Markov Functional Market Model and Standard Market Model

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A dissertation submitted in partial fulfillment for the degree of

Master of Science in Mathematical and Computational Finance

Trinity 2008

This dissertation is dedicated to
My parents

Acknowledgements

First of all, I would like to thank Dr. Ben Hambly for taking me to do this project. His supervision of the project as well as his comments on the draft has been invaluable. Of course, it goes without saying that his fixed income lectures during the master course were also excellent.
There are also many friends who I want to thank for their help throughout the master course at Oxford, particularly my thanks goes to Wu Chen and Li Shanshan for their continuous encouragement and support on both my study and job seeking. I thank Jonathan Buckler for being a nice flatmate and for his help on checking the grammar and spelling. Also, I would like to express my thanks to Matthew Clarke, who is a Manchester alumnus of mine, for his comment on the draft of the paper and for, simply, being a great friend at Oxford.
Last, but not least, I would like to thank my parents for their understanding, trust and more importantly for their love.

Abstract

The introduction of so called Market Models (BGM) in 1990s has developed the world of interest rate modelling into a fresh period. The obvious advantages of the market model have generated a vast amount of research on the market model and recently a new model, called Markov functional market model, has been developed and is becoming increasingly popular. To be clearer between them, the former is called standard market model in this paper.

Both standard market models and Markov functional market models are practically popular and the aim here is to explain theoretically how each of them works in practice. Particularly, implementation of the standard market model has to rely on advanced numerical techniques since Monte Carlo simulation does not work well on path-dependent derivatives. This is where the strength of the Longstaff-Schwartz algorithm comes in. The successful application of the Longstaff-Schwartz algorithm with the standard market model, more or less, adds another weight to the fact that the Longstaff-Schwartz algorithm is extensively applied in practice.

Contents

Chapter 1

Introduction

The trading volume in interest rate derivatives, in both the over-the-counter (OTC) and exchange-traded markets, has been growing rapidly since the 1980s. Pricing interest rate derivatives accurately, however, is usually more difficult than valuing equity and foreign exchange derivatives; one of the reasons is because an individual interest rate has a more complicated behaviour than that of a stock price or an exchange rate. It is, thus, fundamentally important to model interest rate, the non-traded underlying asset in the fixed income market, effectively in the hope of correctly pricing its derivatives. Traditionally, there are three perspectives in modelling interest rates, namely, short rate models, instantaneous forward rate modelling and market models. So far, there has been a large number of classical short rate models, such as Vasicek Model [37], Cox, Ingersoll & Ross (CIR) Model [9], Ho-Lee Model [17] and Hull-White-Vasicek Model [18], which have attracted much attention from both the academic and practitioners due to their tractability and transparency (see Section 2.1). As a result, short rate models have been widely used and it is still being popular in the banking industry. On the other hand, most short rate models involve only one source of uncertainty, namely one-factor short rate models, making all rates perfectly correlated and hence they are not accurate in modelling shifts in the yield curve that are significantly different at different maturities [38].This drawback of the short rate model tends to be more obvious for complex products which may well depend on the difference between yields of different maturities; even though extended two-factor short rate models can, to some extent, allow for a richer yield curve structure, there is another more severe weakness of short rate models. The volatility structure in the short rate models, after being made time-dependent, is nonstationary, which, consequently, leads to model calibration inconsistent; this apparently is significant from a practical point of view [19].

A major breakthrough in arbitrage-free modelling of interest rate was the approach

to the term structure modelling proposed in [16] by Heath, Jarrow and Morton and it is now often referred to as the HJM modelling framework. The distinguishing feature of the HJM framework is that it covers a large number of previously proposed models and that instead of modelling a short-term interest rate, instantaneous forward rates are modelled; hence the difficulty of calibration that short rate models have is resolved naturally [39]. Though it is even possible to take real data for the random movement of the forward rates and incorporate into pricing derivatives, a vital weakness of the HJM modelling framework is that it can be relatively inefficient to price derivatives especially for callable products, as it requires a certain degree of smoothness with respect to the tenor of the bond prices and their volatilities [38](see Section 2.2).
An alternative way of modelling interest rates in an arbitrage-free bond market that has been increasingly popular is to take market traded rates as the underlying variables in the model. The foundation of this construction was built in [35] where the focus was on the effective annual interest rate. The idea was further developed in [6] by Brace, Gatarek and Musiela focusing directly on modelling forward LIBORs, which is often considered as a milestone in so called the **Market Model**. In the meantime, similar development was independently done in [26] by Jamshidian but more attention instead was paid to modelling swap rates. Generally speaking, Market Models, also known as BGM models, are essentially arbitrage-free term structure models which are formulated directly in terms of market observable rates, like LIBORs and swap rates, their volatilities and correlations. By enforcing the log-normality of the forward LIBOR (or swap) rate under the corresponding forward martingale measure, market models are then compatible with the common practice of pricing standard fixed income products, such as caps and swaptions, through justified Black's formula (see Section 2.3.4).
Whereas it is easy to specify the standard Market Models so as to have market prices fitted exactly and model calibration is therefore trivial, Market Models do have a rather undesirable characteristic. To accurately implement a Market Model, it has to be done by Monte Carlo simulation because of the high dimensionality of the model caused by each LIBOR (or swap) rate typically having its own stochastic driver [23]. This is consequently problematic for pricing even non-callable, path-dependent products since it is computationally expensive to generate enough Monte Carlo paths to get a sufficiently accurate price so that the 'Greeks' (risk sensitivities) will be accurately usable in risk management. Not surprisingly, this problem becomes more serious for callable products because simulation is usually poor in performing calculations backwards in time. Moreover, in the case of currencies, such as Yen, with very

low interest rates, market option prices cannot be simply given by Black's formula, consequently, calibrating Market Models is almost as cumbersome as the case of short rate models (see, for example, [1]).

By now, there have been several approaches proposed to overcome the practical difficulties standard Market Models face (see, for example, [33] [13] [27]). This paper partially focuses on another recent developed model which cannot only fit the observed prices of liquid instruments similarly as in the standard Market Models but which also enjoys a low-dimensional property in pricing derivatives. This approach, primarily proposed in [24] [21] [20], is termed the **Markov-Functional Market Model** since its defining characteristic is that zero-coupon bond (ZCB), also called pure discount bond, prices are at any time a function of some low-dimensional Markovian process in some martingale measure. This then implies an efficient implementation as it need only track the driving Markov process. The second main goal of this paper lies on explaining the usage of the **Longstaff-Schwartz algorithm** [29], one of the most successful algorithms developed to price American style products, together with the standard Market Model to price high dimensional fixed income instruments .

The organisation of this paper is as follows. While the first two sections of Chapter 2 are devoted to concisely describing short rate modelling and the HJM approach, section 3 examines the standard Market Model with more details. Though it is possible to use the Markov-Functional Market Model to price European style products, the focus of whole Chapter 3 is predominatingly on pricing multi-temporal (such as American style) products. Chapter 4 succinctly describes the famous Longstaff-Schwartz algorithm which is a powerful tool in pricing multi-temporal products, indicating that the implementation of the standard market model pricing complex interest rate derivatives is indeed possible. In Chapter 5, implementation of the standard market model and the Markov-Functional Market Model will be presented with some numerical results, pricing a Bermudan swaption as an example, in order to make the theoretical comparison into numerical. Finally, the conclusion is in Chapter 6.

Chapter 2

Interest Rate Modelling

2.1 Short rate modelling

The class of short rate models is, in fact, a special case of arbitrage-free models of the term structure for which the short rate $(r_t)_{t\geq 0}$ is, in the risk neutral measure $\mathbb{Q}$, a (time-inhomogeneous) Markov process [22]. Normally, though not necessarily, short rate models are driven by a univariate Brownian motion and this class of models, due to their convenient numerical implementation property, has been significantly important historically.

Almost all the short rate models are specified through a stochastic differential equation (SDE)

$$dr_t = \mu(t, r)dt + \sigma(t, r)dW_t,$$

where W is a Brownian motion in the risk neutral measure $\mathbb{Q}$ and functions μ and σ are carefully chosen to make the model particularly tractable, in a sense that the solution process is, most often, a Gaussian process and therefore the model can be analytically developed further, and arbitrage-free.

By the risk neutral pricing formula, the price of a zero-coupon bond (ZCB) $P(t, T)$, maturing at T, at time t $(t < T)$ is given by

$$P(t, T) = E_{\mathbb{Q}}\left[\exp\Big(-\int_t^T r_s\ ds\Big)\Big|\,\mathcal{F}_t\right], \tag{2.1}$$

where $\mathcal{F}_t$ is the augmented natural filtration generated by the Brownian motion W. The Markov property of r ensures that (2.1) is a function of the triple (r_t, t, T) for all pairs (t, T). In other words, the state of the market at t is completely determined by the pair (r_t, t). It is this property that allows one to price derivatives by most standard numerical methods such as simulation and finite-difference algorithms (see

[10] [11] [15] in this regard). A simplified version of the Hull-White-Vasicek (HWV) model takes the form

$$dr_t = \big(\theta(t) - kr_t\big)dt + \sigma dW_t, \tag{2.2}$$

where k, σ are constants but θ, the mean reversion level, is a deterministic function of time. This simplified form is a suitable candidate for a quick example showing the imperfect calibration property of the short rate models as well as the tractability of short rate models in terms of the exsitence of close-form bond prices.

Theorem 2.1. *The bond prices in short rate model with*

$$\mu(t,r) = \alpha(t)r + \beta(t),$$

$$\sigma^2(t,r) = \gamma(t)r + \delta(t),$$

are of the form

$$P(t,T) = exp\Big(A(t,T) - B(t,T)r_t\Big), \tag{2.3}$$

where equations $A(t,T)$ and $B(t,T)$, respectively, satisfy

$$A_t - \beta B + \frac{1}{2}\delta(t)B^2 = 0,$$

and

$$B_t + \alpha(t)B - \frac{1}{2}\gamma(t)B^2 + 1 = 0.$$

A_t, B_t denote the first derivative of A, B with respect to t, with the boundary condtions $A(T,T) = 0,\ B(T,T) = 0$.

By **Theorem 2.1**[1], in the simplified HWV model, it immediately follows that

$$\alpha(t) = -k, \quad \beta(t) = \theta(t).$$

$$\gamma(t) = 0, \quad \delta(t) = \sigma^2.$$

Whence for equations A and B they become

$$A_t = \theta(t)B(t,T) - \frac{1}{2}\sigma^2 B(t,T)^2, \tag{2.4}$$

$$B_t = kB(t,T) - 1. \tag{2.5}$$

[1]For the proof of this standard result see, for example, Chapter 21 of [5] or Chapter 3 of [7]

Solving (2.5) with $B(T,T)=0$ gives

$$B(t,T) = \frac{1-e^{-k(T-t)}}{k}. \tag{2.6}$$

Substituting (2.6) into (2.4) gives

$$A(T,T) - A(t,T) = \int_t^T \Big[\theta(u)B(u,T) - \frac{\sigma^2 B(u,T)^2}{2}\Big] du. \tag{2.7}$$

Solving (2.7) with $A(T,T)=0$ leads to

$$A(t,T) = \int_t^T \Big[-\theta(u)B(u,T) + \frac{\sigma^2 B(u,T)^2}{2}\Big] du. \tag{2.8}$$

Fitting the observed initial forward price $f^*(0,T)$ results

$$f^*(0,T) = -\frac{\partial}{\partial T}\log P^*(0,T) = -A_T(0,T) + B_T(0,T)r_0, \tag{2.9}$$

where A_T, B_T denote the first derivative of A, B with respect to T.
From (2.6) and (2.8) respectively, it is easy to, differentiating with respect to T, get

$$B_T = e^{-k(T-t)},$$

and

$$\begin{aligned}
A_T &= \frac{\partial}{\partial T}\Big(\int_t^T \Big[-\theta(u)B(u,T) + \frac{\sigma^2 B(u,T)^2}{2}\Big] du\Big) \\
&= -\theta(T)B(T,T) + \frac{\sigma^2 B(T,T)^2}{2} + \int_0^T \big[-\theta(u)B_T(u,T) + \sigma^2 B_T(u,T)B(u,T)\big]\, du \\
&= \int_0^T \big[-\theta(u)B_T(u,T) + \sigma^2 B_T(u,T)B(u,T)\big] du.
\end{aligned}$$

Hence (2.9) becomes

$$\begin{aligned}
f^*(0,T) &= r_0 B_T(0,T) - A_T(0,T) \\
&= r_0 e^{-kT} - \frac{\sigma^2}{k}\int_0^T e^{-k(T-u)}(1-e^{-k(T-u)})du + \int_0^T \theta(u)e^{-k(T-u)}du \\
&= r_0 e^{-kT} - \frac{\sigma^2}{2k^2}(1-e^{-kT})^2 + \int_0^T \theta(u)e^{-k(T-u)}du \\
&= r_0 e^{-kT} - \frac{\sigma^2}{2}B(0,T)^2 + \int_0^T \theta(u)e^{-k(T-u)}du
\end{aligned}$$

Setting

$$x(T) =: r_0 e^{-kT} + \int_0^T \theta(u)e^{-k(T-u)}du,$$

then

$$x(T) = f^*(0,T) + \frac{\sigma^2}{2} B(0,T)^2. \tag{2.10}$$

Observe that

$$\frac{dx}{dT} = -kr_0 e^{-kT} + \theta(T) - k\int_0^T \theta(u) e^{-k(T-u)}\, du = -kx(T) + \theta(T),$$

i.e.

$$\theta(T) = \frac{dx}{dT} + kx(T).$$

So using (2.10) gives

$$\begin{aligned}
\theta(T) &= \frac{\partial}{\partial T}\big(f^*(0,T) + \frac{\sigma^2}{2} B(0,T)^2\big) + k\big[f^*(0,T) + \frac{\sigma^2}{2} B(0,T)^2\big] \\
&= f^*_T(0,T) + \sigma^2 B_T(0,T) B(0,T) + k\big[f^*(0,T) + \frac{\sigma^2}{2} B(0,T)^2\big].
\end{aligned}$$

Thus even in this simplified case, $\{\theta(t)\}_{0 \le t \le T}$ could be found but not so straightforwardly from observed forward rate curve and hence bond prices will match the observed market prices at anytime t^*, $t^* = 0$ in this case, before the maturity. In practice, to better calibrate the model, k, σ will be allowed to be time-dependent as well; consequently calibration would have to employ numerical techinques which are often computationally intensive and unstable. Nevertheless, substituting the expression for θ into (2.8), simplifying algebraically, gives

$$A(t,T) = \log\Big(\frac{P^*(0,T}{P^*(0,t)}\Big) + B(t,T) f^*(0,t) - \frac{\sigma^2}{4k} B(t,T)^2 (1 - e^{-2kt}). \tag{2.11}$$

Whence the close-form of ZCB price, in this special case, follows naturally by substituting (2.11) into (2.3)

$$P(t,T) = \frac{P^*(0,T)}{P^*(0,t)} \exp\big[B(t,T) f^*(0,t) - \frac{\sigma^2}{4k} B(t,T)^2 (1 - e^{-2kt}) - r_t B(t,T)\big],$$

where $B(t,T)$ can be found from (2.6) and $P^*(0,T), P^*(0,t)$ are usually observable from the market.

2.2 HJM modelling framework

The HJM modelling framework relies on exogenously specifying the dynamics of instantaneous continuously compounded forward rates $f(t,T)$. For any fixed maturity $T < T^*$, the dynamics of the forward rate $f(t,T)$ are

$$df(t,T) = \alpha(t,T) dt + \sigma(t,T) dW_t,$$

where $\alpha(t,T) \in \mathbb{R}$, $\sigma(t,T) \in \mathbb{R}^d$ are adapted stochastic processes and W is a d-dimensional standard Brownian motion with respect to the underlying real probability measure $\mathbb{P}$. It is also assumed that

$$\int_0^T \alpha(t,T)\ dt < \infty,$$

and

$$\int_0^T \sigma_i^2(t,T)\ dt < \infty \quad \forall\ 1 \le i \le d.$$

Hence, it is equivalent, for every fixed $T < T^*$ where $T^* > 0$ is the horizon date, to have

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)\ ds + \int_0^t \sigma(s,T)\ dW_s, \tag{2.12}$$

for some Borel-measurable function $f(0,\cdot) : [0,T^*] \to \mathbb{R}$ and stochastic processess $\alpha(\cdot,T)$ and $\sigma(\cdot,T)$. It is worthwhile noticing that in the HJM setting, for any fixed maturity $T < T^*$, the initial condition $f(0,T)$ is determined by the current yield curve, which can be estimated using observed market prices of bonds and/or other relevant instruments; this is exactly why calibration in this setting becomes trivial.
As in Section (2.1), $P(t,T)$ denotes the price at time $t < T$ of a unit ZCB maturing at time $T < T^*$. By the definition of the forward rate, $P(t,T)$ can be recovered from the formula

$$P(t,T) = \exp\Big(-\int_t^T f(t,u)\ du\Big). \tag{2.13}$$

Theorem 2.2 (HJM Drift Condition Theorem)**.** *In the HJM forward rate modelling framework, the bond market is arbitrage free under the risk neutral measure* $\mathbb{Q}$ *if*

$$\alpha(t,T) = \sigma(t,T) \cdot \int_t^T \sigma(t,s)\ ds \qquad \forall\ 0 \le t \le T. \tag{2.14}$$

A more general form of this result is

$$\alpha(t,T) = \sum_{i=1}^{d} \Big(\sigma_i(t,T) \cdot \int_t^T \sigma_i(t,s)\ ds\Big) \qquad \forall\ 0 \le t \le T, \tag{2.15}$$

where d is the number of stochastic drivers.

Proof. The proof[2] of (2.14) begins with deriving the bond prices dynamics in the real measure $\mathbb{P}$.
It is easy to see that (2.13) can also be written in the following two forms

$$\log P(t,T) = -\int_t^T f(t,u)\ du, \tag{2.16}$$

[2]Proving (2.15) is relatively straightforward based on proof of (2.14)

$$-\int_0^t f(0,u)\ du = \log P(t,T) + \int_t^T f(0,u)\ du. \tag{2.17}$$

Substituting (2.12) into (2.16) gives

$$\begin{aligned}
\log P(t,T) &= -\int_t^T f(t,u)\ du \\
&= -\int_t^T \Big(f(0,u) + \int_0^t \alpha(s,u)\ ds + \int_0^t \sigma(s,u)\ dW_s \Big)\ du \\
&= -\int_t^T f(0,u)\ du - \int_t^T \int_0^t \alpha(s,u)\ ds\ du - \int_t^T \int_0^t \sigma(s,u)\ dW_s\ du.
\end{aligned}$$

Then substituting (2.17) into the above expression gives

$$\begin{aligned}
\log P(t,T) &= -\int_t^T f(0,u)\ du - \int_t^T \int_0^t \alpha(s,u)\ ds\ du - \int_t^T \int_0^t \sigma(s,u)\ dW_s\ du \\
&= \int_0^t f(0,u)\ du - \int_0^T \int_0^t \alpha(s,u)\ ds\ du + \int_0^t \int_0^{t\wedge u} \alpha(s,u)\ ds\ du \\
&\quad + \log P(0,T) - \int_0^T \int_0^t \sigma(s,u)\ dW_s\ du + \int_0^t \int_0^{t\wedge u} \sigma(s,u)\ dW_s\ du.
\end{aligned}$$

By Fubine's Theorem

$$\int_0^T \int_0^t \alpha(s,u)\ ds\ du = \int_0^t \int_s^T \alpha(s,u)\ du\ ds,$$

$$\int_0^T \int_0^t \sigma(s,u)\ dW_s\ du = \int_0^t \int_s^T \sigma(s,u)\ du\ dW_s.$$

Hence

$$\begin{aligned}
\log P(t,T) &= \int_0^t \underbrace{f(0,u)\ du + \int_0^t \int_0^{t\wedge u} \alpha(s,u)\ ds\ du + \int_0^t \int_0^{t\wedge u} \sigma(s,u)\ dW_s}_{\bigstar}\ du \\
&\quad + \log P(0,T) - \int_0^t \int_s^T \alpha(s,u)\ du\ ds - \int_0^t \int_s^T \sigma(s,u)\ du\ dW_s
\end{aligned}$$

Note that $\bigstar = f(u,u) = r_u$ therefore

$$\begin{aligned}
\log P(t,T) &= \log P(0,T) + \int_0^t r_u\ du - \int_0^t \int_s^T \alpha(s,u)\ du\ ds - \int_0^t \int_s^T \sigma(s,u)\ du\ dW_s \\
&:= \log P(0,T) + \int_0^t r_u\ du + \int_0^t A(s,T)\ ds + \int_0^t S(s,T)\ dW_s
\end{aligned}$$

Now write $P(t,T) = e^{\log P(t,T)} = e^{X_t}$ and applying It$\hat{o}$ formula gives

$$\begin{aligned}
dP(t,T) &= e^{X_t} dX + \frac{1}{2} e^{X_t} d < X >_t \\
&= P(t,T)\Big(r_t + A(t,T)dt + S(t,T)dW_t \Big) + \frac{1}{2} P(t,T) ||S(t,T)||^2 dt
\end{aligned}$$

Hence in the HJM setting, the bond prices dynamics follow

$$dP(t,T) = P(t,T)\Big(r_t + A(t,T) + \frac{1}{2}||S(t,T)||^2\Big)dt + P(t,T)S(t,T)dW_t, \quad (2.18)$$

where $A(t,T) = -\int_t^T \alpha(t,s)\ ds$ and $S(t,T) = -\int_t^T \sigma(t,s)\ ds$.
It is important to note that the truth of (2.18) is independent of the measure used, then under the risk neutral measure $\mathbb{Q}$, where the discounted bond prices are martingales, the bond prices have the short rate r as the drift. Namely, under $\mathbb{Q}$ (2.18) is reduced to

$$dP(t,T) = P(t,T)\Big(r(t)dt + S(t,T) \cdot d\tilde{W}_t\Big), \quad (2.19)$$

where $\tilde{W}_t$ is a $\mathbb{Q}$-martingale. Meanwhile, it is also true, under $\mathbb{Q}$, to have

$$A(t,T) + \frac{1}{2}\,||S(t,T)||^2 = 0.$$

Equivalently

$$-\int_t^T \alpha(t,s)\ ds + \frac{1}{2}\int_t^T \sigma(t,s)\ ds \cdot \int_t^T \sigma(t,s)\ ds = 0.$$

Differentiating w.r.t T gives the result

$$\alpha(t,T) = \sigma(t,T) \cdot \int_t^T \sigma(t,s)\ ds$$

□

It is, thus, obvious to see that in an arbitrage-free market the drift of the forward rate is completely determined by the volatility. This, however, causes some paths of forward rate, except some special cases where the coefficient σ follows a deterministic function, to explode if log-normality is embedded in forward rates [36]. That is to say the HJM framework can easily lead to non-Markovian forward rate models, which strongly limits its practical application in pricing interest rate derivatives.

2.3 Standard market model

The introduction of market models presented an extraordinarily fresh way of thinking, one that directly models the market interest rates. As a result, when the option price is given by Black's formula (see Section 2.3.3), the link between the SDE governing the evolution of the appropriate market interest rates and the terminal distributions of these rates is clear; this deduces an easy specification of market models such that they can exactly match market prices [22]. Describing LIBOR market model serves as a good example of explaining standard market models since that was what the first market models did.

2.3.1 LIBOR market model (LMM)

Let $T_0 < T_1 < \ldots < T_n$ be a sequence of fixed dates for $i = 1, \ldots, n$ and $\delta_i = T_i - T_{i-1}$, the corresponding forward LIBORs are defined as

$$L_i(t) = \frac{P(t, T_{i-1}) - P(t, T_i)}{\delta P(t, T_i)}. \tag{2.20}$$

To be able to use $P(\cdot, T_n)$ as a numeraire later, define

$$\tilde{P}_i(t) := \frac{P(t, T_i)}{P(t, T_n)} \quad \forall\ i = 0, 1, \ldots, n \tag{2.21}$$

and

$$\pi_i(t) := \prod_{j=1}^{i} (1 + \delta_j L_j(t)). \tag{2.22}$$

In (2.22), since the product over the empty set is unity so $\pi_0 = 1$. For convenience, without loss of generality, also define $\tilde{P}_{n+1} \equiv 1$ and $L_{n+1} \equiv 0$. Then from (2.20) and (2.21) it immediately follows that

$$\tilde{P}_i(t) = \Big(1 + \delta_{i+1} L_i(t)\Big) \tilde{P}_{i+1}(t). \tag{2.23}$$

Furthermore by using (2.22)

$$\tilde{P}_i(t) = \prod_{j=i+1}^{n} \big(1 + \delta_j L_j(t)\big) = \frac{\pi_n(t)}{\pi_i(t)}.$$

Let $\{W_t^i\}_{t \geq 0}$, $i = 1, \ldots, n$, be a set of n correlated Brownian motions, with $dW_t^i dW_t^j = \rho_{ij} dt$, under the forward measures $\mathbb{F}$ where, if choose $P(t, T_n)$ as a numeraire, all the tradable discounted by $P(t, T_n)$ are martingales, namely, all $\tilde{P}_i(t)$ are martingales. Then under $\mathbb{F}$,the forwrad LIBORs $L_i(t)$ must satisfy

$$dL_i(t) = \mu_i(t, L) L_i(t) dt + \sigma_i(t) L_i(t) dW_t^i. \tag{2.24}$$

Since $\tilde{P}_i(t)$ are martingales under $\mathbb{F}$, applying Itô to (2.23) gives

$$d\tilde{P}_i(t) = [1 + \delta_{i+1} L_{i+1}(t)] d\tilde{P}_{i+1}(t) + \delta_{i+1} \tilde{P}_{i+1}(t) dL_{i+1}(t) + \delta_{i+1} d\tilde{P}_{i+1}(t) dL_{i+1}(t) \tag{2.25}$$

Then substituting (2.24) into (2.25) yields

$$\begin{aligned}
d\tilde{P}_i(t) &= [1 + \delta_{i+1} L_{i+1}(t)] d\tilde{P}_{i+1}(t) + \delta_{i+1} \tilde{P}_{i+1}(t) \Big(\mu_{i+1}(t, L) L_{i+1}(t) dt + \sigma_{i+1}(t) L_{i+1}(t) dW_t^{i+1}\Big) \\
&\quad + \delta_{i+1} \sigma_i(t) L_{i+1}(t) dW_t^{i+1} d\tilde{P}_{i+1} \\
&= [1 + \delta_{i+1} L_{i+1}(t)] d\tilde{P}_{i+1}(t) + \delta_{i+1} \tilde{P}_{i+1}(t) \sigma_{i+1}(t) L_{i+1}(t) dW_t^{i+1} \\
&\quad + \delta_{i+1} \tilde{P}_{i+1}(t) \mu_{i+1}(t, L) L_{i+1}(t) dt + \delta_{i+1} \sigma_{i+1}(t) L_{i+1}(t) dW_t^{i+1} d\tilde{P}_{i+1}
\end{aligned}$$

Thus for $d\tilde{P}_i(t)$ to be a martingale, it requires,$\forall\ i = 0, \cdots, n-1$,

$$d\tilde{P}_i(t) = \big(1 + \delta_{i+1} L_i(t)\big) d\tilde{P}_{i+1}(t) + \delta_{i+1}\tilde{P}_{i+1}(t)\sigma_{i+1}(t)L_{i+1}(t)dW_t^{i+1} \tag{2.26}$$

and

$$\delta_{i+1}\tilde{P}_{i+1}(t)\mu_{i+1}(t,L)L_{i+1}(t)dt + \delta_{i+1}\sigma_{i+1}(t)L_{i+1}(t)dW_t^{i+1}d\tilde{P}_{i+1} = 0,$$

i.e.

$$\mu_{i+1}(t,L)\tilde{P}_{i+1}(t)dt = -\sigma_{i+1}(t)dW_t^{i+1}d\tilde{P}_{i+1}. \tag{2.27}$$

Now multiplying (2.26) by $\pi_i(t)$, by backward induction, gives

$$\begin{aligned}
\pi_i(t)d\tilde{P}_i(t) &= \pi_i(t)\big(1 + \delta_{i+1}L_i(t)\big)d\tilde{P}_{i+1}(t) + \pi_i(t)\delta_{i+1}\tilde{P}_{i+1}(t)\sigma_{i+1}(t)L_{i+1}(t)dW_t^{i+1} \\
&= \pi_{i+1}(t)d\tilde{P}_{i+1}(t) + \Big(\frac{\pi_{i+1}(t)}{1+\delta_{i+1}L_{i+1}(t)}\Big)\delta_{i+1}L_{i+1}(t)\tilde{P}_{i+1}(t)\sigma_{i+1}(t)dW_t^{i+1} \\
&= \sum_{j=i+1}^{n} \pi_j(t)\tilde{P}_j(t)\Big(\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\Big)\sigma_j(t)dW_t^j
\end{aligned}$$

thus

$$d\tilde{P}_i(t) = \tilde{P}_i(t)\sum_{j=i+1}^{n}\frac{\pi_j(t)\tilde{P}_j(t)}{\pi_i(t)\tilde{P}_i(t)}\Big(\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\Big)\sigma_j(t)dW_t^j,$$

i.e.

$$d\tilde{P}_i(t) = \tilde{P}_i(t)\sum_{j=i+1}^{n}\Big(\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\Big)\sigma_j(t)dW_t^j. \tag{2.28}$$

Now substituting (2.28) into (2.27), by backward induction again, yields

$$\mu_i(t,L)\tilde{P}_i(t)dt = -\sigma_i(t)dW_t^i\tilde{P}_i(t)\sum_{j=i+1}^{n}\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\sigma_j(t)dW_t^j.$$

Then the drift condition in LMM is

$$\mu_i(t,L) = -\sum_{j=i+1}^{n}\Big(\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\Big)\sigma_i(t)\sigma_j(t)\rho_{ij}.$$

Finally, the original SDE (2.24) becomes

$$dL_i(t) = \Big[-\sum_{j=i+1}^{n}\Big(\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\Big)\sigma_i(t)\sigma_j(t)\rho_{ij}\Big]L_i(t)dt + \sigma_i(t)L_i(t)dW_t^i. \tag{2.29}$$

The procedure for deriving the swap-rate market models (SMM) is identical to that for LIBOR market models except that the algebra is slightly more complicated. The result is stated below (for a detailed derivation see Chapter 18 of [22]).

For each $i = 1, \cdots, n$, the forward par swap rates y^i, in the forward measure $\mathbb{F}$, satisfy the SDE of the form

$$dy_t^i = -\Big(\sum_{j=i+1}^{n} \frac{\Gamma_t^{j-1}\tilde{P}_t^j}{\Gamma_t^{i-1}\tilde{P}_t^i}\big(\frac{\delta_{j-1}y_t^j}{1+\delta_{j-1}y_t^j}\big)\sigma_t^i\sigma_t^j\rho_{ij}\Big)y_t^i dt + \sigma_t^i y_t^i dW_t^i, \tag{2.30}$$

where as always $P(t,T)$ denotes ZCBs, $dW_t^i dW_i^t = \rho_{ij}dt$
and

$$\tilde{P}_t^i := \sum_{j=i}^{n} \delta_j \frac{P(t,T_j)}{P(t,T_n)},$$

and, for $1 \le i \le n$

$$y_t^i := \frac{P(t,T_{i-1}) - P(t,T_n)}{\sum_{j=i}^{n}\delta_j P(t,T_j)},$$

$$\Gamma_t^i := \prod_{j=1}^{i}(1+\delta_j y_t^{j+1}).$$

Also, $\tilde{P}^{n+1} \equiv y^{n+1} :\equiv 0$ and $\Gamma^0 \equiv 1$.

2.3.2 Existence of arbitrage-free strong Markov market model

To show that the market model is strong Markov and consistent with a full arbitrage-free term structure model, a few general SDE theories are stated below. These results are so classical that almost any Stochastic Calculus text contains their proof, see, for example, [22] [28]

Definition 2.3. *On a filtered probability space $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P})$, $\mathbb{R}^n$ is adapted to $\{\mathcal{F}_t\}$. X is a strong Markov process if, given any almost surely (a.s.) finite $\{\mathcal{F}_t\}$ stopping time τ, any $\Gamma \in \mathcal{B}(\mathbb{R}^n)$, and any $t \ge 0$,*

$$\mathbb{P}(X_{\tau+t} \in \Gamma | \mathcal{F}_\tau) = \mathbb{P}(X_{\tau+t} \in \Gamma | X_\tau) \quad a.s. \tag{2.31}$$

An equivalent formulation to (2.31) is the following standard result which is appealing when verifying the strong Markov property.

Theorem 2.4. *The process X is strong Markov if and only if, for a.s. finite $\{\mathcal{F}_t\}$ stopping times τ and all $t > 0$,*

$$\mathbb{E}[f(X_{\tau+t})|\mathcal{F}_\tau] = \mathbb{E}[f(X_{\tau+t})|X_\tau] \tag{2.32}$$

for all bounded continuous functions f.

In fact, the strong Markov property of the solution process for a locally Lipschitz SDE (σ, b) is inherited from the strong Markov property of the driving Brownian motion. The following theorem confirms this connection.

Theorem 2.5. *If the SDE (σ, b) is locally Lipschitz and let $(\Omega, \{\mathcal{F}_t\}, \mathcal{F}, \mathbb{P}, W, X)$ be some solution. Then the solution process X is strong Markov, i.e. (2.32) holds, for all bounded continuous functions f and all a.s. finite $\mathcal{F}_t$ stopping time τ.*

The following theorem shows that there are indeed processes L and y satisfying the SDE (2.29) and (2.30) and hence it is a necessary condition for the model to be arbitrage-free.

Theorem 2.6. *Suppose that, for $i = 1, \cdots, n$, the functions $\sigma_i : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ are bounded on any time interval $[0, t]$. Then strong existence and pathwise uniqueness hold for the SDE (2.29) and (2.30). Furthermore, the solution processes L and y are strong Markov processes.*

The sufficient condition is easily verified by noting that from (2.28) $\tilde{P}_i$ can be written as a Dolean exponential,

$$\tilde{P}_i(t) = \tilde{P}_i(0) \exp\Big(\int_0^t \sum_{j=i+1}^{n} \big(\frac{\delta_j L_j(s)}{1 + \delta_j L_j(s)}\big)\sigma_j(s) dW_s^j\Big)$$

and similarly for SMM

$$\tilde{P}_t^i = \tilde{P}_0^i \exp\Big(\int_0^t \sum_{j=i+1}^{n} \frac{\Gamma_s^{j-1}\tilde{P}_s^j}{\Gamma_s^{i-1}\tilde{P}_s^i}\big(\frac{\delta_{j-1} y_s^j}{1 + \delta_{j-1} y_s^j}\big)\sigma_s^j dW_s^j\Big)$$

and observe that the exponential term has bounded quadratic variation over any time interval $[0, t]$, hence, by Novikov's condition, $\tilde{P}_i(t)$ and $\tilde{P}_t^i$ are indeed true martingales.

2.3.3 Change of Numeraire

Changing numeraire is a very important technique in mathematical finance and it, most often, can dramatically simplify the calculation especially when pricing complex products. This section gives a rather brief examination of changing numeraire which will be used frequently throughout the rest of the paper (for details of change of numeraire, see Chapter 9 in [36]).

A **numerarie** is the unit of account in which other assets are denominated [36]. In principle, any positively priced asset can be taken as a numeriare and hence all other assets are denominated by the chosen numeraire. In a fixed income market, a convenient choice of numeraire is a ZCB maturing at time T and the associated risk neutral measure is often called the T-forward measure.

Theorem 2.7 (Change of numeraire). *Let $N(t)$ be a numeraire and Q^N be the associated measure equivalent to the real world measure $\mathbb{P}$ such that the asset prices $\frac{S_t}{N_t}$ are Q^N martingales. Then for an arbitrary numeraire U, there exists an equivalent measure Q^U such that any contingent claim X_T has price*

$$V(t, S_t) = U_t \mathbb{E}^{Q^U} \big[\frac{X_T}{U_T} \,|\mathcal{F}_t \big]$$

and moreover

$$\frac{dQ^U}{dQ^N}|_{\mathcal{F}_t} = \frac{U_T N_0}{N_T U_0},$$

and $(S/U)_t$ are martingales under Q^U.

Theorem 2.8 (Change of risk neutral measure). *Let $M(t)$ and $N(t)$ be the prices of two assets denominated in a common currency and let $\sigma(t) = (\sigma_1(t), \cdots, \sigma_d(t))$ and $\nu(t) = (\nu_1(t), \cdots, \nu_d(t))$ denote their respective volatility vector process:*

$$d(D(t)M(t)) = D(t)M(t)\sigma(t) \cdot dW(t),$$

$$d(D(t)N(t)) = D(t)N(t)\nu(t) \cdot dW(t),$$

where $D(t) := \exp(-\int_0^t r(s)\ ds)$ is called the discount process.
If taking $N(t)$ as the numeraire then

$$dS^N(t) = S^N(t)[\sigma(t) - \nu(t)] \cdot dW^N(t).$$

2.3.4 Valuation in the standard market model

Based on the results in the last two sections, the common market practice of pricing vanilla style (path independent) products is to assume $\forall\ t \in [0, T_i], \delta = T_i - T_{i-1}$

$$dL_i(t, T_i) = L_i(t, T_i)\sigma_i(t) d\tilde{W}^i,$$

$$dy_n^k(t) = y_n^k(t)\sigma_{n,k}(t) d\tilde{W}_n^k,$$

where $\tilde{W}$ is a 1-dimensional $\mathbb{Q}^i$ Brownian motion, and $\sigma_i(t), \sigma_{n,k}(t)$ is some deterministic function. Then a caplet, one leg of a cap, at $t \in [0, T_i]$, with strike K, is priced by

Proposition 2.9 (Black's formula[3]).

$$Capl_i(t) = \delta P(t, T_i)\Big(L_i(t, T_i)N(d_1) - KN(d_2)\Big), \tag{2.33}$$

[3]proof, based on changing numeraire, is standard, see, for example,[36]

where

$$d_{1,2} = \frac{\log \frac{L_i(t,T_i)}{K} \pm \frac{1}{2}\Sigma_i^2(t,T_i)}{\Sigma_i(t,T_i)}$$

and

$$\Sigma_i^2(t,T_i) = \int_t^{T_i} \sigma_i^2(s)\ ds$$

with N being the standard normal cumulative distribution function

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-z^2/2}\ dz, \quad \forall\ x \in \mathbb{R}.$$

Whence a cap settled in arreas at times $T_i, i = 0, \cdots, n$ where $T_i - T_{i-1} = \delta_i, T_0 = T$ is priced, by definition,

$$Cap(t) = \sum_{i=1}^{n} Capl_i(t) = \sum_{i=1}^{n} \delta_i P(t,T_i)\Big(L_i(t,T_{i-1})N(d_3) - KN(d_4)\Big), \tag{2.34}$$

where for every $i = 0, \cdots, n-1$

$$d_{3,4} = \frac{\log \frac{L_i(t,T_{i-1})}{K} \pm \frac{1}{2}\Sigma_i^2(t,T_{i-1})}{\Sigma_i(t,T_{i-1})}$$

and

$$\Sigma_i^2(t,T_{i-1}) = \int_t^{T_{i-1}} \sigma_i^2(s)\ ds.$$

Then by cap-floor parity, which is an immediate consequence of the no-arbitrage property,

$$Cap(t) - Floor(t) = \sum_{i=1}^{n} \big(P(t,T_{i-1}) - (1 + k\delta_i)P(t,T_i)\big),$$

the price of the floor is easily calculated.
In an almost identical fashion, Black's formula for a payer's swaption V for the period between $[k,n]$, struck at K with swap rates $y_n^k(t)$, is

$$V_n^k(t) = S_n^k(t)(y_n^k(t)N(d_+) - KN(d_-)),$$

where

$$d_\pm = \frac{\log \frac{y_n^k(t)}{K} \pm \frac{1}{2}\Sigma_{k,n}^2}{\Sigma_{k,n}}$$

and

$$\Sigma_{k,n}^2 = \int_t^{T_k} \sigma_{k,n}^2(s)\ ds.$$

Obviously, pricing non-callable fixed income products by Black's formula is just like using the Black-Scholes formula for pricing vanilla products in the equity market. In both cases, a rather simple structure of volatility of the underlying variable is a major assumption, without which the valuation has to switch to numerical methods. While generally this switch works relatively well in the equity market for most common exotic products, it does not work so well in the fixed income market. This is because the full market model (i.e. SDE (2.29) or (2.30)) has to be used if the exotic product depends on multiple market interest rates (LIBOR or swap rate) in a non-linear way, which then leads to a very high dimensional problem. For example, pricing a 10 year Bermudan swaption exercisable quarterly involves solving a 39-dimensional problem. Thereby, Monte Carlo simulation, the only feasible numerical method left in this high dimensional case, could be inefficient without specific numerical technique, especially when pricing and hedging strong path dependent products.

This practical drawback of the standard market model has generated considerable research interest and so far it has been solved relatively well. In Section 5.2.2, it will show how to use the Longstaff-Schwartz algorithm to make the standard market model implementation practically possible. By no means is the Longstaff-Schwartz algorithm the only way of doing this, it just, to a certain degree, tends to be more popular than other methods proposed in [3], [2] and [8].

Chapter 3

Markov functional market model

Generally speaking, a 'good' pricing model for derivatives should, at least from a practical perspective, have the following properties:

1. arbitrage-free;
2. well-calibrated, accurately pricing as many relevant liquid instruments as possible without overfitting;
3. be realistic and transparent in its properties;
4. allows an efficient implementation [20].

As can be seen from Chapter 2, short rate modelling, the forward rate modelling in the HJM framework and standard market model have not been able to meet all these four criterion. Motivated by this observation, a general class of Markov-functional interest rate models has been introduced and received growing attention particularly from practitioners. It is because the Markov-Functional Market Model complements short rate models and standard market models in a way that it allows an efficient implementation and permits accurate calibration of the model through more freedom in choosing the functional form. In addition, the remaining freedom to specify the law of the driving Markov process enables the model to be realistic. The vital assumption in the Markov-Functional Market Model is that the uncertainity can be captured by some low dimensional (time-inhomogeneous) Markov process $\{m_t : 0 \leq t \leq \alpha^*\}$, in that, for any t, the state of the economy at t is summarised via m_t and clearly this is the defining feature of any practically implementable model [20]. α^* is some time on which the value of the derivative, V_{α^*}, will have been determined from the evolution of the asset prices hence only prior evolution of the economy up to α^* need be considered.

3.1 Definition

Let $(N, \mathbf{M})$ be a numeraire pair for the economy $\mathscr{E}$ where the numeraire N, itself a price process, is of the form

$$N_t = N_t(m_t) \quad 0 \leq t \leq \alpha^*$$

and the measure $\mathbf{M}$, often called the martingale measure, is equivalent to the real world measure $\mathbb{P}$ and such that (P_{tT}/N_t) is martingale. Assume that the process m is a Markov process under the measure $\mathbf{M}$ and that ZCBs are of the form

$$P_{t,S} = P_{t,S}(m_t), \quad 0 \leq t \leq \alpha_S \leq S$$

for some boundary curve $\alpha_S : [0, \alpha^*] \to [0, \alpha^*]$.

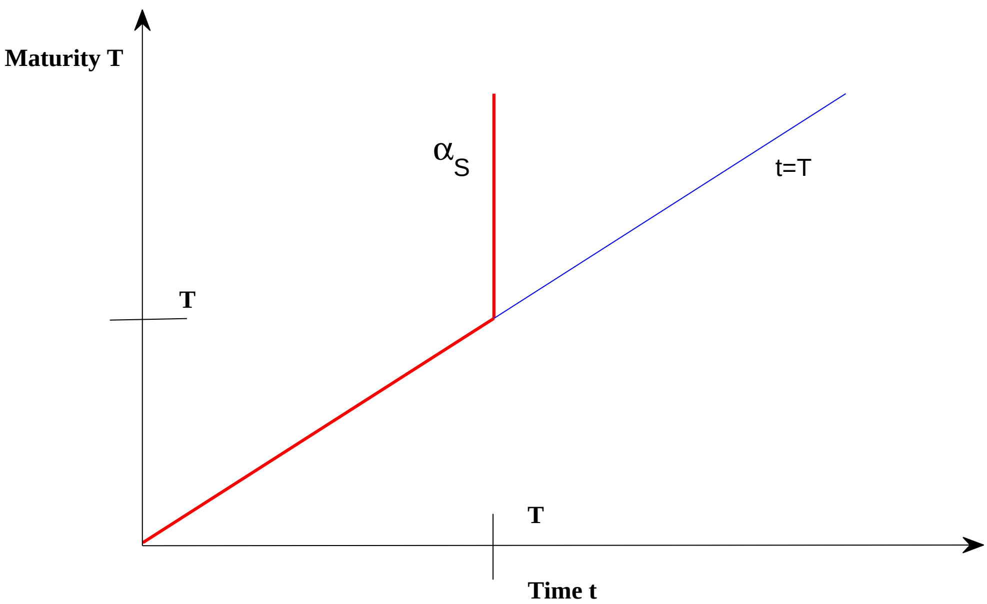

Figure 3.1: Boundary curve

For almost all practical applications, the boundary curve (see Figure 3.1) is appropriately chosen to be of the form

$$\alpha_S = \begin{cases} S, & \text{if } S \leq T, \\ T, & \text{if } S > T, \end{cases} \tag{3.1}$$

for some constant T so that the model need not be defined over the whole time domain $0 \leq t \leq S < \infty$ [20].

Then by the fundamental asset pricing formula the value of a derivative, with payoff V_T at T, at any time t prior to α^* is given by

$$V_t = N_t \mathbf{E}[N_T^{-1} V_T | \mathscr{F}_t] \tag{3.2}$$

for any $t \leq T \leq \alpha^*$
Under these assumptions, it is sufficient to completely specify the Markov-Functional Market Model with the knowledge of

- the law of the process m under $\mathbf{M}$,
- $P_{\alpha_S S}(m_{\alpha_S})$, for $S \in [0, \alpha^*]$, the functional form of the discount factors on the boundary α_S
- the functional form of the numeraire $N_t(m_t)$ for $0 \leq t \leq \alpha^*$.

That is to say it is not necessary to explicitly specify the functional form of discount factors on the interior of the region bounded by α_S. Thus, via the martingale property for numeraire-rebased assets under $\mathbf{M}$, discount factors on the interior of the region bounded by α_S can be recovered by

$$P_{tS}(m_t) = N_t(m_t) \mathbf{E}_{\mathbf{M}} \Big[\frac{P_{\alpha_S S}(m_{\alpha_S})}{N_{\alpha_S}(m_{\alpha_S})} \Big| \mathscr{F}_t \Big]. \tag{3.3}$$

3.2 Implying the functional form of the numeraire

Defining the payment dates for the swap associated with the rate y^i by $S_j^i, j = 1, 2, \cdots, m_i$; though not strictly necessary, for convenience it is assumed that, for all i, j, either $S_j^i > T_n$ or $S_j^i = T_k$, for some $k > i$. This assumption generally holds for many common practical products and in the case where it does not hold one can always introduce auxiliary swap rates y^k to make it hold. To construct a one-dimensional Markov-Functional Market Model which correctly prices options on the swaps associated with these forward rates, we need also to assume that the ith forward rate at T_i, $y_{T_i}^i$, is a monotonic increasing function of the variable m_{T_i}. To simplify calculation, PVBP-digital swaptions, which have a simple payoff structure, are used because calibrating the model to vanilla swaptions is equivalent to calibrating it to the inferred market prices of digital swaptions [12]. The PVBP-digital swaption corresponding to y^i, with strike K, has payoff at T_i of

$$\tilde{V}_{T_i}^i(K) = B_{T_i}^i \mathbb{I}_{\{y_{T_i}^i > K\}}$$

where

$$B_t^i := \sum_{j=i}^{n} \delta_j P_{tS_j}$$

is called the **present value of a basis point** (PVBP) of the swap corresponding to the swap rate y^i and it represents the value of fixed leg of the swap if the fixed leg were unity [22]. Applying (3.2), its value at time zero is given by

$$\tilde{V}_0^i(K) = N_0(m_0)\mathbf{E_M}\left[\hat{B}_{T_i}^i(m_{T_i})\mathbb{I}_{\{y_{T_i}^i(m_{T_i})>K\}}\right], \tag{3.4}$$

where

$$\hat{B}_{T_i}^i(m_{T_i}) = \frac{B_{T_i}^i(m_{T_i})}{N_{T_i}(m_{T_i})}.$$

Then to determine the functional form of $N_{T_i}(m_{T_i})$, it involves working backward iteratively from the terminal time T_n. It is natural to assume that $N_{T_k}(m_{T_k})$, $k = i+1, \cdots, n$, have been already determined and to assume

$$\hat{P}_{T_i S}(m_{T_i}) = \frac{P_{T_i S}(m_{T_i})}{N_{T_i}(m_{T_i})}$$

for relevant $S > T_i$, is known, having been determined by (3.3) and known (conditional) distributions of m_{T_k}, $k = i, \cdots, n$. This then implies that $\hat{B}_{T_i}^i$ is also known. Now consider $y_{T_i}^i$ which can be written as

$$y_{T_i}^i = \frac{N_{T_i}^{-1} - P_{T_i S_{n_i}^i} N_{T_i}^{-1}}{P_{T_i}^i N_{T_i}^{-1}}. \tag{3.5}$$

Simplifying (3.5) algebraically gives

$$N_{T_i}(m_{T_i}) = \frac{1}{\hat{B}_{T_i}^i(m_{T_i}) y_{T_i}^i(m_{T_i}) + \hat{P}_{T_i S_{n_i}^i}(m_{T_i})}. \tag{3.6}$$

Hence, finding the functional form $y_{T_i}^i(m_{T_i})$ will be sufficient to determine $N_{T_i}(m_{T_i})$. Since $y_{T_i}^i$ is assumed to have monotonicity with respect to m_{T_i}, there exists a unique value of K, say $K^i(m^*)$, such that the following holds

$$\{m_{T_i} > m^*\} = \{y_{T_i}^i > K^i(m^*)\}. \tag{3.7}$$

Now define

$$J_0^i(m^*) = N_0(m_0)\mathbf{E_M}\big[\hat{B}_{T_i}^i(m_{T_i})\mathbb{I}_{\{m_{T_i}>m^*\}}\big]. \tag{3.8}$$

Then, for any given m^*, the value of $J_0^i(m^*)$ can be calculated using the known distribution of m_{T_i} under $\mathbf{M}$. Moreover, the value of K can be found using market prices such that

$$J_0^i(m^*) = \tilde{V}_0^i(K). \tag{3.9}$$

It is not hard to see that the value of K satisfying (3.9) is precisely $K^i(m^*)$ by comparing (3.4) and (3.8). Finally, the functional form of $y^i_{T_i}(m_{T_i})$ can be obtained by noticing that it is equivalent to knowing $K^i(m^*)$ for any m^* from (3.7).
Standard market practice is to use Black's formula (**Proposition 2.9**) to find swaption prices $V_0^i(K)$. In fact, the techniques above can be applied more generally, especially for currencies with a large volatility skew, meaning volatility is highly dependent on the strike K, these techniques are still applicable. This is one of the major strengths of the Markov functional market model, working well for currencies such as yen in which it is not suitable to model rates through a log-normal process.

3.3 Swap Markov functional model

This section takes the swap Markov functional model, suitable for pricing swap based products, as an example to show generally how to construct the Markov-Functional Market Model. To keep the notation simple, a special case of a cancellable swap is considered for which the ith forward swap rate y^i starts on date T_1 and has coupons precisely at dates $S_1, \cdots, S_n$ with exercise times at $T_1, \cdots, T_n$. As before, denote by δ_i the accrual factor for the period $[T_i, S_i]$. Then it follows that

$$y_t^i = \frac{P_{tT_i} - P_{tS_n}}{B_t^i},$$

where B_t^i is, as before, the present value of a basis point (PVBP) of the swap. It is worthwhile to note that in this case the last par swap rate y^n is just the forward LIBOR, L^n, for the period $[T_n, S_n]$. To be consistent with Black's formula, assume that y^n is a log-normal martingale under the swaption measure $\mathbb{S}^n$, i.e.

$$dy_t^n = \sigma_t^n y_t^n dW_t, \tag{3.10}$$

where W is a standard Brownian motion under $\mathbb{S}^n$ and σ^n is some deterministic function. From (3.10), it is equivalent to have

$$y_t^n = y_0^n \Big(-\frac{1}{2} \int_0^t (\sigma_u^n)^2 \, du + m_t \Big),$$

where m, a deterministic time-change of a Brownian motion, satisfies

$$dm_t = \sigma_t^n dW_t. \tag{3.11}$$

That is to say m is taken as the driving Markov process of the model, which is the first stage to completely specify the model. As previously indicated, the boundary curve α_S, for this case, is exactly of the form in (3.1) and we only need the functional form of $P_{T_iT_i}(m_{T_i})$ for $i = 1, 2, \cdots, n$, namely the unit map, and $P_{T_nS_n}(m_{T_n})$ on the boundary. In this case, by definition, it follows that

$$P_{T_nS_n}(m_{T_n}) = \frac{1}{1 + \delta_n y_{T_n}^n},$$

and this immediately yields

$$P_{T_nS_n} = \frac{1}{1 + \delta_n y_0^n \Big(-\frac{1}{2} \int_0^t (\sigma_u^n)^2 \ du + m_t \Big)},$$

which then completes the second stage of specifying the swap Markov functional market model. To find the functional form of the numeraire $P_{\cdot S_n}$ at times $T_i, i = 1, \cdots, n-1$, we need only follow the procedures in Section 3.2. For this new model, the value of a PVBP-digital swaption with strike K and corresponding to y^i is given by

$$\tilde{V}_0^i(K) = P_{0S_n}(m_0)\mathbb{E}_{\mathbb{S}^n}\Big[\frac{B_{T_i}^i(m_{T_i})}{P_{T_iS_n(m_{T_i})}}\mathbb{I}_{\{y_{T_i}^i(m_{T_i})>K\}}\Big].$$

Assuming the market price obtained from the Black's formua yields

$$\tilde{V}_0^i(K) = B_0^i(m_0)N(d_2), \tag{3.12}$$

where

$$d_2 = \frac{\log(y_0^i/K)}{\hat{\sigma}^i\sqrt{T_i}} - \frac{1}{2}\hat{\sigma}^i\sqrt{T_i}.$$

Proceeding as in Section 3.2, let $m^* \in \mathbb{R}$ and for $i < n$, evaluate by numerical integration

$$\begin{aligned}
J_0^i(m^*) &= P_{0S_n}(x_0)\mathbb{E}_{\mathbb{S}^n}\Big[\frac{B_{T_i}^i(m_{T_i})}{P_{T_iS_n}(m_{T_i})}\mathbb{I}_{\{m_{T_i}>m^*\}}\Big] \\
&= P_{0S_n}(x_0)\mathbb{E}_{\mathbb{S}^n}\Big[\mathbb{E}_{\mathbb{S}^n}\big[\frac{B_{T_{i+1}}^i(m_{T_{i+1}})}{P_{T_{i+1}S_n}(m_{T_{i+1}})}\Big|\mathbf{F}_{T_i}\big]\mathbb{I}_{\{m_{T_i}>m^*\}}\Big] \\
&= P_{0S_n}(x_0)\int_{m^*}^{\infty}\Big[\int_{-\infty}^{\infty}\frac{B_{T_{i+1}}^i(u)}{P_{T_{i+1}S_n}(u)}\phi_{m_{T_{i+1}}|m_{T_i}}(u)\ du\Big]\phi_{m_{T_i}}(v)\ dv
\end{aligned}$$

where $\phi_{m_{T_i}}$ denotes the transition density function of m_{T_i} and according to (3.11), $\phi_{m_{T_{i+1}}|m_{T_i}}$ denotes the normal conditional density function of $m_{T_{i+1}}$ given m_{T_i} with mean m_{T_i} and variance $\int_{T_i}^{T_{i+1}} (\sigma_u^n)^2 \, du$.
Then

$$y_{T_i}^i(m^*) = K^i(m^*),$$

where $K^i(m^*)$ solves

$$J_0^i(m^*) = \tilde{V}_0^i(K^i(m^*)). \tag{3.13}$$

Whence, having found $J_0^i(m^*)$ numerically, $K^i(m^*)$ can be recovered from (3.12)

$$y_{T_i}^i(m^*) = y_0^i exp\Big[-\frac{1}{2}(\tilde{\sigma}^i)^2 T_i - \tilde{\sigma}^i \sqrt{T_i} N^{-1}\Big(\frac{J_0^i(m^*)}{B_0^i(m_0)}\Big)\Big].$$

Finally, the value of $P_{T_i S_n}(m^*)$ can now be calculated by using (3.6).
Here, the focus is on the case of one-dimensional Markov process m_t, which is sufficient for most important interest rate derivatives. The generalisation to the multi-dimension case is not difficult and necessary for some particular products, for example, Bermudan callable spread option; however working in the multi-dimensional case is still at a relative early stage and some details can be found in [23] [22].
Since the Markov functional market model has successfully transformed the high dimensional standard market model into a low dimensional (1-dim here) problem, the numerical methods do not have to rely on Monte Carlo simulation only. Numerical results of, employing one-dimensional Markov functional market model, pricing Bermudan swaptions will be presented in Section 5.3.5.

Chapter 4

Longstaff-Schwartz algorithm

The Longstaff-Schwartz algorithm is, so far, arguably the most widely adopted method for pricing multi-dimensional American-style financial instruments in both equity and fixed income market. This is mainly because on one hand it can be applied to a large number of common exotic derivatives; on the other hand it has been demonstrated that it is rather effective in numerical implementation (see, for instance, [25] [32]). It is the use of least squares to estimate the conditional expected payoff to the option-holders from continuation that makes the approach a worthwhile substitute for traditional finite difference methods when pricing high-dimensional products. Discussion here focuses on describing the general valuation framework using the Longstaff-Schwartz algorithm but the argument is equally well applicable to any specific product with some minor modification.

4.1 Notation

As before, the framework is based on an underlying complete probability space $(\Omega, \mathcal{F}, P)$ with finite time horizon $[0, T]$. To be consistent with the no arbitrage paradigm, it is assumed that there exists an equivalent martingale measure $\mathbb{Q}$ for the economy; also define $F = \{\mathcal{F}_t : t \in [0, t]\}$ to be the augmented filtration generated by the the relevant price processes for the securities and assume $\mathcal{F} = \mathcal{F}_T$. Let K be the strike price with discrete exercisable times $0 < t_1 \leq t_2 \leq \cdots \leq t_k = T$; in case of continuously exercisable products the method can also be used by taking sufficiently large K. In addition, let $I(w, s; t, T)$ denote the path of cash flows generated by the security, conditional on the product not being exercised at or prior to time t and on the holder of the security pursuing the optimal stopping strategy for all s, $t < s \leq T$.

4.2 Valuation algorithm

At each exercisable time t_k, investors are able to know the cash flow from immediate exercise and the value of immediate exercise simply equals this cash flow. Of course, the continuation cash flows are not known at t_k, however, the fundamental asset pricing formula implies that the value of continuation can be obtained by taking the expectation, in the risk neutral measure $\mathbb{Q}$, of the remaining discounted cash flows $I(w, s; t_k, T)$. More specifically, the value of continuation $C(w; t_k)$ at time t_k is simply

$$C(w;t_k) = \mathbb{E}_{\mathbb{Q}}\Big[\sum_{j=k+1}^{K} exp\big(-\int_{t_k}^{t_j} r(w,s)\ ds\big) I(w,t_j;t_k,T)\,|\mathcal{F}_{t_k}\Big], \tag{4.1}$$

where $r(w,t)$ is the riskless interest rate, possibly in a stochastic form. Hence, the problem of optimal exercise is reduced to comparing the immediate exercise value $I(w, s; t, T)$ and the continuation value $C(w; t_k)$ in the sense that exercise occurs as soon as $I \geq C > 0$. As mentioned earlier, in the Longstaff-Schwartz algorithm least squares are used, working backwards, to approximate $C(w; t_k)$ at $t_{K-1}, t_{K-2}, \cdots, t_1$. To be more specific, it is assumed[1] that the unknown functional form of $C(w; t_{K-1})$ in (4.1) can be expressed as a linear combination of a countable set of $\mathcal{F}_{t_{K-1}}$ measurable basis functions.

4.3 A numerical example

To quickly show an example of how the Longstaff-Schwartz algorithm works, results of pricing an American put option using the Longstaff-Schwartz algorithm are compared with that of using an implicit finite difference technique, a popular method of great accuracy in pricing low-dimensional path dependent products.

In finite difference, 60,000 time steps and 1000 stock price steps are used to discretize the Black-Scholes PDE. The L-S simulation is based on 10,000 paths and 50 exercise points. As shown in the table, the difference between the two methods is quite small and it is believed that the results will be even closer if more simulation paths are used. It is worthwhile noting that the differences in early exercise value could be either positive or negative, which indicates that Longstaff-Schwartz algorithm is capable of replacing the finite-difference to price path-dependent products. This is probably why L-S algorithm is being used intensively in practice when pricing high-dimensional path-dependent derivatives. In Chapter 5, it will become clearer

[1] This assumption can be formally justified, for details, see the original work [29]

that the Longstaff-Schwartz algorithm is powerful yet simple enough to price multi-dimensional path-dependence interest rate products such as Bermudan swaption.

S	σ	**T**	**FD American**	**LS American**	**Analytical European**	**Difference**
16	0.25	1	4.153	4.069	3.653	0.084
16	0.25	2	4.294	4.258	3.583	0.037
16	0.45	1	5.035	5.080	4.853	-0.045
16	0.45	2	5.593	5.801	5.381	-0.208
18	0.25	1	2.652	2.610	2.399	0.041
18	0.25	2	2.975	2.970	2.581	0.005
18	0.45	1	3.890	3.997	3.832	-0.107
18	0.45	2	4.575	4.876	4.555	-0.301
20	0.25	1	1.610	1.596	1.492	0.013
20	0.25	2	2.031	2.055	1.826	-0.024
20	0.45	1	3.175	3.114	3.007	0.061
20	0.45	2	3.973	4.125	3.861	-0.152
22	0.25	1	0.933	0.925	0.886	0.007
22	0.25	2	1.367	1.403	1.274	-0.036
22	0.45	1	2.439	2.408	2.349	0.031
22	0.45	2	3.339	3.468	3.279	-0.129

Table 4.1: Comparison of Finite Difference and Longstaff-Schwartz algorithm

As always, S denotes the spot price, T denotes the maturity and σ denotes the volatility. Other parameters in this comparison are interest rate $r = 0.05$, strike price $K = 20$. The 'Difference' column refers to the difference in early exercise value between two methods and early exercise value is the difference between American option value and analytical European option value. The benefit of employing the Longstaff-Schwartz algorithm here may not be so obvious, indeed, the major strength of Longstaff-Schwartz algorithm is to price multi-dimensional path dependent products; a detailed example of this case is in Section 5.2.2.

Chapter 5

Model implementation and numerical result

Having focused on the theoretical development of the standard Market Model and the Markov Functional Market Model, we are ready to carry out model implementations and present some numerical results, based on pricing an important fixed income derivative Bermudan swaption. The aim is to show that with the help of the Longstaff-Schwartz algorithm, implementing the standard Market Model (LMM/BGM) to price Bermudan swaption, a high-dimensional problem, is indeed possible. Meanwhile, the Markov functional market model, as will be seen, reaches a strong agreement with the BGM model on valuation results.

5.1 Bermudan swaption

A financial instrument is called Bermudan if it has multiple exercise dates, namely, there are times T_i at which the holder of a Bermudan may choose between different payments or underlying products. A Bermudan swaption is a swaption that has a maturity date equal to the last reset date of the underlying swap and that has an initial lockout period in which exercise is prohibited. Effectively speaking, a Bermudan swaption is equivalent to a Bermudan option on a coupon bond with strike equal to the par value of the bond and, as an option on a coupon bond, a Bermudan swaption clearly has positive probability of early exercise.
Let $0 = T_0 < T_1 < \cdots < T_n = T$ denote a given tenor structure and $V(T_1, \cdots, T_n; T_1)$ denote the price of a Bermudan swaption initiated at T_1. Then by definition

$$V(T_i, \cdots, T_n; T_i) := \max\left(V(T_i, \cdots, T_n; T_i), \hat{V}(T_i, \cdots, T_n; T_i)\right) \quad i = 0, \cdots, n$$

where $\hat{V}(T_i, \cdots, T_n; T_i)$ denotes the value of a swap with fixing dates $T_i, \cdots, T_{n-1}$ and payment dates $T_{i+1}, \cdots, T_n$, observed at T_i ; and $V(T_n; T_n) := 0$. Moreover, with a given numeraire N and a corresponding equivalent martingale measure $\mathbb{Q}^N$

$$V(T_{i+1}, \cdots, T_n; T_i) = N(T_i)\mathbb{E}^{\mathbb{Q}^N}\Big(\frac{V(T_{i+1}, \cdots, T_n; T_{i+1})}{N(T_{i+1})}|\mathcal{F}_{T_i}\Big).$$

5.2 Implementation of LMM

This section is to show, step by step, how to price a Bermudan swaption in LMM using Monte Carlo simulation with the application of the Longstaff-Schwartz algorithm. The volatility structure can simply be flat but more complex volatility term structure can be obtained from principal component analysis (PCA) of correlation matrix and adjusting to calibrated volatilities (see [34] on this topic).

5.2.1 Simulating the LIBOR rate

Recall the SDE (2.29) that LIBOR rate follows under forward measure $\mathbb{F}$

$$dL_i(t) = \Big[-\sum_{j=i+1}^{n}\Big(\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\Big)\sigma_i(t)\sigma_j(t)\rho_{ij}\Big]L_i(t)dt + \sigma_i(t)L_i(t)dW_t^i.$$

Since Bermudan swaptions are path-dependent and SDE (2.29) cannot be integrated exactly, the Euler-Maruyama method (Euler scheme) needs to be applied here to simulate the LIBOR rate path [15]. For a better discretization, it is necessary to apply the Euler scheme to $\log L(t)$; applying It$\hat{o}$'s lemma to the above SDE (2.29) gives

$$d\log L_i(t) = \Big[-\sum_{j=i+1}^{n}\Big(\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\Big)\sigma_i(t)\sigma_j(t)\rho_{ij} - \frac{\sigma_i^2}{2}\Big]dt + \sigma_i(t)dW_t^i$$

which is then suitable to be discretized, using the Euler scheme, as

$$L_{i+1}(t) = L_i(t)\exp\Big[-\sum_{j=i+1}^{n}\Big(\frac{\delta_j L_j(t)}{1+\delta_j L_j(t)}\sigma_j(t)\rho_{ij}\Big)\sigma_i(t)h - \frac{\sigma_i^2}{2} + \sigma_i(t)\sqrt{h}Z_t^i\Big]$$

where, following the same notation as in Section 2.3, $i = 0, 1, \cdots, n$, and $Z^1, Z^2, \cdots Z^n$ are independent n-dimensional standard normal random vectors; h is fixed time step [14].

5.2.2 Longstaff Schwartz algorithm

When applying the Longstaff-Schwartz algorithm in the process of pricing a Bermudan swaption, the procedures are divided into the following steps.

1. Simulating a large number of paths (D) of the underlying LIBOR rates so that values of regression coefficients are smooth. To control the discretization bias and to approximate continuous exercise, the number of simulation steps (N) is chosen equal to the number of exercise dates. Let $disc(t)$ be the discrete discounting factor at time t. Consider a payer Bermudan swaption $V_{s,n}(T_i)$ with lockout date T_s exercise dates T_i, $i = s, \cdots, n-1, T_n = T$ and $\delta = T_{i+1} - T_i$ then, by definition,

$$V_{s,n}(T_i, T) = \sum_{k=i}^{n-1} P(T_i, T_{k+1})\delta[L_k(T_i) - K],$$

   where $L_k(T_i)$ is the forward LIBOR rate observed at T_i for period (T_k, T_{k+1}), K is the strike price and $P(T_k, T_{k+1})$ is the ZCB price at T_i for period (T_k, T_{k+1}).

2. To find the Bermudan swaption price, it is necessary to carry out dynamic programming backward from the final exercise time T_{n-1} as a Bermudan swaption is strongly path-dependent. Let $I(T_{n-1})$, at time T_{n-1}, be the maximum of the value of exercising the option and zero, i.e.

$$I(T_{n-1}) = \max(V_{n-1,n}(T_{n-1}), 0).$$

   Furthermore, define stop rule (sr) as the optimal stopping time along a given path (d) of the LIBOR rate process and the stop rule firstly is set equal to the final exercise time, $sr = T_{n-1}$.

3. Working backwards, at time T_{n-1}, make a regression of the basis functions of state variables at that time on $Y(d)$ where

$$Y(T_{n-1}) = \frac{I(sr) \times disc(T_i)}{disc(sr)}.$$

   Again, stop rule (sr) is the next stopping time along a given path. Basis functions, denoted by $X_j(T_i)$ $j = 1 \cdots J$, are chosen to be quadratic functions of the current value of the underlying swap $V_{i,n}(T_i)$ and discounting factor $disc(T_i)$. Regression coefficients, calculated from ordinary least square regression, of $X_j(T_i)$ are called $\beta_j(T_i)$.

4. Then we are ready to compare the continuation value $C(T_i) = \sum \beta_j(T_i)X_j(T_i)$, corresponding to the estimated conditional expectation of the the payoff, with the immediate exercise value $I(T_i) = \max(V_{i,n}(T_i), 0)$. If $I(T_i) > C(T_i)$, then present time is an optimal stopping time and I is set to $V_{i,n}(T_i)$ and stop rule is set to T_{n-1} i.e.
$$I(sr) = V_{i,n}(T_i),$$
$$sr(d) = T_{n-1}.$$

5. Steps 3 and 4 are repeated for all $(T_{n-1} - 1) \leq T_i \leq T_s$, T_s is the lockout date, until one reaches the first exercise time of the swaption and all coefficients $\beta_j(T_i)$ have been calculated.

6. Finally, the value of the swaption can be calculated by discounting the value at the optimal stopping time back to present time and it is calculated as
$$\frac{1}{D}\sum_{d=1}^{D} W(sr)/disc(sr).$$
Once again, (sr) is the variable that keeps tracking optimal stopping time along a given path.

5.3 Implementation of Markov Functional model

The implementation of the Markov Functional model replies heavily on numerical integration, as seen from Section 3.3, when calculating expectations. It is, however, not advisable to apply simple numerical integration schemes such as trapezoid rule or Simpson's rule on a grid of fixed spacing for the Markov process m because, though yielding reasonably accurate prices, Greeks would become very unstable [30]. Moreover, such simple numerical integration scheme on a fixed grid would lead to spiking integrands, in which case the numerical integration is inaccurate when the calculation date is approaching a fixing date. To overcome these problems, Hunt and Kennedy introduced an idea by firstly fitting a polynomial to the payoff function defined on the grid and then calculate analytically the integral of the polynomial against the Gaussian distribution [30]. This is better as the only error in the integration comes from the polynomial fit and the fitting error can be controlled, by choosing a sufficiently high order of polynomial, as the integration of the polynomial is done analytically [30].

5.3.1 Polynomial fitting

There are many ways of fitting a polynomial but in this case it is better to use Neville's algorithm[1] as suggested in [30].
Given a number of points of m_i and a set of functions values f_i a polynomial that passes through these values can be computed recursively. Let $P_{i,\cdots,i+n}$ denote the polynomial defined using the points $m_i, \cdots, m_{i+n}$. Then a high order polynomial is generated by

$$P_{i,\cdots,i+n} = \frac{(m - m_{i+n-1})P_{i,\cdots,i+n} + (m_i - m)P_{i,\cdots,i+n}}{m_i - m_{i+n}} \tag{5.1}$$

and $P_i = f_i$. Each polynomial can then be written as $P_{i,\cdots,i+n} = \sum_{k=0}^{n} c_{i,k} m^k$. Using (5.1) a recurrence formula for the coefficients $c_{i,k}$ is as follows:

$$c_{i,n} = \frac{c_{i,n-1} - c_{i+1,n-1}}{m_i - m_{i+n}},$$

$$c_{i,k} = \frac{m_i c_{i+1,k} - m_{i+n} c_{i,k} + c_{i,k-1} - c_{i+1,k-1}}{m_i - m_{i+n}} \quad \forall k \in [1, n-1],$$

$$c_{i,0} = \frac{m_i c_{i+1,0} - m_{i+n} c_{i,0}}{m_i - m_{i+n}}.$$

5.3.2 Integrating against Gaussian

Recall the Markov process m defined in (3.11)

$$dm_t = \sigma_t^n dW_t$$

has Gaussian density functions. Whence, calculating integrals against a Gaussian density can be broken down to evaluating for different powers m^k of polynomial P in the following integral

$$G(k; h, \mu, \sigma) = \int_{\infty}^{h} m^k \frac{\exp\left[-\frac{1}{2}(\frac{m-\mu}{\sigma^2})^2\right]}{\sigma\sqrt{2\pi}}\, dm.$$

Then using integration by part, the following recurrence relation [2] for G in terms of k can be found

$$G(k) = \mu G(k-1) + (k-1)\sigma^2 G(k-2) - \sigma^2 h^{k-1} \frac{\exp\left[-\frac{1}{2}(\frac{h-\mu}{\sigma^2})^2\right]}{\sigma\sqrt{2\pi}}$$

with[3] $G(0) = N(\frac{h-\mu}{\sigma})$ and $G(-1) = 0$

[1] for details on this algorithm see, for example, [14]
[2] more details can be found in [14]
[3] As in Proposition 2.9, N is the standard normal cumulative distribution function

5.3.3 Expectation calculation

Given a grid on which the Markov process is defined, option values can be calculated by taking expectations of the value function against the Gaussian density. Suppose several option values have already been calculated at time T_{n+1} at grid points m_j, then the following procedure can lead to calculating option values at time T_n for grid points m_i.

- given an order M, the approximating polynomial $P_{(j-M/2),\cdots,(j+1+M/2)}$ for the interval $[m_j, m_{j+1}]$ is fitted through the points $m_{j-M/2}, \cdots, m_{j+1+M/2}$, where $M/2$ denotes the integer division;

- Then calculate the expectation $\mathbb{E}(f(m, T_{n+1})|m_i)$ by adding the integrals of the approximating polynomials against the Gaussian density over all the intervals $[m_j, m_{j+1}]$;

- finally, doing this for all m_i [30].

The fitting of polynomials generally works well for approximating smooth function. Some option payoffs, however, are determined as the maximum of two functions, implying that the payoff is smooth except at the crossover point where the payoff function may have a kink (a non-differentiable point). Since polynomials are "stiff" they will tend to fit functions with a kink poorly; the way to resolve this is to fit the polynomials to both underlying functions and to split the integration interval at the crossover point, with the use of suitable approximating polynomial on either side of the crossover point [30].

5.3.4 Non-parametric implementation

Implying functional form of the numeraire discount bond in the Markov functional model can also be done by, in econometric term, **non-parametric** fit of the functional forms. The implementation procedure presented above (and in Section 3.3) is of high accuracy for most Markov functional models (including equity Markov functional models). It is when fitting derivatives with very long maturities (50 or more) there might be a problem of fitting a large number of functional forms, usually 200 or more. In this case, the accumulation of numerical error can be problematic.
An alternative approach for determining functional form is to use semi-parametric functional form, meaning fixing a functional form, with several free parameters, which is flexible enough to provide a good fit to the observed market price [30]. In addition,

for a suitable choice of the functional form, the prices of discount bonds and options on discount bonds can be calculated analytically, eliminating a source of errors in the calibration procedure. It is beyond the content of this project to show exactly how this approach works and some details can be found in [30].

5.3.5 Numerical results

Below it compares the Bermudan swaption pricing results using different models. Since Bermudan swaptions are not liquid derivatives, it is hard to obtain its latest quoted market prices. To ensure that the models give sensible correct answers, the practical product data used here with strike $2\%, 3\%, 4\%, 5\%$ is the same as in [31], which is a five year semi-annually exercisable USD Bermudan swaption, evaluated in 2003. The data about the swaption with strike 6.24% is quoted from [20], which is a 30-year DEM (German Mark) Bermudan swaption, evaluated in 1998, exercisable every five years. But the main point is, as can be seen, the price difference between different models is generally small and even neglectable to a certain degree.

Strike	**European**	**Bermudan(SMM)**	**Bermudan(LMM)**	**Bermudan(MF)**
2%	37.060	42.230	40.542	42.095
3%	74.050	105.320	103.410	107.200
4%	187.450	244.330	242.270	245.200
5%	446.380	506.580	500.930	504.800
6.24%	482.500	569.800	572.500	566.900

Table 5.1: Bermudan swaption prices (in basis point)

This can show that the Markov functional model is a qualified substitute for the standard market model to price exotic interest rate derivatives of high dimensionality. Without the help of advanced numerical/computational technique improving the efficiency of the standard market model, given the large number of paths having to be simulated , it might be appealing to price multi exercisable strong path dependent interest rate derivatives using the Markov functional model.

Chapter 6

Conclusion

It is, with little doubt, becoming increasingly important to effectively model interest rate given the continuous booming of the fixed income market with a growing number of complicated interest derivatives being traded. Short rate modelling is still playing a part, though with obvious drawbacks, since that is how people firstly start modelling interest rate traditionally and by now it has been relatively well understood. More importantly, it is practically easy to implement achieving a high efficiency especially with modern advanced computing technique, such as parallel computing. The HJM modelling framework looks attractive from a theoretical point of view but its critical practical limitation makes it infeasible to use in reality. Both the standard market model (BGM) and the Markov functional model have obvious advantage of pricing complex derivatives over short rate models and the HJM modelling framework hence it is not surprising that they have received a lot of attention especially from the banking industry.

BGM model used to be impractical to price American style products due to Monte Carlo simulation's low speed and incompatibility with backward calculation; the introduction of the Longstaff-Schwartz algorithm, a regression-based approach, has successfully made this practically possible. The success of the Longstaff Schwartz algorithm is largely due to its general suitability combined with its applicability in both high-dimensional models and multiple exercise times. As a result, it has further increased the popularity of standard market model, at least, in terms of interest rate derivatives pricing.

Motivated by the reduction of dimension of the BGM model, the Markov functional model has been developed with the aim of pricing interest rate derivatives more effectively and efficiently. Having managed to reduce the dimension of the model, the Markov-Functional Market Model generally enjoys a low dimension of one or two and still prices multi-temporal products fairly accurately, having little difference from the standard market model.

As has been seen, the one-factor Markov functional model and multi-factor standard market model are very similar in terms of pricing and dynamics, which agrees with the result shown in [4]. With respect to efficiency, Markov functional model, arguably, tends to slightly outperform the standard market model due to the low dimensional property. Advanced numerical and modern powerful computing techniques (see [32][15] in this regard) have, however, made this computational efficiency gap between these two models neglectably narrow [31].

Currently, much attention has been paid to the derivative pricing itself; analysing financial instruments is not about pricing only and hedging, to some extent, is more important than pricing particularly from a practical point of view. So a possible future work could be looking at the hedging performance comparison between the Markov functional model and the standard market model. This is particularly of interest because implementation for both models might have to be adjusted to improve accuracy and efficiency when calculating Greeks. In addition, whether increasing the number of model factor significantly affects the hedge performance is still up to debate; some recent general results in this area can be found in [31]. This is probably why neither the Markov functional model nor the standard market model has been extensively applied in practical risk management.

The Markov functional model, nevertheless, represents a creatively fresh idea of modelling interest rate by trying to eliminate the weakness of other available models while retaining the strength. The study of the Markov functional model has just begun in recent years, though having already attracted much research attention, there is still much more to be analysed theoretically and numerically. It is quite possible, with further development, the Markov functional model can play a leading role in the study of modelling interest rate and fixed income products.

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