree of $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{P})$ is D, and $\hat{f}$ is a coarse embedding with control functions ρ_- and ρ_+, then (when the right-hand side is defined) we have*

$$\|f\|_B \geq \frac{1}{4} \rho_-\left(\frac{1}{\log(D)} \log\left(\frac{|\mathcal{P}|}{2Q}\right) - 1\right).$$

Proof. Let $C = \|f\|_B$. Note that the set $X_{2C} = \{x \in X \mid \|f(x)\|_E \leq 2C\}$ has measure $\nu(X_{2C}) > \frac{1}{2}$.

For any $r \geq 0$ and $R \in \mathcal{P}$, let $\mathcal{N}_r(R) \subseteq X$ denote the union of all the regions $R' \in \mathcal{P}$ with $d(R, R') \leq r$. Then it follows from our hypotheses that

$$\nu\left(\mathcal{N}_r(R)\right) = \frac{\nu\left(\mathcal{N}_r(R)\right)}{\nu(X)} \leq Q \frac{D^{r+1}}{|\mathcal{P}|}.$$

In particular, if we let

$$r = \frac{1}{\log(D)} \log\left(\frac{|\mathcal{P}|}{2Q}\right) - 1,$$

then $\nu(\mathcal{N}_r(R)) \leq \frac{1}{2}$.

Choose any region $R \subseteq X_{2C}$. By construction, there must exist another region $R' \subseteq X_{2C}$ with $d(R, R') > r$. Therefore, we have

$$\rho_-(r) \leq \rho_-\left(d(R, R')\right) \leq \left\|\hat{f}(R) - \hat{f}(R')\right\|_E \leq \left\|\hat{f}(R)\right\|_E + \left\|\hat{f}(R')\right\|_E \leq 4C,$$

whence the required inequality. □

Now, let $\rho_n \colon \Gamma_n \curvearrowright (X_n, \nu_n)$ be a sequence of probability measure preserving actions, let $\mathcal{P}_n$ be measurable partitions of X_n, all with Q-bounded measure ratios and such that all the measure (or topology) approximating graphs $\mathcal{G}(\Gamma_n \curvearrowright X_n; \mathcal{P}_n)$ have degree at most D. Combining Lemma 5.4.1 and Lemma 5.4.2 yields the following result.

Proposition 5.4.3. *If the actions $\rho_n \colon \Gamma_n \curvearrowright (X_n, \nu_n)$ of above all have E-spectral gap with a uniform constant $\epsilon > 0$ and $|\mathcal{P}_n| \to \infty$, then neither the measure approximating graphs $\mathcal{G}(\Gamma_n \curvearrowright X_n; \mathcal{P}_n)$ nor the topology approximating graphs $\widetilde{\mathcal{G}}(\Gamma_n \curvearrowright X_n; \mathcal{P}_n)$ can be uniformly coarsely embedded into E.*

Proof. Let $\hat{f}_n \colon \mathcal{P}_n \to E$ be any sequence of functions and $f_n \colon X_n \to E$ the measurable functions associated with them.

Note that, subtracting from $\hat{f}_n$ (and f_n) the average value $\int_X f_n d\nu \in E$ if necessary, we can assume that f_n has average 0 (*i.e.* belongs to $L^2_0(X, \nu; E)$) for every $n \in \mathbb{N}$.

The uniform E-valued spectral gap condition now implies that there is a $\delta > 0$ such that

$$\sum_{s \in S^{\pm}} \left\| f_n - s^{-1} \cdot f_n \right\|_B^2 \geq \delta \|f_n\|_B$$

for every $n \in \mathbb{N}$; from which we can conclude that the f_n's cannot be uniform coarse embeddings. Indeed, if they were coarse embeddings we would have $\|f_n\|_B \to \infty$ by Lemma 5.4.2, while Lemma 5.4.1 would imply that the left hand side is bounded by $|S^{\pm}|\rho_+(1)$. □

Remark 5.4.4. Since Cayley graphs can be realized as approximating graphs (see the discussion at the end of Subsection 6.1.2), we can use Proposition 5.4.3 to reprove the fact that Lafforgue expanders do not coarsely embed into any Banach space with non-trivial type (Subsection 2.8.3).

Indeed, if Λ is a group with Lafforgue's strong Banach property (T) (Subsection 2.5.7) then its actions on cosets sets Λ/Λ_i are all ergodic and hence have uniform E-valued spectral gap for every Banach space E of non-trivial type.

Chapter 6

Constructing expanding actions

In this chapter we exhibit concrete examples of expanding actions.

6.1 A spectral criterion for expansion

In the following, $\Gamma \curvearrowright (X, \nu)$ will be a probability measure preserving action of a finitely generated group.

6.1.1 Expansion vs. almost invariant vectors

Let $1 \leq p < \infty$ and recall that L^p_0 is the space of zero average L^p-integrable complex functions. In this subsection we prove that a measure preserving action is expanding in measure if and only if it has a spectral gap. This will be very useful, because the notion of spectral gap has been thoroughly studied and we will hence be able to use a number of sophisticated results as black boxed in order to produce expanders. We begin with a simple lemma:

Lemma 6.1.1. *For every function $f \in L^p_0(X)$ and every constant $c \in \mathbb{C}$ we have*

$$\big\|f + c\big\|_p \geq \frac{\|f\|_p}{2}.$$

Proof. Let g be any function in $L^p(X)$. Applying Jensen inequality we have

$$\left|\int_X g(x)d\nu(x)\right|^p \leq \left(\int_X |g(x)|d\nu(x)\right)^p \leq \int_X |g(x)|^p d\nu(x).$$

Denote by $\nu(g)$ the average $\int_X g(x)d\nu(x)$. Then we have:

$$\big\|g - \nu(g)\big\|_p \leq \big\|g\big\|_p + \|\nu(g)\|_p \leq 2\big\|g\big\|_p. \tag{6.1}$$

Now, for any constant c and any $f \in L^2_0$, the average $\nu(f + c)$ is equal to c. Thus inequality (6.1) reads as $\|f\|_p \leq 2\|f + c\|_p$. □

Recall from Subsection 2.4.4 that a probability measure preserving action $\rho\colon \Gamma \curvearrowright X$ induces a unitary representation $\pi_\rho\colon \Gamma \curvearrowright L^p_0(X)$ be precomposition, and that ρ has a spectral gap in L^p_0 if there exists a $\delta > 0$ such that

$$\sum_{s\in S^\pm} \|s\cdot f - f\| \geq \delta \|f\|$$

for every $f \in L^p_0$.

Proposition 6.1.2. *Let $\rho\colon \Gamma \curvearrowright (X,\nu)$ be a probability measure preserving action of a finitely generated group. Then, for any $1 \leq p < \infty$, the action ρ has a spectral gap in L^p_0 if and only it is expanding in measure.*

Proof. If the action is not expanding in measure, then there exists a sequence of measurable sets A_n with measure $\nu(A_n) \leq 1/2$ and $\nu(S^\pm_e \cdot A_n)/\nu(A_n) \to 1$. Looking at the symmetric difference, we deduce that $\nu\big((s\cdot A_n)\triangle A_n\big)/\nu(A_n) \to 0$ for every $s \in S^\pm$. Denote by $\mathbb{1}_{A_n}$ the indicator function of the set A_n and let $f_n(x) \coloneqq \mathbb{1}_{A_n}(x) - \nu(A_n)$. The sequence $\{f_n\}_{n\in\mathbb{N}}$ lies in $L^p_0(X)$ and we have

$$\|s\cdot f_n - f_n\|_p^p = \nu(A_n \smallsetminus s\cdot A_n) + \nu(s\cdot A_n \smallsetminus A_n) = \nu\big((s\cdot A_n)\triangle A_n\big)$$

while

$$\begin{aligned}\|f_n\|_p^p &= \nu(A_n)\big(1-\nu(A_n)\big)^p + \big(1-\nu(A_n)\big)\nu(A_n)^p \\ &\geq \nu(A_n)\big(1-\nu(A_n)\big)^p \\ &\geq \frac{1}{2^p}\nu(A_n).\end{aligned}$$

It follows that (f_n) is a sequence of almost invariant vectors in L^p_0.

For the converse implication, fix any $1 \leq p < \infty$. We need to show that if ρ is ϵ-expanding then there is a constant $\delta > 0$ so that for every function $f \in L^p_0(X)$ we have $\sum_{s\in S^\pm}\|s\cdot f - f\|_p \geq \delta\|f\|_p$. We prove it first for real valued functions $f \in L^p_0(X;\mathbb{R})$. By density, it is then enough to prove the statement for scale functions of the form

$$f(x) = \sum_{i=0}^{N} \alpha_i \mathbb{1}_{A_i}(x)$$

with $\alpha_i \in \mathbb{R}$ and $A_N \subseteq A_{N-1} \subseteq \cdots A_0$.

There exists a constant c such that both the set $\big\{x \;\big|\; f(x) > c\big\}$ and $\big\{x \;\big|\; f(x) < c\big\}$ have measure smaller or equal than $1/2$. Let $g \coloneqq f - c$, then by Lemma 6.1.1 we have that $\|g\|_p \geq \frac{1}{2}\|f\|_p$. Changing sign if necessary, we may assume that $\|g^+\|_p \geq \frac{1}{4}\|f\|_p$ where $g^+ = \max\{g, 0\}$.

Clearly $\|s \cdot f - f\|_p = \|s \cdot g - g\|_p \geq \|s \cdot g^+ - g^+\|_p$, thus we only need to find a lower bound for the latter. The function g^+ is still a scale function

$$g^+(x) = \sum_{i=0}^{n} \beta_i \mathbb{1}_{B_i}(x)$$

with $B_{i+1} \subseteq B_i$, but this time we can also assume $\beta_i > 0$ and $\nu(B_i) \leq 1/2$ for every $i = 0, \ldots, n$.

Since the B_i's are nested, we have

$$\begin{aligned} \big|s \cdot g^+ - g^+\big|(x) &\geq \sum_{i=0}^{n} \beta_i \big|\mathbb{1}_{B_i}(s^{-1}x) - \mathbb{1}_{B_i}(x)\big| \\ &\geq \sum_{i=0}^{n} \beta_i \big(\mathbb{1}_{B_i \cup sB_i}(x) - \mathbb{1}_{B_i}(x)\big) \\ &= h_s(x) - g^+(x) \end{aligned}$$

where

$$h_s(x) \coloneqq \sum_{i=0}^{n} \beta_i \mathbb{1}_{B_i \cup sB_i}(x).$$

Note that

$$\sum_{s \in S^\pm} \|h_s\|_p^p = \int_{\mathbb{R}^+} \sum_{s \in S^\pm} \nu\big(\{x \mid h_s(x)^p \geq r\}\big) dr.$$

Since the action is expanding we have

$$\sum_{s \in S^\pm} \nu\big(\{x \mid h_s(x)^p \geq r\}\big) \geq \big(\big|S^\pm\big| + \epsilon\big) \nu\big(\{x \mid g^+(x)^p \geq r\}\big);$$

thus we get:

$$\sum_{s \in S^\pm} \|h_s\|_p^p \geq \int_{\mathbb{R}^+} \big(\big|S^\pm\big| + \epsilon\big) \nu\big(\{x \mid g^+(x)^p \geq r\}\big) dr = \big(\big|S^\pm\big| + \epsilon\big) \big\|g^+\big\|_p^p$$

whence we deduce that there exists $s \in S^\pm$ such that $\|h_s\|_p \geq \big(1 + \epsilon/|S^\pm|\big)^{1/p} \|g^+\|_p$.

Let $\delta' \coloneqq \big(1 + \epsilon/|S^\pm|\big)^{1/p} - 1$, then for the same $s \in S^\pm$ we have

$$\big\|s \cdot g^+ - g^+\big\|_p \geq \big\|h_s - g^+\big\|_p \geq \|h_s\|_p - \big\|g^+\big\|_p \geq \delta' \big\|g^+\big\|_p.$$

and thus we obtain:

$$\|s \cdot f - f\|_p \geq \big\|s \cdot g^+ - g^+\big\|_p \geq \delta' \big\|g^+\big\|_p \geq \frac{\delta'}{4} \|f\|_p.$$

That is, f is not an almost-invariant vector. A fortiori, we obtain the desired inequality: $\sum_{s \in S^\pm} \|s \cdot f - f\|_p \geq \frac{\delta'}{4} \|f\|_p$.

To finish the proof of the proposition, we now need to deal with complex valued functions. Let $f(x) = f_1(x) + if_2(x)$ be the decomposition of f into its real and imaginary part. Both f_1 and f_2 have zero average and thus belong to $L^p_0(X;\mathbb{R})$. Note also that the action respects the decomposition into the real and imaginary part: $s \cdot f = s \cdot f_1 + i(s \cdot f_2)$. Then we have

$$\begin{aligned}
\|s \cdot f - f\|_p &= \Big[\int_X \big((s \cdot f_1 - f_1)^2 + (s \cdot f_2 - f_2)^2\big)^{\frac{p}{2}} d\nu\Big]^{\frac{1}{p}} \\
&\geq \Big[\int_X \Big(\frac{1}{2}\big(|s \cdot f_1 - f_1| + |s \cdot f_2 - f_2|\big)^2\Big)^{\frac{p}{2}} d\nu\Big]^{\frac{1}{p}} \\
&\geq \frac{1}{\sqrt{2}}\Big[\int_X |s \cdot f_1 - f_1|^p + |s \cdot f_2 - f_2|^p d\nu\Big]^{\frac{1}{p}} \\
&\geq \frac{1}{\sqrt{2}}\Big(\|s \cdot f_1 - f_1\|_p^p + \|s \cdot f_2 - f_2\|_p^p\Big)^{\frac{1}{p}} \\
&\geq \frac{1}{\sqrt{2}} 2^{-\frac{p-1}{p}} \Big(\|s \cdot f_1 - f_1\|_p + \|s \cdot f_2 - f_2\|_p\Big).
\end{aligned}$$

For the last step we used the inequality $|a+b|^p \leq 2^{p-1}(|a|^p + |b|^p)$. Thus we obtain

$$\begin{aligned}
\Big(\sum_{s \in S^{\pm}} \|s \cdot f - f\|_p\Big) &\geq \frac{1}{\sqrt{2}} 2^{-\frac{p-1}{p}} \sum_{s \in S^{\pm}} \big(\|s \cdot f_1 - f_1\|_p + \|s \cdot f_2 - f_2\|_p\big) \\
&\geq \frac{\delta'}{4\sqrt{2}} 2^{-\frac{p-1}{p}} \big(\|f_1\|_p + \|f_2\|_p\big) \\
&\geq \frac{\delta'}{4\sqrt{2}} 2^{-\frac{p-1}{p}} \|f\|_p
\end{aligned}$$

as desired.

□

6.1.2 Some remarks and consequences

In this subsection we collect some consequences of Proposition 6.1.2. To begin with, note that its proof provides explicit bounds on the expansion constant ϵ in terms of the spectral gap constant δ and vice versa.

Moreover we obtain as a corollary an elementary proof of Lemma 2.4.7 in the special case of actions of finitely generated groups:

Corollary 6.1.3. *The existence of a spectral gap in L^p_0 for a probability measure preserving action of a finitely generated group $\Gamma \curvearrowright X$ does not depend on $1 \leq p < \infty$.*

It was a classical problem (attributed to Ruziewicz) to decide whether the Lebesgue measure was the only finitely additive probability measure on $\mathbb{S}^{n-1}$ invariant under the

action of $\mathrm{SO}(n,\mathbb{R})$ for $n \geq 3$. This was solved affirmatively ([Mar80, Sul81, Dri84]) using the following:

Theorem 6.1.4 (Rosenblatt [Ros81], Schmidt [Sch81])**.** *Given a probability measure preserving action of a countable group $\rho\colon \Gamma \curvearrowright (X,\nu)$ and fixed any $1 \leq p < \infty$, ν is the unique Γ-invariant finitely additive probability measure on X if and only if ρ has a spectral gap in $L^p_0(X,\nu)$.*

Combining this result with Proposition 6.1.2 we obtain the following:

Corollary 6.1.5. *A probability measure preserving action of a finitely generated group Γ is expanding in measure if and only if it admits a unique finitely additive Γ-invariant probability measure.*

Remark 6.1.6. There is a one-to-one correspondence between finitely additive (invariant) probability measures on X and (invariant) *means* on $L^\infty(X)$.

Finally, Proposition 6.1.2 provides us with a proof of Theorem 2.8.3. Let Λ_i be a sequence of finite index subgroups of $\Lambda = \langle S \rangle$ with increasing index. Endow the set of left cosets Λ/Λ_i with the uniform probability measure. Then Λ acts on Λ/Λ_i by left multiplication and, considering the complete partition, we have that the approximating graph $\mathcal{G}(\Lambda \curvearrowright \Lambda/\Lambda_i)$ coincides with the right Schreier graph $\overline{\mathrm{Schr}}^{\mathrm{r}}(\Lambda/\Lambda_i, S)$.

Considering the sequence of actions $\rho_i\colon \Lambda \curvearrowright \Lambda/\Lambda_i$, it follows from Proposition 6.1.2 and Lemma 5.2.3 that the approximating graphs are expanders if and only if[1] all those actions have a uniform spectral gap; which is the statement of Theorem 2.8.3.

Remark 6.1.7. Here we used the uniform aspects of our results to characterise expanders. If one preferred not to do so and to stick with the formalism of a unique action on a measure space (*e.g.* in the hope to apply Proposition 5.2.7), then one could proceed as follows: if the groups Λ_i form a normal filtration (are normal and nested), then the right Schreier graphs can be obtained as graphs approximating the action on the profinite limit $\varprojlim \Lambda/\Lambda_i$ equipped with the probability measure assigning to a Λ_i-coset (seen as a subset of $\varprojlim \Lambda/\Lambda_i$) probability $1/[\Lambda : \Lambda_i]$. Expansion can hence by checked by studying the action $\Lambda \curvearrowright \varprojlim \Lambda/\Lambda_i$.

[1] For the proof of the 'only if' part of the statement it is necessary to use the uniform bound on the spectral gap in term of the expansion constant. This should be used together with the fact that in this setting it is clear that a lower bound to the Cheeger constant is (uniformly) equivalent to a lower bound on the expansion constant.

6.2 Explicit examples via Kazhdan sets

We can now capitalize over the spectral criterion from Section 6.1 to produce a number of concrete examples of expanding actions using the machinery of Kazhdan sets and Kazhdan property (T).

In the following, let G be a locally compact second countable Hausdorff topological group. We will use facts and conventions from Section 2.5.

6.2.1 Expansion and Kazhdan pairs

Let S be a finite subset of the topological group G and denote by $\Gamma \coloneqq \langle S \rangle$ the subgroup of G generated by S. If (X, ν) is a probability space and $\rho\colon G \curvearrowright X$ is a measure preserving action, we can investigate expansion properties of the restriction of ρ to Γ and we obtain the following:

Proposition 6.2.1. *The following are equivalent:*

(i) *the restriction $\rho|_\Gamma\colon \Gamma \curvearrowright (X, \nu)$ is expanding in measure;*

(ii) *the representation $\pi_\rho|_\Gamma\colon \Gamma \curvearrowright L^2_0(X)$ does not weakly contain the trivial representation I_Γ;*

(iii) *there exists a constant $\epsilon > 0$ such that (S, ϵ) is a Kazhdan pair for the representation $\pi_\rho\colon G \curvearrowright L^2_0(X)$.*

Proof. By Proposition 6.1.2 we know that $\rho|_\Gamma$ is expanding in measure if and only if $\pi_{\rho|\Gamma}\colon \Gamma \curvearrowright L^2_0(X)$ has a spectral gap. Note that happens if and only if S is a Kazhdan set for $\pi_{\rho|\Gamma}$ (Remark 2.5.21).

Moreover, we know that when $\pi_{\rho|\Gamma}$ admits a Kazhdan set, then S must be a Kazhdan set as well because it generates Γ (Remark 2.5.23). Therefore, *(i)⇔(ii)* follows from Lemma 2.5.19.

To prove *(i)⇔(iii)* it is now enough to note that, since $\pi_{\rho|\Gamma} = (\pi_\rho)|_\Gamma$ and $S \subseteq \Gamma$, we have have that (S, ϵ) is a Kazhdan pair for $\pi_{\rho|\Gamma}$ if and only if it is a Kazhdan pair for π_ρ as well. □

Remark 6.2.2. When (S, ϵ) is a Kazhdan pair for π_ρ, one can retrieve explicit bounds on the expansion constant of ρ in term of the Kazhdan constant ϵ and vice versa following the proof of Proposition 6.1.2.

In the rest of this section we describe some consequences of Proposition 6.2.1 (we refer the reader to [CG11, Section 2] for more examples of actions of finitely generated groups on measure spaces that have a spectral gap).

6.2.2 A characterisation of Kazhdan sets

Schmidt, Connes and Weiss characterised groups with Kazhdan's property (T) in term of their ergodic actions. Specifically, they proved that G has property (T) if and only if for every measure preserving ergodic action on a probability space $\rho\colon G \curvearrowright (X, \nu)$ the induced unitary representation $\pi_\rho\colon G \curvearrowright L^2_0(X)$ admits a Kazhdan pair.

Using Remark 2.5.25, one can adapt the proof of the Schmidt-Connes-Weiss theorem given in [BdlHV08, Theorem 6.3.4] to prove the following more precise statement:

Fact. *Let G be a locally compact Hausdorff second countable group. Then a compact subset $K \subseteq G$ is a Kazhdan set of G if and only it is a Kazhdan set for every representation π_ρ induced from an ergodic action $\rho\colon G \curvearrowright (X, \nu)$.*

Therefore, Proposition 6.2.1 implies the following:

Theorem 6.2.3. *Let $S \subset G$ be a finite subset of a locally compact Hausdorff second countable group and let $\Gamma := \langle S \rangle \subset G$. Then, S is a Kazhdan set of G if and only if the restriction to Γ of every ergodic action $\rho\colon G \curvearrowright (X, \nu)$ is expanding in measure.*

Moreover, when S is a Kazhdan set of G, all the Γ-actions obtained as restrictions of ergodic G-actions share a lower bound on their expansion constants depending only on the Kazhdan constant of S in G.

Remark 6.2.4. Note that (the easy implication of) Theorem 6.2.3 implies in particular that the restriction of an ergodic action of G to a subgroup Γ generated by a finite Kazhdan set is again ergodic.

6.2.3 Non-compact Lie groups

In the setting of non-compact Lie groups, the work of Y.Shalom provides very general means of proving spectral gap properties. In particular, he proved the following.

Theorem 6.2.5 ([Sha00], Theorem C)**.** *Let $G = \prod_{i=1}^n G_i$ be a semisimple Lie group with finite centre and let π be a unitary G-representation and $H < G$ a closed non-amenable subgroup. If either of the following is true:*

(*a*) *$I_{G_i} \not\prec \pi|_{G_i}$ for every simple factor G_i;*

(*b*) *$I_G \not\prec \pi$ and for every $i = 1, \dots, n$ the closure of the projection of H on G_i is not an amenable subgroup;*

then $I_H \not\prec \pi|_H$. Moreover, if μ is a measure on G such that the closure of the group generated by the support $\mathrm{supp}(\mu)$ *satisfies* (a) *or* (b)*, then the spectral radius of $\pi(\mu)$ is strictly less than* 1.[2]

Remark 6.2.6. Note that the condition $I_{G_i} \not\prec \pi|_{G_i}$ for every $i = 1, \dots, n$ is stronger than $I_G \not\prec \pi$ because almost invariant vectors for π are a fortiori almost invariant vectors of $\pi|_{G_i}$ for each G_i.

As a corollary one can produce a multitude of examples of expanding actions: let $\rho\colon G \curvearrowright (X, \nu)$ be a measure preserving action of a semisimple Lie group with finite centre and let $\pi\colon G \curvearrowright L^2_0(X)$ be the induced unitary representation. Assume that $I_G \not\prec \pi|_{G_i}$ for every G_i (resp. $I_G \not\prec \pi$) and that $S \subset G$ is a finite set such that the closure of $\Gamma \coloneqq \langle S \rangle$ in G is not amenable (resp. the closures of the projections of Γ to the G_i's are not amenable), and consider the probability measure $\mu_{S^\pm} \coloneqq \frac{1}{|S^\pm|} \sum_{s \in S^\pm} \delta_s$.

The support of $\mu_{S^\pm}$ is precisely $S^\pm$ and hence it generates the group $\Gamma < G$—which we are assuming to satisfy the hypotheses of Theorem 6.2.5. In particular, it follows from said theorem that the averaging operator $\pi(\mu_{S^\pm})$ has spectral radius strictly less than 1. Since the set $S^\pm$ is symmetric, it is easy to check that $\pi(\mu_{S^\pm})$ is a self-adjoint operator and hence its operator norm is equal to the spectral radius. In particular, we have $\|\pi(\mu_{S^\pm})\| < 1$ and hence S is a Kazhdan set for π (Proposition 2.5.22). Then the restriction $\rho|_\Gamma\colon \Gamma \curvearrowright (X, \mu)$ is expanding in measure by Proposition 6.2.1.[3]

[2] Some extra explaination is in order. The *support* of a Borel measure on a topological space is the set of points such that have a basis of neigbourhoods of positive measure

$$\mathrm{supp}(\mu) = \{x \in X \mid \mu(U) > 0 \text{ for every open neighbourhood } x \in U \subset X\}.$$

The support is a closed set, and if X is lcoally compact Hausdorff and μ is Radon, then its complement $X \smallsetminus \mathrm{supp}(\mu)$ has measure 0 (in this case $\mathrm{supp}(\mu)$ can be defined as the smallest closed set satisfying this condition).

In Shalom's wording, to deduce that the spectral radius of $\pi(\mu)$ is less than 1 it is enough that "μ is not supported on a closed amenable subgroup [of G]" or that "the projection of μ to every simple factor is not contained on a closed amenable subgroup".

In order to obtain the statement of Theorem 6.2.5 from Shalom's original theorem, one needs to note that if $H < G$ is a closed subgroup and μ is supported in H, then $\mathrm{supp}(\mu) \subset H$ because $\mathrm{supp}(\mu)$ is the smallest closed set where μ is supported. Therefore H contains the subgroup generated by $\mathrm{supp}(\mu)$ and its closure $\overline{\langle \mathrm{supp}(\mu) \rangle}$. Closed subgroups of amenable groups are amenable, and therefore we deduce that when $\overline{\langle \mathrm{supp}(\mu) \rangle}$ is non-amenable then μ is not supported on a closed amenable subgroup of G. Similarly, if the projection of μ to a factor G_i is supported on a closed group H, then H must contain the closure of the projection of (the closure of) the subgroup generated by $\mathrm{supp}(\mu)$.

[3] For our applications we really need the 'moreover' statement of Theorem 6.2.5. In fact, if we try to use only the statement concerning weak containements we deduce that, letting $H \coloneqq \overline{\Gamma} < G$, we have $I_H \not\prec \pi|_H$. Still, this does not (a priori) imply that $I_\Gamma \not\prec \pi|_\Gamma$ (which is what we need in order to apply Proposition 6.2.1).

Example 6.2.7. If a simple Lie group G has Kazhdan property (T) and finite centre and $\rho\colon G \curvearrowright (X,\nu)$ is any measure preserving ergodic action then $I_G \not\prec \pi|_G$. If S generates a discrete subgroup $\Gamma < G$, then $\Gamma = \overline{\Gamma}$ and hence $\rho|_\Gamma$ is expanding as soon as Γ is not amenable. Note that, by the Tits Alternative Theorem, Γ is non-amenable if and only if it contains a non-abelian free subgroup. A typical example of ergodic action of G is the action by left multiplication $G \curvearrowright G/\Lambda$ where $\Lambda < G$ is a lattice. More generally, if $G < G'$ where G' is a finite product of connected, non compact, simple Lie groups with finite centre and Λ is any irreducible lattice of G', then Moore's Ergodicity Theorem implies that the action by left multiplication $G \curvearrowright G'/\Lambda$ is ergodic if and only if the closure of G in G' is not compact.

Remark 6.2.8. We wish to note (for later use), that any right invariant Riemannian metric on G' descends to a Riemannian metric on G'/Λ whose volume form is (a multiple of) the restriction of the Haar measure and the action on the left $G' \curvearrowright G'/\Lambda$ is by bilipschitz diffeomorphisms. In the example above, it follows that if the irreducible lattice Λ is also uniform, then the actions $\Gamma \curvearrowright G'/\Lambda'$ are nice actions on compact manifold and will hence allow us to construct expanders.

Remark 6.2.9. In the above example, we assumed G to have property (T), but the subgroup Γ does not need to have it (nor does G'). Indeed, taking Γ to be any discrete non-abelian free group would do. This is a very interesting feature, as it allows us to build expanders out of actions of free groups and, more generally, of a-T-menable groups.

In the same paper, Shalom constructed explicitly finite Kazhdan sets for algebraic groups and he was also able to compute their Kazhdan constants [Sha00, Theorem A]. More precisely, he finds Kazhdan sets of m elements whose Kazhdan constant is

$$\epsilon = \sqrt{2 - 2(\sqrt{2m-1}/m)}$$

(and we already noted (Remark 6.2.2) that these estimates immediately translate in estimates for the Cheeger constants of the approximating graphs).

Remark 6.2.10. As a concrete example, Shalom proves that the matrices

$$\left(\begin{array}{cc|c} 1 & 2 & \\ 0 & 1 & 0 \\ \hline & 0 & I_{n-2} \end{array}\right), \left(\begin{array}{cc|c} 1 & 0 & \\ 2 & 1 & 0 \\ \hline & 0 & I_{n-2} \end{array}\right)$$

form a Kazhdan set of two elements for $\mathrm{SL}(n,\mathbb{R})$ for every $n \geq 3$.

6.2.4 Compact Lie groups

Theorem 6.2.5 can only be applied to non-compact Lie groups (because compact Lie groups are amenable). Still, the case of compact Lie groups is all but devoid of interest. In fact, one can show that every simple, connected, non-abelian, compact Lie group admits finite Kazhdan sets [Sha99, Theorem 5.17].

This immediately provides us with examples of actions that are expanding in measure. Indeed for any compact Lie group G equipped with its Haar measure ν we can consider its action on itself by left multiplication. Since this action is ergodic, it follows that for every group $\Gamma < G$ generated by a finite Kazhdan set S of G, the action $\Gamma \curvearrowright (G, m)$ is expanding in measure. It is interesting to note that in this case Lemma 2.5.27 implies that the converse is also true:

Corollary 6.2.11. *Let G be a compact Lie group and $\Gamma = \langle S \rangle < G$ a finitely generated subgroup. Then the action $\Gamma \curvearrowright G$ is expanding in measure if and only if S is a Kazhdan set of G.*

Explicit examples are provided by Bourgain and Gamburd in [BG07]. There they prove that if k elements $g_1, \dots, g_k \in \mathrm{SU}(2, \mathbb{C})$ generate a free subgroup of $\mathrm{SU}(2, \mathbb{C})$ and they satisfy a non-abelian Diophantine property then they form a Kazhdan set of $\mathrm{SU}(2, \mathbb{C})$. In particular, they show that when two matrices with algebraic entries $a, b \in \mathrm{SU}(2, \mathbb{C}) \cap \mathrm{GL}(2, \overline{\mathbb{Q}})$ freely generate a free group $\Gamma < \mathrm{SU}(2, \mathbb{C})$, then every ergodic action of $\mathrm{SU}(2, \mathbb{C})$ restricts to an expanding action of Γ.

An obvious example of an ergodic action of $\mathrm{SU}(2, \mathbb{C})$ on a compact space is the action by left multiplication of $\mathrm{SU}(2, \mathbb{C})$ on itself. Alternatively, note that $\mathrm{SU}(2, \mathbb{C})$ is the double cover of $\mathrm{SO}(3, \mathbb{R})$ and the action of the latter on the sphere $\mathbb{S}^2$ is ergodic. Thus, we obtain the following:

Corollary 6.2.12. *Let a and b be two independent rotations of $\mathbb{S}^2$ whose matrices have algebraic entries and let $F_2 = \langle a, b \rangle$ be the generated subgroup of $\mathrm{SO}(3, \mathbb{R})$. Then the action $F_2 \curvearrowright \mathbb{S}^2$ is expanding in measure.*

Remark 6.2.13. The existence of actions by rotations on the sphere $\mathbb{S}^2$ that are expanding in measure has already been successfully used in relation to the Ruziewicz problem and to the problem of constructing finite equidistributed subsets of $\mathbb{S}^2$ (see [Lub10] and Subsection 6.1.2).

The results of [BG07] have been later extended to $\mathrm{SU}(n)$ for any $n \geq 2$ in [BG10] and subsequently to all compact simple Lie groups in [BdS14]. These works build

on the notion of *non-abelian Diophantine property* introduced in [GJS99] in order to study spectral gap properties for generic subgroups of rotations.

For a generic k-tuple of elements in $\mathrm{SU}(2,\mathbb{C})$, it is known that the action on $\mathbb{S}^2$ of the generated subgroup $\Gamma < \mathrm{SU}(2,\mathbb{C})$ is ergodic; and it is conjectured in [GJS99] that the action of Γ should also have a spectral gap (which is a much stronger property). A partial result is due to Fisher [Fis06] who managed to prove that if the conjecture is false then the set of k-tuples inducing actions with spectral gap must have null measure. It is unknown whether the group generated by a generic k-tuple has the non-abelian Diophantine property (an affirmative answer to the latter would clearly imply the conjecture).

More generally, it is unknown whether the action by left multiplication $\Gamma \curvearrowright G$ of a generic finitely generated dense subgroup Γ of a compact simple Lie group G has a spectral gap.

6.3 Some actions with Banach spectral gap

In this section we briefly wish to provide some very concrete example of actions on nice measure spaces that have E-valued spectral gap for every uniformly convex Banach space. This will be used in conjunction with Proposition 5.4.3 to produce explicit examples of superexpanders.

The idea is to mimic what is done with Kazhdan sets. That is, we showed already that any ergodic measure preserving action of a finitely generated group with Kazhdan property (T) must be expanding in measure. Similarly, if one can find an ergodic action of a group with Lafforgue's strong Banach property (T), then Proposition 2.5.30 will directly imply that such action must have the required Banach valued spectral gap (it actually implies that it has E-valued spectral gap for every Banach space E of non-trivial type).

In order to find examples of such actions, we use the approach that Margulis used to solve the Banach-Ruziewicz problem [Mar80].

Lemma 6.3.1. *For $d \geq 5$, let Γ_d consist of matrices in $\mathrm{SO}(d,\mathbb{R})$ whose entries are elements of $\mathbb{Z}[\frac{1}{5}]$ (the subring of $\mathbb{Q}$ generated by the element $\frac{1}{5}$). Then Γ_d is an infinite group with Lafforgue's strong Banach property (T).*

Proof. Consider the diagonal embedding of Γ_d into $G_d = \mathrm{SO}(d,\mathbb{Q}_5) \times \mathrm{SO}(d,\mathbb{R})$. Then Γ_d is a cocompact lattice in G_d [Bor63].

Whenever $d \geq 5$, $\mathrm{SO}(d, \mathbb{Q}_5)$ is an almost simple algebraic group of higher rank (see [Mar80]) and hence it has Lafforgue's strong Banach property (T) (see [Lia14]). This implies that also G_d has strong Banach property (T), since $\mathrm{SO}(d, \mathbb{R})$ is a compact group. Lafforgue proved that strong Banach property (T) passes to cocompact lattices [Laf08], which implies that Γ_d has strong Banach property (T).

Moreover, Γ_d is infinite because it is a lattice in $\mathrm{SO}(d, \mathbb{Q}_5)$, which is a group of higher rank. □

The nice thing with Γ_d is that it sits inside a compact Lie group and can hence be used to produce measure expanding actions on Riemannian manifolds:

Lemma 6.3.2. *Every continuous measure preserving ergodic action of* $\mathrm{SO}(d, \mathbb{R})$ *with* $d \geq 5$ *on a compact Riemannian manifold* M *restricts to an action of* Γ_d *that has* E*-valued spectral gap for every Banach space* E *of non-trivial type.*

Proof. The group Γ_d is in fact a dense subgroup of $\mathrm{SO}(d, \mathbb{R})$ (this is true for every $d \geq 2$ and can be proved by induction). We claim that $G \curvearrowright M$ is a continuous ergodic action of a Lie group on a Riemannian manifold and $\Gamma < G$ is a dense subgroup, then the restriction of the action to Γ must be ergodic.

Indeed, assume by contradiction that $A \subseteq M$ s a Γ-invariant subset that has measure $0 < \nu(A) < \nu(M)$. Since the action is measure preserving, we deduce by ergodicity that there exists a $g \in G$ such that $\nu(g(A) \cap A) < \nu(A) - \epsilon$ for some ϵ small enough.

Since ν is a Radon measure, for every $\delta > 0$ there exists a compact set $K \subset A$ with $\nu(K) > \nu(A) - \delta$. Since $g(K)$ is compact, we have

$$\nu\big(N_r(g(K)) \cap K\big) \xrightarrow{r \to 0} \nu\big(g(K) \cap K\big) \leq \nu\big(g(A) \cap A\big),$$

and we can therefore choose a radius r small enough so that $\nu\big(N_r(g(K)) \cap K\big) \leq \nu\big(g(A) \cap A\big) + \frac{\epsilon}{2}$.

Since Γ is dense in G, the action is continuous and K is compact, there exists a $\gamma \in \Gamma$ close enough to $g \in G$ such that $d(\gamma(x), g(x)) < r$ for every $x \in K$. In particular we have $\gamma(K) \subseteq N_r(g(K))$. We thus obtain a chain of inequalities

$$\begin{aligned}
\nu(\gamma(A) \cap A) &\leq \nu(\gamma(K) \cap K) + 2\delta \\
&\leq \nu\big(N_r(g(K)) \cap K\big) + 2\delta \\
&\leq \nu(\gamma(A) \cap A) + \frac{\epsilon}{2} + 2\delta \\
&\leq \nu(A) - \frac{\epsilon}{2} + 2\delta.
\end{aligned}$$

If we had chosen δ to be small enough, we would deduce that $\nu(\gamma(A) \cap A) < \nu(A)$, against the assumption of Γ-invariance for A.

We have thus proved that the action $\Gamma_d \curvearrowright M$ is ergodic. It follows that this action must have the required Banach spectral gap because Γ_d has strong property (T) (Proposition 2.5.30).

□

To find examples of ergodic actions on Riemannian manifolds it is enough to equip $\mathrm{SO}(d, \mathbb{R})$ with any left-invariant Riemannian metric. The action by left translation $\mathrm{SO}(d, \mathbb{R}) \curvearrowright \mathrm{SO}(d, \mathbb{R})$ is clearly ergodic. Also the natural action by rotations on the sphere $\mathrm{SO}(d, \mathbb{R}) \curvearrowright \mathbb{S}^{d-1}$ is ergodic. Thus we obtain:

Corollary 6.3.3. *For every $d \geq 5$, the actions $\Gamma_d \curvearrowright \mathrm{SO}(d, \mathbb{R})$ and $\Gamma_d \curvearrowright \mathbb{S}^{d-1}$ have E-spectral gap for every Banach space E with trivial type.*

See [FNvL17] for (uncountably many) more examples of actions of groups on manifolds that have Banach valued spectral gap.

Chapter 7

Constructing appropriate partitions

In this chapter we introduce two fairly general strategies for producing examples of metric measure spaces with good partitions where we can apply the results obtained so far.

7.1 Voronoi tessellations

A convenient way for defining sufficiently regular measurable partitions on a metric space (X, d) is given by the Voronoi tessellations.

7.1.1 Voronoi partitions and nets

Let Y be a countable discrete subset of a metric space X.

Definition 7.1.1. The *Voronoi tessellation* associated with $Y \subset X$ is the family $\mathcal{V}(Y) \coloneqq \{R(y) \mid y \in Y\}$ where where $R(y)$ is the set of points of X that are closer to y than to any other point of Y:

$$R(y) \coloneqq \big\{x \in X \;\big|\; d(x,y) < d(x,y') \text{ for all } y' \in Y,\ y' \neq y\big\}.$$

Remark 7.1.2. If the metric space X is nice enough, the regions $R(y)$ will be disjoint open sets. In general these regions need not be open (*e.g.* X is not proper). Still, since Y is countable, they will always be measurable sets.

Let X be equipped with a Borel measure ν. In case that for any couple of points $y \neq y'$ of Y the hyperplane $P(y,y') \coloneqq \{x \in X \mid d(x,y) = d(x,y')\}$ has measure zero, the Voronoi tessellation $\mathcal{V}(Y)$ covers a conull subset of X and it is therefore a measurable partition of X.

Now the hope is that, if the space X is reasonable enough and Y is nicely distributed, the Voronoi tessellation should provide us with nice measurable partitions to which we can apply the approximating graph construction to produce expanders.

Assume that Y is an (r, ϵ)-net (Subsection 2.1.3). It follows that for every $y \in Y$ the region $R(y)$ is contained in the ball $B(y, r)$ and contains the ball $B\big(y, \frac{\epsilon}{2}\big)$. If we know that the space X satisfies some sort of 'uniform doubling' condition, so that the ratios $\nu(B(x, r))/\nu(B_x, \epsilon)$ are bounded in function of r/ϵ, we can then apply Theorem 5.2.12 to any sequence of Voronoi tessellations $\mathcal{V}(Y_n)$ where the Y_n are (r_n, ϵ_n)-nets such that $r_n \to 0$ and r_n/ϵ_n is bounded (*e.g.* r_n-nets).

Moreover, if $\Gamma \curvearrowright X$ is an action by Lipschitz maps and the space X has bounded geometry (Definition 2.1.10), then we would also immediately obtain that the approximating graphs $\mathcal{G}\big(\Gamma \curvearrowright X; \mathcal{V}(Y_n)\big)$ and $\widetilde{\mathcal{G}}\big(\Gamma \curvearrowright X; \mathcal{V}(Y_n)\big)$ have bounded degrees (Remark 5.3.4). Note that in the general case the bound on the degree depends on the function f_ϵ of the definition of bounded geometry. Still, for well behaved spaces (*e.g.* doubling) one expects to be able to construct uniform bounds.

7.1.2 Regularity of Voronoi partitions of manifolds

Typical examples of well-behaved metric spaces are the Riemannian manifolds. Let (M, ϱ) be a Riemannian manifold with its Riemannian metric and volume (Section 2.3).

To begin with, note that in the setting of complete Riemannian manifolds Voronoi tessellations do provide us with measurable partitions. In fact, every hyperplane $P(y, y') \coloneqq \{x \in X \mid d(x, y) = d(x, y')\}$ has measure 0 by Lemma 2.3.11.

Let $Y_n \subset M$ be a sequence of (r_n, ϵ_n)-nets with $r_n \to 0$ and such that the ratios r_n/ϵ_n are uniformly bounded by a constant ξ (one can always choose such a sequence). Let also Γ be a finitely generated group acting on M by homeomorphisms. We can then apply the approximating procedure to the Voronoi tessellations and obtain the following:

Theorem 7.1.3. *The action $\Gamma \curvearrowright M$ is expanding in measure if and only if the measure approximating graphs $\mathcal{G}\big(\Gamma \curvearrowright M; \mathcal{V}(Y_n)\big)$ and/or the topology approximating graphs $\widetilde{\mathcal{G}}\big(\Gamma \curvearrowright M; \mathcal{V}(Y_n)\big)$ share a uniform lower bound on their Cheeger constants.*

In particular, if the action is by quasi-symmetric homeomorphisms with bounded measure distortion, then $\Gamma \curvearrowright M$ is expanding if and only if $\big(\mathcal{G}\big(\Gamma \curvearrowright M; \mathcal{V}(Y_n)\big)\big)_{n\in\mathbb{N}}$ and $\big(\widetilde{\mathcal{G}}\big(\Gamma \curvearrowright M; \mathcal{V}(Y_n)\big)\big)_{n\in\mathbb{N}}$ are families of expanders.

Proof. The Riemannian volume is a Radon measure. For every $y \in Y_n$ the tile $R(y) \in \mathcal{V}(Y_n)$ has diameter bounded by $2r_n$ and hence $\operatorname{mesh}\big(\mathcal{V}(Y_n)\big) \to 0$ as n grows

to infinity. We already noted that for every $y \in Y_n$ we have $B\big(y, \frac{\epsilon}{2}\big) \subseteq R(y) \subseteq B(y, r_n)$, thus, in the notation of Subsection 2.3.3, for every $y, y' \in Y_n$ we have:

$$\frac{\nu(R(y))}{\nu(R(y'))} \leq \frac{\nu\big(B(y, r_n)\big)}{\nu\big(B\big(y', \frac{\epsilon_n}{2}\big)\big)} \leq \frac{V_M(r_n)}{v_M\big(\frac{\epsilon_n}{2}\big)},$$

and the latter is bounded by a constant depending only on M and $\frac{2r_n}{\epsilon_n}$ (and hence ξ) by Lemma 2.3.14. It follows that the Voronoi tessellations have uniformly bounded measure ratios and we can hence apply Theorem 5.2.12 to prove the first part of the statement.

It now remains to find a uniform upper bound on the degrees to deduce the equivalence of expansion in measure and the construction of expander graphs. Note that every tile has eccentricity bounded above by 2ξ and that M is a doubling metric measure space (Corollary 2.3.15). When we assume that the action be by quasi-symmetric homeomorphisms with bounded measure distortion, we are then under the hypotheses of Proposition 5.3.2, which completes the proof. □

Since diffeomorphisms of compact Riemannian manifolds are bi-Lipschitz (Corollary 2.3.3), combining Theorem 7.1.3 with Proposition 6.1.2 yields the following:

Corollary 7.1.4. *If $\Gamma \curvearrowright M$ is an action by measure preserving diffeomorphisms on a compact manifold, then the measure (or topology) approximating graphs associated with the Voronoi tessellations $\mathcal{V}(Y_n)$ are expanders if and only if the action has a spectral gap.*

Note that a number of the concrete examples of expanding actions that we gave in Chapter 6 satisfy the hypotheses of Corollary 7.1.4 and can hence be immediately used to construct families of expanders. In particular, we report the following:

Corollary 7.1.5. *Let $S \subset \mathrm{SO}(3, \mathbb{R})$ be a finite set of matrices with algebraic coeffients that generates a free subgroup $F_S < \mathrm{SO}(3, \mathbb{R})$. Then the approximating graphs $\mathcal{G}\big(F_S \curvearrowright \mathbb{S}^2; \mathcal{V}(Y_n)\big)$ are expanders.*

Corollary 7.1.6. *Let $\Gamma_d \subset \mathrm{SO}(d, \mathbb{R})$ be as in Corollary 6.3.3, then for every $d \geq 5$, the approximating graphs $\mathcal{G}\big(\Gamma_d \curvearrowright \mathrm{SO}(d, \mathbb{R}); \mathcal{V}(Y_n)\big)$ and $\mathcal{G}\big(\Gamma_d \curvearrowright \mathbb{S}^{d-1}; \mathcal{V}(Y_n)\big)$ are superexpanders.*

7.1.3 A note on cardinalities

If $Y_r \subset M$ is an r-net in a compact Riemannian manifold, we have that the balls $B(y, \frac{r}{2})$ with $y \in Y_r$ are disjoint, while the balls $B(y, r)$ cover M. It follows that

$$v_M\Big(\frac{r}{2}\Big)|Y_r| \leq \mathrm{Vol}(M) \leq V_M(r)|Y_r|.$$

Recall that the ratio $V_M(2r)/v_M(r)$ is uniformly bounded (Lemma 2.3.14). This, together with Theorem 7.1.3, yields the following:

Corollary 7.1.7. *Let $\rho\colon \Gamma \curvearrowright M$ be a measure expanding action by quasi-symmetric homeomorphisms on a compact Riemannian manifold. Then there exists a constant C depending only on M such that for every $n \in \mathbb{N}$ there exists a partition $\mathcal{P}_n$ of M for which*

$$n \leq \Big|\widetilde{\mathcal{G}}\big(\Gamma \curvearrowright M; \mathcal{P}_n\big)\Big| \leq Cn.$$

and such that $\widetilde{\mathcal{G}}\big(\Gamma \curvearrowright M; \mathcal{P}_n\big)$ is a sequence of expanders.

Proof. The function $V_M(r)$ is continuous (and for small values it behaves like $r^{\dim(M)}$), thus there exists an appropriate value $r_n > 0$ such that $V_M(r_n) = \mathrm{Vol}(M)/n$. Let Y_n be an r_n-net and $\mathcal{P}_n = \mathcal{V}(Y_n)$ the Voronoi tessellation. Then we have

$$n = \frac{\mathrm{Vol}(M)}{V_M(r)} \leq |\mathcal{P}_n| \leq \frac{\mathrm{Vol}(M)}{v_M(\frac{r}{2})} \leq Cn$$

as desired. □

If one is willing to allow worse Cheeger constant, then an even sharper control on the cardinalities of the vertex sets can be obtained.

Proposition 7.1.8. *Let $\rho\colon \Gamma \curvearrowright M$ be a measure expanding action by Lipschitz homeomorphisms on a compact Riemannian manifold. Then for every sequence of natural numbers $(k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ with $k_n \to \infty$, there exists a sequence of partitions $\mathcal{P}_n$ of M with $|\mathcal{P}_n| = k_n$ and such that the approximating graphs $\widetilde{\mathcal{G}}\big(\Gamma \curvearrowright M; \mathcal{P}_n\big)$ are expanders.*

Proof. By Theorem 7.1.3, it is enough to show that for every $k \in \mathbb{N}$ there exists a (r_k, ϵ_k)-net $Y_k \subset M$ of cardinality k and such that the ratios r_k/ϵ_k are uniformly bounded.

This is done quite easily. Fix any $r > 0$ and let Y_r be a r-net. By Zorn's lemma, it can be extended to a $\frac{r}{2}$ net $Y_{\frac{r}{2}} \supset Y_r$. Note that every intermediate set Z

$$Y_r \subseteq Z \subseteq Y_{\frac{r}{2}}$$

is r-dense and $\frac{r}{2}$ separated, *i.e.* a $(r, r/2)$-net. Adding all the points one at the time we thus obtain $(r, r/2)$-nets of every cardinality between $|Y_r|$ and $\left|Y_{\frac{r}{2}}\right|$. Repeating the process produces nets of any cardinality. □

Corollary 7.1.9. *It is possible to construct expanders and superexpanders whose graphs have arbitrary cardinality.*

7.2 Actions on sequences of covers

It this section we show how to obtain approximating graphs with bounded degrees by looking at actions on sequences of covers. In some sense what we are going to do is a complementary approach to what has been done so far. That is, we have been fixing an action on a nice space and obtained a sequence of graphs by considering finer and finer partition; now we change perspective and we will obtain sequences of graphs by approximating actions on larger and larger spaces while the size of the tiles stays the same. What we have in mind is the following:

Example 7.2.1. Consider the natural action by translations $\mathbb{R}^2 \curvearrowleft \mathbb{Z}^2$ and let X_n be the quotient $\mathbb{R}^2/(n\mathbb{Z})^2$. We thus obtain a sequence of coverings $X_0 \longleftarrow X_1 \longleftarrow \cdots$. Note that natural tiling of $\mathbb{R}^2$ by unit squares descends to a tiling $\mathcal{T}_n$ of X_n into n^2 tiles of area 1.

Consider now the natural left action of $\mathrm{SL}(2,\mathbb{Z})$ on $\mathbb{R}^2$. For every $n \in \mathbb{N}$, this action descends to a continuous action on X_n because the lattice $(n\mathbb{Z})^2$ is preserved. We can thus look at the (measure) approximating graphs $\mathcal{G}(\mathrm{SL}(2,\mathbb{Z}) \curvearrowright X_n; \mathcal{T}_n)$ and it is simple to observe that these graphs have uniformly bounded degree.

7.2.1 Generalities on fundamental domains

We begin with some definitions.[1] Let (X, d, ν) be a connected topological space with a Borel measure and let Λ be a countable group with a right action $(X, \nu) \curvearrowleft \Lambda$ by homeomorphisms.

Definition 7.2.2. A closed subset $\overline{\Delta} \subseteq X$ is a *regular fundamental domain* for the action of Λ if

- $\overline{\Delta}$ is the closure of a connected open subset $\Delta \subset X$;
- $\nu(\partial\Delta) = 0$;

[1] The conventions that we use here are not standard.

- $\overline{\Delta}$ intersects the orbit $x \cdot \Lambda$ for every $x \in X$;
- $\overline{\Delta}$ can intersect its homeomorphic copies under the Λ-action only on its boundary: $\overline{\Delta} \cap (\overline{\Delta} \cdot h) \subseteq \partial\Delta$ for every $h \in \Lambda \smallsetminus \{e\}$.

Remark 7.2.3. Not every action admits a regular fundamental domain. In particular, note that any point of X with non trivial stabiliser can only lie on the boundary of translates of a regular fundamental domain.

From now on, assume that the action $X \curvearrowleft \Lambda$ admits a regular fundamental domain and let $\overline{\Delta}$ be a fixed such domain. Note in particular that, since $\nu(\partial\Delta) = 0$, the set $\{\Delta \cdot h \mid h \in \Lambda\}$ is a measurable partition of X. We will denote such partition by $\mathcal{T}$. we will generally denote a generic region in $\mathcal{T}$ by R, while we keep the symbol Δ for the fixed (open) fundamental domain. That is, a generic region $R \in \mathcal{T}$ will be equal to $\Delta \cdot h$ for a unique $h \in \Lambda$.

Let $\Lambda = \Lambda_0 >_f \Lambda_1 >_f \Lambda_2 >_f \cdots$ be a residual filtration of Λ (*i.e.* such that $\bigcap_{i\in\mathbb{N}} \Lambda_i = \{e\}$) and let $X_i \coloneqq X/\Lambda_i$ be the quotient space. This gives a sequence of finite index surjections:

$$X_0 \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots$$

These surjections need not be covers, as we do not require the action to be free. Since they still enjoy some of the properties of coverings of spaces, we call such maps *singular covers*. Note that they are open maps (*i.e.* they send open set to open sets).

Let $\pi_i \colon X \to X_i$ denote the singular covering map and let R be any region in $\mathcal{T}$. Then the restriction of π_i to R is injective for every $i \in \mathbb{N}$. Moreover, if R' is a second region in $\mathcal{T}$, we have that whenever their images under π_i intersect non trivially we must in fact have $\pi_i(R) = \pi_i(R')$. Let $\mathcal{T}_i$ be the set of subsets of X_i that are images under π_i of regions in $\mathcal{T}$. It follows from the above discussion that the regions in $\mathcal{T}_i$ are disjoint open sets such that the union of their closure covers the whole of X_i.

Note also that there is a natural bijection between regions in $\mathcal{T}_i$ and left cosets of Λ_i in Λ. In fact, the preimage of a region $\pi_i(\Delta \cdot h) \in \mathcal{T}_i$ is equal to the disjoint union of the regions $\Delta \cdot (hk)$ with $k \in \Lambda_i$, *i.e.* it the union of the images of (the interior of) the fundamental domain under the elements in the coset $h\Lambda_i \in \Lambda/\Lambda_i$.

Assume now that the action $(X, \nu) \curvearrowright \Lambda$ be measure preserving. Then ν induces a natural measure ν_i on X_i by imposing that the restriction of ν_i to a region in $\mathcal{T}_i$ coincides with the restriction of ν to with a region in $\mathcal{T}$. That is, if $A \subseteq \pi_i(R) = \pi(\Delta \cdot h)$ is a measurable subset, we let

$$\nu_i(A) \coloneqq \nu\big(\pi_i^{-1}(A) \cap R\big) = \nu\big(\pi_i^{-1}(A) \cdot h^{-1} \cap \Delta\big)$$

(this is well-defined because the action is measure-preserving). For a general measurable set $A \subseteq X_i$ we define $\pi_i(A)$ as the sum $\sum_{R \in \mathcal{T}_i} \nu(A \cap R)$.

Note that since $\nu(\partial\Delta) = 0$ we have that $\mathcal{T}_i$ is a measurable partition (*i.e.* partitions a set of co-null measure) for every $i \in \mathbb{N}$. Moreover, if $X_i \to X_j$ is a (singular) cover of index D then the volume of the preimage of any set $A \subseteq X_j$ will be $D\nu_j(A)$ (the *degree* of these singular covers can be defined as the cardinality of the preimage of a point lying in one of the regions R. Such definition does not depend on the specific point within a fixed region, and it is constant when varying the region because the cover is obtained from a quotient by a group action).

7.2.2 Compatible actions

Again, let $(X, \nu) \curvearrowleft \Lambda$ be a measure preserving action with regular fundamental domain Δ, $(\Lambda_i)_{i\in\mathbb{N}}$ a filtration and $(X_i)_{i\in\mathbb{N}}$ the quotients with their measurable partitions $\mathcal{T}_i$. This time we also assume that the measure ν is *strictly positive*, *i.e.* it is such that every open set has strictly positive measure.

Let also $\Gamma = \langle S \rangle$ be a finitely generated group with a left measure-class preserving action $\Gamma \curvearrowright (X, \nu)$. We say that such action is *compatible with the filtration* $(\Lambda_i)_{i\in\mathbb{N}}$ if for every $g \in \Gamma$, $h \in \Lambda_i$ and $x \in X$ there exists an $h' \in \Lambda_i$ so that $g(x \cdot h) = g(x) \cdot h'$. That is, ρ is compatible if it induces a quotient action on X/Λ_i for every $i \in \mathbb{N}$. Let ρ_i denote the induced action on X_i. Note that ρ_i still preserves the measure class of ν_i.

Given a compatible action ρ we can now consider the (measure) approximating graphs $\mathcal{G}\big(\rho_i \colon \Gamma \curvearrowright X_i \,;\, \mathcal{T}_i\big)$ and we thus obtain an infinite sequence of graphs of increasing cardinality. Note that two regions $R, R' \in \mathcal{T}_i$ form an edge in $\mathcal{G}(\Gamma \curvearrowright X_i; \mathcal{T}_i)$ if and only if there exists an $s \in S^{\pm}$ such that $s(R) \cap R' \neq \emptyset$. This is because ν_i is a strictly positive measure on X_i and the regions R and R' are open.

We will say that a graph morphism $f \colon \mathcal{G} \to \mathcal{G}'$ is *locally surjective* if the link of every vertex v of $\mathcal{G}$ surjects onto the link of the image of v. That is, for every vertex $w \in \mathcal{G}'$ linked to $f(v)$ by an edge there exists a vertex $w' \in \mathcal{G}$ linked to v by an edge and such that $w = f(w')$.

Lemma 7.2.4. *The quotient map $\Lambda \to \Lambda/\Lambda_i$ induces a surjective and locally surjective graph morphism $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{T}) \to \mathcal{G}(\Gamma \curvearrowright X_i; \mathcal{T}_i)$.*

Proof. Since $\mathcal{T}$ and $\mathcal{T}_i$ are in natural bijection with Λ and Λ/Λ_i, the quotient map does define a surjection of the vertex sets of the approximating graphs. We need to show that this is a graph morphism.

Since ρ is a compatible action, for every $g \in \Gamma$ and $A \subseteq X$ we have $g \cdot \pi_i(A) = \pi_i(g \cdot A)$. Note that if (R, R') is an edge in $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{T})$, then $\nu\big(s(R) \cap R'\big) > 0$ for some $s \in S^{\pm}$. Thus, we have $\nu_i\big(s(\pi_i(R)) \cap \pi_i(R')\big) = \nu_i\big(\pi_i(s(R)) \cap \pi_i(R')\big) \geq \nu\big(s(R) \cap R'\big)$ is positive and hence $\big(\pi_i(R), \pi_i(R')\big)$ is an edge of $\mathcal{G}(\Gamma \curvearrowright X_i; \mathcal{T}_i)$.

To prove that the morphism is locally surjective, let $R \in \mathcal{T}$ be any region. If $\big(\pi_i(R), \pi_i(R')\big)$ is an edge of $\mathcal{G}(\Gamma \curvearrowright X_i; \mathcal{T}_i)$ then $\nu_i\big(\pi_i(s(R)) \cap \pi_i(R')\big)$ must be positive. Since $\pi_i^{-1}\big(\pi_i(s(R))\big) = s(R) \cdot \Lambda_i$, it follows from the definition of ν_i that there must exist an $h \in \Lambda_i$ so that $\nu\big(s(R) \cap (R' \cdot h)\big) > 0$, therefore $\big(\pi_i(R), \pi_i(R')\big)$ is the image of the edge $(R, R' \cdot h)$ of $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{T})$. □

It follows that as soon as the graph $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{T})$ has bounded degree then all the graphs $\mathcal{G}(\Gamma \curvearrowright X_i; \mathcal{T}_i)$ have uniformly bounded degree as well. In particular, if one knows that $\mathcal{G}(\Gamma \curvearrowright X; \mathcal{T})$ has bounded degree and that the actions $\Gamma \curvearrowright X_i$ are uniformly expanding in measure, then the graphs $\mathcal{G}(\Gamma \curvearrowright X_i; \mathcal{T}_i)$ are a sequence of expanders.

For example, let G be an (non-compact) connected Lie group and $\Lambda < G$ a cocompact lattice. Fix any right-invariant[2] Riemannian metric on G then the right action $G \curvearrowleft \Lambda$ admits a compact regular fundamental domain $\overline{\Delta}$ (*e.g.* consider the Voronoi tiling associated with $\Lambda \subset G$), and we thus obtain a partition $\mathcal{T}$.

Let now $S \subset G$ be any finite subset and $\Gamma = \langle S \rangle$ the generated subgroup. The action on the left $\Gamma \curvearrowright G$ is by bi-Lipschitz diffeomorphisms (the differentials at any two points are conjugated by the right-action of G and they hence have the same norm). It follows that the graph $\mathcal{G}(\Gamma \curvearrowright G; \mathcal{T})$ has bounded degree.

Moreover, the left action $\Gamma \curvearrowright G$ commutes with the right action $G \curvearrowleft \Lambda$ and it is hence compatible with any filtration of Λ. Choose now any sequence $\Lambda_i < \Lambda$. The left action $G \curvearrowright G/\Lambda_i$ is always ergodic, therefore if S is a Kazhdan set of G we deduce that the actions $\Gamma \curvearrowright G/\Lambda_i$ have uniform spectral gap. It follows that the graphs $\mathcal{G}(\Gamma \curvearrowright G/\Lambda_i; \mathcal{T}_i)$ are expanders.

It is also known that the actions described in Example 7.2.1 are uniformly expanding in measure. We thus obtain expanders in this case as well. We wish to remark that these expanders are not new: these are in fact (equivalent to) the first known examples of expanders originally discoverd by Margulis in [Mar73] (see also [GG81]).

[2] I am sorry.

Chapter 8

Warped cones and warped systems

In this chapter we explore the strong connection between approximating graphs and warped metrics and use it to prove various rigidity results.

8.1 Roe's warped cone

I originally learned about John Roe's warped cone construction through his paper on property A [Roe05], but he had actually introduced this construction some time before. Indeed, he writes about it in [Roe96, Chapter 2] and [Roe95]. We shall follow the exposition given in [Roe05].

Remark 8.1.1. It is quite interesting to go back and read those original works, as the current research focus concerning warped cones is fairly different from the original flavour that they had in Roe's original work.

8.1.1 Warped cones of manifolds

We begin by giving Roe's definition as in [Roe05]. Let (M, ϱ) be a compact Riemannian manifold, the open cone on M is the space $\mathcal{O}(M) \coloneqq M \times [1, \infty)$ with the metric $d_{\mathcal{O}}$ induced by the Riemannian metric $\varrho_{\mathcal{O}} \coloneqq t^2\varrho + dt^2$, where dt^2 is the standard Euclidean metric on $\mathbb{R}$.

Remark 8.1.2. If $M = \mathbb{S}^n$ is the standard sphere, then the open cone $\mathcal{O}(\mathbb{S}^n)$ is simply the truncated Euclidean cone. That is, $\mathcal{O}(\mathbb{S}^n)$ is isometric to the space $\mathbb{R}^{n+1} \setminus B(0,1)$ equipped with its path-metric (not the subset metric coming from $\mathbb{R}^{n+1} = \mathbb{E}^{n+1}$). With this identification, the level set $\mathbb{S}^n \times \{t\}$ with $t \geq 1$ is mapped onto the sphere $\partial B(0,t) \subseteq \mathbb{R}^{n+1} \setminus B(0,1)$.

If one only cares about spaces up to bi-Lipschitz equivalence (which we do), one can avoid mentioning Riemannian manifolds and simply do as follows: let $M \hookrightarrow \mathbb{S}^n$ be

any smooth embedding (there always is such an embedding for n large enough), then define the open cone $\mathcal{O}(M)$ as the subset of $\mathbb{R}^{n+1} \smallsetminus B(0,1)$ consisting of rays passing through $M \subseteq \mathbb{S}^n = \partial B(0,1)$. One can then check that, as long as M is compact, different choices of embeddings produce bi-Lipschitz equivalent spaces.

Remark 8.1.3. As currently defined, the cone $\mathcal{O}(M)$ is a Riemannian manifold with boundary. One could have also considered the analogous cone metric on the set $M \times (0,\infty)$ to obtain a (non complete) Riemannian manifold without boundary; or the quotient of $M \times [0,\infty)$ collapsing the set $M \times \{0\}$ to a point in order to obtain a (complete) pseudo-manifold with a singularity at 0. As we are mainly concerned with coarse geometry, all the approaches of above are equivalent as long as the manifold M has finite diameter (which is always the case if M is compact).

Let now $\Gamma = \langle S \rangle$ be a finitely generated group and $\gamma \curvearrowright M$ an action by homeomorphism. Letting $g \cdot (x,t) \coloneqq (g \cdot x, t)$ naturally induces an action of Γ on the cone $\mathcal{O}(M)$ by homeomorphisms that fix the coordinate t.

Definition 8.1.4. The *warped cone* $\mathcal{O}_\Gamma(M)$ of the manifold M under the action of Γ is the metric space $\big(M \times [1,\infty), \delta_\Gamma\big)$, where $\delta_\Gamma = \delta_S$ is the metric obtained warping the cone metric $d_\mathcal{O}$ of $\mathcal{O}(M) = M \times [1,\infty)$.

Note that the definition of warped cone depends on the choice of generating set. Still, different generating set produce coarsely equivalent (in fact, bi-Lipschitz) warped cones.

8.1.2 Warped metrics and Lipschitz conjugations

We now wish to study how the coarse geometry of a warped space depends on (the conjugacy class of) the action. We prove the following:

Lemma 8.1.5. *Let (X_1,d_1) and (X_2,d_2) be metric spaces, $\Gamma_1 = \langle S_1 \rangle$ and $\Gamma_2 = \langle S_2 \rangle$ be finitely generated groups, and $\Gamma_1 \curvearrowright X_1$ and $\Gamma_2 \curvearrowright X_2$ actions by homeomorphisms. Assume that there exist a (L,A)-coarsely Lipschitz map $\varphi\colon \Gamma_1 \to \Gamma_2$ and a φ-equivariant L-Lipschitz map $F\colon (X_1,d_1) \to (X_2,d_2)$. Then $F\colon (X_1,\delta_{S_1}) \to (X_2,\delta_{S_2})$ is $(L+A)$-Lipschitz.*

Proof. By Lemma 2.2.5 for every $x,y \in X_1$ we have:

$$\delta_{S_1}(x,y) = \inf\Big\{ n + \sum_{i=0}^{n} d(x_i,y_i) \Big\}$$

where the infimum is taken over $n \in \mathbb{N}$ and $(n+1)$-tuples $x_0, \dots, x_n$ and $y_0, \dots, y_n$ such that $x = x_0$, $y = y_n$ and $x_i = s_i(y_{i-1})$ for some $s_i \in (S_1)^{\pm}$.

Using the triangle inequality and the fact that F is φ-equivariant, we obtain:

$$\begin{aligned}
\delta_{S_2}\big(F(x), F(y)\big) &\leq \sum_{i=0}^{n} \delta_{S_2}\big(F(x_i), F(y_i)\big) + \sum_{i=0}^{n-1} \delta_{S_2}\big(F(y_i), F(s_i \cdot y_i)\big) \\
&\leq \sum_{i=0}^{n} d_2\big(F(x_i), F(y_i)\big) + \sum_{i=0}^{n-1} \delta_{S_2}\big(F(y_i), \varphi(s_i) \cdot F(y_i)\big) \\
&\leq \sum_{i=0}^{n} L d_1(x_i, y_i) + \sum_{i=0}^{n-1} L + A \\
&\leq (L + A)\Big(n + \sum_{i=0}^{n} d(x_i, y_i)\Big)
\end{aligned}$$

and hence the claim follows. □

Corollary 8.1.6. *Let X_1, X_2, Γ_1, Γ_2, φ and F be as in Lemma 8.1.5. If φ and F are L-bi-Lipschitz equivalences, then F is a L-bi-Lipschitz equivalence also with respect to the warped metrics δ_{S_1} and δ_{S_2}.*

Proof. It is enough to note that, as we required both φ and F to be bi-Lipschitz equivalences, the inverse map F^{-1} is φ^{-1}-equivariant (because ϕ is a bijection). We can apply thus apply Lemma 8.1.5 to both F and its inverse F^{-1} and we deduce that they are L-Lipschitz maps. □

Corollary 8.1.7. *If two actions on manifolds $\rho_1 \colon \Gamma \curvearrowright M_1$ and $\rho_2 \colon \Gamma \curvearrowright M_2$ are conjugated by a bi-Lipschitz equivalence $F \colon M_1 \to M_2$ (*i.e. *F is an equivariant bi-Lipschitz equivalence), then $\mathcal{O}_\Gamma(M_1)$ and $\mathcal{O}_\Gamma(M_2)$ are bi-Lipschitz equivalent.*

Proof. It is enough to notice that F extends to an equivariant bi-Lipschitz equivalence $\mathcal{O}(M_1) \to \mathcal{O}(M_2)$ given by $(x, t) \mapsto (F(x), t)$. □

Remark 8.1.8. Looking at the statement of Lemma 8.1.5, one would expect that when φ is a (L, A)-quasi-isometry then F should provide us with a $(L + A)$-bi-Lipschitz equivalence between the warped metrics. This is not the case, because F^{-1} needs not be $\overline{\varphi}$-equivariant (where $\overline{\varphi}$ is the coarse inverse). In fact, letting $X_1 = X_2$, $\Gamma_1 = \{e\}$ and Γ_2 any finite group with an action X_2 with unbounded displacement, one can see that the identity map $X_1 \to X_2$ does not produce a coarse equivalence with respect to the warped metrics.

It follows that to produce a statement implying that F induces a quasi-isometric equivalence between the warped metrics one also needs to require that the inverse F^{-1} be *close* to be $\overline{\varphi}$-equivariant; *i.e.* we need that $F^{-1}(x_2)$ and $F^{-1}(s \cdot x_2)$ be at bounded distance in (X_1, δ_{S_1}) for every $s \in (S_2)^{\pm}$ and $x_2 \in X_2$.

It is interesting to compare this issue with the fact that there are examples of graphs (and groups) that are quasi-isometric but not bi-Lipschitz equivalent (see [DPT15] and references therein).

8.1.3 Extending the definition to general metric spaces

We now wish to extend the definition of warped cones to actions on more general metric spaces. Since the warping procedure is well-defined for every metric space, the only thing that needs to be decided is how to define the space that has to play the role of the open cone $\mathcal{O}(M)$.

Since J. Roe was mostly concerned with rather nice spaces (*e.g.* compact manifold and finite simplicial complexes), his idea was to exploit what noted in Remark 8.1.2. That is, if a metric space X admits a bi-Lipschitz embedding into the sphere $\mathbb{S}^n$, then one can define $\mathcal{O}(X)$ as the subset $X \times [1, \infty) \subset \mathbb{S}^n \times [1, \infty) = \mathcal{O}(\mathbb{S}^n)$. Note that this notion of $\mathcal{O}(X)$ is only defined up to bi-Lipschitz equivalence.

A more general and intrinsic approach is that of Druţu–Nowak [DN17], which is also extensively used by D. Sawicki. That is, let (X, d_X) be any metric space such that $\operatorname{diam}(X) \leq 2$ and define $d_{X\mathbb{R}} \colon \big(X \times [1, \infty)\big)^2 \to \mathbb{R}$ by

$$d_{X\mathbb{R}}\big((x,t),(x't')\big) \coloneqq \min\{t,t'\} d_X(x,x') + |t - t'|. \tag{8.1}$$

It is then simple to check (see [Saw15]) that the $d_{X\mathbb{R}}$ defines a metric on $X \times [1, \infty)$.

Remark 8.1.9. Note that $d_{X\mathbb{R}}$ does not satisfy the triangle inequality if $\operatorname{diam}(X) > 2$. Indeed, any two points on the same level $X \times \{t\}$ should have distance $t d_X(x, x')$. If we choose an appropriate third point on the level $X \times \{1\}$, we should obtain by triangle inequality $t d_X(x, x') \leq d_X(x, x') + 2(t - 1)$; whence $(t - 1)d(x, x') \leq 2(t - 1)$ and hence $d(x, x') \leq 2$.

Still, since we are only concerned about spaces up to bi-Lipschitz equivalences, we are always entitled to rescale a metric space and we can thus use (8.1) to define a metric on every metric space with bounded diameter.

First off, we wish to show that when M is a compact manifold with $\operatorname{diam}(M) \leq 2$, using the metric $d_{M\mathbb{R}}$ instead of the cone metric $d_{\mathcal{O}}$ produces equivalent results. Given a metric d, let td denote the metric d rescaled by t. We begin with the following:

Lemma 8.1.10. *Let (M, ϱ) be a compact Riemannian manifold and d its induced Riemannian metric. Then there exists a constant L depending only on* $\operatorname{diam}(M)$ *such that for every (x,t) and (x',t') in $M \times [1,\infty)$ we have*

$$d_{\mathcal{O}}\big((x,t),(x't')\big) \leq d_{X\mathbb{R}}\big((x,t),(x't')\big) \leq L d_{\mathcal{O}}\big((x,t),(x't')\big).$$

The statement holds also when $\operatorname{diam}(M) > 2$ *and $d_{X\mathbb{R}}$ is not a metric.*

Proof. Assume without loss of generality that $t \geq t'$, so that

$$d_{X\mathbb{R}}\big((x,t),(x',t')\big) = (t-t') + t' d(x,x').$$

Then $d_{X\mathbb{R}}\big((x,t),(x',t')\big)$ is equal to the length of the path going from (x,t) to (x,t') moving on a straight line and then proceeding to (x',t') following a geodesic of M. By the definition of the Riemannian distance, it immediately follows that $d_{\mathcal{O}} \leq d_{X\mathbb{R}}$.

Conversely, for every $(x,t),(x',t') \in M \times [1,\infty)$ there exists a geodesic $\gamma\colon [0,1] \to \mathcal{O}(M)$ such that $\|\gamma\| = d_{\mathcal{O}}\big((x,t),(x',t')\big)$. Let $\widetilde{\gamma}\colon [0,\ell] \to M$ be the arc-length reparametrisation of the projection of the path γ to the base set set $M \times \{1\}$. Note that, since γ is a geodesic of $\big(\mathcal{O}(M), d_{\mathcal{O}}\big)$, then $\widetilde{\gamma}$ must be a geodesic of M (both Riemannian and metric). In particular, $\ell = d(x,x') \leq \operatorname{diam}(M)$.

Define the map

$$\begin{aligned} H\colon \quad [0,\ell] \times [1,\infty) &\longrightarrow \mathcal{O}(M) \\ (s,t) &\longmapsto (\widetilde{\gamma}(s),t). \end{aligned}$$

Since the pull back of a Riemannian metric tensor through a smooth curve α is equal to the Euclidean length ds^2 rescaled by the (square of) the speed of α, then the pull-back of the Riemannian metric tensor $\varrho_{\mathcal{O}}$ through H is just a standard cone metric

$$H^* \varrho_{\mathcal{O}} = t^2 ds^2 + dt^2,$$

and H is an isometric embedding of $[0,\ell] \times [1,\infty)$ with respect to the induced Riemannian metric. We thus have that the geodesic γ gives us a geodesic in $[0,\ell]\times[1,\infty)$ as well and that $d_{\mathcal{O}}\big((x,t),(x',t')\big) = d_{[0,\ell]\times[1,\infty)}\big((0,t),(\ell,t')\big) = \|\gamma\|$.

Computing the distance between $(0,t)$ and (ℓ,t') in the cone is now a straightforward exercise of Euclidean geometry and one can show that it satisfies:

$$d_{[0,\ell]\times[1,\infty)}\big((0,t),(\ell,t')\big) \geq \begin{cases} \big(t^2 + (t')^2 - 2tt'\cos(\ell)\big)^{\frac{1}{2}} & \text{if } L \leq \pi \\ t + t' & \text{if } \ell \geq \pi \end{cases}$$

(the inequality comes from the fact that we removed the tip of the cone $[0,\ell]\times(0,1)$, otherwise equality would hold).

Now, if $\ell \geq \pi$ we have

$$d_{\mathcal{O}}\big((x,t),(x't')\big) \geq t + t' \geq 2t' = \frac{2t'}{\ell} d(x,x') \geq \frac{2}{\ell} t' d(x,x').$$

For $\ell < \pi$, we have

$$\begin{aligned}
d_{\mathcal{O}}\big((x,t),(x't')\big) &\geq \Big(t^2 + (t')^2 - 2tt'\cos(\ell)\Big)^{\frac{1}{2}} \\
&\geq \Big((t-t')^2 + 2tt'(1-\cos(\ell))\Big)^{\frac{1}{2}} \\
&\geq \frac{1}{\sqrt{2}}\Big(|t-t'| + \sqrt{2tt'}\sqrt{1-\cos(\ell)}\Big) \\
&= \frac{1}{\sqrt{2}}\Big(t - t' + \sqrt{2tt'}\sqrt{2\sin^2\Big(\frac{\ell}{2}\Big)}\ \Big) \\
&\geq \frac{1}{\sqrt{2}}\Big(\sqrt{2tt'}\sqrt{2\sin^2\Big(\frac{\ell}{2}\Big)}\ \Big) \\
&\geq \frac{1}{\sqrt{2}}\Big(t'\sqrt{2}\cdot\frac{\sqrt{2}}{\pi}\ell\Big) = \frac{\sqrt{2}}{\pi} t' d(x,x')
\end{aligned}$$

(where we used that $\sin(x) \geq 2x/\pi$ for $x \leq \pi/2$. We deduce that

$$d_{X\mathbb{R}}\big((x,t),(x't')\big) \leq L d_{\mathcal{O}}\big((x,t),(x't')\big)$$

for $L \coloneqq \max\Big(\frac{\pi}{\sqrt{2}}, \frac{\operatorname{diam}(M)}{2}\Big)$. □

Corollary 8.1.11. *With the notation of Lemma 8.1.10, we have that for every $t_0 \geq 1$, the level set $M \times \{t_0\}$ with the restriction of the metric $d_{\mathcal{O}}$ is L-bi-Lipschitz equivalent to the metric space $(M, t_0 d)$.*

Remark 8.1.12. A different way to put a metric on the set $X \times [1,\infty)$ that is more reminiscent of the cone construction for manifold is to consider the 0-cone over X as defined in [BH13, Chapter I.5]. This is a more natural generalisation because the 0-cone over a compact Riemannian manifold M should be equal to the pseudo manifold $M \times [0,\infty)$ equipped with the cone Riemannian metric.

Remark 8.1.13. If one wishes to use a definition such as (8.1) even for a geodesic metric space X with diameter larger than 2, a natural choice would be to define the distance as the path-distance generated by said expression. This construction could then be applied to spaces with unbounded diameter as well, and a modification of

Lemma 8.1.10 should imply that the maps sending the metric spaces $(X, t_0 d)$ to the level set $X \times \{t_0\}$ are uniform coarse embeddings (note that here we are not claiming that they should be quasi-isometric embedding).

Lemma 8.1.10 entitles us to give the following:

Definition 8.1.14. Let $\Gamma = \langle S \rangle$ be a finitely generated group acting on a general metric space X with $\mathrm{diam}(X) \leq 2$. The *warped cone* is the metric space $\mathcal{O}_\Gamma(X) \coloneqq \big(X \times [1, \infty), \delta_\Gamma\big)$ where $\delta_\Gamma = \delta_S$ is the warping of the metric $d_{X\mathbb{R}}$.

Again, the warped cone (without specifying a generating set) is only defined up to bi-Lipschitz equivalence, and, thanks to Lemma 8.1.5 and Lemma 8.1.10, we have that this definition is compatible with the definition of warped cones for compact manifolds.

8.2 Warped systems

In most situations, working with the actual warped cone can be quite cumbersome, while what one really cares about is just the behaviour of the level sets $X \times \{t\}$ as t grows to infinity. For this reason we find it convenient to introduce another piece of terminology to better describe these.

8.2.1 A few definitions

For a metric space (X, d) and a parameter $t \geq 1$, we denote by $d^t \coloneqq td$ the rescaling of the metric d by t. If S is a finite set of homeomorphisms of X, we denote by δ_S^t the warping of the metric d^t along S.

Definition 8.2.1. Given a finite set S of homeomorphisms of a metric space (X, d) (equivalently, an action of the free group $F_S \curvearrowright X$), its *warped system* $\mathrm{WSys}\big(F_S \curvearrowright (X, d)\big)$ is the data of the family of metric spaces $\big\{(X, \delta_S^t) \;\big|\; t \in [1, \infty)\big\}$ together with the set of generating homeomorphisms S (we will usually drop the distance d from the notation).

We say that a warped system *satisfies a property P asymptotically* if if there exists a parameter t_0 large enough so that (X, δ_S^t) satisfies P for every $t \geq t_0$.

Remark 8.2.2. Note that warped systems are completely well-defined, not only up to bi-Lipschitz equivalent because the set S is explicit. Note also that even if a set S' is obtained from S only by taking some duplicates of some homeomorphism, then the warped systems $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ and $\mathrm{WSys}\big(F_{S'} \curvearrowright X\big)$ are formally different (even though the metric spaces coincide).

In the sequel we will want to study the (uniform) coarse geometry of warped systems. Since we are using the convention (Subsection 2.6.3) that a coarse equivalence between two sequences of metric spaces $(X_n)_{n\in\mathbb{N}}$ and $(Y_n)_{n\in\mathbb{N}}$ is a sequence of uniform coarse equivalences $X_n \sim Y_n$, it is natural to say that two warped systems $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ and $\mathrm{WSys}\big(F_{S'} \curvearrowright X'\big)$ are *coarsely equivalent* if there exists a family of uniform coarse equivalences $(X, \delta_S^t) \sim (X', \delta_{S'}^t)$ for every $t \geq 1$.

This notion of coarse equivalence is usually too strong for us, especially because we will also be interested in comparing warped system with (countable) sequences of metric spaces. We thus introduce a weaker notion of coarse equivalence:

Definition 8.2.3. Let $\big(X_i, d_{X_i}\big)_{i\in I}$ and $\big(Y_j, d_{Y_j}\big)_{j\in J}$ be two families of metric spaces indexed over some directed poset $(I, \leq)$ and $(J, \leq)$. We say that they are *coarsely sub-equivalent* if there exist cofinal sequences[1] $(i_n)_{n\in\mathbb{N}} \subseteq I$ and $(j_n)_{n\in\mathbb{N}} \subseteq J$ such that the sequences of metric spaces $\big(X_{i_n}, d_{X_{i_n}}\big)_{n\in\mathbb{N}}$ and $\big(Y_{j_n}, d_{Y_{j_n}}\big)_{n\in\mathbb{N}}$ are coarsely equivalent.

If two families of metric spaces are not coarsely sub-equivalent, we say that they are *coarsely disjoint*.[2]

Remark 8.2.4. Note that the relation of coarse sub-equivalence is *not* an equivalence relation, as it need not be transitive.

In our settings, two warped systems $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ and $\mathrm{WSys}\big(F_{S'} \curvearrowright X'\big)$ are coarsely sub-equivalent if there exist two unbounded sequences $t_n \to \infty$, $s_n \to \infty$ such that $(X, \delta_S^{t_n}) \sim (X', \delta_{S'}^{s_n})$ uniformly on n. A warped system $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ is coarsely sub-equivalent to a sequence of metric spaces $\big(Y_n, d_{Y_n}\big)_{n\in\mathbb{N}}$ if there is an unbounded sequence $t_n \to \infty$ such that $(X, \delta_\Gamma^{t_n})$ is coarsely equivalent to a subsequence of $\big(Y_n, d_{Y_n}\big)_{n\in\mathbb{N}}$.

8.2.2 Warped systems and level sets

Let (X, d) be a metric space with $\mathrm{diam}(X) \leq 2$, $\Gamma = \langle S\rangle$ a finitely generated group acting on X by homeomorphisms and $\mathcal{O}_\Gamma(X)$ the associated warped cone. Then for every $t \geq 1$ the metric space (X, δ_S^t) is isometric to the level set $X \times \{t\} \subset \mathcal{O}_\Gamma(X)$ (with the subset metric). That is, the warped system coincides with the family of level sets of the warped cone together with the extra piece of information about the generating set.

[1] The sequence $(i_n)_{n\in\mathbb{N}}$ is *cofinal* in $(I, \leq)$ if $i_n \leq i_m$ for every $n \leq m$ and for every $j \in I$ there exists an $n \in \mathbb{N}$ such that $j \leq i_n$.

[2] This definition agrees with the notion of coarse disjointness given in [FNvL17].

Note that if M is a compact manifold with $\operatorname{diam}(M) > 2$, Lemmata 8.1.10 and 8.1.5 still imply directly that (X, δ_S^t) is bi-Lipschitz equivalent to the level set $M \times \{t\} \subset \mathcal{O}_\Gamma(M)$ (while one would need to rescale the metric first in order to talk about the warped cone in the generalised sense).

If we are only interested on a warped system $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ up to bi-Lipschitz equivalence and the set of homeomorphism S comes from an action of a non-free group Γ, *i.e.* $\Gamma = \langle S \rangle$ and we are given an action $\Gamma \curvearrowright X$, then we will sometime use the notation $\mathrm{WSys}\big(\Gamma \curvearrowright X\big)$. We will do so especially when we want to prove that some coarse geometric properties of the warped system depend on the group acting. This is convention is equivalent to our standard convention for warped cones, where the generating set is always dropped from the notation.

As a warning, we wish to remark that the notion of coarse equivalence for warped cones and warped systems (a priori) differ. That is, if $f\colon \mathcal{O}_\Gamma(X) \to \mathcal{O}_\Lambda(Y)$ is a coarse equivalence, we cannot immediately deduce that the warped systems $\mathrm{WSys}\big(\Gamma \curvearrowright X\big)$ and $\mathrm{WSys}\big(\Lambda \curvearrowright Y\big)$ are coarsely equivalent because we are not given that f sends level sets near (corresponding) level sets. Vice versa, even if we know that two warped systems $\mathrm{WSys}\big(\Gamma \curvearrowright X\big)$ and $\mathrm{WSys}\big(\Lambda \curvearrowright Y\big)$ are coarsely equivalent we cannot trivially deduce that the warped cones $\mathcal{O}_\Gamma(X)$ and $\mathcal{O}_\Lambda(Y)$ are coarsely equivalent because it might be the case that the family of coarse equivalences between level sets are not compatible and do not glue nicely.

This inequivalence is one of the reason why we decided to introduce the notion of warped systems. Indeed, most of our results concern mainly the geometry of the level sets, which made it artificial to maintain the whole cone as a metric space.

Remark 8.2.5. What is obvious is that a fixed map $f\colon X \to Y$ induces a coarse equivalence of warped systems if and only if the trivial extension $f \times \mathrm{id}\colon X \times [1,\infty) \to Y \times [1,\infty)$ is a coarse equivalence.[3]

8.2.3 Warped systems and approximating graphs

We will now link warped systems on manifolds with (topology) approximating graphs. Most of what follows is also be true for more general metric spaces where Voronoi tiles and nets are sufficiently well-behaved. We will now use the material from Subsection 2.6.2.

Let (M, d) be a compact Riemannian manifold and S a finite set of homeomorphisms. For every $t \geq 1$, let $Y_t \subset (M, \delta_\Gamma^t)$ be a $\frac{1}{3}$-net and let $\mathcal{X}_t \coloneqq \mathcal{VR}(2, Y_t)$ be the

[3]It is interesting to note that if f is known to be a homeomorphism and it induces a coarse-equivalence then it must conjugate the two actions [Saw17b].

associated Vietoris-Rips graph (*i.e.* with edges between points of Y_t with warped distance less than 2). By Lemma 2.6.6, we deduce that the inclusion $Y_t \subset M$ induces a $(3,1)$-quasi-isometry[4] between the graph $\mathcal{X}_t$ and (M, δ_S^t) for every $t \geq 1$.

Now, consider the Voronoi tessellation $\mathcal{V}(Y_t)$ of the manifold (M,d) associated with the subset Y_t, and the induced topology approximating graph $\widetilde{\mathcal{G}}\big(F_S \curvearrowright M; \mathcal{V}(Y_t)\big)$. Note that the Vietoris-Rips graph $\mathcal{X}_t$ and the approximating graph $\widetilde{\mathcal{G}}\big(F_S \curvearrowright M; \mathcal{V}(Y_t)\big)$ have the same vertex set; the next lemma shows that this natural identification is a coarse-equivalence.

Lemma 8.2.6. *For every $t \geq 1$ the topology approximating graph $\widetilde{\mathcal{G}}\big(F_S \curvearrowright M; \mathcal{V}(Y_t)\big)$ is contained in the graph $\mathcal{X}_t$ and the inclusion is a (L,A)-quasi-isometry where the constants L and A depend only on the geometry of M.*

Proof. If (y,y') is an edge in $\widetilde{\mathcal{G}}\big(F_S \curvearrowright M; \mathcal{V}(Y_t)\big)$, by definition there must exist an element $x \in X$ such that $x \in \overline{R(y)}$ and $s \cdot x \in \overline{R(y')}$ for some $s \in S_e^{\pm}$. Then we have

$$\delta_\Gamma^t(y,y') \leq \delta_\Gamma^t(y,x) + \delta_\Gamma^t(x, s\cdot x) + \delta_\Gamma^t(s\cdot x, y') \leq \frac{1}{3} + 1 + \frac{1}{3} < 2$$

thus (y,y') is also an edge of $\mathcal{X}_t$.

Conversely, if $\delta_\Gamma^t(y,y') < 2$ then either we also have $d^t(y,y') < 2$ or there exists a point $x \in X$ with $d^t(y,x) < 1$ and $d^t(s\cdot x, y') < 1$. It is hence enough to bound the distance in $\widetilde{\mathcal{G}}\big(F_S \curvearrowright M; \mathcal{V}(Y_t)\big)$ of two vertices y, y' with $d^t(y,y') < 2$.

Picking a geodesic path γ in (M, d^t) between y and y' we can define a sequence of vertices $y = y_0, y_1, \dots, y_n = y'$ by keeping track of which regions of $\mathcal{V}(Y_t)$ are traversed by γ. Then each couple (y_i, y_{i+1}) is an edge of $\widetilde{\mathcal{G}}\big(F_S \curvearrowright M; \mathcal{V}(Y_t)\big)$. We can bound n using the geometry of M because all the regions $R(y_i)$ are contained in a ball of radius 3 of (M, d^t) and thus one can obtain the required uniform bound using volume estimate techniques (as in the proof of Lemma 2.3.14). □

This simple lemma is actually a pivotal point of this manuscript. Indeed, we now have a complete correspondence between (topology) approximating graphs and warped systems. This is a very useful asset, because the first are rather simple to study from a dynamical point of view, while the latter have some clearly defined geometric structure for one to work with.

We can now join the results of this chapter with those of Chapters 5 and 7 (recall that a family of metric expanders is a family of metric spaces uniformly coarsely equivalent to expanders).

[4] One can check that in this case it actually is a $(2,1)$-quasi-isometry

Theorem 8.2.7. *Let S be a finite set of quasi-symmetric homeomorphisms with bounded measure distortion on a compact Riemannian manifold (M, ϱ). Then the following are equivalent:*

(i) the warped system $\mathrm{WSys}\big(F_S \curvearrowright M\big)$ *is a family of metric expanders;*

(ii) the warped system is coarsely sub-equivalent to a sequence of expanders;

(iii) the action $F_S \curvearrowright M$ is expanding in measure with respect to the Riemannian volume.

Proof. (i)⇒(ii) is obvious. To prove *(ii)⇒(iii)*, first note that the topology approximating graphs $\widetilde{\mathcal{G}}\big(F_S \curvearrowright M; \mathcal{V}(Y_t)\big)$ have uniformly bounded degrees by Proposition 5.3.2 (the full argument is given in the proof of Theorem 7.1.3). Assume that $\mathcal{G}_n$ is a sequence of expanders that is coarsely equivalent to a subsequence $(M, \delta_S^{t_n})$ with $t_n \to \infty$. Then, by Lemma 2.7.5, the topology approximating graphs $\widetilde{\mathcal{G}}\big(F_S \curvearrowright M; \mathcal{V}(Y_{t_n})\big)$ must be a sequence of expanders as well. We are hence under the hypotheses of Theorem 7.1.3 and thus we deduce that the action $F_S \curvearrowright M$ is expanding in measure.

Finally, since we already remarked that the topology approximating graphs have uniformly bounded degrees and are uniformly quasi isometric to the spaces (M, δ_S^t), to prove *(iii)⇒(i)* it is enough to apply Lemma 5.2.3 for every $t \geq 1$.

□

Corollary 8.2.8. *If an action $F_S \curvearrowright M$ as in Theorem 8.2.7 is also measure-preserving, then (i), (ii) and (iii) are equivalent to the action $F_S \curvearrowright M$ having spectral gap.*

Corollary 8.2.9. *Let G be a compact Lie group and let $\Gamma = \langle S \rangle$ where $S \subseteq G$ is a finite subset. Choose any Riemannian metric on G. Then the warped system* $\mathrm{WSys}\big(\Gamma \curvearrowright G\big)$ *is a family of metric expanders if and only if S is a Kazhdan set of G.*

Proof. If the volume form induced by the Riemannian metric coincides with the Haar measure then the statement follows from Theorem 8.2.7 and Corollary 6.2.11. Any other Riemannian volume form equals $f(x)\nu(x)$ for some strictly positive smooth function f. Since G is a compact, there are constants $0 < c < C$ so that $c < f(x) < C$ $\forall x \in G$ and it follows that $\Gamma \curvearrowright G$ is expanding in measure with respect to ν if and only if it is expanding in measure with respect to $f(x)\nu(x)$.

Alternatively, it follows from Corollary 8.1.6 that choosing a different Riemannian metric on G produces a coarsely equivalent warped system. The statement follows from the case where the volume coincides with the Haar measure because the definition of metric expanders is rigid under coarse equivalences. □

Since warped systems are (families of spaces) coarsely equivalent to level sets of warped cones, applying Theorem 2.7.9 and Proposition 5.4.3 we obtain the following:[5]

Corollary 8.2.10. *If an action $F_S \curvearrowright M$ as in Theorem 8.2.7 is expanding in measure, then the warped cone $\mathcal{O}_{F_S}(M)$ does not coarsely embed into any L^p space.*

Moreover, if the action is measure preserving and it has E-spectral gap for a Banach space E then $\mathcal{O}_{F_S}(M)$ does not coarsely embed into E.

Note in particular that we can produce examples of warped cones arising from the action by left multiplication $\Gamma \curvearrowright G$ of a subgroup Γ of a compact Lie group G that do not coarsely embed into Hilbert spaces even if the warping group Γ has the Haagerup property (*e.g.* any $F_2 < \mathrm{SU}(2, \mathbb{C})$ with spectral gap). Before [NS17], this used to be an open question attributed to Roe himself.[6]

8.3 Local rigidity

We now wish to study the coarse geometry of warped systems/approximating graphs. The first reason for doing so is to show that the expanders that we can construct using this machinery are something really new and not just a light modification of some previously known examples.

Secondly, we will be able to prove some coarse geometric rigidity results that will allow us to prove that we can construct *many* inequivalent families of expanders.

It is also conceivable that some of the results concerning the coarse geometry of warped cones turn out to be helpful as invariants to study properties of dynamical actions.

In this section we will be concerned with geometric information coming from the local structure of warped cones over manifolds. For doing this, we will have to restrict to actions by isometries.

[5] Corollary 8.2.10 was independently proved in [NS17].

[6] It is folklore to attribute this question to J. Roe, as he proved in [Roe05] that if a warped cone warped cone arising from the action by multiplication of a subgroup Γ of a compact Lie group coarsely embed into a Hilbert space then Γ must have the Haagerup property. Still, we could not find this question written anywhere. Chances are that it was asked in private communication.

8.3.1 Local geometry of warped systems by isometries

Let $\Gamma \curvearrowright M$ be an action by isometries on a Riemannian manifold and S a fixed generating set of Γ. In particular, Γ is equipped with the induced *right*[7] word metric $|\cdot|$ and left and right Cayley graphs (Subsection 2.2.3). These are fixed, because so is the generating set. To lighten the notation, in what follows we will denote the ball $B_{(\Gamma,|\cdot|)}(e,r) \subset \Gamma$ simply by $B_\Gamma(r)$.

Following our convention, for every subset $Y \subseteq M$ we denote by $B_\Gamma(r) \cdot Y$ the union of the images of Y under the elements in $B_\Gamma(r)$:

$$B_\Gamma(r) \cdot Y \coloneqq \bigcup \{\gamma \cdot Y \mid \gamma \in \Gamma,\ |\gamma| < r\}.$$

We define the set $\chi^t_\Gamma(r) \subset M$ as follows:

$$\chi^t_\Gamma(r) \coloneqq \Big\{ x \in M \;\Big|\; \exists \gamma \in \Gamma,\ |\gamma| \leq 6r \text{ such that } d(x, \gamma \cdot x) \leq \frac{6r}{t} \Big\}.$$

That is, $\chi^t_\Gamma(r)$ is the set of points that are 'almost fixed' (up to an error of the order of r/t) by some element of Γ of length at most $6r$.

The rationale for defining $\chi^t_\Gamma(r)$ comes from the fact that we already know that for every point $x \in M$ the orbit map $\gamma \mapsto \gamma(x)$ induces a 1-Lipschitz embedding of the right Cayley graph $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Gamma, S)$ to (M, δ^t_Γ) for every t. Moreover, if x has trivial stabilizer, we expect that this embedding $\overline{\mathrm{Cay}}^{\mathrm{r}}(\Gamma, S) \hookrightarrow (M, \delta^t_\Gamma)$ tends to be a (locally) bi-Lipschitz embedding when t goes to infinity. In order to prove this, one needs some control on the neighbourhoods of a fixed radius r in the warped system. Since the action is by isometries, the warped distance between two points $x, y \in (X, \delta^t_\Gamma)$ can be expressed as

$$\delta^t_\Gamma(x,y) = \inf_{\gamma \in \Gamma} \big[d^t(x, \gamma \cdot y) + |\gamma| \big] = \inf_{\gamma \in \Gamma} \big[t d(x, \gamma \cdot y) + |\gamma| \big] \tag{8.2}$$

(Lemma 2.2.7), and thus we have the following inclusion of neighbourhoods:

$$N_{(M,\delta^t_\Gamma)}(Y, r) \subseteq B_\Gamma(r) \cdot N_{(M,d)}\Big(Y, \frac{r}{t}\Big) = N_{(M,d)}\Big(B_\Gamma(r) \cdot Y\,,\ \frac{r}{t}\Big).$$

In particular, the set $\chi^t_\Gamma(r)$ contains the set of 'bad points' of M whose neighbourhood of radius r with respect to the warped metric δ^t_Γ could self-intersect in unexpected ways (the number 6 appears for technical reasons).

We can now (asymptotically) characterise balls in warped systems over isometric actions up to bi-Lipschitz equivalence. Equip the direct product $\Gamma \times \mathbb{E}^k$ is with the ℓ^1-distance (*i.e.* defined by $d\big((\gamma_1, v_1), (\gamma_2, v_2)\big) \coloneqq \big|\gamma_1 \gamma_2^{-1}\big| + \|v_1 - v_2\|_2$). Then we have the following:[8]

[7] We generally prefer to use the left word metric, but all the results of this section are natural to state in the right word metric.

[8] Similar observations were made in [SW17, Lemma 3.8].

Lemma 8.3.1. *Let $L > 1$ and $r > 0$ be fixed and $\Gamma \curvearrowright M$ an action by isometries. Then there exists a t_0 large enough so that for every $t \geq t_0$ and for every $x_0 \notin \chi^t_\Gamma(r)$, the ball $B_{(M,\delta^t_\Gamma)}(x_0, r)$ is L-bi-Lipschitz equivalent to the ball of radius r in the product $\Gamma \times \mathbb{E}^{\dim(M)}$.*

In particular, for any point $x \notin \chi^t_\Gamma(r)$ for some $t \in \mathbb{R}$, the ball of radius r around x in the warped system $\mathrm{WSys}(\Gamma \curvearrowright M)$ is asymptotically L-bi-Lipschitz to the r-ball in $\Gamma \times \mathbb{E}^{\dim(M)}$.

Proof. Fix $t > 1$ and $x_0 \notin \chi^t_\Gamma(r)$. By definition of $\chi^t_\Gamma(r)$, it follows that the balls $B_{(M,d^t)}(\gamma \cdot x_0, 3r)$ with $\gamma \in B_\Gamma(3r)$ are disjoint. In fact, we already noted that the image $\gamma \cdot B_{(M,d^t)}(x_0, 3r)$ coincides with the ball $B_{(M,d^t)}(\gamma \cdot x_0, 3r)$. Thus, if two balls $B_{(M,d^t)}(\gamma_1 \cdot x_0, 3r)$ and $B_{(M,d^t)}(\gamma_2 \cdot x_0, 3r)$ intersect, then $B_{(M,d^t)}(x_0, 3r) \cap \gamma_1^{-1}\gamma_2 \cdot B_{(M,d^t)}(x_0, 3r) \neq \emptyset$ and hence $d^t(x_0, \gamma_1^{-1}\gamma_2(x_0)) \leq 6r$.

Since M is compact, the infimum in the equality (8.2) is actually a minimum. Therefore, for every two points $x, y \in B_{(M,\delta^t_\Gamma)}(x_0, r)$, there exist $\gamma_x, \gamma_y \in B_\Gamma(r)$ so that $\delta^t_\Gamma(x, x_0) = |\gamma_x| + d^t(\gamma_x \cdot x_0, x)$ and $\delta^t_\Gamma(y, x_0) = |\gamma_y| + d^t(\gamma_y \cdot x_0, y)$. Again by (8.2), there exists a $\gamma \in \Gamma$ with $|\gamma| \leq 2r$ so that

$$2r \geq \delta^t_\Gamma(x, y) = |\gamma| + d^t(\gamma \cdot x, y).$$

It follows that the point y belongs to both $\gamma\gamma_x \cdot B_{(M,d^t)}(x_0, 3r)$ and $\gamma_y \cdot B_{(M,d^t)}(x_0, r)$ and therefore, by construction, we must have $\gamma = \gamma_y\gamma_x^{-1}$.

As a consequence, we deduce that the ball of radius r centred at x_0 in (M, δ^t_Γ) is actually isometric to the ball of radius r centred at (x_0, e) in the direct product $\Gamma \times (M, d^t)$ equipped with the ℓ^1-distance. In fact, consider the map $\phi\colon \Gamma \times (M, d^t) \to M$ defined by $(\gamma, z) \mapsto \gamma(z)$. For every pair $(\gamma, z) \in B_{\Gamma\times(M,d^t)}\big((e, x_0), r\big)$ we have $\delta^t_\Gamma(x_0, \gamma(z)) \leq |\gamma| + d^t(\gamma(x_0), \gamma(z)) \leq r$ so that ϕ sends the relevant ball into the right ball. Vice versa, given $x \in B_{(M,\delta^t_\Gamma)}(x_0, r)$, we deduce as that there exists a unique $\gamma_x \in B_\Gamma(r)$ such that $x \in \gamma_x \cdot B_{(M,d^t)}(x_0, r)$ and hence letting $z = \gamma^{-1}(x)$ we have $x = \phi(\gamma_x, z)$. We thus obtained a bijection, and from the preceding paragraph we deduce that for any two points $(\gamma_x, z_x), (\gamma_y, z_y) \in \Gamma \times (B, d^t)$ we have

$$\delta^t_\Gamma\big(\phi(\gamma_x, z_x), \phi(\gamma_y, z_y)\big) = \big|\gamma_y\gamma_x^{-1}\big| + d^t\big(\gamma_y\gamma_x^{-1} \cdot \gamma_x(z_x), \gamma_y(z_y)\big) = \big|\gamma_y\gamma_x^{-1}\big| + d^t(z_x, z_y)$$

and the latter is precisely the ℓ^1-distance that we are working with (*i.e.* using the *right* word metric).

The statement of the lemma thus reduces to proving that $B_{\Gamma\times(M,d^t)}\big((e, x_0), r\big)$ is L-bi-Lipschitz equivalent to $B_{\Gamma\times\mathbb{E}^{\dim(M)}}\big((e, 0), r\big)$ for t large enough. Since in both

cases we are considering the ℓ^1-distance on the product, it is enough to prove that $B_{(M,d^t)}(x_0, r)$ and $B_{\mathbb{E}^{\dim(M)}}(0, r)$ are L-bi-Lipschitz equivalent.

Note that the ball $B_{(M,d^t)}(x_0, r)$ is isometric to the ball $B_{(M,d)}(x_0, \frac{r}{t})$ with the metric rescaled by t, and similarly we have that $B_{\mathbb{E}^{\dim(M)}}(0, r)$ is isometric to $B_{\mathbb{E}^{\dim(M)}}(0, \frac{r}{t})$ with the metric rescaled by t. Rescaling by t on both sides it is hence enough to show that there is a bi-Lipschitz equivalence between $B_{\mathbb{E}^{\dim(M)}}(0, \frac{r}{t})$ and $B_{(M,d)}(x_0, \frac{r}{t})$ when t is large enough. This equivalence is given to us by the Riemannian exponential (Corollary 2.3.9).

□

Lemma 8.3.1 completely describes the local structure of warped systems outside from a set of 'bad points'. This will be used in the next section to prove a coarse rigidity result, but before doing that we will have to show that if the set of bad points of M is small and $\mathrm{WSys}(\Gamma \curvearrowright M)$ is coarsely sub-equivalent to $\mathrm{WSys}(\Lambda \curvearrowright N)$ then the coarse equivalences will send the set of bad points to small sets.

More precisely, recall that an action on a measure space is *essentially free* if the set of points with nontrivial stabilizer has measure zero. Then we prove the following:

Lemma 8.3.2. *Let $\Gamma \curvearrowright M$ and $\Lambda \curvearrowright N$ be essentially free actions by isometries on compact Riemannian manifolds and let $L, A, r > 0$ be fixed. If there exist increasing unbounded sequences $(t_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$, and (L, A)-quasi-isometries $f_k \colon (M, \delta_\Gamma^{t_k}) \to (N, \delta_\Lambda^{s_k})$, then for every k large enough, there exists a point $x_k \in M \setminus \chi_\Gamma^{t_k}(r)$ whose image $f_k(x_k)$ is* not *in $\chi_\Lambda^{s_k}(r)$.*

Proof. The rough idea is to show that, since the action is essentially free, the measure of both $\chi_\Gamma^{t_k}(r)$ (and its image) and $\chi_\Lambda^{s_k}(r)$ tends to 0 as k goes to infinity. This would be easy to do if the maps f_k were nice and regular, but since they are only quasi-isometries we will have to work a bit harder to control the size of the image of $\chi_\Gamma^{t_k}(r)$.

Without loss of generality, we renormalize the Riemannian metrics so that M and N have volume 1. Let $Y_k \subset (M, \delta_\Gamma^{t_k})$ be an $L(A+1)$-net subset. Note that the balls $B_{(M,\delta_\Gamma^{t_k})}\left(y, L\frac{A+1}{2}\right)$ with $y \in Y_k$ are disjoint, and (in the notation of Subsection 2.3.3) they have volume bounded between $v_M\left(L\frac{A+1}{2t_k}\right)$ and $V_M\left(L\frac{A+1}{2t_k}\right)|B_\Gamma(L\frac{A+1}{2})|$.

Let $Z_k \subseteq Y_k$ be the subset of those points which are close to $\chi_\Gamma^{t_k}(r)$:

$$Z_k \coloneqq \left\{ y \in Y_k \;\middle|\; \delta_\Gamma^{t_k}\left(y, \chi_\Gamma^{t_k}(r)\right) < L(A+1) \right\}$$

and let $\Omega_k := N_{(M,\delta_\Gamma^{t_k})}\big(Z_k, L(A+1)\big)$. Then $\chi_\Gamma^{t_k}(r)$ is contained in Ω_k and Ω_k is contained in a "small" neighbourhood of $\chi_\Gamma^{t_k}(r)$:

$$\Omega_k \subseteq N_{(M,\delta_\Gamma^{t_k})}\big(\chi_\Gamma^{t_k}(r), 2L(A+1)\big) \subseteq B_\Gamma(2L(A+1)) \cdot N_{(M,d)}\Big(\chi_\Gamma^{t_k}(r), 2L\frac{A+1}{t_k}\Big).$$

Note that the measure of the right-hand side tends to 0 as k tends to infinity, because the sets $\chi_\Gamma^{t_k}(r)$ form a sequence of closed nested subsets that converge (in measure) to the union of the sets of fixed points of finitely many elements of Γ.

Combining the two inequalities

$$|Z_k| v_M\Big(L\frac{A+1}{2t_k}\Big) \leq \mathrm{Vol}(\Omega_k) \to 0,$$
$$|Y_k||B_\Gamma(L(A+1))| V_M\Big(L\frac{A+1}{t_k}\Big) \geq \mathrm{Vol}(M) = 1$$

with with the uniform bound on volumes of balls of small radii (Lemma 2.3.14), we obtain that the ratios $|Z_k|/|Y_k|$ tend to 0 as k goes to infinity.

Now, since f_k is an (L, A)-quasi-isometry, the image $f_k(Y_k)$ is a 1-separated $(L^2(A+1)+2A)$-dense subset of $(N, \delta_\Lambda^{s_k})$ (see Lemma 2.6.3) and we also have

$$f_k\big(\chi_\Gamma^{t_k}(r)\big) \subseteq f_k(\Omega_k) \subseteq N_{(N,\delta_\Lambda^{s_k})}\big(f_k(Z_k), L^2(A+1)+A\big).$$

We deduce that the volume of (a neighbourhood of) $f_k(\Omega_k)$ is bounded above by

$$V_N\Big(\frac{L^2(A+1)+A}{s_k}\Big)|B_\Lambda(L^2(A+1)+A)||Z_k|. \tag{8.3}$$

On the other hand, 1-separatedness gives us an upper bound on $|Y_k|$ in terms of v_N:

$$|Y_k| v_N\Big(\frac{1}{2s_k}\Big) \leq \mathrm{Vol}(N) = 1. \tag{8.4}$$

Since $|Z_k|/|Y_k|$ tends to 0 as k tends to infinity, combining the estimates (8.3) and (8.4) and applying once more Lemma 2.3.14 implies that also the measure of (a neighbourhood of) $f_k(\Omega_k)$ tends to 0. As we also have that the volume of $\chi_\Lambda^{s_k}(r) \subseteq N$ tends to 0, the statement of the lemma follows trivially. □

8.3.2 Stable rigidity for warped systems

We can now prove the quasi-isometric rigidity result (see Subsection 2.8.2 for the definition of filtrations and box spaces). The following theorem is inspired by [KV17, Theorem 7]

Theorem 8.3.3. *Let $\Gamma \curvearrowright (M, d)$ be an essentially free action by isometries on a compact Riemannian manifold and Λ a group generated by a finite set S'.*

(i) If Λ acts essentially freely by isometries on a compact Riemannian manifold N and the warped systems $\mathrm{WSys}(\Gamma \curvearrowright M)$ and $\mathrm{WSys}(\Lambda \curvearrowright N)$ are coarsely sub-equivalent, then $\Lambda \times \mathbb{Z}^{\dim(N)}$ is quasi-isometric to $\Gamma \times \mathbb{Z}^{\dim(M)}$.

(ii) If there exists a normal residual filtration $(\Lambda_k)_{k\in\mathbb{N}}$ of Λ so that the box space $\square_{(\Lambda_k)}\Lambda$ is coarsely sub-equivalent to $\mathrm{WSys}(\Gamma \curvearrowright M)$, then Λ is quasi-isometric to $\Gamma \times \mathbb{Z}^{\dim(M)}$.

Proof. Consider the first assertion. Suppose that $(t_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ are increasing unbounded sequences such that the levels $(M, \delta_\Gamma^{t_k})$ and $(N, \delta_\Lambda^{s_k})$ are uniformly coarsely equivalent. Then there exists a sequence of quasi-isometries $f_k \colon (M, \delta_\Gamma^{t_k}) \to (N, \delta_\Lambda^{s_k})$ that are all (L, A)-quasi-isometries for some fixed constants L and A. In particular, by Lemma 2.6.3 they are (L, A)-quasi-isometric embeddings with A-dense image.

Fix an integer radius $r \in \mathbb{N}$. By Lemma 8.3.2, for every k large enough there exists a point $x_k \in M \setminus \chi_\Gamma^{t_k}(r)$ such that $f_k(x_k)$ is not in $\chi_\Lambda^{s_k}(r)$. Let $y_k \coloneqq f_k(x_k)$. Fix $\epsilon > 0$ small. By Lemma 8.3.1, we also have that there exists a $k = k(r)$ large enough so that the balls $B_{(M,\delta_\Gamma^{t_k})}(x_{k(r)}, r)$ and $B_{(N,\delta_\Lambda^{s_k})}(y_{k(r)}, Lr + A)$ are $(1+\epsilon)$-bi-Lipschitz equivalent to $B_{\Gamma\times\mathbb{E}^m}(r)$ and $B_{\Lambda\times\mathbb{E}^n}(Lr + A)$ respectively, where $m = \dim(M)$ and $n = \dim(N)$.

Note that the inclusion $\mathbb{Z}^d \hookrightarrow \mathbb{E}^d$ is a $(\sqrt{d}, \sqrt{d})$-quasi-isometry and that the restriction of f_k to $B_{(M,\delta_\Gamma^{t_k})}(x_k, r)$ is an (L, A)-quasi-isometric embedding into $B_{(N,\delta_\Lambda^{s_k})}(y_k, Lr + A)$. We then have a concatenation of quasi-isometric embeddings as depicted in the following diagram:

$$\begin{array}{ccccc}
B_{\Gamma\times\mathbb{Z}^m}(r) & \xhookrightarrow{(\sqrt{m},\sqrt{m})} & B_{\Gamma\times\mathbb{E}^m}(r) & \xleftrightarrow{(1+\epsilon,0)} & B_{(M,\delta_\Gamma^{t_k})}(x_{k(r)}, r) \\
\big\downarrow \hat{f}_r & & & & \big\downarrow f_{k(r)} \\
B_{\Lambda\times\mathbb{Z}^n}\big(\sqrt{n}(Lr + A + 1)\big) & \xhookleftarrow{(\sqrt{n},\sqrt{n})} & B_{\Lambda\times\mathbb{E}^n}(Lr + A) & \xleftrightarrow{(1+\epsilon,0)} & B_{(N,\delta_\Lambda^{s_k})}(y_{k(r)}, Lr + A),
\end{array}$$

where $\hat{f}_r$ is defined as the composition and the labels represent the quasi-isometry constants. Then $\hat{f}_r$ is a (L', A')-quasi-isometric embedding where $L' = \sqrt{nm}L$ and $A' = \sqrt{n}(\sqrt{m}L + A + 1)$ (if the ϵ coming from the bi-Lipschitz map is small enough, we can ignore it altogether because the distances in $\Gamma \times \mathbb{Z}^m$ and $\Lambda \times \mathbb{Z}^n$ take integer values).

We thus obtained a sequence of uniform quasi-isometric embeddings $\hat{f}_r$. Note that, by construction, $\hat{f}_r$ sends the identity element of $\Gamma \times \mathbb{Z}^m$ to the identity element of $\Lambda \times \mathbb{Z}^n$. It follows that for every fixed vertex $v \in \Gamma$, the image $\hat{f}_r(v)$ can only take finitely many values in $\Lambda \times \mathbb{Z}^n$ and hence there exists a subsequence $\hat{f}_{r_l}$ such that $\hat{f}_{r_l}(v)$ is constant.

Using a diagonal argument, we can further pass to a subsequence $\hat{f}_{r_i}$ such that for every $i > j$ the restriction of $\hat{f}_{r_i}$ to the ball $B_{\Gamma\times\mathbb{Z}^m}(j)$ coincides with $\hat{f}_{r_j}$. It follows that setting $\hat{f}|_{B_{\Gamma\times\mathbb{Z}^m}(i)} \coloneqq \hat{f}_{r_i}$ gives a well-defined (L', A')-quasi-isometric embedding $\hat{f}\colon \Gamma \times \mathbb{Z}^m \to \Lambda \times \mathbb{Z}^n$.

By Lemma 2.6.3, it only remains to show that $\hat{f}$ is coarsely surjective. This is easily done, because if g is any quasi-isometry and R is any radius, then there exists an $R' \geq R$ large enough so that the image $g\big(B(x, R')\big)$ is coarsely dense in $B\big(g(x), R\big)$. As $\hat{f}_r$ is defined as a composition of (restrictions of) quasi-isometries, it follows that for every $R > 0$ the image of $\hat{f}_r$ is coarsely dense in $B_{\Lambda\times\mathbb{Z}^n}(R)$ for every r large enough and therefore the same holds true for $\hat{f}$.

The proof of (ii) follows the same lines. Indeed, assume that there exist uniform (L, A)-quasi-isometries $f_k\colon (M, \delta_S^{t_k}) \to \mathrm{Cay}(\Lambda/\Lambda_k, S')$ for some sequence $t_k \to \infty$. Since the normal subgroups Λ_k are a nested sequence with trivial intersection, for every $r \in \mathbb{N}$ there is a $k = k(r)$ large enough so that the ball of radius $Lr + A$ in the Cayley graph $\mathrm{Cay}(\Lambda/\Lambda_k, S')$ and the ball of radius $Lr + A$ in $\mathrm{Cay}(\Lambda, S')$ are isometric. One can hence fix some points $x_k \in M \setminus \chi_\Gamma^{t_k}(r)$ and consider the diagram

$$\begin{array}{ccccc}
B_{\Gamma\times\mathbb{Z}^m}(r) & \xrightarrow{(\sqrt{m},\sqrt{m})} & B_{\Gamma\times\mathbb{E}^m}(r) & \xleftarrow{(1+\epsilon,0)} & B_{(M,\delta_\Gamma^{t_k})}(x_{k(r)}, r) \\
\big\downarrow \hat{f}_r & & & & \big\downarrow f_{k(r)} \\
B_\Lambda(Lr + A) & & \xleftarrow{\qquad\cong\qquad} & & B_{\Lambda_k\backslash\Lambda}(Lr + A)
\end{array}$$

and argue as above. □

Remark 8.3.4. For the proof of (ii) of Theorem 8.3.3 we needed to restrict to normal subgroups $\Lambda_k \lhd \Lambda$ because for non-normal subgroups it is not in general true that balls of radius $Lr + A$ in the Schreier graphs $\mathrm{Schr}(\Lambda_k\backslash\Lambda, S')$ are isometric to the ball of radius $Lr + A$ in $\mathrm{Cay}(\Lambda, S')$ for large k. Still, it is true that for large k the ball of radius $Lr + A$ centred on the coset $\Lambda_k \in \Lambda_k\backslash\Lambda$ in the Schreier graph $\mathrm{Schr}(\Lambda_k\backslash\Lambda, S')$ is isometric to the ball of radius $Lr + A$ in $\mathrm{Cay}(\Lambda, S')$. One can easily modify the proof of (ii) to prove the analogous statement in the case of non-normal residual nested

box spaces assuming that $\chi_\Gamma^{t_k}(r)$ be empty for large k (*i.e.* assuming that the action $\Gamma \curvearrowright M$ be free).

8.3.3 First examples of coarsely disjoint superexpanders

Theorem 8.3.3 is already enough to prove that some classes of examples that we constructed are coarsely disjoint (*i.e.* they are not coarsely sub-equivalent). Still, to do so one needs to employ quasi-isometric rigidity results for groups in order them to distinquish groups up to stabilisation by $\mathbb{Z}^n$.

As an example, it follows from the Splitting Theorem in [KL97] (see also [KKL98]) that if Γ and Λ are cocompact lattices in a semisimple algebraic group with no rank-one simple factors and $\Gamma \times \mathbb{Z}^m$ is quasi-isometric to $\Lambda \times \mathbb{Z}^n$, then $m = n$ and Γ is quasi-isometric to Λ. Note in particular that the Splitting Theorem applies to the group $\Gamma_d = \mathrm{SO}(d, \mathbb{Z}[\frac{1}{5}])$ with $d \geq 5$ (Section 6.3).

Then Theorems 8.3.3 and 8.2.7 imply the following:

Corollary 8.3.5. *The metric superexpanders* $\mathrm{WSys}(\Gamma_d \curvearrowright \mathrm{SO}(d, \mathbb{R}))$ *and* $\mathrm{WSys}(\Gamma_{d'} \curvearrowright \mathrm{SO}(d', \mathbb{R}))$ *(see Definition 2.7.6 and Corollary 7.1.6) are coarsely disjoint if* $d \neq d'$*. Moreover, the metric superexpanders* $\mathrm{WSys}(\Gamma_d \curvearrowright \mathrm{SO}(d, \mathbb{R}))$ *and* $\mathrm{WSys}(\Gamma_d \curvearrowright \mathbb{S}^{d-1})$ *(with the same* d*) are also coarsely disjoint.*

The class of cocompact lattices in a semisimple algebraic group over a non-Archimedean local field, in which every simple factor is of higher rank, is quasi-isometrically rigid [KL97] (see also [EF97]). It then follows from Theorem 8.3.3 that any normal residual nested box space of such a group Λ cannot be coarsely sub-equivalent to any warped system $\mathrm{WSys}(\Gamma \curvearrowright M)$ (where the action is essentially free and by isometries on a compact manifold). Indeed, if there existed a normal residual nested box space $\square_{(\Lambda_k)}\Lambda$ that is coarsely sub-equivalent to $\mathrm{WSys}(\Gamma \curvearrowright M)$ then Theorem 8.3.3 would imply that $\Gamma \times \mathbb{Z}^d$ is quasi isometric to Λ and, by the quasi-isometrc rigidity, we would deduce that $\Gamma \times \mathbb{Z}^d$ must be a group of the same sort (cocompact lattice with higher rank factors etc.). In particular, $\Gamma \times \mathbb{Z}^d$ should have Kazhdan's property (T), which it clearly does not. This implies the following theorem.

Theorem 8.3.6. *If* Γ *is a group with Lafforgue strong Banach property (T) with an ergodic essentially free action by isometries on a compact Riemannian manifold* M*,*

then the warped system $\mathrm{WSys}(\Gamma \curvearrowright M)$ *is a metric superexpander that is not coarsely sub-equivalent to a Lafforgue expander (Definition 2.8.5).*[9]

Remark 8.3.7. In Corollary 8.3.5 we used Theorem 8.3.3 to show that some families of (metric) superexpanders are not coarsely sub-equivalent. Still, we actually know something more *i.e.* that they are not even "locally" coarsely sub-equivalent. For instance, if we assume the actions to be free we can avoid using Lemma 8.3.2 and then the proof of Theorem 8.3.3.(i) works verbatim with the following (weaker) assumptions: there exist sequences of points $(x_k)_{k\in\mathbb{N}}$ in M and $(y_k)_{k\in\mathbb{N}}$ in N, increasing unbounded sequences $t_k \to \infty$ and $s_k \to \infty$, and neighbourhoods $A_k \subset (M, \delta_\Gamma^{t_k})$ and $B_k \subset (N, \delta_\Lambda^{s_k})$ of x_k and y_k, respectively, such that A_k and B_k are uniformly quasi-isometric and for every $r > 0$ there exists a k large enough so that the balls $B_{(M,\delta_\Gamma^{t_k})}(x_k, r)$ and $B_{(N,\delta_\Lambda^{s_k})}(y_k, r)$ are contained in A_k and B_k, respectively.

8.4 Discrete fundamental groups of warped systems

We will now turn towards more global coarse invariants of warped systems. In particular, we want to study their discrete fundamental groups (Section 3.1). Our main tools will come from specialising to warped systems the general results obtained in Chapter 4. For this reason, in this section we will restrict our attention to a metric space X that has homotopy rectifiable paths and, for our main result, we will also need to assume that X is compact.

8.4.1 Computing the discrete fundamental group

Let $F_S \curvearrowright (X, d)$ be a continuous action and recall that δ_S^t denotes the rescaled metric d^t on X warped by F_S. Note that if X is compact and $(X, \delta_S^{t_0})$ is jumping-geodesic, then also (X, δ_S^t) is jumping-geodesic for every $t > t_0$ (in particular, the warped system is asymptotically jumping-geodesic). Indeed, by compactness, for every $x, x' \in X$ there exist $y_1, \dots, y_n \in X$ and $\vec{s}_1 \dots \vec{s}_n \in S \sqcup S^{-1}$ so that

$$\delta_S^t(x, x') = d^t(x, y_1) + 1 + d^t\big(\vec{s}_1(y_1), y_2\big) + 1 + \dots + d^t\big(\vec{s}_n(y_n), x'\big)$$

(Lemma 2.2.5). We also know that between x and y_1 there is a jumping-geodesic in $(X, \delta_S^{t_0})$. Note that this jumping-geodesic cannot have any jump, otherwise performing the same jump on the level t we would obtain a path between x and y_1 that is strictly

[9] In our definition of Lafforgue expanders we insist that the filtration be normal. Still, if the action $\Gamma \curvearrowright M$ is free, we can deduce that $\mathrm{WSys}(\Gamma \curvearrowright M)$ is not even coarsely subequivalent to non-normal Lafforgue expanders (see Remark 8.3.4).

shorter than $d^t(x, y_1)$. We deduce that the jumping-geodesic between x and y_1 (in the level t_0) is actually a continuous path and it is hence a continuous geodesic in (X, d) (and (X, d^t)). The same argument holds for all the other pairs of points (y_n, y_{n+1}), and gluing together these continuous paths we obtain the required jumping-geodesic between x and x' in (X, δ_S^t).

Let $\mathrm{Ell}_\theta \coloneqq \{w \in F_S \mid |w| \leq 4\theta,\ \mathrm{Fix}(w) \neq \emptyset\}$ be the set of *elliptic elements* (*i.e.* elements with fixed points) of length at most 4θ (in general this set is not closed under conjugation because of the condition $|w| \leq 4\theta$). Note that if $y \in X$ is a fixed point of w and β is a continuous path joining x_0 to y, then $\alpha_w w(\beta)\beta^*$ is a closed loop in X.

Theorem 8.4.1. *Let X be compact, (X, δ_S) asymptotic jumping-geodesic and fix $\theta \in \mathbb{N}$. Then there exists a t_0 large enough so that for every $t \geq t_0$ and $w \in F_S$ there exists a path γ so that $([\gamma], w) \in \ker\big(\widehat{\Phi}_S \colon \pi_1(X) \rtimes_{\phi_S} F_S \to \pi_{1,\theta}(X, \delta_S^t)\big)$ if and only if $w \in \langle\langle \mathrm{Ell}_\theta \rangle\rangle$.*

Moreover, if X is semi-locally simply connected we can choose t_0 large enough so that for every $t \geq t_0$ we have

$$\pi_{1,\theta}\big((X, \delta_S^t), x_0\big) \cong \big(\pi_1(X, x_0) \rtimes_{\phi_S} F_S\big) \big/ \langle\langle K_\theta \rangle\rangle$$

where

$$K_\theta \coloneqq \big\{\big([\beta w(\beta^*)\alpha_w^*], w\big) \mid w \in \mathrm{Ell}_\theta,\ \beta(0) = x_0,\ \beta(1) \in \mathrm{Fix}(w)\big\}.$$

Proof. Following the proof of Lemma 4.3.1 it is easy to see that $K_\theta \subseteq \ker \widehat{\Phi}_S$ for t large enough. Therefore, for every $w \in \langle\langle \mathrm{Ell}_\theta \rangle\rangle$ we have explicitly exhibited the required path γ so that $([\gamma], w)$ is in the kernel.

We now prove the converse. First of all, in the sequel we always assume that t_0 is large enough so that $(X, \delta_S^{t_0})$ is jumping-geodesic (we will not mention this anymore).

Given a word $w = \vec{s}_1 \cdots \vec{s}_{|w|}$, a point $x \in X$ and a radius $r \geq 0$, we define a sequence of sets as follows: $C_w^{(0)}(x, r)$ is the ball $B(x, r)$ in (X, d) and for $1 \leq i \leq |w|$ we let

$$C_w^{(i)}(x, r) \coloneqq N_r\Big(\vec{s}_i\big(C_w^{(i-1)}(x, r)\big)\Big)$$

where N_r is the neighbourhood of radius r with respect to the distance d. Finally, let $C_w(x, r) \coloneqq C_w^{(|w|)}(x, r)$.

Note that as r tends to zero, the set $C_w^{(i)}(x, r)$ converges to the single point $\vec{s}_i \cdots \vec{s}_1(x)$ and in particular $C_w(x, r)$ converges to $w_{\mathrm{rev}}(x)$. By compactness, if w_{rev} does not have fixed points there exists a radius $r_w > 0$ so that $x \notin C_w(x, r)$ for every $x \in X$ and $r \leq r_w$. We let

$$t_0 \coloneqq \max\left\{\frac{4\theta}{r_w} \;\middle|\; |w| \leq 4\theta,\ \mathrm{Fix}(w_{\mathrm{rev}}) = \emptyset\right\} \cup \{1\}.$$

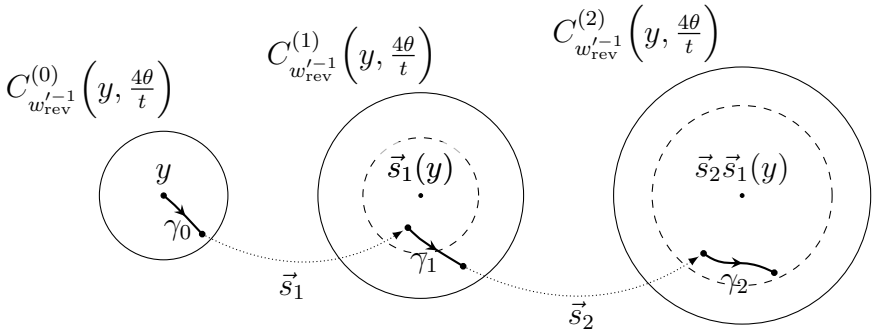


Figure 8.1: The jumping-path $\vec{\boldsymbol{\gamma}}'$ is contained in the sets $C^{(i)}_{w'^{-1}_{\mathrm{rev}}}\left(y, \frac{4\theta}{t}\right)$, where w'^{-1}_{rev} is the word $\vec{s}_1 \cdots \vec{s}_n$.

Fix $t \geq t_0$. By Theorem 4.2.2, the kernel of the discretisation map is $\langle \mathrm{FT}_\theta \rangle$; it is therefore enough to show that if $\Phi_S([\gamma], w) \in \mathrm{FT}_\theta$ then w is conjugate to an element of Ell_θ.

Let $([\gamma], w)$ be such a pair and let $\vec{\boldsymbol{\gamma}} = \gamma \psi_S(w) = \gamma \alpha_w \overset{w_{\mathrm{rev}}^{-1}}{\rightsquigarrow} x_0$. By hypothesis, there exists a jumping-path $\vec{\boldsymbol{\gamma}}' \in \mathcal{T}_\theta$ which is freely-homotopic to $\vec{\boldsymbol{\gamma}}$. Tracing the base point under the free homotopy, we thus obtain a jumping-path $\vec{\boldsymbol{\beta}}$ so that $\vec{\boldsymbol{\gamma}}$ is homotopic to $\vec{\boldsymbol{\beta}}\vec{\boldsymbol{\gamma}}'\vec{\boldsymbol{\beta}}^*$. Choose a continuous path ξ going from x_0 to $\vec{\boldsymbol{\gamma}}'(0)$ and let $([\gamma'], w') = \Phi_S^{-1}\big(\xi \vec{\boldsymbol{\gamma}}' \xi^*\big)$, where we require that the word w' matches exactly the sequence of (inverses of) jumps in the path $\vec{\boldsymbol{\gamma}}'$. Then $|w'| \leq 4\theta$ and w' is conjugated to w, as $([\gamma], w)$ is conjugated to $([\gamma'], w')$ by $\Phi_S^{-1}(\vec{\boldsymbol{\beta}}\xi^*)$. To prove our claim it is hence enough to show that w' is in Ell_θ *i.e.* that it has a fixed point in X.

Let $\vec{\boldsymbol{\gamma}}' = \gamma_0 \overset{\vec{s}_1}{\rightsquigarrow} \cdots \overset{\vec{s}_n}{\rightsquigarrow} \gamma_n$ (so that $w' = \vec{s}_1^{-1} \cdots \vec{s}_n^{-1}$) and let $y \coloneqq \gamma_0(0)$. It is easy to show by induction that the image of γ_i is contained in $C^{(i)}_{w'^{-1}_{\mathrm{rev}}}\big(y, 4\theta/t\big)$ (see Figure 8.1). If w' (and hence w'^{-1}) did not have fixed points, by construction we would have $y \notin C_{w'^{-1}_{\mathrm{rev}}}\big(y, 4\theta/t\big)$ because $4\theta/t \leq r_{w'^{-1}_{\mathrm{rev}}}$ by definition. Still, $\vec{\boldsymbol{\gamma}}'$ is closed and therefore $\gamma_n(1) = y$ is in $C_{w'^{-1}_{\mathrm{rev}}}\big(y, 4\theta/t\big)$, a contradiction.

Assume now that X is semi-locally simply connected. Since it is compact, there exists a constant $\epsilon > 0$ small enough so that every path contained in a ball of radius ϵ is homotopic in X to a constant path. Moreover, by compactness there exist $\epsilon' \geq \epsilon'' > 0$ so that:

- for every $w \in \mathrm{Ell}_\theta$ and $z \in \mathrm{Fix}(w)$ we have $w\big(B(z, \epsilon')\big) \subseteq B(z, \epsilon)$;
- any two points in $B(x, \epsilon'')$ can be joined with a continuous path contained in $B(x, \epsilon')$ (recall that X is locally path connected).

Finally, we can further enlarge t_0 so that if $w \in \mathrm{Ell}_\theta$ then for every $y \in X$ such that $y \in C_{w_{\mathrm{rev}}^{-1}}\big(y, 4\theta/t_0\big)$ there exists a $z \in \mathrm{Fix}(w)$ so that

$$C_{w_{\mathrm{rev}}^{-1}}\Big(y, \frac{4\theta}{t_0}\Big) \subseteq B(z, \epsilon''). \tag{8.5}$$

Let again $t \geq t_0$ and $\Phi_S([\gamma], w) \in \mathrm{FT}_\theta$ and let $\vec{\boldsymbol{\gamma}}', [\gamma'], w', y$ and ξ be as above; it will be enough to show that $([\gamma'], w') \in K_\theta$. By the previous argument, we know that $w' \in \mathrm{Ell}_\theta$, therefore we only need to understand the continuous part of $\vec{\boldsymbol{\gamma}}'$. Note also that we already know that y belongs to $C_{w_{\mathrm{rev}}^{-1}}\big(y, 4\theta/t_0\big)$ because $\vec{\boldsymbol{\gamma}}$ is a closed jumping-path, and hence by (8.5) there exists a fixed point $z \in \mathrm{Fix}(w')$ so that y is in $B(z, \epsilon'')$.

It follows from Lemma 4.1.7 that $\vec{\boldsymbol{\gamma}}'$ is homotopic to $\gamma'' \overset{w_{\mathrm{rev}}'^{-1}}{\rightsquigarrow} y$, where γ'' is an appropriate continuous path that we can suppose to be completely contained in $w'^{-1}\big(C_{w_{\mathrm{rev}}'^{-1}}\big(y, 4\theta/t\big)\big)$. Note that the latter is in turn contained in $w'^{-1}\big(B(z, \epsilon'')\big)$ and hence in $B(z, \epsilon)$.

By construction, there exists a continuous path η joining y to z with image contained in $B(z, \epsilon')$. Then both $\eta, w'(\eta)$ and γ'' are contained in $B(z, \epsilon)$ and therefore the closed path $(\gamma'')^*\eta w'(\eta^*)$ is null-homotopic. Note now that by definition $\big([\xi\eta w'(\eta^*\xi^*)\alpha_{w'}^*], w'\big)$ is in K_θ (see Figure 8.2). Since we have

$$\begin{aligned}
\Phi_S\big([\gamma'], w']\big) &= \Big[\xi\gamma'' \overset{w_{\mathrm{rev}}'^{-1}}{\rightsquigarrow} \xi^*\Big] \\
&= \Big[\xi\gamma''(\gamma'')^*\eta w'(\eta^*) \overset{w_{\mathrm{rev}}'^{-1}}{\rightsquigarrow} \xi^*\Big] \\
&= \Big[\xi\eta w'(\eta^*\xi^*) \overset{w_{\mathrm{rev}}'^{-1}}{\rightsquigarrow} x_0\Big] = \Phi_S\big([\xi\eta w'(\eta^*\xi^*)\alpha_{w'}^*], w'\big),
\end{aligned}$$

we can conclude that $[\gamma'] = [\alpha_{w'} w'(\xi\eta)\eta^*\xi^*]$ because Φ_S is injective.

□

The following is just a catchy restatement of the second part of Theorem 8.4.1:

Corollary 8.4.2. *If X is compact and semi-locally simply connected and* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *is asymptotically jumping-geodesic, then its θ-discrete fundamental group is asymptotically isomorphic to* $\big(\pi_1(X, x_0) \rtimes_{\phi_S} F_S\big)/\langle\langle K_\theta \rangle\rangle$.

In particular we are entitled to give the following:

Definition 8.4.3. The *asymptotic θ-discrete fundamental group* of an asymptotically jumping-geodesic warped system $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ over a compact semi-locally simply connected space X is the group:

$$\pi_{1,\theta}\big(F_S \curvearrowright X, x_0\big) \coloneqq \big(\pi_1(X, x_0) \rtimes_{\phi_S} F_S\big)/\langle\langle K_\theta \rangle\rangle,$$

where $K_\theta \coloneqq \big\{\big([\beta w(\beta^*)\alpha_w^*], w\big) \;\big|\; w \in \mathrm{Ell}_\theta,\ \beta(0) = x_0,\ \beta(1) \in \mathrm{Fix}(w)\big\}$.

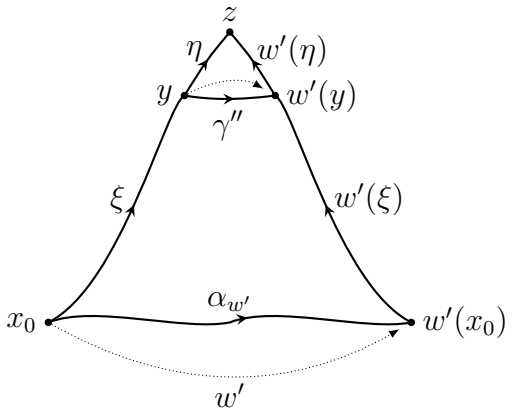


Figure 8.2: Jumping paths in the kernel differ from elements of K_θ by short null-homotopic paths

8.4.2 Some explicit computations

Theorem 8.4.1 immediately allows us to compute the discrete fundamental groups of various warped systems. In particular we have the following:

Corollary 8.4.4. *If F_S acts by rotations on an even dimensional sphere $\mathbb{S}^{2n}$, then we have $\pi_{1,\theta}\big((\mathbb{S}^{2n}, \delta_S^t), x_0\big) = \{0\}$ for every t and $\theta \geq 1$.*

Proof. The point is that every sphere is simply-connected, and every homeomorphism of an even dimensional sphere has fixed points. Note that, to be precise, Theorem 8.4.1 only implies the statement asymptotically. To prove that the statement holds for every t one should notice that this only requires the easy implication of Theorem 8.4.1 which can be deduced directly from Lemma 4.3.1. □

Corollary 8.4.5. *There exist coarsely simply-connected expanders and superexpanders.*

Before doing more computations, note that—when specialised to warped systems—Proposition 4.3.8 yields a much sharper result than it does in the general case. Indeed, let again $\Gamma = \langle S \mid R\rangle$ be a presentation of a (non necessarily finitely presented) finitely generated group, and let Γ_θ be the finitely presented group $\langle S \mid R_\theta\rangle$ where R_θ is the subset of $\langle\langle R\rangle\rangle$ of words of length at most 4θ. Then Theorem 8.4.1 implies the following:

Corollary 8.4.6. *Let X be compact and semi-locally simply connected and let $\Gamma \curvearrowright X$ be a free action of $\Gamma = \langle S \mid R\rangle$ so that $F_S \curvearrowright X$ is asymptotically jumping-geodesic. Then*

$$\pi_{1,\theta}\big(F_S \curvearrowright X, x_0\big) \cong \frac{\pi_1(X) \rtimes_{\phi_S} F_S}{\langle\langle \big\{([\alpha_r^*], r) \;\big|\; r \in R_\theta\big\}\rangle\rangle}.$$

Proof. An element of K_θ is of the form $\big([\beta w(\beta^*)\alpha_w^*], w\big)$ with $w \in \mathrm{Ell}_\theta$, $\beta(0) = x_0$ and $\beta(1) \in \mathrm{Fix}(w)$. Since the action is free, Ell_θ is equal to R_θ, and hence $w = r \in R_\theta$. Moreover, we have $\beta w(\beta^*) = \beta r(\beta^*) = \beta\beta^* \sim x_0$ and hence $\big([\beta w(\beta^*)\alpha_w^*], w\big) = \big([\alpha_r^*], r\big)$. The statement now follows from Theorem 8.4.1. □

The proof of Proposition 4.3.8 immediately implies the following:

Corollary 8.4.7. *Under the hypotheses of Corollary 8.4.6, there is a short exact sequence*

$$1 \to \ \langle\!\langle [\alpha_r] \mid r \in R_\theta \rangle\!\rangle_{\pi_{1,\theta}(X,\delta_S)} \to \ \pi_{1,\theta}\big(F_S \curvearrowright X, x_0\big) \ \to \ \big(G_\theta \rtimes_{\bar\phi} \Gamma_\theta\big) \to 1,$$

where G_θ is the quotient

$$G_\theta \coloneqq \frac{\pi_1(X, x_0)}{\langle\!\langle \big\{ [\alpha_r] \;\big|\; r \in \langle\!\langle R_\theta \rangle\!\rangle \big\} \cup \big\{ [\alpha_r \gamma \alpha_r^* \gamma^{-1}] \;\big|\; [\gamma] \in \pi_1(X, x_0),\ r \in \langle\!\langle R_\theta \rangle\!\rangle \big\} \rangle\!\rangle}.$$

Corollary 8.4.8. *Let Γ be finitely presented and θ large enough so that $\Gamma = \Gamma_\theta$. If $\Gamma \curvearrowright X$ is as in Corollary 8.4.6 and the paths α_r are null-homotopic (*e.g. *if $\Gamma = F_S$ or $\pi_1(X) = \{e\}$), then*

$$\pi_{1,\theta}\big(F_S \curvearrowright X, x_0\big) \cong \pi_1(X, x_0) \rtimes_{\bar\phi} \Gamma.$$

8.5 Global rigidity (via coarse fundamental groups)

We will now employ the discrete fundamental group of warped systems as an invariant of coarse geometry. Since we want to use the tools of Section 8.4, we will work with the following standing assumption:

Convention. From now on, we will always assume that $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ is an asymptotically jumping-geodesic warped system over a compact, semi-locally simply connected space X with homotopy rectifiable paths.

8.5.1 Limits of asymptotic discrete fundamental groups

Recall that from Lemma 3.1.4 we know that the identity map on X induces a surjection $\pi_{1,\theta}\big(F_S \curvearrowright X\big) \to \pi_{1,\theta'}\big(F_S \curvearrowright X\big)$ for every choice of $\theta < \theta'$. That is, the family of asymptotic θ-discrete fundamental groups forms a direct system. We can hence take the direct limit (Section 2.2.5) and define

$$\pi_{1,\infty}\big(F_S \curvearrowright X\big) \coloneqq \varinjlim \pi_{1,\theta}\big(F_S \curvearrowright X\big).$$

Remark 8.5.1. Note that in the definition of $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ it is important that we are taking the limit of *asymptotic* discrete fundamental groups for a *family* of metric spaces. Indeed $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is *not* the direct limit of the discrete fundamental groups of a single metric space, as such a limit would always be trivial (every fixed θ-path will become trivial if looked at with respect to a very large parameter θ').[10]

Remark 8.5.2. Despite being inspired by it, the definition of $\pi_{1,\infty}$ is quite different from the definition of *coarse homology* of [BCW14].

As a consequence of Theorem 8.4.1, we can easily prove the following:

Theorem 8.5.3. *The limit group* $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ *is isomorphic to the quotient* $\big(\pi_1(X,x_0) \rtimes_{\phi_S} F_S\big)/\langle\!\langle K_\infty \rangle\!\rangle$ *where*

$$K_\infty \coloneqq \Big\{ \big([\beta w(\beta^*)\alpha_w^*], w\big) \;\Big|\; \mathrm{Fix}(w) \neq \emptyset,\ \beta(0) = x_0,\ \beta(1) \in \mathrm{Fix}(w) \Big\}.$$

Proof. By Lemma 2.2.9, the direct limit of a nested sequence of quotients $\varinjlim \Gamma/\Gamma_i$ is isomorphic to Γ/Γ_∞—where $\Gamma_\infty = \bigcup_i \Gamma_i$. Then, the proof of the theorem follows easily from Theorem 8.4.1, as $\langle\!\langle K_\infty \rangle\!\rangle = \bigcup_{\theta>1} \langle\!\langle K_\theta \rangle\!\rangle$. □

The interest of Theorem 8.5.3 is that the group $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ can prove to be a useful coarse invariant for $\mathrm{WSys}\big(F_S \curvearrowright X\big)$. For this we need to give another definition:

Definition 8.5.4. We say that the warped system $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ has *stable discrete fundamental group* if there exists a θ large enough so that the natural surjection $\pi_{1,\theta}\big(F_S \curvearrowright X\big) \to \pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is an isomorphism.

It is simple to prove the following:

Lemma 8.5.5. *If* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *is induced by a free action of a finitely generated group* $\Gamma = \langle S \mid R \rangle$*, then it has stable discrete fundamental group if and only if* Γ *is finitely presented.*

Proof. As in Corollary 8.4.6, we let R_θ be the (finite) subset of $\langle\!\langle R \rangle\!\rangle$ of words of length at most 4θ. Then, since the Γ-action is free and every $r \in \langle\!\langle R \rangle\!\rangle$ acts trivially, we have $K_\theta = \big\{ \big([\alpha_r^*], r\big) \;\big|\; r \in R_\theta \big\}$ and $K_\infty = \big\{ \big([\alpha_r^*], r\big) \;\big|\; r \in \langle\!\langle R \rangle\!\rangle \big\}$. In particular, if $\langle\!\langle K_\theta \rangle\!\rangle = \langle\!\langle K_\infty \rangle\!\rangle$ for some $\theta \in \mathbb{N}$, then $\langle\!\langle R \rangle\!\rangle = \langle\!\langle R_\theta \rangle\!\rangle$ and hence Γ is finitely presented.

[10] This is a very different situation from what happens when considering the inverse limit for $\theta \to 0$. Compare with Section 3.2.

Vice versa, if Γ is finitely presented then there exists a $\theta \in \mathbb{N}$ such that $\langle\langle R_\theta \rangle\rangle = \langle\langle R \rangle\rangle$. Following the lines of the proof of Proposition 4.3.8, we claim that K_∞ is contained in $\langle\langle K_\theta \rangle\rangle$. Indeed, if $([\alpha_r^*], r)$ is in $\langle\langle K_\theta \rangle\rangle$, then for every $w \in F_S$ we have

$$\begin{aligned}
([\alpha^*_{wrw^{-1}}], wrw^{-1}) &= \big([w(r(\alpha^*_{w^{-1}})\alpha^*_r)\alpha^*_w], wrw^{-1}\big) \\
&= \big([\alpha_w w(\alpha^*_r)\alpha^*_w], wrw^{-1}\big) \\
&= \big(\phi_S(w)[\alpha^*_r], wrw^{-1}\big) \\
&= (e, w)([\alpha^*_r], r)(e, w)^{-1}
\end{aligned}$$

and hence $([\alpha^*_{wrw^{-1}}], wrw^{-1})$ is in $\langle\langle K_\theta \rangle\rangle$ as well. Moreover, given $([\alpha^*_{r_1}], r_1)$ and $([\alpha^*_{r_2}], r_2)$ in $\langle\langle K_\theta \rangle\rangle$ we have

$$\begin{aligned}
\big([\alpha^*_{r_1 r_2}], r_1 r_2\big) &= \big([r_1(\alpha^*_{r_2})\alpha^*_{r_1}], r_1 r_2\big) \\
&= \big([\alpha^*_{r_2}\alpha^*_{r_1}], r_1 r_2\big) \\
&= \big([\alpha^*_{r_1}\alpha_{r_1} r_1 (\alpha_{r_2})^* \alpha^*_{r_1}], r_1 r_2\big) \\
&= \big([\alpha^*_{r_1}], r_1\big)\big([\alpha^*_{r_2}], r_2\big)
\end{aligned}$$

and the latter is in $\langle\langle K_\theta \rangle\rangle$. The claim easily follows.

□

Remark 8.5.6. Lemma 8.5.5 is enough to show that many interesting examples of warped systems have stable discrete fundamental group. Still, it also gives us means for constructing warped system with unstable discrete fundamental group. For example, it is well-known that there exists a finite set $S \subset F_2 \times F_2$ so that the generated subgroup $\Gamma = \langle S \rangle < F_2 \times F_2$ is not finitely presented (see *e.g.*[BH13, Section III.Γ.5]). Consider now any embedding of $F_2 \times F_2$ in a compact Lie group G (*e.g.* an embedding in $G = \mathrm{SO}(3, \mathbb{R}) \times \mathrm{SO}(3, \mathbb{R})$). This induces an embedding $\Gamma \hookrightarrow G$ and the isometric action by left translation produces a warped system $\mathrm{WSys}\big(\Gamma \curvearrowright G\big) = \mathrm{WSys}\big(F_S \curvearrowright G\big)$ that, by Lemma 8.5.5, does not have stable discrete fundamental group.

8.5.2 Stability of stable discrete fundamental groups

The following lemma will be key for using discrete fundamental groups of warped system (with stable discrete fundamental group) as a coarse invariant.

Lemma 8.5.7. *Let* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *a warped system with stable discrete fundamental group and let* $(Y_k)_{k\in\mathbb{N}}$ *be a sequence of* 1*-geodesic metric spaces. If* $(Y_k)_{k\in\mathbb{N}}$ *is coarsely*

equivalent to an unbounded sequence in $\mathrm{WSys}\big(F_S \curvearrowright X\big)$*, then for every* θ *large enough there exists a* $\bar{k}$ *such that*

$$\pi_{1,\theta}(Y_k) \cong \pi_{1,\infty}\big(F_S \curvearrowright X\big) \tag{8.6}$$

for every $k \geq \bar{k}$*. In particular, if* $(Y_k)_{k\in\mathbb{N}}$ *is coarsely sub-equivalent to* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$*, then for every* θ *large enough* (8.6) *holds for infinitely many* $k \in \mathbb{N}$*.*

Proof. Let $\big(X, \delta_S^{t_k}\big)_{k\in\mathbb{N}}$ with $t_k \to \infty$ be the sequence of spaces coarsely equivalent to $(Y_k)_{k\in\mathbb{N}}$ and let L and A be the uniform constants of the quasi-isometries.

Since $\big(F_S \curvearrowright X\big)$ has stable fundamental group, there exists a $\bar{\theta}$ large enough so that $\pi_{1,\theta}\big(F_S \curvearrowright X\big) \cong \pi_{1,\infty}\big(F_S \curvearrowright X\big)$ for every $\theta \geq \bar{\theta}$. In particular, from Theorem 8.4.1 it follows that for every $\theta \geq \bar{\theta}$ there exists $n(\theta)$ so that

$$\pi_{1,\theta}\big(X, \delta_S^{t_k}\big) \cong \pi_{1,\infty}\big(F_S \curvearrowright X\big)$$

for every $k \geq n(\theta)$. Then, for every $k \in \mathbb{N}$ and $\theta \geq L\bar{\theta} + A$, by Lemma 3.1.4 we have a concatenation of surjections:

$$\pi_{1,\bar{\theta}}\big(X, \delta_S^{t_k}\big) \twoheadrightarrow \pi_{1,\theta}\big(Y_k\big) \twoheadrightarrow \pi_{1,L\theta+A}\big(X, \delta_S^{t_k}\big) \ .$$

If $k \geq n(L\theta + A)$, it follows from the discussion above that in the following diagram the maps are isomorphisms

$$\begin{array}{ccc} \pi_{1,\bar{\theta}}\big(X, \delta_S^{t_k}\big) & \xrightarrow{\ \mathrm{id}_*\ } & \pi_{1,L\theta+A}\big(X, \delta_S^{t_k}\big) \\ & {\scriptstyle\cong}\searrow \quad \swarrow{\scriptstyle\cong} & \\ & \pi_{1,\infty}\big(F_S \curvearrowright X\big) & \end{array}$$

and in particular id_* is an isomorphism. Hence by (iv) of Lemma 3.1.4 we deduce that

$$\pi_{1,\theta}(Y_k) \cong \pi_{1,\infty}\big(F_S \curvearrowright X\big).$$

□

Remark 8.5.8. Note in particular that Lemma 8.5.7 immediately implies that the discrete fundamental group of the topology approximating graphs (with mesh small enough) coincides with the asymptotic discrete fundamental group of the warped system. This fact could also be proved by hand (and does not require the warped system to have stable discrete fundamental group).

The above result can be further specialised in the study of coarse equivalences of warped systems:

Theorem 8.5.9. *Let* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *and* $\mathrm{WSys}\big(F_T \curvearrowright Y\big)$ *be two coarsely sub-equivalent warped systems. If* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *has stable discrete fundamental group, then* $\mathrm{WSys}\big(F_T \curvearrowright Y\big)$ *has stable discrete fundamental group as well and*

$$\pi_{1,\infty}\big(F_S \curvearrowright X\big) \cong \pi_{1,\infty}\big(F_T \curvearrowright Y\big).$$

Proof. Let $(t_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$ be unbounded sequences such that the families of metric spaces $\big(X, \delta_S^{t_k}\big)_{k\in\mathbb{N}}$ and $\big(X, \delta_T^{s_k}\big)_{k\in\mathbb{N}}$ are coarsely equivalent, and let L and A be the quasi-isometry constants of the coarse equivalence. Fix three parameters θ, θ' and θ'' satisfying $\theta' \geq L\theta + A$ and $\theta'' \geq L(L\theta' + A) + A$. For every $k \in \mathbb{N}$, the quasi-isometries induce a concatenation of surjections

$$\begin{array}{ccccc} \pi_{1,\theta}\big(X, \delta_S^{t_k}\big) & & \pi_{1,L\theta'+A}\big(X, \delta_S^{t_k}\big) & & \pi_{1,L\theta''+A}\big(X, \delta_S^{t_k}\big) \\ & \searrow & \nearrow & \searrow & \nearrow \\ & \pi_{1,\theta'}\big(Y, \delta_T^{s_k}\big) & & \pi_{1,\theta''}\big(Y, \delta_T^{s_k}\big) & \end{array} .$$

If θ is large enough so that the projection $\pi_{1,\theta}\big(F_S \curvearrowright X\big) \to \pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is an isomorphism, then we can argue as in the proof of Lemma 8.5.7 to deduce from Lemma 3.1.4 and Theorem 8.4.1 that for every k large enough all the surjections above are actually isomorphisms.

Since the composition map

$$\begin{array}{ccc} & \pi_{1,L\theta'+A}\big(X, \delta_S^{t_k}\big) & \\ \cong \nearrow & & \searrow \cong \\ \pi_{1,\theta'}\big(Y, \delta_T^{s_k}\big) & \dashrightarrow & \pi_{1,\theta''}\big(Y, \delta_T^{s_k}\big) \end{array}$$

is induced by a map that is A-close to the identity, we deduce that for every k large enough $(\mathrm{id}_Y)_* \colon \pi_{1,\theta'}\big(Y, \delta_T^{s_k}\big) \to \pi_{1,\theta''}\big(Y, \delta_T^{s_k}\big)$ is an isomorphism. From this it follows that $\mathrm{WSys}\big(F_T \curvearrowright Y\big)$ also has stable discrete fundamental group.

Now, the fact that $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is isomorphic to $\pi_{1,\infty}\big(F_T \curvearrowright Y\big)$ follows trivially from Lemma 8.5.7. □

Corollary 8.5.10. *Let* $\Gamma = \langle S \mid R\rangle$ *be a finitely presented group and* $\Lambda = \langle T\rangle$ *be finitely generated. If there are free actions* $\Gamma \curvearrowright X$ *and* $\Lambda \curvearrowright Y$ *where* $\pi_1(X) = \pi_1(Y) = \{0\}$ *so that the induced warped systems* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *and* $\mathrm{WSys}\big(F_T \curvearrowright Y\big)$ *are coarsely sub-equivalent, then* Λ *is also finitely presented and* $\Gamma \cong \Lambda$.

Proof. By Lemma 8.5.5 the warped system $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ has stable discrete fundamental group and hence by Theorem 8.5.9 the same is true for $\mathrm{WSys}\big(F_T \curvearrowright Y\big)$ and $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is isomorphic to $\pi_{1,\infty}\big(F_T \curvearrowright Y\big)$. Again by Lemma 8.5.5 we deduce that Λ is finitely presented, and by Corollary 8.4.8 we deduce that $\Gamma \cong \Lambda$. □

Remark 8.5.11. It is not clear to the author whether the group $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is a coarse invariant of warped systems that do not have stable discrete fundamental group.

8.6 Warped systems and box spaces

Also in this section we have the standing assumption that warped systems are asymptotically jumping-geodesic and come from actions on compact semi-locally simply connected metric spaces with homotopy rectifiable paths. Let $\Lambda = \langle T \mid R\rangle$ be a (not necessarily finite) presentation of a finitely generated infinite group and, as before, let R_θ be the subset of $\langle\langle R\rangle\rangle$ of words of length at most 4θ and let Λ_θ be the finitely presented group $\langle T \mid R_\theta\rangle$. Note that we have a natural surjection $\Lambda_\theta \to \Lambda$.

8.6.1 A major obstruction

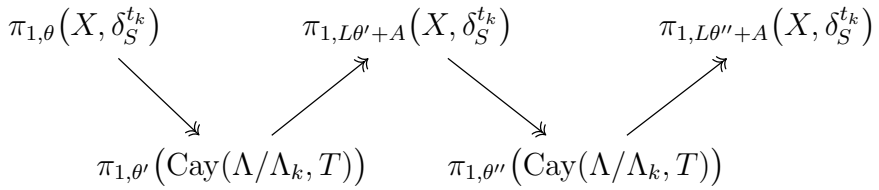
The rather peculiar fact that the discrete fundamental group of a warped system does not depend meaningfully on the parameter t is noticeably dissimilar from what Theorem 4.3.10 implies for box spaces. This suggest us to use Lemma 8.5.7 to prove the following:

Theorem 8.6.1. *Let $\Gamma = \langle S \mid R\rangle$ be an infinite finitely generated group. If a normal residual nested box space $\square_{(\Lambda_k)}\Lambda$ is coarsely sub-equivalent to a warped system $\mathrm{WSys}\big(F_S \curvearrowright X\big)$, then $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ has stable discrete fundamental group if and only if Λ is finitely presented.*

Moreover, when this happens we must have $\Lambda_k \cong \pi_{1,\infty}\big(F_S \curvearrowright X\big)$ for infinitely many $k \in \mathbb{N}$.

Proof. Let $\big(X, \delta_S^{t_k}\big)_{k\in\mathbb{N}}$ be coarsely equivalent to (a subsequence of) the box space. Now the proof follows closely the proof of Theorem 8.5.9: assume that $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ has stable discrete fundamental group, let L and A be the quasi-isometry constants of the coarse equivalence and let $\theta, \theta',\ \theta''$ satisfy $\theta' \geq L\theta + A$ and $\theta'' \geq L(L\theta' + A) + A$.

For every $k \in \mathbb{N}$, the quasi-isometries induce a concatenation of surjections



and just as in Theorem 8.5.9 we can deduce that if θ is large enough so that the θ-discrete fundamental group of $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ stabilised, then for every k large enough $\mathrm{id}_*\colon \pi_{1,\theta'}\big(\mathrm{Cay}(\Lambda/\Lambda_k, T)\big) \to \pi_{1,\theta''}\big(\mathrm{Cay}(\Lambda/\Lambda_k, T)\big)$ is an isomorphism.

From Theorem 4.3.10 we know that $\pi_{1,\theta'}\big(\mathrm{Cay}(\Lambda/\Lambda_k, T)\big) \cong (\Lambda_k)_{\theta'} < \Lambda_{\theta'}$ and from its proof it also follows that the map id_* coincides with the quotient $(\Lambda_k)_{\theta'} \to (\Lambda_k)_{\theta''}$ induced from $\Lambda_{\theta'} \to \Lambda_{\theta''}$.

Now, if $\langle\langle R \rangle\rangle$ was strictly larger than $\langle\langle R_{\theta'} \rangle\rangle$ we could choose a relation $r \in \langle\langle R \rangle\rangle \smallsetminus \langle\langle R_{\theta'} \rangle\rangle$. Choosing a θ'' larger than $|r|/4$, we would find that r denotes an element in the kernel of $\mathrm{id}_*\colon (\Lambda_k)_{\theta'} \to (\Lambda_k)_{\theta''}$ for every k large enough. Still, since the sequence Λ_k is residual, there must be a k large enough so that r is not trivial in $(\Lambda_k)_{\theta'}$, and this contradicts the fact that id_* is an isomorphism.

Since R_θ is finite, we deduce that if $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ has stable discrete fundamental group then Λ is finitely presented. The inverse implication is analogous.

The 'moreover' part of the statement follows immediately from Theorem 4.3.10 and Lemma 8.5.7. □

Remark 8.6.2. In the proof of Theorem 8.6.1 we did not really need that that the normal residual sequence $\Lambda_k \lhd \Lambda$ consists of nested subgroups. Indeed, we only need that for every $\theta \in \mathbb{N}$ there exists an n large enough so that Λ_k consists only of elements of length larger than 4θ for every $k \geq n$.

Moreover, the same proof works also for non-normal box spaces if one knows that for every k large enough the group Λ_k does not have non-trivial elements *conjugate* to elements of less than 4θ.

Theorem 8.6.1 implies that box spaces and warped systems tend to have very different coarse geometry. We wish to give some examples of such differences in the next two subsections.

8.6.2 Box spaces that are not coarsely-equivalent to warped systems

It follows from Theorem 8.6.1 that for a normal residual nested box space of a finitely presented group $\square_{(\Lambda_k)}\Lambda$ to be coarsely sub-equivalent to a warped system it is necessary

that the groups Λ_k be isomorphic for infinitely many k. Up to passing to a subsequence, we can hence assume that they are isomorphic for every k. In the sequel we always make this assumption.

Note that if $\Lambda \cong F_n$ is a free group, then the rank of the subgroup Λ_k is known to be $\mathrm{rk}(\Lambda_k) = (n-1)[F_n : \Lambda_k] + 1$ and hence a bound on the rank of Λ_k implies a bound on the index $[F_n : \Lambda_k]$. It follows that no normal nested residual box space of a free group can be coarsely sub-equivalent to a warped system.

More in general, recall that the *rank gradient* of a residual filtration is defined as

$$\mathrm{RG}(\Lambda, (\Lambda_k)_{k\in\mathbb{N}}) \coloneqq \lim_{k\to\infty} \frac{\mathrm{rk}(\Lambda_k)}{[\Lambda : \Lambda_k]}.$$

Recall also that if Λ has *fixed price* p (for a definition and discussion see *e.g.* [Fur09]), then every normal residual filtration has rank gradient $p-1$.

Corollary 8.6.3. *If Λ is finitely presented and it admits a normal nested residual box space $\square_{(N_k)}\Lambda$ that is coarsely sub-equivalent to a warped system then $\mathrm{RG}(\Lambda, (N_k)_{k\in\mathbb{N}}) = 0$. In particular, if Λ has fixed price $p > 1$, then no such box space coarsely sub-equivalent to a warped system.*

A quite different reason for box spaces to not be coarsely equivalent to warped systems goes as follows. Let Λ be a lattice in a simple Lie group G not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$. Then the finite index subgroups $\Lambda_k <_f \Lambda$ are lattices as well and hence the Mostow Rigidity Theorem applies. That is, if Λ_k is isomorphic to $\Lambda_{k'}$ then Λ_k and Λ'_k are actually conjugate in G and hence G/Λ_k and $G/\Lambda_{k'}$ have the same (finite) volume with respect to the Haar measure. Still, G/Λ_k is a cover of G/Λ of rank $[\Lambda : \Lambda_k]$ and hence it has volume $\mathrm{Vol}(G/\Lambda_k) = [\Lambda : \Lambda_k]\,\mathrm{Vol}(G/\Lambda)$, which is again implying an upper bound on the index in terms of the isomorphism class of Λ_k. We hence proved the following:

Corollary 8.6.4. *If Λ is lattice in a simple Lie group G not locally isomorphic to $\mathrm{SL}(2,\mathbb{R})$, then no normal residual nested box space of Λ is coarsely sub-equivalent to a warped system.*

8.6.3 Warped systems that are not coarsely-equivalent to box spaces

We already noted that the warped system $\mathrm{WSys}\big(F_2 \curvearrowright \mathbb{S}^2\big)$ induced by an action by rotations has stable discrete fundamental group and we have $\pi_{1,\infty}\big(F_2 \curvearrowright \mathbb{S}^2\big) = \{e\}$. It follows from Theorem 8.6.1 that if $\mathrm{WSys}\big(F_2 \curvearrowright \mathbb{S}^2\big)$ was coarsely sub-equivalent to

a normal nested residual box space then the quotienting groups Λ_k should be trivial and hence Λ would have to be finite, a contradiction.

The argument above relies on the observation that, in that specific case, any group having $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ as a finite index subgroup could not admit box spaces coarsely equivalent to warped systems. This strategy can be applied in other cases as well:

Corollary 8.6.5. *Assume that* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *has stable discrete fundamental group. If every group* Λ *containing* $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ *as a finite index subgroup cannot have normal nested residual box spaces coarsely sub-equivalent to a warped system, then* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *is not coarsely equivalent to any normal nested box space.*

In particular, this is the case when $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ *is:*

(a) *a finite group;*

(b) *a non residually finite group;*

(c) *a non finitely presented group;*

(d) *a lattice in a high rank simple Lie group.*

Proof. Case (a) is obvious and case (b) follows from the fact that we insist that box spaces be generated by residual sequences and therefore the group Λ (and its subgroups) would have to be residually finite.

Case (c) holds true as Theorem 8.6.1 implies that the group Λ should be finitely presented, and therefore the same should be true for its finite index subgroups.

Case (d) follows from the fact that lattices in high rank Lie groups are rigid under quasi-isometries [KL97]. This implies that a group Λ containing such a lattice as a finite index subgroup would have to be itself a lattice and hence Corollary 8.6.4 would apply. □

Remark 8.6.6. By Corollary 8.4.7, to find examples of warped systems with stable discrete fundamental group so that $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is not residually finite it would be enough to find a free action of a finitely presented but not residually finite group.

We feel that it should also be possible to find examples of warped systems with stable fundamental group for which $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is not finitely presented. Still, Lemma 8.5.5 implies that we cannot hope to find such an example by considering free actions on 'pleasant' compact spaces.

As already hinted in Remark 8.6.2, many of the results listed so far can be somewhat extended to box spaces that are not necessarily nested, normal or residual. We did not do so to avoid unnecessary complications. Note however that if we are only interested in nested sequences, we immediately have the following stronger result:

Corollary 8.6.7. *Assume that* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *has stable discrete fundamental group. If* $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ *does not contain a finite index normal subgroup isomorphic to* $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ *itself (*i.e. *it is* co-Hopfian*), then* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *is not coarsely equivalent to any normal nested box space.*[11]

We wish to remark here that *many* groups are co-Hopfian. See [vL17] for an exhaustive study of such groups.

Note that if $\Gamma \curvearrowright X$ is an action of a finitely generated group and S and T are two finite sets of generators, then $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ and $\mathrm{WSys}\big(F_T \curvearrowright X\big)$ are naturally coarsely equivalent. Moreover, using Theorem 8.5.3 it is simple to prove that $\pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is naturally isomorphic to $\pi_{1,\infty}\big(F_T \curvearrowright X\big)$.[12] In view of these facts, in what follows we feel justified to simply use the notation $\mathrm{WSys}\big(\Gamma \curvearrowright X\big)$ and $\pi_{1,\infty}\big(\Gamma \curvearrowright X\big)$ to denote the (coarse equivalence class) of the warped system induced by $\Gamma \curvearrowright M$ and the limit of its discrete fundamental groups.

One of the main results of [dLV17] is that the warped system $\mathrm{WSys}\big(\Gamma_d \curvearrowright \mathrm{SO}(d,\mathbb{R})\big)$ is not coarsely equivalent to a box space of a lattice in a high rank semisimple algebraic group (Theorem 8.3.6). We will now complete that result by showing that such a warped system is not coarsely equivalent to any normal nested box space.

Assume that Γ be finitely presented, M has finite fundamental group and that the action $\Gamma \curvearrowright M$ be free and by isometries. Then Corollary 8.4.7 implies that $\pi_{1,\infty}\big(\Gamma \curvearrowright M\big)$ is virtually isomorphic to Γ (recall that two groups are *virtually isomorphic* if they are equivalent under the equivalence relation induced by taking quotients by finite subgroups or passing to a finite index subgroups). If $\mathrm{WSys}\big(\Gamma \curvearrowright M\big)$ is coarsely equivalent to a normal residual nested box space of Λ, it follows that Γ is virtually isomorphic to Λ as well and, by Theorem 8.3.3, it is hence quasi-isometric to $\Gamma \times \mathbb{Z}^{\dim(M)}$, which is often not the case. For example, we immediately get the following:

[11] Here we do not need to ask for the filtration to be residual. In fact, if $\Lambda_k \lhd \Lambda$ was a non-residual filtration so that the box space $\square_{(N_k)}\Lambda$ is coarsely sub-equivalent to the warped system, we would deduce that $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ is coarsely sub-equivalent to a normal nested residual box space of the quotient $\overline{\Lambda} = \Lambda / \cap_{k\in\mathbb{N}} \Lambda_k$.

[12] Here we could not just apply Theorem 8.5.9 because we are not assuming these warped systems to have stable discrete fundamental group.

Corollary 8.6.8. *Let $\Gamma \curvearrowright M$ be a free action by isometries of a finitely presented group on a Riemannian manifold with finite fundamental group. If Γ either*

- *has polynomial growth;*
- *has property (T); or*
- *is Gromov hyperbolic;*

then $\mathrm{WSys}\big(\Gamma \curvearrowright M\big)$ is not coarsely equivalent to any normal residual nested box space.

In particular, all the superexpanders obtained from the warped system $\mathrm{WSys}\big(\Gamma_d \curvearrowright \mathrm{SO}(d,\mathbb{R})\big)$ are not coarsely equivalent to any such box space.

We would like to remark that it is possible to prove the above statement about virtual isomorphisms directly from Theorem 8.4.1 without passing through Corollary 8.4.7 (and hence avoiding Proposition 4.3.8). We wish to do so explicitly, as we think that this technique is interesting in its own right.

Theorem 8.6.9. *Let $\Gamma \curvearrowright M$ be a free action of a finitely generated group on a compact manifold with finite fundamental group. Then $\pi_{1,\infty}\big(\Gamma \curvearrowright M\big)$ is virtually isomorphic to Γ.*

Proof. Let $\langle S \mid R\rangle$ be a presentation of Γ and consider the universal cover $\widetilde{M} \to M$. For every $s \in S$, choose a lift $\tilde{s}$ to the universal cover:

$$\begin{array}{ccc} \widetilde{M} & \xrightarrow{\tilde{s}} & \widetilde{M} \\ \downarrow & & \downarrow \\ M & \xrightarrow{s} & M \end{array}$$

this induces an action $\tilde{\rho}\colon F_S \curvearrowright \widetilde{M}$ and an associated warped system $\mathrm{WSys}\big(F_S \curvearrowright \widetilde{M}\big)$.

Note now that $\tilde{\rho}(R)$ is a subset of the group of deck transformations of $\widetilde{M}$, which is a finite group by hypothesis. It follows that $\ker(\tilde{\rho})$ is a subgroup of finite index of $\langle\!\langle R\rangle\!\rangle \subset F_S$.

Let $\widetilde{\Gamma} \coloneqq F_S/\ker(\tilde{\rho})$ and note that Γ is the quotient of $\widetilde{\Gamma}$ by the finite subgroup $\langle\!\langle R\rangle\!\rangle/\ker(\tilde{\rho})$. Since $\widetilde{\Gamma} \curvearrowright \widetilde{M}$ is a free action on a simply connected manifold, we can apply Corollary 8.4.6 to the warped system $\mathrm{WSys}\big(\widetilde{\Gamma} \curvearrowright \widetilde{M}\big) = \mathrm{WSys}\big(F_S \curvearrowright \widetilde{M}\big)$ to deduce that

$$\pi_{1,\theta}\big(F_S \curvearrowright \widetilde{M}\big) \cong \widetilde{\Gamma}_\theta$$

where $\widetilde{\Gamma}_\theta$ is the group $F_S/\langle\!\langle \{w \in \ker(\tilde{\rho}) \mid |w| \leq 4\theta\}\rangle\!\rangle$.

Note that at the level of jumping-fundamental groups it is simple to mimic the theory of topological covers and deduce that for every $t \in \mathbb{R}_+$ there is an injection

$$F_S \cong \mathcal{J}_S\Pi_1\big(t \cdot \widetilde{M}\big) \longhookrightarrow \mathcal{J}_S\Pi_1\big(t \cdot M\big)$$

whose image is a subgroup of index (at most) $|\pi_1(M)|$ (this map coincides with the natural inclusion $F_S \hookrightarrow \pi_1(M) \rtimes F_S$). In the above we added t to the notation to remember that we are working with metrics scaled by t.

Since the quotient map $\widetilde{M} \to M$ is 1-Lipschitz with respect to the warped metrics δ_S^t, it follows from Theorem 4.2.2 that the above injection descends to a homomorphism between the discrete fundamental groups *via* the discretisation procedure

$$\begin{array}{ccccc}
 & & \mathcal{J}_S\Pi_1\big(t \cdot \widetilde{M}\big) & \longhookrightarrow & \mathcal{J}_S\Pi_1\big(t \cdot M\big) \\
 & & \Big\downarrow \hat{\cdot} & & \Big\downarrow \hat{\cdot} \\
\widetilde{\Gamma}_\theta & \twoheadrightarrow & \pi_{1,\theta}\Big(\widetilde{M}, \delta_S^t\Big) & \dashrightarrow & \pi_{1,\theta}\Big(M, \delta_S^t\Big)
\end{array}$$

and that the image has index (at most) $|\pi_1(M)|$.

Since the above homomorphisms do not depend on t (as long as t is large enough), they induce a homomorphism of the direct systems as θ varies in $\mathbb{N}$, and therefore induce a limit homomorphism

$$\widetilde{\Gamma} \cong \varinjlim \widetilde{\Gamma}_\theta \longrightarrow \varinjlim \pi_{1,\theta}\big(M, \delta_S^t\big) = \pi_{1,\infty}\big(M, \delta_S^t\big)$$

whose image is a finite index subgroup. Moreover, using Theorem 8.4.1 it is easy to check that this limit homomorphism is actually injective, thus completing the proof.

□

8.6.4 Warped systems that are actually coarsely-equivalent to box spaces

Despite all the examples provided above, warped systems over compact manifolds and box spaces *can* be coarsely equivalent. The easiest example is probably the following: let $X = \mathbb{T}^d$ be the d-dimensional torus and consider the trivial warped system $\mathrm{WSys}\big(\{e\} \curvearrowright \mathbb{T}^d\big)$. It is then easy to see that $(\mathbb{T}^d, \delta_S^n)$ is just the torus with the metric rescaled by n and it is hence quasi-isometric to the finite quotient $\big(\mathbb{Z}/n\mathbb{Z}\big)^d \cong \mathbb{Z}^d/(n\mathbb{Z})^d$. That is, $\mathrm{WSys}\big(\{e\} \curvearrowright \mathbb{T}^d\big)$ is coarsely equivalent to a box space of $\mathbb{Z}^d$.

The above example can be made quite more interesting using a result of Kielak and Sawicki. In [Saw17a, Appendix] they show that there exist (uncountably many) actions $\mathbb{Z}^k \curvearrowright \mathbb{T}^d$ by rotations such that $\mathrm{WSys}\big(\mathbb{Z}^k \curvearrowright \mathbb{T}^d\big)$ is coarsely equivalent to $\mathrm{WSys}\big(\{e\} \curvearrowright \mathbb{T}^{d+k}\big)$ and it is hence coarsely equivalent to a box space of $\mathbb{Z}^{d+k}$.

Note that the above result cannot hold for every action by rotation $\mathbb{Z}^k \curvearrowright \mathbb{T}^d$. Indeed, there exist actions by rotations on tori that produce non coarsely equivalent warped systems (see [Kim06] and [Saw17a]).

If we do not require the nested sequence to be normal nor residual, we can obtain other interesting examples such as the following:

Example 8.6.10. Let $\Lambda \coloneqq \mathbb{Z}^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ where $\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{Z}^2$ is the natural action. Note that $k\mathbb{Z}^2$ is a characteristic subgroup of $\mathbb{Z}^2$ and hence $N_k \coloneqq (k\mathbb{Z})^2 \rtimes \mathrm{SL}(2,\mathbb{Z})$ is a (non-normal) subgroup of Λ. Moreover, it is simple to show that $N_k \cong \Lambda$ for every k, so that the box space $\square_{N_k}\Lambda$ could be coarsely equivalent to some warped system, and this is actually the case. Indeed, consider the natural action $\mathrm{SL}(2,\mathbb{Z}^2) \curvearrowright \mathbb{T}^2$. It is then a relatively simple task to check that the spaces $(\mathbb{T}^2, \delta_S^n)$ and $\overline{\mathrm{Schr}}^{\mathrm{r}}(N_k\backslash\Lambda, S)$ are uniformly quasi-isometric.

The interest of this example is that the Schreier graphs $\overline{\mathrm{Schr}}^{\mathrm{r}}(N_k\backslash\Lambda, S)$ form a family of expanders. Indeed, one can check that is nothing but Example 7.2.1 in disguise. In particular, the warped system $\mathrm{WSys}\big(\mathrm{SL}(2,\mathbb{Z}) \curvearrowright \mathbb{T}^2\big)$ is as far as possible from a nicely behaved warped system such as $\mathrm{WSys}\big(\{e\} \curvearrowright \mathbb{T}^d\big)$.

We find it quite suggestive that this example of warped system coarsely equivalent to a box space is obtained in an instance (the action on the torus) where the two standard approaches for constructing approximanting graphs (choosing finer and finer partitions *vs.* looking at sequences of covers) produce the same end result.

Appendix A

Coarsely inequivalent expanders via subsequences

We wish to point out that once an expander $(\mathcal{G}_n)$ is given, one can always construct a continuum of non-coarsely equivalent expanders by carefully choosing subsequences of it. Similar arguments are used in [Hum14, Theorem 2.8] and [KV17, Proposition 2].

Proposition. *Let $(\mathcal{G}_n)_{n\in\mathbb{N}}$ be a family of finite graphs with uniformly bounded degree and $|\mathcal{G}_n| \to \infty$. Then there exists a continuum $\mathcal{I}$ of subsets $I_a \subset \mathbb{N}$ such that for every pair $I_a \neq I_b$ in $\mathcal{I}$, the subsequences $(\mathcal{G}_n)_{n\in I_a}$ and $(\mathcal{G}_n)_{n\in I_b}$ are not uniformly coarsely equivalent.*

Proof. Choosing a subsequence if necessary, we can assume that $|\mathcal{G}_{n+1}| > n|\mathcal{G}_n|$ for every $n \in \mathbb{N}$. We claim that for every choice of control functions ρ_- and ρ_+ there is an n_0 large enough so that for all $n > m > n_0$, the graphs $\mathcal{G}_n$ and $\mathcal{G}_m$ cannot be coarsely equivalent with control functions ρ_- and ρ_+. Indeed, suppose that there exists such a coarse equivalence $f\colon \mathcal{G}_n \to \mathcal{G}_m$, and let $r > 0$ be large enough, so that $\rho_-(r) \geq 1$. Then the pre-image $f^{-1}(v)$ of any vertex $v \in \mathcal{G}_m$ must have diameter at most r, and it follows that $f^{-1}(v)$ has cardinality at most D^{r+1}, where D is the uniform bound on the degree. In particular, we have $m|\mathcal{G}_m| < |\mathcal{G}_n| \leq D^{r+1}|\mathcal{G}_m|$. Hence, to prove the claim, it is sufficient to let $n_0 = D^{r+1}$.

It follows from the above discussion that if I and J are two subsets of $\mathbb{N}$ so that $I \setminus J$ is infinite, then the sequences $(\mathcal{G}_n)_{n\in I}$ and $(\mathcal{G}_n)_{n\in J}$ are not uniformly coarsely equivalent. To conclude the proof, it is enough to observe that there exists an uncountable family of sets $I_a \subset \mathbb{N}$ so that $I_a \setminus I_b$ is infinite for every $a \neq b$. □

If one makes some clever choices, it is possible to find an uncountable family $\mathcal{I}$ of infinite subsets of $\mathbb{N}$ such that for any two $I_a \neq I_b \in \mathcal{I}$ the intersection $I_a \cap I_b$

is finite.[1] In particular, it follows that it is always possible to find a continuum of coarsely disjoint expanders.

Still, these examples are somewhat silly and do not encode what one would think of as 'being different expanders'. One possible way to overcome this issue could be that one can ask a priori to compare only sequences of graphs where the size of the n^{th} graphs are (uniformly) comparable. In this case the vast control over cardinalities allowed by our construction (Proposition 7.1.8) could be useful.

Note also that just by selecting subsequences of a given box space one cannot produce a continuum of expanders that are not coarsely equivalent to any box space (compare with Subsection 8.6), for the silly reason that a subsequence of a box space still is a box space.

[1] There are some nice and concrete examples on Mathoverflow.

Appendix B

Proofs for unified warped cones

We wish to report that some of the rigidity results that we proved for warped systems hold in the case of warped cones (as by Definition 8.1.14) as well.

For example, it is straightforward to modify the proofs of Lemma 8.3.1 and Theorem 8.3.3 in order to prove that under the same hypotheses (*i.e.* essentially free actions by isometries on compact manifolds) if the warped cones $\mathcal{O}_\Gamma(M)$ and $\mathcal{O}_\Lambda(N)$ are quasi-isometric, then $\Gamma \times \mathbb{Z}^{\dim(M)+1}$ is quasi-isometric to $\Lambda \times \mathbb{Z}^{\dim(N)+1}$.

One can hence use the techniques of Subsection 8.3.3 to produce examples of warped cones that are not coarsely equivalent. As already discussed in Subsection 8.2.2, this does not immediately follow from the analoguous results for warped systems.

Also the techniques involving the use of discrete fundamental groups translate fairly well to this setting. Before doing so we need some new notation: as in the introduction, we denote the t-level of a warped cone $X \times \{t\} \subset \mathcal{O}_S(X)$ by $\mathcal{O}_S^t(X)$. Further, for $1 \leq a \leq b < \infty$ we will denote by $\mathcal{O}_S^{[a,b]}(X)$ the subset $X \times [a,b] \subseteq \mathcal{O}_S(X)$ with the induced metric.

As in the the last few sections, we still assume the space X to be a 'nice' compact space and $\mathrm{WSys}\big(S \curvearrowright X\big)$ to be jumping-geodesic. For any fixed $\theta \geq 1$, it is easy to show that for $t \gg 0$ large enough $\pi_{1,\theta}\big(\mathcal{O}_S^t(X)\big) \cong \pi_{1,\theta}\big(X, \delta_S^t\big)$. Moreover it is also simple to prove the following lemma:

Lemma. *For every $\theta \geq 1$ there exists a t_0 large enough so that for every $t_0 \leq a \leq t \leq b \leq \infty$ the natural inclusion and projection*

$$\mathcal{O}_S^t(X) \xhookrightarrow{\ \iota\ } \mathcal{O}_S^{[a,b]}(X) \xrightarrow{\ p\ } \mathcal{O}_S^t(X)$$

induce isomorphisms

$$\pi_{1,\theta}\big(\mathcal{O}_S^t(X)\big) \xrightarrow{\ \iota_*\ } \pi_{1,\theta}\big(\mathcal{O}_S^{[a,b]}(X)\big) \xrightarrow{\ p_*\ } \pi_{1,\theta}\big(\mathcal{O}_S^t(X)\big)\ .$$

This lemma allows us to mimic the proof of Theorem 8.5.9 in the context of warped cones.

Theorem. *If* $\mathrm{WSys}\big(F_S \curvearrowright X\big)$ *has stable discrete fundamental group and* $\mathcal{O}_S(X)$ *is quasi-isometric to* $\mathcal{O}_T(Y)$ *then* $\mathrm{WSys}\big(F_T \curvearrowright Y\big)$ *has stable discrete fundamental group and* $\pi_{1,\infty}\big(F_S \curvearrowright X\big) \cong \pi_{1,\infty}\big(F_T \curvearrowright Y\big)$.

Sketch of proof. Let $f\colon \mathcal{O}_S(X) \to \mathcal{O}_T(Y)$ be an (L, A)-quasi-isometry and let $\bar{f}$ be the coarse inverse. Also, fix three parameters θ, θ' and θ'' satisfying $\theta \geq L + A$, $\theta' \geq L\theta + A$ and $\theta'' \geq L(L\theta' + A) + A$ with θ large enough so that the projection $\pi_{1,\theta}\big(F_S \curvearrowright X\big) \to \pi_{1,\infty}\big(F_S \curvearrowright X\big)$ is an isomorphism.

For every $a \gg 1$ there exists $c, b \gg 1$ such that

$$f\Big(\mathcal{O}_S^{[c,\infty]}(X)\Big) \subseteq \mathcal{O}_T^{[b,\infty]}(Y) \quad \text{and} \quad \bar{f}\Big(\mathcal{O}_T^{[b,\infty]}(Y)\Big) \subseteq \mathcal{O}_S^{[a,\infty]}(X).$$

By the lemma above, we can deduce that both

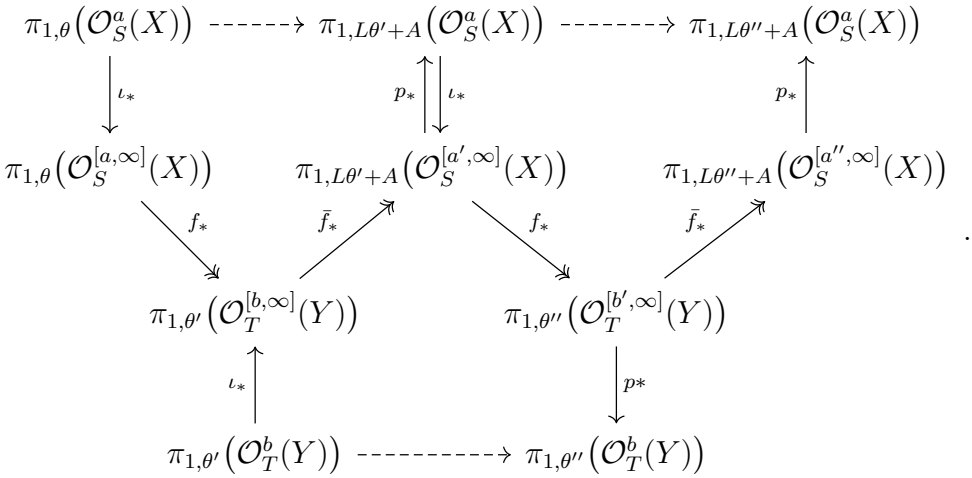
$$f_*\colon \pi_{1,\theta}\big(\mathcal{O}_S^{[c,\infty]}(X)\big) \longrightarrow \pi_{1,\theta'}\big(\mathcal{O}_T^{[b,\infty]}(Y)\big)$$

and

$$\bar{f}_*\colon \pi_{1,\theta'}\big(\mathcal{O}_T^{[b,\infty]}(Y)\big) \longrightarrow \pi_{1,L\theta'+A}\big(\mathcal{O}_S^{[a,\infty]}(X)\big)$$

are surjective. Indeed, every θ'-path Z in $\mathcal{O}_T^{[b,\infty]}(Y)$ is equivalent to a 1-path in a level set which is sufficiently high up so that its image under $\bar{f}$ is a θ-path in $\mathcal{O}_S^{[c,\infty]}(X)$. This θ-path is then is mapped to $[Z]$ by f_*. The same argument works for $\bar{f}_*$ as well.

We can now find parameters $a > a' > a'' \gg 1$ and $b > b' \gg 1$ so that the following composition of maps make sense and it induces a commutative diagram:



To conclude, note that the dashed homomorphisms are induced by functions that are close to the identity and that ι_* and p_* are isomorphisms. Then observe that

Lemma 3.1.4 implies that the maps f_* are also injective and hence all the maps are isomorphisms. $\square$

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