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Overpartitions with parts separated by parity

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Abstract. In this article, we generalize Andrews' partitions separated by parity to overpartitions in two ways. We investigate the generating functions for 16 overpartition families whose parts are separated by parity, and we prove various q-series identities for these functions. These identities include relations to modular forms, q-hypergeometric series, and mock modular forms.

1 Introduction and statement of results

We recall that a *partition* of an integer n is a non-increasing sequence $\lambda = \lambda_1 + \cdots + \lambda_\ell$ of positive integers whose parts, denoted by λ_j $(1 \le j \le \ell)$, sum up to n. Restrictions on partitions concerned with the parity of the parts in the partition have played a major role in the history of partition theory; see, e.g., Andrews' article [3] for an excellent overview of many ways parity has appeared in partition theory. In Andrews' recent works in this direction [4, 5], he introduced the notion of "partitions with parts separated by parity" in connection with mock theta functions. A partition λ has *parts separated by parity*[1] if each even part λ_j is larger than each odd part λ_k, or if each odd part λ_j is larger than each even part λ_k. Andrews used the notation $p_{\mathrm{xy}}^{\mathrm{zw}}(n)$ to count various kinds of partitions of n with parts separated by parity. In his notation, we let $\{\mathrm{x}, \mathrm{z}\} = \{\mathrm{e}, \mathrm{o}\}$, where the case $\mathrm{z} = \mathrm{e}, \mathrm{x} = \mathrm{o}$ (resp., $\mathrm{z} = \mathrm{o}, \mathrm{x} = \mathrm{e}$) signifies that even parts are larger than odd parts (resp., that odd parts are larger than even parts). Andrews then allowed y and w to denote either u or d, which represent unrestricted and distinct, respectively, and signify whether parts of parity x and z are unrestricted or must be distinct, respectively. Andrews studied the eight possible functions built this way; e.g., $p_{\mathrm{od}}^{\mathrm{eu}}(n)$ counts the number of partitions of n with even parts larger than odd parts which have all odd parts distinct.

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[1] Partitions into only odd parts or only into even parts are permitted.

Andrews' papers [4, 5] and the follow-up work of the first author and Jennings–Shaffer [12] focused on the generating functions

$$F_{\mathrm{xy}}^{\mathrm{zw}}(q) := \sum_{n\geq 0} p_{\mathrm{xy}}^{\mathrm{zw}}(n) q^n.$$

These generating functions have connections to a wide variety of modular-type objects. To state some of these results, we define the *q-Pochhammer symbol*

$$(a)_n = (a;q)_n := \prod_{k=0}^{n-1}\left(1-aq^k\right), \quad (a_1,a_2,\ldots,a_m;q)_n := \prod_{k=1}^{m}(a_k;q)_n,$$

for $n \in \mathbb{N}_0 \cup \{\infty\}$, $m \in \mathbb{N}$. As a first example, Andrews proved [4, Equation (2.1)] that

$$F_{\mathrm{eu}}^{\mathrm{ou}}(q) = \frac{1}{(1-q)\left(q^2;q^2\right)_\infty},$$

which is essentially a modular form of weight $-\frac{1}{2}$. The papers [5, 12] both proved (see also notation and comments from [10, 11]) that

$$F_{\mathrm{od}}^{\mathrm{eu}}(q) = \frac{1}{\left(q^2;q^2\right)_\infty}\left(1 - \frac{\sigma(-q)}{2} + \frac{(-q;-q)_\infty}{2}\right),$$

where

$$\sigma(q) := \sum_{n\geq 0} \frac{q^{\frac{n(n+1)}{2}}}{(-q;q)_n}$$

is the famous Ramanujan σ-function from his lost notebook [22]. It was shown by Andrews–Dyson–Hickerson [6] and Cohen [13] that $\sigma(q)$ is related to Maass forms. The variety of generating functions in [5] also connects to modular forms and mock modular forms. Andrews' work has given rise to many other examples, looking at further generating function identities [12], asymptotic formulas [10, 11], and connections with mock theta functions [17]. Although the definitions of each $p_{\mathrm{xy}}^{\mathrm{zw}}(n)$ are quite similar, the properties that arise from their generating functions are very different (including modular forms, mock modular forms, false modular forms, and mock Maass theta functions like $\sigma(q)$); this suggests a rich theory yet to be fully understood.

In this article, we extend this framework to overpartitions. An *overpartition* [14] is a partition where the first instance of each part may possibly be overlined. For instance, there are 8 overpartitions of 3, given by

$$3,\ \overline{3},\ 2+1,\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1},\ 1+1+1,\ \overline{1}+1+1.$$

Overpartitions have played numerous roles in q-series and combinatorics (see, e.g., [8, 20] and the references therein), mathematical physics (see, e.g., [16]), symmetric functions (see, e.g., [9]), representation theory (see, e.g., [19]), and algebraic number theory (see, e.g., [21]). The goal of this article is to pursue this theme in the context of partitions with parts separated by parity.

We define two variations of overpartitions with parts separated by parity. In both cases, we say that an overpartition λ has *parts separated by parity*[2] if each even λ_j is larger than each odd λ_k, or if each odd λ_j is larger than each even λ_k. Because allowing the distinct parts to be overlined introduces powers of two that are both complicated and unenlightening, we introduce an additional constraint in order to create an alternative family of overpartitions. If we impose the restriction that any parts required to be distinct cannot be overlined, then we call these *modified*. We let $\overline{p}_{\mathrm{xy}}^{\mathrm{zw}}(n)$ and $\underline{p}_{\mathrm{xy}}^{\mathrm{zw}}(n)$ denote the functions which count the number of overpartitions with parts separated by parity and the modified variation, respectively. For instance, we have that $\overline{p}_{\mathrm{od}}^{\mathrm{eu}}(3) = 6$ and $\underline{p}_{\mathrm{od}}^{\mathrm{eu}}(3) = 3$, respectively, since the corresponding overpartitions are

$$3,\ 2+1,\ \overline{3},\ \overline{2}+1,\ 2+\overline{1},\ \overline{2}+\overline{1}$$

and

$$3,\ 2+1,\ \overline{2}+1.$$

The difference between the two cases is that the overpartitions counted by $\overline{p}_{\mathrm{od}}^{\mathrm{eu}}(3)$ are permitted to have overlines on both the even and odd parts, whereas the overpartitions counted by $\underline{p}_{\mathrm{od}}^{\mathrm{eu}}(3)$ are only permitted to have overlines on the even parts.

In this article, we consider the 16 generating functions

$$\overline{F}_{\mathrm{xy}}^{\mathrm{zw}}(q) := \sum_{n\geq 0} \overline{p}_{\mathrm{xy}}^{\mathrm{zw}}(n) q^n, \quad \underline{F}_{\mathrm{xy}}^{\mathrm{zw}}(q) := \sum_{n\geq 0} \underline{p}_{\mathrm{xy}}^{\mathrm{zw}}(n) q^n.$$

We also need the *Jacobi theta function*

$$\Theta(\tau) := \sum_{n\in zZ} q^{n^2} \quad (q := e^{2\pi i\tau}).$$

This is a modular form of weight $\frac{1}{2}$. Our first results are formulas for the generating functions $\overline{F}_{\mathrm{xy}}^{\mathrm{zw}}(q)$.

Theorem 1.1 *The following identities hold:*

$$\text{(1.1)} \quad \overline{F}_{\mathrm{eu}}^{\mathrm{ou}}(q) = \frac{(-q^2;q^2)_\infty}{2(q^2;q^2)_\infty}\left(\Theta^2(\tau)+1\right),$$

$$\text{(1.2)} \quad \overline{F}_{\mathrm{ou}}^{\mathrm{eu}}(q) = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty} + \frac{2q}{1-q}\frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}\frac{(-q,q^2;q^2)_n q^{2n}}{(q^3,-q^2;q^2)_n},$$

$$\text{(1.3)} \quad \overline{F}_{\mathrm{ed}}^{\mathrm{od}}(q) = (-2q;q^2)_\infty - \frac{q}{1-q}(-2q^2;q^2)_\infty + \frac{q}{1-q}(-2q;q^2)_\infty,$$

$$\text{(1.4)} \quad \overline{F}_{\mathrm{od}}^{\mathrm{ed}}(q) = (-2q^2;q^2)_\infty + \frac{q}{1-q}\left(3(-2q^2;q^2)_\infty - (-2q;q^2)_\infty\right),$$

$$\text{(1.5)} \quad \overline{F}_{\mathrm{eu}}^{\mathrm{od}}(q) = \frac{(-q^2;q^2)_\infty}{(q^2;q^2)_\infty}\sum_{n\geq 0}\frac{2^n q^{n^2}}{(-q^2;q^2)_n},$$

[2] As in the standard case, overpartitions into only odd parts or only even parts are permitted.

$$\overline{F}^{\mathrm{eu}}_{\mathrm{od}}(q) = -\frac{\left(-2q, q^2; q^2\right)_\infty}{\left(-q^2; q^2\right)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{2n}}{\left(2q; q^2\right)_{n+1}} + 2\sum_{n\geq 0} \frac{(-1)^n 2^n q^{n^2+2n}}{\left(1+q^{2n+2}\right)\left(2q; q^2\right)_{n+1}}, \tag{1.6}$$

$$\overline{F}^{\mathrm{ou}}_{\mathrm{ed}}(q) = \frac{\left(-q^2; q^2\right)_\infty}{\left(q^2; q^2\right)_\infty} + 2q\left(-2q^2; q^2\right)_\infty \sum_{n\geq 0} \frac{(-1)^n q^{2n}}{\left(2q; q^2\right)_{n+1}}$$
$$+ \frac{2q\left(-q; q^2\right)_\infty}{\left(q^3; q^2\right)_\infty} \sum_{n\geq 0} \frac{(-1)^n 2^n \left(-q; q^2\right)_n q^{n^2+3n}}{\left(2q, -q^2; q^2\right)_{n+1}}, \tag{1.7}$$

$$\overline{F}^{\mathrm{ed}}_{\mathrm{ou}}(q) = \left(-2q^2; q^2\right)_\infty + \sum_{n\geq 0} \frac{\left(-q; q^2\right)_n \left(-2q^{2n+2}; q^2\right)_\infty}{\left(q; q^2\right)_{n+1}} q^{2n+1}. \tag{1.8}$$

Remark We note a connection to modular forms. In particular, (1.1) yields a sum of weakly holomorphic modular forms of weights $-\frac{1}{2}$ and $\frac{1}{2}$.

We next turn to $\underline{F}^{\mathrm{zw}}_{\mathrm{xy}}(q)$. We note that in cases where odd and even parts are either both unrestricted or both distinct, the generating functions boil down to either $\overline{F}$ or F, respectively; that is,

$$\underline{F}^{\mathrm{ou}}_{\mathrm{eu}}(q) = \overline{F}^{\mathrm{ou}}_{\mathrm{eu}}(q), \quad \underline{F}^{\mathrm{eu}}_{\mathrm{ou}}(q) = \overline{F}^{\mathrm{eu}}_{\mathrm{ou}}(q), \quad \underline{F}^{\mathrm{od}}_{\mathrm{ed}}(q) = F^{\mathrm{od}}_{\mathrm{ed}}(q), \quad \underline{F}^{\mathrm{ed}}_{\mathrm{od}}(q) = F^{\mathrm{ed}}_{\mathrm{od}}(q).$$

These are treated either in Theorem 1.1 or in previous works [5, 12]. Thus, we focus on the remaining four examples of interest, in which one parity is restricted and one is unrestricted. In order to state this theorem, we recall the third order mock theta function

$$\phi(q) := \sum_{n\geq 0} \frac{q^{n^2}}{\left(-q^2; q^2\right)_n}.$$

We prove the following result on the remaining modified generating functions.

Theorem 1.2 *The following identities hold:*

(1.9)
$$\underline{F}^{\mathrm{od}}_{\mathrm{eu}}(q) = \frac{\left(-q^2; q^2\right)_\infty}{\left(q^2; q^2\right)_\infty} \phi(q),$$

$$\underline{F}^{\mathrm{eu}}_{\mathrm{od}}(q) = \frac{\left(-q^2; q^2\right)_\infty}{\left(q^2; q^2\right)_\infty} - 2q\left(-q; q^2\right)_\infty \sum_{n\geq 0} \frac{q^n}{\left(-q^2; q^2\right)_{n+1}}$$

(1.10)
$$+ 4q\frac{\left(-q^2; q^2\right)_\infty}{\left(q^2; q^2\right)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{n^2+2n}}{\left(1+q^{2n+2}\right)\left(q; q^2\right)_{n+1}},$$

(1.11)
$$\underline{F}^{\mathrm{ou}}_{\mathrm{ed}}(q) = -2q\left(-q^2; q^2\right)_\infty \sum_{n\geq 0} \frac{q^n}{\left(-q^2; q^2\right)_{n+1}} + \frac{\left(-q; q^2\right)_\infty}{\left(q; q^2\right)_\infty} \sum_{n\geq 0} \frac{(-1)^n \left(-q; q^2\right)_{n+1} q^{n^2+n}}{\left(-q^2, q; q^2\right)_{n+1}},$$

(1.12)

$$\underline{F}_{\mathrm{ou}}^{\mathrm{ed}}(q) = \left(-q^2;q^2\right)_\infty \left(1 + 2q \sum_{n\geq 0} \frac{\left(-q;q^2\right)_n q^{2n}}{\left(q,-q^2;q^2\right)_n}\right).$$

The article is organized as follows. In Section 2, we provide several known q-series identities which we use in the proof of our main results. In Sections 3 and 4, we prove the results regarding standard and modified overpartitions with parts separated by parity, respectively. In Section 5, we discuss future directions for work.

2 Preliminaries

In this section, we recall various identities we use in this article. Andrews' book [2] or Fine's book [15] are excellent introductions to q-series transformations of these types.

2.1 The Heine transformation

We begin by stating one of Heine's transformations.

Lemma 2.1 [2, Corollary 2.3] *For* $|q|, |t| < 1$ *and* $0 < |b| < 1$, *we have*

$$\sum_{n\geq 0} \frac{(a,b)_n t^n}{(q,c)_n} = \frac{(b,at)_\infty}{(c,t)_\infty} \sum_{n\geq 0} \frac{\left(\frac{c}{b},t\right)_n b^n}{(q,at)_n}.$$

We also use another Heine transformation[3] (see [18, (III.2)]).

Lemma 2.2 *For* $|q|, |t| < 1$ *and* $0 < |c| < |b| < 1$, *we have*

$$\sum_{n\geq 0} \frac{(a,b)_n t^n}{(q,c)_n} = \frac{\left(\frac{c}{b},bt\right)_\infty}{(c,t)_\infty} \sum_{n\geq 0} \frac{\left(\frac{abt}{c},b\right)_n \left(\frac{c}{b}\right)^n}{(q,bt)_n}.$$

Lemma 2.3 [7, Equation (4.1)] *We have*[4]

$$\sum_{n\geq 0} \frac{(x)_n q^n}{(y)_n} = \frac{q(x)_\infty}{y\left(1-\frac{xq}{y}\right)(y)_\infty} + \frac{1-\frac{q}{y}}{1-\frac{xq}{y}}.$$

2.2 Identities from the lost notebook

A final identity which we require emerges from the early work on Ramanujan's Lost Notebook. In particular, we use the following very general transformation formula due to Andrews that is used frequently to prove identities for partial theta functions.

[3] The analytic conditions can be easily derived from those in Lemma 2.1.

[4] Although we do not specify the range in which this and the following identities hold, they are straightforwardly derived from the proof in [7].

Lemma 2.4 [1, Theorem 1] *We have*

$$\sum_{n\geq 0}\frac{(B,-Abq)_n q^n}{(-aq,-bq)_n}=\frac{-a^{-1}(B,-Abq)_\infty}{(-ab,-bq)_\infty}\sum_{n\geq 0}\frac{\left(A^{-1}\right)_n\left(\frac{Abq}{a}\right)^n}{\left(-\frac{B}{a}\right)_{n+1}}$$
$$+(1+b)\sum_{n\geq 0}\frac{\left(-a^{-1}\right)_{n+1}\left(-\frac{ABq}{a}\right)_n(-b)^n}{\left(-\frac{B}{a},\frac{Abq}{a}\right)_{n+1}}.$$

Letting $A\mapsto\frac{A}{b}$, $B\mapsto -Bq^2$, $q\mapsto q^2$, and then $b\to 0$, we obtain the following.

Lemma 2.5 [1, Equation (3.9)] *We have*

$$\sum_{n\geq 0}\frac{\left(-Bq^2,-Aq^2;q^2\right)_n q^{2n}}{\left(-aq^2;q^2\right)_n}=\frac{-a^{-1}\left(-Bq^2,-Aq^2;q^2\right)_\infty}{\left(-aq^2;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(\frac{Aq^2}{a}\right)^n}{\left(\frac{Bq^2}{a};q^2\right)_{n+1}}$$
$$+\sum_{n\geq 0}\frac{\left(-a^{-1};q^2\right)_{n+1}\left(\frac{AB}{a}\right)^n q^{n^2+3n}}{\left(\frac{Bq^2}{a},\frac{Aq^2}{a};q^2\right)_{n+1}}.$$

3 Proof of Theorem 1.1

In this section, we prove all claimed identities for the functions $\overline{F}_{\mathrm{xy}}^{\mathrm{zw}}$.

3.1 Proof of (1.1)

We may build up a generating function for $\overline{F}_{\mathrm{eu}}^{\mathrm{ou}}$ by summing over the cases where $2n$ is the size of the maximum even part of the overpartition for each $n\in\mathbb{N}$. There are two possibilities: either the overpartition has a marked part of size $2n$ or it only has unmarked parts of size $2n$. Both cases yield the same expression for the generating function, so the construction is a multiple of two. This gives

$$\overline{F}_{\mathrm{eu}}^{\mathrm{ou}}(q)=2\frac{\left(-q;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(-q^2;q^2\right)_{n-1}\left(q;q^2\right)_n q^{2n}}{\left(q^2,-q;q^2\right)_n}$$
$$=\frac{\left(-q;q^2\right)_\infty}{\left(q;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(-1,q;q^2\right)_n q^{2n}}{\left(q^2,-q;q^2\right)_n}.$$

Using Lemma 2.1 with $q\mapsto q^2$, $a=-1$, $b=q$, $c=-q$, and $t=q^2$, we obtain

$$\overline{F}_{\mathrm{eu}}^{\mathrm{ou}}(q)=2\frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\sum_{n\geq 0}\frac{q^n}{1+q^{2n}}. \tag{3.1}$$

By work of Kronecker (see [23])

$$1+4\sum_{n\geq 1}\frac{(-1)^n q^n}{1+q^{2n}}=\frac{(q,q)_\infty}{(-q,-q)_\infty}.$$

Replacing q by $-q$ gives

$$\sum_{n\geq 1} \frac{q^n}{1+q^{2n}} = \frac{1}{4}\left(\Theta^2(\tau)-1\right).$$

Plugging this into (3.1) gives the claim.

3.2 Proof of (1.2)

The construction of the generating function $\overline{F}_{\mathrm{ou}}^{\mathrm{eu}}(q)$ is analogous to that of $\overline{F}_{\mathrm{eu}}^{\mathrm{ou}}(q)$. Namely, we sum over the instances where the largest part is odd, and we distinguish two cases based on whether there exists an overlined part of that size. As with $\overline{F}_{\mathrm{eu}}^{\mathrm{ou}}(q)$, these cases turn out the same generating function. Moreover, we account separately for the situation where there are no odd parts with the leading term. We obtain the identity

$$\overline{F}_{\mathrm{ou}}^{\mathrm{eu}}(q) = \frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} + \frac{2q}{1-q}\frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(-q,q^2;q^2\right)_n q^{2n}}{\left(q^3,-q^2;q^2\right)_n}.$$

This completes the proof.

3.3 Proof of (1.3)

We again follow Section 3.1, splitting into two cases where the largest even part is either marked or not and including a leading term to account for the case where there are no even parts, and we obtain

$$\overline{F}_{\mathrm{ed}}^{\mathrm{od}}(q) = \left(-2q;q^2\right)_\infty + 2q^2\left(-2q^3;q^2\right)_\infty\sum_{n\geq 0}\frac{\left(-2q^2;q^2\right)_n q^{2n}}{\left(-2q^3;q^2\right)_n}.$$

Using Lemma 2.3 with $q\mapsto q^2$ and then $x=-2q^2$ and $y=-2q^3$, gives the claim.

3.4 Proof of (1.4)

We again follow Section 3.1, splitting into two cases depending on whether the largest odd part is either marked or not and including a leading term to account for the case where there are no odd parts, and we obtain

$$\overline{F}_{\mathrm{od}}^{\mathrm{ed}}(q) = \left(-2q^2;q^2\right)_\infty + 2q\left(-2q^2;q^2\right)_\infty\sum_{n\geq 0}\frac{\left(-2q;q^2\right)_n q^{2n}}{\left(-2q^2;q^2\right)_n}.$$

Using Lemma 2.3 with $q\mapsto q^2$, and then $x=-2q$ and $y=-2q^2$, we obtain the claim.

3.5 Proof of (1.5)

We again follow Section 3.1, splitting into two cases depending on whether the largest even part is either marked or not, and obtain

$$\overline{F}_{\mathrm{eu}}^{\mathrm{od}}(q) = 2\sum_{n\geq 0}\frac{\left(-q^2;q^2\right)_{n-1}\left(-2q^{2n+1};q^2\right)_\infty q^{2n}}{\left(q^2;q^2\right)_n} = \left(-2q;q^2\right)_\infty\sum_{n\geq 0}\frac{\left(-1;q^2\right)_n q^{2n}}{\left(-2q,q^2;q^2\right)_n}.$$

Letting $q \mapsto q^2$, $t = -\frac{2q}{a}$, $a \to \infty$, $b = q^2$, and $c = q$ in Lemma 2.1, we conclude the claimed formula for $\overline{F}_{\mathrm{eu}}^{\mathrm{od}}(q)$.

3.6 Proof of (1.6)

We again follow Section 3.1, splitting into two cases depending on whether the largest odd part is either marked or not and including a leading term to account for the case where there are no odd parts. We obtain

$$\begin{aligned}\overline{F}_{\mathrm{od}}^{\mathrm{eu}}(q) &= \frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} + 2\sum_{n\geq 0} \frac{\left(-2q;q^2\right)_n\left(-q^{2n+2};q^2\right)_\infty q^{2n+1}}{\left(q^{2n+2};q^2\right)_\infty}\\ &= \frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} + 2q\frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(-2q,q^2;q^2\right)_n q^{2n}}{\left(-q^2;q^2\right)_n}.\end{aligned}$$

Applying Lemma 2.5 with $a = 1$, $A = -1$, and $B = \frac{2}{q}$, gives (1.6).

3.7 Proof of (1.7)

We again follow Section 3.1, splitting into two cases depending on whether the largest even part is either marked or not and including a leading term to account for the case where there are no even parts. We obtain

$$\begin{aligned}\overline{F}_{\mathrm{ed}}^{\mathrm{ou}}(q) &= \frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} + 2\sum_{n\geq 0} \frac{\left(-2q^2;q^2\right)_n\left(-q^{2n+3};q^2\right)_\infty q^{2n+2}}{\left(q^{2n+3};q^2\right)_\infty},\\ &= \frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} + 2q^2\frac{\left(-q^3;q^2\right)_\infty}{\left(q^3;q^2\right)_\infty}\sum_{n\geq 0}\frac{\left(-2q^2,q^3;q^2\right)_n q^{2n}}{\left(-q^3;q^2\right)_n}.\end{aligned}$$

Using Lemma 2.5 with $B = 2$, $A = -q$, and $a = q$, we obtain

$$\begin{aligned}\sum_{n\geq 0}\frac{\left(-2q^2,q^3;q^2\right)_n q^{2n}}{\left(-q^3;q^2\right)_n} &= -\frac{\left(-2q^2,q^3;q^2\right)_\infty}{q\left(-q^3;q^2\right)_\infty}\sum_{n\geq 0}\frac{(-1)^n q^{2n}}{\left(2q;q^2\right)_{n+1}}\\ &\quad + \left(1+\frac{1}{q}\right)\sum_{n\geq 0}\frac{(-1)^n 2^n\left(-q;q^2\right)_n q^{n^2+3n}}{\left(2q,-q^2;q^2\right)_{n+1}}.\end{aligned}$$

Plugging this into the formula for $\overline{F}_{\mathrm{ed}}^{\mathrm{ou}}(q)$ then gives the claim.

3.8 Proof of (1.8)

We again follow Section 3.1, splitting into two cases depending on whether the largest odd part is either marked or not and including a leading term to account for the case where there are no odd parts. This quickly yields the claimed result, namely

$$\overline{F}_{\rm ou}^{\rm ed}(q) = \left(-2q^2;q^2\right)_\infty + \sum_{n\geq 0} \frac{\left(-q;q^2\right)_n \left(-2q^{2n+2};q^2\right)_\infty q^{2n+1}}{\left(q;q^2\right)_{n+1}}$$
$$= \left(-2q^2;q^2\right)_\infty \left(1 + \frac{2q}{1-q} \sum_{n\geq 0} \frac{\left(-q;q^2\right)_n q^{2n}}{\left(q^3,-2q^2;q^2\right)_n}\right).$$

4 Proof of Theorem 1.2

In this section, we prove Theorem 1.2.

4.1 Proof of (1.9)

We modify our argument from Section 3.1 and obtain

$$\underline{F}_{\rm eu}^{\rm od}(q) = 2\left(-q;q^2\right)_\infty \sum_{n\geq 0} \frac{\left(-q^2;q^2\right)_{n-1} q^{2n}}{\left(q^2,-q;q^2\right)_n} = \left(-q;q^2\right)_\infty \sum_{n\geq 0} \frac{\left(-1;q^2\right)_n q^{2n}}{\left(q^2,-q;q^2\right)_n}.$$

We again use Lemma 2.1 with $q \mapsto q^2$, $t = -\frac{q}{a}$, $a \to \infty$, $b = q^2$, and $c = -q^2$ to conclude the claim.

4.2 Proof of (1.10)

As above, we have

$$\underline{F}_{\rm od}^{\rm eu}(q) = \frac{\left(-q^2;q^2\right)_\infty}{\left(q^2;q^2\right)_\infty} \left(1 + 2q \sum_{n\geq 0} \frac{\left(-q;q^2\right)_n \left(q^2;q^2\right)_n q^{2n}}{\left(-q^2;q^2\right)_n}\right).$$

Using Lemma 2.5 with $B = \frac{1}{q}$, $A = -1$, and $a = 1$, we have

$$\sum_{n\geq 0} \frac{\left(-q,q^2;q^2\right)_n}{\left(-q^2;q^2\right)_n} q^{2n} = -\frac{\left(-q,q^2;q^2\right)_\infty}{\left(-q^2;q^2\right)_\infty} \sum_{n\geq 0} \frac{(-1)^n q^{2n}}{\left(q;q^2\right)_{n+1}} + 2\sum_{n\geq 0} \frac{(-1)^n q^{n^2+2n}}{\left(1+q^{2n+2}\right)\left(q;q^2\right)_{n+1}}.$$

We next claim that the first summation can be written

$$\sum_{n\geq 0} \frac{(-1)^n q^{2n}}{\left(q;q^2\right)_{n+1}} = \sum_{n\geq 0} \frac{q^n}{\left(-q^2;q^2\right)_{n+1}}.$$

Changing variables $q \mapsto q^{\frac{1}{2}}$, the claim is equivalent to

$$\frac{1}{1-q^{\frac{1}{2}}} \sum_{n\geq 0} \frac{(-1)^n q^n}{\left(q^{\frac{3}{2}};q\right)_n} = \frac{1}{1+q} \sum_{n\geq 0} \frac{q^{\frac{n}{2}}}{\left(-q^2;q\right)_n}.$$

Using Lemma 2.2 with $a = 0$, $b = q$, $c = -q^2$, and $t = q^{\frac{1}{2}}$, we obtain the claim. In turn, we complete the proof.

4.3 Proof of (1.11)

We follow the argument in Section 4.1. Namely, we split into two terms based on whether the largest even part is marked or not and include a leading term to account for the case where there are no even parts. This gives

$$\underline{F}_{\text{ed}}^{\text{ou}}(q)=\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\left(1+2\sum_{n\geq1}\frac{(q;q^2)_n(-q^2;q^2)_{n-1}q^{2n}}{(-q;q^2)_n}\right)=\frac{(-q;q^2)_\infty}{(q;q^2)_\infty}\sum_{n\geq0}\frac{(q,-1;q^2)_nq^{2n}}{(-q;q^2)_n}.$$

Using Lemma 2.5 with $B=-\frac{1}{q}$, $A=\frac{1}{q^2}$, and $a=\frac{1}{q}$, we have

$$\sum_{n\geq0}\frac{(q,-1;q^2)_nq^{2n}}{(-q;q^2)_n}=-\frac{q(q,-1;q^2)_\infty}{(-q;q^2)_\infty}\sum_{n\geq0}\frac{q^n}{(-q^2;q^2)_{n+1}}+\sum_{n\geq0}\frac{(-1)^n(-q;q^2)_{n+1}q^{n^2+n}}{(-q^2,q;q^2)_{n+1}},$$

from which the desired result follows.

4.4 Proof of (1.12)

We follow the argument in Section 4.1. Namely, we split into two terms based on whether the largest odd part is marked or not and include a leading term to account for the case where there are no even parts. This directly gives the claim.

5 Future work

In this article, we produce a number of new formulas for overpartitions with parts separated by parity. These new formulas, along with connections to modular and mock modular forms, suggest a number of applications and open questions:

(1) What are the asymptotics growth rates of the various functions counting overpartitions with parts separated by parity?
(2) Which families of overpartitions with parts separated by parity are connected to modular forms, mock modular forms, or some other modular object?
(3) Are there two-variable extensions for these generating functions which have connections to two-variable modular objects like Jacobi forms?
(4) Do any of these overpartition families possess Ramanujan-type congruences?

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