

# Causal and compositional structure in quantum theory



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# Abstract

Quantum theory challenges classical causal explanations for physical phenomena, calling for a quantum generalisation of causal structure. This thesis advances the study of *causal decompositions*, a research programme that aims to bridge two prominent, yet conceptually distinct types of structural assumption found across the quantum causality literature. The first is in terms of the global dependency relations between input and output systems of quantum transformations, which in the unitary case we shall refer to as *causal structure*; the second is in terms of the *compositional structure* of quantum circuits or networks. Concretely, a causal decomposition is a circuit representation of a unitary transformation in which no-influence relations that hold between its inputs and outputs are made manifest in the absence of paths through the circuit connecting them. Applications include translating between different quantum causal modelling frameworks; equipping a framework based on unitary causal structure with a local, bottom-up description via generalised *autonomous causal mechanisms*; and relating notions of compatibility between quantum transformations and *relativistic* causality.

Whether causal decompositions always exist remains an open problem. The contributions of this thesis are threefold.

First, we develop an order-theoretic account of circuit syntax and demonstrate that every circuit with a given input-output connectivity can be rewritten into a one of a canonical form given by the concept lattice construction from lattice theory. This systematises the search for causal decompositions by providing the shape of the quantum circuit one should look for.

Second, we establish a combinatorial condition that fully characterises when causal decompositions into *unitary* quantum circuits exist without requiring the generalisation to *routed* quantum circuits prompted by earlier research. This solves a well-defined sub-problem of the broader quest, and our proof, an interplay of order theory and operator algebra, lays a foundation for future extensions.

Third, we detail the application to relativistic causality. Within a multipartite framework generalising existing literature, we emphasise that the mere lack of superluminal signalling through a quantum channel is not sufficient to conclude its realisability by dynamics that are locally consistent with relativity theory. We conjecture that, on the other hand, a lack of superluminal *causal influences* through its unitary dilations is sufficient—at least up to arbitrarily small errors. We demonstrate this in various special cases and show that the general statement reduces to the existence of a weak version of causal decompositions.



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# Contributions

Chapters 2 to 4 of this thesis are the result of three projects conducted, in reverse chronological order, during the latter half of my DPhil. Versions of Chapters 2 and 3 are on the arXiv (Refs. [65] and [67], respectively) and Chapter 3 is due to be published soon; Chapter 4 will also appear on the arXiv at some point. Chapter 2 is work fully of my own, while Chapter 3 is the result of a collaboration with Robin Lorenz and Chapter 4 of work with Robin as well as Jonathan Barrett, Robert Spekkens, and Augustin Vanrietvelde.

One source of original inspiration for the project leading to Chapter 4 was formed while completing my master’s dissertation [64] in early 2021 under supervision of Jeremy Butterfield, whom I’d like to thank again here. The start of the actual project was marked by a mid-2023 meeting with Robin, Rob, and Augustin about nonlocal quantum computation and its potential integration with causal decompositions. A significant number of meetings with all three as well as Jon ensued in which we attempted to use nonlocal computation techniques to construct new causal decompositions. The text and novel results that eventually made it into Chapter 4 are mostly my own work, but Rob, Robin, Augustin, and Jon have all given valuable feedback on early drafts and have contributed also to the version that will eventually appear separately as a paper.

The initial idea for the research question answered in Chapter 3 originates from Robin, who also suggested a first candidate for the combinatorial condition we identify there. This was investigated, adjusted and refined by both of us over the course of many meetings starting early 2024. Our framework has gone through many iterations, the ultimate one being based on the formalism of Chapter 2 that I developed, and as a result, all text and proofs in this chapter are written by me, with the exception of the occasional paragraph; but Robin contributed very much to its initial development and provided invaluable feedback on later drafts.

The formalism for Chapter 2 naturally sprung out of the research required for Chapter 3. While I did this work mostly alone, I received some valuable comments from several people; in particular I’d like to thank Gabriele Tedeschi and Aleks Kissinger for answering my questions about category theory.

Finally, two other projects I completed with others during the my DPhil but which fall outside the scope of this thesis are:

- van der Lugt, T., Barrett, J. and Chiribella, G. “Device-Independent Certification of Indefinite Causal Order in the Quantum Switch”. *Nature Communications* 14.1 (2023), p. 5811. arXiv: 2208.00719.
- van der Lugt, T. and Ormrod, N. “Possibilistic and Maximal Indefinite Causal Order in the Quantum Switch”. *Quantum* 8 (2024), p. 1543. arXiv: 2311.00557.

# Chapter 1

## Introduction

The deepest understanding of a scientific phenomenon is reached when one can give it an explanation of a causal nature: one that not merely identifies or predicts patterns in correlations, but tells a complete story, giving a structured account of all causal processes at play. Over the past half-century, structural causal models [76, 93] have emerged as a powerful framework for formulating and answering questions about causation as rigorously as those about correlation. These models have not only aided our understanding across the sciences—by mirroring the way we naturally think about the world and make decisions—but have also provided crucial insights for predicting the outcomes of interventions, be they in medical practice or social policy. A key insight of the field is that imposing basic structural assumptions onto a model, guided by existing knowledge of the way the world works, enables significantly stronger conclusions about causal relations than approaches relying on raw data alone.

The same half-century has seen a ‘second quantum revolution’ unfold, accompanied by the emergence of an abstract, informational approach to quantum theory. This approach has aided the development of new applications, from computing to cryptography, but has also helped deepen our foundational grasp of what has the reputation of being, still, the least understood scientific theory of all. Arguably the most significant obstacle to a full understanding of quantum theory is Bell’s theorem [15], which since 1964 has challenged our classical ideas of how a physical theory should explain observations. Indeed, cast in modern terms, the theorem asserts precisely that the correlations predicted by quantum theory and observed in quantum experiment defy satisfactory explanation within the structural causal modelling framework [120]—a framework otherwise so successful across many areas of science.

That quantum theory is in tension with notions of causality is by no means an insight that relies on the formal framework of structural causal models and it has indeed been recognised independently in various contexts, with the term ‘causality’ taking on many different meanings. In physics literature, the term is perhaps most commonly associated with principles of relativity theory; and the conflict between quantum theory and relativistic causality is a famous one that has puzzled physicists ever since at least 1935 [36]. Bell’s theorem itself played no small part in this, prompting a close

consideration of what compatibility or ‘peaceful coexistence’ [89] with relativistic causality should in fact mean. In the modern quantum foundations literature, it has given rise to the development of generalised, ‘post-quantum’ probabilistic theories [10, 80], in which context ‘causality’ often refers to the operational criterion of ‘no signalling outside the future’—whether with respect to a spacetime or a circuit topology [29, 57]. Another field concerned with relativistic causality is quantum field theory, where the compatibility of isolated evolutions with relativity theory is ensured by the *micro-causality* condition of commuting spacelike observables while a still growing body of literature is devoted to how conformity with relativistic causality can be guaranteed in the presence of successive measurements, too [3, 19, 39, 91]. A further example of a research direction highlighting the conflict between causality and quantum theory is that of *indefinite causal order*, inspired in part by quantum superpositions of spacetime geometries [30, 51, 73]. Together, these examples represent only a small fraction of the literature investigating principles of causality, understood in some or other sense, in the quantum-theoretical context.

However, the formulation of Bell’s theorem as a statement of incompatibility with the structural causal modelling paradigm has provided a precise and compelling perspective on how quantum theory—even independently of any spacetime considerations—challenges our classical notion of causality, and with it, our classical way of explaining and understanding the world [22, 23, 120]. As a result, it has led to a recent surge of interest in causality in quantum theory, in particular by raising the question of whether an appropriately generalised notion of *quantum causal model* exists that can accommodate for and perhaps even offer satisfactory explanations for the correlations observed in quantum experiments [4, 11, 27, 33, 42, 44, 52, 78, 79, 83, 101, 111]. Any such framework should in particular specify what it means for a piece of quantum data to have, or be compatible with, a given *causal structure*, most commonly expressed, following the classical framework, in terms of a directed acyclic graph (DAG). Two approaches are worth detailing.

The first is that pioneered by Allen et al. [4] and Barrett, Lorenz and Oreshkov [11] and subsequently developed in e.g. [12, 62, 75, 99, 110]. It takes as a starting point the functional interpretation of classical causal models: that underlying every model is a *functional* causal model in which the dynamics between all variables in the model are deterministic, reducing all stochasticity to ignorance about local noise sources affecting individual variables. Causal structure ultimately refers, in this reading, to the functional (in)dependences between variables satisfied on this fundamental, deterministic level. Generalising this to the quantum case, where fundamental dynamics are described by *unitary* transformations rather than functions, the framework of [4, 11] is based on the notion of *causal influence* between inputs and outputs of unitaries. The data in this framework—quantum channels or, more generally, *process operators* or *higher-order quantum transformations*—are considered compatible with a causal structure described by a DAG if they admit of a unitary dilation whose influences are, roughly, a subset of the directed edges of the DAG. The result is a fully quantum framework that enjoys many direct analogies to the classical one, including a *d*-separation theorem and a

calculus for do-interventions [11]. One might say the type of causal structure imposed by this formalism is ‘top-down’ in nature: it regards a unitary transformation as a black box whose compatibility with a causal structure is expressed completely in terms of its global input-output behaviour. In this sense it can be compared to the notion of *signalling* through generic channels studied elsewhere in quantum causality (although these two notions should not be conflated: an absence of observable signals through a quantum channel does not imply the absence of influence on the fundamental level).

A second type of approach to formalising quantum causal structure is exemplified by the frameworks of Fritz [44] and Henson, Lal and Pusey [52] and widely used in the context of network nonlocality [20, 43, 81, 100, 118]. One way in which these frameworks differ from that of Refs. [4, 11] is that the data under study—that what is to be structured—are not quantum objects but instead probability distributions over classical random variables; it is only the *unobserved* (latent) variables in the model that are upgraded from classical to quantum.<sup>1</sup> This approach therefore enables a direct generalisation of Bell inequalities and their violations to beyond-Bell scenarios. However, a conceptually much more significant departure from [4, 11] is that these works understand causal structure instead in a ‘bottom-up’, *compositional* sense: to impose a causal structure is to assert that the process giving rise to the data can be decomposed into local subprocesses combined into a circuit or network whose form is determined by the DAG. For the purposes of this thesis I will, to avoid confusion, refer to this type of structure as *compositional structure*, reserving the term *causal structure* henceforth for influences through unitary transformations. Beyond Refs. [44, 52], several other approaches to quantum causal modelling with similar compositional flavours have been proposed, including in the works [27, 33, 42, 79].

This second type of approach fits in naturally with recent advances in applied category theory. In addition to a compositional approach to quantum theory [1, 32], these have provided a reformulation of the *classical* causal modelling paradigm which, although completely equivalent<sup>2</sup> to the traditional formulation of [76, 93], emphasises its compositional aspects; see e.g. [45, 54, 63]. If the unitary causal influence approach to quantum causal modelling [4, 11] was guided by the functional interpretation of classical causal models, the approaches of [44, 52] and others instead take as primary the compositional perspective made explicit by these categorical reformulations.<sup>3</sup> Crucially, however, despite the equivalence of the classical formulations, the relation between these different approaches to quantum causal modelling remains anything but clear. At the heart of this lies a highly nontrivial relationship between causal and compositional structure in quantum theory. The aim of this thesis is to advance our understanding of that relation.

Several earlier works have examined the connection between top-down and bottom-up constraints on quantum transformations. One of the first works was that of Beckman

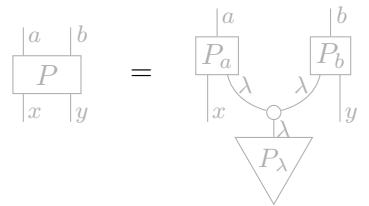
<sup>1</sup>Or even to resources in post-quantum, generalised probabilistic theories or symmetric monoidal categories.

<sup>2</sup>If one specialises to the category  $\mathbf{Mat}(\mathbb{R})$  of positive real matrices, that is.

<sup>3</sup>Not to suggest that [44, 52] and other compositional approaches to quantum causal modelling were in fact inspired by those categorical works; that would be anachronistic (excepting Ref. [42]).

et al. [13], set in a relativistic context. It addressed the question, in two bipartite scenarios, of whether any quantum channel that exhibits no signalling between spacelike-separated inputs and outputs (a top-down requirement related but not identical to the absence of causal influence) can also be realised by a sequence of local interactions and subluminal propagation of systems (a bottom-up requirement). Similar relativistically motivated questions were then studied in, for example, Refs. [8, 35, 86]. Meanwhile, in the setting of quantum cellular automata, a number of works investigated whether automata satisfying global causal conditions can, similarly, be represented by local generating rules: see e.g. Refs. [5–7, 38, 85, 88, 108]. Recently, Lorenz and Barrett [62] initiated the study of *causal decompositions* of general unitary transformations with finitely many inputs and outputs, generalising some of these earlier works. Causal decompositions are circuit representations of the unitary in which global no-influence relations between inputs and outputs are made apparent by the absence of paths connecting them—thus representing the unitary’s causal structure in terms of the circuit’s compositional structure. In this thesis, we further develop the theory of causal decompositions, whose general existence remains an open problem; and, armed with insights from this field, revisit the relativistic question posed by Beckman et al. [13] in a broader context.

Before spelling out in more detail what causal decompositions of unitary transformations are and what they can be used for, it is perhaps insightful to clarify one subtlety that already arises in the classical case. As an example, consider a classical stochastic process described by a conditional probability distribution  $P(ab|xy)$  on four finitely-valued random variables. A global, top-down constraint that this distribution may satisfy is that it is *nonsignalling* in both directions:  $P(a|xy)$  is independent of  $y$  and  $P(b|xy)$  is independent of  $x$ . A closely related, but strictly distinct bottom-up constraint is instead to demand that the process decomposes as  $P(ab|xy) = \sum_{\lambda} P_{\lambda}(\lambda) P_a(a|x\lambda) P_b(b|y\lambda)$  for an appropriate variable  $\lambda$  and stochastic processes  $P_{\lambda}$ ,  $P_a$  and  $P_b$ . In terms of the diagrammatic language employed in e.g. Refs [45, 54, 63], this can be expressed as the existence of a circuit decomposition of the form



$$\begin{array}{c} a \\ | \\ \boxed{P} \\ | \\ x \quad y \end{array} = \begin{array}{c} a \\ | \\ \boxed{P_a} \\ | \\ x \end{array} \quad \begin{array}{c} b \\ | \\ \boxed{P_b} \\ | \\ y \end{array} \quad \begin{array}{c} \lambda \\ \curvearrowright \\ \boxed{P_{\lambda}} \\ \curvearrowleft \\ \lambda \end{array}, \quad (1.1)$$

read from bottom to top; here  $\curvearrowright$  denotes a copy map. (Classical systems and processes are denoted in grey throughout this thesis to distinguish them from (black) quantum ones.) The existence of such a decomposition implies the two nonsignalling relations stated earlier; this is made evident in the diagram by the lack of paths connecting  $x$  to  $b$  and  $y$  to  $a$ . The converse does however not hold: this is a direct consequence of Bell’s theorem. It is not too difficult to convince oneself that in fact not a single classical circuit lacking paths from  $x$  to  $b$  and from  $y$  to  $a$ —whether or not it is of the

specific form in Eq. (1.1)—can express all stochastic processes  $P(ab|xy)$  satisfying the nonsignalling conditions.

This inequivalence between the top-down and bottom-up constraints may in a classical, deterministic theory however be attributed completely to the fact that the nonsignalling constraint is expressed on the level of the *stochastic* process  $P$ , which inherently encodes an observer’s ignorance of any unobserved systems. Indeed, the situation changes as soon as one requires not merely nonsignalling constraints, but causal constraints: functional independences on the level of the underlying deterministic dynamics of *isolated* systems. Indeed, any function  $f : X \times \Lambda \times Y \rightarrow A \times M \times B$  through which  $a \in A$  does not depend on  $y \in Y$  and  $b \in B$  not on  $x \in X$  (regardless of the values of the remaining arguments) can immediately be decomposed as  $f(x, \lambda, y) = (f_a(x, \lambda), f_\mu(x, \lambda, y), f_b(\lambda, y))$ —or, diagrammatically, as

$$\begin{array}{c} |a| \\ | \mu | \\ |b| \\ \hline \boxed{f} \\ \hline |x| \quad | \lambda | \quad |y| \end{array} = \begin{array}{c} \boxed{f_a} \quad \boxed{f_\mu} \quad \boxed{f_b} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \circ \quad \circ \quad \circ \\ |x| \quad | \lambda | \quad |y| \end{array}, \quad (1.2)$$

thereby representing the causal conditions completely in the circuit’s compositional structure.

In quantum theory, on the other hand, expressing causal structure compositionally is nontrivial even in the absence of any ignorance—i.e. with complete access to the fundamental dynamics of isolated systems: in the quantum case, unitary transformations. Consider a three-input, three-output unitary  $U : \mathcal{H}_A \otimes \mathcal{H}_B \otimes \mathcal{H}_C \rightarrow \mathcal{H}_D \otimes \mathcal{H}_E \otimes \mathcal{H}_F$  between tensor products of finite-dimensional Hilbert spaces and two causal no-influence constraints analogous to the above: that **A** does not influence **F** and **C** does not influence **D**, regardless of the state on the remaining systems. Lorenz and Barrett [62] showed that  $U$  satisfies these constraints precisely if it can be decomposed into a unitary circuit of the form

$$\begin{array}{c} |D| \quad |E| \quad |F| \\ \hline \boxed{U} \\ \hline |A| \quad |B| \quad |C| \end{array} = \begin{array}{c} |D| \quad |E| \quad |F| \\ \diagup \quad \diagdown \quad \diagup \quad \diagdown \\ \boxed{W} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \boxed{T_i} \quad \boxed{V_i} \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \boxed{S} \\ |A| \quad |B| \quad |C| \end{array}. \quad (1.3)$$

This circuit reflects the lack of influence from **A** to **F** and from **C** to **D** by a lack of paths between those systems, making it into an example of a *causal decomposition*.<sup>4</sup>

There are two things to note about this particular diagram. First, the index  $i$  on the wires and two of the gates signifies that this is a unitary circuit in the extended quantum circuit paradigm introduced by Lorenz and Barrett [62], which was later developed into the formalism of *routed quantum circuits* [104, 107]. In brief, this circuit

<sup>4</sup>Lorenz and Barrett [62] instead spoke in terms of *causally faithful circuit decompositions*; we will use both terms and regard the latter as a special case of causal decompositions. See Section 3.4.1.

language involves direct sums in addition to the usual tensor products and sequential compositions; in particular, the left- and rightmost gates should be interpreted as block-diagonal unitary transformations  $\bigoplus_i T_i$  and  $\bigoplus_i V_i$ . Lorenz and Barrett [62] found that giving causal decompositions in terms of traditional (‘non-routed’) unitary circuits is not always possible. It is relevant to note that the need for routed circuits however disappears if one allows the gates in the circuit to be general quantum channels, rather than just unitary ones.

Second, note that the shape of the circuit in Eq. (1.3) is different from that in Eq. (1.2). Indeed, not all unitaries  $U$  can be decomposed into circuits with just two ‘layers’ of gates such as in Eq. (1.2) (even with arbitrary quantum channels in place of the copy maps and classical functions). Instead, quantum dynamics generally rely on gates such as  $T_i$  in Eq. (1.3), which lies in the future of both  $\mathbf{A}$  and  $\mathbf{B}$  and in the past of both  $\mathbf{D}$  and  $\mathbf{E}$ , thereby providing the opportunity for all those systems to meet and *interact*. The significance of such interaction gates, also emphasised recently in [25], will be a theme throughout this thesis. It may be regarded as a manifestation of the *holism* pervasive in quantum theory: that properties of systems can generally not be reduced to properties of their subsystems.

More generally, a causal decomposition refers to any circuit decomposition of a unitary transformation that reflects a desired set of global no-influence conditions satisfied by the unitary in terms of absences of paths connecting inputs to outputs. Whether a set of no-influence conditions can always be represented in this way remains an open problem, and one we will work towards solving in this thesis.

Before describing some of the applications of causal decompositions in more detail, it is worth taking a brief moment to compare them to the work of Arrighi, Nesme and Werner [8], who like this thesis and Ref. [62] investigate circuit decompositions of general unitary transformations satisfying global causal conditions. The work shows that any unitary, no matter its causal structure, decomposes into a circuit consisting of commuting unitary gates—one for each output system—as well as state preparations on ancillary systems. By commuting the gates past each other, any desired individual no-influence relationship between a given input and output that holds in  $\mathcal{U}$  can be expressed by the lack of a path connecting them. Similar results have subsequently been shown in different settings in, for example, Refs. [6, 38]. Decompositions of this type do not however provide a single circuit satisfying at once all of a desired set of no-connectivity constraints, which is required of a causal decomposition.

Where known to exist, causal decompositions find many applications. Here follow a couple.

- Causal decompositions provide a bridge between different approaches to quantum causal modelling. Whether one’s interest in quantum causal models is applied or foundational, it is important to appreciate the dissimilarities between these frameworks—especially because they all refer to DAGs as ‘causal structure’, yet in fact interpret them in very different ways. In unitary-influence-based models [4,

11], an edge of the DAG represents a potential causal influence through unitary dynamics; in compositional approaches [44, 52] it represents, roughly speaking, a wire in a circuit. As an example, suppose we are given a quantum causal model in the former sense [4, 11] with DAG

$$G \quad := \quad \begin{array}{ccccc} & \mathbf{D} & & \mathbf{E} & & \mathbf{F} \\ & \uparrow & \nearrow & \uparrow & \nearrow & \uparrow \\ \mathbf{A} & & \mathbf{B} & & \mathbf{C} & \end{array}, \quad (1.4)$$

as well as choices of measurements for each of its six quantum systems. One may ask: are the resulting correlations between measurement outcomes quantum-compatible with some other DAG  $G'$  in the compositional sense of [44, 52]? Contrary to what one might naively expect, the correct answer  $G'$  would in this case not resemble  $G$  at all, due to the nontrivial shape of causal decompositions. (The correct answer would depend on details of the question and the precise compositional formalism [44, 52] adopted; but it would look something like the shape of the circuit in Eq. (1.3)).

In the context of this application it is worth noting that general quantum causal models in the sense of [4, 11] are not described by simple unitary transformations but instead by *quantum process operators*, also known as *higher-order quantum transformations* [28]; and causal structures by DAGs with generally more than two layers as in Eq. (1.4). However, the problem of decomposing such higher-order quantum transformations can always be reduced to that of decomposing first-order unitary transformations, which is why we permit ourselves to focus on the first-order case in this thesis. This reduction is detailed in [62, §7.3]; for our present purposes, suffice it to say that it is achieved by a procedure employed across research on causality known in different contexts as ‘splitting nodes’ [11, 12], constructing the ‘single-world intervention graph’ [82], ‘maximal interruption’ [118], or ‘plugging SWAPs into the higher-order quantum map’ [75].

- Recall that for the example discussed earlier, Lorenz and Barrett [62] showed that *any* tripartite unitary satisfying two global causal constraints ( $\mathbf{A}$  does not influence  $\mathbf{F}$  and  $\mathbf{C}$  does not influence  $\mathbf{D}$ ) admits of a routed unitary causal decomposition of the shape in Eq. (1.3); and conversely, every such circuit implements a unitary satisfying the two constraints. Results of this kind thus provide a *parametrisation* of the set of all those unitaries—the parameters in question being the dimensions of the internal wires and the gates of the circuit—and with it, a constructive way to build any unitary in the set. As for any good parametrisation, the parameters’ domains are simple in structure: natural numbers for the wires and (block-diagonal, in the routed case) unitary matrices for the gates.

Parametrisations like this are particularly relevant for the quantum causal models of [4, 11], where they can be regarded as providing generalised *autonomous causal mechanisms*. In classical causal modelling, the latter is an important concept referring to the stochastic processes  $P(X|\text{Pa}(X))$  that determine each variable of

the model solely in terms of its causal parents (in the deterministic example of Eq. (1.2), they are the functions  $f_a, f_\mu$ , and  $f_b$ ). ‘Autonomy’ refers to the fact that one may intervene on, or otherwise vary, each of the mechanisms individually without affecting other parts of the model or its overall causal structure. Moreover, varying each of them independently allows one to recover any other model with the same causal structure.

Refs. [4, 11] identified a close analogue of these mechanisms for their *quantum* causal models, given by the Choi operators  $\rho_{A|Pa(A)}$  of the model’s marginals. However, while each quantum causal model does correspond uniquely to a choice of such operators, they lack autonomy: to constitute a valid quantum causal model, they must all commute pairwise, a global condition that is easily violated if one varies the operators independently.

The wires and gates in a causal decomposition provide an alternative generalisation of causal mechanisms that do enjoy a high level of autonomy (although overall dimension constraints still have to be satisfied). In light of the nontrivial shape of causal decompositions, one however needs to give up the idea that the role of a mechanism is to determine a single variable in terms of its causal parents (cf. Eqs. (1.2) and (1.3)). Perhaps this should be seen as another manifestation of quantum holism.

- In proofs, compositional structure often provides a more powerful handle than causal structure, especially when employing diagrammatic reasoning. Causal decompositions allow one to use this tool even if the ultimate goal is to prove something about causal structure. An explicit example from the literature is in [12, Theorem 3], where the very decomposition of Eq. (1.3) was used to derive a central result in the study of indefinite causal order.
- Causal decompositions can be used to generalise to more inputs and outputs the relativistic question posed by Beckman et al. [13]: if a quantum channel is consistent with the principle of no-superluminal-signalling imposed by some localisation of its input and output systems in spacetime, does this mean it can also in fact be realised by locally covariant dynamics (a quantum circuit whose wires follow timelike or lightlike curves)? Also here, it is important to make a distinction signalling and influence. We discuss this application in detail in Chapter 4.
- Finally, rather than an evolution of physical systems, one may also regard a unitary as a *passive* transformation, relating different ways of partitioning the same overall quantum system into subsystems. From this viewpoint, causal decompositions clarify how one such partitioning can be obtained from another by a sequence of fine-grainings and coarse-grainings. This perspective is explored in a recent work of [109].

To conclude this introduction, let us outline the rest of this thesis. In Chapter 2, we investigate the syntactical aspects of compositional structure, developing an order-theoretic formalism for circuit syntax especially suited for studying their input-output *connectivity*. We then show that any circuit with a given connectivity  $G$  can be rewritten, using purely syntactical operations, into a particular canonical shape  $L_G$ , which therefore acts as a canonical shape for causal decompositions. The shape of the circuit in Eq. (1.3) is an example. This result answers in the positive a question posed by Lorenz and Barrett [62], which was whether the shape of causal decompositions—when they exist—can always be chosen the same for unitaries satisfying the same causal constraints. Notice that this result is crucial to the first two applications outlined above (for the first because  $G'$  should ideally only depend on the causal structure  $G$ , not the specific causal model; and for the second because such a canonical circuit shape leads to a single parametrisation covering all unitaries that satisfy the causal constraints).

The canonical shape  $L_G$  turns out to be given by the *concept lattice* construction from lattice theory [17, 46]. As we will argue, the fact that it is a lattice is a syntactical manifestation of the significance of *interaction* in quantum circuits, generalising the second comment we made below Eq. (1.3).

The result of Chapter 2 also systematises the search for causal decompositions: with their shapes now determined, what remains is to equip them with appropriate quantum semantics implementing desired unitaries. We do this in Chapter 3, solving a particular subproblem in the search for causal decompositions. More precisely, we fully characterise those causal constraints that can always—i.e. for any unitary satisfying them—be represented by causal decompositions in terms of unitary circuits, *without* requiring the generalisation to routed circuits. The syntactical aspects of our methods rely heavily on the combinatorics of the concept lattice; the semantic ones on the structure of finite-dimensional operator algebras.

In Chapter 4, we detail the spacetime application of causal decompositions mentioned above. We discuss the distinction between *signalling* and *causal influence* in this context, and propose a conjecture: that the absence of superluminal *influences* through unitary dilations of a quantum channel is, up to arbitrarily small errors, necessary and sufficient for realisability by a circuit in the spacetime. We show it to be equivalent to the existence of a weak variant of causal decompositions. In this chapter we again encounter the notion of *interaction*, as its presence is generally constrained by the relativistic structure of a spacetime (a spacetime itself not necessarily being a lattice). We address this constraint with the help of nonlocal quantum computation [14, 21, 34, 71]. To close off the chapter we show that all Clifford unitaries admit causal decompositions, which are of a particular shape relevant to their implementation in spacetime. We conclude in Chapter 5 with a summary and open questions.



## Chapter 2

# Compositional structure

This chapter is about the syntax of *circuits*, or *string diagrams*, which describe parallel and sequential compositions of processes in general symmetric monoidal categories (SMCs), including but not limited to quantum theory and classical probability theory. The syntax of a circuit—i.e. its ‘shape’—alone often implies relevant information about the process (morphism of the SMC) it implements: for example, the absence of a path between an input and an output of the circuit prohibits the flow of causal influence between those systems. Generally, and especially in the context of causal inference, one may wonder about the full class of morphisms in a specified SMC that can be expressed by a circuit with a given shape;<sup>1</sup> a circuit shape Q may then be considered ‘at least as expressive’ as a shape P if it can express at least as many morphisms of the SMC. Sometimes, this is the case simply because any circuit with shape P can be rewritten into one with shape Q using purely syntactical operations valid in any SMC, such as composing two or more gates, treating them as a single one, or adding trivial gates and trivial wires.

In this chapter we set out an order-theoretic formalisation of circuit syntax and syntactical rewrites, taking a particular focus on the input-output *connectivity* of a circuit, a binary relation specifying which inputs can reach which outputs via directed paths through the circuit. We characterise the *concept lattice* from formal concept analysis [17, 46, 115] as the unique smallest circuit shape that every circuit with at most a given connectivity can be syntactically rewritten into and which is therefore most expressive, in the sense described above. It thus provides a universal shape for causal decompositions, dependent only on the desired connectivity and not on the unitary to be decomposed; we will use this result in Chapter 3. The focus of this chapter is however on the order-theoretic aspects, and all results, unless otherwise stated, hold for infinite partially-ordered sets too, which do not have an obvious interpretation as circuit syntax. We conclude by constructing the circuit shape which, dually to the concept lattice, is least expressive with a given connectivity.

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<sup>1</sup>This type of problem is sometimes referred to as *causal compatibility* [119], but in the wider context of this thesis *compositional compatibility* would be more appropriate.

## 2.1 First definitions

A preorder on a set  $P$  is a relation  $\leq \subseteq P \times P$  that is (i) **reflexive**:  $\forall p \in P : p \leq p$  and (ii) **transitive**:  $\forall p, q, r \in P : p \leq q \wedge q \leq r \implies p \leq r$ . A partial order is a preorder that is also (iii) **antisymmetric**:  $\forall p, q \in P : p \leq q \wedge q \leq p \implies p = q$ . For preorders and partial orders alike, we write  $p < q$  iff  $p \leq q$  and  $p \neq q$ ;  $p \# q$  iff  $p \not\leq q$  and  $q \not\leq p$ ; and  $p \triangleleft q$  iff  $p$  **covers**  $q$  from below, that is, if  $p < q$  and there is no  $r \in P$  so that  $p < r < q$ .

We propose to model circuit syntax by partial orders decorated with input-output structure, as follows.

**Definition 2.1.** Let  $A$  and  $B$  be sets. A **circuit shape** from  $A$  to  $B$  is a tuple  $(P, \leq, \lambda, \mu)$  of a set  $P$ , a partial order  $\leq$  on  $P$ , and two functions  $\lambda : A \rightarrow P$  and  $\mu : B \rightarrow P$ . We will often refer to elements of  $P$  as **gates** and to elements of  $A$  and  $B$  as **inputs** and **outputs**, respectively. The letter  $P$  may also refer to the entire tuple  $(P, \leq, \lambda, \mu)$ ; in this case we write  $\leq$  or  $\leq_P$  and  $\lambda_P, \mu_P$  for the order and maps constituting the circuit shape on the set  $P$ .<sup>2</sup>

**Definition 2.2.** We call a circuit shape  $P$  **finite** if the underlying set  $P$  as well as the sets  $A$  and  $B$  are finite. Finite circuit shapes may be conveniently depicted by diagrams. First of all, the *Hasse diagram* of the partial order  $(P, \leq)$  is the directed acyclic graph on  $P$  obtained by drawing an edge from  $p$  to  $q$  iff  $p < q$ . The elements of  $P$  are laid out on the page so that all edges point upward; the latter are then drawn as undirected edges. The **circuit diagram** of  $(P, \leq, \lambda, \mu)$  is obtained by adding nodes for each  $a \in A$  and  $b \in B$  and upward-directed edges from  $a$  to  $\lambda(a)$  and from  $\mu(b)$  to  $b$ . We will often leave the gates of the circuit shape unlabelled, in which case the diagram determines the circuit shape up to isomorphisms (defined imminently).<sup>3</sup>

**Example 2.3.** The diagram



<sup>2</sup>A circuit shape as defined here can be regarded as a special kind of *cospan* in the category of partial orders. Cospans in other categories have been used in various contexts in applied category theory to model processes with inputs and outputs [18, 40, 41]. It is perhaps worth noting that our notion of morphism from Definition 2.4 below does not quite match what would usually be called a morphism between cospans, which would instead have equalities in items (ii) and (iii).

<sup>3</sup>Other treatments of circuits and string diagram syntax are based on directed (acyclic) graphs, or similar structures, from the outset (see e.g. [18]). Such an approach allows for a distinction between circuit shapes with identical transitive reductions, i.e. that induce the same partial order among their gates. In Example 2.3, for instance, one could allow for an additional wire directly from gate  $q$  to gate  $s$ . Our order-based formalism here does not make such distinctions. They are not relevant to the question of expressivity described in the opening of this chapter, since the wire from  $q$  to  $s$  can just as well be incorporated into the gate  $r$ .

denotes the circuit shape  $(P, \leq, \lambda, \mu)$  where  $P = \{p, q, r, s\}$ ;  $\leq$  is the unique partial order satisfying  $q < p$ ,  $q < r < s$ ,  $p \# r$  and  $p \# s$ ; the sets of inputs and outputs are  $A = \{a_1, a_2, a_3\}$  and  $B = \{b_1, b_2\}$ ; and, finally, the input map  $\lambda : A \rightarrow P$  acts as  $a_1 \mapsto p, a_2 \mapsto r, a_3 \mapsto s$  and the output map  $\mu : B \rightarrow P$  as  $b_1 \mapsto p$  and  $b_2 \mapsto s$ . //

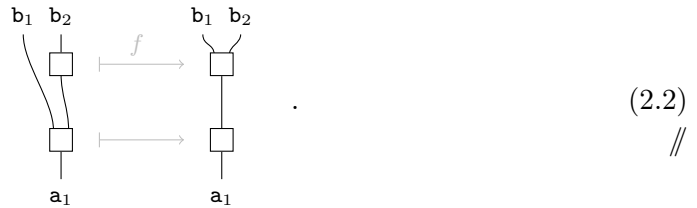
**Definition 2.4.** Let  $P, Q$  be circuit shapes from  $A$  to  $B$ . A **morphism** of circuit shapes  $f : P \rightarrow Q$  is a map between the underlying sets  $P, Q$  satisfying the following properties.

- (i)  $f$  is **order-preserving**: for all  $p, p' \in P$ ,  $p \leq p' \implies f(p) \leq f(p')$ .
- (ii)  $f$  **respects inputs**: for all  $a \in A$ ,  $f(\lambda_P(a)) \geq \lambda_Q(a)$ .
- (iii)  $f$  **respects outputs**: for all  $b \in B$ ,  $f(\mu_P(b)) \leq \mu_Q(b)$ .

Moreover, we say  $f$  is an **embedding**, and write  $f : P \hookrightarrow Q$ , if it is a morphism of circuit shapes and an order embedding (i.e.  $f(p) \leq f(p') \implies p \leq p'$ ). An **isomorphism**  $f : P \xrightarrow{\sim} Q$  is a bijective morphism whose inverse is also a morphism.

**Remark 2.5.** We regard the sets of inputs and outputs  $A$  and  $B$  as (arbitrary but) fixed; two circuit shapes can be isomorphic only if they are defined on identical input and output sets. Moreover, two circuit shapes that differ only by a permutation on  $A$  or  $B$  may not be isomorphic.

Furthermore, if  $f : P \rightarrow Q$  is bijective and an embedding, then it is automatically an isomorphism of partial orders; however, its inverse  $f^{-1}$  is only a morphism of circuit shapes—and  $f$  therefore only an isomorphism of *circuit shapes*—if it also satisfies the *equalities*  $f \circ \lambda_P = \lambda_Q$  and  $f \circ \mu_P = \mu_Q$ . The map  $f$  illustrated below is an example of a bijective embedding that fails this property: it satisfies  $f(\mu_P(b_1)) < \mu_Q(b_1)$  so its inverse is not a morphism.



## 2.2 Morphisms as syntactical circuit rewrites

In the next chapter, finite circuit shapes  $(P, \leq, \lambda, \mu)$  will form the syntactical basis for *quantum circuits*, obtained by equipping  $P$  with quantum semantics. Every such quantum circuit expresses a particular overall quantum channel whose inputs are labelled by  $A$  and outputs by  $B$ . Similarly, one can consider the expressivity of circuit shapes  $P$  in arbitrary symmetric monoidal categories beyond quantum theory. If a particular circuit shape  $Q$  is more expressive than  $P$  in all SMCs—including freely generated ones—then that fact should be derivable by a procedure that only uses the basic axioms of SMCs. I will therefore refer to such procedures as *syntactical circuit rewrites*.

The main aim of this section is to argue that morphisms between finite circuit shapes model such syntactical circuit rewrites on the order-theoretic level. In particular, I show in Theorem 2.16 that any morphism factorises into morphisms of particular elementary types, all of which can clearly be interpreted as valid syntactical rewrites. I shall however refrain from making any formal connection with symmetric monoidal categories, keeping the focus instead on the order theory, and all notions and results, with the exception of Proposition 2.10, are defined and hold in the infinite case as well. Let's start with some background on quotients of partial orders [58, 96–98, 116], which are the order-theoretic representation of syntactical rewrites involving only (parallel as well as sequential) compositions of gates.

**Definition 2.6.** Let  $(P, \leq)$  be a partial order and  $\theta$  an equivalence relation on  $P$ . Denote by  $[p]$  the equivalence class of  $p \in P$  under  $\theta$  and define the binary relation  $\leq_\theta$  on the quotient set  $P/\theta$  by

$$[p] \leq_\theta [q] \quad :\Longleftrightarrow \quad \exists p' \in [p], q' \in [q] \text{ such that } p' \leq q'. \quad (2.3)$$

In other words, it is the minimal relation on  $P/\theta$  making the quotient map  $\pi_\theta : P \rightarrow P/\theta, p \mapsto [p]$  order-preserving.

While the relation  $\leq_\theta$  so defined is always reflexive, it may not be antisymmetric or transitive, and thus not necessarily a partial order.

**Example 2.7.** If  $P = \{p, q, r\}$ ,  $p < q < r$ , and the equivalence classes of  $\theta$  are  $\{p, r\}$  and  $\{q\}$ , then  $\leq_\theta$  is not antisymmetric, as we have  $\{p, r\} \leq_\theta \{q\} \leq_\theta \{p, r\}$ . Moreover, if  $P = \{p, q, r, s\}$ ,  $p < q$ ,  $r < s$ , and  $\{p, q\}$  and  $\{r, s\}$  are  $\leq$ -incomparable, then the relation  $\leq_\theta$  defined by the equivalence relation  $\theta$  with equivalence classes  $\{p\}, \{r, s\}, \{q\}$  is not transitive. //

The following notion has been studied in detail in Refs. [58, 96–98], amongst other works.

**Definition 2.8.** An equivalence relation  $\theta$  on  $(P, \leq)$  is a **compatible congruence**<sup>4</sup> if the transitive closure  $\overrightarrow{\leq_\theta}$  of  $\leq_\theta$  is antisymmetric and hence a partial order. When  $\theta$  is a compatible congruence, we will henceforth equip the set  $P/\theta$  with the order  $\overrightarrow{\leq_\theta}$ . Moreover, when  $(P, \leq, \lambda, \mu)$  is a circuit, we denote by  $P/\theta$  the circuit  $(P/\theta, \overrightarrow{\leq_\theta}, \pi_\theta \circ \lambda, \pi_\theta \circ \mu)$ , where  $\pi_\theta : P \rightarrow P/\theta$  is the quotient map.

Which equivalences  $\theta$  are compatible congruences? First, call a subset  $S \subseteq P$  **convex** if for all  $s, t \in S$  and  $p \in P$ ,  $s \leq p \leq t \implies p \in S$ . If  $\theta$  is a compatible congruence on  $P$ , then every equivalence class  $[p]$  is necessarily convex [96, §36]; this is required to avoid failure of antisymmetry of  $\leq_\theta$  (cf. Example 2.7). However, not every equivalence

<sup>4</sup>We take this terminology from [94, 116]; it has also simply been called an *order congruence* [58].

relation  $\theta$  on  $\mathbf{P}$  whose equivalence classes are convex is a compatible congruence. As an example, consider the order with Hasse diagram

$$\begin{array}{cc} c & d \\ | & \diagdown \\ & \times \\ | & \diagup \\ a & b \end{array} \quad (2.4)$$

and the equivalence relation  $\theta$  whose equivalence classes are  $\{a, d\}$  and  $\{b, c\}$ . Once one identifies  $a$  and  $d$ , the set  $\{b, c\}$  becomes non-convex, resulting in a relation  $\leq_\theta$  that is not antisymmetric. However, as long as one performs a sequence of quotients that identify the elements of a *single* convex subset at a time, the resulting relation will remain a partial order. In the finite case, every quotient by a compatible congruence arises in this way.

**Definition 2.9.** Let  $S \subseteq \mathbf{P}$  be a convex set. Denote by  $\theta_S$  the equivalence relation on  $\mathbf{P}$  that has equivalence classes  $S$  and  $\{p\}$  for  $p \in \mathbf{P} \setminus S$ . If  $S$  contains precisely two elements, that is,  $S = \{p, q\}$  where  $p \triangleleft q$  or  $p \# q$ , then we call  $\theta_S$  an **atomic compatible congruence**.

**Proposition 2.10.** *If  $S \subseteq \mathbf{P}$  is convex, then  $\theta_S$  is a compatible congruence. Moreover, if  $\mathbf{P}$  is finite then an equivalence relation  $\theta$  on  $\mathbf{P}$  is a compatible congruence if and only if the quotient map  $\pi_\theta : \mathbf{P} \rightarrow \mathbf{P}/\theta$  is, up to order isomorphisms, a finite composition of quotients by atomic compatible congruences.*

We omit the (straightforward) proof (see also [97, §7]). When interpreting a finite circuit shape  $\mathbf{P}$  as the syntax of a circuit in an SMC, each quotient by an atomic compatible congruence  $\theta_{\{p,q\}}$  can be regarded as the result of an elementary ‘gate composition’ procedure: that is, the sequential composition (if  $p \triangleleft q$  or  $q \triangleleft p$ ) or parallel composition (monoidal product; if  $p \# q$ ) of two gates  $p$  and  $q$ . The connection with circuit shape morphisms is due to the following.

**Definition 2.11.** The **kernel** of a map  $f : \mathbf{P} \rightarrow \mathbf{Q}$ , denoted  $\ker f$ , is the equivalence relation on  $\mathbf{P}$  whose equivalence classes are the fibres of  $f$ : that is,  $p(\ker f)p' \iff f(p) = f(p')$  for  $p, p' \in \mathbf{P}$ .

**Proposition 2.12** ([98, §2], [116, Prop. 4.10]). *An equivalence  $\theta$  on  $\mathbf{P}$  is a compatible congruence if and only if it is the kernel of an order-preserving map  $f : \mathbf{P} \rightarrow \mathbf{Q}$  for some partially ordered set  $\mathbf{Q}$ .*

**Proposition 2.13.** *Let  $f : \mathbf{P} \rightarrow \mathbf{Q}$  be a morphism of circuit shapes. Let  $\pi_f : \mathbf{P} \twoheadrightarrow \mathbf{P}/\ker f$  be the quotient map and  $[f] : \mathbf{P}/\ker f \rightarrow \mathbf{Q}$  the (well-defined) injective map  $[p] \mapsto f(p)$ .*

Then  $\pi_f$  and  $[f]$  are morphisms and  $f = [f] \circ \pi_f$ :

$$\begin{array}{ccc} P & \xrightarrow{\pi_f} & P/\ker f \\ & \searrow f & \downarrow [f] \\ & & Q \end{array} \quad (2.5)$$

*Proof.* The only nontrivial statement to verify is that  $[f] : P/\ker f \rightarrow Q$  is a morphism of circuit shapes. Recall that  $P/\ker f$  is ordered by  $\leq_{\ker f}$  and has input and output maps  $\lambda_{P/\ker f} = \pi_f \circ \lambda_P$  and  $\mu_{P/\ker f} = \pi_f \circ \mu_P$ , respectively. To show that  $[f]$  is order-preserving, first suppose that  $[p], [q] \in P/\ker f$  and  $[p] \leq_{\ker f} [q]$ . This means that there exist  $p', q' \in P$  so that  $[p] = [p']$ ,  $p' \leq_P q'$ , and  $[q'] = [q]$ . By construction, then,

$$[f]([p]) = f(p) = f(p') \leq_Q f(q') = f(q) = [f]([q]). \quad (2.6)$$

Thus,  $[p] \leq_{\ker f} [q] \implies [f]([p]) \leq_Q [f]([q])$ . Since  $\leq_Q$  is transitive, we also have  $[p] \leq_{\ker f} [q] \implies [f]([p]) \leq_Q [f]([q])$ , so  $[f]$  is order-preserving. That it also respects inputs is immediate:  $[f] \circ \lambda_{P/\ker f} = [f] \circ \pi_f \circ \lambda_P = f \circ \lambda_P \geq \lambda_Q$  pointwise for all  $\mathbf{a} \in \mathbf{A}$ , where the last inequality follows from the fact that  $f$  is itself a morphism. Similarly,  $[f]$  respects outputs.  $\square$

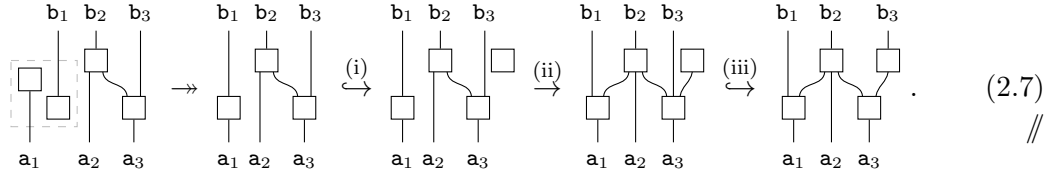
It remains to argue that in addition to the quotient map  $\pi_f$ , the injective morphism  $[f]$  also models a valid syntactical circuit rewrite.<sup>5</sup>

**Definition 2.14.** The following are three types of injective morphisms of circuit shapes, each of which corresponds to a certain elementary type of circuit rewrite.

- (i) Let  $(P, \leq_P, \lambda, \mu)$  be a circuit shape from  $\mathbf{A}$  to  $\mathbf{B}$  and  $S$  an arbitrary set. Write  $P \sqcup S$  for the disjoint union of the sets  $P$  and  $S$ . Let  $\leq$  be the partial order on  $P \sqcup S$  that satisfies  $\leq \cap (P \times P) = \leq_P$  and  $s \# q$  for any  $s \in S$  and  $q \in P \sqcup S$ . Denote by  $P \sqcup S$  the circuit shape from  $\mathbf{A}$  to  $\mathbf{B}$  given by  $(P \sqcup S, \leq, \lambda, \mu)$ . Say that a morphism  $f : P \rightarrow Q$  **adds isolated gates** if  $Q = P \sqcup S$  for some set  $S$  and  $f$  is the inclusion map  $P \hookrightarrow P \sqcup S$ . Each such map is in fact an embedding.
- (ii) A morphism that **adds wires** is one of the form  $\text{id}_P : (P, \leq, \lambda, \mu) \rightarrow (P, \leq', \lambda, \mu)$ ,  $p \mapsto p$ , where  $\leq, \leq'$  are such that  $p \leq q \implies p \leq' q$ .
- (iii) A morphism that **advances inputs and delays outputs** is one of the form  $\text{id}_P : (P, \leq, \lambda, \mu) \rightarrow (P, \leq, \lambda', \mu')$ ,  $p \mapsto p$  where  $\lambda(\mathbf{a}) \geq \lambda'(\mathbf{a})$  for all  $\mathbf{a} \in \mathbf{A}$  and  $\mu(\mathbf{b}) \leq \mu'(\mathbf{b})$  for all  $\mathbf{b} \in \mathbf{B}$ . This is always an embedding.

<sup>5</sup>Note that in disanalogy to the isomorphism theorems for algebraic structures, the map  $[f] : P/\ker f \rightarrow \text{im}(f) \subseteq Q$  is not necessarily an isomorphism: it is injective but not always an order embedding.

**Example 2.15.** The first morphism below is a quotient map; the other three are of the types just introduced.



**Theorem 2.16.** A map  $f : P \rightarrow Q$  is a morphism of circuit shapes if and only if it is, up to isomorphisms, the composition of a finite sequence of maps of the following four types.

- quotients by compatible congruences (which, in case  $P$  is finite, are in turn always finite compositions of quotients by atomic compatible congruences, by Proposition 2.10);
- morphisms that add isolated gates;
- morphisms that add wires;
- morphisms that advance inputs and delay outputs.

*Proof.* Due to Proposition 2.13 it suffices to show that  $[f] : P/\ker f \rightarrow Q$  factorises into morphisms of the latter three types. Denote by  $\tilde{Q}$  the circuit shape  $(Q, \leq_Q, f \circ \lambda_P, f \circ \mu_P)$  and consider

$$\begin{aligned}
 \pi_f : P &\twoheadrightarrow P/\ker f \\
 p &\mapsto [p]; \\
 \text{id}_{P/\ker f} : P/\ker f &\hookrightarrow P/\ker f \sqcup (Q \setminus \text{im}(f)) \\
 [p] &\mapsto [p]; \\
 g : P/\ker f \sqcup (Q \setminus \text{im}(f)) &\rightarrow \tilde{Q} \\
 [p] &\mapsto [f]([p]) \quad \text{for } p \in P; \\
 q &\mapsto q \quad \text{for } q \in Q \setminus \text{im}(f); \\
 \text{id}_Q : \tilde{Q} &\hookrightarrow Q \\
 q &\mapsto q.
 \end{aligned} \tag{2.8}$$

Recall from Definition 2.14 that  $Q \setminus \text{im}(f)$  is discretely ordered in the context of the circuit  $P/\ker f \sqcup (Q \setminus \text{im}(f))$ . It is clear that all four maps are, up to isomorphism, morphisms of the required types and that  $f$  factorises as

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & & & [f] & & & \\
 & & \nearrow & & \searrow & & \\
 P & \xrightarrow{\pi_f} & P/\ker f & \xrightarrow{\text{id}_{P/\ker f}} & P/\ker f \sqcup (Q \setminus \text{im}(f)) & \xrightarrow{g} & \tilde{Q} & \xrightarrow{\text{id}_Q} & Q \\
 & \searrow & & & & & & & \\
 & & & f & & & & & 
 \end{array}
 \end{array} \tag{2.9}$$

□

In Proposition 3.3 of the next chapter, we will use Theorem 2.16 to show what will now be clear: if a morphism  $f : P \rightarrow Q$  exists, then any quantum channel that can be expressed by a circuit of shape  $P$  can also be expressed by a circuit of shape  $Q$ . This may be straightforwardly generalised to arbitrary SMCs, including freely generated ones; therefore, these morphisms are a sound representation of circuit rewrites in arbitrary SMCs. Theorem 2.16 makes it reasonable to expect that the representation is complete, too: that every circuit rewrite valid in all, including free, SMCs is described on the order-theoretic level by a morphism of circuit shapes. As the focus of this thesis lies on quantum circuits, specifically, I will not make this formal. It however motivates the following definition and terminology, which will provide a useful way to state consequences of some of the results to follow.

**Definition 2.17.** Letting  $A$  and  $B$  be sets, denote by  $\text{Circ}_{AB}$  the category of circuit shapes from  $A$  to  $B$  and morphisms between them. Define a preorder  $\preceq$  on the objects of  $\text{Circ}_{AB}$  by

$$P \preceq Q \iff \exists f : P \rightarrow Q. \quad (2.10)$$

We denote by  $\text{Circ}_{AB}^{\preceq}$  the resulting preordered proper class, or equivalently, thin category. Two circuit shapes  $P$  and  $Q$  are **(syntactically) equivalent** if  $P \preceq Q$  and  $Q \preceq P$ .

**Example 2.18.** The two circuit shapes in Eq. (2.2) are not isomorphic but they are equivalent. A morphism from left to right is the one illustrated in Eq. (2.2) itself; a morphism in the other direction is given by mapping both elements of the circuit shape on the right to the bottom element of the circuit shape on the left. Both of these shapes are moreover equivalent to the single-gate shape with one input and two outputs. Indeed, it is clear that all these three circuit shapes have equal expressivity no matter what SMC one interprets them in (all being able to express any SMC morphism with one input and two outputs). //

**Remark 2.19.** In addition to syntactical circuit rewrites, the morphisms from Definition 2.4 can also be used to model implementations of circuits in a spacetime. More precisely, looking ahead to Chapter 4, where such implementations will play a central role, consider what we there call a *spacetime situation*: a tuple  $\mathcal{M} := (M, \preceq_g, p, q)$  consisting of a time-oriented Lorentzian manifold  $M$  without closed causal curves; the partial order  $\preceq_g$  on its points induced by the Lorentzian metric; and two functions  $p : A \rightarrow M, q : B \rightarrow M$  specifying localisations of input and output systems in the spacetime. In the terminology of this chapter,  $\mathcal{M}$  is nothing more than a special circuit shape. A morphism  $f : P \rightarrow \mathcal{M}$  from a finite circuit shape  $P$  into  $\mathcal{M}$  provides a localisation of each of  $P$ 's gates at points in spacetime in such a way that appropriate causal curves are guaranteed to exist for each  $P$ 's wires (edges of its circuit diagram). The observation that circuits and spacetimes can be treated on an equal footing in this way has also been made in Ref. [84]. //

### 2.3 Connectivity and interaction

The following notion is central to causal decompositions, the subject of Chapters 3 and 4.

**Definition 2.20.** Let  $P$  be a circuit shape from  $A$  to  $B$ . The **(input-output) connectivity** of  $P$  is the binary relation  $G_P \subseteq A \times B$  satisfying

$$\forall a \in A, b \in B : \quad a G_P b \iff \lambda_P(a) \leq \mu_P(b). \quad (2.11)$$

**Example 2.21.** Let  $A := \{a_1, a_2, a_3\}$  and  $B := \{b_1, b_2, b_3\}$  and consider the relation  $C_3 \subseteq A \times B$  defined as

$$C_3 := \begin{array}{c} b_1 \quad b_2 \quad b_3 \\ \uparrow \quad \times \quad \uparrow \quad \times \quad \uparrow \\ a_1 \quad a_2 \quad a_3 \end{array}; \quad (2.12)$$

this relation will play an important role in Chapter 3. The three circuit shapes

$$B_{C_3} := \begin{array}{c} b_1 \quad b_2 \quad b_3 \\ \square \quad \square \quad \square \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \square \quad \square \quad \square \\ \uparrow \quad \uparrow \quad \uparrow \\ a_1 \quad a_2 \quad a_3 \end{array}, \quad \text{LOSEC}_{C_3} := \begin{array}{c} b_1 \quad b_2 \quad b_3 \\ \square \quad \square \quad \square \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \square \quad \square \quad \square \\ \uparrow \quad \uparrow \quad \uparrow \\ a_1 \quad a_2 \quad a_3 \end{array}, \quad \text{and} \quad L_{C_3} := \begin{array}{c} b_1 \quad b_2 \quad b_3 \\ \square \quad \square \quad \square \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \square \quad \square \quad \square \\ \uparrow \quad \uparrow \quad \uparrow \\ a_1 \quad a_2 \quad a_3 \end{array} \quad (2.13)$$

all have connectivity  $C_3$ .  $B_{C_3}$  and  $L_{C_3}$  will be familiar from Chapter 1 and  $\text{LOSEC}_{C_3}$  will feature in Chapter 4.

There exist morphisms  $f : B_{C_3} \rightarrow \text{LOSEC}_{C_3}$  and  $g : \text{LOSEC}_{C_3} \rightarrow L_{C_3}$ .  $f$  is just the obvious inclusion map and  $g$  is a quotient by a compatible congruence, whose equivalence classes are highlighted below:

$$\begin{array}{c} b_1 \quad b_2 \quad b_3 \\ \square \quad \square \quad \square \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \square \quad \square \quad \square \\ \uparrow \quad \uparrow \quad \uparrow \\ a_1 \quad a_2 \quad a_3 \end{array} \xrightarrow{g} \begin{array}{c} b_1 \quad b_2 \quad b_3 \\ \square \quad \square \quad \square \\ \diagdown \quad \diagup \quad \diagdown \quad \diagup \\ \square \quad \square \quad \square \\ \uparrow \quad \uparrow \quad \uparrow \\ a_1 \quad a_2 \quad a_3 \end{array}. \quad (2.14)$$

(Recall from Definition 2.2 that we always depict circuit shapes by their Hasse diagrams, which are transitively reduced; this is why the wire from  $\lambda(a_2)$  to  $\mu(b_2)$  present in the left diagram seems to disappear. One may think of it as being incorporated into either the leftmost or rightmost gate of the right diagram.) As a result, the three circuit shapes are ordered as  $B_{C_3} \preceq \text{LOSEC}_{C_3} \preceq L_{C_3}$ . No morphisms exist in any of the opposite directions, however, as we will see shortly. //

If two circuit shapes have distinct connectivity relations, this provides an immediate indication of their syntactical inequivalence. Indeed:

**Proposition 2.22.** *Let  $P$  and  $Q$  be circuit shapes from  $A$  to  $B$ . If there exists a morphism  $f : P \rightarrow Q$  then  $G_P \subseteq G_Q$ . In other words, the map  $G_{(-)} : \text{Circ}_{AB}^{\prec} \rightarrow \mathcal{P}(A \times B)$  is order-preserving.<sup>6</sup> In particular, if  $P$  and  $Q$  are syntactically equivalent then  $G_P = G_Q$ .*

*Proof.* Let  $a \in A$  and  $b \in B$  be such that  $aG_P b$ , that is,  $\lambda_P(a) \leq_P \mu_P(b)$ . By the fact that  $f$  is a circuit shape morphism, it follows that  $\lambda_Q(a) \leq_Q f(\lambda_P(a)) \leq_Q f(\mu_P(b)) \leq_Q \mu_Q(b)$ , meaning that also  $aG_Q b$ .  $\square$

In order to prove that two circuit shapes with *identical* connectivity are syntactically inequivalent, on the other hand, the following will be useful.

**Definition 2.23.** Let  $(P, \leq, \lambda, \mu)$  be a circuit shape. For  $p \in P$ , we write

$$p^- := \{a \in A : \lambda(a) \leq p\} \quad \text{and} \quad p^+ := \{b \in B : \mu(b) \geq p\}. \quad (2.15)$$

Thus,  $p^-$  are the inputs that lie to  $p$ 's past, and  $p^+$  the outputs lying to  $p$ 's future.

**Proposition 2.24.** *If  $f : P \rightarrow Q$  is a morphism of circuit shapes then for any  $p \in P$ ,  $p^- \subseteq f(p)^-$  and  $p^+ \subseteq f(p)^+$ .*

*Proof.* Suppose  $a \in p^-$ : that is,  $\lambda_P(a) \leq p$ . Since  $f$  is a morphism, we have  $\lambda_Q(a) \leq f(\lambda_P(a)) \leq f(p)$ . Therefore  $a \in f(p)^-$ . The inclusion  $p^+ \subseteq f(p)^+$  follows similarly.  $\square$

**Example 2.25.** Consider again the circuit shapes  $B_{C_3}$ ,  $\text{LOSEC}_{C_3}$ , and  $L_{C_3}$  from Example 2.21. Note that  $L_{C_3}$ 's left-most gate—call it  $p \in L_{C_3}$ —satisfies  $p^- = \{a_1, a_2\}$  and  $p^+ = \{b_1, b_2\}$ . Neither of the circuit shapes  $B_{C_3}$  and  $\text{LOSEC}_{C_3}$  has a gate  $q$  with  $\{a_1, a_2\} \subseteq q^-$  and  $\{b_1, b_2\} \subseteq q^+$ , on the other hand; by Proposition 2.24, this means there is no morphism  $L_{C_3} \rightarrow \text{LOSEC}_{C_3}$  or  $L_{C_3} \rightarrow B_{C_3}$ , so that  $L_{C_3}$  is syntactically inequivalent to both other circuit shapes.

A physical intuition for this syntactical inequivalence is to regard  $p$  as a gate where all four systems  $a_1, a_2, b_1$  and  $b_2$  can meet and interact. In Chapters 3 and 4 we will see that as opposed to the classical case, the presence of such *interaction gates* can be crucial when it comes to expressing quantum channels, and that the syntactical inequivalence is witnessed by the fact that  $L_{C_3}$  can indeed express strictly the most tripartite quantum channels.

Proposition 2.24 can also be used to prove syntactical inequivalence of  $B_{C_3}$  and  $\text{LOSEC}_{C_3}$ , so in conclusion,  $B_{C_3} \prec \text{LOSEC}_{C_3} \prec L_{C_3}$ .  $\parallel$

In Section 2.4, we construct the analogue  $L_G$  of  $L_{C_3}$  for general relations  $G \subseteq A \times B$ , which we show in fact admits incoming morphisms from *all* other circuit shapes  $P$  with connectivity  $G_P \subseteq G$ . We generalise  $B_{C_3}$  in Section 2.5, which satisfies the dual property, admitting outgoing morphisms into any circuit shape  $Q$  with connectivity  $G_Q \supseteq G$ .

<sup>6</sup>In other words still, it defines a functor  $G_{(-)} : \text{Circ}_{AB} \rightarrow \mathcal{P}(A \times B)$  when regarding  $\mathcal{P}(A \times B)$  as a preorder category. However, some of our results below (esp. Corollaries 2.38 and 2.43) are phrased more naturally in terms of the preorder category  $\text{Circ}_{AB}^{\prec}$ .

## 2.4 Characterisation of the concept lattice

Let  $A$  and  $B$  be arbitrary sets. From each binary relation  $G \subseteq A \times B$  one may construct a complete lattice via a construction first realised by Birkhoff [17] and extensively studied in formal concept analysis [46, 115]. We adopt the terminology from [46, 115] and call this the *concept lattice*. We will briefly recall this construction and then show, in Theorem 2.33, that with the natural choice of input and output maps, it constitutes the smallest circuit shape with connectivity  $G$  that admits incoming morphisms from every other circuit shape with at most the same connectivity.

We will use the notation

$$G(a) := \{b \in B \mid aGb\} \quad \text{for } a \in A \quad \text{and} \quad G^{-1}(b) := \{a \in A \mid aGb\} \quad \text{for } b \in B. \quad (2.16)$$

In the construction below, an important role is played by intersections of sets of the form  $G(a)$  and  $G^{-1}(b)$ . Let us start by giving some intuition for why these intersections are relevant when constructing a circuit shape that is as ‘expressive’ as possible. Imagine that  $P$  is a circuit shape with connectivity  $G$  and consider a gate  $q \in P$ . Let  $q^- \subseteq A$  be the set of overall input wires that lie to  $q$ ’s past, as in Definition 2.23. Consider also the outputs  $q^+ \subseteq B$  lying to  $q$ ’s future. Because of the fact that  $P$  has connectivity  $G$ , we have  $aGb$  for each  $a \in q^-, b \in q^+$ : in other words,  $q^- \subseteq \bigcap_{b \in q^+} G^{-1}(b)$ . Denote the latter set by  $\mathbf{p}(q^+)$ . Suppose, now, that the inclusion  $q^- \subseteq \mathbf{p}(q^+)$  is strict: that there is  $a' \in \mathbf{p}(q^+) \setminus q^-$ . Then  $a'Gb$  for all  $b \in q^+$ . This means that adding a path from  $a'$  to the gate  $q$  yields a circuit shape whose connectivity remains  $G$ . Doing this for all  $a' \in \mathbf{p}(q^+) \setminus q^-$  we obtain a new circuit shape  $P'$  with respect to which  $q^- = \mathbf{p}(q^+)$  and which is at least as expressive as  $P$ . By a dual procedure, we can ensure that also the inclusion  $q^+ \subseteq \bigcap_{a \in q^-} G(a) =: \mathbf{c}(q^-)$  is saturated. In agreement with this intuition, each gate in the circuit shape constructed below corresponds precisely to a pair  $\alpha \subseteq A, \beta \subseteq B$  so that  $\alpha = \mathbf{p}(\beta)$  and  $\beta = \mathbf{c}(\alpha)$ .

Formally, now, denote by  $\mathcal{P}(A)$  and  $\mathcal{P}(B)$  the powersets of  $A$  and  $B$  and consider the maps<sup>7</sup>

$$\mathbf{c} : \mathcal{P}(A) \rightarrow \mathcal{P}(B) :: \alpha \mapsto \bigcap_{a \in \alpha} G(a) = \{b \in B \mid \forall a \in \alpha : aGb\} \quad (2.17)$$

$$\text{and } \mathbf{p} : \mathcal{P}(B) \rightarrow \mathcal{P}(A) :: \beta \mapsto \bigcap_{b \in \beta} G^{-1}(b) = \{a \in A \mid \forall b \in \beta : aGb\}. \quad (2.18)$$

In particular,  $\mathbf{c}(\emptyset) = B$  and  $\mathbf{p}(\emptyset) = A$ . With  $\mathcal{P}(A)$  and  $\mathcal{P}(B)$  ordered under inclusion, these maps are order-reversing:

$$\begin{aligned} \forall \alpha, \alpha' \subseteq A : \alpha \subseteq \alpha' &\implies \mathbf{c}(\alpha) \supseteq \mathbf{c}(\alpha') \\ \text{and } \forall \beta, \beta' \subseteq B : \beta \subseteq \beta' &\implies \mathbf{p}(\beta) \supseteq \mathbf{p}(\beta'). \end{aligned} \quad (2.19)$$

They also satisfy

$$\forall \alpha \subseteq A : \alpha \subseteq \mathbf{p}\mathbf{c}(\alpha) \quad \text{and} \quad \forall \beta \subseteq B : \beta \subseteq \mathbf{c}\mathbf{p}(\beta). \quad (2.20)$$

<sup>7</sup>The letters  $\mathbf{c}$  and  $\mathbf{p}$  we use here stand for ‘children’ and ‘parents’, respectively—customary terminology in causality parlance, where we will ultimately apply these constructions.

Eqs. (2.19) and (2.20) make the pair  $(\mathbf{c}, \mathbf{p})$  into an (antitone) **Galois connection** between the partially ordered sets  $\mathcal{P}(\mathbf{A})$  and  $\mathcal{P}(\mathbf{B})$  [72].<sup>8</sup> A direct consequence is that

$$\mathbf{p}\mathbf{c}\mathbf{p} = \mathbf{p} \quad \text{and} \quad \mathbf{c}\mathbf{p}\mathbf{c} = \mathbf{c}. \quad (2.21)$$

Moreover, the map  $\mathbf{p}\mathbf{c} : \mathcal{P}(\mathbf{A}) \rightarrow \mathcal{P}(\mathbf{A})$  acts as a *closure operator* on  $\mathcal{P}(\mathbf{A})$ , meaning it is

- (i) extensive:  $\forall \alpha \subseteq \mathbf{A} : \alpha \subseteq \mathbf{p}\mathbf{c}(\alpha)$ ;
- (ii) idempotent:  $\mathbf{p}\mathbf{c}\mathbf{p}\mathbf{c} = \mathbf{p}\mathbf{c}$ ; and
- (iii) order-preserving:  $\forall \alpha, \alpha' \subseteq \mathbf{A} : \alpha \subseteq \alpha' \implies \mathbf{p}\mathbf{c}(\alpha) \subseteq \mathbf{p}\mathbf{c}(\alpha')$ .

We therefore say that  $\mathbf{p}\mathbf{c}(\alpha)$  is the **closure** of  $\alpha \subseteq \mathbf{A}$  and that  $\alpha$  is **closed** if  $\mathbf{p}\mathbf{c}(\alpha) = \alpha$ . It follows from (i)–(iii) that the closure of a set is precisely the smallest closed set containing it and that the intersection of (arbitrarily many) closed sets is again closed. Moreover, by virtue of Eq. (2.21), the closed subsets of  $\mathbf{A}$  are precisely the sets  $\alpha$  that lie in the image of  $\mathbf{p}$ .

**Example 2.26.** With respect to the relation  $C_3$  defined in Example 2.21, the closure of  $\{\mathbf{a}_1\} \subseteq \mathbf{A}$  is  $\{\mathbf{a}_1, \mathbf{a}_2\}$ , while  $\{\mathbf{a}_2\}$  is itself closed. //

**Definition 2.27.** Let  $(\mathbf{P}, \leq)$  be a partial order and  $S \subseteq \mathbf{P}$  a subset. A **greatest lower bound** or **meet** of  $S$  is an element  $p \in \mathbf{P}$  satisfying

$$\forall q \in \mathbf{P} : (\forall s \in S : q \leq s) \iff q \leq p. \quad (2.22)$$

If it exists, the meet is unique and denoted by  $\bigwedge S$  or  $\bigwedge_{s \in S} s$ . The **least upper bound** or **join**  $\bigvee S \equiv \bigvee_{s \in S} s$  is defined dually. A **lattice** is a partial order  $(\mathbf{P}, \leq)$  in which each finite subset  $S$  (or, equivalently, each pair of elements) has a meet as well as a join; a **complete lattice** is one for which this holds for arbitrary (potentially infinite) subsets.

Let  $\mathcal{P}^c(\mathbf{A}) \subseteq \mathcal{P}(\mathbf{A})$  be the set of closed subsets of  $\mathbf{A}$ .  $\mathcal{P}^c(\mathbf{A})$  is closed under arbitrary intersections; this implies that it forms a complete lattice when ordered under inclusion [17]. The meets and joins are given, respectively, by

$$\bigwedge_{\alpha \in S} \alpha = \bigcap_{\alpha \in S} \alpha \quad \text{and} \quad \bigvee_{\alpha \in S} \alpha = \mathbf{p}\mathbf{c} \left( \bigcup_{\alpha \in S} \alpha \right). \quad (2.23)$$

Dually, the map  $\mathbf{c}\mathbf{p} : \mathcal{P}(\mathbf{B}) \rightarrow \mathcal{P}(\mathbf{B})$  satisfies conditions analogous to (i)–(iii) above and is therefore a closure operator on  $\mathcal{P}(\mathbf{B})$ . Let  $\mathcal{P}^c(\mathbf{B})$  be the set of closed subsets of  $\mathbf{B}$  with respect to this closure operator. Just like  $\mathcal{P}^c(\mathbf{A})$ ,  $(\mathcal{P}^c(\mathbf{B}), \subseteq)$  forms a complete lattice; so does the opposite order  $\mathcal{P}^c(\mathbf{B})^{\text{op}} := (\mathcal{P}^c(\mathbf{B}), \supseteq)$ . Note that  $\mathcal{P}^c(\mathbf{A})$  and  $\mathcal{P}^c(\mathbf{B})$  are the images of the operators  $\mathbf{p}$  and  $\mathbf{c}$ , respectively; therefore, Eq. (2.21) implies that  $\mathbf{c}$  and  $\mathbf{p}$  restrict to a pair of inverse maps  $\mathbf{c} : \mathcal{P}^c(\mathbf{A}) \rightarrow \mathcal{P}^c(\mathbf{B})^{\text{op}}$  and  $\mathbf{p} : \mathcal{P}^c(\mathbf{B})^{\text{op}} \rightarrow \mathcal{P}^c(\mathbf{A})$ , which due to Eq. (2.19) are in fact order isomorphisms. The concept lattice, defined as follows, takes a symmetric approach, incorporating both  $\mathcal{P}^c(\mathbf{A})$  and  $\mathcal{P}^c(\mathbf{B})^{\text{op}}$ .

<sup>8</sup>In categorical terms, this means the maps form a dual adjunction between the orders  $\mathcal{P}(\mathbf{A})$  and  $\mathcal{P}(\mathbf{B})$  seen as thin categories.

**Definition 2.28** ([46, 115]). Let  $G \subseteq \mathbf{A} \times \mathbf{B}$ . The **concept lattice**<sup>9</sup>  $\mathbf{L}_G$  is the set

$$\mathbf{L}_G := \{\langle \alpha, \beta \rangle \in \mathcal{P}^c(\mathbf{A}) \times \mathcal{P}^c(\mathbf{B}) \mid \alpha = \mathbf{p}(\beta) \text{ and } \beta = \mathbf{c}(\alpha)\} \quad (2.24)$$

ordered by

$$\langle \alpha, \beta \rangle \leq \langle \alpha', \beta' \rangle \quad :\Longleftrightarrow \quad \alpha \subseteq \alpha' \text{ or, equivalently, } \beta \supseteq \beta'. \quad (2.25)$$

For arbitrary  $v \in \mathbf{L}_G$ , we will denote by  $\alpha_v \subseteq \mathbf{A}$  and  $\beta_v \subseteq \mathbf{B}$  the closed sets so that  $v = \langle \alpha_v, \beta_v \rangle$ .  $\mathbf{L}_G$  forms a complete lattice: the meet and join of a subset  $S \subseteq \mathbf{L}_G$  are, respectively, given by

$$\bigwedge_{v \in S} v = \left\langle \bigcap_{v \in S} \alpha_v, \mathbf{cp} \left( \bigcup_{v \in S} \beta_v \right) \right\rangle \quad \text{and} \quad \bigvee_{v \in S} v = \left\langle \mathbf{pc} \left( \bigcup_{v \in S} \alpha_v \right), \bigcap_{v \in S} \beta_v \right\rangle. \quad (2.26)$$

Together with the maps

$$\begin{aligned} \lambda_{\mathbf{L}_G} : \mathbf{A} &\rightarrow \mathbf{L}_G, & \mathbf{a} &\mapsto \langle \mathbf{pc}(\{\mathbf{a}\}), \mathbf{c}(\{\mathbf{a}\}) \rangle \\ \text{and } \mu_{\mathbf{L}_G} : \mathbf{B} &\rightarrow \mathbf{L}_G, & \mathbf{b} &\mapsto \langle \mathbf{p}(\{\mathbf{b}\}), \mathbf{cp}(\{\mathbf{b}\}) \rangle, \end{aligned} \quad (2.27)$$

it defines a circuit shape  $\mathbf{L}_G \equiv (\mathbf{L}_G, \leq, \lambda_{\mathbf{L}_G}, \mu_{\mathbf{L}_G})$ , which we shall also refer to as the concept lattice.

**Remark 2.29.**  $\mathcal{P}^c(\mathbf{A})$ ,  $\mathcal{P}^c(\mathbf{B})^{\text{op}}$ , and  $\mathbf{L}_G$  defined above provide three different, isomorphic perspectives on the concept lattice. Denote by  $\pi_{\mathbf{A}} : \mathbf{L}_G \rightarrow \mathcal{P}^c(\mathbf{A})$  the map  $v \mapsto \alpha_v$ ; this is an order isomorphism whose inverse is  $\alpha \mapsto \langle \alpha, \mathbf{c}(\alpha) \rangle$ . If we equip  $\mathcal{P}^c(\mathbf{A})$  with the input and output maps  $\lambda_{\mathcal{P}^c(\mathbf{A})} := \pi_{\mathbf{A}} \circ \lambda_{\mathbf{L}_G} = \mathbf{pc}(\{-\})$  and  $\mu_{\mathcal{P}^c(\mathbf{A})} := \pi_{\mathbf{A}} \circ \mu_{\mathbf{L}_G} = \mathbf{c}(\{-\})$ , then  $\pi_{\mathbf{A}}$  becomes an isomorphism of circuit shapes. We can do the same to  $\mathcal{P}^c(\mathbf{B})^{\text{op}}$  using the map  $\pi_{\mathbf{B}} : \mathbf{L}_G \rightarrow \mathcal{P}^c(\mathbf{B})^{\text{op}}$  defined by  $v \mapsto \beta_v$ , which has inverse  $\beta \mapsto \langle \mathbf{p}(\beta), \beta \rangle$ . The following diagram of circuit shape isomorphisms then commutes.

$$\begin{array}{ccc} & \mathbf{L}_G & \\ \pi_{\mathbf{A}} \swarrow & & \searrow \pi_{\mathbf{B}} \\ \mathcal{P}^c(\mathbf{A}) & \xrightleftharpoons[\mathbf{p}]{\mathbf{c}} & \mathcal{P}^c(\mathbf{B})^{\text{op}} \end{array} \quad (2.28)$$

//

**Example 2.30.** It may be verified that  $\mathbf{L}_{C_3}$  in Example 2.21 is the concept lattice associated with relation  $C_3$ . Here we illustrate the construction of  $\mathbf{L}_G$  in a slightly more

<sup>9</sup>In formal concept analysis,  $\mathbf{A}$  is a set of objects,  $\mathbf{B}$  a set of properties, and  $G$  the relation specifying which objects have which properties; together, this triple is referred to as a *formal context*. An element  $\langle \alpha, \beta \rangle \in \mathbf{L}_G$  is then called a *concept*: it defines a set of objects ( $\alpha$ ) that is uniquely characterised by the set of properties ( $\beta$ ) that they share (i.e.  $\alpha = \mathbf{p}(\beta)$  and  $\beta = \mathbf{c}(\alpha)$ ).

involved case. Consider the relation between  $\mathbf{A} = \{1, 2, 3, 4\}$  and  $\mathbf{B} = \{a, b, c, d, e\}$  given by

$$G := \begin{array}{ccccc} & a & b & c & d & e \\ & \swarrow & \searrow & \swarrow & \searrow & \swarrow \\ 1 & & 2 & & 3 & & 4 \end{array} . \quad (2.29)$$

We will abbreviate subsets like  $\{2, 3, 4\} \subseteq \mathbf{A}$  by ‘234’. Recall that the closed subsets of  $\mathbf{A}$  are those of the form  $\mathbf{p}(\beta)$  for some  $\beta \subseteq \mathbf{B}$ ; these are precisely the set  $\mathbf{A}$  itself (1234), the sets of the form  $G^{-1}(\mathbf{b})$  for  $\mathbf{b} \in \mathbf{B}$  (12, 234, 34, and 4), and all their intersections (2 and  $\emptyset$ ). Taken together, we get  $\mathcal{P}^c(\mathbf{A}) = \{\emptyset, 2, 12, 4, 34, 234, 1234\}$ . We can now obtain  $\mathcal{P}^c(\mathbf{B})$  by applying the map  $\mathbf{c}$  to each of the closed sets  $\alpha \in \mathcal{P}^c(\mathbf{A})$ ; we get  $\mathcal{P}^c(\mathbf{B}) = \{abcde, abc, ab, cde, cd, c, \emptyset\}$ , respectively.

Ordering  $\mathcal{P}^c(\mathbf{A})$  under  $\subseteq$  and  $\mathcal{P}^c(\mathbf{B})$  under  $\supseteq$ , these sets are represented by the Hasse diagrams

$$(\mathcal{P}^c(\mathbf{A}), \subseteq) = \begin{array}{c} 1234 \\ \swarrow \quad \searrow \\ 12 \quad 234 \\ \swarrow \quad \searrow \\ 2 \quad 34 \\ \swarrow \quad \searrow \\ \emptyset \quad 4 \end{array} \quad \text{and} \quad (\mathcal{P}^c(\mathbf{B}), \supseteq) = \begin{array}{c} \emptyset \\ \swarrow \quad \searrow \\ ab \quad c \\ \swarrow \quad \searrow \\ abc \quad cd \\ \swarrow \quad \searrow \\ abcde \quad cde \end{array} \quad (2.30)$$

with  $\mathbf{p}, \mathbf{c}$  acting as mutually inverse order-isomorphisms between the two posets. The corresponding concept lattice for  $G$  is therefore

$$\begin{array}{c} \langle 1234, \emptyset \rangle \\ \swarrow \quad \searrow \\ \langle 12, ab \rangle \quad \langle 234, c \rangle \\ \swarrow \quad \searrow \\ \langle 2, abc \rangle \quad \langle 34, cd \rangle \\ \swarrow \quad \searrow \\ \langle \emptyset, abcde \rangle \quad \langle 4, cde \rangle \\ \swarrow \quad \searrow \\ \langle \emptyset, abcde \rangle \end{array} . \quad (2.31)$$

We additionally need the maps  $\lambda_{L_G}$  and  $\mu_{L_G}$  providing the location of the input and output wires. To calculate  $\lambda_{L_G}(3)$ , for instance, one could explicitly compute the closure  $\mathbf{pc}(\{3\}) = \mathbf{p}(\{c, d\}) = \{3, 4\}$ , or use the fact that  $\lambda_{L_G}(3)$  is the smallest element  $v \in L_G$  so that  $3 \in \alpha_v$  (cf. Proposition 2.32 below). From Eq. (2.31) this is easily read off to be the pair  $v = \langle 34, cd \rangle$ . Similarly, for  $\mathbf{b} \in \mathbf{B}$ ,  $\mu_{L_G}(\mathbf{b})$  is the largest element  $w \in L_G$  that satisfies  $\mathbf{b} \in \beta_w$ . The resulting circuit diagram of the concept lattice  $L_G$  is

$$L_G = \begin{array}{c} \begin{array}{ccccc} a & b & & c & \\ & & & & d \\ & & & & e \end{array} \\ \begin{array}{c} \boxed{1234, \emptyset} \\ \swarrow \quad \searrow \\ \boxed{12, ab} \quad \boxed{234, c} \\ \swarrow \quad \searrow \\ \boxed{2, abc} \quad \boxed{34, cd} \\ \swarrow \quad \searrow \\ \boxed{\emptyset, abcde} \quad \boxed{4, cde} \\ \swarrow \quad \searrow \\ \boxed{\emptyset, abcde} \end{array} \end{array} \quad (2.32) \quad //$$

**Remark 2.31.** In the next chapter, where the boxes in circuit diagrams will be interpreted as completely positive, trace-preserving quantum gates, the only possibility for a gate without output wires, like  $\langle 1234, \emptyset \rangle$  above, is to assign it a discarding operation. This box therefore does not add to the expressivity of the circuit shape. The same holds for the bottom element  $\langle \emptyset, abcde \rangle$  in the case of unitary circuits. However,  $L_G$ , by virtue of being a complete lattice, always contains a minimum and maximum, and to avoid having to make unnecessary case distinctions we will leave these gates in even if they semantically useless. //

The following Proposition lists some properties of the concept lattice useful to have at hand and which, in the finite case, are all manifested as intuitive properties of the circuit diagram.

**Proposition 2.32.** *Let  $G \subseteq A \times B$  be a binary relation and  $L_G$  the canonical circuit shape.*

(i) *For all  $\mathbf{a} \in A$ ,  $\mathbf{b} \in B$ , and  $v \in L_G$ , we have*

$$\lambda_{L_G}(\mathbf{a}) \leq v \iff \mathbf{a} \in \alpha_v \quad \text{and} \quad v \leq \mu_{L_G}(\mathbf{b}) \iff \mathbf{b} \in \beta_v. \quad (2.33)$$

*In other words, using the notation of Eq. (2.15),  $v^- = \alpha_v$  and  $v^+ = \beta_v$ . As a consequence,  $\lambda_{L_G}(\mathbf{a}) = \min\{v \in L_G \mid \mathbf{a} \in \alpha_v\}$  and  $\mu_{L_G}(\mathbf{b}) = \max\{v \in L_G \mid \mathbf{b} \in \beta_v\}$ .*

(ii)  *$L_G$  has connectivity  $G$ .*

(iii) *For all  $v \in L_G$ , we have*

$$v = \bigvee_{\mathbf{a} \in \alpha_v} \lambda_{L_G}(\mathbf{a}) = \bigwedge_{\mathbf{b} \in \beta_v} \mu_{L_G}(\mathbf{b}); \quad (2.34)$$

*in particular, these meets and joins exist.*<sup>10</sup>

(iv) *For all  $v \in L_G$ , we have*

$$\alpha_v = \lambda_{L_G}^{-1}(v) \cup \bigcup_{u < v} \alpha_u \quad \text{and} \quad \beta_v = \mu_{L_G}^{-1}(v) \cup \bigcup_{w > v} \beta_w. \quad (2.35)$$

*In terms of the circuit diagram, each  $\mathbf{b} \in \beta_v$  is either an output wire of the gate  $v$  itself, or an output that can be reached from  $v$  by a path passing through some  $w > v$ .*

Item (iv) will be useful to keep in mind in Chapter 3.

*Proof.* We have

$$\lambda_{L_G}(\mathbf{a}) \leq v \iff \mathbf{pc}(\{\mathbf{a}\}) \subseteq \alpha_v \iff \{\mathbf{a}\} \subseteq \alpha_v \iff \mathbf{a} \in \alpha_v. \quad (2.36)$$

<sup>10</sup>That these meets and joins exist is of course immediate for the complete lattice  $L_G$ , but not for  $\hat{L}_G$ , which is generally not a lattice.

The second equivalence here follows from extensivity (2.20) (left to right) and idempotency (right to left) of the closure operator  $\mathbf{pc}$ , as well as the fact that  $\alpha_v$  is closed. This, and a dual argument for  $\mu_{L_G}(\mathbf{b})$ , proves (i).

Item (ii) is a restatement of a basic result of formal concept analysis [46, Thm. 3], and follows from (i):

$$\lambda_{L_G}(\mathbf{a}) \leq \mu_{L_G}(\mathbf{b}) \iff \mathbf{a} \in \alpha_{\mu_{L_G}(\mathbf{b})} \iff \mathbf{a} \in \mathbf{p}(\{\mathbf{b}\}) \iff \mathbf{a}G\mathbf{b}. \quad (2.37)$$

As a consequence of Eq. (2.36), any  $v \in L_G$  is an upper bound to the set  $\{\lambda_{L_G}(\mathbf{a}) \mid \mathbf{a} \in \alpha_v\}$  and must therefore be at least as large as the least upper bound  $\bigvee_{\mathbf{a} \in \alpha_v} \lambda_{L_G}(\mathbf{a})$ . The converse statement that  $v \leq \bigvee_{\mathbf{a} \in \alpha_v} \lambda_{L_G}(\mathbf{a})$  is equivalent to saying that  $\alpha_v \subseteq \mathbf{pc}(\bigcup_{\mathbf{a} \in \alpha_v} \mathbf{pc}(\{\mathbf{a}\}))$ , which is immediate from extensivity of  $\mathbf{pc}$ . This, together with a dual argument, proves (iii). Finally, (iv) follows directly from (i).  $\square$

We now state and prove the main result of this chapter.

**Theorem 2.33.** *Let  $\mathbf{A}$  and  $\mathbf{B}$  be sets and  $G \subseteq \mathbf{A} \times \mathbf{B}$ . The concept lattice  $L_G$  is, up to circuit shape isomorphisms, the unique circuit shape satisfying the following three properties.*

- (i)  $L_G$  has connectivity  $G_{L_G} = G$ .
- (ii) For any circuit shape  $P$  with connectivity  $G_P \subseteq G$ , there exists a morphism  $f : P \rightarrow L_G$ .<sup>11</sup>
- (iii) If  $K$  is any other circuit shape satisfying (i) and (ii), then there is an injective morphism  $L_G \rightarrow K$ .

In fact, for any circuit shape  $Q$  with connectivity  $G$  and morphism  $g : L_G \rightarrow Q$ ,  $g$  is an embedding.

The concept lattice is not uniquely characterised by only properties (i) and (ii): in fact, any circuit shape whose underlying order is a complete lattice satisfies them. Intuitively, the joins and meets of such complete lattices act as the ‘interaction gates’ necessary for this to be the case (cf. Example 2.25).

**Lemma 2.34.** *Let  $\mathbf{A}$  and  $\mathbf{B}$  be sets,  $G \subseteq \mathbf{A} \times \mathbf{B}$  a relation, and  $L$  a circuit shape from  $\mathbf{A}$  to  $\mathbf{B}$  which has connectivity  $G$  and whose underlying partial order  $(L, \leq_L)$  is a complete lattice. Then  $L$  admits incoming morphisms  $f : P \rightarrow L$  from any other circuit shape  $P$  with connectivity  $G_P \subseteq G$ .*

*Proof.* Let  $P$  be a circuit shape with connectivity  $G_P \subseteq G$  and consider the map

$$f : P \rightarrow L, \quad p \mapsto \bigvee_{\mathbf{a} : \lambda_P(\mathbf{a}) \leq p} \lambda_L(\mathbf{a}) = \bigvee_{\mathbf{a} \in p^-} \lambda_L(\mathbf{a}), \quad (2.38)$$

<sup>11</sup>Changing the inclusion  $G_P \subseteq G$  in (ii) to an equality (as would be more natural from the point of view of causally *faithful* decompositions to be discussed in Section 3.4.1) would not change validity of the theorem and while it might seem to make the uniqueness statement stronger, it is not difficult to see that this in fact not the case.

where  $\bigvee$  is the join in  $L$  and  $p^-$  is as in Eq. (2.15). This map is order-preserving:

$$p \leq_P q \implies p^- \subseteq q^- \implies \bigvee_{a \in p^-} \lambda_L(a) \leq \bigvee_{a \in q^-} \lambda_L(a). \quad (2.39)$$

It also respects inputs: for all  $a \in A$ ,

$$f(\lambda_P(a)) = \bigvee_{a': \lambda_P(a') \leq \lambda_P(a)} \lambda_L(a') \geq \lambda_L(a). \quad (2.40)$$

Finally,  $f$  respects outputs: since  $G_P \subseteq G = G_L$ , we have  $\lambda_P(a) \leq \mu_P(b) \implies \lambda_L(a) \leq \mu_L(b)$ , so that for all  $b \in B$ ,

$$f(\mu_P(b)) = \bigvee_{a: \lambda_P(a) \leq \mu_P(b)} \lambda_L(a) \leq \bigvee_{a: \lambda_L(a) \leq \mu_L(b)} \lambda_L(a) \leq \mu_L(b). \quad (2.41)$$

We conclude that  $f : P \rightarrow L$  is a circuit shape morphism. A dual construction, which instead sets  $f(p) := \bigwedge_{b: \mu_P(b) \geq p} \mu_L(b)$ , yields a generally distinct morphism.  $\square$

The following will allow us to single out the concept lattice in particular.

**Lemma 2.35.** *Every circuit shape endomorphism of  $L_G$  is trivial:  $\text{End}(L_G) = \{\text{id}_{L_G}\}$ .*

*Proof.* Let  $f : L_G \rightarrow L_G$  be a morphism. Since  $f$  is order-preserving, we have, for all  $S \subseteq L_G$ ,

$$f\left(\bigvee S\right) \geq \bigvee f(S) \quad \text{and} \quad f\left(\bigwedge S\right) \leq \bigwedge f(S). \quad (2.42)$$

Moreover, since  $f$  is a morphism, it satisfies  $f(\lambda_{L_G}(a)) \geq \lambda_{L_G}(a)$  and  $f(\mu_{L_G}(b)) \leq \mu_{L_G}(b)$  for all  $a \in A$ ,  $b \in B$ . Combining these facts with Proposition 2.32(iii) gives that for  $v \in L_G$ ,

$$f(v) = f\left(\bigvee_{a \in \alpha_v} \lambda_{L_G}(a)\right) \geq \bigvee_{a \in \alpha_v} f(\lambda_{L_G}(a)) \geq \bigvee_{a \in \alpha_v} \lambda_{L_G}(a) = v \quad (2.43)$$

and, dually,

$$f(v) = f\left(\bigwedge_{b \in \beta_v} \mu_{L_G}(b)\right) \leq \bigwedge_{b \in \beta_v} f(\mu_{L_G}(b)) \leq \bigwedge_{b \in \beta_v} \mu_{L_G}(b) = v. \quad (2.44)$$

Therefore  $f(v) = v$ .  $\square$

*Proof of Theorem 2.33.* The fact that  $L_G$  has connectivity  $G$  was shown in Proposition 2.32, and that (ii) is satisfied by  $L_G$  is a direct consequence of Lemma 2.34 above. Item (iii) follows from the final claim in the theorem statement, so let us prove the latter. Let  $Q$  be a circuit shape with connectivity  $G$  and  $g : L_G \rightarrow Q$  be a morphism. By (ii), there also exists a morphism  $f : Q \rightarrow L_G$ . The composite  $f \circ g$  is an endomorphism of  $L_G$  and is therefore, by Lemma 2.35, the identity morphism  $\text{id}_{L_G}$ . Thus,  $g$  has a left-inverse  $f$  which is, in particular, order-preserving; this means that  $g$  must be an order embedding and thus a circuit shape embedding.

It remains to show that  $L_G$  is the unique circuit shape, up to isomorphism, satisfying (i)–(iii). The proof is similar: Suppose that  $K$  also satisfies these properties; then there exist morphisms  $f : K \rightarrow L_G$  and  $g : L_G \rightarrow K$  which are now both injective. The composite  $f \circ g : L_G \rightarrow L_G$  is an endomorphism of  $L_G$  and therefore, again, equal to the identity  $\text{id}_{L_G}$ .  $f$  must therefore be surjective; since it is also injective, it is a bijection, meaning  $g$  is not only a right-inverse but also a two-sided inverse. As  $f$  and  $g$  are both circuit shape morphisms, they are in fact circuit shape isomorphisms.  $\square$

**Remark 2.36.** The Basic Theorem of Formal Concept Analysis [115], [46, Thm. 3] tells us that any complete lattice  $(L, \leq)$  arises as the concept lattice of *some* binary relation  $G$ . More precisely, a circuit shape  $(L, \leq, \lambda, \mu)$  whose underlying order  $(L, \leq)$  is a complete lattice is isomorphic to the concept lattice  $L_G$  iff the connectivity of  $L$  is  $G$  and the images  $\lambda(A), \mu(B) \subseteq L$  are join- and meet-dense in  $L$ , respectively, which in the case of  $L_G$  itself is guaranteed by Proposition 2.32(iii). //

**Remark 2.37.** Another natural circuit shape whose connectivity is  $G \subseteq A \times B$  is the one with underlying partial order  $\mathcal{P}(A)$ , input map  $\lambda : a \mapsto \{a\}$  and output map  $\mu : b \mapsto G^{-1}(b)$ . A dual construction yields a (generally non-isomorphic) circuit shape on the partial order  $\mathcal{P}(B)^{\text{op}}$ . Both these circuit shapes have connectivity  $G$  and have, like the concept lattice  $L_G$ , incoming morphisms from every other shape with connectivity at most  $G$  (after all, powersets are complete lattices); they are therefore syntactically equivalent to  $L_G$ . However, these circuit shapes generally contain many redundant gates:  $L_G$  embeds into both of them (both by construction and as a consequence of Theorem 2.33). //

From a category-theoretic point of view, the formulation of Theorem 2.33 may perhaps seem a little unsatisfactory: one might have hoped that the morphisms in items (ii) or (iii) were unique, in which case the theorem could have been compactly stated as a characterisation of  $L_G$  as an appropriate (co)limit in  $\text{Circ}_{AB}$ . Circuit shape morphisms are however generally highly non-unique (see for instance the proof of Lemma 2.34). The consequence of Theorem 2.33 for the preorder  $\text{Circ}_{AB}^{\preceq}$  from Definition 2.17, on the other hand—obtained by forgetting the distinction between any parallel morphisms—can be naturally stated as the fact that  $L_G$  is a maximum of the subclass of circuit shapes with connectivity at most  $G$ . We can actually say a bit more:

**Corollary 2.38.** *For any circuit shape  $P$  and relation  $G \subseteq A \times B$ , we have*

$$G_P \subseteq G \iff P \preceq L_G. \quad (2.45)$$

*As a result, the map  $L_{(-)} : \mathcal{P}(A \times B) \rightarrow \text{Circ}_{AB}^{\preceq}$  is order-preserving and forms an upper Galois adjoint to the map  $G_{(-)} : \text{Circ}_{AB}^{\preceq} \rightarrow \mathcal{P}(A \times B)$ .*

*Proof.* The left-to-right direction follows from Theorem 2.33; the right-to-left direction from Proposition 2.22.  $\square$

For the same reason as described above (the category  $\text{Circ}_{\mathbf{A}\mathbf{B}}$  having too many morphisms) this Galois adjunction cannot be promoted to an adjunction of categories  $\mathcal{P}(\mathbf{A} \times \mathbf{B}) \rightarrow \text{Circ}_{\mathbf{A}\mathbf{B}}$ . Note that the statement of Corollary 2.38 characterises  $\mathbf{L}_G$  up to syntactical equivalence (Definition 2.17), but not up to isomorphism; the latter requires information about morphisms, like item (iii) of Theorem 2.33, not present in the preorder  $\text{Circ}_{\mathbf{A}\mathbf{B}}^{\leq}$ .

## 2.5 A dual construction

To conclude this chapter, we construct another circuit shape with a given connectivity relation  $G \subseteq \mathbf{A} \times \mathbf{B}$ , denoted  $\mathbf{B}_G$ , which generalises  $\mathbf{B}_{C_3}$  from Example 2.21 and is relevant to the expression of classical functions (Eq. (1.2)). Dually to  $\mathbf{L}_G$ , it admits morphisms *into* all other circuit shapes with connectivity *at least*  $G$ . There is one obvious candidate: the circuit shape  $(\mathbf{A} \sqcup \mathbf{B}, \leq, \iota_{\mathbf{A}}, \iota_{\mathbf{B}})$  whose underlying set is the disjoint union  $\mathbf{A} \sqcup \mathbf{B}$ ; whose order  $\leq$  satisfies  $p < q$  iff  $p \in \mathbf{A}, q \in \mathbf{B}$  and  $pGq$ ; and whose input and output maps are the inclusion maps  $\iota_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbf{A} \sqcup \mathbf{B}$  and  $\iota_{\mathbf{B}} : \mathbf{B} \rightarrow \mathbf{A} \sqcup \mathbf{B}$ , respectively. For a general relation  $G$ , however, this shape may contain unnecessarily many gates. We would like  $\mathbf{B}_G$  to be the *smallest* circuit shape with connectivity  $G$  admitting morphisms into all other shapes with the same connectivity, i.e. satisfying a condition analogous to Theorem 2.33(iii). This requires taking some edge-cases into account, leading us to the following definition.

**Definition 2.39.** Let  $G \subseteq \mathbf{A} \times \mathbf{B}$  and define

$$\bar{\mathbf{A}} := \{\mathbf{a} \in \mathbf{A} \mid |G(\mathbf{a})| \neq 1\} \quad \text{and} \quad (2.46)$$

$$\bar{\mathbf{B}} := \{\mathbf{b} \in \mathbf{B} \mid |G^{-1}(\mathbf{b})| \neq 1 \text{ or } \forall \mathbf{a} \in G^{-1}(\mathbf{b}) : |G(\mathbf{a})| = 1\}. \quad (2.47)$$

Denote by  $\mathbf{B}_G$  the circuit shape  $(\bar{\mathbf{A}} \sqcup \bar{\mathbf{B}}, \leq, \lambda_{\mathbf{B}_G}, \mu_{\mathbf{B}_G})$ , where  $\leq$  is the partial order satisfying  $p < q$  iff  $p \in \bar{\mathbf{A}}, q \in \bar{\mathbf{B}}$ , and  $pGq$ ; and where  $\lambda_{\mathbf{B}_G} : \mathbf{A} \rightarrow \bar{\mathbf{A}} \sqcup \bar{\mathbf{B}}$  and  $\mu_{\mathbf{B}_G} : \mathbf{B} \rightarrow \bar{\mathbf{A}} \sqcup \bar{\mathbf{B}}$  are the maps defined by

$$\lambda_{\mathbf{B}_G}(\mathbf{a}) = \begin{cases} \mathbf{a} & \text{if } \mathbf{a} \in \bar{\mathbf{A}} \\ \mathbf{b} & \text{if } G(\mathbf{a}) = \{\mathbf{b}\} \end{cases} \quad \text{and} \quad \mu_{\mathbf{B}_G}(\mathbf{b}) = \begin{cases} \mathbf{b} & \text{if } \mathbf{b} \in \bar{\mathbf{B}} \\ \mathbf{a} & \text{if } G^{-1}(\mathbf{b}) = \{\mathbf{a}\} \end{cases}. \quad (2.48)$$

**Example 2.40.** If  $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5\}$ ,  $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3, \mathbf{b}_4, \mathbf{b}_5\}$ , and

$$G = \begin{array}{ccccc} & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{a}_1 & \nearrow & \uparrow & \times & \uparrow & \nearrow \\ & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array}, \quad \text{then} \quad \mathbf{B}_G = \begin{array}{ccccc} & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 & \mathbf{b}_4 & \mathbf{b}_5 \\ & \boxed{\mathbf{b}_1} & \boxed{\mathbf{b}_2} & \boxed{\mathbf{b}_3} & & \boxed{\mathbf{b}_5} \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ \mathbf{a}_1 & & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \\ & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & & & & \end{array}. \quad (2.49)$$

//

We have the following analogues of Lemma 2.35 and Theorem 2.33. (Lemma 2.41 is generally *not* satisfied by the circuit shape on  $\mathbf{A} \sqcup \mathbf{B}$  discussed above Definition 2.39.)

**Lemma 2.41.** *Every circuit shape endomorphism of  $B_G$  is trivial:  $\text{End}(B_G) = \{\text{id}_{B_G}\}$ .*

*Proof.* Let  $f : B_G \rightarrow B_G$  be a circuit shape morphism and let  $\mathbf{a} \in \bar{A}$ . We show that  $f(\mathbf{a}) = \mathbf{a}$ . First of all, we have  $f(\mathbf{a}) = f(\lambda_{B_G}(\mathbf{a})) \geq \lambda_{B_G}(\mathbf{a}) = \mathbf{a}$ . Suppose for contradiction that this inequality is strict: that  $f(\mathbf{a}) > \mathbf{a}$ . By construction of  $B_G$ , this means that  $f(\mathbf{a}) = \mathbf{b}$  for some  $\mathbf{b} \in G(\mathbf{a}) \cap \bar{B}$ . Since  $\mathbf{a} \in \bar{A}$ , this means  $|G(\mathbf{a})| > 1$ . Now take any  $\mathbf{b}' \in G(\mathbf{a})$  distinct from  $\mathbf{b}$ . Then  $\lambda_{B_G}(\mathbf{a}) \leq \mu_{B_G}(\mathbf{b}')$ ; this is a consequence of the fact that  $B_G$  has connectivity  $G$ , which can be straightforwardly verified. We therefore have  $\mathbf{b} = f(\mathbf{a}) = f(\lambda_{B_G}(\mathbf{a})) \leq f(\mu_{B_G}(\mathbf{b}')) \leq \mu_{B_G}(\mathbf{b}')$ . Since  $\mathbf{b}$  is by construction a maximal element of  $B_G$ , we must in fact have  $\mathbf{b} = \mu_{B_G}(\mathbf{b}')$ , which implies  $\mathbf{b} = \mathbf{b}'$ , a contradiction. We conclude that  $f(\mathbf{a}) = \mathbf{a}$ .

Next, suppose that  $\mathbf{b} \in \bar{B}$ . We wish to show that  $f(\mathbf{b}) = \mathbf{b}$ . To start, we have  $f(\mathbf{b}) = f(\mu_{B_G}(\mathbf{b})) \leq \mu_{B_G}(\mathbf{b}) = \mathbf{b}$ . Suppose that in fact  $f(\mathbf{b}) < \mathbf{b}$ . Then  $f(\mathbf{b}) = \mathbf{a}$  for some  $\mathbf{a} \in G^{-1}(\mathbf{b}) \cap \bar{A}$ . Take any  $\mathbf{a}' \in G^{-1}(\mathbf{b})$ ; we have  $\lambda_{B_G}(\mathbf{a}') \leq \mu_{B_G}(\mathbf{b})$ , so  $\lambda_{B_G}(\mathbf{a}') \leq f(\lambda_{B_G}(\mathbf{a}')) \leq f(\mu_{B_G}(\mathbf{b})) = f(\mathbf{b}) = \mathbf{a}$ , which by construction means that  $\mathbf{a}' = \mathbf{a}$ . Therefore,  $|G^{-1}(\mathbf{b})| = 1$ . Since  $\mathbf{b} \in \bar{B}$  by assumption, we must have  $|G(\mathbf{a})| = 1$ . This is in contradiction with the fact that  $\mathbf{a} \in \bar{A}$ . Therefore  $f(\mathbf{b}) = \mathbf{b}$ .  $\square$

**Theorem 2.42.** *Let  $A$  and  $B$  be sets and  $G \subseteq A \times B$ . Then  $B_G$  is, up to isomorphisms, the unique circuit shape from  $A$  to  $B$  satisfying the following three properties.*

- (i)  $B_G$  has connectivity  $G_{B_G} = G$ .
- (ii) For any circuit shape  $P$  with connectivity  $G_P \supseteq G$ , there exists a morphism  $f : B_G \rightarrow P$ .
- (iii) If  $Q$  is any other circuit shape satisfying (i) and (ii), then there is an injective morphism  $B_G \rightarrow Q$ .

*In fact, for any circuit shape  $Q$  satisfying (i) and (ii) and morphism  $f : B_G \rightarrow Q$ ,  $f$  is an embedding.*

*Proof.* We only show that (ii) holds for  $B_G$ . (i) can be straightforwardly verified and all other claims of the theorem follow in a way similar to the proof of Theorem 2.33, now using Lemma 2.41 instead of Lemma 2.35.

To show (ii), let  $P$  be a circuit shape with connectivity  $G_P \supseteq G$ . Define the map  $f : B_G \rightarrow P$  by

$$\begin{aligned} f(\mathbf{a}) &:= \lambda_P(\mathbf{a}) \quad \text{for } \mathbf{a} \in \bar{A}; \\ f(\mathbf{b}) &:= \mu_P(\mathbf{b}) \quad \text{for } \mathbf{b} \in \bar{B}. \end{aligned} \tag{2.50}$$

It follows from the construction of  $B_G$  and the connectivity of  $P$  that  $f$  is order-preserving.  $f$  also respects inputs: if  $\mathbf{a} \in \bar{A}$  then  $f(\lambda_{B_G}(\mathbf{a})) = f(\mathbf{a}) = \lambda_P(\mathbf{a})$ , while if  $\mathbf{a} \in A \setminus \bar{A}$ , then  $G(\mathbf{a}) = \{\mathbf{b}\}$  for some  $\mathbf{b} \in \bar{B}$  and  $f(\lambda_{B_G}(\mathbf{a})) = f(\mathbf{b}) = \mu_P(\mathbf{b}) \geq \lambda_P(\mathbf{a})$ . The final inequality here is due to the connectivity of  $P$ . Similarly,  $f(\mu_{B_G}(\mathbf{b})) \leq \mu_P(\mathbf{b})$  for all  $\mathbf{b} \in B$ . In conclusion,  $f$  is a circuit shape morphism  $B_G \rightarrow P$ .  $\square$

In terms of the preorder  $\preceq$  from Definition 2.17, it follows that  $B_G$  is a minimum of the class of circuit shapes with connectivity at least  $G$ . Together with Proposition 2.22, this implies that dually to Corollary 2.38,  $B_{(-)}$  is *lower* Galois adjoint to connectivity and is in this sense the ‘free’ circuit with connectivity  $G$ .

**Corollary 2.43.** *For any circuit shape  $P$  and relation  $G \subseteq A \times B$ , we have*

$$G \subseteq G_P \iff B_G \preceq P. \quad (2.51)$$

*In other words, the map  $B_{(-)} : \mathcal{P}(A \times B) \rightarrow \text{Circ}_{AB}^{\preceq}$  is order-preserving and forms a lower Galois adjoint to the map  $G_{(-)} : \text{Circ}_{AB}^{\preceq} \rightarrow \mathcal{P}(A \times B)$ .*

Combining Corollaries 2.38 and 2.43, we see that for any circuit  $P$  and relation  $G$ ,

$$G_P = G \iff B_G \preceq P \preceq L_G. \quad (2.52)$$



## Chapter 3

# Causal and compositional structure

In this chapter we turn to causal decompositions, which aim to express *causal* constraints on unitary transformations—of the ‘top-down’ type discussed in Chapter 1—in terms of the *compositional* structure of quantum circuits, whose syntactical aspects were set out in Chapter 2.

More precisely, the *causal structure* of a unitary transformation  $\mathcal{U}$  whose input and output systems are labelled by the elements of finite sets  $\mathbf{A}$  and  $\mathbf{B}$ , respectively, is captured by a relation  $G_{\mathcal{U}} \subseteq \mathbf{A} \times \mathbf{B}$  specifying which inputs  $\mathbf{a} \in \mathbf{A}$  influence which outputs  $\mathbf{b} \in \mathbf{B}$  through  $\mathcal{U}$ . A quantum circuit  $\mathbf{P}$  with the same inputs and outputs defines, on the other hand, a *connectivity* relation  $G_{\mathbf{P}} \subseteq \mathbf{A} \times \mathbf{B}$ , indicating the existence of directed paths connecting inputs to outputs. It encodes one aspect of the circuit’s compositional structure. If  $\mathcal{U}$  admits a decomposition into a circuit  $\mathbf{P}$ , then  $G_{\mathcal{U}} \subseteq G_{\mathbf{P}}$ : the presence of causal influence through  $\mathcal{U}$  requires a path through the circuit representation that can mediate it. The *no-influence* relations satisfied by  $\mathcal{U}$  by virtue of the inclusion  $G_{\mathcal{U}} \subseteq G_{\mathbf{P}}$  are thus reflected in the circuit representation  $\mathbf{P}$  by the absence of such mediating paths.

Now, it turns out that every unitary that satisfies a single no-influence constraint, say,  $(\mathbf{a}, \mathbf{b}) \notin G_{\mathcal{U}}$ , admits of a circuit decomposition reflecting that property by the absence of a path, i.e. satisfying  $(\mathbf{a}, \mathbf{b}) \notin G_{\mathbf{P}}$  [35]. A natural question is whether also *multiple* no-influence constraints can be reflected at once in the compositional structure of a single circuit: that is, whether the constraint that  $G_{\mathcal{U}} \subseteq G$  for a given relation  $G \subseteq \mathbf{A} \times \mathbf{B}$  implies the existence of a circuit decomposition  $\mathbf{P}$  with connectivity  $G_{\mathbf{P}}$  satisfying  $G_{\mathcal{U}} \subseteq G_{\mathbf{P}} \subseteq G$ . A *causal decomposition* is a circuit decomposition achieving this for some given relation  $G$ . As a special case, a *causally faithful* decomposition is one that satisfies  $G_{\mathcal{U}} = G_{\mathbf{P}}$  and thus reflects through the absence of paths precisely *all* no-influence relations satisfied by  $\mathcal{U}$ . Causal and causally faithful circuit decompositions thereby form a bridge between causal and compositional structure.

The question of the existence of such decompositions was first addressed in full generality by Lorenz and Barrett [62], who showed for a number of relations  $G \subseteq \mathbf{A} \times \mathbf{B}$  that *any* unitary  $\mathcal{U}$  with causal structure  $G_{\mathcal{U}} \subseteq G$  indeed admits a circuit decomposition

with connectivity  $G$ .<sup>1</sup> Moreover, the shape of these circuits was uniquely determined by  $G$  and independent of the unitary  $\mathcal{U}$ . This type of result finds many applications, some of which were outlined in Chapter 1. First of all, it offers a constructive description of the entire class of unitaries  $\mathcal{U}$  with  $G_{\mathcal{U}} \subseteq G$ , parametrising it by the gates and wires in the circuit. Such a compositional parametrisation acts as a useful information-theoretic tool, giving a structural handle when proving claims about (top-down) causal structure (see e.g. [12, 75, 108, 110]). Moreover, the parameters may be independently varied without leaving the class of unitaries with  $G_{\mathcal{U}} \subseteq G$ ; in this sense they constitute a generalisation of the *autonomous causal mechanisms* from classical causal modelling [76] and thereby find an important application to the quantum causal modelling approach based on unitary causal influences developed in [4, 11, 12]. Moreover, causal decompositions are instrumental in comparing the latter with other approaches to quantum causal modelling such as those of Refs. [33, 44, 52], which instead take compositional structure as their starting point. A further application is to the compatibility of quantum channels with *relativistic* causality, the subject of Chapter 4; and a final one pertains to gaining insight into how different *partitionings* of the same global quantum system into subsystems relate to one another [109].

Despite this wealth of applications (and the mathematical beauty of the problem!) the general existence of causal and causally faithful decompositions of unitary transformations remains unknown. For many of the motivations mentioned above it is natural, or even essential, to require that the gates in causal decompositions themselves be unitary, too. Ref. [62] showed that not all unitary transformations admit such *unitary* causally faithful decompositions, and addressed this by introducing an extended paradigm for unitary quantum circuits that incorporates direct-sum structure and non-factor algebras, in addition to the usual tensor and sequential products. This has since been developed into the framework of *routed quantum circuits* [107]. As Ref. [62] showed, strictly more unitary transformations admit causal decompositions in terms of *routed unitary circuits* than in terms of traditional unitary circuits.

The focus of this chapter will however be on decompositions into (traditional, non-routed) unitary circuits—i.e. those whose wires represent tensor-product factors of Hilbert spaces, and whose gates are unitary transformations between them. Our main result is the identification of a combinatorial condition on a binary relation  $G \subseteq \mathbf{A} \times \mathbf{B}$ —the  $C_3$  *exclusion property*—which we prove is satisfied by  $G$  if and only if all unitaries  $\mathcal{U}$  satisfying  $G_{\mathcal{U}} \subseteq G$  admit a *unitary* circuit decomposition  $\mathbf{P}$  with connectivity  $G_{\mathbf{P}} \subseteq G$ . In other words, the condition characterises those causal constraints that can always (i.e. regardless of the unitary  $\mathcal{U}$  that satisfies them) be expressed in terms of the compositional structure of unitary circuits, without requiring the generalisation to routed circuits. Our proof makes heavy use of the concept lattice  $L_G$  from Chapter 2, which is the shape of the unitary causal decompositions we show exist in case the  $C_3$  exclusion property is satisfied. In fact, we show that the property can equivalently be expressed in terms of various insightful properties of the concept lattice: in particular, it corres-

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<sup>1</sup>The results of [62] were mostly phrased in terms of causally *faithful* circuit decompositions, particularly, but their proofs also show the existence of what we here call causal decompositions.

ponds to the requirement that  $L_G$  has no more than one path between any input and output. This formalises the intuition suggested by [62] that routed circuits only offer an advantage in the presence of non-unique paths.

After settling some notation in Section 3.1, we start in Section 3.2 by introducing quantum compositional structure, defining a *quantum circuit* as a circuit shape equipped with quantum semantics and formulating the semantic consequences of some of the results in Chapter 2. Section 3.3 introduces causal structure; Section 3.4 then presents causal decompositions and goes in some detail on our reason for focussing on decomposing unitary channels, rather than generic CPTP maps. In Section 3.5 we state our main result and give its proof, which relies on algebraic facts proven in Section 3.6.

### 3.1 Quantum systems and quantum channels

Let us start by setting out some terminology and notation. By a **(quantum) system** we mean the full algebra  $\mathbf{A} = \mathcal{L}(\mathcal{H}_{\mathbf{A}})$  of operators on some finite-dimensional Hilbert space  $\mathcal{H}_{\mathbf{A}}$ . The identity operator is denoted by  $\mathbf{1}_{\mathbf{A}}$ , and the **commutant** of a subset  $\mathbf{X} \subseteq \mathbf{A}$  is the subalgebra

$$\mathbf{X}' := \{a \in \mathbf{A} \mid \forall x \in \mathbf{X} : ax = xa\}. \quad (3.1)$$

Given quantum systems  $\mathbf{A}_1$  and  $\mathbf{A}_2$ , we will often denote the composite system  $\mathbf{A}_1 \otimes \mathbf{A}_2 = \mathcal{L}(\mathcal{H}_{\mathbf{A}_1}) \otimes \mathcal{L}(\mathcal{H}_{\mathbf{A}_2}) \cong \mathcal{L}(\mathcal{H}_{\mathbf{A}_1} \otimes \mathcal{H}_{\mathbf{A}_2})$  by  $\mathbf{A}_1 \mathbf{A}_2$ , and, when clear from context, we will abuse notation by letting  $\mathbf{A}_1$  also denote the subalgebra  $\mathbf{A}_1 \otimes \{\mathbf{1}_{\mathbf{A}_2}\} = \{a_1 \otimes \mathbf{1}_{\mathbf{A}_2} \mid a_1 \in \mathbf{A}_1\} \subseteq \mathbf{A}_1 \mathbf{A}_2$ . With this notation, we have  $\mathbf{A}'_1 = \mathbf{A}_2$ .

Denote by  $\mathcal{L}(\mathcal{H}_{\mathbf{A}}, \mathcal{H}_{\mathbf{B}})$  the space of linear maps  $\mathcal{H}_{\mathbf{A}} \rightarrow \mathcal{H}_{\mathbf{B}}$  between two Hilbert spaces. We use an asterisk to denote the adjoint  $(-)^* : \mathcal{L}(\mathcal{H}_{\mathbf{A}}, \mathcal{H}_{\mathbf{B}}) \rightarrow \mathcal{L}(\mathcal{H}_{\mathbf{B}}, \mathcal{H}_{\mathbf{A}})$ , which in the case of  $\mathcal{L}(\mathcal{H}_{\mathbf{A}}, \mathcal{H}_{\mathbf{A}})$  constitutes the involution of the  $C^*$ -algebra  $\mathbf{A}$ . Operator algebras  $\mathbf{A}$  and  $\mathbf{B}$  themselves are equipped with the Hilbert-Schmidt inner product  $\langle a, a' \rangle := \text{Tr}(a^* a')$ , and we reserve the dagger  $(-)^{\dagger} : \mathcal{L}(\mathbf{A}, \mathbf{B}) \rightarrow \mathcal{L}(\mathbf{B}, \mathbf{A})$  for the adjoint of superoperators with respect to the Hilbert-Schmidt inner product.

Transformations or **(quantum) channels** are superoperators  $\mathcal{E} : \mathbf{A} \rightarrow \mathbf{B}$  that are completely positive and trace-preserving ( $\text{Tr}_{\mathbf{B}} \circ \mathcal{E} = \text{Tr}_{\mathbf{A}}$ ). These conditions are equivalent to saying that the adjoint  $\mathcal{E}^{\dagger} : \mathbf{B} \rightarrow \mathbf{A}$  is completely positive and unital, i.e. satisfies  $\mathcal{E}^{\dagger}(\mathbf{1}_{\mathbf{B}}) = \mathbf{1}_{\mathbf{A}}$ . This is due to the fact that the Hilbert-Schmidt adjoint of the trace is

$$(\text{Tr}_{\mathbf{A}} : \mathbf{A} \rightarrow \mathbb{C})^{\dagger} = (\mathbb{C} \rightarrow \mathbf{A}, 1 \mapsto \mathbf{1}_{\mathbf{A}}), \quad (3.2)$$

the preparation of the unit.  $\mathcal{E}$  is **unitary** if it is of the form  $U(\cdot)U^*$  for some unitary map  $U : \mathcal{H}_{\mathbf{A}} \rightarrow \mathcal{H}_{\mathbf{B}}$ . This is the case precisely if  $\mathcal{E}$  is a  $*$ -isomorphism of the operator algebras (see Proposition 3.30).

Throughout this chapter and the next, we will strive to keep a healthy separation between *syntactic* notions (such as labels of quantum systems, relations between labels, shapes of circuits) and *semantic* ones (such as quantum systems, unitary channels, commutation of subalgebras). Mathematically speaking, the former are combinatorial objects, the latter operator-algebraic. We will often use a typewriter font for symbols

relating to syntactic notions, as we have already started doing in Chapter 2, and a bold font for semantic ones. On the syntactic side, our general setup will consist of two finite sets  $\mathbf{A}$  and  $\mathbf{B}$  whose elements should now be viewed as *labels* of input and output quantum systems, respectively. Semantically, meanwhile, we consider quantum systems  $\mathbf{A}_{\mathbf{a}}$  labelled by  $\mathbf{a} \in \mathbf{A}$  and  $\mathbf{B}_{\mathbf{b}}$  labelled by  $\mathbf{b} \in \mathbf{B}$ . Given such a choice of quantum systems and a subset  $\alpha \subseteq \mathbf{A}$ , we write

$$\mathbf{A}_{\alpha} := \bigotimes_{\mathbf{a} \in \alpha} \mathbf{A}_{\mathbf{a}}, \quad \text{and thus} \quad \mathbf{A}_{\mathbf{A}} := \bigotimes_{\mathbf{a} \in \mathbf{A}} \mathbf{A}_{\mathbf{a}}; \quad (3.3)$$

$\mathbf{B}_{\beta}$  and  $\mathbf{B}_{\mathbf{B}}$  are defined similarly. In particular,  $\mathbf{A}_{\emptyset} = \mathbf{B}_{\emptyset} = \mathcal{L}(\mathbb{C}) \cong \mathbb{C}$ , the trivial quantum system. As above, we will often use the same notation  $\mathbf{A}_{\alpha}$  and  $\mathbf{B}_{\beta}$  to refer instead to the subalgebras  $\mathbf{A}_{\alpha} \otimes \{\mathbf{1}_{\mathbf{A}_{\mathbf{A} \setminus \alpha}}\} \subseteq \mathbf{A}_{\mathbf{A}}$  and  $\mathbf{B}_{\beta} \otimes \{\mathbf{1}_{\mathbf{B}_{\mathbf{B} \setminus \beta}}\} \subseteq \mathbf{B}_{\mathbf{B}}$ , respectively, whose commutants are then given by

$$\mathbf{A}'_{\alpha} = \mathbf{A}_{\mathbf{A} \setminus \alpha} \quad \text{and} \quad \mathbf{B}'_{\beta} = \mathbf{B}_{\mathbf{B} \setminus \beta}. \quad (3.4)$$

Our main objects of study are the multi-input, multi-output quantum channels  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$ .

## 3.2 Compositional structure of quantum channels

Here we define *quantum circuits* as circuit shapes equipped with quantum semantics and investigate the semantic consequences of some of the order-theoretic results of Chapter 2. From here onwards we assume that circuit shapes  $(\mathbf{P}, \leq, \lambda, \mu)$  are finite, unless otherwise stated. Recall from Definition 2.2 that this means that the set  $\mathbf{P}$  as well as the sets of input and output labels  $\mathbf{A}, \mathbf{B}$ —the domains of  $\lambda$  and  $\mu$ —are finite. Moreover, for two elements  $p, q$  of a partial order  $(\mathbf{P}, \leq)$ , recall that we write  $p \triangleleft q$  if  $q$  covers  $p$  from above: that is, if  $p < q$  and there are no  $r \in \mathbf{P}$  satisfying  $p < r < q$ . In the below we will use the notation

$$\text{Pa}(p) := \{q \in \mathbf{P} \mid q \triangleleft p\} \quad \text{and} \quad \text{Ch}(p) := \{r \in \mathbf{P} \mid p \triangleleft r\}. \quad (3.5)$$

**Definition 3.1.** A **(quantum) circuit** with input systems  $\mathbf{A}_{\mathbf{a}}$  for  $\mathbf{a} \in \mathbf{A}$  and output systems  $\mathbf{B}_{\mathbf{b}}$  for  $\mathbf{b} \in \mathbf{B}$  is a tuple  $\mathbf{P} = (\mathbf{P}, \{\mathbf{Z}_p^q\}_{p \triangleleft q}, \{\mathcal{E}_p\}_p)$  of a circuit shape  $\mathbf{P} \equiv (\mathbf{P}, \leq, \lambda, \mu)$  from  $\mathbf{A}$  to  $\mathbf{B}$  together with a choice of

$$\text{quantum systems} \quad \mathbf{Z}_p^q \quad \text{for each } p, q \in \mathbf{P} \text{ so that } p \triangleleft q \quad (3.6)$$

$$\text{and quantum channels} \quad \mathcal{E}_p : \mathbf{A}_{\lambda^{-1}(p)} \mathbf{Z}_{\text{Pa}(p)}^p \rightarrow \mathbf{Z}_p^{\text{Ch}(p)} \mathbf{B}_{\mu^{-1}(p)} \quad \text{for each } p \in \mathbf{P}. \quad (3.7)$$

Here  $\mathbf{Z}_{\text{Pa}(p)}^p := \bigotimes_{q \in \text{Pa}(p)} \mathbf{Z}_q^p$  and  $\mathbf{Z}_p^{\text{Ch}(p)} := \bigotimes_{r \in \text{Ch}(p)} \mathbf{Z}_p^r$ . For typing reasons, we assume that all  $\mathbf{Z}_p^q$  systems are distinct; however, they may of course be isomorphic. In a **unitary (quantum) circuit**, all gates  $\mathcal{E}_p$  are unitary channels.

When we say **gate**, this may refer either to the elements  $p$  of  $\mathbf{P}$  or to their associated channels  $\mathcal{E}_p$ .

**Definition 3.2.** Let  $\mathbf{P} = (\mathbf{P}, \{\mathbf{Z}_p^q\}_{p \leq q}, \{\mathcal{E}_p\}_p)$  be a quantum circuit and  $T \subseteq \mathbf{P}$  a subset of gates. We denote by  $\mathcal{E}_T$  the quantum channel obtained by composing the gates  $\mathcal{E}_p$  for  $p \in T$  along all matching  $\mathbf{Z}_q^r$ . Note that this can only be done in one way, since each  $\mathbf{Z}_q^r$  system appears precisely once as an input to a gate  $\mathcal{E}_p$  for  $p \in \mathbf{P}$ , and precisely once as an output. The channel  $\mathcal{E}_T$  obtained in this way has domain and codomain

$$\mathcal{E}_T : \mathbf{A}_{\lambda^{-1}(T)} \otimes \bigotimes_{p \in T} \mathbf{Z}_{\text{Pa}(p) \setminus T}^p \rightarrow \bigotimes_{p \in T} \mathbf{Z}_p^{\text{Ch}(p) \setminus T} \otimes \mathbf{B}_{\mu^{-1}(T)}. \quad (3.8)$$

In particular, the composition of all gates in the circuit yields the channel

$$\mathcal{E}_{\mathbf{P}} := \mathcal{E}_{\mathbf{P}} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}. \quad (3.9)$$

If an arbitrary channel  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  satisfies  $\mathcal{E} = \mathcal{E}_{\mathbf{P}}$ , then we say  $\mathbf{P}$  is a **circuit decomposition** of  $\mathcal{E}$ . A **unitary circuit decomposition** is one in which each gate  $\mathcal{E}_p$  is itself a unitary channel.

In Section 2.2 we argued, on the basis of Theorem 2.16, that morphisms of circuit shapes constitute an appropriate notion of syntactical circuit rewrite and can therefore never decrease the expressivity of a circuit shape. More precisely, in the quantum case:

**Proposition 3.3.** *Let  $\mathbf{P}$  and  $\mathbf{Q}$  be circuit shapes from  $\mathbf{A}$  to  $\mathbf{B}$ . If there exists a circuit morphism  $f : \mathbf{P} \rightarrow \mathbf{Q}$  then any channel  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  that has a (unitary) circuit decomposition of shape  $\mathbf{P}$  also has a (unitary) circuit decomposition of shape  $\mathbf{Q}$ .*

*Proof.* By Theorem 2.16 it suffices to show this under the assumption that  $f$  is any of the elementary types of circuit morphisms listed there. We will omit the straightforward proofs.  $\square$

**Definition 3.4.** The **connectivity**  $G_{\mathbf{P}} \subseteq \mathbf{A} \times \mathbf{B}$  of a circuit  $\mathbf{P} = (\mathbf{P}, \{\mathbf{Z}_p^q\}_{p \leq q}, \{\mathcal{E}_p\}_p)$  is the connectivity  $G_{\mathbf{P}}$  of its underlying circuit shape.

**Corollary 3.5.** *Let  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  be a quantum channel and  $G \subseteq \mathbf{A} \times \mathbf{B}$ .  $\mathcal{E}$  has a circuit decomposition  $\mathbf{P}$  with connectivity  $G_{\mathbf{P}} \subseteq G$  if and only if  $\mathcal{E}$  has a circuit decomposition of shape  $L_G$ .*

*Proof.* The ‘only if’ direction follows from Theorem 2.33 and Proposition 3.3. The ‘if’ direction is immediate, since  $L_G$  has connectivity relation  $G_{L_G} = G$ .  $\square$

A natural question now arises: Is the full generality of the concept lattice  $L_G$  actually needed in the quantum case, or is there a ‘simpler’ circuit shape—one strictly below  $L_G$  in the syntactical preorder on circuit shapes from Definition 2.17—that also satisfies the first claim of Corollary 3.5? Recall first of all that the maximum and/or minimum of  $L_G$  may in some cases be removed without impacting its expressivity (see

Remark 2.31). One may wonder whether more interesting simplifications can be made, however. Consider for instance the case of multivariate classical functions: every function  $f$  that has a (quantum or classical) circuit decomposition with connectivity  $G$  has one not only of the shape  $L_G$ , but also of the syntactically much weaker shape  $B_G$  from Section 2.5, due to the existence of copy maps (cf. Eq. (1.2)). The ‘interaction gates’ provided by the joins and meets in  $L_G$  are not needed to express these functions.

The situation is different in the quantum case: the quantum no-broadcasting theorem, and the general fact that properties of composite quantum systems often do not reduce to properties of their composites, suggest that interaction is crucial for the implementation of quantum channels. The below illustrates this by way of a simple example. A more general investigation of the question whether the full generality of  $L_G$  (up to potential removal of its minimum and/or maximum) is needed is left for future work; but a further indication is that the shapes of all causal decompositions constructed in Lorenz and Barrett [62] were indeed of the shape  $L_G$  with appropriately removed minimum and/or maximum.

For our example, let  $A = B = \{1, 2\}$  and consider the relation and circuit shapes

$$G_2 := \begin{array}{c} 1 \quad 2 \\ \nearrow \quad \searrow \\ 1 \quad 2 \end{array}, \quad B_{G_2} := \begin{array}{c} 1 \quad 2 \\ \boxed{c} \quad \boxed{d} \\ \swarrow \quad \searrow \\ \boxed{a} \quad \boxed{b} \\ 1 \quad 2 \end{array} \quad \text{and} \quad L_{G_2} := \begin{array}{c} 1 \quad 2 \\ \swarrow \quad \searrow \\ \boxed{q} \\ \swarrow \quad \searrow \\ 1 \quad 2 \end{array}. \quad (3.10)$$

By an argument similar to that in Example 2.25, there is no morphism from  $L_{G_2} \rightarrow B_{G_2}$ :  $L_{G_2}$  has a gate that allows for interaction between  $a_1, a_2, b_1$  and  $b_2$ , while  $B_{G_2}$  does not. Therefore,  $L_{G_2}$  is strictly above  $B_{G_2}$  in the syntactical order  $\preceq$ . This is reflected by the existence of a quantum channel that has a decomposition of shape  $L_{G_2}$ , but not of shape  $B_{G_2}$ .

**Proposition 3.6.** *Let  $A_1 = A_2 = B_1 = B_2 = \mathcal{L}(\mathbb{C}^2)$  be qubits. The quantum CNOT channel  $\text{CNOT} : A_1 A_2 \rightarrow B_1 B_2$  has no circuit decomposition of the shape  $B_{G_2}$  in Eq. (3.10).*

This was also noted in [87], though without proof. Our proof makes use of the following variant of the quantum no-broadcasting theorem [9, 59].

**Lemma 3.7.** *Let  $\mathcal{E} : A \rightarrow BC$  be a quantum channel. Suppose that there exists a quantum channel  $\mathcal{F} : B \rightarrow A$  so that*

$$\begin{array}{c} A \\ \boxed{\mathcal{F}} \\ B \quad \overline{\overline{\mathcal{C}}} \\ \boxed{\mathcal{E}} \\ A \end{array} = \begin{array}{c} A \\ | \\ A \end{array}, \quad (3.11)$$

*the identity channel  $\text{id}_A : A \rightarrow A$ . Here  $\overline{\overline{\mathcal{C}}}$  denotes the partial trace. In that case, there*

is a state  $\rho$  on  $\mathcal{C}$  so that

$$\begin{array}{c} \text{---} \overline{\mathcal{B}} \\ | \\ \boxed{\mathcal{E}} \\ | \\ \mathcal{A} \end{array} \begin{array}{c} \mathcal{C} \\ | \\ \mathcal{C} \end{array} = \begin{array}{c} \text{---} \overline{\mathcal{A}} \\ | \\ \boxed{\rho} \\ | \\ \mathcal{A} \end{array} \begin{array}{c} \mathcal{C} \\ | \\ \mathcal{C} \end{array}. \quad (3.12)$$

*Proof of Proposition 3.6.* Suppose, for contradiction, that

$$\begin{array}{c} \begin{array}{cc} \mathcal{B}_1 & \mathcal{B}_2 \\ | & | \\ \boxed{\mathcal{C}} & \boxed{\mathcal{D}} \\ | & | \\ \text{---} \overline{\mathcal{W}} & \text{---} \overline{\mathcal{X}} \\ | & | \\ \boxed{\mathcal{A}} & \boxed{\mathcal{B}} \\ | & | \\ \mathcal{A}_1 & \mathcal{A}_2 \end{array} \\ \text{---} \overline{\mathcal{W}} \text{---} \overline{\mathcal{X}} \text{---} \overline{\mathcal{Y}} \text{---} \overline{\mathcal{Z}} \end{array} = \begin{array}{c} \mathcal{B}_1 \quad \mathcal{B}_2 \\ | \quad | \\ \bullet \quad \oplus \\ | \quad | \\ \mathcal{A}_1 \quad \mathcal{A}_2 \end{array} \quad (3.13)$$

for some quantum systems  $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$  and quantum channels  $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ . Preparing the state  $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2}$  on  $\mathcal{A}_2$  and tracing out  $\mathcal{B}_2$  yields

$$\begin{array}{c} \mathcal{B}_1 \\ | \\ \boxed{\mathcal{C}} \\ | \\ \text{---} \overline{\mathcal{A}} \\ | \\ \mathcal{A}_1 \end{array} \begin{array}{c} \text{---} \overline{\mathcal{B}} \\ | \\ \boxed{\mathcal{B}} \\ | \\ \text{---} \overline{\mathcal{A}} \\ | \\ \mathcal{A}_1 \end{array} = \begin{array}{c} \mathcal{B}_1 \quad \text{---} \overline{\mathcal{A}} \\ | \quad | \\ \bullet \quad \oplus \\ | \quad | \\ \mathcal{A}_1 \quad \text{---} \overline{\mathcal{A}} \\ | \quad | \\ \mathcal{A}_1 \quad \mathcal{A}_1 \end{array} = \begin{array}{c} \mathcal{B}_1 \\ | \\ \mathcal{B}_1 \\ | \\ \mathcal{A}_1 \end{array}, \quad (3.14)$$

the identity channel from  $\mathcal{A}_1$  to  $\mathcal{B}_1$  (both qubits). Tracing out  $\mathcal{B}_1$  from Eq. (3.13) and applying Lemma 3.7 to the channel  $\mathcal{A}$  then implies that

$$\begin{array}{c} \text{---} \overline{\mathcal{B}_1} \quad \mathcal{B}_2 \\ | \quad | \\ \bullet \quad \oplus \\ | \quad | \\ \mathcal{A}_1 \quad \mathcal{A}_2 \end{array} = \begin{array}{c} \text{---} \overline{\mathcal{B}_1} \quad \mathcal{B}_2 \\ | \quad | \\ \boxed{\mathcal{C}} \quad \boxed{\mathcal{D}} \\ | \quad | \\ \text{---} \overline{\mathcal{W}} \quad \text{---} \overline{\mathcal{X}} \\ | \quad | \\ \boxed{\mathcal{A}} \quad \boxed{\mathcal{B}} \\ | \quad | \\ \mathcal{A}_1 \quad \mathcal{A}_2 \end{array} = \begin{array}{c} \text{---} \overline{\mathcal{B}_1} \quad \mathcal{B}_2 \\ | \quad | \\ \boxed{\mathcal{D}} \\ | \quad | \\ \text{---} \overline{\mathcal{W}} \quad \text{---} \overline{\mathcal{X}} \\ | \quad | \\ \boxed{\mathcal{A}} \quad \boxed{\mathcal{B}} \\ | \quad | \\ \mathcal{A}_1 \quad \mathcal{A}_2 \end{array} = \begin{array}{c} \text{---} \overline{\mathcal{B}_1} \quad \mathcal{B}_2 \\ | \quad | \\ \boxed{\rho} \quad \boxed{\mathcal{B}} \\ | \quad | \\ \text{---} \overline{\mathcal{A}_1} \quad \text{---} \overline{\mathcal{A}_2} \\ | \quad | \\ \mathcal{A}_1 \quad \mathcal{A}_2 \end{array} \quad (3.15)$$

for some state  $\rho$  on  $\mathcal{X}$ . This is a contradiction, as the marginal output on  $\mathcal{B}_2$  depends on the input  $\mathcal{A}_1$  in the CNOT on the left-hand side, but not in the circuit on the right-hand side.  $\square$

As an immediate consequence of this result, also the circuit shapes  $\mathcal{B}_{C_3}$  and  $\mathcal{L}_{C_3}$  from Example 2.21 are quantum inequivalent, with  $\mathcal{L}_{C_3}$  able to express strictly more quantum channels.<sup>2</sup> The significance of interaction gates for the implementation of quantum channels is closely related to the recent work of Centeno and Wolfe [25], who demonstrate the significance of *intermediate latents* in quantum networks.

<sup>2</sup>The circuit shape  $\text{LOSEC}_{C_3}$  discussed there is also strictly weaker in quantum channel expressivity than  $\mathcal{L}_{C_3}$ , but in a much more subtle way, as we will discuss in Chapter 4.

The top-down properties of signalling and (in the unitary case) causal influence can be defined in multiple intuitive ways, whose equivalence has been studied before in e.g. [35, 75, 86]. Here we state some, focussing first on signalling through generic channels.

- (i)  $\mathcal{E}^\dagger(\mathbf{B}_\beta) \subseteq \mathbf{A}'_\alpha = \mathbf{A}_{\mathbf{A} \setminus \alpha}$ .
- (ii) There exists a quantum channel  $\mathcal{D} : \mathbf{A}_{\mathbf{A} \setminus \alpha} \rightarrow \mathbf{B}_\beta$  so that  $\text{Tr}_{\mathbf{B}_{\mathbf{B} \setminus \beta}} \circ \mathcal{E} = \text{Tr}_{\mathbf{A}_\alpha} \otimes \mathcal{D}$ , or diagrammatically,

Circuit diagrams like these are always to be read from bottom to top; the symbol  $\bar{\text{tr}}$  denotes the partial trace.

$$\begin{array}{c} B_{B \setminus \beta} \quad B_{\beta} \\ \vdots \quad \vdots \\ \boxed{\mathcal{E}} \\ \vdots \quad \vdots \\ A_{\alpha} \quad A_{A \setminus \alpha} \end{array} = \begin{array}{c} B_{B \setminus \beta} \quad B_{\beta} \\ \vdots \quad \vdots \\ \boxed{\mathcal{G}} \\ \vdots \quad \vdots \\ A_{\alpha} \quad A_{A \setminus \alpha} \end{array} \begin{array}{c} \text{---} X \text{---} \\ \vdots \quad \vdots \\ \boxed{\mathcal{F}} \\ \vdots \quad \vdots \\ A_{\alpha} \quad A_{A \setminus \alpha} \end{array} . \quad (3.17)$$

The first, algebraic, condition, which echoes the microcausality axiom from quantum field theory, is most natural for the algebraic proofs in this chapter. Condition (ii), on the other hand, has a more transparent operational interpretation: it expresses that the marginal output state on  $\mathbf{B}_\beta$  does not depend on the state of the input system  $\mathbf{A}_\alpha$ . It directly implies (is in fact equivalent to) an even more direct operational condition: that any trace-preserving operation performed on  $\mathbf{A}_\alpha$  prior to application of  $\mathcal{E}$  makes no difference for observations at the marginal output  $\mathbf{B}_\beta$ .

The final condition in item (iii) gives a preview of the notions central to the rest of



**Proposition 3.10** (Time symmetry [8, 62]). *Let  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  be a unitary channel and  $\mathcal{U}^\dagger : \mathbf{B}_{\mathbf{B}} \rightarrow \mathbf{A}_{\mathbf{A}}$  its inverse. For any  $\alpha \subseteq \mathbf{A}$  and  $\beta \subseteq \mathbf{B}$ , we have  $\mathbf{A}_\alpha \rightarrow_{\mathcal{U}} \mathbf{B}_\beta$  if and only if  $\mathbf{B}_\beta \rightarrow_{\mathcal{U}^\dagger} \mathbf{A}_\alpha$ .*

*Proof.* Since  $\mathcal{U}$  is a \*-isomorphism (see Proposition 3.30) and hence commutes with taking the commutant, we have  $\mathbf{A}_\alpha \rightarrow_{\mathcal{U}} \mathbf{B}_\beta \Leftrightarrow \mathcal{U}^\dagger(\mathbf{B}_\beta) \subseteq \mathbf{A}'_\alpha \Leftrightarrow \mathbf{B}_\beta \subseteq \mathcal{U}(\mathbf{A}'_\alpha) \Leftrightarrow \mathbf{B}_\beta \subseteq \mathcal{U}(\mathbf{A}_\alpha)' \Leftrightarrow \mathcal{U}(\mathbf{A}_\alpha) \subseteq \mathbf{B}''_\beta \Leftrightarrow \mathcal{U}^{\dagger\dagger}(\mathbf{A}_\alpha) \subseteq \mathbf{B}_\beta \Leftrightarrow \mathbf{B}_\beta \rightarrow_{\mathcal{U}^\dagger} \mathbf{A}_\alpha$ .<sup>3</sup>  $\square$

**Proposition 3.11** (Causal atomicity [11, 75, 117]). *Let  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  be a unitary channel. For any  $\alpha \subseteq \mathbf{A}$  and  $\beta \subseteq \mathbf{B}$ ,  $\mathbf{A}_\alpha \rightarrow_{\mathcal{U}} \mathbf{B}_\beta$  if and only if there are  $\mathbf{a} \in \alpha$  and  $\mathbf{b} \in \beta$  such that  $\mathbf{A}_{\mathbf{a}} \rightarrow_{\mathcal{U}} \mathbf{B}_{\mathbf{b}}$ .*

*Proof.* We prove the contrapositives. Suppose that  $\mathbf{A}_\alpha \not\rightarrow_{\mathcal{U}} \mathbf{B}_\beta$ , that is,  $\mathcal{U}^\dagger(\mathbf{B}_\beta) \not\subseteq \mathbf{A}_{\mathbf{A} \setminus \alpha}$ . In that case we have, for any  $\mathbf{a} \in \alpha$  and  $\mathbf{b} \in \beta$ ,  $\mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) \subseteq \mathcal{U}^\dagger(\mathbf{B}_\beta) \subseteq \mathbf{A}_{\mathbf{A} \setminus \alpha} \subseteq \mathbf{A}_{\mathbf{A} \setminus \{\mathbf{a}\}}$ , so  $\mathbf{A}_{\mathbf{a}} \not\rightarrow_{\mathcal{U}} \mathbf{B}_{\mathbf{b}}$  either. This direction of course holds for generic channels, too.

For the other direction, suppose that for all  $\mathbf{a} \in \alpha$  and  $\mathbf{b} \in \beta$ , we have  $\mathbf{A}_{\mathbf{a}} \rightarrow_{\mathcal{U}} \mathbf{B}_{\mathbf{b}}$ . Then for all  $\mathbf{b} \in \beta$ ,  $\mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) \subseteq \bigcap_{\mathbf{a} \in \alpha} \mathbf{A}_{\mathbf{A} \setminus \{\mathbf{a}\}} = \mathbf{A}_{\mathbf{A} \setminus \alpha}$ . Now,  $\mathbf{B}_\beta$  is just the algebraic span  $\bigvee_{\mathbf{b} \in \beta} \mathbf{B}_{\mathbf{b}}$ ,<sup>4</sup> and  $\mathcal{U}^\dagger$ , being a \*-isomorphism, preserves algebraic spans: therefore,

$$\mathcal{U}^\dagger(\mathbf{B}_\beta) = \mathcal{U}^\dagger\left(\bigvee_{\mathbf{b} \in \beta} \mathbf{B}_{\mathbf{b}}\right) = \bigvee_{\mathbf{b} \in \beta} \mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) \subseteq \mathbf{A}_{\mathbf{A} \setminus \alpha}, \quad (3.19)$$

meaning  $\mathbf{A}_\alpha \rightarrow_{\mathcal{U}} \mathbf{B}_\beta$ .  $\square$

Causal atomicity states, in other words, that all multi-system causal influence relations through a unitary channel are determined by the single-system ones. Although simple to prove, it is a remarkable fact, seeing as many other properties of quantum systems are not reducible to properties of their subsystems. Causal atomicity allows us to focus fully on the single-system influences, captured by the following.

**Definition 3.12.** The **causal structure** of  $\mathcal{U}$  is the relation  $G_{\mathcal{U}} \subseteq \mathbf{A} \times \mathbf{B}$  satisfying

$$(\mathbf{a}, \mathbf{b}) \in G_{\mathcal{U}} \iff \mathbf{A}_{\mathbf{a}} \rightarrow_{\mathcal{U}} \mathbf{B}_{\mathbf{b}}. \quad (3.20)$$

### 3.4 Causal decompositions

Causal decompositions relate the compositional structure (from Chapter 2 and Section 3.2) to the causal structure (from Section 3.3) of unitary transformations. One direction is simple: compositional structure constrains signalling and causal structure in the following way.

<sup>3</sup>Recall that we are working only with finite-dimensional factor algebras, so that  $\mathbf{X}'' = \mathbf{X}$  is always satisfied.

<sup>4</sup>The algebraic span  $\bigvee_{\mathbf{b} \in \beta} \mathbf{B}_{\mathbf{b}}$  is given by the double commutant  $\left(\bigcup_{\mathbf{b} \in \beta} \mathbf{B}_{\mathbf{b}}\right)''$ .

**Proposition 3.13** (Soundness of no-connectivity for no-signalling). *Let  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  be a quantum channel with a circuit decomposition  $\mathbf{P}$ . If  $\alpha \subseteq \mathbf{A}$  and  $\beta \subseteq \mathbf{B}$  are such that  $G_{\mathbf{P}} \cap (\alpha \times \beta) = \emptyset$ , then  $\mathbf{A}_{\alpha} \rightarrow_{\mathcal{E}} \mathbf{B}_{\beta}$ .*

*Proof.* Let  $\mathbf{P} \equiv (\mathbf{P}, \leq, \lambda, \mu)$  be the circuit shape underlying the quantum circuit  $\mathbf{P}$  and write

$$\downarrow \mu(\beta) := \{p \in \mathbf{P} \mid \exists \mathbf{b} \in \beta : p \leq \mu(\mathbf{b})\}. \quad (3.21)$$

Recalling the notation from Definition 3.2, consider the two channels  $\mathcal{E}_{\downarrow \mu(\beta)}$  and  $\mathcal{E}_{\mathbf{P} \setminus \downarrow \mu(\beta)}$ . Note that

$$\lambda^{-1}(\downarrow \mu(\beta)) = \{\mathbf{a} \in \mathbf{A} \mid \exists \mathbf{b} \in \beta : \lambda(\mathbf{a}) \leq \mu(\mathbf{b})\} = G_{\mathbf{P}}^{-1}(\beta); \quad (3.22)$$

therefore the domains of these channels are (cf. Eq. (3.8))

$$\mathcal{E}_{\downarrow \mu(\beta)} : \mathbf{A}_{G_{\mathbf{P}}^{-1}(\beta)} \rightarrow \mathbf{Z} \mathbf{B}_{\mu^{-1}(\downarrow \mu(\beta))} \quad (3.23)$$

$$\text{and } \mathcal{E}_{\mathbf{P} \setminus \downarrow \mu(\beta)} : \mathbf{A}_{\mathbf{A} \setminus G_{\mathbf{P}}^{-1}(\beta)} \mathbf{Z} \rightarrow \mathbf{B}_{\mathbf{B} \setminus \mu^{-1}(\downarrow \mu(\beta))}, \quad (3.24)$$

where  $\mathbf{Z} := \bigotimes_{p < q; p \in \downarrow \mu(\beta); q \notin \downarrow \mu(\beta)} \mathbf{Z}_p^q$ .  $\mathbf{P}$  is a circuit decomposition of  $\mathcal{E}$ , so  $\mathcal{E}$  is the composition of these two gates:

$$\begin{array}{c} \begin{array}{|c|} \hline \dots \\ \hline \mathcal{E} \\ \hline \dots \\ \hline \end{array} = \begin{array}{ccc} & \begin{array}{c} \mathbf{B}_{\mathbf{B} \setminus \mu^{-1}(\downarrow \mu(\beta))} \quad \mathbf{B}_{\mu^{-1}(\downarrow \mu(\beta))} \\ \dots \\ \mathcal{E}_{\mathbf{P} \setminus \downarrow \mu(\beta)} \end{array} & \\ \begin{array}{c} \mathbf{A}_{\mathbf{A} \setminus G_{\mathbf{P}}^{-1}(\beta)} \\ \dots \\ \mathcal{E}_{\downarrow \mu(\beta)} \end{array} & \begin{array}{c} \mathbf{Z} \\ \dots \end{array} & \begin{array}{c} \mathcal{E}_{\downarrow \mu(\beta)} \\ \dots \\ \mathbf{A}_{G_{\mathbf{P}}^{-1}(\beta)} \end{array} \end{array} \quad (3.25)$$

Item (iii) of Definition 3.8 tells us that this is precisely the no-signalling condition

$$\mathbf{A}_{\mathbf{A} \setminus G_{\mathbf{P}}^{-1}(\beta)} \rightarrow_{\mathcal{E}} \mathbf{B}_{\mu^{-1}(\downarrow \mu(\beta))}. \quad (3.26)$$

The premise that  $G_{\mathbf{P}} \cap (\alpha \times \beta) = \emptyset$  implies that  $\alpha \subseteq \mathbf{A} \setminus G_{\mathbf{P}}^{-1}(\beta)$ ; it is also the case that  $\beta \subseteq \mu^{-1}(\downarrow \mu(\beta))$ . The no-signalling condition of Eq. (3.26) therefore also implies  $\mathbf{A}_{\alpha} \rightarrow_{\mathcal{E}} \mathbf{B}_{\beta}$ , as desired.  $\square$

Due to causal atomicity (Proposition 3.11), the special case of the above Proposition for unitary channels can be stated compactly.

**Corollary 3.14** (Soundness of no-connectivity for no-influence). *If a unitary channel  $\mathcal{U}$  has a circuit decomposition  $\mathbf{P}$ , then  $G_{\mathcal{U}} \subseteq G_{\mathbf{P}}$ .*

One way of regarding a circuit decomposition is therefore as a possible more ‘fine-grained’ description of the process giving rise to  $\mathcal{U}$ , in which the lack of causal influence between certain input and output systems is made immediately clear through the lack of causal mediators (systems along paths connecting those inputs and outputs). Now, as we saw in Definition 3.8, if a unitary  $\mathcal{U}$  satisfies a no-influence constraint of the form  $\mathbf{A}_{\alpha} \rightarrow_{\mathcal{U}} \mathbf{B}_{\beta}$ , then this can always be reflected in a no-connectivity relation of

an appropriate circuit decomposition (Eq. (3.17)). Considering instead a more general causal constraint expressed by the fact that  $G_{\mathcal{U}} \subseteq G$  for some relation  $G \subseteq \mathbf{A} \times \mathbf{B}$ , a naturally arising question, and one motivated by the applications described earlier, is whether also the constraint  $G_{\mathcal{U}} \subseteq G$  can always be reflected by corresponding no-connectivity relations in a single circuit—without making any further assumptions on  $\mathcal{U}$ . This gives rise to the definition below.

**Definition 3.15.** Let  $G \subseteq \mathbf{A} \times \mathbf{B}$  be a relation and  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  a unitary channel that satisfies the causal constraint  $G_{\mathcal{U}} \subseteq G$ . We say that a circuit decomposition  $\mathbf{P}$  of  $\mathcal{U}$  **represents** this constraint if its connectivity satisfies  $G_{\mathbf{P}} \subseteq G$ . After all, in this case it directly implies the causal constraint  $G_{\mathcal{U}} \subseteq G$  by the chain of inclusions  $G_{\mathcal{U}} \subseteq G_{\mathbf{P}} \subseteq G$ , appealing to soundness (Corollary 3.14). A **causal decomposition** is a circuit decomposition representing a given causal constraint; a **unitary causal decomposition** one all of whose gates are themselves unitary.

Moreover, we say that a relation  $G \subseteq \mathbf{A} \times \mathbf{B}$  **implies (unitary) causal decompositions** if for each choice of quantum systems  $\mathbf{A}_{\mathbf{a}}$  for  $\mathbf{a} \in \mathbf{A}$  and  $\mathbf{B}_{\mathbf{b}}$  for  $\mathbf{b} \in \mathbf{B}$ , every unitary channel  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  satisfying the causal constraint  $G_{\mathcal{U}} \subseteq G$  admits of a (unitary) causal decomposition representing the constraint. By Corollary 3.5 the latter condition is equivalent to requiring the existence of a (unitary) circuit decomposition of  $\mathcal{U}$  with shape  $\mathbf{L}_G$ .

In other words, a causal decomposition diagrammatically exposes the no-influence constraints  $G_{\mathcal{U}} \subseteq G$  by factoring the inclusion through  $G_{\mathbf{P}}$  as  $G_{\mathcal{U}} \subseteq G_{\mathbf{P}} \subseteq G$ .

In the next section we shall characterise those  $G$  that imply *unitary* causal decompositions, specifically. Our motivation for requiring gates to be unitary is twofold. First of all, unitary transformations  $\mathcal{U}$  play a fundamental role in physics by describing the evolution of isolated quantum systems. If the purpose of a circuit decomposition of  $\mathcal{U}$  is to describe the physics of such an evolution on a more fine-grained level, for instance to explain no-influence by a lack of mediating systems, then it is natural to require that its constituent gates also correspond to evolutions of isolated systems. Secondly, if  $G$  implies unitary causal decompositions, then the systems  $\mathbf{Z}_p^q$  and gates  $\mathcal{E}_p$  constituting the circuit decompositions with shape  $\mathbf{L}_G$  parametrise the class of unitary channels  $\mathcal{U}$  satisfying  $G_{\mathcal{U}} \subseteq G$ , thereby providing generalised autonomous causal mechanisms for the quantum causal models of [4, 11], as described in Chapter 1. For this application it is of course crucial that any combination of parameters in fact specifies a channel that belongs to the class—i.e. one that is, in particular, unitary. Since inserting generic channels into the boxes of a circuit shape generally yields non-unitary channels, it makes sense to focus on circuit decompositions with only unitary gates.

### 3.4.1 Causally faithful decompositions

In light of some ambiguity of the term ‘causal structure’ in the literature, it is worth making a comment about the different roles that binary relations  $G \subseteq \mathbf{A} \times \mathbf{B}$  can play. As a bottom-up notion, a relation  $G \subseteq \mathbf{A} \times \mathbf{B}$  appears as the *connectivity relation* of a circuit;

as a top-down notion, a relation is instead about causal influences through unitary transformations. Importantly, however, for the second kind there are two distinct views on what  $G$  is taken to represent. Loosely speaking, on a ‘positive view’, to consider a relation  $G$  is to assert a set of causal influences that *are present*; on a ‘negative view’, it is instead to assert a set of causal influences that *may be present*. We reserve the term *causal structure* for the former view. Every unitary channel  $\mathcal{U}$  thus has a unique, well-defined causal structure: it is the relation  $G_{\mathcal{U}}$  from Definition 3.12. In contrast, the ‘negative view’ sees  $G$  as encoding *no-influence constraints*; compatibility of a unitary  $\mathcal{U}$  with these constraints is the statement that  $G_{\mathcal{U}} \subseteq G$ .

The connectivity of a circuit decomposition is best seen as imposing constraints on the causal structure of a unitary in the negative sense. Indeed, while the absence of a path connecting  $\mathbf{a}$  to  $\mathbf{b}$  implies the absence of causal influence, the *presence* of one does in no way guarantee the presence of influence. For this reason, the role we have assigned to  $G$  in the discussion about causal decompositions above is the negative one. It is also most relevant to the parametrisation application (the class parametrised by the gates and wires of  $L_G$  is that of the unitaries satisfying  $G_{\mathcal{U}} \subseteq G$ , not  $G_{\mathcal{U}} = G$ ) as well as to the spacetime application to be discussed in Chapter 4 (indeed, spacelike separation rules out causal influence, but timelike separation does not necessitate it).

However, one of the advantages of the top-down approach to quantum causal influences from [4, 11, 12, 61, 62] is that it allows one to speak of causation also in the positive sense. Rather than something imposed by an external causal framework such as a relativistic spacetime or the topology of a circuit or network, this enables a more primitive view of causation as a property of unitary dynamics itself, which is helpful in a foundational context (see e.g. [74]). The following special type of causal decomposition then becomes relevant.

**Definition 3.16.** A circuit decomposition  $P$  of  $\mathcal{U}$  is **causally faithful** if  $G_P = G_{\mathcal{U}}$ ; that is, if it reflects precisely all no-influence relations satisfied by  $\mathcal{U}$ . We say that the relation  $G \subseteq A \times B$  **implies (unitary) causally faithful decompositions** if for each choice of quantum systems  $A_{\mathbf{a}}$  for  $\mathbf{a} \in A$  and  $B_{\mathbf{b}}$  for  $\mathbf{b} \in B$ , every unitary channel  $\mathcal{U} : A_A \rightarrow B_B$  with causal structure  $G_{\mathcal{U}} = G$  admits a (unitary) causally faithful circuit decomposition, or, equivalently, a (unitary) circuit decomposition with shape  $L_G$ .

Most results of Lorenz and Barrett [62] were formulated as the statement that  $G$  implies causally faithful decompositions, but actually proved the at least as general statement that  $G$  implies causal decompositions. In the rest of this chapter we will keep both views into account and we will see in Theorem 3.23 that implying unitary causal decompositions is in fact equivalent to implying unitary causally faithful decompositions.

### 3.4.2 Existing results and routed quantum circuits

The question now becomes: Which relations  $G \subseteq A \times B$  imply unitary causal, or causally faithful, decompositions? Lorenz and Barrett [62] proved a number of positive results,

but also found counterexamples. The simplest counterexample, which turns out crucial for the treatment of the general case in Section 3.5, is the following.

**Example 3.17.** Let  $\mathbf{A} = \mathbf{B} = \{1, 2, 3\}$  and consider the 3-qubit unitary channel

$$\begin{array}{|c|c|c|} \hline \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \hline \boxed{\mathcal{U}_3} \\ \hline \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \hline \end{array} := \begin{array}{|c|c|c|} \hline \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \hline \oplus & \bullet & \oplus \\ | & | & | \\ \hline \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \hline \end{array}, \quad (3.27)$$

consisting of two CNOT gates, where  $\mathbf{A}_i \cong \mathbf{B}_i \cong \mathcal{L}(\mathbb{C}^2)$  for  $i \in \{1, 2, 3\}$ . The causal structure of this unitary channel is  $G_{\mathcal{U}_3} = C_3$  from Eq. (2.12), which we reproduce here for convenience (and with appropriate labels):

$$C_3 = \begin{array}{ccc} 1 & 2 & 3 \\ \uparrow & \times & \uparrow \\ 1 & 2 & 3 \end{array}. \quad (3.28)$$

Indeed, it is clear that  $\mathbf{A}_3 \not\rightarrow_{\mathcal{U}_3} \mathbf{B}_1$  since there is no path between those systems in the circuit representation of  $\mathcal{U}_3$  given above (cf. Proposition 3.13). Since the two CNOTs commute, we also have  $\mathbf{A}_1 \not\rightarrow_{\mathcal{U}_3} \mathbf{B}_3$ . Meanwhile, there is influence between all other systems.

However,  $\mathcal{U}_3$  does not admit a unitary causally faithful circuit decomposition. If it did, then by Corollary 3.5 it would have to have one of the particular shape  $L_{C_3}$  from Eq. (2.13). That is however easily seen to be impossible, i.e.

$$\begin{array}{|c|c|c|} \hline \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \hline \oplus & \bullet & \oplus \\ | & | & | \\ \hline \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \hline \end{array} \neq \begin{array}{|c|c|c|} \hline \mathbf{B}_1 & \mathbf{B}_2 & \mathbf{B}_3 \\ \hline \boxed{\mathcal{Z}} \\ \hline \mathcal{X} & \begin{array}{c} \xrightarrow{E} \\ \xrightarrow{F} \end{array} & \mathcal{Y} \\ \hline \mathcal{C} & \mathcal{D} & \\ \hline \mathcal{W} \\ \hline \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{A}_3 \\ \hline \end{array} \quad (3.29)$$

for any choice of systems  $\mathbf{C}, \mathbf{D}, \mathbf{E}, \mathbf{F}$  and unitary channels  $\mathcal{W}, \mathcal{X}, \mathcal{Y}, \mathcal{Z}$ . Indeed, since  $\mathbf{A}_2$  is a qubit and  $\mathcal{W}$  is unitary, either of the two systems  $\mathbf{C}, \mathbf{D}$  would have to be trivial, contradicting the fact that  $\mathcal{U}_3$  has influence  $\mathbf{A}_2 \rightarrow_{\mathcal{U}_3} \mathbf{B}_1$  as well as  $\mathbf{A}_2 \rightarrow_{\mathcal{U}_3} \mathbf{B}_3$ .

As a result,  $C_3$  does not imply unitary causal or causally faithful decompositions. //

It was shown in [62] that the situation changes when considering an extended quantum circuit paradigm that, in addition to tensor products and sequential composition, incorporates direct sums. This extended paradigm has been formalised in the framework of *routed quantum circuits* [104, 107], which turn out to naturally model phenomena outside the context of causal decompositions as well [74, 75, 104, 106, 107, 110]. Routed circuits come with a graphical circuit language in which the direct sum structure is represented by indices on the wires. This circuit language satisfies soundness of no-connectivity for no-signalling and no-influence, i.e. statements analogous to

Proposition 3.13 and Corollary 3.14 (see [62, App. A.10] and [107]). It therefore makes sense to consider causal and causally faithful decompositions in terms of routed circuits as well, and in particular—for the same reasons as described above—in terms of *unitary* routed circuits. Lorenz and Barrett [62] showed that some unitaries  $\mathcal{U}$  may admit unitary routed circuit decompositions of a given connectivity  $G$  even if they do not admit unitary circuit decompositions with that connectivity in the traditional, non-routed circuit formalism: as an example,  $\mathcal{U}_3$  from Eq. (3.27) admits a unitary routed circuit decomposition of the shape in Eq. (3.29). In fact, they showed that every tripartite unitary channel  $\mathcal{U}$  with  $G_{\mathcal{U}} \subseteq C_3$  does [62, Theorem 3]: in other words,  $C_3$  implies unitary routed causal decompositions even though it does not imply unitary causal decompositions.

It is worth noting that any unitary routed circuit may be converted into a traditional quantum circuit of the same shape that expresses the same overall channel; however, in the process, one generally has to give up unitarity of the gates. Indeed, as far as causal decompositions are concerned, the merit of generalising to routed circuits predominantly lies in preserving unitarity of the gates in the decomposition. (The generalisation makes much sense from a mathematical perspective, too: traditional unitary quantum circuits can be regarded as decompositions of  $*$ -isomorphisms with the special property that the *intermediate*  $C^*$ -algebras (inner wires in the circuit) are all factors (algebras with trivial centre; see Section 3.6), while *routed* unitary decompositions also allow for non-factor algebras.)

One may thus consider, for each relation  $G \subseteq \mathbf{A} \times \mathbf{B}$  and unitary channel  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  satisfying  $G_{\mathcal{U}} \subseteq G$ , the statements

- (i)  $\mathcal{U}$  admits a *unitary circuit decomposition* with connectivity  $G$  (i.e. into unitary gates in the traditional, non-routed circuit setting);
- (ii)  $\mathcal{U}$  admits a *unitary routed circuit decomposition* with connectivity  $G$ ;
- (iii)  $\mathcal{U}$  admits a *circuit decomposition* with connectivity  $G$  (i.e. into generic CPTP gates in the traditional circuit setting).

We have (i)  $\Rightarrow$  (ii)  $\Rightarrow$  (iii) while, as witnessed by Example 3.17 and [62, Theorem 3], the direction (i)  $\Leftarrow$  (ii) generally fails.

Lorenz and Barrett [62] proved that

**Theorem 3.18** ([62]). *If  $|\mathbf{A}| \leq 3$  or  $|\mathbf{B}| \leq 3$ , then any  $G \subseteq \mathbf{A} \times \mathbf{B}$  implies routed unitary causal decompositions, and therefore also implies causal decompositions (into generic channels).*

They showed the same for a handful of relations  $G$  in the case where  $|\mathbf{A}| = |\mathbf{B}| = 4$ . Future work will however show that the direction (ii)  $\Leftarrow$  (iii) above also fails in general, and that some four-input, four-output unitaries do not admit causally faithful routed unitary decompositions (of type (ii)), even if they do still admit causally faithful decompositions (of type (iii)). The following remains an open problem.

**Conjecture 3.19.** *Every relation  $G \subseteq \mathbf{A} \times \mathbf{B}$  implies causal decompositions. Equivalently, every unitary  $\mathcal{U}$  admits a causally faithful circuit decomposition.*

### 3.4.3 Decompositions of generic quantum channels

Even prior to the question of whether to allow the gates in causal decompositions to be generic quantum channels or only unitary ones, one might wonder why we require that the overall channel  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$ —that which is to be decomposed—be unitary to begin with. Indeed, many of the notions we have defined in the preceding sections were stated for general channels  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$ : they have circuit decompositions, as defined in Definition 3.2, and satisfy (no-)signalling relations as in Definition 3.8; moreover, Proposition 3.13 showed that no-connectivity implies no-signalling through general channels. One could define a relation  $G_{\mathcal{E}} \subseteq \mathbf{A} \times \mathbf{B}$  specifying which single-system signalling relations  $\mathbf{A}_{\mathbf{a}} \rightarrow_{\mathcal{E}} \mathbf{B}_{\mathbf{b}}$  hold in  $\mathcal{E}$ , and wonder whether, or in what special cases, there exist circuit decompositions  $\mathbf{P}$  of  $\mathcal{E}$  that explain a given collection of no-signalling relations  $G_{\mathcal{E}} \subseteq G$  by factoring the inclusion through as  $G_{\mathcal{E}} \subseteq G_{\mathbf{P}} \subseteq G$ .

One can generally however not expect such decompositions to exist. One reason is that signalling through generic channels is not *atomic* in the sense of Proposition 3.11: if  $\mathcal{E}$ , for instance, satisfies the no-signalling relations  $\mathbf{A}_{\mathbf{a}} \nrightarrow_{\mathcal{E}} \mathbf{B}_{\mathbf{b}_1}$  and  $\mathbf{A}_{\mathbf{a}} \nrightarrow_{\mathcal{E}} \mathbf{B}_{\mathbf{b}_2}$  for some  $\mathbf{a} \in \mathbf{A}$  and  $\mathbf{b}_1, \mathbf{b}_2 \in \mathbf{B}$ , then it may still exhibit signalling to the composite system  $\mathbf{A}_{\mathbf{a}} \rightarrow_{\mathcal{E}} \mathbf{B}_{\{\mathbf{b}_1, \mathbf{b}_2\}}$ . An example is the qubit channel

$$\begin{array}{c} \mathbf{B}_{\mathbf{b}_1} \mathbf{B}_{\mathbf{b}_2} \\ \boxed{\mathcal{E}} \\ \mathbf{A}_{\mathbf{a}} \end{array} := \begin{array}{c} \mathbf{B}_{\mathbf{b}_1} \mathbf{B}_{\mathbf{b}_2} \\ \text{---} \oplus \text{---} \\ \triangle \Phi^+ \\ \mathbf{A}_{\mathbf{a}} \end{array}, \quad (3.30)$$

where  $\Phi^+ := (|00\rangle + |11\rangle) / \sqrt{2}$  is the maximally entangled state. The connectivity of a quantum circuit, on the other hand, *is* atomic, in the sense that the absence of paths from  $\mathbf{a}$  to  $\mathbf{b}_1$  and from  $\mathbf{a}$  to  $\mathbf{b}_2$  through a circuit decomposition of  $\mathcal{E}$  implies the absence of a path from  $\mathbf{a}$  to the joint system  $\{\mathbf{b}_1, \mathbf{b}_2\}$ —which in turn necessarily implies, by Proposition 3.13, the no-signalling relation  $\mathbf{A}_{\mathbf{a}} \nrightarrow_{\mathcal{E}} \mathbf{B}_{\{\mathbf{b}_1, \mathbf{b}_2\}}$ . Equality  $G_{\mathcal{E}} = G_{\mathbf{P}}$  for the example above can thus not be reached: any circuit decomposition satisfies the strict inclusion  $G_{\mathcal{E}} \subsetneq G_{\mathbf{P}}$ .

A more fundamental understanding of this failure can be reached by considering unitary dilations of  $\mathcal{E}$ —unitary channels  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \mathbf{E} \rightarrow \mathbf{B}_{\mathbf{B}} \mathbf{F}$  satisfying  $\mathcal{E}(-) = \text{Tr}_{\mathbf{F}} \mathcal{U}(- \otimes \rho)$  for some state  $\rho \in \mathbf{E}$ —and their causal structure  $G_{\mathcal{U}} \subseteq (\mathbf{A} \cup \{\mathbf{e}\}) \times (\mathbf{B} \cup \{\mathbf{f}\})$ . Note first of all the following, which can be straightforwardly verified.

**Proposition 3.20.** *If  $\mathcal{E}$  has a unitary dilation  $\mathcal{U}$ , then for any  $\alpha \subseteq \mathbf{A}, \beta \subseteq \mathbf{B}$ , we have  $\mathbf{A}_{\alpha} \nrightarrow_{\mathcal{U}} \mathbf{B}_{\beta} \implies \mathbf{A}_{\alpha} \nrightarrow_{\mathcal{E}} \mathbf{B}_{\beta}$ . In particular,  $G_{\mathcal{E}} \subseteq G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B})$ .*

In other words, the absence of *influence* on the fundamental, unitary level implies the absence of *signalling* on the operational level. The converse however generally fails:

influences on the unitary level may become unobservable on the operational level due to a fine-tuned choice of dilation state  $\rho$  (see Section 4.1.2). In fact, *every* unitary dilation  $\mathcal{U}$  of the example channel in Eq. (3.30) satisfies the strict inclusion  $G_{\mathcal{E}} \subsetneq G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B})$ . Indeed, suppose that  $G_{\mathcal{E}} = G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B})$ , so that in particular  $\mathbf{A}_{\mathbf{a}} \not\rightarrow_{\mathcal{U}} \mathbf{B}_{\mathbf{b}_1}$  and  $\mathbf{A}_{\mathbf{a}} \not\rightarrow_{\mathcal{U}} \mathbf{B}_{\mathbf{b}_2}$ ; causal atomicity, which does hold for  $\mathcal{U}$  (Proposition 3.11), would then imply that also  $\mathbf{A}_{\mathbf{a}} \not\rightarrow_{\mathcal{U}} \mathbf{B}_{\{\mathbf{b}_1, \mathbf{b}_2\}}$ . This would in turn mean that  $\mathbf{A}_{\mathbf{a}} \not\rightarrow_{\mathcal{E}} \mathbf{B}_{\{\mathbf{b}_1, \mathbf{b}_2\}}$  (Proposition 3.20), contradicting the definition given in Eq. (3.30).

Now, the connectivity of a circuit decomposition does in fact not only impose no-signalling relations through  $\mathcal{E}$ , but also tells us about no-influence relations through its unitary dilations:

**Proposition 3.21.** *If a channel  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  has a circuit decomposition  $\mathbf{P}$  with connectivity  $G_{\mathbf{P}}$ , then it admits of a unitary dilation  $\mathcal{U} : \mathbf{A}_{\mathbf{A}}\mathbf{E} \rightarrow \mathbf{B}_{\mathbf{B}}\mathbf{F}$  that itself has a unitary circuit decomposition  $\mathbf{P}_{\mathcal{U}}$  whose connectivity relation  $G_{\mathbf{P}_{\mathcal{U}}} \subseteq (\mathbf{A} \cup \{\mathbf{e}\}) \times (\mathbf{B} \cup \{\mathbf{f}\})$  satisfies  $G_{\mathbf{P}_{\mathcal{U}}} \cap (\mathbf{A} \times \mathbf{B}) = G_{\mathbf{P}}$ . Overall, then, we have*

$$G_{\mathcal{E}} \subseteq G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B}) \subseteq G_{\mathbf{P}_{\mathcal{U}}} \cap (\mathbf{A} \times \mathbf{B}) = G_{\mathbf{P}}. \quad (3.31)$$

*Proof.* For each  $p \in \mathbf{P}$ , the circuit shape underlying  $\mathbf{P}$ , dilate the gate  $\mathcal{E}_p : \mathbf{A}_{\lambda^{-1}(p)} \mathbf{Z}_{\mathbf{Pa}(p)}^p \rightarrow \mathbf{Z}_p^{\text{Ch}(p)} \mathbf{B}_{\mu^{-1}(p)}$  to a unitary  $\mathcal{U}_p : \mathbf{A}_{\lambda^{-1}(p)} \mathbf{Z}_{\mathbf{Pa}(p)}^p \mathbf{E}_p \rightarrow \mathbf{Z}_p^{\text{Ch}(p)} \mathbf{B}_{\mu^{-1}(p)} \mathbf{F}_p$  with dilating systems  $\mathbf{E}_p$  and  $\mathbf{F}_p$  and dilation state  $\rho_{\mathbf{E}_p} \in \mathbf{E}_p$ . Collect all dilating systems together into  $\mathbf{E} := \bigotimes_{p \in \mathbf{P}} \mathbf{E}_p$  and  $\mathbf{F} := \bigotimes_{p \in \mathbf{P}} \mathbf{F}_p$  and define the state  $\rho_{\mathbf{E}} := \bigotimes_{p \in \mathbf{P}} \rho_{\mathbf{E}_p}$ . Let  $\mathbf{P}_{\mathcal{U}}$  be the circuit consisting of the unitary gates  $\mathcal{U}_p$  and  $\mathcal{U} : \mathbf{A}_{\mathbf{A}}\mathbf{E} \rightarrow \mathbf{B}_{\mathbf{B}}\mathbf{F}$  the unitary it implements; then  $\mathcal{U}$  is a unitary dilation of  $\mathcal{E}$  and  $\mathbf{P}_{\mathcal{U}}$  satisfies the required connectivity constraint.  $\square$

The observation about the example  $\mathcal{E}$  in Eq. (3.30) we made above—that every circuit decomposition  $\mathbf{P}$  of  $\mathcal{E}$  necessarily satisfies  $G_{\mathcal{E}} \subsetneq G_{\mathbf{P}}$ —can thus be understood as a consequence of the combination of two facts: first, that all of  $\mathcal{E}$ ’s unitary dilations satisfy  $G_{\mathcal{E}} \subsetneq G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B})$ , and second, that the circuit decomposition  $\mathbf{P}$  implies the existence of a unitary dilation with  $G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B}) \subseteq G_{\mathbf{P}}$ . More generally speaking, *signalling* and *influence* are notions that should not be conflated, and out of the two, it is influence that is more closely related to compositional structure (see Eq. (3.31)).

Finally, even in cases where equality  $G_{\mathcal{E}} = G_{\mathbf{P}}$  cannot be reached, one may nevertheless be interested in determining whether  $\mathcal{E}$  admits a circuit decomposition with some other given connectivity  $G_{\mathbf{P}}$ . However, as Proposition 3.21 shows, the existence of such a decomposition is equivalent to the existence of an appropriate circuit decomposition of one of  $\mathcal{E}$ ’s unitary dilations. It is for these reasons that in the rest of this chapter, we will focus on the purely unitary case.

### 3.5 Characterisation of unitary causal decompositions

We now present the main contribution of this chapter: a combinatorial characterisation of those relations  $G \subseteq \mathbf{A} \times \mathbf{B}$  that imply unitary causal and causally faithful decompos-

itions. Recall from Example 3.17 that the relation  $C_3$  does not satisfy this property. The general condition turns out to boil down to the absence of a copy of  $C_3$ .

**Definition 3.22.** A relation  $G \subseteq A \times B$  satisfies the  $C_3$  **exclusion property**, abbreviated ( $C_3$ -**EP**), if for all  $a_1, a_2, a_3 \in A$  and  $b_1, b_2, b_3 \in B$ , we have

$$G \cap (\{a_1, a_2, a_3\} \times \{b_1, b_2, b_3\}) \neq \begin{array}{ccc} b_1 & b_2 & b_3 \\ \uparrow & \nearrow & \uparrow \\ & a_2 & \\ \uparrow & & \uparrow \\ a_1 & & a_3 \end{array}. \quad (C_3\text{-EP})$$

**Theorem 3.23.** Let  $A$  and  $B$  be finite sets and  $G \subseteq A \times B$ . The following are equivalent.

- (i)  $G$  satisfies ( $C_3$ -**EP**).
- (ii)  $G$  implies unitary causal decompositions: that is, for all choices of quantum systems  $A_a$  for  $a \in A$  and  $B_b$  for  $b \in B$ , every unitary channel  $U : A_A \rightarrow B_B$  with causal structure  $G_U \subseteq G$  admits a unitary circuit decomposition  $P$  with connectivity  $G_P \subseteq G$ .
- (iii)  $G$  implies unitary causally faithful decompositions: that is, for all choices of quantum systems  $A_a$  for  $a \in A$  and  $B_b$  for  $b \in B$ , every unitary channel  $U : A_A \rightarrow B_B$  with causal structure  $G_U = G$  admits a unitary circuit decomposition with connectivity  $G_P = G$ , that is, a unitary causally faithful decomposition.

*Proof.* (i)  $\Rightarrow$  (ii) is proven in Theorem 3.26; (ii)  $\Rightarrow$  (iii) is immediate; and (iii)  $\Rightarrow$  (i) is proven in Proposition 3.28.  $\square$

It is important to stress what Theorem 3.23 does *not* show. For one, it is not the case that if  $G$  fails ( $C_3$ -**EP**), then *no* unitary channel  $U$  with  $G_U \subseteq G$  admits a unitary causal decomposition reflecting  $G_U \subseteq G$ . Indeed, there exist unitary channels with causal structure  $G_U = C_3$  that, in contrast to  $U_3$  in Example 3.17, do have a unitary causally faithful decomposition [62]: consider for instance the circuit

$$\begin{array}{ccc} B_{b_1} & B_{b_2} & B_{b_3} \\ \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \\ \begin{array}{c} \diagup \\ \diagdown \end{array} & \begin{array}{c} \diagdown \\ \diagup \end{array} & \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \\ \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} & \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \\ A_{a_1} & A_{a_2} & A_{a_3} \end{array} \quad (3.32)$$

consisting only of qubits and identity gates. It is a unitary causally faithful decomposition for the unitary channel that it defines.

Moreover, Theorem 3.23 characterises which  $G$  imply *unitary* causal decompositions, specifically. As discussed below Example 3.17, every unitary with  $G_U \subseteq C_3$  still admits a *routed* unitary causal decomposition reflecting that constraint (i.e. of the type (ii) mentioned there), and therefore also a causally faithful decomposition into generic CPTP maps (i.e. of type (iii)).

### 3.5.1 Overview of the proof

We start in Section 3.5.2 with a purely syntactical theorem describing a number of equivalent formulations of the  $C_3$  exclusion property. We show, in particular, that the property manifests itself in a simple property of the circuit diagram of  $L_G$ , namely that it does not have more than one path between any given input and output. Note that  $L_{C_3}$  indeed violates this property (see Eq. (3.29)); an example satisfying the property is in Eq. (2.32). Readers familiar with the routed circuit formalism or the results from Lorenz and Barrett [62] might find this result intuitive, as in a routed circuit, any introduction of nontrivial route indices (e.g. at gate  $\mathbf{W}$  in Eq. (3.29)) is necessarily followed by a recombination (at gate  $\mathbf{Z}$ ).

In Section 3.5.3 we use this syntactical result to prove the direction (i)  $\Rightarrow$  (ii) in Theorem 3.23: that is, we prove that if  $G_{\mathcal{U}} \subseteq G$  and  $G$  satisfies  $(C_3\text{-EP})$ , then  $\mathcal{U}$  admits a decomposition with connectivity  $G$ . We do this by induction on  $L_G$ , providing it with appropriate quantum semantics from the bottom upwards. Each induction step crucially relies on an operator-algebraic result (Lemma 3.27 proven in Section 3.6) about the representation of commuting subalgebras on tensor-product factors; after all, a unitary gate  $\mathcal{V}_q$  is nothing but a refactorisation of its input space into a tensor product of output systems. The circuit shape  $L_G$  turns out to be very useful here: not only does it provide the right circuit syntax, but the specific way it was constructed in Chapter 2—as consisting of pairs  $\langle \alpha, \beta \rangle$  of subsets  $\alpha$  of inputs and  $\beta$  of outputs—also acts as a blueprint for the semantic proof, telling us at each induction step which commuting subalgebras need to be considered and where they should go. Moreover, it is the combinatorial properties of  $L_G$  implied by  $(C_3\text{-EP})$  from Section 3.5.2 that guarantee the premises of the operator-algebraic Lemma 3.27, which in turn yields the desired decompositions into tensor products of factor algebras—as opposed to direct sums over tensor products as in Proposition 3.34, which may in some cases instead be used for *routed* circuit decompositions.

The converse direction  $\neg(\text{i}) \Rightarrow \neg(\text{ii})$  of Theorem 3.23 is a more or less straightforward generalisation of Example 3.17, where we exhibited a unitary channel  $\mathcal{U}_3$  with causal structure  $G_{\mathcal{U}_3} = C_3$  that has no unitary circuit decomposition with connectivity  $C_3$ . Showing the stronger claim that  $\neg(\text{i}) \Rightarrow \neg(\text{iii})$  however requires more work. We do this in Section 3.5.4: given an arbitrary relation  $G$  that fails  $(C_3\text{-EP})$ , we construct a unitary channel with causal structure  $G_{\mathcal{U}} = G$  and show that it does not admit a unitary causally faithful circuit decomposition. The constructed unitary  $\mathcal{U}$  contains  $\mathcal{U}_3$  as a factor, and the proof is ultimately, like Example 3.17, based on a dimensionality argument.

### 3.5.2 The $C_3$ exclusion property in terms of the concept lattice

The theorem below proves a number of alternative formulations of the  $C_3$  exclusion property, some in terms of the concept lattice  $L_G$ . Recall that

$$C_3 = \begin{array}{ccc} & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ & \nearrow & \nwarrow & \nearrow \\ \mathbf{a}_1 & & \mathbf{a}_2 & & \mathbf{a}_3 \\ & \nwarrow & \nearrow & \nwarrow \end{array} \quad \text{and} \quad L_{C_3} = \begin{array}{ccc} & \mathbf{b}_1 & \mathbf{b}_2 & \mathbf{b}_3 \\ & | & | & | \\ \square & & \square & \\ & | & | & | \\ \square & & \square & \\ & | & | & | \\ \mathbf{a}_1 & & \mathbf{a}_2 & & \mathbf{a}_3 \end{array}; \quad (3.33)$$

and, from Proposition 2.32, that for any  $v \in L_G$ ,

$$\beta_v = \mu_K^{-1}(v) \cup \bigcup_{w \in \text{Ch}(v)} \beta_w, \quad (3.34)$$

where  $\text{Ch}(v) := \{w \in L_G \mid v \leq w\}$ . Moreover, an **order embedding** is a map  $\varphi : P \rightarrow Q$  between partial orders  $P, Q$  that satisfies  $p \leq_P q \Leftrightarrow \varphi(p) \leq_Q \varphi(q)$ , and a **path** from  $\mathbf{a}$  to  $\mathbf{b}$  through a circuit shape  $P$  is a sequence  $p_1, p_2, \dots, p_n \in P$  such that  $\lambda(\mathbf{a}) = p_1 \leq p_2 \leq \dots \leq p_n = \mu(\mathbf{b})$ . In case  $P$  is finite, there exists such a path iff  $\mathbf{a} G_P \mathbf{b}$ .

**Theorem 3.24.** *Let  $\mathbf{A}$  and  $\mathbf{B}$  be finite sets and  $G \subseteq \mathbf{A} \times \mathbf{B}$ . The following are equivalent.<sup>5</sup>*

- (i)  *$G$  satisfies the  $C_3$  exclusion property ( $C_3$ -EP).*
- (ii) *The circuit shape  $L_{C_3}$  does not embed into  $L_G$ , in the following sense: denoting by  $\perp, \top \in L_{C_3}$  respectively the minimum and maximum of  $L_{C_3}$ , there is no order embedding  $\varphi : L_{C_3} \rightarrow L_G$  such that both  $\alpha_{\varphi(\perp)}$  and  $\beta_{\varphi(\top)}$  are nonempty.*
- (iii) *The circuit shape  $L_G$  has, for each  $\mathbf{a} \in \mathbf{A}$  and  $\mathbf{b} \in \mathbf{B}$ , no more than one path from  $\mathbf{a}$  to  $\mathbf{b}$ .*
- (iv) *For all  $v \in L_G$  so that  $\alpha_v \neq \emptyset$  and distinct  $w, w' \in \text{Ch}(v)$ ,  $\beta_w \cap \beta_{w'} = \emptyset$ : that is, the union in Eq. (3.34) is disjoint whenever  $\alpha_v \neq \emptyset$ . (Or, equivalently, the dual statement.)*
- (v) *For all  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathbf{B}$ , either  $\mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\}) \cap \mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\}) = \emptyset$  or  $\mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\}) \subseteq \mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\})$  or  $\mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\}) \supseteq \mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\})$ .*

Condition (ii) is closely analogous to the  $C_3$  exclusion property itself in that it asserts the absence of a particular substructure, albeit now on the level of the canonical circuit shape. Condition (iii) is the condition that is graphically easiest to check.<sup>6</sup> Note that since  $L_G$  has connectivity  $G$ , another way of phrasing (iii) is that  $L_G$  has exactly one

<sup>5</sup>In case  $\mathbf{A}$  and  $\mathbf{B}$  are infinite, items (i), (ii) and (v) are still equivalent, as can be seen from the proof below. However, as far as the application of this theorem to quantum circuits in Section 3.5.3 is concerned, we can assume that  $\mathbf{A}$  and  $\mathbf{B}$  are finite.

<sup>6</sup>Condition (iii) is also meaningful in the context of routed circuits. In a routed circuit, any nontrivial sectorial correlations (repeated indices) created at a bifurcation in the circuit must later be recombined

path between each  $\mathbf{a}$  and  $\mathbf{b}$  such that  $\mathbf{a}G\mathbf{b}$ . Condition (iv) is most directly useful for our proof in Section 3.5.3. We have included condition (v) for completeness.

*Proof.* Many of these implications are most straightforwardly proven in their contrapositive form. To avoid an overabundance of subscripts we will write  $\lambda := \lambda_{L_G}$  and  $\mu := \mu_{L_G}$  for the maps defined in Eq. (2.27).

Our proof for the direction  $\neg(i) \Rightarrow \neg(ii)$  builds on a general result in formal concept analysis [46, Proposition 38] which states that if  $G \subseteq A \times B$ ,  $A' \subseteq A$ ,  $B' \subseteq B$ , and  $G' := G \cap (A' \times B')$ , then the concept lattice  $L_{G'}$  embeds into  $L_G$ . One such order embedding is given by  $\varphi : L_{G'} \rightarrow L_G, v \mapsto \langle \mathbf{pc}(\alpha_v), \mathbf{c}(\alpha_v) \rangle$ . Here  $(\mathbf{c}, \mathbf{p})$  is the Galois connection derived from  $G$ ; the relation  $G'$  comes with its own Galois connection  $(\mathbf{c}', \mathbf{p}')$  between  $\mathcal{P}(A')$  and  $\mathcal{P}(B')$ . For completeness we will show that  $\varphi$  is indeed an order embedding. This amounts to demonstrating that for all  $\alpha_1, \alpha_2 \subseteq A'$  that are closed with respect to  $\mathbf{p}'\mathbf{c}'$ , we have

$$\alpha_1 \subseteq \alpha_2 \iff \mathbf{pc}(\alpha_1) \subseteq \mathbf{pc}(\alpha_2). \quad (3.35)$$

The left-to-right direction is immediate from Eq. (2.19). For the right-to-left direction, note that  $\mathbf{pc}(\alpha_1) \subseteq \mathbf{pc}(\alpha_2)$  implies  $\mathbf{c}(\alpha_1) \supseteq \mathbf{c}(\alpha_2)$  (by Eqs. (2.19) and (2.21)), which in particular means that  $\mathbf{c}(\alpha_1) \cap B' \supseteq \mathbf{c}(\alpha_2) \cap B'$ . Since  $\mathbf{c}'(-) = \mathbf{c}(-) \cap B'$ , which can be easily verified, this means that  $\mathbf{c}'(\alpha_1) \supseteq \mathbf{c}'(\alpha_2)$ , and therefore  $\mathbf{p}'\mathbf{c}'(\alpha_1) \subseteq \mathbf{p}'\mathbf{c}'(\alpha_2)$ . But since  $\alpha_1, \alpha_2$  were closed with respect to  $\mathbf{p}'\mathbf{c}'$ , this just means that  $\alpha_1 \subseteq \alpha_2$ .

Now, assume  $\neg(i)$ , that is, that there exist  $A' := \{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \subseteq A$  and  $B' := \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\} \subseteq B$  so that  $G \cap (A' \times B') = C_3$ . To show  $\neg(ii)$ , it remains to show that  $\alpha_{\varphi(\perp)} \neq \emptyset$  and  $\beta_{\varphi(\top)} \neq \emptyset$ . For the former, note (e.g. from the diagram for  $L_{C_3}$  above) that  $\alpha_{\perp} \neq \emptyset$ ; therefore the closure  $\mathbf{pc}(\alpha_{\perp})$  is nonempty too and thus  $\alpha_{\varphi(\perp)} = \mathbf{pc}(\alpha_{\perp}) \neq \emptyset$ . For the latter, note similarly that  $\mathbf{c}(\alpha_{\top}) \supseteq \mathbf{c}'(\alpha_{\top}) = \beta_{\top} \neq \emptyset$ , so that  $\beta_{\varphi(\top)} = \mathbf{c}(\alpha_{\top}) \neq \emptyset$  either.

Next, let us show that  $\neg(ii) \Rightarrow \neg(i)$ .  $\neg(ii)$  tells us that there exist  $v, w, w', x \in L_G$  so that  $\alpha_v \neq \emptyset \neq \beta_x$ ,  $v < w < x$ ,  $v < w' < x$ , and  $w \# w'$  (the latter meaning  $w$  and  $w'$  are incomparable in the order  $\leq$  on  $L_G$ ). Pick arbitrary  $\mathbf{a}_2 \in \alpha_v$  and  $\mathbf{b}_2 \in \beta_x$ . We claim that it is possible to pick  $\mathbf{a}_1 \in \alpha_w$  and  $\mathbf{b}_3 \in \beta_{w'}$  so that  $\lambda(\mathbf{a}_1) \not\leq \mu(\mathbf{b}_3)$ . Indeed, if all  $\mathbf{a}_1 \in \alpha_w, \mathbf{b}_3 \in \beta_{w'}$  satisfied  $\lambda(\mathbf{a}_1) \leq \mu(\mathbf{b}_3)$ , then we would have  $\bigvee_{\mathbf{a}_1 \in \alpha_w} \lambda(\mathbf{a}_1) \leq \bigwedge_{\mathbf{b}_3 \in \beta_{w'}} \mu(\mathbf{b}_3)$ , which by Proposition 2.32(iii) just means  $w \leq w'$ , in contradiction with the assumption that  $w \# w'$ . Similarly, it is possible to pick  $\mathbf{a}_3 \in \alpha_{w'}$  and  $\mathbf{b}_1 \in \beta_w$  so that  $\lambda(\mathbf{a}_3) \not\leq \mu(\mathbf{b}_1)$ .

Proving  $\neg(i)$  is now a matter of verifying that on the inputs  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  and outputs  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ ,  $G$  restricts to the relation  $C_3$ . First of all,  $\mathbf{a}_1 G \mathbf{b}_1$  follows from the facts that  $L_G$  has connectivity  $G$  and that  $\lambda(\mathbf{a}_1) \leq w \leq \mu(\mathbf{b}_1)$  (by Proposition 2.32(ii) and (i)). Likewise,  $\mathbf{a}_1 G \mathbf{b}_2$  since  $\lambda(\mathbf{a}_1) \leq w < x \leq \mu(\mathbf{b}_2)$ . Meanwhile, we have  $\neg(\mathbf{a}_1 G \mathbf{b}_3)$  by

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(see for instance the circuits of Lorenz and Barrett [62]). If condition (iii) holds, no bifurcations are followed by recombinations in the circuit shape  $L_G$ , except possibly at trivial top and bottom gates as for instance in Eq. (2.32)—and therefore routed unitary circuits are in this case equally powerful to traditional unitary circuits. What this and Theorem 3.23 thus tell us is that whenever the canonical circuit shape *does* allow for genuinely routed circuits (i.e. in case non-unique paths exist), then traditional circuits do not suffice for causal decompositions. (In some such cases routed circuits may suffice, but this has not been proven in general; recall Section 3.4.2).

construction. Similarly it can be shown that  $\mathbf{a}_2 G \mathbf{b}_i$  for all of  $i \in \{1, 2, 3\}$ , and  $\neg(\mathbf{a}_3 G \mathbf{b}_1)$  while  $\mathbf{a}_3 G \mathbf{b}_2$  and  $\mathbf{a}_3 G \mathbf{b}_3$ . This concludes the proof of  $\neg(\text{ii}) \Rightarrow \neg(\text{i})$ .

We omit the proof of equivalence of (ii) and (iii), which is straightforward. Let's prove equivalence of (iii) and (iv). To see that (iii)  $\Rightarrow$  (iv), let  $v \in L_G$  be such that  $\alpha_v \neq \emptyset$  and suppose  $w, w' \in \text{Ch}(v)$  are distinct. Pick  $\mathbf{a} \in \alpha_v$ . If there were  $\mathbf{b} \in \beta_w \cap \beta_{w'}$ , then there would be two distinct paths from  $\mathbf{a}$  to  $\mathbf{b}$ : one through  $v$  and  $w$  and another through  $v$  and  $w'$ .

Conversely, to show that  $\neg(\text{iii}) \Rightarrow \neg(\text{iv})$ , suppose that there are two distinct paths from  $\mathbf{a} \in A$  to  $\mathbf{b} \in B$  in  $L_G$ ; denote them by  $\{p_i\}_{i=1}^s$  and  $\{p'_i\}_{i=1}^{s'} \subseteq L_G$ . Let  $k \in \{1, \dots, \min\{s, s'\}\}$  be the first index where they diverge, so that  $p_k = p'_k$  but  $p_{k+1} \neq p'_{k+1}$ . Let  $v := p_k = p'_k$ ,  $w := p_{k+1}$ , and  $w' := p'_{k+1}$ . Note that  $\alpha_v$  is non-empty, as it contains  $\mathbf{a}$ . We also have  $w = p_{k+1} \leq p_s = \mu(\mathbf{b})$ , meaning  $\mathbf{b} \in \beta_w$ . Similarly,  $\mathbf{b} \in \beta_{w'}$ . This establishes  $\neg(\text{iv})$ .

It remains to verify that (i) and (v) are equivalent. First of all, if  $\neg(\text{v})$ , then there exist  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  so that none of  $\mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\}) \setminus \mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\})$ ,  $\mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\}) \cap \mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\})$ , and  $\mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\}) \setminus \mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\})$  are empty. Let  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  be elements of each of those sets, respectively; then  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$  witness failure of  $(C_3\text{-EP})$ , establishing  $\neg(\text{i})$ . Conversely, if  $G$  contains  $C_3$ , say on the inputs  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$  and the outputs  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ , then  $\mathbf{a}_1 \in \mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\}) \setminus \mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\})$ ,  $\mathbf{a}_2 \in \mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\}) \cap \mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\})$ , and  $\mathbf{a}_3 \in \mathbf{p}(\{\mathbf{b}_2, \mathbf{b}_3\}) \setminus \mathbf{p}(\{\mathbf{b}_1, \mathbf{b}_2\})$ , establishing  $\neg(\text{v})$ .  $\square$

The following consequence of (iv) above will also be useful in Section 3.5.3.

**Lemma 3.25.** *Let  $G$  satisfy  $(C_3\text{-EP})$ , let  $v \in L_G$  be such that  $\alpha_v \neq \emptyset$  and let  $w, w' \in \text{Ch}(v)$  be distinct. We have*

$$G^{-1}(\beta_w) \cap G^{-1}(\beta_{w'}) = \alpha_v. \quad (3.36)$$

*Proof.* It suffices to show that for all  $\mathbf{b} \in \beta_w$  and  $\mathbf{b}' \in \beta_{w'}$ ,  $G^{-1}(\mathbf{b}) \cap G^{-1}(\mathbf{b}') = \mathbf{p}(\{\mathbf{b}, \mathbf{b}'\}) = \alpha_v$ . To do so, note first of all that  $\alpha_v \subseteq \alpha_w = \mathbf{p}(\beta_w) \subseteq \mathbf{p}(\{\mathbf{b}\})$ ; similarly,  $\alpha_v \subseteq \mathbf{p}(\{\mathbf{b}'\})$ . This implies that  $\alpha_v \subseteq \mathbf{p}(\{\mathbf{b}, \mathbf{b}'\})$ , that is,  $v \leq x$  where  $x := \langle \mathbf{p}(\{\mathbf{b}, \mathbf{b}'\}), \mathbf{cp}(\{\mathbf{b}, \mathbf{b}'\}) \rangle \in L_G$ . Suppose that  $v < x$ ; then there is  $w'' \in \text{Ch}(v)$  so that  $v < w'' \leq x$ . We have  $w \neq w''$  or  $w' \neq w''$ ; without loss of generality, say that the former holds. By (iv) in Theorem 3.24, we must have  $\beta_w \cap \beta_{w''} = \emptyset$ . However, both  $\beta_w$  and  $\beta_{w''}$  contain  $\mathbf{b}$  ( $\mathbf{b} \in \beta_w$  by assumption and  $\mathbf{b} \in \mathbf{cp}(\{\mathbf{b}, \mathbf{b}'\}) = \beta_x \subseteq \beta_{w''}$ ), yielding a contradiction. Therefore  $v < x$  cannot hold, and we must have  $v = x$ , or in other words,  $\alpha_v = \mathbf{p}(\{\mathbf{b}, \mathbf{b}'\})$ .  $\square$

### 3.5.3 Sufficiency of the $C_3$ exclusion property

In this section we prove the direction (i)  $\Rightarrow$  (ii) of Theorem 3.23, showing that every unitary channel  $\mathcal{U} : \mathbf{A}_A \rightarrow \mathbf{B}_B$  whose causal structure satisfies  $G_{\mathcal{U}} \subseteq G$  for some relation  $G$  satisfying the  $C_3$  exclusion property admits a unitary circuit decomposition with connectivity  $G$ . In Corollary 3.5 we saw that  $L_G$  is a good candidate for the shape of such a decomposition. What remains is therefore to provide it with a suitable unitary

quantum semantics that implements  $\mathcal{U}$ . For the theorem below, which achieves precisely that, recall our notation from Section 3.1; in particular, for  $\beta \subseteq \mathbf{B}$ , we write  $\mathbf{B}_\beta$  to denote the tensor product  $\bigotimes_{\mathbf{b} \in \beta} \mathbf{B}_{\mathbf{b}}$  but often also let it refer to the corresponding subalgebra  $\mathbf{B}_\beta \otimes \{\mathbf{1}_{\mathbf{B}_{\mathbf{B} \setminus \beta}}\}$  of  $\mathbf{B}_{\mathbf{B}}$ . Furthermore, we write, for  $v \in \mathbf{L}_G$  (recalling Eq. (3.5)),

$$\text{Pa}(v) := \{u \in \mathbf{L}_G \mid u \prec v\}; \quad (3.37)$$

$$\text{Ch}(v) := \{w \in \mathbf{L}_G \mid v \prec w\}; \quad (3.38)$$

$$\downarrow v := \{u \in \mathbf{L}_G \mid u \leq v\}; \quad (3.39)$$

$$\downarrow^* v := \{u \in \mathbf{L}_G \mid u < v\}. \quad (3.40)$$

As in the previous subsection, we abbreviate

$$\lambda := \lambda_{\mathbf{L}_G} \quad \text{and} \quad \mu := \mu_{\mathbf{L}_G}. \quad (3.41)$$

**Theorem 3.26.** *Let  $G \subseteq \mathbf{A} \times \mathbf{B}$  be a binary relation satisfying (C<sub>3</sub>-EP),  $\mathbf{A}_{\mathbf{a}}$  for  $\mathbf{a} \in \mathbf{A}$  and  $\mathbf{B}_{\mathbf{b}}$  for  $\mathbf{b} \in \mathbf{B}$  be quantum systems, and  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  be a unitary channel whose causal structure satisfies  $G_{\mathcal{U}} \subseteq G$ . There exist quantum systems*

$$\mathbf{Z}_v^w \quad \text{for all } v, w \in \mathbf{L}_G \text{ such that } v \prec w \quad (3.42)$$

and unitary channels

$$\mathcal{V}_v : \mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v \xrightarrow{\sim} \mathbf{Z}_v^{\text{Ch}(v)} \mathbf{B}_{\mu^{-1}(v)} \quad \text{for all } v \in \mathbf{L}_G \quad (3.43)$$

such that, defining  $\mathcal{V}_T$ , for any  $T \subseteq \mathbf{L}_G$ , to be the composition of  $\{\mathcal{V}_v \mid v \in T\}$  along the systems  $\{\mathbf{Z}_v^w \mid v, w \in T, v \prec w\}$ , we have that for all  $v \in \mathbf{L}_G$

$$\mathbf{B}_{\mathbf{b}} = \mathcal{V}_{\downarrow v} \mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) \quad \text{for } \mathbf{b} \in \mu^{-1}(v); \quad (3.44a)$$

$$\mathbf{Z}_v^w \subseteq \mathcal{V}_{\downarrow v} \mathcal{U}^\dagger(\mathbf{B}_{\beta_w}) \quad \text{for } w \in \text{Ch}(v). \quad (3.44b)$$

Moreover, the overall unitary channel  $\mathcal{V}_{\mathbf{L}_G}$  is identical to  $\mathcal{U}$  up to local unitary channels on the output systems  $\{\mathbf{B}_{\mathbf{b}}\}_{\mathbf{b} \in \mathbf{B}}$ . Therefore,  $\mathcal{U}$  has a unitary circuit decomposition of shape  $\mathbf{L}_G$ .

We will prove Theorem 3.26 by induction on  $\mathbf{L}_G$ , providing it with a quantum semantics from the bottom upwards. At each  $v \in \mathbf{L}_G$ , we are given incoming systems  $\mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v$  which need to be refactored into appropriate outgoing systems  $\mathbf{Z}_v^{\text{Ch}(v)} \mathbf{B}_{\mu^{-1}(v)}$  by making use of the causal constraint  $G_{\mathcal{U}} \subseteq G$ . (All of these algebras may be trivial if  $v$  is  $\mathbf{L}_G$ 's minimal or maximal element.) Since the latter can be expressed in terms of commutation relations between the sets of algebras  $\{\mathbf{A}_{\mathbf{a}}\}_{\mathbf{a}}$  and  $\{\mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}})\}_{\mathbf{b}}$ , the proof of each induction step reduces to a problem of representing subalgebras subject to commutation relations on appropriate tensor product factors. Specifically, we will rely on Lemma 3.27 below. As this result exposes the operator-algebraic core of our methods, we temporarily adopt the mathematicians' terminology

of ‘finite-dimensional factor  $C^*$ -algebra’ and ‘ $*$ -isomorphism’ where we otherwise have said ‘quantum system’ and ‘unitary channel’, respectively. This is also the terminology used in Section 3.6, where Lemma 3.27 is proven.

**Lemma 3.27.** *Let  $\mathbf{A}, \mathbf{X}_1, \dots, \mathbf{X}_n$  be finite-dimensional factor  $C^*$ -algebras and let  $\mathbf{B}_k \subseteq \mathbf{A}\mathbf{X}_1\mathbf{X}_2 \cdots \mathbf{X}_n$  for  $k \in \{1, \dots, n\} =: [n]$  be unital factor  $*$ -subalgebras of their tensor product so that*

- (i)  $\mathbf{B}_k \subseteq (\mathbf{B}_l)'$  for all  $k, l \in [n], k \neq l$ ;
- (ii)  $\mathbf{B}_k \subseteq \mathbf{A}\mathbf{X}_k = (\mathbf{X}_{[n] \setminus \{k\}})'$  for all  $k \in [n]$ ; and
- (iii)  $\mathbf{A} \subseteq \bigvee_{k \in [n]} \mathbf{B}_k$ , where  $\bigvee$  denotes the algebraic span.

*Then there are finite-dimensional factor  $C^*$ -algebras  $\mathbf{Z}_k$  for each  $k \in [n]$  and a  $*$ -isomorphism*

$$\mathcal{V} : \mathbf{A} \xrightarrow{\sim} \mathbf{Z}_1\mathbf{Z}_2 \cdots \mathbf{Z}_n \quad (3.45)$$

*so that, writing  $\mathcal{V}$  for  $\mathcal{V} \otimes \text{id}_{\mathbf{X}_1 \cdots \mathbf{X}_n}$ , for each  $k$  we have*

$$\mathcal{V}(\mathbf{B}_k) \subseteq \mathbf{Z}_k\mathbf{X}_k \text{ and} \quad (3.46a)$$

$$\mathbf{Z}_k = \mathcal{V}(\mathbf{A} \cap \mathbf{B}_k). \quad (3.46b)$$

*Proof.* See Section 3.6. An updated, nicer proof can be found in the standalone paper based on this chapter [67].  $\square$

It is worth noting that without assumption (iii), a similar representation can be given, with the exception that the right-hand side of Eq. (3.45) will generally be an algebra of operators on a *direct sum over* tensor products of Hilbert spaces: see Proposition 3.34. Such a representation can be used, in some cases, to provide proofs of the *routed* unitary circuit decompositions shown in Lorenz and Barrett [62].<sup>7</sup> The assumption of ( $C_3$ -EP), however, allows us to focus entirely on cases where assumption (iii) is satisfied, therefore yielding the tensor product factorisations in Eq. (3.45) required to construct our (non-routed) unitary circuit.

*Proof of Theorem 3.26.* We proceed by induction on  $L_G$ . The base case—i.e. when  $v$  is the minimal element of  $L_G$ —separates into two cases, where  $\alpha_v = \emptyset$  or  $\alpha_v \neq \emptyset$ .

Suppose first that  $\alpha_v = \emptyset$ . In this case  $\mathbf{A}_{\lambda^{-1}(v)} = \mathbf{A}_\emptyset = \mathbb{C}$  is trivial, as is each  $\mathbf{B}_b$  for  $b \in \mu^{-1}(v)$ , since  $G^{-1}(b) = \alpha_v$  so that  $\mathcal{U}^\dagger(\mathbf{B}_b) \subseteq \mathbf{A}_{\alpha_v} = \mathbf{A}_\emptyset = \mathbb{C}$ . Therefore it suffices to set  $\mathbf{Z}_v^w := \mathbb{C}$  for each  $w \in \text{Ch}(v)$  and to let  $\mathcal{V}_v$  be the trivial transformation between trivial systems. This choice satisfies Eq. (3.44).

The base case where  $\alpha_v \neq \emptyset$  is covered by the induction case (apart from some of the algebras below being trivial), which we turn to now.

So let  $v \in L_G$  be such that  $\alpha_v \neq \emptyset$ ; and assume that there exist systems  $\mathbf{Z}_u^{u'}$  for all  $u, u' \in L_G$  such that  $u \leq u' \leq v$  and, for each  $u < v$ , a unitary channel

<sup>7</sup>The proofs used in Lorenz and Barrett [62] are however of a different nature.

$\mathcal{V}_u : \mathbf{A}_{\lambda^{-1}(u)} \mathbf{Z}_{\text{Pa}(u)}^u \xrightarrow{\sim} \mathbf{Z}_u^{\text{Ch}(u)} \mathbf{B}_{\mu^{-1}(u)}$  such that  $\mathcal{V}_{\downarrow u}$  satisfies (3.44). Consider the composition of these unitary channels  $\mathcal{V}_u$  for  $u < v$ ,

$$\mathcal{V}_{\downarrow^* v} : \mathbf{A}_{\alpha_v \setminus \lambda^{-1}(v)} \rightarrow \mathbf{Z}_{\text{Pa}(v)}^v \mathbf{W}, \quad (3.47)$$

where

$$\mathbf{W} := \bigotimes_{u < v} \mathbf{Z}_u^{\text{Ch}(u) \setminus \downarrow v} \mathbf{B}_{\mu^{-1}(u)}. \quad (3.48)$$

(In the base case, let  $\mathcal{V}_{\downarrow^* v}$  denote the identity channel on the trivial system  $\mathbf{A}_{\alpha_v \setminus \lambda^{-1}(v)} = \mathbf{A}_\emptyset = \mathbb{C}$ .) From now on we will, abusing notation as always, let  $\mathcal{V}_{\downarrow^* v}$  denote instead the extended unitary

$$\mathcal{V}_{\downarrow^* v} \equiv \mathcal{V}_{\downarrow^* v} \otimes \text{id}_{\mathbf{A} \setminus (\alpha_v \setminus \lambda^{-1}(v))} : \mathbf{A}_{\mathbf{A}} \xrightarrow{\sim} \mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v \mathbf{W} \mathbf{A}_{\mathbf{A} \setminus \alpha_v}. \quad (3.49)$$

We now wish to construct systems  $\mathbf{Z}_w^v$  for  $w \in \text{Ch}(v)$  and a unitary channel  $\mathcal{V}_v : \mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v \xrightarrow{\sim} \mathbf{Z}_v^{\text{Ch}(v)} \mathbf{B}_{\mu^{-1}(v)}$  so that Eq. (3.44) is satisfied for  $\mathcal{V}_{\downarrow v}$ . We will do this by an application of Lemma 3.27. To this end, consider the following subalgebras of  $\mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v \mathbf{W} \mathbf{A}_{\mathbf{A} \setminus \alpha_v}$ :

$$\tilde{\mathbf{B}}_{\beta_w} := \mathcal{V}_{\downarrow^* v} \mathcal{U}^\dagger(\mathbf{B}_{\beta_w}) \quad \text{for } w \in \text{Ch}(v) \quad (3.50a)$$

$$\text{and } \tilde{\mathbf{B}}_{\mathbf{b}} := \mathcal{V}_{\downarrow^* v} \mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) \quad \text{for } \mathbf{b} \in \mu^{-1}(v). \quad (3.50b)$$

We prove the following claims necessary for the application of Lemma 3.27. Keep in mind that  $\mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v$  is to be the input to the unitary  $\mathcal{V}_v$  that we are looking to construct, and thus plays the role of the factor  $\mathbf{A}$  in Lemma 3.27.

(i) All subalgebras defined in Eq. (3.50) commute pairwise:

$$\tilde{\mathbf{B}}_{\mathbf{b}} \subseteq (\tilde{\mathbf{B}}_{\mathbf{b}'})' \quad \text{for all distinct } \mathbf{b}, \mathbf{b}' \in \mu^{-1}(v); \quad (3.51a)$$

$$\tilde{\mathbf{B}}_{\mathbf{b}} \subseteq (\tilde{\mathbf{B}}_{\beta_w})' \quad \text{for all } \mathbf{b} \in \mu^{-1}(v) \text{ and } w \in \text{Ch}(v); \quad (3.51b)$$

$$\text{and } \tilde{\mathbf{B}}_{\beta_w} \subseteq (\tilde{\mathbf{B}}_{\beta_{w'}})' \quad \text{for all distinct } w, w' \in \text{Ch}(v). \quad (3.51c)$$

(ii) We have

$$\tilde{\mathbf{B}}_{\mathbf{b}} \subseteq \mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v = (\mathbf{W} \mathbf{A}_{\mathbf{A} \setminus \alpha_v})' \quad \text{for } \mathbf{b} \in \mu^{-1}(v); \quad (3.52a)$$

$$\tilde{\mathbf{B}}_{\beta_w} \subseteq \mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v \mathbf{A}_{G^{-1}(\beta_w) \setminus \alpha_v} = (\mathbf{W} \mathbf{A}_{\mathbf{A} \setminus G^{-1}(\beta_w)})' \quad \text{for } w \in \text{Ch}(v), \quad (3.52b)$$

where  $G^{-1}(\beta_w) := \bigcup_{\mathbf{b} \in \beta_w} G^{-1}(\mathbf{b})$ . Moreover, for any two distinct  $w, w' \in \text{Ch}(v)$ , the sets  $G^{-1}(\beta_w) \setminus \alpha_v$  and  $G^{-1}(\beta_{w'}) \setminus \alpha_v$  are disjoint. Therefore, the only tensor factors of  $\mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v \mathbf{W} \mathbf{A}_{\mathbf{A} \setminus \alpha_v}$  on which more than one of the subalgebras from Eq. (3.50) act non-trivially are  $\mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v$ .

(iii) Our algebra of interest  $\mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v$  is spanned by the subalgebras under consideration:

$$\mathbf{A}_{\lambda^{-1}(v)} \mathbf{Z}_{\text{Pa}(v)}^v \subseteq \left( \bigvee_{\mathbf{b} \in \mu^{-1}(v)} \tilde{\mathbf{B}}_{\mathbf{b}} \right) \vee \left( \bigvee_{w \in \text{Ch}(v)} \tilde{\mathbf{B}}_{\beta_w} \right). \quad (3.53)$$

Once (i)–(iii) are established, Lemma 3.27 will give quantum systems  $Z_v^w$  for  $w \in \text{Ch}(v)$  and  $\hat{B}_{\mathbf{b}}$  for  $\mathbf{b} \in \mu^{-1}(v)$ , as well as a unitary channel  $\mathcal{V}_v : A_{\lambda^{-1}(v)} Z_{\text{Pa}(v)}^v \rightarrow Z_v^{\text{Ch}(v)} \hat{B}_{\mu^{-1}(v)}$ . Eq. (3.46b) entails that for  $\mathbf{b} \in \mu^{-1}(v)$ ,

$$\hat{B}_{\mathbf{b}} = \mathcal{V}_v(A_{\lambda^{-1}(v)} Z_{\text{Pa}(v)}^v \cap \tilde{B}_{\mathbf{b}}) = \mathcal{V}_v(\tilde{B}_{\mathbf{b}}) = \mathcal{V}_{\downarrow v} \mathcal{U}^\dagger(B_{\mathbf{b}}), \quad (3.54)$$

where in the second equality we have applied Eq. (3.52a). This means in particular that  $\hat{B}_{\mathbf{b}} \cong B_{\mathbf{b}}$ , so we can choose  $\hat{B}_{\mathbf{b}}$  and  $\mathcal{V}_v$  so that in fact  $\hat{B}_{\mathbf{b}} = B_{\mathbf{b}}$ . This establishes the induction hypothesis Eq. (3.44a).

To establish the other part (3.44b), note that according to Eq. (3.46b),

$$Z_v^w = \mathcal{V}_v(A_{\lambda^{-1}(v)} Z_{\text{Pa}(v)}^v \cap \tilde{B}_{\beta_w}) \subseteq \mathcal{V}_v(\tilde{B}_{\beta_w}) = \mathcal{V}_{\downarrow v} \mathcal{U}^\dagger(B_{\beta_w}). \quad (3.55)$$

It remains to prove (i)–(iii) and the final statement of Theorem 3.26.

*Proof of (i).* Note that these algebras commute iff the same holds for the respective algebras without tildes (see Eq. (3.50)), considered as subalgebras of  $B_{\mathbf{B}}$ . It is clear that  $B_{\mathbf{b}} \subseteq (B_{\mathbf{b}'})' = B_{\mathbf{B} \setminus \{\mathbf{b}'\}}$  for distinct  $\mathbf{b}, \mathbf{b}' \in \mu^{-1}(v)$ , establishing (3.51a). For (3.51b), it suffices to note that if  $\mathbf{b} \in \mu^{-1}(v)$  and  $w \in \text{Ch}(v)$ , then  $\mathbf{b} \notin \beta_w$ . This follows from Proposition 2.32(i), which shows that  $\mathbf{b} \in \mu^{-1}(v)$  iff  $v$  is the largest element of  $L_G$  so that  $\mathbf{b} \in \beta_v$ ; for any  $w \in \text{Ch}(v)$ , we have  $w > v$ , which implies that  $\mathbf{b}$  cannot be in  $\beta_w$ . Finally, for Eq. (3.51c) it suffices to show that  $\beta_w \cap \beta_{w'} = \emptyset$  for distinct  $w, w' \in \text{Ch}(v)$ . This follows by assumption of ( $C_3$ -EP) and Theorem 3.24(iv).

*Proof of (ii).* Recall that

$$W = \bigotimes_{u < v} Z_u^{\text{Ch}(u) \setminus \downarrow v} B_{\mu^{-1}(u)}. \quad (3.56)$$

First we show that

$$W \subseteq \mathcal{V}_{\downarrow^* v} \mathcal{U}^\dagger(B_{\mathbf{B} \setminus \beta_v}). \quad (3.57)$$

Fix  $u \in L_G$  so that  $u < v$  and  $\alpha_u \neq \emptyset$ . (If  $\alpha_u = \emptyset$  then  $Z_u^{\text{Ch}(u) \setminus \downarrow v} B_{\mu^{-1}(u)}$  is trivial so need not be considered.) It follows from the induction hypotheses (3.44) that

$$Z_u^{\text{Ch}(u) \setminus \downarrow v} B_{\mu^{-1}(u)} \subseteq \mathcal{V}_{\downarrow u} \mathcal{U}^\dagger \left( B_{\beta_{\text{Ch}(u) \setminus \downarrow v} \cup \mu^{-1}(u)} \right) \quad (3.58)$$

where  $\beta_{\text{Ch}(u) \setminus \downarrow v} := \bigcup \{ \beta_w \mid w \in \text{Ch}(u) \setminus \downarrow v \}$ . Now,  $\mathcal{V}_{\downarrow^* v} \mathcal{U}^\dagger = \mathcal{V}_{\downarrow^* v \setminus \downarrow u} \circ \mathcal{V}_{\downarrow u} \circ \mathcal{U}^\dagger$ , but  $\mathcal{V}_{\downarrow^* v \setminus \downarrow u}$  acts trivially on  $Z_u^{\text{Ch}(u) \setminus \downarrow v} B_{\mu^{-1}(u)}$ . Therefore we also have

$$Z_u^{\text{Ch}(u) \setminus \downarrow v} B_{\mu^{-1}(u)} \subseteq \mathcal{V}_{\downarrow^* v} \mathcal{U}^\dagger \left( B_{\beta_{\text{Ch}(u) \setminus \downarrow v} \cup \mu^{-1}(u)} \right). \quad (3.59)$$

To establish Eq. (3.57) it remains to show that

$$\beta_{\text{Ch}(u) \setminus \downarrow v} \cap \beta_v = \emptyset \quad \text{and} \quad \mu^{-1}(u) \cap \beta_v = \emptyset. \quad (3.60)$$

The first statement relies on ( $C_3$ -EP). Let  $w \in \text{Ch}(u) \setminus \downarrow v$ . Let  $w'$  be such that  $u < w' \leq v$ . By ( $C_3$ -EP), in particular the formulation of Theorem 3.24(iv), it follows

that  $\beta_w \cap \beta_{w'} = \emptyset$ , and since  $w' \leq v$ , we have  $\beta_v \subseteq \beta_{w'}$  so  $\beta_w \cap \beta_v = \emptyset$  too. This holds for all  $w \in \text{Ch}(u) \setminus \downarrow v$ , which implies that  $\beta_{\text{Ch}(u) \setminus \downarrow v} \cap \beta_v = \emptyset$ .

To show the second equation in (3.60), suppose that  $\mathbf{b} \in \mu^{-1}(u)$ . In this case,  $u$  is the greatest element of  $L_G$  satisfying  $\mathbf{b} \in \beta_u$  (see Proposition 2.32(i)). Since  $v > u$ , we can't have  $\mathbf{b} \in \beta_v$ . This completes the proof of (3.60) and hence of (3.57).

To show Eq. (3.52a), it suffices to show that  $\tilde{\mathbf{B}}_{\mathbf{b}}$ , for  $\mathbf{b} \in \mu^{-1}(v)$ , commutes with  $\mathbf{W}$  and with  $\mathbf{A}_{\mathbf{A} \setminus \alpha_v}$ . Commutation with  $\mathbf{W}$  follows from Eq. (3.57) and the fact that  $\mathbf{b} \in \mu^{-1}(v) \subseteq \beta_v$ . To show commutation with  $\mathbf{A}_{\mathbf{A} \setminus \alpha_v}$ , note that since  $\mathcal{U}$  has causal structure  $G_{\mathcal{U}} \subseteq G$ , we have  $\mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) \subseteq \mathbf{A}_{G_{\mathcal{U}}^{-1}(\mathbf{b})} \subseteq \mathbf{A}_{G^{-1}(\mathbf{b})} = \mathbf{A}_{\mathbf{p}(\{\mathbf{b}\})} = \mathbf{A}_{\alpha_{\mu(\mathbf{b})}} = \mathbf{A}_{\alpha_v}$ . Applying  $\mathcal{V}_{\downarrow^* v}$ , which acts trivially on  $\mathbf{A}_{\mathbf{A} \setminus \alpha_v}$ , to both sides yields  $\tilde{\mathbf{B}}_{\mathbf{b}} = \mathcal{V}_{\downarrow^* v} \mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) \subseteq \mathbf{A}_{\alpha_v}$ , so  $\tilde{\mathbf{B}}_{\mathbf{b}}$  must commute with  $\mathbf{A}_{\mathbf{A} \setminus \alpha_v}$ .

Similarly, for Eq. (3.52b) we need to show that  $\tilde{\mathbf{B}}_{\beta_w}$  commutes with  $\mathbf{W}$  and with  $\mathbf{A}_{\mathbf{A} \setminus G^{-1}(\beta_w)}$ . The first fact follows from Eq. (3.57) and the fact that  $\beta_w \subseteq \beta_v$  for  $w \in \text{Ch}(v)$ . For the second fact, note that since  $G_{\mathcal{U}} \subseteq G$ ,  $\mathcal{U}^\dagger(\mathbf{B}_{\beta_w}) \subseteq \mathbf{A}_{G_{\mathcal{U}}^{-1}(\beta_w)} \subseteq \mathbf{A}_{G^{-1}(\beta_w)}$ . This implies that  $\tilde{\mathbf{B}}_{\beta_w}$  commutes with  $\mathbf{A}_{\mathbf{A} \setminus G^{-1}(\beta_w)}$  by an argument similar to the above.

The final claim that  $G^{-1}(\beta_w) \setminus \alpha_v$  and  $G^{-1}(\beta_{w'}) \setminus \alpha_v$  are disjoint for distinct  $w, w' \in \text{Ch}(v)$  follows directly from Lemma 3.25, which established that  $G^{-1}(\beta_w) \cap G^{-1}(\beta_{w'}) = \alpha_v$ .

*Proof of (iii).* Observe that by Eq. (3.34), the algebra on the right-hand side of Eq. (3.53) is just  $\tilde{\mathbf{B}}_{\beta_v} := \mathcal{V}_{\downarrow^* v} \mathcal{U}^\dagger(\mathbf{B}_{\beta_v})$ . For each  $\mathbf{a} \in \lambda^{-1}(v)$ ,  $G(\mathbf{a}) = \beta_v$  and therefore  $\mathbf{A}_{\lambda^{-1}(v)} \subseteq \mathcal{U}^\dagger(\mathbf{B}_{\beta_v})$ .  $\mathcal{V}_{\downarrow^* v}$  acts trivially on  $\mathbf{A}_{\lambda^{-1}(v)}$ , so we get  $\mathbf{A}_{\lambda^{-1}(v)} \subseteq \mathcal{V}_{\downarrow^* v} \mathcal{U}^\dagger(\mathbf{B}_{\beta_v}) = \tilde{\mathbf{B}}_{\beta_v}$ . It remains to show that  $\mathbf{Z}_u^v \subseteq \tilde{\mathbf{B}}_{\beta_v}$  for  $u \in \text{Pa}(v)$ . This follows from the induction hypothesis (3.44b) applied to  $\mathbf{Z}_u^v$  and the fact that  $\mathcal{V}_{\downarrow^* v \setminus \downarrow u}$  acts trivially on  $\mathbf{Z}_u^v$ .

Let us now prove the final statement of the theorem. Consider the unitary  $\mathcal{V}_{L_G} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  and let  $\mathbf{b} \in \mathbf{B}$ . Note that  $\mathcal{V}_{L_G} = \mathcal{V}_{L_G \setminus \downarrow \mu(\mathbf{b})} \circ \mathcal{V}_{\downarrow \mu(\mathbf{b})}$  and that  $\mathcal{V}_{L_G \setminus \downarrow \mu(\mathbf{b})}$  acts trivially on  $\mathbf{B}_{\mathbf{b}}$ . Therefore, by Eq. (3.44a),

$$\mathcal{V}_{L_G} \mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) = \mathcal{V}_{L_G \setminus \downarrow \mu(\mathbf{b})} \left( \mathcal{V}_{\downarrow \mu(\mathbf{b})} \mathcal{U}^\dagger(\mathbf{B}_{\mathbf{b}}) \right) = \mathcal{V}_{L_G \setminus \downarrow \mu(\mathbf{b})}(\mathbf{B}_{\mathbf{b}}) = \mathbf{B}_{\mathbf{b}}. \quad (3.61)$$

Let  $\mathcal{W}_{\mathbf{b}} : \mathbf{B}_{\mathbf{b}} \rightarrow \mathbf{B}_{\mathbf{b}}$  be the \*-automorphism defined by  $\mathcal{W}_{\mathbf{b}} := \left( \mathcal{U} \mathcal{V}_{L_G}^\dagger \right) |_{\mathbf{B}_{\mathbf{b}}}$ ; then  $\mathcal{W}_{\mathbf{b}} \mathcal{V}_{L_G} \mathcal{U}^\dagger |_{\mathbf{B}_{\mathbf{b}}} = \text{id}_{\mathbf{B}_{\mathbf{b}}}$ . Since this holds for each  $\mathbf{b} \in \mathbf{B}$ , we have  $\left( \bigotimes_{\mathbf{b} \in \mathbf{B}} \mathcal{W}_{\mathbf{b}} \right) \mathcal{V}_{L_G} \mathcal{U}^\dagger = \text{id}_{\mathbf{B}_{\mathbf{B}}}$ , so that

$$\left( \bigotimes_{\mathbf{b} \in \mathbf{B}} \mathcal{W}_{\mathbf{b}} \right) \mathcal{V}_{L_G} = \mathcal{U}. \quad (3.62)$$

$\mathcal{V}_{L_G}$  has a unitary circuit decomposition of shape  $L_G$ ; by incorporating the local unitaries  $\mathcal{W}_{\mathbf{b}}$  into the existing gates, we see that  $\mathcal{U}$  itself also admits a unitary circuit decomposition of the same shape, completing the proof of Theorem 3.26.  $\square$

### 3.5.4 Necessity of the $C_3$ exclusion property

Having shown that  $(C_3\text{-EP})$  is sufficient, we now show that it is also a necessary condition on a relation  $G \subseteq \mathbf{A} \times \mathbf{B}$  to imply unitary causal decompositions, thus proving the direction (ii)  $\Rightarrow$  (i) in Theorem 3.23. In fact we prove the stronger implication (iii)  $\Rightarrow$  (i). We show, specifically, that whenever  $G \subseteq \mathbf{A} \times \mathbf{B}$  fails  $(C_3\text{-EP})$ , then there exists a choice of quantum systems  $\mathbf{A}_{\mathbf{a}}$  for  $\mathbf{a} \in \mathbf{A}$  and  $\mathbf{B}_{\mathbf{b}}$  for  $\mathbf{b} \in \mathbf{B}$  as well as a unitary channel  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  which has causal structure  $G_{\mathcal{U}} = G$ , yet no unitary circuit decomposition with connectivity  $G$ —that is, no causally faithful unitary decomposition. Recall from our discussion below Example 3.17 that this does not mean that  $\mathcal{U}$  has no causally faithful decompositions at all; a causally faithful unitary *routed* circuit decomposition might still exist [62], or a causally faithful circuit decomposition in terms of non-unitary quantum channels, although this has not been established in general.

We have already seen an example in Example 3.17: we defined

$$\begin{array}{|c|} \hline \bar{\mathbf{B}}_1 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{B}}_2 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{B}}_3 \\ \hline \end{array} \begin{array}{|c|} \hline \mathcal{U}_3 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{A}}_1 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{A}}_2 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{A}}_3 \\ \hline \end{array} := \begin{array}{|c|} \hline \bar{\mathbf{B}}_1 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{B}}_2 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{B}}_3 \\ \hline \end{array} \begin{array}{|c|} \hline \oplus \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{A}}_1 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{A}}_2 \\ \hline \end{array} \begin{array}{|c|} \hline \bar{\mathbf{A}}_3 \\ \hline \end{array}, \quad (3.63)$$

where  $\bar{\mathbf{A}}_i = \bar{\mathbf{B}}_i = \mathcal{L}(\mathbb{C}^2)$  for  $i \in \{1, 2, 3\}$ , saw that it has causal structure  $C_3$ , which fails  $(C_3\text{-EP})$ , and also showed that  $\mathcal{U}_3$  has no causally faithful unitary decomposition. Our proof in this section generalises this observation by showing that whenever a unitary channel  $\mathcal{U}$  contains  $\mathcal{U}_3$  as a factor on those systems where  $G_{\mathcal{U}} = G$  contains  $C_3$ , by virtue of its violation of  $(C_3\text{-EP})$ , then  $\mathcal{U}$  does not admit a causally faithful unitary decomposition either.

More precisely, we have the following, which directly implies (iii)  $\Rightarrow$  (i) in Theorem 3.23.

**Proposition 3.28.** *Suppose  $G \subseteq \mathbf{A} \times \mathbf{B}$  fails  $(C_3\text{-EP})$ ; without loss of generality, write  $\mathbf{A} = \{1, \dots, n\}$  and  $\mathbf{B} = \{1, \dots, m\}$  so that  $G \cap (\{1, 2, 3\} \times \{1, 2, 3\}) = C_3$ . Let  $\mathcal{V} : \mathbf{A}_1 \cdots \mathbf{A}_n \rightarrow \mathbf{B}_1 \cdots \mathbf{B}_m$  be a unitary channel between quantum systems  $\mathbf{A}_1, \dots, \mathbf{A}_n$  and  $\mathbf{B}_1, \dots, \mathbf{B}_m$  so that  $G_{\mathcal{V}} = G$ .<sup>8</sup> In addition, let  $\bar{\mathbf{A}}_i = \bar{\mathbf{B}}_i = \mathcal{L}(\mathbb{C}^2)$  for  $i \in \{1, 2, 3\}$  be qubits and  $\mathcal{U}_3 : \bar{\mathbf{A}}_1 \bar{\mathbf{A}}_2 \bar{\mathbf{A}}_3 \rightarrow \bar{\mathbf{B}}_1 \bar{\mathbf{B}}_2 \bar{\mathbf{B}}_3$  be the unitary channel from Eq. (3.63). Finally, define*

$$\mathcal{U} := \mathcal{U}_3 \otimes \mathcal{V} : (\bar{\mathbf{A}}_1 \mathbf{A}_1)(\bar{\mathbf{A}}_2 \mathbf{A}_2)(\bar{\mathbf{A}}_3 \mathbf{A}_3) \mathbf{A}_4 \cdots \mathbf{A}_n \rightarrow (\bar{\mathbf{B}}_1 \mathbf{B}_1)(\bar{\mathbf{B}}_2 \mathbf{B}_2)(\bar{\mathbf{B}}_3 \mathbf{B}_3) \mathbf{B}_4 \cdots \mathbf{B}_m, \quad (3.64)$$

regarded as a unitary channel with  $n$  inputs and  $m$  outputs:

$$(3.65)$$

Then  $\mathcal{U}$  has causal structure  $G$ , yet no unitary circuit decomposition with connectivity  $G$ .

**Lemma 3.29.** Let  $|\psi\rangle \in \mathcal{H}_X \otimes \mathcal{H}_Y$  be a bipartite state with marginals  $\rho_X := \text{Tr}_Y |\psi\rangle \langle \psi|$  and  $\rho_Y = \text{Tr}_X |\psi\rangle \langle \psi|$ , supported on  $\text{supp } \rho_X \subseteq \mathcal{H}_X$  and  $\text{supp } \rho_Y \subseteq \mathcal{H}_Y$ . Then  $|\psi\rangle \in \text{supp } \rho_X \otimes \text{supp } \rho_Y$ .

*Proof.* Let  $|\psi\rangle = \sum_{i=1}^r c_i |\alpha_i\rangle \otimes |\beta_i\rangle$  be the Schmidt decomposition, where all coefficients  $c_i$  are nonzero. Then  $\text{supp } \rho_X$  is the span of the vectors  $\{|\alpha_i\rangle\}_{i=1}^r$  and  $\text{supp } \rho_Y$  the span of  $\{|\beta_i\rangle\}_{i=1}^r$ . Clearly, then,  $|\psi\rangle \in \text{supp } \rho_X \otimes \text{supp } \rho_Y$ .  $\square$

*Proof of Proposition 3.28.* It is immediate from the construction that  $\mathcal{U}$  (with respect to the grouping of systems as in Eq. (3.64)) has causal structure  $G$ . Suppose, for contradiction, that  $\mathcal{U}$  has a unitary causally faithful circuit decomposition: i.e. a circuit decomposition  $\mathbf{P}$  with connectivity  $G$ . Then  $\mathbf{P}$  has, in particular, no path from  $\bar{A}_1 A_1$  to  $\bar{B}_3 B_3$ , and no path from  $\bar{A}_3 A_3$  to  $\bar{B}_1 B_1$ .

Now abbreviate  $\hat{A}_4 := A_2 A_4 \cdots A_n$  and  $\hat{B}_4 := B_2 B_4 \cdots B_m$  and regard  $\mathcal{U}$  as a four-input, four-output unitary channel

$$\mathcal{U} : (\bar{A}_1 A_1)(\bar{A}_2)(\hat{A}_4)(\bar{A}_3 A_3) \rightarrow (\bar{B}_1 B_1)(\bar{B}_2)(\hat{B}_4)(\bar{B}_3 B_3). \quad (3.66)$$

The circuit  $\mathbf{P}$ , now interpreted as a four-input, four-output circuit, then satisfies

$$G_{\mathbf{P}} \subseteq C_4 := \begin{array}{cccc} 1 & 2 & 4 & 3 \\ \uparrow & \swarrow & \nearrow & \uparrow \\ & 1 & 2 & 4 \\ \downarrow & \swarrow & \nearrow & \downarrow \\ 1 & 2 & 4 & 3 \end{array}. \quad (3.67)$$

By Corollary 3.5, then,  $\mathcal{U}$  also has a unitary circuit decomposition of the canonical shape  $L_{C_4}$ . One may compute this shape, much like  $L_{C_3}$  in Eq. (3.33), to be

$$L_{C_4} = \begin{array}{cccc} 1 & 2 & 4 & 3 \\ | & | & | & | \\ \square & & \square & \\ | & | & | & | \\ \square & & \square & \\ | & | & | & | \\ 1 & 2 & 4 & 3 \end{array}. \quad (3.68)$$

<sup>8</sup>Such a choice of  $\mathcal{V}$  always exists for appropriate choices of quantum systems; however,  $G$  may constrain the dimensions of these systems. See e.g. Eq. (3.32) and Ref. [62, §7.2].

$$\begin{array}{c} \bar{B}_1 B_1 \\ \bar{B}_2 \\ \bar{B}_4 \\ \bar{B}_3 B_3 \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \mathcal{U} \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} A_1 A_1 \\ A_2 \\ A_4 \\ A_3 A_3 \end{array} = \begin{array}{c} \bar{B}_1 B_1 \\ \bar{B}_2 \\ \bar{B}_4 \\ \bar{B}_3 B_3 \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \mathcal{Z} \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} E \\ F \end{array} \begin{array}{c} \mathcal{X} \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} C \\ D \end{array} \begin{array}{c} \mathcal{Y} \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \bar{A}_1 A_1 \\ \bar{A}_2 \\ \bar{A}_4 \\ \bar{A}_3 A_3 \end{array} \begin{array}{c} \downarrow \\ \downarrow \\ \downarrow \\ \downarrow \end{array} \begin{array}{c} \mathcal{W} \end{array} \quad (3.69)$$

Denote by  $\{|0\rangle, |1\rangle\}$  the basis of  $\mathbb{C}^2$  with respect to which the CNOTs in the construction of  $\mathcal{U}_3$  are defined. Also choose a basis  $\{|j\rangle\}_{j=0}^{d-1}$  for the Hilbert space  $\mathcal{H}_{\mathbf{A}_4}$ . For  $i \in \{0, 1\}$  and  $j \in \{0, \dots, d-1\}$ , define the states

Let us first consider the states  $\rho_C^{ij}$  on system  $C$ . We claim that for any  $j, j' \in \{0, \dots, d-1\}$ , the states  $\rho_C^{0j}$  and  $\rho_C^{1j'}$  have orthogonal supports, i.e. are perfectly distinguishable. To see this intuitively, consider the qubits  $\bar{A}_1, \bar{A}_2$ , and  $\bar{B}_1$ . From the CNOTs in the definition of  $\mathcal{U}$  in Eq. (3.65), it is clear that if  $\bar{A}_1$  is prepared in state  $|0\rangle$ , the information about the computational degree of freedom  $i$  of  $\bar{A}_2$  ends up on  $\bar{B}_1$ . Since every path from  $\bar{A}_2$  to  $\bar{B}_1$  in Eq. (3.69) passes through  $C$ , that information must also be perfectly recoverable from  $C$ .

$$\begin{aligned}
& \text{Diagram 1: } \bar{B}_1 \text{ (top wire), } \bar{B}_1 \text{ (input to } \mathcal{X} \text{), } \bar{B}_1 \text{ (output of } \mathcal{X} \text{), } \bar{B}_1 \text{ (input to } \mathcal{W} \text{), } \bar{B}_1 \text{ (output of } \mathcal{W} \text{). } \mathcal{X} \text{ has inputs } \bar{A}_1, A_1 \text{ and output } C. \mathcal{W} \text{ has inputs } \bar{A}_2, \bar{A}_4 \text{ and outputs } i, j. \\
& = \text{Diagram 2: } \bar{B}_1 \text{ (top wire), } \bar{B}_1 \text{ (input to } \mathcal{X} \text{), } \bar{B}_1 \text{ (output of } \mathcal{X} \text{), } \bar{B}_1 \text{ (input to } \mathcal{W} \text{), } \bar{B}_1 \text{ (output of } \mathcal{W} \text{). } \mathcal{X} \text{ has inputs } \bar{A}_1, A_1 \text{ and output } C. \mathcal{W} \text{ has inputs } \bar{A}_2, \bar{A}_4 \text{ and outputs } i, j. \\
& = \text{Diagram 3: } \bar{B}_1 \text{ (top wire), } \bar{B}_1 \text{ (input to } \mathcal{U} \text{), } \bar{B}_1 \text{ (output of } \mathcal{U} \text{), } \bar{B}_1 \text{ (input to } \mathcal{W} \text{), } \bar{B}_1 \text{ (output of } \mathcal{W} \text{). } \mathcal{U} \text{ has inputs } \bar{A}_1, A_1, \bar{A}_2, \bar{A}_4 \text{ and outputs } i, j. \\
& = \text{Diagram 4: } \bar{B}_1 \text{ (top wire), } \bar{B}_1 \text{ (input to } \mathcal{U} \text{), } \bar{B}_1 \text{ (output of } \mathcal{U} \text{), } \bar{B}_1 \text{ (input to } \mathcal{W} \text{), } \bar{B}_1 \text{ (output of } \mathcal{W} \text{). } \mathcal{U} \text{ has inputs } \bar{A}_1, A_1, \bar{A}_2, \bar{A}_4 \text{ and outputs } i, j. \\
& = \text{Diagram 5: } \bar{B}_1 \text{ (top wire), } \bar{B}_1 \text{ (input to } \mathcal{U} \text{), } \bar{B}_1 \text{ (output of } \mathcal{U} \text{), } \bar{B}_1 \text{ (input to } \mathcal{W} \text{), } \bar{B}_1 \text{ (output of } \mathcal{W} \text{). } \mathcal{U} \text{ has inputs } \bar{A}_1, A_1, \bar{A}_2, \bar{A}_4 \text{ and outputs } i, j.
\end{aligned}
\tag{3.71}$$

The circuit fragment enclosed in a dashed box on the left-hand side forms a CPTP map  $\mathcal{E} : \mathcal{C} \rightarrow \bar{\mathcal{B}}_1$ . Being CPTP,  $\mathcal{E}$  cannot increase distinguishability; in particular, if  $\rho_{\mathcal{C}}^{0j}$  and  $\rho_{\mathcal{C}}^{1j'}$  had non-orthogonal supports, then  $\mathcal{E}(\rho_{\mathcal{C}}^{0j})$  and  $\mathcal{E}(\rho_{\mathcal{C}}^{1j'})$  would have non-orthogonal supports. In light of the right-hand side of Eq. (3.71), the latter is obviously untrue; therefore,  $\rho_{\mathcal{C}}^{0j}$  and  $\rho_{\mathcal{C}}^{1j'}$  must have orthogonal supports.

Let  $\mathcal{H}_{\mathcal{C}}^0 := \left( \bigcup_{j=0}^{d-1} \text{supp } \rho_{\mathcal{C}}^{0j} \right)^{\perp\perp} \subseteq \mathcal{H}_{\mathcal{C}}$  be the subspace of  $\mathcal{H}_{\mathcal{C}}$  spanned by the supports of  $\rho_{\mathcal{C}}^{0j}$  for  $j \in \{0, \dots, d-1\}$ . Moreover, let  $\mathcal{H}_{\mathcal{C}}^1 := (\mathcal{H}_{\mathcal{C}}^0)^{\perp}$ . It follows from the claim we just proved that then

$$\text{supp } \rho_{\mathcal{C}}^{ij} \subseteq \mathcal{H}_{\mathcal{C}}^i \quad (3.72)$$

for all  $i \in \{0, 1\}$  and  $j \in \{0, \dots, d-1\}$ . By a symmetric argument concerning instead the states  $\rho_{\mathcal{D}}^{ij}$  on  $\mathcal{D}$ , we can construct orthogonal subspaces  $\mathcal{H}_{\mathcal{D}}^0, \mathcal{H}_{\mathcal{D}}^1 \subseteq \mathcal{H}_{\mathcal{D}}$  so that

$$\text{supp } \rho_{\mathcal{D}}^{ij} \subseteq \mathcal{H}_{\mathcal{D}}^i \quad (3.73)$$

for  $i \in \{0, 1\}$  and  $j \in \{0, \dots, d-1\}$ .

By Lemma 3.29 and Eqs. (3.72) and (3.73) and the fact that  $\rho_{\mathcal{CD}}^{ij}$  is pure,

$$\text{supp } \rho_{\mathcal{CD}}^{ij} \subseteq \mathcal{H}_{\mathcal{C}}^i \otimes \mathcal{H}_{\mathcal{D}}^i. \quad (3.74)$$

Moreover, for fixed  $i$ , the  $d$  states  $\rho_{\mathcal{CD}}^{i0}, \rho_{\mathcal{CD}}^{i1}, \dots, \rho_{\mathcal{CD}}^{i(d-1)}$  are all pairwise orthogonal by their construction in Eq. (3.70) (and the fact that  $\mathcal{W}$  is a unitary channel). These two facts imply that

$$\dim(\mathcal{H}_{\mathcal{C}}^i \otimes \mathcal{H}_{\mathcal{D}}^i) \geq d \quad (3.75)$$

for each  $i \in \{0, 1\}$ . We conclude that

$$\dim(\mathcal{H}_{\mathcal{C}} \otimes \mathcal{H}_{\mathcal{D}}) = \dim \mathcal{H}_{\mathcal{C}}^0 \dim \mathcal{H}_{\mathcal{D}}^0 + \dim \mathcal{H}_{\mathcal{C}}^0 \dim \mathcal{H}_{\mathcal{D}}^1 \quad (3.76)$$

$$\begin{aligned} &+ \dim \mathcal{H}_{\mathcal{C}}^1 \dim \mathcal{H}_{\mathcal{D}}^0 + \dim \mathcal{H}_{\mathcal{C}}^1 \dim \mathcal{H}_{\mathcal{D}}^1 \\ &\geq 2d + \dim \mathcal{H}_{\mathcal{C}}^0 \dim \mathcal{H}_{\mathcal{D}}^1 + \dim \mathcal{H}_{\mathcal{C}}^1 \dim \mathcal{H}_{\mathcal{D}}^0 > 2d, \end{aligned} \quad (3.77)$$

which is in contradiction with the fact that  $\mathcal{W}$  is a unitary channel and  $\dim(\mathbb{C}^2 \otimes \mathcal{H}_{\mathcal{A}_4}) = 2d$ .  $\mathcal{U}$  can therefore have no causally faithful unitary decomposition.  $\square$

### 3.6 Appendix: Operator algebra and proof of lemma

In Section 3.1 we defined a quantum system as the algebra of operators  $\mathcal{L}(\mathcal{H})$  on a given finite-dimensional Hilbert space. In what follows, an algebra-first approach is more useful, even though we will stick to the finite-dimensional case throughout.

A **\*-algebra** is a complex algebra  $\mathbf{A}$  with unit  $\mathbf{1}_{\mathbf{A}}$ , equipped with an **involution**, i.e. a map  $(\cdot)^* : \mathbf{A} \rightarrow \mathbf{A}$  satisfying, for all  $a, b \in \mathbf{A}$  and  $\lambda \in \mathbb{C}$ :

$$(a + b)^* = a^* + b^*; \quad (\lambda a)^* = \bar{\lambda} a^*; \quad (ab)^* = b^* a^*; \quad a^{**} = a. \quad (3.78)$$

A  **$C^*$ -algebra** is a \*-algebra  $\mathbf{A}$  equipped with a norm  $\|\cdot\|$  in which  $\mathbf{A}$  is complete and which satisfies the identity

$$\|a^* a\| = \|a\|^2 \quad \text{for all } a \in \mathbf{A}. \quad (3.79)$$

In the finite-dimensional case, completeness of a normed  $*$ -algebra  $\mathbf{A}$  is a vacuous statement, and  $\mathbf{A}$  admits a norm making it into a  $C^*$ -algebra precisely if the involution is **positive**, meaning

$$a^*a = 0 \iff a = 0 \quad \text{for all } a \in \mathbf{A}; \quad (3.80)$$

moreover, in this case the norm making  $\mathbf{A}$  into a  $C^*$ -algebra is unique [37].

In the following, then, let  $\mathbf{A}$  be a finite-dimensional  $C^*$ -algebra. A  **$*$ -subalgebra**  $\mathbf{X}$  of  $\mathbf{A}$  is a subalgebra closed under the involution, and a **unital**  $*$ -subalgebra is one that contains  $\mathbf{1}_{\mathbf{A}}$ . The **commutant** of  $\mathbf{X}$ , when understood as a  $*$ -subalgebra of  $\mathbf{A}$ , is

$$\mathbf{X}' := \{a \in \mathbf{A} \mid \forall x \in \mathbf{X} : ax = xa\}. \quad (3.81)$$

Moreover, the **centre** of  $\mathbf{X}$  is  $\mathcal{Z}(\mathbf{X}) := \mathbf{X} \cap \mathbf{X}'$ , and  $\mathbf{X}$  is a **factor** if it has trivial centre,  $\mathcal{Z}(\mathbf{X}) = \{\lambda \mathbf{1}_{\mathbf{X}} \mid \lambda \in \mathbb{C}\}$ . If  $\mathbf{A}$  itself is a factor and  $\mathbf{X}$  is an arbitrary  $*$ -subalgebra of it, we always have  $\mathbf{X}'' = \mathbf{X}$ . (For this latter fact the finite-dimensionality of  $\mathbf{A}$  is essential.)

A  **$*$ -isomorphism** is an algebra isomorphism that preserves the involution. Every algebra  $\mathcal{L}(\mathcal{H})$  of linear operators on a finite-dimensional Hilbert space  $\mathcal{H}$  forms a  $C^*$ -algebra under the Hermitian adjoint, and every finite-dimensional  $C^*$ -algebra  $\mathbf{A}$  is  $*$ -isomorphic to a  $*$ -subalgebra of  $\mathcal{L}(\mathcal{H})$  for some finite-dimensional  $\mathcal{H}$ .  $\mathbf{A}$  is a factor iff it is isomorphic to a full matrix algebra  $\mathcal{L}(\mathcal{H})$ . The algebras ('quantum systems') we considered in the previous sections of this chapter were therefore all factor algebras.

For the proof of Lemma 3.27 below—in particular, for Proposition 3.34—it is however useful to also discuss the representation of non-factor  $C^*$ -algebras (which are related to the generalisation to *routed* unitary circuits discussed below Example 3.17). In the finite-dimensional case, the Wedderburn-Artin theorem tells us that every  $C^*$ -algebra is isomorphic to a finite direct sum of factors [37]. In fact, if  $\mathbf{A}$  is any  $*$ -subalgebra of  $\mathcal{L}(\mathcal{H})$  and  $m := \dim \mathcal{Z}(\mathbf{A})$ , then there are  $\mathcal{H}^i, \mathcal{K}^i$  and a unitary map  $U : \mathcal{H} \rightarrow \bigoplus_{i=1}^m \mathcal{H}^i \otimes \mathcal{K}^i$  defining a  $*$ -isomorphism

$$U(\cdot)U^* : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}\left(\bigoplus_{i=1}^m \mathcal{H}^i \otimes \mathcal{K}^i\right) \quad (3.82)$$

so that

$$U\mathbf{A}U^* = \bigoplus_{i=1}^m \mathcal{L}(\mathcal{H}^i) \otimes \{\mathbf{1}_{\mathcal{K}^i}\} \quad \text{and} \quad U\mathbf{A}'U^* = \bigoplus_{i=1}^m \{\mathbf{1}_{\mathcal{H}^i}\} \otimes \mathcal{L}(\mathcal{K}^i). \quad (3.83)$$

The  $m$  projectors  $\pi^i := U^*(\mathbf{1}_{\mathcal{H}^i \otimes \mathcal{K}^i} \oplus 0)U \in \mathbf{A}$  form a basis for the centre  $\mathcal{Z}(\mathbf{A})$ .

More generally, if  $\mathbf{A} \subseteq \mathcal{L}(\mathcal{H})$  is a  $*$ -subalgebra and  $\{\pi^i\}_{i \in [m]} \subseteq \mathcal{L}(\mathcal{H})$  any complete family of mutually orthogonal projectors, then we say  $\mathbf{A}$  is **block-diagonal** with respect to this family if  $\mathbf{A} \subseteq \bigoplus_{i \in [m]} \mathcal{L}(\pi^i \mathcal{H}) \subseteq \mathcal{L}\left(\bigoplus_{i \in [m]} \pi^i \mathcal{H}\right) = \mathcal{L}(\mathcal{H})$ . This is the case precisely if each  $\pi^i$  commutes with every element of  $\mathbf{A}$ .

According to the following result, wherever we said 'unitary channel' in the previous sections, we could just as well have said ' $*$ -isomorphism'.

**Proposition 3.30.** *A map between factors  $\varphi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$  is a  $*$ -isomorphism iff it is of the form  $\varphi(a) = UaU^*$  for some unitary map  $U : \mathcal{H} \rightarrow \mathcal{K}$ .*

*Proof.* For  $U : \mathcal{H} \rightarrow \mathcal{K}$ , write  $\text{Ad}_U : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$  for the mapping  $a \mapsto UaU^*$ . It is clear that this is always a  $*$ -isomorphism. For the converse direction, suppose  $\varphi : \mathcal{L}(\mathcal{H}) \rightarrow \mathcal{L}(\mathcal{K})$  is a  $*$ -isomorphism, pick an arbitrary unitary map  $V : \mathcal{K} \rightarrow \mathcal{H}$  and let  $\psi := \text{Ad}_V \circ \varphi$ , so that  $\psi$  is a  $*$ -automorphism of  $\mathcal{L}(\mathcal{H})$ . Every  $*$ -automorphism of a factor is *inner*, i.e. is of the form  $\psi = \text{Ad}_W$  for some unitary  $W \in \mathcal{L}(\mathcal{H})$ . A proof of this fact for the finite-dimensional case follows, for instance, from [37, Proposition 5.5]. We now have  $\varphi = \text{Ad}_{V^*} \circ \text{Ad}_W = \text{Ad}_{V^*W}$ , where  $V^*W : \mathcal{H} \rightarrow \mathcal{K}$ .  $\square$

We will now build up to a proof of Lemma 3.27. We start with the following straightforward generalisation of Eq. (3.83) in the case of factor algebras. For  $n \in \mathbb{N}$ , we write  $[n] := \{1, 2, \dots, n\}$ . As in the preceding sections of this chapter, the notation  $\mathbf{Z}_1 \mathbf{Z}_2 \cdots \mathbf{Z}_n$  and  $\mathbf{Z}_{[n]}$  refers to the tensor product  $\mathbf{Z}_1 \otimes \mathbf{Z}_2 \otimes \cdots \otimes \mathbf{Z}_n$  and depending on the context,  $\mathbf{Z}_k$  may also refer to the unital  $*$ -subalgebra  $\mathbf{Z}_k \otimes \{\mathbf{1}_{\mathbf{Z}_{[n] \setminus \{k\}}}\} \subseteq \mathbf{Z}_{[n]}$ .

**Proposition 3.31.** *Let  $\mathbf{A}$  be a finite-dimensional factor  $C^*$ -algebra and  $\mathbf{B}_1, \mathbf{B}_2, \dots, \mathbf{B}_n \subseteq \mathbf{A}$  pairwise commuting, unital  $*$ -subalgebras that are themselves factors. Then there are factor algebras  $\mathbf{Z}_1, \mathbf{Z}_2, \dots, \mathbf{Z}_n$  and a  $*$ -isomorphism*

$$\mathcal{V} : \mathbf{A} \xrightarrow{\sim} \mathbf{Z}_1 \mathbf{Z}_2 \cdots \mathbf{Z}_n \quad (3.84)$$

so that for each  $k \in [n] := \{1, 2, \dots, n\}$ ,

$$\mathcal{V}(\mathbf{B}_k) \subseteq \mathbf{Z}_k. \quad (3.85)$$

Moreover, if  $\bigvee_{k \in [n]} \mathbf{B}_k = \mathbf{A}$  then all of these inclusions are equalities.

If  $\mathbf{A} = \mathcal{L}(\mathcal{H}_{\mathbf{A}})$ , then by Proposition 3.30 this statement asserts precisely that there are  $\mathcal{H}_{\mathbf{Z}_k}$  for  $k \in [n]$  and a unitary map

$$V : \mathcal{H}_{\mathbf{A}} \xrightarrow{\sim} \mathcal{H}_{\mathbf{Z}_1} \otimes \mathcal{H}_{\mathbf{Z}_2} \otimes \cdots \otimes \mathcal{H}_{\mathbf{Z}_n} \quad (3.86)$$

so that for each  $k \in [n]$ ,

$$V \mathbf{B}_k V^* \subseteq \mathcal{L}(\mathcal{H}_{\mathbf{Z}_k}) \otimes \{\mathbf{1}_{\mathbf{Z}_{[n] \setminus \{k\}}}\}. \quad (3.87)$$

*Proof.* By Eq. (3.83), and since  $\mathbf{B}_1$  is a unital factor  $*$ -subalgebra of the factor  $\mathbf{A}$ , there is a factor  $\mathbf{X}_1$  and a  $*$ -isomorphism  $\varphi_1 : \mathbf{A} \xrightarrow{\sim} \mathbf{B}_1 \otimes \mathbf{X}_1$  such that  $\varphi_1(\mathbf{B}_1) = \mathbf{B}_1 \otimes \{\mathbf{1}_{\mathbf{X}_1}\}$ . Since  $\mathbf{B}_2$  commutes with  $\mathbf{B}_1$ , under this isomorphism  $\varphi_1(\mathbf{B}_2) \subseteq (\mathbf{B}_1 \otimes \{\mathbf{1}_{\mathbf{X}_1}\})' = \{\mathbf{1}_{\mathbf{B}_1}\} \otimes \mathbf{X}_1$  so  $\mathbf{B}_2$  corresponds to a unital factor  $*$ -subalgebra of  $\mathbf{X}_1$ , meaning there is a  $*$ -isomorphism  $\varphi_2 : \mathbf{X}_1 \xrightarrow{\sim} \mathbf{B}_2 \otimes \mathbf{X}_2$  under which it corresponds to  $\mathbf{B}_2 \otimes \{\mathbf{1}_{\mathbf{X}_2}\}$ . Iterating this procedure yields a  $*$ -isomorphism

$$\mathcal{V} : \mathbf{A} \xrightarrow{\sim} \mathbf{B}_1 \otimes \mathbf{B}_2 \otimes \cdots \otimes \mathbf{B}_{n-1} \otimes \mathbf{X}_{n-1} \quad (3.88)$$

so that  $\mathcal{V}(\mathbf{B}_k) = \mathbf{B}_k$  for  $k = 1, \dots, n-1$  and  $\mathcal{V}(\mathbf{B}_n) \subseteq \mathbf{X}_{n-1}$ . Finally, if  $\bigvee_{k \in [n]} \mathbf{B}_k = \mathbf{A}$  then  $\bigvee_{k \in [n]} \mathcal{V}(\mathbf{B}_k) = \mathcal{V}(\mathbf{A})$  so we must have equality  $\mathcal{V}(\mathbf{B}_n) = \mathbf{X}_{n-1}$ .  $\square$

The following two Lemmas will be useful for proving Proposition 3.34 below.

**Lemma 3.32.** *Let  $\mathbf{A}, \mathbf{B}$  be factors and  $\mathbf{X} \subseteq \mathbf{A} \otimes \mathbf{B}$  a unital  $*$ -subalgebra so that  $\mathbf{A} \otimes \{\mathbf{1}_\mathbf{B}\} \subseteq \mathbf{X}$ . Then there is a unital  $*$ -subalgebra  $\mathbf{Y} \subseteq \mathbf{B}$  so that  $\mathbf{X} = \mathbf{A} \otimes \mathbf{Y}$ .*

*Proof.* The condition  $\mathbf{A} \otimes \{\mathbf{1}_\mathbf{B}\} \subseteq \mathbf{X}$  implies that  $\mathbf{X}' \subseteq (\mathbf{A} \otimes \{\mathbf{1}_\mathbf{B}\})' = \{\mathbf{1}_\mathbf{A}\} \otimes \mathbf{B}$ . Therefore  $\mathbf{X}' = \{\mathbf{1}_\mathbf{A}\} \otimes \mathbf{Z}$  for some unital  $*$ -subalgebra  $\mathbf{Z}$  of  $\mathbf{B}$ . Finally, then,  $\mathbf{X} = \mathbf{X}'' = (\{\mathbf{1}_\mathbf{A}\} \otimes \mathbf{Z})' = \mathbf{A} \otimes \mathbf{Z}'$ .  $\square$

The Lemma below essentially states that if an algebra  $\mathbf{X}$  is block-diagonal in some Hilbert space representation and is nonzero in each block, then it can only be a factor algebra if it is the same in all blocks.

**Lemma 3.33.** *Let  $\mathcal{H}^j$  for  $j = 1, \dots, m$  be finite-dimensional Hilbert spaces and  $\mathbf{X} \subseteq \bigoplus_{j \in [m]} \mathcal{L}(\mathcal{H}^j)$  be a unital  $*$ -subalgebra which is a factor. For  $i \in [m]$ , denote by  $\pi^i := \mathbf{1}_{\mathcal{H}^i} \oplus 0 \in \bigoplus_j \mathcal{L}(\mathcal{H}^j)$  the projector onto the  $i$ -th Hilbert space. Then for each  $i$ , right-multiplication by  $\pi^i$  is a  $*$ -isomorphism of  $\mathbf{X}$  onto  $\mathbf{X}\pi^i$ .*

*Proof.* It suffices to show that  $(\cdot)\pi^i : \mathbf{X} \rightarrow \mathbf{X}\pi^i$  is an injective  $*$ -homomorphism (it is clearly surjective). That it is a  $*$ -homomorphism follows immediately from the fact that  $\mathbf{X}$  commutes with  $\pi^i$  and  $\pi^i$  is self-adjoint. Injectivity surmounts to the proposition that  $x\pi^i = 0 \Rightarrow x = 0$  for  $x \in \mathbf{X}$ . Note that  $\mathbf{X}$  can be seen as a unital factor  $*$ -subalgebra of  $\mathcal{L}(\bigoplus_j \mathcal{H}^j)$ ; therefore there is another factor  $\mathbf{Y}$  and a  $*$ -isomorphism  $\varphi : \mathcal{L}(\bigoplus_j \mathcal{H}^j) \xrightarrow{\sim} \mathbf{X} \otimes \mathbf{Y}$  so that  $\varphi(x) = x \otimes \mathbf{1}_\mathbf{Y}$  for  $x \in \mathbf{X}$ . On the other hand,  $\pi^i \in \mathbf{X}'$  so we must have  $\varphi(\pi^i) = \mathbf{1}_\mathbf{X} \otimes \tilde{\pi}^i$  for some  $\tilde{\pi}^i \in \mathbf{Y}$ . Suppose that  $x\pi^i = 0$ ; then  $0 = \varphi(x)\varphi(\pi^i) = x \otimes \tilde{\pi}^i$ . Since  $\pi^i$  is nonzero, then,  $x$  must be zero.  $\square$

We can now prove the following variant of Lemma 3.27. It has one fewer premise, which comes at the cost of requiring a direct sum in the Hilbert space representation. As we mentioned in Section 3.5.3, this result can be used to prove causal decompositions in terms of *routed* unitary circuits [107] for some relations  $G$  that do not satisfy  $(C_3\text{-EP})$ , providing an alternative to the proofs in Lorenz and Barrett [62].

**Proposition 3.34.** *Let  $\mathbf{A}, \mathbf{X}_1, \dots, \mathbf{X}_n$  be factors and let  $\mathbf{B}_k \subseteq \mathbf{A}\mathbf{X}_1 \cdots \mathbf{X}_n$  for  $k \in [n]$  be (potentially non-factor) unital  $*$ -subalgebras that satisfy (i) and (ii) in Lemma 3.27, which for convenience we restate here:*

- (i)  $\mathbf{B}_k \subseteq (\mathbf{B}_l)'$  for all  $k, l \in [n], k \neq l$ ;
- (ii)  $\mathbf{B}_k \subseteq \mathbf{A}\mathbf{X}_k = (\mathbf{X}_{[n] \setminus \{k\}})'$  for all  $k \in [n]$ .

*Then there are  $m \in \mathbb{N}_{>0}$ , finite-dimensional Hilbert spaces  $\mathcal{H}_{\mathbf{Z}_k^i}$  for all  $k \in [n]$  and*

$i \in [m]$ , and a  $*$ -isomorphism

$$\mathcal{V} : \mathbf{A} \xrightarrow{\sim} \mathcal{L} \left( \bigoplus_{i \in [m]} \mathcal{H}_{\mathbf{Z}_1^i} \otimes \cdots \otimes \mathcal{H}_{\mathbf{Z}_n^i} \right) \quad (3.89)$$

so that, writing  $\mathcal{V}$  for  $\mathcal{V} \otimes \text{id}_{\mathbf{X}_1 \dots \mathbf{X}_n}$ , for each  $k \in [n]$

$$\mathcal{V}(\mathbf{B}_k) \subseteq \left( \bigoplus_{i \in [m]} \mathcal{L}(\mathcal{H}_{\mathbf{Z}_k^i}) \otimes \{ \mathbf{1}_{\mathbf{Z}_{[n] \setminus \{k\}}^i} \} \right) \otimes \mathbf{X}_k \otimes \{ \mathbf{1}_{\mathbf{X}_{[n] \setminus \{k\}}} \}. \quad (3.90)$$

*Proof.* We first simplify the statement by showing that we can assume each of the  $\mathbf{X}_k$  to be a trivial algebra. For this, consider, for each  $k$ , the algebraic span  $\mathbf{B}_k \vee \mathbf{X}_k \subseteq \mathbf{A} \otimes \mathbf{X}_k$ ; by Lemma 3.32, we have

$$\mathbf{B}_k \vee \mathbf{X}_k = \tilde{\mathbf{B}}_k \otimes \mathbf{X}_k \quad (3.91)$$

for some unital  $*$ -subalgebra  $\tilde{\mathbf{B}}_k$  of  $\mathbf{A}$ . It now suffices to prove the proposition for  $\tilde{\mathbf{B}}_k$ , i.e. to show the existence of  $\mathcal{V}$  as in Eq. (3.89) so that

$$\mathcal{V}(\tilde{\mathbf{B}}_k) \subseteq \bigoplus_{i \in [m]} \mathcal{L}(\mathcal{H}_{\mathbf{Z}_k^i}) \otimes \{ \mathbf{1}_{\mathbf{Z}_{[n] \setminus \{k\}}^i} \}. \quad (3.92)$$

After all, this would imply that

$$(\mathcal{V} \otimes \text{id}_{\mathbf{X}_1 \dots \mathbf{X}_n})(\mathbf{B}_k) \subseteq (\mathcal{V} \otimes \text{id}_{\mathbf{X}_1 \dots \mathbf{X}_n})(\mathbf{B}_k \vee \mathbf{X}_k) = \mathcal{V}(\tilde{\mathbf{B}}_k) \otimes \mathbf{X}_k \otimes \{ \mathbf{1}_{\mathbf{X}_{[n] \setminus \{k\}}} \} \quad (3.93)$$

demonstrating Eq. (3.90).

Now, even if the algebras  $\mathbf{B}_k$  are factors,  $\tilde{\mathbf{B}}_k$  might not necessarily be. To be able to apply Proposition 3.31, therefore, we first project down using the central projectors of  $\tilde{\mathbf{B}}_k$ . We will henceforth drop the tildes on  $\tilde{\mathbf{B}}_k$ , redefining  $\mathbf{B}_k := \tilde{\mathbf{B}}_k$ . We also assume, without loss of generality, that  $\mathbf{A} = \mathcal{L}(\mathcal{H}_{\mathbf{A}})$  for a Hilbert space  $\mathcal{H}_{\mathbf{A}}$ .

Fix  $k \in [n]$ . As noted after Eq. (3.83), the centre  $\mathcal{Z}(\mathbf{B}_k)$  admits a basis  $\{ \pi_k^{i_k} \}_{i_k \in [m_k]} \subseteq \mathcal{Z}(\mathbf{B}_k)$  consisting of mutually orthogonal projectors  $\pi_k^{i_k} : \mathcal{H}_{\mathbf{A}} \rightarrow \mathcal{H}_{\mathbf{A}}$ . Here  $m_k$  is the dimension of  $\mathcal{Z}(\mathbf{B}_k)$ . Each algebra  $\mathbf{B}_k \pi_k^{i_k}$  is a factor (see Eq. (3.83)) and  $\sum_{i_k \in [m_k]} \pi_k^{i_k} = \mathbf{1}_{\mathbf{A}}$ .

Since all  $\mathbf{B}_k, \mathbf{B}_l$  commute, we have  $[\pi_k^{i_k}, \pi_l^{i_l}] = 0$  for all  $k, l, i_k, i_l$ . Hence, if we define  $\pi^{\vec{i}} := \pi_1^{i_1} \pi_2^{i_2} \cdots \pi_n^{i_n}$  for  $\vec{i} = (i_1, i_2, \dots, i_n)$ , then  $\{ \pi^{\vec{i}} \}_{\vec{i}}$  forms a complete family of mutually orthogonal projectors on  $\mathcal{H}_{\mathbf{A}}$ . This induces a subspace decomposition

$$\mathcal{H}_{\mathbf{A}} = \bigoplus_{\vec{i}} \mathcal{H}_{\mathbf{A}^{\vec{i}}}, \quad (3.94)$$

where  $\mathcal{H}_{\mathbf{A}^{\vec{i}}} := \pi^{\vec{i}} \mathcal{H}_{\mathbf{A}}$ . Moreover, for each  $\vec{i} = (i_1, i_2, \dots, i_n)$  and  $k \in [n]$ , we have  $\pi^{\vec{i}} \in \mathbf{B}'_k$ . Therefore, if we define

$$\mathbf{B}_k^{\vec{i}} := \mathbf{B}_k \pi^{\vec{i}}, \quad \text{then we have} \quad \mathbf{B}_k \subseteq \bigoplus_{\vec{i}} \mathbf{B}_k^{\vec{i}} : \quad (3.95)$$

in other words,  $\mathbf{B}_k$  is block-diagonal in the decomposition defined by Eq. (3.94).

Now fix  $\vec{i} = (i_1, i_2, \dots, i_n)$  and  $k \in [n]$ . We claim that  $\mathbf{B}_k^{\vec{i}}$  is a factor. To see this, note first that  $\mathbf{B}_k^{\vec{i}} = \mathbf{B}_k \pi^{\vec{i}} = \mathbf{B}_k \pi_k^{i_k} \pi^{\vec{i}}$ . As we noted above, the algebra  $\mathbf{B}_k \pi_k^{i_k}$  is a factor. Moreover, like  $\mathbf{B}_k$  itself, it commutes with all  $\pi^{\vec{j}}$  and is therefore block-diagonal in the decomposition of Eq. (3.94). Lemma 3.33 thus tells us that  $\mathbf{B}_k^{\vec{i}} = \mathbf{B}_k \pi_k^{i_k} \pi^{\vec{i}}$  is  $*$ -isomorphic to  $\mathbf{B}_k \pi_k^{i_k}$ ; in particular, it is also a factor.

We thus have  $n$  pairwise-commuting factor unital  $*$ -subalgebras  $\mathbf{B}_1^{\vec{i}}, \dots, \mathbf{B}_n^{\vec{i}}$  of  $\mathcal{L}(\mathcal{H}_{\mathbf{A}^{\vec{i}}})$ . By Proposition 3.31, then, there exists a unitary

$$V^{\vec{i}} : \mathcal{H}_{\mathbf{A}^{\vec{i}}} \xrightarrow{\sim} \mathcal{H}_{\mathbf{Z}_1^{\vec{i}}} \otimes \dots \otimes \mathcal{H}_{\mathbf{Z}_n^{\vec{i}}} \quad (3.96)$$

that factorises  $\mathcal{H}_{\mathbf{A}^{\vec{i}}}$  into finite-dimensional Hilbert spaces  $\mathcal{H}_{\mathbf{Z}_k^{\vec{i}}}$  so that for each  $k$ ,

$$V^{\vec{i}} \mathbf{B}_k^{\vec{i}} (V^{\vec{i}})^* \subseteq \mathcal{L}(\mathcal{H}_{\mathbf{Z}_k^{\vec{i}}}) \otimes \left\{ \mathbf{1}_{\mathbf{Z}_{[n] \setminus \{k\}}^{\vec{i}}} \right\}. \quad (3.97)$$

Finally, defining  $V := \bigoplus_{\vec{i}} V^{\vec{i}} : \mathcal{H}_{\mathbf{A}} \xrightarrow{\sim} \bigoplus_{\vec{i}} \bigotimes_k \mathcal{H}_{\mathbf{Z}_k^{\vec{i}}}$  and using Eq. (3.95), we get

$$V \mathbf{B}_k V^* \subseteq V \left( \bigoplus_{\vec{i}} \mathbf{B}_k^{\vec{i}} \right) V^* = \bigoplus_{\vec{i}} V^{\vec{i}} \mathbf{B}_k^{\vec{i}} (V^{\vec{i}})^* \subseteq \bigoplus_{\vec{i}} \mathcal{L}(\mathcal{H}_{\mathbf{Z}_k^{\vec{i}}}) \otimes \left\{ \mathbf{1}_{\mathbf{Z}_{[n] \setminus \{k\}}^{\vec{i}}} \right\}, \quad (3.98)$$

which is the desired result (3.92).  $\square$

We can now prove Lemma 3.27, which we restate for convenience.

**Lemma 3.27.** *Let  $\mathbf{A}, \mathbf{X}_1, \dots, \mathbf{X}_n$  be finite-dimensional factor  $C^*$ -algebras and let  $\mathbf{B}_k \subseteq \mathbf{A} \mathbf{X}_1 \mathbf{X}_2 \dots \mathbf{X}_n$  for  $k \in \{1, \dots, n\} =: [n]$  be unital factor  $*$ -subalgebras of their tensor product so that*

- (i)  $\mathbf{B}_k \subseteq (\mathbf{B}_l)'$  for all  $k, l \in [n], k \neq l$ ;
- (ii)  $\mathbf{B}_k \subseteq \mathbf{A} \mathbf{X}_k = (\mathbf{X}_{[n] \setminus \{k\}})'$  for all  $k \in [n]$ ; and
- (iii)  $\mathbf{A} \subseteq \bigvee_{k \in [n]} \mathbf{B}_k$ , where  $\bigvee$  denotes the algebraic span.

*Then there are finite-dimensional factor  $C^*$ -algebras  $\mathbf{Z}_k$  for each  $k \in [n]$  and a  $*$ -isomorphism*

$$\mathcal{V} : \mathbf{A} \xrightarrow{\sim} \mathbf{Z}_1 \mathbf{Z}_2 \dots \mathbf{Z}_n \quad (3.45)$$

*so that, writing  $\mathcal{V}$  for  $\mathcal{V} \otimes \text{id}_{\mathbf{X}_1 \dots \mathbf{X}_n}$ , for each  $k$  we have*

$$\mathcal{V}(\mathbf{B}_k) \subseteq \mathbf{Z}_k \mathbf{X}_k \text{ and} \quad (3.46a)$$

$$\mathbf{Z}_k = \mathcal{V}(\mathbf{A} \cap \mathbf{B}_k). \quad (3.46b)$$

*Proof.* First of all, assumptions (i) and (ii) and Proposition 3.34 above give us Hilbert spaces  $\mathcal{H}_{\mathbf{Z}_k^i}$  and a  $*$ -isomorphism  $\mathcal{V} : \mathbf{A} \xrightarrow{\sim} \mathcal{L}(\bigoplus_{i \in [n]} \bigotimes_{k \in [n]} \mathcal{H}_{\mathbf{Z}_k^i})$  satisfying Eq. (3.90).

By assumption (iii),

$$\mathcal{L} \left( \bigoplus_{i \in [m]} \bigotimes_{k \in [n]} \mathcal{H}_{\mathbf{Z}_k^i} \right) = \mathcal{V}(\mathbf{A}) \subseteq \bigvee_{k \in [n]} \mathcal{V}(\mathbf{B}_k) \subseteq \bigoplus_{i \in [m]} \mathcal{L} \left( \bigotimes_{k \in [n]} \mathcal{H}_{\mathbf{Z}_k^i} \right) \otimes \mathbf{X}_{[n]} \quad (3.99)$$

where for the final inclusion we have used Eq. (3.90). Thus, *every* operator on the Hilbert space  $\bigoplus_{i \in [m]} \bigotimes_{k \in [n]} \mathcal{H}_{\mathbf{A}_k^i}$  is block-diagonal in the direct sum indexed by  $i$ , which can only be the case if the direct sum is trivial: that is, if there is only one  $i \in [m]$  for which  $\bigotimes_{k \in [n]} \mathcal{H}_{\mathbf{Z}_k^i}$  has positive dimension. We can therefore drop the direct sum and the index  $i$ , yielding Hilbert spaces  $\mathcal{H}_{\mathbf{Z}_k}$  for  $k = 1, \dots, n$  and a \*-isomorphism  $\mathcal{V} : \mathbf{A} \xrightarrow{\sim} \bigotimes_{k \in [n]} \mathbf{Z}_k$ , where  $\mathbf{Z}_k := \mathcal{L}(\mathcal{H}_{\mathbf{Z}_k})$ , satisfying Eq. (3.46a).

It remains to prove Eq. (3.46b). Fix  $k \in [n]$ . We have  $\mathbf{Z}_k \subseteq \mathcal{V}(\mathbf{A}) \subseteq \bigvee_{l \in [n]} \mathcal{V}(\mathbf{B}_l)$  by assumption (iii). Moreover, by Eq. (3.46a), we have  $\mathcal{V}(\mathbf{B}_l) \subseteq \mathbf{Z}_k'$  whenever  $k \neq l$ ; or, equivalently,  $\mathbf{Z}_k \subseteq (\mathcal{V}(\mathbf{B}_l))'$ . Therefore  $\mathbf{Z}_k \subseteq \left( \bigvee_{l \in [n] \setminus \{k\}} \mathcal{V}(\mathbf{B}_l) \right)'$ . By Proposition 3.31, there is a \*-isomorphism  $\varphi : \bigvee_{l \in [n]} \mathcal{V}(\mathbf{B}_l) \xrightarrow{\sim} \bigotimes_{l \in [n]} \mathcal{V}(\mathbf{B}_l)$  under which  $\varphi(\mathcal{V}(\mathbf{B}_l)) = \mathcal{V}(\mathbf{B}_l) \otimes \{\mathbf{1}_{\mathcal{V}(\mathbf{B}_{[n] \setminus \{l\}})}\}$  for each  $l$ . Overall, then, we have  $\varphi(\mathbf{Z}_k) \subseteq \left( \bigotimes_{l \in [n] \setminus \{k\}} \mathcal{V}(\mathbf{B}_l) \right)' \cap \bigotimes_{l \in [n]} \mathcal{V}(\mathbf{B}_l) = \mathcal{V}(\mathbf{B}_k) \otimes \{\mathbf{1}_{\mathcal{V}(\mathbf{B}_{[n] \setminus \{k\}})}\}$ , and therefore  $\mathbf{Z}_k \subseteq \mathcal{V}(\mathbf{B}_k)$ .

Since also  $\mathbf{Z}_k \subseteq \mathcal{V}(\mathbf{A})$ , we have  $\mathbf{Z}_k \subseteq \mathcal{V}(\mathbf{A} \cap \mathbf{B}_k)$ . For the inverse inclusion, note that by Eq. (3.46a),

$$\mathcal{V}(\mathbf{A} \cap \mathbf{B}_k) = \mathcal{V}(\mathbf{A}) \cap \mathcal{V}(\mathbf{B}_k) \subseteq (\mathbf{Z}_1 \mathbf{Z}_2 \cdots \mathbf{Z}_n) \cap (\mathbf{Z}_k \mathbf{X}_k) = \mathbf{Z}_k. \quad (3.100)$$

This completes the proof.  $\square$



## Chapter 4

# Relativistic, causal, and compositional structure

A transformation of quantum systems may be deemed compatible with the relativistic causal structure of a spacetime in several different senses. A global, top-down perspective dictates that the transformation must not enable signalling between spacelike separated regions, a requirement we will call *signal consistency*. A stronger, bottom-up requirement is *realisability*, meaning the transformation can be implemented by dynamics that are locally consistent with relativity theory. Realisability implies signal consistency, and it is natural to ask whether the converse also holds: *Is the absence of superluminal signalling sufficient to conclude that a transformation is in fact realisable?*

This question has been studied in various special cases in the quantum-information-theoretic literature [8, 13, 35, 86], where transformations are quantum channels between tensor products of input and output systems. Signal consistency and realisability are properties of a channel that hold relative to a specification of a spacetime  $M$  as well as localisations of the channel’s inputs and outputs at points in  $M$ ; we shall refer to such a specification as a *spacetime situation*  $\mathcal{M}$ . In this setting, realising a channel means implementing it by a quantum circuit whose gates are at spacetime points and whose wires follow causal curves.

The answer to the question in italics turns out to depend on the particular spacetime situation under consideration. Consider, for instance, the case of two input systems  $\mathbf{A}_1, \mathbf{A}_2$  and two output systems  $\mathbf{B}_1, \mathbf{B}_2$ . If  $\mathbf{B}_1$  lies in the future lightcones of both  $\mathbf{A}_1$  and  $\mathbf{A}_2$ , while  $\mathbf{B}_2$  lies in the future lightcone of  $\mathbf{A}_2$  but is spacelike separated from  $\mathbf{A}_1$  (Fig. 4.1a), then signal consistency implies realisability: indeed, any bipartite channel  $\mathcal{E} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$  that features no signalling from  $\mathbf{A}_1$  to  $\mathbf{B}_2$  can be implemented without communication between those systems (see Definition 3.8 and [35]).<sup>1</sup> However, in a Bell-like situation where  $\mathbf{B}_1$  is also spacelike separated from  $\mathbf{A}_2$ , the implication fails [13].<sup>2</sup> The paradigmatic example is the Popescu-Rohrlich (PR) box: it cannot

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<sup>1</sup>In the terminology of [13, 35], “semicausal operations are semilocalisable.”

<sup>2</sup>“Not all causal operations are localisable.”

be used to signal in either direction, yet any quantum circuit realisation would require communication in at least one direction.

Constraints on communication between spacelike separated inputs and outputs are, however, not the only spacetime-imposed limits on realisable channels. Consider instead a setting where both  $\mathbf{B}_1$  and  $\mathbf{B}_2$  are in the future lightcones of both  $\mathbf{A}_1$  and  $\mathbf{A}_2$ . In this case, signal consistency is a trivial constraint, but realisability may still be obstructed by the absence of a point in the *intersection* of the future lightcones of  $\mathbf{A}_1, \mathbf{A}_2$  and the past lightcones of  $\mathbf{B}_1, \mathbf{B}_2$ , where all systems can meet and interact. This constraint on interaction has been studied extensively in the context of position-based quantum cryptography and holography and has led to the development of protocols for *non-local quantum computation* [14, 21, 34, 69, 92]. These provide realisations of arbitrary quantum channels without relying on such interaction, instead making use of a shared entanglement resource. This can generally, however, only be achieved approximately, up to an arbitrarily small error [71].

In this chapter we examine how these ideas generalise to arbitrary, multipartite spacetime situations  $\mathcal{M}$  that may impose both communication (*connectivity*) constraints (as in [8, 13, 35, 86]) and interaction constraints (as in [14, 21, 34, 69, 92]). We argue for the significance of yet a third notion of compatibility with relativistic causality: *influence consistency*, which asserts the existence of a *unitary dilation*  $\mathcal{U}$  of  $\mathcal{E}$  that itself features no influences between spacelike separated inputs and outputs. Thus, while signal consistency is an operational criterion that merely rules out violations of relativistic causality on the epistemic level of the channel  $\mathcal{E}$ , influence consistency demands compatibility of  $\mathcal{E}$  with an underlying fundamental, unitary description  $\mathcal{U}$  that is itself consistent with relativistic causality. Influence consistency is necessary for realisability, and we conjecture that it, unlike signal consistency, is also sufficient for approximate realisability. We show this in a number of special cases and make precise the connection with causal decompositions.

We begin in Section 4.1 by reviewing the bipartite cases discussed above in more detail and showing how considering influences through unitary dilations can shed light on some phenomena encountered in the quantum-relativistic causality literature. In particular, we take a focus, in Section 4.1.2, on the Bell-like scenario with two-way constraints from [13], where we show influence consistency is indeed equivalent to realisability. The fact that channels like the PR box are, although signal-consistent, not realisable can thus be explained by the fact that any hypothetical implementation would necessarily involve superluminal influences on the fundamental, unitary level, even though those influences would not manifest as superluminal signalling on the observed level. We argue that the latter can only be the consequence of a fine-tuned environment state. In addition, in Section 4.1.3 we discuss how also the phenomenon of *jamming*—encountered elsewhere in the quantum relativistic causality literature [49, 53, 112–114]—while generally regarded compatible with signal consistency, cannot arise when imposing influence consistency.

In Section 4.2, we define signal consistency, influence consistency, and realisability for general, multipartite spacetime situations  $\mathcal{M}$ , and state our main conjecture: that

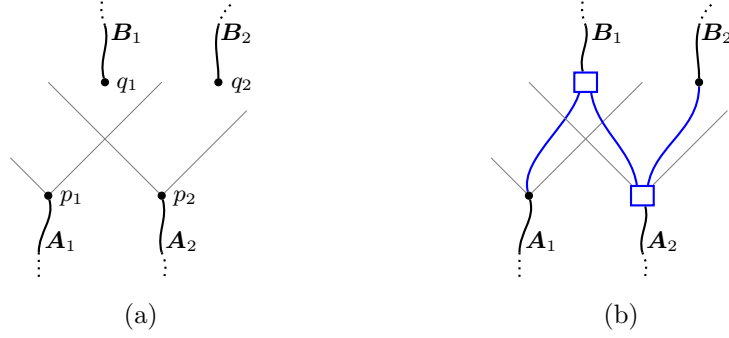


Figure 4.1: (a) The spacetime situation  $\mathcal{M}_1$  considered in Section 4.1.1. Time flows upwards. Input systems  $\mathbf{A}_1, \mathbf{A}_2$  are localised at spacetime points  $p_1, p_2$ , output systems  $\mathbf{B}_1, \mathbf{B}_2$  at  $q_1, q_2$ . (b) A realisation of a quantum channel  $\mathcal{E} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$ . The gates are performed within negligibly small regions of spacetime and the wires follow causal trajectories (i.e. everywhere time- or lightlike).

influence consistency with  $\mathcal{M}$  is sufficient for approximate realisability in  $\mathcal{M}$ . We allow these realisations to be approximate rather than exact since this is all one can achieve in the general case, as noted earlier in the discussion of nonlocal quantum computation. The conjecture boils down to the existence of approximate causal decompositions of a particular shape: those consisting of local operations, shared entanglement, and one round of non-interactive communication among a subset  $G$  of input-output pairs (denoted  $\text{LOSEC}_G$ ). We show using methods of nonlocal quantum computation [34] that their existence, and the main conjecture, is completely equivalent to the existence of general approximate causal decompositions (of any shape). We conclude by showing that all Clifford unitaries admit exact causally faithful decompositions of the  $\text{LOSEC}_G$  form.

## 4.1 Bipartite case studies

### 4.1.1 One-way constraint

Consider the situation depicted in Fig. 4.1a, inspired by Ref. [13]. At two points  $p_1$  and  $p_2$  in a spacetime  $M_1$ , highly localised quantum systems arrive with internal degrees of freedom described by finite-dimensional Hilbert spaces  $\mathcal{H}_{\mathbf{A}_1}$  and  $\mathcal{H}_{\mathbf{A}_2}$ . At two other points  $q_1 \in M_1$ , in the joint future of  $p_1$  and  $p_2$ , and  $q_2 \in M_1$ , in the future of  $p_2$  but spacelike separated from  $p_1$ , two quantum systems  $\mathcal{H}_{\mathbf{B}_1}$  and  $\mathcal{H}_{\mathbf{B}_2}$  leave. We will refer to the combination of this spacetime  $M_1$ , the quantum systems  $\mathcal{H}_{\mathbf{A}_1}, \mathcal{H}_{\mathbf{A}_2}, \mathcal{H}_{\mathbf{B}_1}, \mathcal{H}_{\mathbf{B}_2}$ , and their localisations  $p_1, p_2, q_1, q_2$  as the *spacetime situation*  $\mathcal{M}_1$ . Any other quantum systems that might be present in the spacetime are not specified by  $\mathcal{M}_1$ , neither is the dynamics taking place between the incoming and outgoing systems. The restriction of the dynamics to our four systems of interest, however, should no matter what be describable by a quantum channel  $\mathcal{E} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$ . Recall from Chapter 3 that we write  $\mathbf{X}$  for  $\mathcal{L}(\mathcal{H}_{\mathbf{X}})$ , the algebra of linear operators on a finite-dimensional Hilbert

space  $\mathcal{H}_X$ , and  $\mathbf{XY} := \mathbf{X} \otimes \mathbf{Y} = \mathcal{L}(\mathcal{H}_X) \otimes \mathcal{L}(\mathcal{H}_Y) \cong \mathcal{L}(\mathcal{H}_X \otimes \mathcal{H}_Y)$ .

The channel  $\mathcal{E} : \mathbf{A}_1 \mathbf{A}_2 \rightarrow \mathbf{B}_1 \mathbf{B}_2$  describes only the nonrelativistic, information-theoretic facets of the dynamics concerning the systems' internal degrees of freedom; it is silent about any spatiotemporal aspects. The spacetime situation  $\mathcal{M}_1$ , however, constrains the possible dynamics  $\mathcal{E}$  that can take place. First of all, relativistic causality imposes that it should not be possible to use  $\mathcal{E}$  for superluminal signalling. We introduced the notion of *signalling* between inputs and outputs of a channel in Chapter 3 (Definition 3.8); this defines a property purely of the channel  $\mathcal{E}$ , not a priori related to relativistic notions. However, since in the present spacetime context,  $\mathbf{A}_1, \mathbf{B}_2$  is the only input-output pair whose localisations in  $\mathcal{M}_1$  are spacelike separated, we say that  $\mathcal{E}$  is **signal-consistent with  $\mathcal{M}_1$**  precisely if  $\mathbf{A}_1 \not\rightarrow_{\mathcal{E}} \mathbf{B}_2$ . Denote by

$$\text{SC}_{\mathcal{M}_1} := \{\mathcal{E} : \mathbf{A}_1 \mathbf{A}_2 \rightarrow \mathbf{B}_1 \mathbf{B}_2 \mid \mathcal{E} \text{ is CPTP and } \mathbf{A}_1 \not\rightarrow_{\mathcal{E}} \mathbf{B}_2\} \quad (4.1)$$

the set of channels satisfying this constraint. (The constraint is called *semicausality* in [13, 35].<sup>3</sup>)

Signal consistency is a global, top-down constraint on the overall channel  $\mathcal{E}$ ; another, conceptually distinct requirement is that  $\mathcal{E}$  can in fact be *realised* by dynamics that are everywhere, locally, consistent with relativity theory. Following in spirit e.g. Refs. [13, 14, 21, 34, 35, 69, 102], we restrict our attention to realisations expressed as finite-dimensional quantum circuits embedded in the spacetime by placing all gates at spacetime points and wires along causal curves. In the situation  $\mathcal{M}_1$  of Fig. 4.1a, the most general channel that can arise in this way is one implemented by a circuit of the form depicted in Fig. 4.1b. Let us therefore call a channel  $\mathcal{E} : \mathbf{A}_1 \mathbf{A}_2 \rightarrow \mathbf{B}_1 \mathbf{B}_2$  **realisable in  $\mathcal{M}_1$**  if it admits of a circuit decomposition of that form, a property referred to as *semilocalisability* in [13, 35]. Denote the set of these channels by

$$\text{Real}_{\mathcal{M}_1} := \left\{ \begin{array}{c} \begin{array}{c} \mathbf{B}_1 \quad \mathbf{B}_2 \\ \boxed{\mathcal{G}} \quad \mathbf{X} \\ \mathbf{A}_1 \quad \mathbf{A}_2 \end{array} \quad \left| \quad \begin{array}{l} \mathcal{H}_X \text{ finite-} \\ \text{dimensional;} \\ \mathcal{F}, \mathcal{G} \text{ CPTP} \end{array} \right. \end{array} \right\}. \quad (4.2)$$

How are signal consistency with and realisability in  $\mathcal{M}_1$  related? First of all, if our notion of realisability is any good, then no realisable channels should ever lead to superluminal signalling. We saw in Section 3.3 that if a channel has a circuit decomposition of the form in Eq. (4.2), then it indeed satisfies the no-signalling relation  $\mathbf{A}_1 \not\rightarrow_{\mathcal{E}} \mathbf{B}_2$ . Moreover, thanks to [35], the converse also holds (see Definition 3.8). This purely non-relativistic result about quantum channels therefore tells us that with respect to the spacetime situation  $\mathcal{M}_1$ , a channel is realisable in  $\mathcal{M}_1$  precisely if it is signal consistent with  $\mathcal{M}_1$ :

$$\text{Real}_{\mathcal{M}_1} = \text{SC}_{\mathcal{M}_1}. \quad (4.3)$$

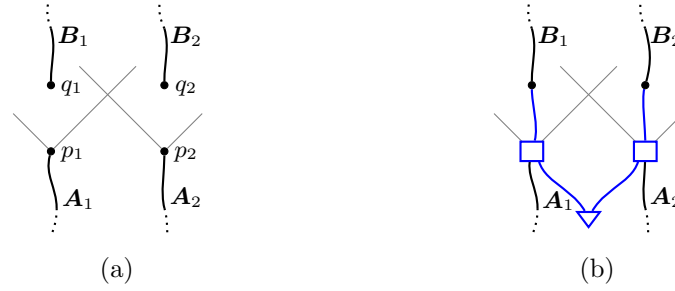


Figure 4.2: (a) The spacetime situation  $\mathcal{M}_2$  considered in Section 4.1.2. Similar to  $\mathcal{M}_1$  in Fig. 4.1a, except that  $B_1$  is now localised at a point  $q_1$  outside the future lightcone of  $p_2$ . (b) A realisation of a channel  $\mathcal{E} : A_1 A_2 \rightarrow B_1 B_2$  by local operations and shared entanglement.

#### 4.1.2 Two-way constraints; the significance of unitary dilations

The same does not go through for other configurations of our four systems in spacetime. Consider the spacetime situation  $\mathcal{M}_2$  depicted in Fig. 4.2a: here, not only are  $A_1$  and  $B_2$  localised at spacelike-separate points, but  $A_2$  and  $B_1$  are, too. This setup is thus akin to the standard setup of a Bell experiment—although the inputs and outputs here are quantum systems, rather than classical variables. The appropriate notion of **signal consistency with  $\mathcal{M}_2$**  requires of a channel  $\mathcal{E}$  that it features no signalling between either of these two input-output pairs:<sup>4</sup>

$$\text{SC}_{\mathcal{M}_2} := \{\mathcal{E} : A_1 A_2 \rightarrow B_1 B_2 \mid \mathcal{E} \text{ is CPTP and } A_1 \nrightarrow_{\mathcal{E}} B_2 \text{ and } A_2 \nrightarrow_{\mathcal{E}} B_1\}. \quad (4.4)$$

The channels  $\mathcal{E}$  that are **realisable in  $\mathcal{M}_2$** ,<sup>5</sup> on the other hand, are those that can be implemented by local operations and shared entanglement (LOSE), as depicted in Fig. 4.2b. They form the set

$$\text{Real}_{\mathcal{M}_2} := \left\{ \begin{array}{c} B_1 \quad B_2 \\ \square \quad \square \\ \diagdown \quad \diagup \\ A_1 \quad A_2 \end{array} \right\}. \quad (4.5)$$

Here, and in the following, we implicitly quantify over finite-dimensional systems for the internal wires of the circuit (in this case the two systems of the shared entanglement resource) and over channels for its gates (in this case two channels and a state), similarly to Eq. (4.2).

It is immediate that  $\text{Real}_{\mathcal{M}_2} \subseteq \text{SC}_{\mathcal{M}_2}$ : this follows from soundness of no-connectivity for no-signalling (Proposition 3.13) and the absence of paths connecting  $A_1$  to  $B_2$  and  $A_2$  to  $B_1$  through the circuit in Eq. (4.5). This time, however, the converse inclusion fails.

<sup>3</sup>Note however that while Refs. [13, 35] focus on cases with identical ingoing and outgoing systems ( $A_1 = B_1$  and  $A_2 = B_2$ ), we allow them to be distinct and of potentially different dimensions.

<sup>4</sup>This property is called *causality* of  $\mathcal{E}$  in Refs. [13, 35], to contrast with the notion of semicausality from Eq. (4.1).

<sup>5</sup>Or *localisable*, in the terms of Refs. [13, 35].

**Proposition 4.1** (Beckman et al. [13]). *There exist channels  $\mathcal{E} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$  that feature no signalling  $\mathbf{A}_1 \nrightarrow_{\mathcal{E}} \mathbf{B}_2$  and  $\mathbf{A}_2 \nrightarrow_{\mathcal{E}} \mathbf{B}_1$ , but that nevertheless admit no decomposition into local operations and shared entanglement. As a consequence,*

$$\text{Real}_{\mathcal{M}_2} \subsetneq \text{SC}_{\mathcal{M}_2}. \quad (4.6)$$

One example provided in Ref. [13] is the Popescu-Rohrlich (PR) box [80]. This is a conditional probability distribution  $P_{\text{PR}}(ab|xy)$  over four binary random variables, or, in other words, a classical stochastic channel with two binary inputs  $x, y$  and two binary outputs  $a, b$ ; but like any classical channel, it can equally well be considered as a two-qubit quantum channel  $\mathcal{E}_{\text{PR}} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$  by pre- and post-composing it with, respectively, measure and prepare channels in a fixed basis. This channel cannot be realised by shared entanglement and local operations. Indeed, the PR box violates the Tsirelson bound, which cannot be achieved by local operations on a shared entangled state [31].

Signal consistency of a channel alone is thus not sufficient to conclude realisability.<sup>6</sup> Are there any stronger restrictions on the channel, then, that *are* sufficient? One sufficient additional restriction noted in [13, 77] is unitarity: any channel  $\mathcal{U} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$  that is unitary and that satisfies  $\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2$  and  $\mathbf{A}_2 \nrightarrow_{\mathcal{U}} \mathbf{B}_1$  factorises into a tensor product  $\mathcal{U} = \mathcal{U}_1 \otimes \mathcal{U}_2$  of local unitaries  $\mathcal{U}_1 : \mathbf{A}_1 \rightarrow \mathbf{B}_1$  and  $\mathcal{U}_2 : \mathbf{A}_2 \rightarrow \mathbf{B}_2$  and can thus be realised in  $\mathcal{M}_2$  even without using any shared entanglement.<sup>7</sup> To find a condition that is both necessary and sufficient for general channels, we look instead at their unitary *dilations*. It will be useful to establish some notation.

**Definition 4.2.** For a finite-dimensional system  $\mathcal{H}_E$ , denote by  $\mathfrak{D}(E)$  the set of density operators on  $\mathcal{H}_E$ . For a unitary channel  $\mathcal{U} : \mathbf{A}_1\mathbf{A}_2\mathbf{E} \rightarrow \mathbf{B}_1\mathbf{B}_2\mathbf{F}$  and a state  $\rho \in \mathfrak{D}(E)$ , write  $\mathcal{E}_{(\mathcal{U}, \rho)}$  for the channel  $\text{Tr}_F \mathcal{U}(- \otimes \rho) : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$ , or, diagrammatically,

$$\begin{array}{|c|c|} \hline \mathbf{B}_1 & \mathbf{B}_2 \\ \hline \mathcal{E}_{(\mathcal{U}, \rho)} \\ \hline \mathbf{A}_1 & \mathbf{A}_2 \\ \hline \end{array} \quad := \quad \begin{array}{|c|c|c|} \hline \mathbf{B}_1 & \mathbf{B}_2 & \overline{\mathbf{F}} \\ \hline \mathcal{U} \\ \hline \mathbf{A}_1 & \mathbf{A}_2 & \begin{array}{c} \mathbf{E} \\ \triangle \rho \end{array} \\ \hline \end{array}. \quad (4.7)$$

The pair  $(\mathcal{U}, \rho)$  thus forms a unitary dilation of a channel (CPTP map)  $\mathcal{E} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$  if  $\mathcal{E} = \mathcal{E}_{(\mathcal{U}, \rho)}$ . Every channel has a unitary dilation by the classic result of Stinespring [95].

Whilst a channel  $\mathcal{E}$  describes only the epistemic, operational aspects of a physical process, a unitary dilation describes a possible underlying, fundamental dynamics. As

<sup>6</sup>At this point it is worth emphasising that our interest lies in *quantum* circuit realisations. In the literature one occasionally runs into (hypothetical) generalised, post-quantum theories that can realise PR correlations even with circuits of the form of Eq. (4.5) [10, 24, 29], but we do not consider such theories here.

<sup>7</sup>Note that this can be regarded as a special case of our Theorem 3.23.

before in Chapter 3, we shall refer to signalling relations through such a unitary channel as *(causal) influence*, rather than signalling. Crucially, a channel  $\mathcal{E}$  admits many different unitary dilations: indeed, distinct dynamics on the fundamental level may be indistinguishable on the observed level (without access to the environment system). In particular, therefore, the presence or absence of causal influence on the unitary level is not a property of  $\mathcal{E}$  itself, but rather of one of its particular unitary dilations. Nevertheless, causal influences have consequences for signalling; we have, for instance, the following extension of Proposition 3.20.

**Proposition 4.3.** *Let  $\mathcal{U} : \mathbf{A}_1 \mathbf{A}_2 \mathbf{E} \rightarrow \mathbf{B}_1 \mathbf{B}_2 \mathbf{F}$  be a unitary channel. Then for any  $\rho \in \mathfrak{D}(\mathbf{E})$ , no-influence relations through  $\mathcal{U}$  imply no-signalling relations through  $\mathcal{E}_{(\mathcal{U}, \rho)}$ : in particular,*

$$\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2 \implies \mathbf{A}_1 \nrightarrow_{\mathcal{E}_{(\mathcal{U}, \rho)}} \mathbf{B}_2. \quad (4.8)$$

*The converse implication may, however, fail. Moreover, in case it fails, i.e. if  $\mathbf{A}_1 \rightarrow_{\mathcal{U}} \mathbf{B}_2$  but there exists  $\rho \in \mathfrak{D}(\mathbf{E})$  so that  $\mathbf{A}_1 \nrightarrow_{\mathcal{E}_{(\mathcal{U}, \rho)}} \mathbf{B}_2$ , then the set of states  $\rho$  for which  $\mathbf{A}_1 \nrightarrow_{\mathcal{E}_{(\mathcal{U}, \rho)}} \mathbf{B}_2$  forms a lower-dimensional surface in  $\mathfrak{D}(\mathbf{E})$ , and therefore is of zero Haar measure.*

In other words, influence present on the fundamental level is not necessarily observable as signalling on the operational level—but when that is the case, it can only be the result of *fine-tuning* of the environment state  $\rho$ .

*Proof.* The implication in Eq. (4.8) is a special case of Proposition 3.20; the fact that the converse can fail is witnessed by the channel in Eq. (3.30). For the final statement, consider the set  $S \subseteq \mathfrak{D}(\mathbf{E})$  of states  $\rho$  that satisfy the no-signalling condition  $\mathbf{A}_1 \nrightarrow_{\mathcal{E}_{(\mathcal{U}, \rho)}} \mathbf{B}_2$ . This is a linear constraint on  $\rho$  (see Definition 3.8), so  $S$  forms an affine subspace of  $\mathfrak{D}(\mathbf{E})$ . Now, the assumption that  $\mathbf{A}_1 \rightarrow_{\mathcal{U}} \mathbf{B}_2$  implies that there exists at least one  $\rho \in \mathfrak{D}(\mathbf{E}) \setminus S$ . This is enough to conclude that  $S$  is a strict, and hence lower-dimensional, subspace of  $\mathfrak{D}(\mathbf{E})$ .  $\square$

Let us now return to the context of the spacetime situation  $\mathcal{M}_2$  in Fig. 4.2. Rather than only imposing that a channel  $\mathcal{E}$  cannot be used to signal superluminally, it makes sense to also consider the stronger requirement that it is consistent with fundamental unitary dynamics featuring no superluminal influences. After all, superluminal influences are at odds with the spirit of relativistic causality even if they are hidden from observation by a fine-tuned environment; and by the above, even the slightest perturbations to the environment would almost surely<sup>8</sup> reveal them. We will therefore consider a channel  $\mathcal{E} : \mathbf{A}_1 \mathbf{A}_2 \rightarrow \mathbf{B}_1 \mathbf{B}_2$  to be **influence-consistent with  $\mathcal{M}_2$**  if it admits of a unitary dilation  $(\mathcal{U}, \rho)$  satisfying the no-influence conditions  $\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2$  and  $\mathbf{A}_2 \nrightarrow_{\mathcal{U}} \mathbf{B}_1$ .<sup>9</sup>

<sup>8</sup>i.e. with probability 1.

<sup>9</sup>The term *consistent* has two meanings here: it expresses the requirement that  $\mathcal{E}$  has unitary dilation  $\mathcal{U}$ , i.e. is consistent with having arisen from fundamental dynamics  $\mathcal{U}$  (even though  $\mathcal{E}$  also has many other unitary dilations) and also that  $\mathcal{U}$  is, in turn, consistent with relativistic causality in the context of the spacetime situation  $\mathcal{M}_2$ .

Denote the set of such channels by

$$\text{IC}_{\mathcal{M}_2} := \{\mathcal{E} : \mathbf{A}_1 \mathbf{A}_2 \rightarrow \mathbf{B}_1 \mathbf{B}_2 \mid \mathcal{E} \text{ has a unitary dilation } (\mathcal{U}, \rho) \text{ satisfying} \\ \mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2 \text{ and } \mathbf{A}_2 \nrightarrow_{\mathcal{U}} \mathbf{B}_1\}. \quad (4.9)$$

Contrary to signal consistency, influence consistency turns out to be both necessary and sufficient for realisability in  $\mathcal{M}_2$ .

**Theorem 4.4.** *A channel  $\mathcal{E} : \mathbf{A}_1 \mathbf{A}_2 \rightarrow \mathbf{B}_1 \mathbf{B}_2$  has a unitary dilation  $(\mathcal{U}, \rho)$  satisfying  $\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2$  and  $\mathbf{A}_2 \nrightarrow_{\mathcal{U}} \mathbf{B}_1$  if and only if it has a decomposition into a local operations and shared entanglement. In other words,*

$$\text{Real}_{\mathcal{M}_2} = \text{IC}_{\mathcal{M}_2}. \quad (4.10)$$

*Proof.* Suppose first that  $\mathcal{E}$  admits a decomposition into local operations  $\mathcal{F}, \mathcal{G}$  and shared entanglement  $\sigma$ . Dilate  $\mathcal{F}$  and  $\mathcal{G}$  to unitary channels  $\mathcal{U}_1$  and  $\mathcal{U}_2$ :

$$\begin{array}{c} \mathbf{B}_1 \quad \mathbf{B}_2 \\ \boxed{\mathcal{E}} \\ \mathbf{A}_1 \quad \mathbf{A}_2 \end{array} = \begin{array}{c} \mathbf{B}_1 \quad \mathbf{B}_2 \\ \boxed{\mathcal{F}} \quad \boxed{\mathcal{G}} \\ \mathbf{A}_1 \quad \mathbf{A}_2 \end{array} = \begin{array}{c} \mathbf{B}_1 \quad \mathbf{B}_2 \\ \boxed{\mathcal{U}_1} \quad \boxed{\mathcal{U}_2} \\ \mathbf{A}_1 \quad \mathbf{A}_2 \end{array}. \quad (4.11)$$

The unitary  $\mathcal{U} := \mathcal{U}_1 \otimes \mathcal{U}_2$  then forms a dilation for the channel  $\mathcal{E}$ , whose dilation state is the tensor product of  $\sigma$  and the two dilation states arising from the dilations of  $\mathcal{F}$  and  $\mathcal{G}$ .  $\mathcal{U}$  being a product unitary, it clearly has no influence  $\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2$  and  $\mathbf{A}_2 \nrightarrow_{\mathcal{U}} \mathbf{B}_1$ , and thus demonstrates influence consistency of  $\mathcal{E}$  with  $\mathcal{M}_2$ .

Now suppose, conversely, that  $\mathcal{E}$  is influence-consistent with  $\mathcal{M}_2$ , i.e. has some unitary dilation  $(\mathcal{U} : \mathbf{A}_1 \mathbf{A}_2 \mathbf{E} \rightarrow \mathbf{B}_1 \mathbf{B}_2 \mathbf{F}, \rho)$  satisfying  $\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2$  and  $\mathbf{A}_2 \nrightarrow_{\mathcal{U}} \mathbf{B}_1$ . Tripartite unitaries with precisely this combination of no-influence constraints have been studied in detail in [4, 62]. As we discussed in Section 3.4.2, Ref. [62] showed that each such  $\mathcal{U}$  admits a causal decomposition of shape  $\text{L}_{C_3}$ ; as a result, there exist systems  $C, D, E, F$  and channels  $\mathcal{D}, \mathcal{F}, \mathcal{G}, \mathcal{H}$  so that

$$\begin{array}{c} \mathbf{B}_1 \quad \mathbf{B}_2 \\ \boxed{\mathcal{U}} \\ \mathbf{A}_1 \quad \mathbf{A}_2 \end{array} = \begin{array}{c} \mathbf{B}_1 \quad \mathbf{B}_2 \\ \boxed{\mathcal{H}} \\ \mathbf{E} \quad \mathbf{F} \\ \boxed{\mathcal{F}} \quad \boxed{\mathcal{G}} \\ \mathbf{C} \quad \mathbf{D} \\ \boxed{\mathcal{D}} \\ \mathbf{A}_1 \quad \mathbf{A}_2 \end{array} = \begin{array}{c} \mathbf{B}_1 \quad \mathbf{B}_2 \\ \boxed{\mathcal{F}} \quad \boxed{\mathcal{G}} \\ \mathbf{C} \quad \mathbf{D} \\ \boxed{\mathcal{D}} \\ \mathbf{A}_1 \quad \mathbf{A}_2 \end{array}, \quad (4.12)$$

where we have re-ordered the systems of  $\mathcal{U}$  for clarity and have used, for the last equality, the fact that  $\mathcal{H}$  is trace-preserving. (See [4, §III-D] for an alternative proof.)  $\square$

What does this result mean for channels like  $\mathcal{E}_{\text{PR}}$ , which we saw is signal-consistent with  $\mathcal{M}_2$  yet not realisable? By Theorem 4.4,  $\mathcal{E}_{\text{PR}}$  is also not influence-consistent. Of



Figure 4.3: (a) The spacetime situation  $\mathcal{M}_3$  of Section 4.1.3. The shaded area is the intersection of the future lightcones of  $q_1$  and  $q_2$ . (b) A realisation of a channel in  $\mathcal{M}_3$ . The system  $A$  can never reach  $B_1$  or  $B_2$  via causal trajectories, so can be traced out immediately.

course, it still has unitary dilations, as any channel does. What failure of influence consistency tells us is that no matter what the actual fundamental unitary interaction underlying a particular occurrence of  $\mathcal{E}_{\text{PR}}$  was, it must have involved causal influences that in the context of  $\mathcal{M}_2$  would count as superluminal—even though those influences were not observed on the operational level as signalling through  $\mathcal{E}_{\text{PR}}$  itself. Moreover, by Proposition 4.3, the latter fact must have been a consequence of fine-tuning of the environment state. Had the environment state been even slightly different, a channel would almost surely have occurred that was not signal-consistent with  $\mathcal{M}_2$ —and thus obviously not realisable in it, either. The same can be said for any other channel that, like  $\mathcal{E}_{\text{PR}}$ , is a member of  $\text{SC}_{\mathcal{M}_2} \setminus \text{Real}_{\mathcal{M}_2}$ .<sup>10</sup>

Before continuing, it is worth noting that the distinction between signalling and influence, and the resulting separation between signal consistency and realisability, is by no means unique to quantum theory. In a classical theory described by deterministic functions on the fundamental and stochastic channels on the epistemic level, the presence of influence through a functional dilation of a stochastic channel is perfectly compatible with an absence of signalling through the channel itself (the PR box  $P_{\text{PR}}(ab|xy)$  being an example). The distinction between signalling and influence has also been highlighted in the context of post-quantum correlations, especially in relation to so-called *jamming* scenarios [112–114], which we now turn to.

### 4.1.3 Jamming and causal atomicity

The relevance of unitary dilations also becomes apparent in another simple spacetime situation, which we will briefly discuss here. Consider the situation  $\mathcal{M}_3$  depicted in Fig. 4.3a, involving only a single input  $A_1$  and two outputs  $B_1, B_2$ . Denoting by  $J^+(p)$  the future lightcone of a spacetime point  $p$ , we see that  $q_1, q_2 \notin J^+(p_1)$ . Therefore, signal consistency of a channel  $\mathcal{E} : A_1 \rightarrow B_1 B_2$  with  $\mathcal{M}_3$  should imply that  $A_1 \rightarrow_{\mathcal{E}} B_1$

<sup>10</sup>Note that while all channels in the set  $\text{SC}_{\mathcal{M}_2} \setminus \text{IC}_{\mathcal{M}_2}$  must have arisen from a fine-tuned process, in the sense just described, the set itself is of nontrivial size: relatively in  $\text{SC}_{\mathcal{M}_2}$ , it has a non-empty interior (it includes a ball around  $\mathcal{E}_{\text{PR}}$  that is sufficiently small such that all channels in the ball still can be used to violate the Tsirelson bound). Then again, in light of the present discussions, one may argue that  $\text{SC}_{\mathcal{M}_2}$  is not an appropriate set to measure against.

$$\begin{array}{c} B_1 \quad B_2 \\ | \quad | \\ \boxed{\mathcal{E}} \\ | \\ A_1 \end{array} \quad := \quad \begin{array}{c} B_1 \quad B_2 \\ | \quad | \\ \oplus \\ | \quad | \\ \nabla \Phi^+ \\ | \quad | \\ A_1 \end{array} \quad (4.13)$$
$$\text{SC}_{\mathcal{M}_3} := \{\mathcal{E} : A_1 \rightarrow B_1 B_2 \mid \mathcal{E} \text{ is CPTP and } A_1 \nrightarrow_{\mathcal{E}} B_1 \text{ and } A_1 \nrightarrow_{\mathcal{E}} B_2\}. \quad (4.14)$$
$$\text{Real}_{\mathcal{M}_3} := \left\{ \begin{array}{c|l} \begin{array}{c} B_1 \\ | \\ A_1 \\ | \\ \vdots \end{array} & \begin{array}{c} B_2 \\ | \\ \rho \\ \triangle \end{array} \end{array} \middle| \rho \in \mathfrak{D}(B_1 B_2) \right\}. \quad (4.15)$$
$$\text{IC}_{\mathcal{M}_3} := \{\mathcal{E} : \mathbf{A}_1 \rightarrow \mathbf{B}_1 \mathbf{B}_2 \mid \mathcal{E} \text{ has a unitary dilation } (\mathcal{U}, \rho) \text{ satisfying} \\ \mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_1 \text{ and } \mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2\}. \quad (4.16)$$

Just as the case for signal consistency, we have not a priori enforced the no-influence relation  $\mathbf{A}_1 \not\rightarrow_{\mathcal{U}} \mathbf{B}_1 \mathbf{B}_2$ . However, in the case of unitary transformations like  $\mathcal{U}$ , the

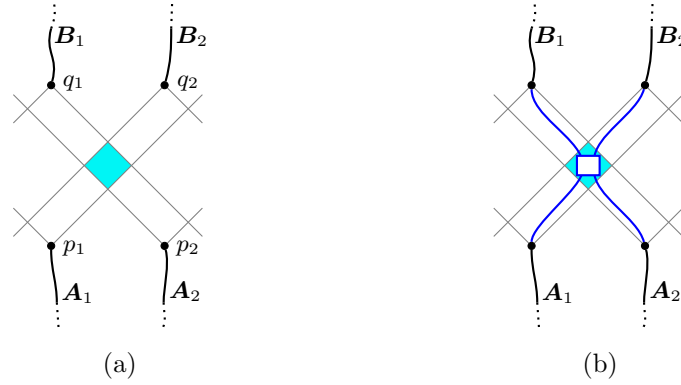


Figure 4.4: (a) The spacetime situation  $\mathcal{M}_4$  from Section 4.1.4. The shaded area is the intersection of the future lightcones of  $p_1$  and  $p_2$  and the past lightcones of  $q_1$  and  $q_2$ . (b) Any bipartite channel can be realised simply at a point within the intersection of the four lightcones.

property of *causal atomicity* we encountered in Section 3.3.1 means that it is implied by the single-system influence relations:

$$\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_1 \mathbf{B}_2 \iff \mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_1 \text{ and } \mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2. \quad (4.17)$$

Causal atomicity thus rules out jamming-like behaviour as in Eq. (4.13) when influence consistency is imposed. More precisely, we have the following.

**Proposition 4.5.**  $\text{Real}_{\mathcal{M}_3} = \text{IC}_{\mathcal{M}_3} \subsetneq \text{SC}_{\mathcal{M}_3}$ .

*Proof.* To show that  $\text{IC}_{\mathcal{M}_3} \subseteq \text{Real}_{\mathcal{M}_3}$ , suppose that  $\mathcal{E}$  has some unitary dilation  $\mathcal{U} : \mathbf{A}_1 \mathbf{E} \rightarrow \mathbf{B}_1 \mathbf{B}_2 \mathbf{F}$  satisfying the no-influence conditions  $\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_1$  and  $\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_2$ . Causal atomicity tells us that in that case,  $\mathbf{A}_1 \nrightarrow_{\mathcal{U}} \mathbf{B}_1 \mathbf{B}_2$  either. As a result of Proposition 4.3,  $\mathcal{E}$  itself then also has no signalling  $\mathbf{A}_1 \nrightarrow_{\mathcal{E}} \mathbf{B}_1 \mathbf{B}_2$ . The fact that  $\mathcal{E}$  has a circuit decomposition as in Eq. (4.15) is nothing more than a restatement of that fact.

For the converse direction  $\text{Real}_{\mathcal{M}_3} \subseteq \text{IC}_{\mathcal{M}_3}$ , note that any channel  $\mathcal{E} : \mathbf{A}_1 \rightarrow \mathbf{B}_1 \mathbf{B}_2$  with a circuit decomposition as in Eq. (4.15) has a unitary dilation  $(\mathcal{U}, \rho)$  with dilating systems  $\mathbf{E} := \mathbf{B}_1 \mathbf{B}_2$  and  $\mathbf{F} := \mathbf{A}_1$ : it is given by the unitary  $\mathcal{U} = \text{id} : \mathbf{A}_1 \mathbf{B}_1 \mathbf{B}_2 \rightarrow \mathbf{A}_1 \mathbf{B}_1 \mathbf{B}_2$  and dilating state  $\rho \in \mathfrak{D}(\mathbf{E}) = \mathfrak{D}(\mathbf{B}_1 \mathbf{B}_2)$ . This unitary clearly satisfies the required no-influence conditions from Eq. (4.16).

The strict inclusion  $\text{IC}_{\mathcal{M}_3} \subsetneq \text{SC}_{\mathcal{M}_3}$  is now a consequence of what we already noted after Eq. (4.15).  $\square$

#### 4.1.4 Nonlocal quantum computation

Before we can move on to the general, multipartite case in Section 4.2, we will need to consider two final bipartite examples  $\mathcal{M}_4$  and  $\mathcal{M}_5$  in Figs. 4.4a and 4.5a. The relativistic structure of each of these situations is deceptively simple: each output  $\mathbf{B}_1, \mathbf{B}_2$  is in the

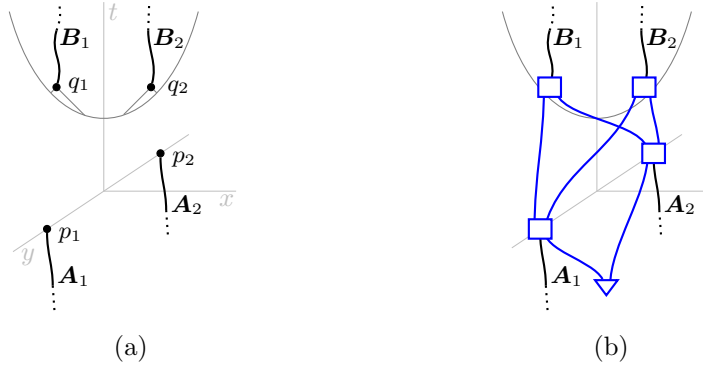


Figure 4.5: (a) The spacetime situation  $\mathcal{M}_5$  from Section 4.1.4. The future lightcones of  $J^+(p_1)$  and  $J^+(p_2)$  are themselves not illustrated, but their intersection forms a region which has an edge in the shape of the hyperbola in the  $(x, t)$ -plane. The points  $q_1$  and  $q_2$  are localised within this intersection, but so close to the hyperbola that their own past lightcones do not intersect within  $J^+(p_1) \cap J^+(p_2)$ . As a result, the intersection of these four lightcones is empty. (b) A realisation of a channel in  $\mathcal{M}_5$  by local operations, shared entanglement, and one round of non-interactive communication.

future lightcone of each input  $\mathbf{A}_1, \mathbf{A}_2$ . Therefore the top-down constraints of signal consistency and influence consistency become trivial and are satisfied by all channels:

$$\text{SC}_{\mathcal{M}_4} = \text{SC}_{\mathcal{M}_5} = \text{IC}_{\mathcal{M}_4} = \text{IC}_{\mathcal{M}_5} = \{\mathcal{E} : \mathbf{A}_1 \mathbf{A}_2 \rightarrow \mathbf{B}_1 \mathbf{B}_2 \mid \mathcal{E} \text{ is CPTP}\}. \quad (4.18)$$

Whether all channels are also realisable however depends on the more fine-grained structure of the spacetime.

In the  $(1+1)$ -dimensional Minkowski spacetime situation  $\mathcal{M}_4$ , any channel  $\mathcal{E}$  can be realised:

$$\text{Real}_{\mathcal{M}_4} = \text{SC}_{\mathcal{M}_4}. \quad (4.19)$$

This can be done simply by sending both inputs  $\mathbf{A}_1$  and  $\mathbf{A}_2$  to a point in the diamond-shaped central region; realising  $\mathcal{E}$  there; and sending the resulting systems  $\mathbf{B}_1$  and  $\mathbf{B}_2$  out to  $q_1$  and  $q_2$ , respectively (see Fig. 4.4b). This protocol relies on the fact that this central region, which is the intersection of the future lightcones of  $p_1$  and  $p_2$  and the past lightcones of  $q_1$  and  $q_2$ , is non-empty.<sup>11</sup>

Consider now the  $(2+1)$ -dimensional Minkowski spacetime  $\mathcal{M}_5$  of Fig. 4.5a. Just like in  $\mathcal{M}_4$ , each of the outputs is localised in the future lightcones of both inputs. However,  $\mathcal{M}_5$  lacks a central region as in  $\mathcal{M}_4$ : that is, the intersection of the future lightcones of  $p_1$  and  $p_2$  and the past lightcones of  $q_1$  and  $q_2$  is empty (see the caption of Fig. 4.5). A protocol as in  $\mathcal{M}_4$ , involving an interaction between all four systems simultaneously, is therefore impossible. Direct, non-interactive communication between

<sup>11</sup>This can be seen as a consequence of the fact that  $q_1$  and  $q_2$  are both in the future of (read: are upper bounds of)  $p_1$  and  $p_2$ , combined with the fact that the metric-induced order on  $(1+1)$ -dimensional Minkowski spacetime is a lattice. Crucially, the latter property is not satisfied by  $(d+1)$ -dimensional Minkowski spacetime when  $d > 1$ .

the inputs and outputs can still take place, however, and also pre-shared entanglement can be used: see Fig. 4.5b. The realisable channels are therefore those with circuit decompositions of the following shape.

$$\text{Real}_{\mathcal{M}_5} := \left\{ \begin{array}{c} \begin{array}{cc} \mathbf{B}_1 & \mathbf{B}_2 \\ \square & \square \\ | & | \\ \square & \square \\ | & | \\ \mathbf{A}_1 & \mathbf{A}_2 \end{array} \end{array} \right\}. \quad (4.20)$$

The question thus arises: can every channel that is influence consistent with  $\mathcal{M}_5$ —that is, just any channel  $\mathcal{E} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$ —be realised by a circuit of this shape? This has been extensively studied in the literature on position-based quantum cryptography and holography [14, 21, 34, 69, 92], where circuit implementations of this shape are referred to as *nonlocal quantum computation*.<sup>12</sup> The answer is subtle, as it depends on whether one is content with *approximate* realisations.

**Proposition 4.6** (Beigi and König [14], Buhrman et al. [21] and Miller and Alnawakhtha [71]). *Any channel  $\mathcal{E} : \mathbf{A}_1\mathbf{A}_2 \rightarrow \mathbf{B}_1\mathbf{B}_2$  can be arbitrarily closely approximated, in the diamond norm, by nonlocal quantum computations (circuits of the form of Eq. (4.20)). However, some channels can only be realised approximately, and not exactly. As a result of this and Eq. (4.18),*

$$\text{Real}_{\mathcal{M}_5} \subsetneq \overline{\text{Real}_{\mathcal{M}_5}} = \text{IC}_{\mathcal{M}_5} = \text{SC}_{\mathcal{M}_5}, \quad (4.21)$$

where  $\overline{\text{Real}_{\mathcal{M}_5}}$  is the closure of  $\text{Real}_{\mathcal{M}_5}$  in the diamond norm.

The diamond norm provides a well-motivated metric of operational distinguishability on the space of quantum channels [2, 47]. The fact that all channels can be approximated by nonlocal quantum computation was first proven in generality by Buhrman et al. [21], expanding on a technique for nonlocal measurements by Vaidman [103] that uses  $N$  rounds of teleportation, where  $N$  grows rapidly with the inverse of the desired error level. Beigi and König [14] subsequently described a protocol based instead on port-based teleportation which generally requires less entanglement for the same error level. Significant work has since been devoted to bounding the amount of required shared entanglement. Some particular classes of channels are known to be *exactly* realisable by nonlocal quantum computation with a finite amount of shared entanglement (see

<sup>12</sup>That term is usually reserved for implementations where the communication between  $\mathbf{A}_1$  and  $\mathbf{B}_2$  and between  $\mathbf{A}_2$  and  $\mathbf{B}_1$ —i.e. that going along the diagonal wires in Eq. (4.20)—is purely classical. However, any protocol as in Eq. (4.20) with quantum communication can be replaced by one with classical communication by increasing the amount of shared entanglement and using teleportation. For the purposes of this work, we put no bounds on the amount of available entanglement, and wherever communication is allowed (i.e. by the spatiotemporal structure) we do not impose that it be classical. Therefore, we will refer also to protocols with full quantum communication as nonlocal quantum computation.

e.g. [48, 92, 121] and Section 4.5), but Miller and Alnawakhtha [71] recently showed this not to be the case for all channels [71].

This last result implies the strict inclusion in Eq. (4.21) and shows that in general spacetime situations  $\mathcal{M}$ , one may only expect influence consistency with  $\mathcal{M}$  to be sufficient for *approximate* realisability in  $\mathcal{M}$ .

## 4.2 Definitions and conjecture in the multipartite case

In the previous section, we have considered, in five simple scenarios, whether demanding that a quantum channel is *signal-consistent*—i.e. that it displays no superluminal signalling relative to a given localisation of its input and output systems—is sufficient for it to be *realisable*. We recalled from previous literature that in general, it is not. We proposed a stronger condition called *influence consistency*, expressing compatibility of the channel with an underlying *unitary* dynamics that itself has no superluminal influences. We saw that influence consistency can be sufficient for realisability in cases where signal consistency is not (in  $\mathcal{M}_2$  and  $\mathcal{M}_3$ ). However, we also saw that not all influence-consistent channels can always be realised *exactly*, and that one needs to be content with arbitrarily close approximations (in  $\mathcal{M}_5$ ). Here we will pose a conjecture that generalises these bipartite cases to configurations of arbitrarily many input and output systems in spacetime. Let us start by generalising some of the notions encountered in the previous section.

### 4.2.1 Preliminary definitions

**Definition 4.7.** A **spacetime situation** is a tuple of the form

$$\mathcal{M} = (M, \preceq_g, p : \mathbf{A} \rightarrow M, q : \mathbf{B} \rightarrow M), \quad (4.22)$$

where  $M$  is a spacetime (time-oriented Lorentzian manifold without closed causal curves),  $\preceq_g$  is the partial order induced on its points by the spacetime metric,  $\mathbf{A}$  and  $\mathbf{B}$  are arbitrary *finite* sets and  $p, q$  are arbitrary functions. We will write  $p_{\mathbf{a}} := p(a)$  and  $q_{\mathbf{b}} := q(\mathbf{b})$  for  $\mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}$ , and refer to these spacetime points as the *input* and *output points*, respectively. For simplicity, we assume that there exists a point  $p \in M$  that lies to the common past of all input and output points. This is to accommodate for shared entanglement in Section 4.3.<sup>13</sup>

In the terminology of Chapter 2,  $\mathcal{M}$  is a particular kind of ‘circuit shape’. As such, it defines a ‘connectivity’ relation  $G_{\mathcal{M}} \subseteq \mathbf{A} \times \mathbf{B}$ , in this case encoding the relativistic causal relations between all input and output points:

$$\forall \mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B} : (\mathbf{a}, \mathbf{b}) \in G_{\mathcal{M}} \iff p_{\mathbf{a}} \preceq_g q_{\mathbf{b}}. \quad (4.23)$$

<sup>13</sup>The notion defined here is closely related to that of a *relativistic quantum task* considered in Refs. [34, 55, 69]. A relativistic quantum task additionally includes the specification of a quantum channel or set of quantum channels that one desires to realise, and the problem is to solve whether a relativistic quantum

The quantum transformations of interest, constrained by the spacetime situation, are quantum channels  $\mathcal{E} : \mathbf{A}_a \rightarrow \mathbf{B}_b$  defined with respect to choices of (finite-dimensional) quantum systems  $\mathbf{A}_a$  for  $a \in \mathbf{A}$  and  $\mathbf{B}_b$  for  $b \in \mathbf{B}$  (recall our notation from Section 3.1).

**Remark 4.8.** In contrast to the special cases discussed in Section 4.1, in the definition of a general spacetime situation above we have only included the specification of quantum system *labels* (elements of  $\mathbf{A}$  and  $\mathbf{B}$ ) and not of quantum systems themselves. We do this to draw a closer parallel to Chapters 2 and 3. As a result, the classes of channels  $\text{SC}_{\mathcal{M}}$ ,  $\text{IC}_{\mathcal{M}}$ , and  $\text{Real}_{\mathcal{M}}$  defined below are proper classes rather than sets. However, to be able to compactly state the implications between signal consistency, influence consistency, and realisability, we will continue to speak in terms of inclusions between these classes. Moreover, we will use the closure notation  $\overline{\text{Real}_{\mathcal{M}}}$  to denote the proper class of channels approximately realisable in  $\mathcal{M}$  in the diamond norm, even though this is strictly speaking no longer a closure of a set in the topological sense. //

Recall from Section 4.1.3 that defining signal-consistency with respect to a spacetime situation generally requires a consideration of signalling relations between *collections* of input and output systems.

**Definition 4.9.** Let  $\mathcal{M}$  be a spacetime situation defined on the sets  $\mathbf{A}$  and  $\mathbf{B}$ , let  $\mathbf{A}_a$  and  $\mathbf{B}_b$  for  $a \in \mathbf{A}, b \in \mathbf{B}$  be quantum systems and let  $\mathcal{E} : \mathbf{A}_a \rightarrow \mathbf{B}_b$  be a quantum channel.  $\mathcal{E}$  is **signal-consistent with  $\mathcal{M}$**  if

$$\forall a \in \mathbf{A}, \beta \subseteq \mathbf{B} : \mathbf{A}_a \rightarrow_{\mathcal{E}} \mathbf{B}_{\beta} \implies \bigcap_{b \in \beta} J^+(q_b) \subseteq J^+(p_a), \quad (4.24)$$

where  $J^+(p)$  is the (closed) future lightcone of  $p \in M$ . Denote the class of all signal-consistent channels, for any choice of quantum systems  $\mathbf{A}_a$  and  $\mathbf{B}_b$  for  $a \in \mathbf{A}$  and  $b \in \mathbf{B}$ , by  $\text{SC}_{\mathcal{M}}$ .

This is operationally motivated as follows: if  $\mathbf{A}_a \rightarrow_{\mathcal{E}} \mathbf{B}_{\beta}$ , then an agent located at or before  $p_a$  and with access to input system  $\mathbf{A}_a$  can send a signal that can generally only be decyphered by an agent with access to all of the systems  $\mathbf{B}_b$  for  $b \in \beta$ . The earliest those systems can be brought together is in the region  $\bigcap_{b \in \beta} J^+(p_b)$ , so a necessary requirement for subluminality of signals is that that region lies within the future lightcone of  $p_a$ . If a signal could in fact already be detected with access to fewer systems, that is, if  $\mathbf{A}_a \rightarrow_{\mathcal{E}} \mathbf{B}_{\beta'}$  for some minimal  $\beta' \subsetneq \beta$ , then the required subluminality is guaranteed by Eq. (4.24) applied to  $\beta'$ .

One might wonder whether a stronger constraint could be obtained by also considering composite *input* systems  $\mathbf{A}_{\alpha}$  ( $\alpha \subseteq \mathbf{A}$ ). However, an inspection of the proof of Proposition 3.11 reveals that generic channels satisfy a weak version of atomicity, which has been called *domain atomicity* [117]: for any channel  $\mathcal{E} : \mathbf{A}_a \rightarrow \mathbf{B}_b$  and subsets

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task is possible or not. Here, we instead aim to characterise the entire class of channels that can be approximately realised in a given spacetime situation.

$\alpha \subseteq \mathbf{A}, \beta \subseteq \mathbf{B}$ , there is signalling  $\mathbf{A}_\alpha \rightarrow_{\mathcal{E}} \mathbf{B}_\beta$  if and only if  $\mathbf{A}_a \rightarrow_{\mathcal{E}} \mathbf{B}_\beta$  for at least one  $a \in \alpha$ . The condition in Eq. (4.24) does therefore not need to be strengthened.

We move on to influence consistency. The fact that causal influences through unitary channels are atomic in their codomain as well as their domain allows the definition to be further simplified.

**Definition 4.10** (Influence consistency).  $\mathcal{E} : \mathbf{A}_\mathbf{A} \rightarrow \mathbf{B}_\mathbf{B}$  is **influence-consistent** with  $\mathcal{M}$  if it admits of a unitary dilation—a unitary channel  $\mathcal{U} : \mathbf{A}_\mathbf{A} \mathbf{E} \rightarrow \mathbf{B}_\mathbf{B} \mathbf{F}$  and state  $\rho \in \mathfrak{D}(\mathbf{E})$  so that  $\mathcal{E} = \text{Tr}_{\mathbf{F}} \mathcal{U}(- \otimes \rho)$ —satisfying

$$\forall a \in \mathbf{A}, \beta \subseteq \mathbf{B} : \mathbf{A}_\alpha \rightarrow_{\mathcal{U}} \mathbf{B}_\beta \implies \bigcap_{b \in \beta} J^+(q_b) \subseteq J^+(p_a). \quad (4.25)$$

By causal atomicity (in the domain as well as the codomain; see Proposition 3.11), this is equivalent to the condition that  $\forall a \in \mathbf{A}, b \in \mathbf{B} : \mathbf{A}_a \rightarrow_{\mathcal{U}} \mathbf{B}_b \implies p_a \preceq_g q_b$ , or, stated even more simply,

$$G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B}) \subseteq G_{\mathcal{M}}, \quad (4.26)$$

where  $G_{\mathcal{U}}$  is the causal structure of  $\mathcal{U}$  from Definition 3.12.

Denote by  $\text{IC}_{\mathcal{M}}$  the class of all quantum channels influence-consistent with  $\mathcal{M}$ . Let  $\overline{\text{IC}_{\mathcal{M}}}$  moreover denote the class of quantum channels obtained as limits  $\lim_{n \rightarrow \infty} \mathcal{E}_n : \mathbf{A}_\mathbf{A} \rightarrow \mathbf{B}_\mathbf{B}$  of diamond-norm-converging sequences of channels  $\{\mathcal{E}_n : \mathbf{A}_\mathbf{A} \rightarrow \mathbf{B}_\mathbf{B}\}_{n=1}^\infty \subseteq \text{IC}_{\mathcal{M}}$ .

**Definition 4.11.** Recall from Definition 3.1 that a *quantum circuit*  $\mathbf{P}$  is a *finite* circuit shape  $\mathbf{P}$  equipped with a quantum semantics. We will say that  $\mathbf{P}$  is **implementable** in a spacetime situation  $\mathcal{M}$  if there exists a circuit shape morphism  $f : \mathbf{P} \rightarrow \mathcal{M}$ , and that a quantum channel  $\mathcal{E} : \mathbf{A}_\mathbf{A} \rightarrow \mathbf{B}_\mathbf{B}$  is **realisable** in  $\mathcal{M}$  if it admits of a circuit decomposition  $\mathbf{P}$  that is implementable in  $\mathcal{M}$ . Denote by  $\text{Real}_{\mathcal{M}}$  the class of channels realisable in  $\mathcal{M}$ . Moreover,  $\mathcal{E}$  is **approximately realisable** in  $\mathcal{M}$  if  $\mathcal{E}$  is a limit  $\mathcal{E} = \lim_{n \rightarrow \infty} \mathcal{E}_n$  of a diamond-norm-converging sequence of realisable channels  $\{\mathcal{E}_n : \mathbf{A}_\mathbf{A} \rightarrow \mathbf{B}_\mathbf{B}\}_{n=1}^\infty \subseteq \text{Real}_{\mathcal{M}}$ . The class of approximately realisable channels is denoted by  $\overline{\text{Real}_{\mathcal{M}}}$ .

Recall from Definition 2.4 that a circuit shape morphism  $f : \mathbf{P} \rightarrow \mathcal{M}$  is a function mapping gates (elements of  $\mathbf{P}$ ) to spacetime points (elements of the spacetime  $\mathcal{M}$  underlying  $\mathcal{M}$ ) such that

- $f$  is order-preserving: for  $r, s \in \mathbf{P}$ ,  $r \leq_{\mathbf{P}} s \implies f(r) \preceq_g f(s)$ ;
- for all  $a \in \mathbf{A}$ ,  $f(\lambda_{\mathbf{P}}(a)) \preceq_g p_a$ ;
- for all  $b \in \mathbf{B}$ ,  $f(\mu_{\mathbf{P}}(b)) \preceq_g q_b$ .

These three conditions respectively guarantee the existence of causal curves for all internal wires of the circuit, all input wires of the circuit, and all output wires of the

circuit. The existence of such a morphism therefore indeed captures the right notion of implementability of a circuit in the spacetime situation  $\mathcal{M}$ .

Finally, note that by virtue of Proposition 2.22, the connectivity relation  $G_{\mathbf{P}}$  of an implementable quantum circuit always satisfies

$$G_{\mathbf{P}} \subseteq G_{\mathcal{M}}. \quad (4.27)$$

### 4.2.2 Main conjecture

Having now defined notions of signal consistency, influence consistency, and realisability for general spacetime situations, what can we say about their relationship? The straightforward directions are the following.

**Proposition 4.12.** *Let  $\mathcal{M}$  be any spacetime situation. We have*

$$\text{Real}_{\mathcal{M}} \subseteq \text{IC}_{\mathcal{M}} \subseteq \text{SC}_{\mathcal{M}}. \quad (4.28)$$

*Proof.* Suppose that  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  is a quantum channel realisable in  $\mathcal{M}$ , and let  $\mathbf{P}$  be the circuit decomposition of  $\mathcal{E}$  implementable in  $\mathcal{M}$ . By Proposition 3.21,  $\mathcal{E}$  has a unitary dilation  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \mathbf{E} \rightarrow \mathbf{B}_{\mathbf{B}} \mathbf{F}$  satisfying  $G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B}) \subseteq G_{\mathbf{P}}$ . Since implementability implies that  $G_{\mathbf{P}} \subseteq G_{\mathcal{M}}$ , this means that  $G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B}) \subseteq G_{\mathcal{M}}$  so that  $\mathcal{E}$  is influence-consistent with  $\mathcal{M}$  by Eq. (4.26). This proves the first inclusion. The second inclusion follows from Proposition 3.20.  $\square$

More interesting are the opposite inclusions. Let us first recall what the examples in Section 4.1 had to say about this. First of all, the cases of  $\mathcal{M}_1$  and  $\mathcal{M}_4$  witnessed equality of all three sets in Eq. (4.28): in these cases, both influence consistency and signal consistency were necessary and sufficient for realisability. In  $\mathcal{M}_2$  and  $\mathcal{M}_3$ , on the other hand, we found channels—exemplified by the PR box and the channel in Eq. (4.13)—that were signal-consistent but not influence-consistent: as a result the second inclusion in Eq. (4.28) was strict, and signal consistency was insufficient for realisability. In those same cases we however saw that *influence* consistency *was* sufficient: the first inclusion in Eq. (4.28) was saturated (Theorem 4.4 and Proposition 4.5). Finally, in  $\mathcal{M}_5$  even influence-consistency was not sufficient for realisability itself, but only for *approximate* realisability (Proposition 4.6). This leads us to pose the following conjecture for the general case.

**Conjecture 4.13.** *Let  $\mathcal{M}$  be a spacetime situation. We have*

$$\text{IC}_{\mathcal{M}} \subseteq \overline{\text{Real}_{\mathcal{M}}} : \quad (4.29)$$

*that is, any channel that is influence-consistent with  $\mathcal{M}$  is approximately realisable in  $\mathcal{M}$ .*

This is equivalent to  $\overline{\text{IC}_{\mathcal{M}}} \subseteq \overline{\text{Real}_{\mathcal{M}}}$  and therefore, by Proposition 4.12, to

$$\overline{\text{Real}_{\mathcal{M}}} = \overline{\text{IC}_{\mathcal{M}}}; \quad (4.30)$$

that is, ‘up to arbitrarily small errors’, influence consistency is both a necessary and a sufficient criterion for a channel to be realisable in a spacetime situation.

**Remark 4.14.** It is worth noting that the class  $\text{SC}_{\mathcal{M}}$  is always closed in the diamond norm, since the absence of signalling between subsystems is a linear equality constraint on channels.  $\text{Real}_{\mathcal{M}}$ , on the other hand, is not always closed: counterexamples are  $\mathcal{M}_5$ , as a consequence of [71] (see Section 4.1.4), and  $\mathcal{M}_2$ , as a consequence of results from [60, 90]. Since Theorem 4.4 showed that  $\text{Real}_{\mathcal{M}_2} = \text{IC}_{\mathcal{M}_2}$ , we see that the set  $\text{IC}_{\mathcal{M}}$  is generally not closed either. //

### 4.3 Characterising approximate realisability

As in the bipartite cases we investigated in Section 4.1, proving realisability of a channel in a given spacetime situation  $\mathcal{M}$  can always be reduced to showing the existence of an appropriate circuit decomposition whose shape is constrained by  $\mathcal{M}$ . One way in which a spacetime situation constrains a circuit is via its connectivity, which necessarily satisfies  $G_{\mathbf{P}} \subseteq G_{\mathcal{M}}$ . If  $\mathcal{M}$  is a lattice (Definition 2.27), as  $(1+1)$ -d Minkowski spacetime for instance is, then this is in fact the only constraint.

**Proposition 4.15.** *If the partial order underlying spacetime situation  $\mathcal{M}$  is a lattice, then a quantum circuit  $\mathbf{P}$  is implementable in  $\mathcal{M}$  if and only if  $G_{\mathbf{P}} \subseteq G_{\mathcal{M}}$ .*

*Proof.* The ‘only if’ direction was already noted in Eq. (4.27). For the ‘if’ direction, let  $\mathbf{P}$  be the circuit shape underlying  $\mathbf{P}$ ; by definition of a quantum circuit,  $\mathbf{P}$  must be finite. We need to show that there exists a circuit shape morphism  $f : \mathbf{P} \rightarrow \mathcal{M}$ . This follows by a proof similar to that of Lemma 2.34. For that result, we assumed that the codomain of  $f$  is a *complete* lattice (see Definition 2.27). However, since  $\mathbf{P}$  is finite, the joins and meets in the proof of Lemma 2.34 are all finite, too, so assuming that  $\mathcal{M}$  is a (potentially non-complete) lattice suffices.  $\square$

More general spacetimes however, like the  $(2+1)$ -d case in Fig. 4.5, also constrain the presence of *interaction gates* in implementable circuits, which turned out so crucial for the study of causal decompositions in Chapters 2 and 3. The circuit shape that incorporates both connectivity and interaction constraints in the general case is the following.

**Definition 4.16.** Let  $G \subseteq \mathbf{A} \times \mathbf{B}$ . The circuit shape consisting of **local operations, shared entanglement and one round of non-interactive communication along  $G$** , abbreviated by  $\text{LOSEC}_G$ , is the tuple

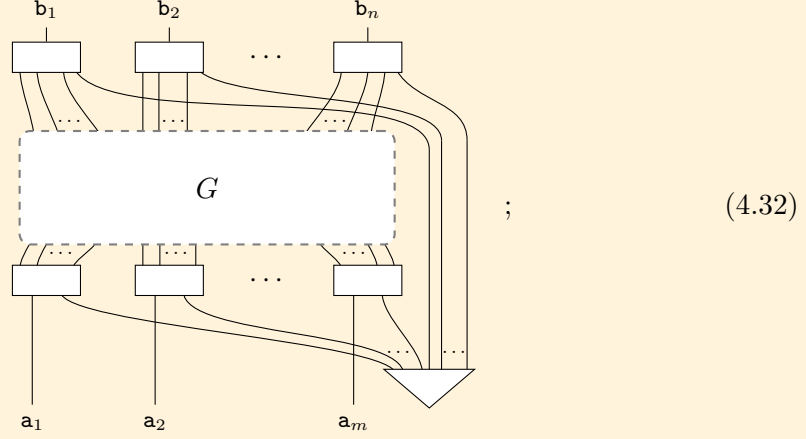
$$\text{LOSEC}_G := (\mathbf{A} \sqcup \mathbf{B} \sqcup \{r\}, \leq, \lambda, \mu) \quad (4.31)$$

where  $\leq$  is the minimal order on  $\mathbf{A} \sqcup \mathbf{B} \sqcup \{r\}$  satisfying

- $r < \mathbf{a}$  and  $r < \mathbf{b}$  for all  $\mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}$

- $\mathbf{a} < \mathbf{b}$  for  $\mathbf{a} \in \mathbf{A}, \mathbf{b} \in \mathbf{B}$  satisfying  $\mathbf{a}G\mathbf{b}$ .

Writing  $\mathbf{A} = \{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m\}$  and  $\mathbf{B} = \{\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n\}$ , it has circuit diagram



the obscured part labelled by  $G$  contains a wire from  $\mathbf{a}$  to  $\mathbf{b}$  iff  $\mathbf{a}G\mathbf{b}$ , and no gates.

Note that the circuits in Figs. 4.1b, 4.2b, 4.3b and 4.5b in Section 4.1 are all of the  $\text{LOSEC}_{G_{\mathcal{M}_i}}$  form for  $i = 1, 2, 3, 5$ , respectively, although in the case of  $\mathcal{M}_1$  in Fig. 4.1b no shared entanglement is in fact needed.

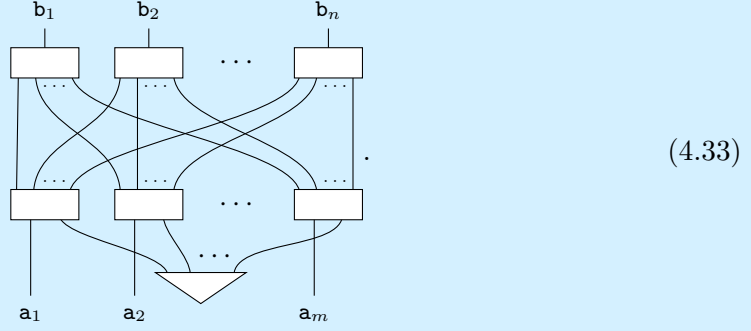
Similarly to Corollary 2.43 about the similar circuit shape  $\mathbf{B}_G$ , a  $\text{LOSEC}_G$  circuit can be implemented in a spacetime situation  $\mathcal{M}$  precisely if  $G \subseteq G_{\mathcal{M}}$  (whether or not  $\mathcal{M}$  is a lattice). In particular, the gate  $r$  should be localised at a point in the past of all its input and output points  $p_{\mathbf{a}}$  and  $q_{\mathbf{b}}$ , whose existence we have assumed in Definition 4.7. For a channel  $\mathcal{E}$ , having approximate  $\text{LOSEC}_{G_{\mathcal{M}}}$  decompositions is thus a sufficient condition on a channel for being approximately realisable in  $\mathcal{M}$ . By the following result, it is also necessary. This relies heavily on the shared entanglement provided by  $r$ . The result was first shown by Dolev [34], whose proof described in detail the particular teleportation procedures required. Here we give an alternative, diagrammatic proof.

**Theorem 4.17.** *If a quantum channel  $\mathcal{E}$  has a circuit decomposition  $\mathbf{P}$  with connectivity  $G_{\mathbf{P}}$ , then it is approximated by  $\text{LOSEC}_{G_{\mathbf{P}}}$  circuits. As a result, the following are equivalent for any quantum channel  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  and spacetime situation  $\mathcal{M}$  defined on  $\mathbf{A}$  and  $\mathbf{B}$ .*

- (i)  $\mathcal{E}$  is approximately realisable in  $\mathcal{M}$ .
- (ii)  $\mathcal{E}$  is approximated by circuits  $\mathbf{P}$  with connectivity  $G_{\mathbf{P}} \subseteq G_{\mathcal{M}}$ .
- (iii)  $\mathcal{E}$  is approximated by circuits with shape  $\text{LOSEC}_{G_{\mathcal{M}}}$ .

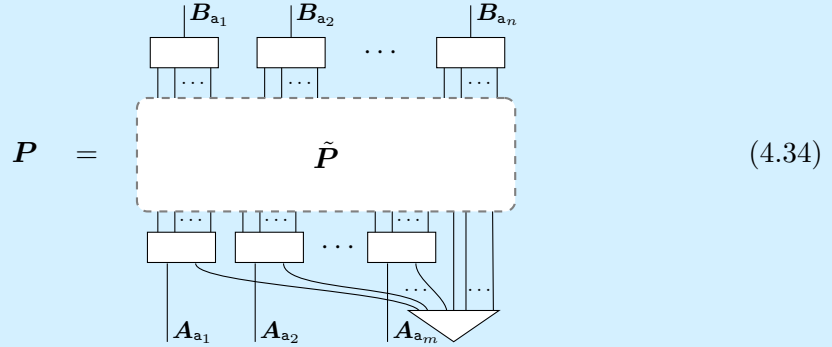
**Lemma 4.18** (Multipartite nonlocal quantum computation). *Any quantum channel  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  can be arbitrarily closely approximated, in the diamond norm, by circuits of*

the shape  $\text{LOSEC}_{A \times B}$ , i.e. the circuit shape with diagram



*Proof.* A variant of this result is shown explicitly in Buhrman et al. [21, Appendix B] by a back-and-forth teleportation strategy; an alternative protocol using port-based teleportation can be obtained by generalising the bipartite case described by Beigi and König [14].  $\square$

**Lemma 4.19.** Suppose that  $\mathcal{E} : A_A \rightarrow B_B$  has a circuit decomposition  $P$  which is of the form



where  $\tilde{P}$  is a quantum circuit with  $n > 0$  gates. Then for any  $\varepsilon > 0$ ,  $\mathcal{E}$  is  $\varepsilon$ -approximated, in the diamond norm, by a quantum circuit  $P'$  which is also of the form of Eq. (4.34), but where  $\tilde{P}$  has  $n - 1$  gates. Moreover, the connectivity relation remains unchanged:  $G_{P'} = G_P$ .

*Proof.* Assume that  $\mathcal{E}$  has a decomposition  $P$  of the form in Eq. (4.34); let  $\varepsilon > 0$ ; and consider an arbitrary gate  $\mathcal{D}$  of  $\tilde{P}$ . By Lemma 4.18,  $\mathcal{D}$  is  $\varepsilon$ -approximated by a circuit of the form in Eq. (4.33). Applying this approximation within the context of the circuit

$\mathbf{P}$ , we get

The approximation above still holds to an error level not greater than  $\varepsilon$ , as pre- and postcomposition with CPTP maps never increases the diamond distance [47]. In the final step, we have merely re-organised the position of the gates on the page. A new circuit of the form of Eq. (4.34) can now be constructed by considering each of the circuit fragments encapsulated in a dashed box in Eq. (4.35) as single gates, and incorporating the shared entangled state at the bottom of the diagram into the shared entanglement resource of Eq. (4.34). By construction, the resulting circuit  $\mathbf{P}'$  has one fewer internal gate but unchanged connectivity  $G_{\mathbf{P}'} = G_{\mathbf{P}}$ .  $\square$

*Proof of Theorem 4.17.* Let  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  be a channel with circuit decomposition  $\mathbf{P}$ . Our aim is to show that  $\mathcal{E}$  is approximated by  $\text{LOSEC}_{G_{\mathbf{P}}}$  circuits. First, we bring  $\mathbf{P}$  into the form of Eq. (4.34). To that end, add onto each individual overall input and output wire  $\mathbf{A}_{\mathbf{a}}$  or  $\mathbf{B}_{\mathbf{b}}$  of  $\mathbf{P}$  a gate corresponding to the identity transformation on that system. This yields a circuit

which is of the form of Eq. (4.34) with  $\tilde{\mathbf{P}} = \mathbf{P}$ . Clearly,  $\mathbf{P}'$  still implements  $\mathcal{E}$  and has connectivity relation  $G_{\mathbf{P}'} = G_{\mathbf{P}}$ .

Let  $\varepsilon > 0$ . If  $\mathbf{P}$  had  $n$  gates, apply Lemma 4.19  $n$  times, each time constructing an approximation to within  $\varepsilon/n$ . The resulting circuit is an approximation of  $\mathcal{E}$  to within  $\varepsilon$ . It has connectivity  $G_{\mathbf{P}}$  and is of the form of Eq. (4.34) for a circuit  $\tilde{\mathbf{P}}$  that has zero gates; in other words, it is a  $\text{LOSEC}_{G_{\mathbf{P}}}$  circuit. This completes the proof of the first statement of Theorem 4.17.

The equivalence of conditions (i)–(iii) is now immediate: (i)  $\Rightarrow$  (ii) follows directly from Eq. (4.27), (ii)  $\Rightarrow$  (iii) from the first claim of Theorem 4.17 just proven, and (iii)  $\Rightarrow$  (i) from the discussion preceding the Theorem.  $\square$

**Example 4.20.** Any circuit with shape  $L_{C_3}$  can be approximately rewritten into the  $\text{LOSEC}_{C_3}$  form:

The  $\text{LOSEC}_{C_3}$  circuit on the right can be implemented in any spacetime situation  $\mathcal{M}$  with  $C_3 \subseteq G_{\mathcal{M}}$  while the  $L_{C_3}$  circuit on the left cannot necessarily: no appropriate localisations exist for its leftmost and rightmost gates in spacetimes that, like  $\mathcal{M}_2$  in Fig. 4.5, lack the required four-way intersections of lightcones. //

## 4.4 Equivalence to causal decompositions

Conjecture 4.13 posed that influence consistency with a spacetime situation  $\mathcal{M}$  is a sufficient criterion for approximate realisability in  $\mathcal{M}$ . By Theorem 4.17 above, deciding the latter boils down to deciding the existence of approximate circuit decompositions with appropriate connectivity. Since, on the other hand, influence consistency is a property of a quantum channel expressed in terms of influence relations through *unitary* dilations, this means that Conjecture 4.13 is closely related to the existence of *causal decompositions* of unitary transformations that we studied in Chapter 3. More precisely, consider the following weakening of the notion from Definition 3.15.

**Definition 4.21.** Say that a relation  $G \subseteq A \times B$  **implies approximate causal decompositions** if for each choice of quantum systems  $A_a$  for  $a \in A$  and  $B_b$  for  $b \in B$ , every unitary channel  $\mathcal{U} : A_A \rightarrow B_B$  with causal structure  $G_{\mathcal{U}} \subseteq G$  is the limit, in the diamond norm, of quantum circuits  $P$  satisfying  $G_P \subseteq G$ . Note that the gates in these circuits need not be unitary themselves: that is not a relevant requirement in the context of the spacetime realisability question of Conjecture 4.13.

Of course, if a relation implies (exact) causal decompositions in the sense of Definition 3.15 then it also implies approximate causal decompositions. This allows us to carry over results from the study of causal decompositions to solve special cases of Conjecture 4.13. In particular, Theorem 4.17 leads to the following.

**Definition 4.22.** For any binary relation  $G \subseteq A \times B$ , let the **augmented** relation  $G^\# \subseteq (A \cup \{e\}) \times (B \cup \{f\})$  be

$$G^\# := G \cup (\{e\} \times B) \cup (A \times \{f\}) \cup \{(e, f)\} \quad (4.38)$$

for some  $\mathbf{e} \notin \mathbf{A}$  and  $\mathbf{f} \notin \mathbf{B}$ . That is, it is the maximal relation on  $(\mathbf{A} \cup \{\mathbf{e}\}) \times (\mathbf{B} \cup \{\mathbf{f}\})$  so that  $G^\sharp \cap (\mathbf{A} \times \mathbf{B}) = G$ .

**Corollary 4.23.** *Let  $\mathcal{M}$  be a spacetime situation. If  $G^\sharp_{\mathcal{M}}$  implies approximate causal decompositions, then  $\text{IC}_{\mathcal{M}} \subseteq \overline{\text{Real}_{\mathcal{M}}}$ .*

*Proof.* Suppose that  $\mathcal{E} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  is influence-consistent with  $\mathcal{M}$ , meaning it has a unitary dilation  $\mathcal{U} : \mathbf{A}_{\mathbf{A}}\mathbf{E} \rightarrow \mathbf{B}_{\mathbf{B}}\mathbf{F}$  with dilation state  $\rho \in \mathfrak{D}(\mathbf{E})$  satisfying  $G_{\mathcal{U}} \cap (\mathbf{A} \times \mathbf{B}) \subseteq G_{\mathcal{M}}$  (see Eq. (4.26)). If we associate the quantum systems  $\mathbf{E}$  and  $\mathbf{F}$  with the labels  $\mathbf{e}$  and  $\mathbf{f}$ , respectively, this last statement says precisely that  $G_{\mathcal{U}} \subseteq G^\sharp_{\mathcal{M}}$ . Since by assumption  $G^\sharp_{\mathcal{M}}$  implies approximate causal decompositions, this means  $\mathcal{U}$  is the limit  $\lim_n \mathcal{E}_{P_n^\sharp}$  of quantum channels  $\mathcal{E}_{P_n^\sharp} : \mathbf{A}_{\mathbf{A}}\mathbf{E} \rightarrow \mathbf{B}_{\mathbf{B}}\mathbf{F}$  implemented by circuits  $P_n^\sharp$  that satisfy  $G_{P_n^\sharp} \subseteq G$ . For each  $n$ , preparing the state  $\rho$  on the input wire  $\mathbf{E}$  of  $P_n^\sharp$  and tracing out  $\mathbf{F}$  yields a circuit  $P_n$  that instead implements a channel  $\mathcal{E}_{P_n} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$ . The resulting circuits  $P_n$  satisfy  $G_{P_n} \subseteq G_{P_n^\sharp} \cap (\mathbf{A} \times \mathbf{B}) \subseteq G_{\mathcal{M}}$ . Moreover, since preparing  $\rho$  and tracing  $\mathbf{F}$  cannot increase the diamond distance, we have  $\mathcal{E} = \lim_n \mathcal{E}_{P_n}$ . The channel  $\mathcal{E}$  thus satisfies (ii) of Theorem 4.17, meaning it is also approximately realisable in  $\mathcal{M}$ .  $\square$

One way to arrive at a class of special cases of Conjecture 4.13 is thus to use our main result of Chapter 3, which entails that any relation  $G$  satisfying the  $C_3$  exclusion property ( $C_3$ -EP) implies (unitary and exact) causal decompositions. Unfortunately, relations that are of the augmented form and also satisfy the  $C_3$  exclusion property from Chapter 3 are somewhat limited. This becomes especially apparent when formulating the condition in terms of the concept lattice  $L_{G^\sharp}$ .

**Proposition 4.24.** *Let  $\mathbf{A}, \mathbf{B}$  be finite sets and  $G \subseteq \mathbf{A} \times \mathbf{B}$ . The following are equivalent.*

- (i)  $G^\sharp$  satisfies the  $C_3$  exclusion property ( $C_3$ -EP).
- (ii) For all  $\mathbf{a}_1, \mathbf{a}_2 \in \mathbf{A}$  and  $\mathbf{b}_1, \mathbf{b}_2 \in \mathbf{B}$ , we have

$$G \cap (\{\mathbf{a}_1, \mathbf{a}_2\} \times \{\mathbf{b}_1, \mathbf{b}_2\}) \neq \begin{array}{cc} \mathbf{b}_1 & \mathbf{b}_2 \\ \uparrow & \uparrow \\ \mathbf{a}_1 & \mathbf{a}_2 \end{array}. \quad (4.39)$$

- (iii) The lattice  $L_{G^\sharp}$  is a total order.

*Proof.* If  $\neg(\text{i})$ , then there are  $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbf{A} \cup \{\mathbf{e}\}$  and  $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathbf{B} \cup \{\mathbf{f}\}$  so that  $G^\sharp \cap (\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3\} \times \{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}) = C_3$ . By construction of  $G^\sharp$ , we must have  $\mathbf{e} \notin \{\mathbf{a}_1, \mathbf{a}_3\}$  and  $\mathbf{f} \notin \{\mathbf{b}_1, \mathbf{b}_3\}$ . Therefore  $G \cap (\{\mathbf{a}_1, \mathbf{a}_3\} \times \{\mathbf{b}_1, \mathbf{b}_3\}) = G^\sharp \cap (\{\mathbf{a}_1, \mathbf{a}_3\} \times \{\mathbf{b}_1, \mathbf{b}_3\}) = C_3 \cap (\{\mathbf{a}_1, \mathbf{a}_3\} \times \{\mathbf{b}_1, \mathbf{b}_3\})$ , which means that  $\neg(\text{ii})$ . For the opposite direction  $\neg(\text{ii}) \Rightarrow \neg(\text{i})$ , suppose that we have equality in Eq. (4.39); then  $G^\sharp \cap (\{\mathbf{a}_1, \mathbf{e}, \mathbf{a}_2\} \times \{\mathbf{b}_1, \mathbf{f}, \mathbf{b}_2\})$  is of the form of  $C_3$ , so  $G^\sharp$  violates ( $C_3$ -EP).

Now let us prove that (i)  $\Leftrightarrow$  (iii). By Theorem 3.24, (i) is equivalent to the absence in  $L_{G^\#}$  of any non-unique paths between inputs  $\mathbf{a} \in \mathbf{A}$  and outputs  $\mathbf{b} \in \mathbf{B}$ . Note that by construction of  $G^\#$ ,  $\mathbf{e}$  is an input to the minimum element of  $L_{G^\#}$ , while  $\mathbf{f}$  is an output of the maximum element. The absence of non-unique paths between any input-output pair in  $L_{G^\#}$  is therefore equivalent to the absence of non-unique paths between  $\mathbf{e}$  and  $\mathbf{f}$  in particular. This is in turn equivalent to  $L_{G^\#}$  being a total order, concluding the proof.  $\square$

Together with the causal decompositions shown by Lorenz and Barrett [62] (see Theorem 3.18), this allows us to establish a number of classes of special cases of Conjecture 4.13, going beyond the cases  $\mathcal{M}_1$ – $\mathcal{M}_5$  discussed in Section 4.1.

**Corollary 4.25.** *The inclusion  $\text{IC}_{\mathcal{M}} \subseteq \overline{\text{Real}_{\mathcal{M}}}$  is satisfied for any spacetime situation  $\mathcal{M}$  defined on sets of system labels  $\mathbf{A}$  and  $\mathbf{B}$  that satisfies any of the following conditions.*

- (a)  $|\mathbf{A}| \leq 2$ ;
- (b)  $|\mathbf{B}| \leq 2$ ;
- (c)  $|\mathbf{A}| = 3$  and there is  $\mathbf{a} \in \mathbf{A}$  so that  $G_{\mathcal{M}}(\mathbf{a}) = \mathbf{B}$ ;
- (d)  $|\mathbf{B}| = 3$  and there is  $\mathbf{b} \in \mathbf{B}$  so that  $G_{\mathcal{M}}^{-1}(\mathbf{b}) = \mathbf{A}$ ;
- (e)  $G_{\mathcal{M}}$  satisfies any of the conditions in Proposition 4.24.

*Proof.* (a) and (b) follow directly from Theorem 3.18 and Corollary 4.23. For (c), suppose that  $|\mathbf{A}| = 3$  and  $G_{\mathcal{M}}(\mathbf{a}) = \mathbf{B}$ . The augmented relation  $G_{\mathcal{M}}^\#$  is defined on  $|\mathbf{A}| + 1$  input labels  $\mathbf{A} \cup \{\mathbf{e}\}$ , but  $\mathbf{a}$  and  $\mathbf{e}$  are  $G_{\mathcal{M}}^\#$ -related to the same outputs. For the purposes of causal decompositions they can therefore be treated as a single system, bringing us back to a relation defined on  $|\mathbf{A}| \leq 3$  inputs to which we can apply Theorem 3.18. (d) follows similarly and (e) is immediate.  $\square$

Now, how might one hope to prove more general cases of Conjecture 4.13? One natural line of attack is to focus on the problem of causal decompositions itself and prove cases of Conjecture 3.19 beyond those solved by [62] and Chapter 3. However, there are two reasons why proving Conjecture 3.19 in its full generality might *prima facie* not be necessary in case one is ultimately only interested in applications to Conjecture 4.13. First, Conjecture 4.13 only asks for approximate decompositions, whereas Conjecture 3.19 posed the existence of exact ones. It is worth noting that the only reason for the relevance of approximate decompositions in the spacetime context is because of the need for decompositions of the  $\text{LOSEC}_G$  form; the case for the existence of exact causal decompositions of shape  $L_G$  is not in any way weakened by the considerations in this chapter. Nevertheless, for completeness let's weaken Conjecture 3.19:

**Conjecture 4.26.** *Every binary relation  $G \subseteq \mathbf{A} \times \mathbf{B}$  implies approximate causal decompositions. Equivalently, every unitary channel  $\mathcal{U}$  is approximated by quantum circuits*

$\mathbf{P}$  with connectivity  $G_{\mathbf{P}} = G_{\mathcal{U}}$ .

The second reason why Conjecture 3.19, and, for that matter, Conjecture 4.26, might seem too strong is that Conjecture 4.13 only requires us to prove the existence of approximate causal decompositions for relations  $G$  of the form  $G_{\mathcal{M}}^{\sharp}$  for some spacetime situation  $\mathcal{M}$ . This however turns out not to be a restriction if we allow our spacetimes to have arbitrary dimension, as we did in Conjecture 4.13.

**Lemma 4.27.** *Let  $G \subseteq \mathbf{A} \times \mathbf{B}$  be an arbitrary relation. There exists a spacetime situation  $\mathcal{M} = (\mathbb{R}^{d+1}, \preceq_{\eta}, p, q)$  on  $(d+1)$ -dimensional Minkowski spacetime  $(\mathbb{R}^{d+1}, \eta)$ , where  $d := \min(\{m-1, n-1\})$ , so that  $G = G_{\mathcal{M}}$ .*

*Proof.* This is a consequence of [70, Proposition 1].  $\square$

**Theorem 4.28.** *Conjectures 4.13 and 4.26 are equivalent.*

*Proof.* The direction  $4.26 \Rightarrow 4.13$  is immediate due to Corollary 4.23. For the opposite direction, suppose  $\mathcal{U} : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  is a unitary channel with causal structure  $G_{\mathcal{U}}$ . Applying Lemma 4.27, let  $\mathcal{M}$  be a spacetime situation such that  $G_{\mathcal{M}} = G_{\mathcal{U}}$ . Since  $\mathcal{U}$  is already unitary, we have  $\mathcal{U} \in \text{IC}_{\mathcal{M}}$ , so that by Conjecture 4.13,  $\mathcal{U} \in \overline{\text{Real}_{\mathcal{M}}}$  also. In other words,  $\mathcal{U}$  is approximated by circuits  $\mathbf{P}$  which are implementable in  $\mathcal{M}$ , and which therefore in particular satisfy  $G_{\mathbf{P}} \subseteq G_{\mathcal{M}}$ . Every such circuit  $\mathbf{P}$  can be trivially rewritten into one with connectivity  $G_{\mathbf{P}} = G_{\mathcal{M}}$ . This establishes Conjecture 4.26.  $\square$

**Remark 4.29.** A natural question is whether all binary relations  $G$  may also arise from spacetimes satisfying desirable physical requirements, for example, those that are (3+1)-dimensional and globally hyperbolic. If not, then a full solution to approximate causal decompositions might not be necessary to solve Conjecture 4.13 in those cases that are physically relevant. As a first step in this direction, we note that work by Meyer [70, Theorem 5] implies that under the assumptions that

- the spacetime underlying  $\mathcal{M}$  is (3+1)-dimensional Minkowski spacetime  $(\mathbb{R}^{3+1}, \eta)$ ;
- all relations  $p_{\mathbf{a}} \preceq_{\eta} q_{\mathbf{b}}$  or  $p_{\mathbf{a}} \not\preceq_{\eta} q_{\mathbf{b}}$  are robust under sufficiently small perturbations of  $q_{\mathbf{b}}$ ,

there exist binary relations not realisable as  $G_{\mathcal{M}}$ . An explicit example is the relation  $G$  between  $\mathbf{A} = \{1, 2, \dots, 9\}$  and  $\mathbf{B} = \mathcal{P}(\mathbf{A})$  defined by  $\mathbf{a}G\mathbf{b} \Leftrightarrow \mathbf{a} \in \mathbf{b}$ , but smaller examples might also exist. //

## 4.5 The Clifford case

Until now we have focussed on one particular way of proving special cases of Conjectures 3.19 and 4.13: fix a binary relation  $G$  and show that  $G$  implies causal decompositions, or, respectively, fix a spacetime situation  $\mathcal{M}$  and show that  $\text{IC}_{\mathcal{M}} \subseteq \overline{\text{Real}_{\mathcal{M}}}$ . Another possible avenue is to start from some non-causal assumption on a unitary channel  $\mathcal{U}$  and derive from it that  $\mathcal{U}$  admits causally faithful circuit decompositions, regardless of its causal structure. To close this chapter, we derive such a result, showing that every Clifford unitary admits a causally faithful circuit decomposition. The decomposition is of the  $\text{LOSEC}_{G_{\mathcal{U}}}$  form, meaning it can be readily implemented in any spacetime situation  $\mathcal{M}$  with  $G_{\mathcal{U}} \subseteq G_{\mathcal{M}}$ . It is also an exact decomposition, and thus generalises the known result that any Clifford unitary can be simulated exactly by nonlocal quantum computation without constraints on connectivity (see e.g. [26, 92]).

**Definition 4.30.** Let  $\mathbf{A}_{\mathbf{a}} = \mathcal{L}(\mathcal{H}_{\mathbf{a}})$  be a quantum system with finite dimension  $d_{\mathbf{a}} := \dim \mathcal{H}_{\mathbf{a}}$  and fix some basis  $\{|k\rangle_{\mathbf{A}_{\mathbf{a}}}\}_{k=1}^{d_{\mathbf{a}}}$ . Denote by  $\mathcal{G}_{\mathbf{a}}$  the set of generalised Pauli operators

$$\mathcal{G}_{\mathbf{a}} := \{Z^a X^b \mid a, b = 0, \dots, d_{\mathbf{a}} - 1\} \subseteq \mathbf{A}_{\mathbf{a}} \quad (4.40)$$

where

$$Z := \sum_k e^{2\pi i k/d} |k\rangle \langle k| \quad \text{and} \quad X := \sum_k |k \oplus 1\rangle \langle k|. \quad (4.41)$$

For composite systems  $\mathbf{A}_{\mathbf{A}} = \bigotimes_{\mathbf{a} \in \mathbf{A}} \mathbf{A}_{\mathbf{a}}$ , we write

$$\mathcal{G}_{\mathbf{A}} := \left\{ \bigotimes_{\mathbf{a} \in \mathbf{A}} g_{\mathbf{a}} \mid \forall \mathbf{a} \in \mathbf{A} : g_{\mathbf{a}} \in \mathcal{G}_{\mathbf{a}} \right\}. \quad (4.42)$$

A unitary map  $U : \bigotimes_{\mathbf{a} \in \mathbf{A}} \mathcal{H}_{\mathbf{a}} \rightarrow \bigotimes_{\mathbf{b} \in \mathbf{B}} \mathcal{H}_{\mathbf{b}}$  and its associated channel  $\mathcal{U} = U(\cdot)U^\dagger : \mathbf{A}_{\mathbf{A}} \rightarrow \mathbf{B}_{\mathbf{B}}$  are called **Clifford** if  $U\mathcal{G}_{\mathbf{A}}U^\dagger \subseteq \mathcal{G}_{\mathbf{B}}$ , or, in other words, if there is a function of finite sets  $f_U : \mathcal{G}_{\mathbf{A}} \rightarrow \mathcal{G}_{\mathbf{B}}$  such that<sup>14</sup>

$$f_U(g)U = Ug \quad \text{for all } g \in \mathcal{G}_{\mathbf{A}}. \quad (4.43)$$

In the following we will write  $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_m$  for the elements of  $\mathbf{A}$  and  $\mathbf{b}_1, \mathbf{b}_2, \dots, \mathbf{b}_n$  for those of  $\mathbf{B}$ . Note that there is a natural identification of the set  $\mathcal{G}_{\mathbf{A}}$  of tensor products of Pauli operators with the Cartesian product

$$\mathcal{G}_{\mathbf{A}} \cong \bigtimes_{\mathbf{a} \in \mathbf{A}} \mathcal{G}_{\mathbf{a}}, \quad g_{\mathbf{a}_1} \otimes g_{\mathbf{a}_2} \otimes \dots \otimes g_{\mathbf{a}_m} \mapsto (g_{\mathbf{a}_1}, g_{\mathbf{a}_2}, \dots, g_{\mathbf{a}_m}). \quad (4.44)$$

The same holds for  $\mathcal{G}_{\mathbf{B}}$ . We may therefore regard  $f_U : \mathcal{G}_{\mathbf{A}} \rightarrow \mathcal{G}_{\mathbf{B}}$  as a multi-input, multi-output classical deterministic process, and by treating it as a multipartite quantum

<sup>14</sup>Normally when talking about Clifford unitaries, one assumes that the dimensions of all elementary systems  $\mathbf{A}_{\mathbf{a}}$  and  $\mathbf{B}_{\mathbf{b}}$  are equal and that the number of in- and output wires of the Clifford unitaries match. For our purposes it makes sense to consider the slightly more general class of unitaries defined here.

channel with dephased inputs and outputs, we can incorporate it into the same diagrammatic language we have been using for quantum circuits. We denote classical systems and processes in grey so depict  $f_U$  as

$$\begin{array}{c} \mathcal{G}_{b_1} \mid \mathcal{G}_{b_2} \cdots \mathcal{G}_{b_n} \\ \boxed{f_U} \\ \mathcal{G}_{a_1} \mid \mathcal{G}_{a_2} \cdots \mathcal{G}_{a_m} \end{array} . \quad (4.45)$$

Write  $\mathcal{G}_a \rightarrow_{f_U} \mathcal{G}_b$  if the  $\mathcal{G}_b$ -component of the output of  $f_U$  functionally depends on the  $\mathcal{G}_a$ -component of the input. In analogy to the causal structure  $G_U$  of a unitary channel  $U$ , let the **causal structure** of the function  $f_U$  be the relation  $G_{f_U} \subseteq A \times B$  defined by

$$\forall a \in A, b \in B : (a, b) \in G_{f_U} \iff \mathcal{G}_a \rightarrow_{f_U} \mathcal{G}_b. \quad (4.46)$$

Classical functions like  $f_U$  always admit circuit decompositions of the  $\text{LOSEC}_{G_{f_U}}$  form:

$$\begin{array}{c} \mathcal{G}_{b_1} \mid \mathcal{G}_{b_2} \cdots \mathcal{G}_{b_n} \\ \boxed{f_U} \\ \mathcal{G}_{a_1} \mid \mathcal{G}_{a_2} \cdots \mathcal{G}_{a_m} \end{array} = \begin{array}{c} \begin{array}{c} \mathcal{G}_{b_1} \\ \boxed{f_U} \\ \mathcal{G}_{a_1} \end{array} \quad \begin{array}{c} \mathcal{G}_{b_2} \\ \boxed{f_U} \\ \mathcal{G}_{a_2} \end{array} \quad \cdots \quad \begin{array}{c} \mathcal{G}_{b_m} \\ \boxed{f_U} \\ \mathcal{G}_{a_m} \end{array} \\ \text{---} G_{f_U} \text{---} \end{array} . \quad (4.47)$$

Here  $\begin{array}{c} \text{---} \\ \text{---} \\ \text{---} \end{array} : X \rightarrow X^n$  denotes the copy map on a classical system  $X$ . One copy of each input is sent to each output site  $G_{f_U}$ -related to the input. At each output site  $f_U$  is then computed separately on all received inputs, filling in any missing,  $G_{f_U}$ -unrelated inputs with arbitrary (and irrelevant) states  $\nabla$ . Only the relevant outputs are kept.

As the following lemma shows,  $f_U$  inherits its causal structure  $G_{f_U}$  entirely from the Clifford unitary  $U$  that defines it.

**Lemma 4.31.** *For any Clifford unitary  $U = U(\cdot)U^\dagger : A_A \rightarrow B_B$ , we have  $G_{f_U} = G_U$ .*

*Proof.* Recall that we write  $A_i := \mathcal{L}(\mathcal{H}_{A_i})$  and  $B_j := \mathcal{L}(\mathcal{H}_{B_j})$ . Suppose that  $U = U(\cdot)U^\dagger$  satisfies the no-influence condition  $A_a \nrightarrow_U B_b$ , defined in Definition 3.8. This means that

$$U(A_a \otimes \{1_{A_{A \setminus \{a\}}}\}) \subseteq \{1_{B_b}\} \otimes B_{B \setminus \{b\}}. \quad (4.48)$$

Since  $\mathcal{G}_a \subseteq A_a$  and  $f_U$  is just the restriction of  $U$  to  $\mathcal{G}_A$ , this shows that for all  $g_a \in A_a$  the  $\mathcal{G}_b$  component of  $f_U(g_a \otimes 1_{A_{A \setminus \{a\}}})$  is always  $1_{B_b} \in \mathcal{G}_b$ , so is in particular independent of  $g_a$ . Using the fact that  $f_U$  preserves matrix multiplication, we have that on general  $\bigotimes_{a' \in A} g_{a'} \in \mathcal{G}_A$ ,  $f_U$  evaluates as

$$f_U \left( g_a \otimes \bigotimes_{a' \in A \setminus \{a\}} g_{a'} \right) = f_U \left( g_a \otimes 1_{A_{A \setminus \{a\}}} \right) f_U \left( 1_{A_a} \otimes \bigotimes_{a' \in A \setminus \{a\}} g_{a'} \right) \quad (4.49)$$

whose  $\mathcal{G}_b$  component is then independent of  $g_a$  too. Therefore  $\mathcal{G}_a \rightarrow_{f_U} \mathcal{G}_b$ .

Conversely, if  $\mathcal{G}_a \rightarrow_{f_U} \mathcal{G}_b$  then

$$\mathcal{U}(\mathcal{G}_a \otimes \{\mathbf{1}_{A_{\mathbb{A} \setminus \{a\}}}\}) = f_U(\mathcal{G}_a \otimes \{\mathbf{1}_{A_{\mathbb{A} \setminus \{a\}}}\}) \subseteq \{\mathbf{1}_{B_b}\} \otimes B_{B \setminus \{b\}}. \quad (4.50)$$

Since the Pauli operators in  $\mathcal{G}_a$  linearly span the full matrix algebra  $A_a$ , this means that also Eq. (4.48) is satisfied, so that  $A_a \rightarrow_{\mathcal{U}} B_b$ .  $\square$

**Theorem 4.32.** *Every Clifford unitary  $\mathcal{U}$  admits an exact  $\text{LOSEC}_{G_U}$  decomposition that uses one maximally entangled state for each input system.*

*Proof.* Let  $\mathcal{P}$  denote the classically controlled gate that applies the Pauli gate specified by the state of its control system  $\mathcal{G}_a$ :

$$\begin{array}{c} |A_a \\ \boxed{\mathcal{P}} \\ \swarrow \mathcal{G}_a \\ \triangle g \end{array} |A_a = \begin{array}{c} |A_a \\ \boxed{g(\cdot)g^\dagger} \\ |A_a \end{array} \quad \text{for each } g \in \mathcal{G}_a. \quad (4.51)$$

Teleportation [16] of a system  $A_a$  can then be represented diagrammatically by

$$\begin{array}{c} |A_a \\ \boxed{\mathcal{P}} \\ \swarrow \mathcal{G}_a \\ \boxed{M} \\ |A_a \end{array} |A_a = \begin{array}{c} |A_a \\ |A_a \\ |A_a \end{array}, \quad (4.52)$$

where  $\bigcup_{A_a A_a} := |\Phi^+\rangle\langle\Phi^+|_{A_a A_a}$  denotes the maximally entangled state

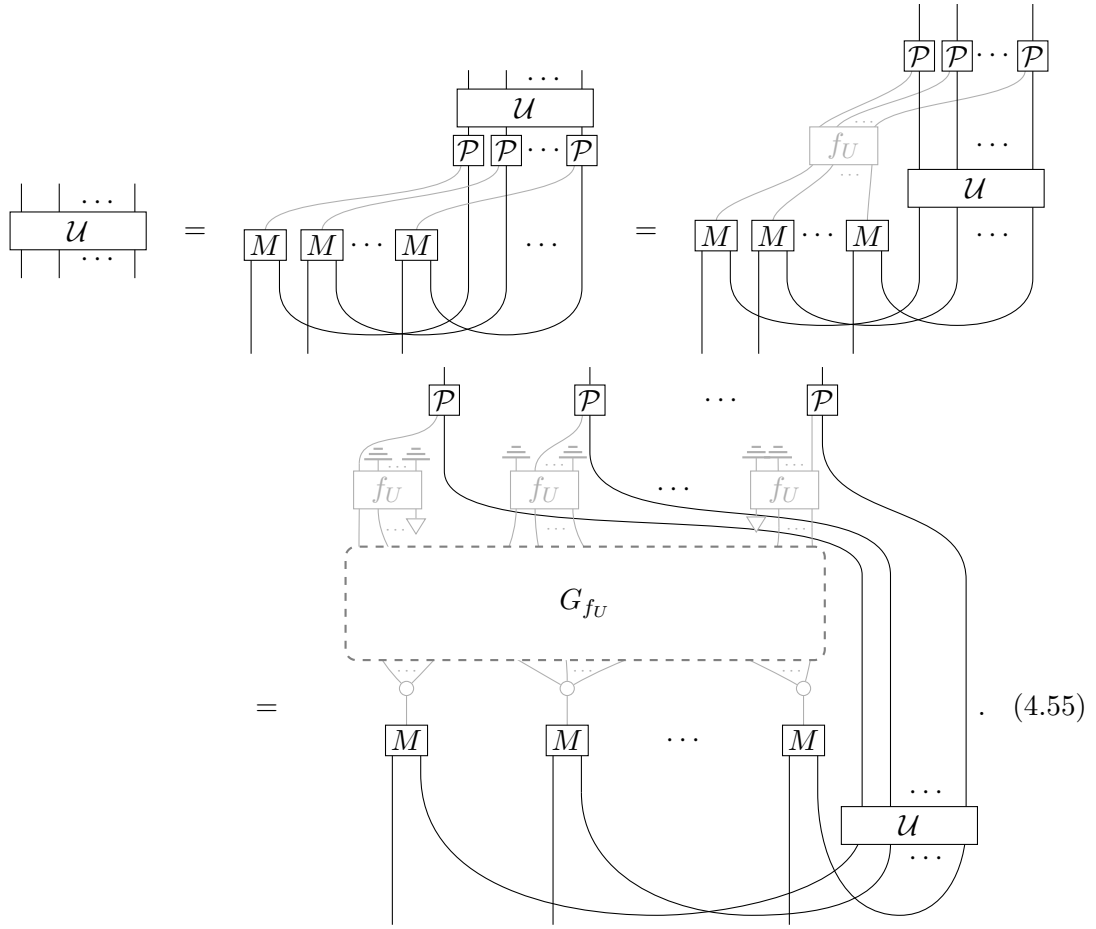
$$|\Phi^+\rangle_{A_a A_a} := \frac{1}{\sqrt{d_a}} \left( \sum_i |i\rangle_{A_a} \otimes |i\rangle_{A_a} \right), \quad (4.53)$$

and  $M$  is a measurement of  $A_a \otimes A_a$  in the basis  $\{(g_a \otimes \mathbf{1}_{A_a}) |\Phi^+\rangle \mid g_a \in \mathcal{G}_a\}$  whose classical outcomes are labelled by elements of  $\mathcal{G}_a$ .

The property of  $\mathcal{U}$  being a Clifford unitary, i.e. Eq. (4.43), can meanwhile be expressed diagrammatically as

$$\begin{array}{c} \dots \\ \boxed{\mathcal{U}} \\ \boxed{\mathcal{P}} \boxed{\mathcal{P}} \dots \boxed{\mathcal{P}} \end{array} = \begin{array}{c} \boxed{\mathcal{P}} \boxed{\mathcal{P}} \dots \boxed{\mathcal{P}} \\ \swarrow \swarrow \dots \swarrow \\ \boxed{f_U} \boxed{\mathcal{U}} \end{array}. \quad (4.54)$$

Consequently, any Pauli errors arising from teleportation procedures can be delayed until after application of the unitary, but at the cost of needing to compute the classical function  $f_U$ . Using the  $\text{LOSEC}_{G_{f_U}}$  decomposition of  $f_U$  from Eq. (4.47), then, we get



Since  $G_{f_U} = G_U$  by Lemma 4.31, the resulting circuit constitutes a  $\text{LOSEC}_{G_U}$  decomposition of  $U$ .  $\square$



## Chapter 5

# Conclusion and outlook

In this thesis we have studied the relation between compositional, causal, and relativistic structure in the context of quantum transformations. We started in Chapter 2 by introducing an order-theoretic framework for circuit syntax especially suited to the study of input-output connectivity. We showed that a circuit has connectivity  $G_p \subseteq G$  precisely if it can be syntactically rewritten into a circuit of a particular canonical shape  $L_G$ , which is therefore the most expressive circuit shape with that connectivity. In doing so, we solved an open problem posed by Lorenz and Barrett in their foundational work on causal decompositions [62], which was to decide—even if assuming that all unitaries have a causally faithful circuit decomposition—whether there exists a universal circuit shape for all unitaries with a given causal structure  $G$ , or whether the required shape might depend on the specific unitary under consideration. Having such a universal shape is crucial to roles that causal decompositions play as a bridge between different approaches to quantum causal modelling as well as in providing a constructive local parametrisation of the entire set of unitaries satisfying given global causal constraints. The shape  $L_G$  we showed to be universal turned out to be a complete lattice, which we argued is physically intuitive as joins and meets provide the opportunity for systems to interact before being sent to the output locations.

In Chapter 3 we used the tools developed in Chapter 2 to make progress on the open problem of the existence of causal decompositions of unitary transformations. We started by outlining the compositional structure of quantum transformations, furnishing the order-theoretic circuit syntax from Chapter 2 with quantum semantics and arguing by way of a simple example (Proposition 3.6) that the full generality of the lattice  $L_G$ , with the maximal interaction it provides, may generally be required to express quantum channels. We explained why the focus of causal decompositions lies on decomposing *unitary* transformations, specifically, the reason being that the relationship between a generic quantum channel’s compositional structure and its signalling relations is always mediated by the causal structure of an appropriate unitary dilation. We also noted that for some applications of causal decompositions—in particular that of parametrising the unitary transformations satisfying a global causal constraint—it is relevant to require the gates in the decompositions themselves to be unitary, too. Our main contribution was then a combinatorial characterisation of those relations  $G$  that ‘imply’ such *unitary*

causal decompositions, or, equivalently (as we found), imply unitary causally faithful decompositions; that is, those global causal constraints that can be fully represented in terms of the compositional structure of unitary circuits. The combinatorial condition turned out to be naturally expressible both in terms of the relation  $G$  and in terms of the circuit shape  $L_G$ , in a way that made precise an intuition about routed circuits: that they only provide an advantage in the presence of non-unique paths between inputs and outputs.

In Chapter 4 we turned our attention to relativistic causality, where the first results along the lines of causal decompositions in fact found their roots [13, 35, 86]. These works investigated the relation between (what we called) *signal consistency* and *realisability* of quantum channels relative to a spacetime localisation of their input and output systems. We saw that for a general such *spacetime situation*  $\mathcal{M}$ , we have

$$\text{Real}_{\mathcal{M}} \subseteq \overline{\text{Real}_{\mathcal{M}}} \subseteq \overline{\text{IC}_{\mathcal{M}}} \subseteq \overline{\text{SC}_{\mathcal{M}}} = \text{SC}_{\mathcal{M}},$$

and that equivalence between signal consistency ( $\text{SC}_{\mathcal{M}}$ ) and realisability ( $\text{Real}_{\mathcal{M}}$ ) can be obstructed in at least two distinct ways. First, because *influence consistency* ( $\text{IC}_{\mathcal{M}}$ ) is strictly stronger than signal consistency yet necessary for realisability, making the third inclusion above generally strict. We saw how some phenomena from the literature—the PR box and jamming correlations—can indeed be ruled out by demanding that not signalling through channels, but causal influences through their unitary dilations conform to relativistic causality. In addition, we argued that imposing influence consistency is natural because any unobserved superluminal influences on the unitary level would almost surely become observable upon only small perturbations of the fine-tuned environment state.

The second obstruction for the equivalence of signal consistency and realisability that we saw arises in situations previously studied in the context of position-based cryptography and holography, where the spacetime geometry limits the interaction that can take place between systems due to the absence of intersections of lightcones. While this can mostly be overcome by nonlocal quantum computation, simulating interaction with shared entanglement up to arbitrarily small error, some channels remain impossible to realise exactly due to the recent result of Miller and Alnawakhtha [71], implying strictness of the first inclusion above. This left open the possibility for equality in the second inclusion, which we posed as Conjecture 4.13. We saw that an answer to this conjecture reduced to a combination of established nonlocal computation techniques with a full solution to the existence of causal decompositions of unitary transformations—albeit of a weaker kind than those featuring in Chapter 3: first, because the gates are allowed to be general channels, rather than unitaries, and second, because approximations are allowed. We finished with a proof that all Clifford unitaries admit causally faithful circuit decompositions, weak in the first sense (they use shared entanglement) but not in the second (the decompositions are exact).

The research programme pursued in this thesis is far from complete, and each of the chapters has left open natural avenues for further investigation. I will highlight some, starting with a problem arising from Chapter 2 and Section 3.2, purely about quantum *compositional* structure.

*Find sufficient and necessary conditions for the quantum equivalence of circuit shapes.*

Here I consider two circuit shapes *quantum-equivalent* if a quantum channel can be expressed (exactly) by the one if and only if it can be expressed by the other. This notion is to be contrasted with *syntactical equivalence* from Definition 2.17. It follows from Proposition 3.3 that syntactical equivalence implies quantum equivalence. The converse is not quite true for one obvious reason: in the way we have defined it, all gates of a quantum circuit must be trace-preserving, meaning the only possible semantics for a gate with no output wires, i.e. an element  $p \in \mathbf{P}$  that is maximal and satisfies  $\mu^{-1}(p) = \emptyset$ , is a partial trace, which always factorises along its input systems. Any circuit shape  $\mathbf{P}$  is thus quantum equivalent—but not necessarily syntactically equivalent—to a reduced shape  $\tilde{\mathbf{P}} \preceq \mathbf{P}$  obtained by factorising and/or removing appropriate such maximal elements. We may however ask: *If any two reduced circuit shapes  $\tilde{\mathbf{P}}$  and  $\tilde{\mathbf{Q}}$  are syntactically inequivalent, are they necessarily quantum-inequivalent, too?*<sup>1</sup>

Due to the possibility of nonlocal quantum computation, the answer is still in the negative if one considers not exact but approximate circuit decompositions: the shapes  $\mathbf{L}_G$  and  $\mathbf{LOSEC}_G$ , for instance, can approximate the same quantum channels but are generally syntactically inequivalent. Nevertheless, as a consequence of Miller and Alnawakhtha [71] (Proposition 4.6), whenever  $\mathbf{L}_G$  and  $\mathbf{LOSEC}_G$  are syntactically inequivalent, there remain quantum channels that can be *exactly* expressed by  $\mathbf{L}_G$  but not by  $\mathbf{LOSEC}_G$ . As far as I am aware, no counterexamples to the *exact* question in italics are known. If the answer is positive, this would reveal an interesting fact about quantum theory, starkly contrasting it with classical theory, where the hierarchy of circuit shapes instead largely collapses (see also [84]). Whatever the answer, a precise characterisation of the quantum-inequivalent circuit shapes would have potential applications to the study of quantum networks (where similar research into the significance of *intermediate latents* is being done [25]) as well as to position-based quantum cryptography, where attention has been largely limited to the simple bipartite case from Proposition 4.6. An answer to the question whether exact causal decompositions must generally always be of the syntactically most expressive shape  $\mathbf{L}_G$  (up to the potential removal of its top and/or bottom) would also follow from such a characterisation.

Relatedly, the order-theoretic formalism for circuit syntax, which we put forward in Chapter 2 motivated mostly by its application to causal decompositions, also warrants further investigation outside the quantum context. This partly to help solve the problem of quantum equivalence but also out of independent interest. First, it would

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<sup>1</sup>In process-theoretic jargon: does inequivalence between  $\mathbf{P}$  and  $\mathbf{Q}$  with respect to any ‘process-terminal’ or ‘causal’ process theory [29, 56, 57, 84] imply inequivalence in the theory of quantum systems and CPTP maps?

be worth formalising the connection with free symmetric monoidal categories, which I deferred in Section 2.2 for lack of relevance to the main subject of this thesis, but which would provide better ground for regarding morphisms of circuit shapes as a complete model for syntactical circuit rewrites. Moreover, there is much more to be said about the category-theoretic structure of  $\text{Circ}_{\mathbf{A}, \mathbf{B}}$ , given fixed sets  $\mathbf{A}, \mathbf{B}$ , and the ensuing order-theoretic structure of  $\text{Circ}_{\mathbf{A}, \mathbf{B}}^{\leq}$ . A line of interesting results was recently presented in independent work of Salzger and Selby [84], who propose an order-theoretic framework for circuit syntax very similar to that developed in Chapter 2: their definitions of circuits and morphisms (there called *framed partial orders* and *frame- and order-preserving maps*) are distinct from, but for most practical purposes equivalent to ours.<sup>2</sup> Notably, the work characterises all reduced circuit shapes (as in  $\tilde{\mathbf{P}}$  above) up to syntactical inequivalence in several special cases where  $\mathbf{A}$  and  $\mathbf{B}$  have small numbers of elements. This line of investigation is worth extending in future work.

The results presented in this thesis also suggest next steps towards a complete understanding of causal decompositions. One is to

*find a combinatorial condition characterising those relations  $G \subseteq \mathbf{A} \times \mathbf{B}$  that imply unitary routed causal decompositions. The condition should be expressed as a natural property of  $G$  and/or its concept lattice  $\mathbf{L}_G$ , perhaps similar in nature to the  $C_3$  exclusion property.*

Lorenz and Barrett [62] showed the existence of routed unitary causal decompositions<sup>3</sup> for a number of relations on a case-by-case basis and recent work of Vanrietvelde, Mestoudjian and Arrighi [108] solves another class of cases. There however also exist counterexamples: there are four-input, four-output unitaries that admit no causally faithful routed unitary circuit decomposition [105]. The tools developed in Chapters 2 and 3 may prove useful for finding a complete characterisation: the concept lattice provides the definitive syntax for causal decompositions and Proposition 3.34 may be a good starting point for constructing the quantum semantics of general routed decompositions. As for the combinatorial form of the condition, it is not difficult to see that implying unitary routed causal decompositions is a hereditary property: that is, if  $G \subseteq \mathbf{A} \times \mathbf{B}$  satisfies it, then any restriction  $G \cap (\mathbf{A}' \times \mathbf{B}')$ , where  $\mathbf{A}' \subseteq \mathbf{A}$  and  $\mathbf{B}' \subseteq \mathbf{B}$ , satisfies it too.<sup>4</sup> This implies that the condition can be formulated as an ‘exclusion property’ much like  $(C_3\text{-EP})$ , requiring that  $G$  restricts nowhere to any out of a set of forbidden relations  $X$ . Whether  $X$  is a singleton (as for  $(C_3\text{-EP})$ ), a finite set, or an infinite set remains to be seen.

Moreover, in those cases where routed unitary circuits do not suffice, future research should point out whether causal decompositions in terms of generic channels still do

<sup>2</sup>Precisely, the category of framed partial orders and frame- and order-preserving maps is isomorphic to a strict subcategory of our  $\text{Circ}_{\mathbf{A}, \mathbf{B}}$ , but the approaches become equivalent after quotienting out by syntactical equivalence.

<sup>3</sup>There called ‘extended’, rather than routed.

<sup>4</sup>It is not immediately clear whether the same is true about the property of implying unitary routed *causally faithful* decompositions. Indeed, this was why in the non-routed case, the direction (iii)  $\Rightarrow$  (i) of Theorem 3.23 required a non-trivial proof (Section 3.5.4).

exist. Such decompositions would be relevant to the spacetime context, solving special cases of Conjecture 4.13, but less useful for the application of providing parametrisations and autonomous causal mechanisms. To serve the latter application, it would be worth investigating whether an even more general *unitary* quantum circuit formalism exists, generalising routed unitary quantum circuits and allowing for strictly more causal decompositions.

When it comes to spacetime application of causal decompositions, also *geometric* conditions on binary relations are worth investigating:

*Determine which binary relations  $G \subseteq \mathbf{A} \times \mathbf{B}$  arise as the metric-induced relations  $G_{\mathcal{M}}$  of spacetime situations  $\mathcal{M}$  that have  $3+1$  dimensions and/or satisfy other physically desirable properties, such as global hyperbolicity; and investigate whether such geometric constraints imply algebraic properties that facilitate the derivation of causal decompositions required for the physically relevant cases of Conjecture 4.13.*

Recall from Lemma 4.27 that any relation can occur as  $G_{\mathcal{M}}$  in Minkowski spacetimes of unrestricted dimension, but that physically reasonable (though perhaps too strong) geometric restrictions can indeed impose nontrivial constraints on the realisable relations (Remark 4.29). A connection between geometric constraints and algebraic properties as suggested above is speculative, but would be very interesting.

Let me close by mentioning another, more ambitious avenue generalising the problem setting of Chapter 4. Even though the framework we proposed there can be regarded as generalising a question asked in previous literature [13, 35, 86] to arbitrarily many input and output systems localised at arbitrary points in spacetime, it is still restricted to the limit of finite-dimensional quantum systems travelling along highly localised worldlines through spacetime. A natural generalisation would be to quantum field theories, with the algebraic formalism perhaps being most suitable [50]. In this context, an appropriate notion of *realisability* of quantum operations in local spacetime regions has only recently been proposed [39]—of which the quantum circuits in spacetime from Chapter 4 can be considered a limiting case [64]—and it was shown in [19] that those operations never lead to superluminal signalling or superluminal causal influences. It would be worth investigating, also in this more general context, what can be said about the converse.



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