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## Schrödinger's ants: a continuous description of Kirman's recruitment model

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## Abstract

We show how the approach to equilibrium in Kirman's ants model can be fully characterized in terms of the spectrum of a Schrödinger equation with a Pöschl–Teller ( $\tan^2$ ) potential. Among other interesting properties, we have found that in the bimodal phase where ants visit mostly one food site at a time, the switch time between the two sources only depends on the 'spontaneous conversion' rate and *not* on the recruitment rate. More complicated correlation functions can be computed exactly, and involve higher and higher eigenvalues and eigenfunctions of the Schrödinger operator, which can be expressed in terms of hypergeometric functions.

## 1. Introduction

Kirman's *ant model* [1] undoubtedly stands among the most inspiring toy models in the behavioural economics literature. While initially inspired by the experiment described below, its conclusions have implications much beyond collective animal behaviour, as it has been used to model shifts in sentiment of economic agents, trend reversal in financial markets, herding and social influence, etc. It is also akin to another famous model in population dynamics with competing species: the Moran model [2].

Several decades ago, entomologists were puzzled by the behaviour of ants who, able to choose between two identical and inexhaustible food sources  $A$  and  $B$ , tended to concentrate on one of them for a while, but occasionally switched collectively to the other without any apparent reason [3, 4]. Such intermittent herding behaviour is very similar to that observed in humans choosing between equivalent restaurants [5], or in financial markets [6–8], and is consistent with the notion that most large fluctuations are of purely endogenous nature: in contrast with the views of standard economic choice theory, the asymmetric exploitation observed in ants does not seem to correspond to the equilibrium state of an isolated representative ant with rational expectations. These observations suggest instead that such phenomena should rather be explained in terms of interactions between individual agents, ants in this particular setting, through what biologists call recruitment dynamics. To account for such intricate behaviour, Kirman proposed a simple yet insightful model [1] based on tandem recruitment that we describe below.

Consider the  $N$  ants from the entomologists' experiment and denote by  $k(t) \in [0, N]$  the number of ants feeding on source  $A$  at time  $t$ . The model states that when two ants meet, one of them converts the other with probability  $\mu/N$ , but each ant may in addition change its own mind spontaneously with probability  $\varepsilon$ . Within such a simple setting, Kirman was able to show that, in the large  $N$  limit, the stationary state depends only on a parameter  $\alpha := \varepsilon/\mu$ . When  $\alpha > 1$  the stationary distribution of  $k$  is unimodal, with a maximum at  $k = N/2$ , whereas for  $\alpha < 1$  it is bimodal, with the most probable states being  $k = 0$  and  $k = N$  (corresponding to the situation observed experimentally). Remarkably, the interesting  $\alpha < 1$  regime can be obtained even for weakly persuasive agents (small  $\mu$ ), provided the self-conversion rate  $\varepsilon$  is also low.

The most important point to be raised is that in the  $\alpha < 1$  regime none of the  $k$  states is, alone, an equilibrium state. Although the system can spend a long time at  $k = 0, N$  (local stationarity), these states cannot be considered as equilibria: every possible state is always revisited, and there is no convergence to any one of them in particular, discarding also the possibility of having multiple equilibria. Rather, there is perpetual change, and the system's natural endogenous dynamic is only in a *statistical equilibrium*. Contrary to this notion, most economic models focus on finding the equilibrium state to which the system under study will finally converge, and say that the latter may only be knocked off its path by large exogenous shocks.

Yet financial markets, and even larger economic and social systems, display a number of regular large switches (correlations, mood of investors etc) which do not always seem to be driven by exogenous shocks. In Kirman's stylised setting such switches can be understood endogenously, as in similar models where agents are allowed to interact and influence their peers' behaviour, thereby causing the emergence of non-trivial auto-reinforcing social dynamics [9, 10]. Several extensions of Kirman's model have been proposed [8, 11, 12], and the basic dynamics of imitation coupled with random switching can also be used to describe agents making political choices—say voting to the left or to the right—while being influenced by their peers, as described in the so-called voter model, see e.g. [13].

In the present paper we present a continuous description of the ant model which notably allows us to derive the typical switching time, using classical methods from statistical physics and quantum mechanics.

## 2. Master equation

As mentioned above the original model describes  $N$  ants faced with two identical food sources, with the relevant dynamical variable being  $k$ , the number of ants feeding on—say—source  $A$ . Each time step allows an ant to either switch randomly to the other food source with probability proportional to  $\varepsilon$ , or to get recruited by another ant from the other food source with probability proportional to  $\mu$ .

Defining the unit of time as the time required for all the ants to make a decision leads to  $dt = 1/N$  as the infinitesimal time unit. It is also clear that, to remain intensive in the large  $N$  limit, the probability to interact with another ant should be proportional to  $1/N$ . Altogether, we may write a Master equation for the evolution of the probability  $\mathbf{P}(k, t)$  that there are  $k$  ants feeding at source  $A$  at time  $t$ :

$$\mathbf{P}\left(k, t + \frac{1}{N}\right) - \mathbf{P}(k, t) = \frac{1}{N} \{W(k+1 \rightarrow k)\mathbf{P}(k+1, t) + W(k-1 \rightarrow k)\mathbf{P}(k-1, t) - [W(k \rightarrow k-1) - W(k \rightarrow k+1)]\mathbf{P}(k, t)\}, \quad (1)$$

where the transition rates are given by:

$$W(k \rightarrow k+1) = \left(1 - \frac{k}{N}\right) \left(\varepsilon + \frac{\mu}{N} \frac{k}{N-1}\right) \quad (2a)$$

$$W(k \rightarrow k-1) = \frac{k}{N} \left(\varepsilon + \frac{\mu}{N} \frac{N-k}{N-1}\right). \quad (2b)$$

Note that this specification only differs from Kirman's original one in the rescaling of the recruitment rate by  $N$ . With the notations of [1],  $1 - \delta = \mu/N$ .

## 3. Continuous description and Fokker–Planck equation

Here we follow Kirman's original paper [1] and derive a proper continuous-time Fokker–Planck equation in the limit  $N \rightarrow \infty$ .

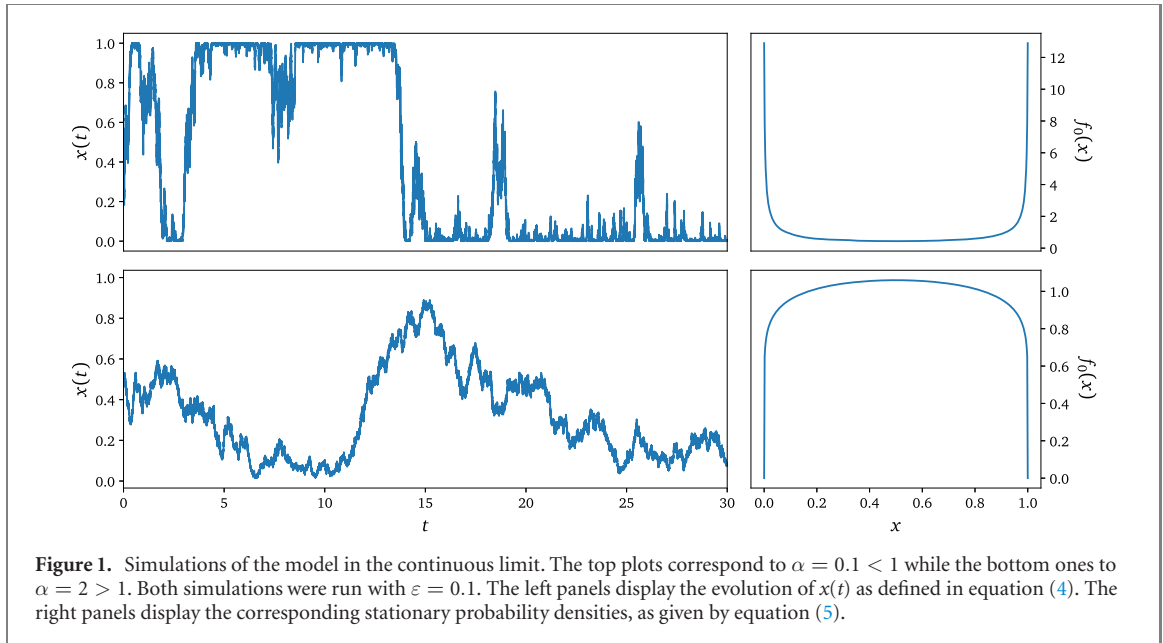
We define the variable  $x = \frac{k}{N} \in [0; 1]$  together with its probability density function  $f(x, t)$ . Taking the continuous limit  $N \rightarrow \infty$  of equation (1) leads to the following Fokker–Planck equation [14]:

$$\partial_t f = \partial_x J^f, \quad \text{with } J^f(x, t) = -\varepsilon(1-2x)f(x, t) + \mu \partial_x [x(1-x)f(x, t)], \quad (3)$$

the probability flux, see appendix A for the details of the calculations and the first  $1/N$  corrections. The conservation of the number of ants in the model is ensured by the condition  $J^f(x, t) = 0$  at the boundaries  $x = 0$  and  $x = 1$  at all times. Equation (3) corresponds to the following stochastic process for  $x$ :

$$\dot{x} = \varepsilon(1-2x) + \sqrt{2\mu x(1-x)}\eta(t), \quad (4)$$

with  $\eta$  a Gaussian white noise with unit variance. One can note that while the drift term  $\varepsilon(1-2x)$  is maximal at the boundaries and tends to pull  $x$  towards  $1/2$ , the noise term has the opposite effect. The diffusion constant is proportional to  $\sqrt{2\mu x(1-x)}$  and is maximal at  $x = 1/2$  and so tends to push the system away from  $x = 1/2$ .



**Figure 1.** Simulations of the model in the continuous limit. The top plots correspond to  $\alpha = 0.1 < 1$  while the bottom ones to  $\alpha = 2 > 1$ . Both simulations were run with  $\varepsilon = 0.1$ . The left panels display the evolution of  $x(t)$  as defined in equation (4). The right panels display the corresponding stationary probability densities, as given by equation (5).

Note that this stochastic process is very similar to the Moran model of genetic population dynamics [2]—with the same diffusion term  $\propto \sqrt{x(1-x)}$ —where the analogue of the number of ants at each food source is the proportion of genes from two competing alleles ( $A$  or  $B$ ) [15]. The  $\varepsilon$  term corresponds to spontaneous mutations. When  $\varepsilon = 0$ , there is a non zero probability that the whole population becomes of type  $A$  or  $B$  after a finite time, corresponding to  $\delta(x)$  or  $\delta(1-x)$  contributions to  $f(x, t)$  with a time dependent weight, see [16], and [17] for a recent thorough discussion. Interestingly, one can associate different payoffs to picking  $A$  or  $B$ , leading to a game-theoretical extension of this family of models, see [18] for a development in the context of evolutionary games. It is also equivalent to a model describing noise-induced bistability transitions in chemical reactions [19], as shown in [20], where the equation describing the process is equivalent to equation (3). Finally, the same diffusion term  $\propto \sqrt{x(1-x)}$  also emerges naturally within the so-called voter model, that describes the opinion dynamics of peer-influenced voters [13].

When  $\varepsilon > 0$ , one can check that the normalised stationary distribution  $f_0(x)$ , obtained by setting  $J^f(x, t) = 0$ , writes:

$$f_0(x) = \frac{\Gamma(2\alpha)}{\Gamma^2(\alpha)} [x(1-x)]^{\alpha-1}, \quad \text{with } \alpha := \frac{\varepsilon}{\mu}. \quad (5)$$

This result is the same as that obtained by Föllmer and Kirman in [1].

Upon looking at the behaviour of the solution, shown in figure 1, one can see that there is a clear transition in the behaviour of the model at  $\alpha_c = 1$ . For  $\alpha > \alpha_c$ , the stationary density in equation (5) is maximal at  $x = 1/2$ , and the dynamics show that  $x(t)$  fluctuates around  $1/2$ , corresponding to a situation where the ants are, on average, evenly distributed across both food sources. For  $\alpha < \alpha_c$  the density  $f_0$  diverges at the boundaries. The top left panel in figure 1 shows that this corresponds to a very different picture, in which nearly all of the ants choose either one of the sources for a certain amount of time, until a noise-induced ‘avalanche’ causes a switch over to the other source. It is also easy to check that in the absence of noise (and  $\alpha \rightarrow 0$ ) the long-time stationary density is given by  $f_0(x) = p\delta(x) + (1-p)\delta(x-1)$ , a situation discussed at length in [17].

Having this in mind, a natural question to ask is: given a certain initial condition  $f(x, 0) = \delta(x - x_0)$ , how long does it take for the system to converge to the stationary state, or equivalently, how long does it take for the ants to switch from one source to the other in the  $\alpha < 1$  regime?

#### 4. Schrödinger’s equation and general solution

Here we obtain a full dynamical solution in terms of the eigenvalues and eigenfunctions of a certain quantum mechanical Hamiltonian.

Using the Ito rule [21], one can see that introducing a change of variables  $\varphi(x)$  in equation (4) yields a noise term proportional to  $\sqrt{x(1-x)}\varphi'(x)$ , and so motivates a choice satisfying  $\varphi'(x) = 1/\sqrt{x(1-x)}$ . We therefore define a new, more convenient, variable  $\varphi \in [-\pi/2, \pi/2]$  as:

$$\sin \varphi = 2x - 1. \quad (6)$$

The corresponding Fokker–Planck equation for its probability density  $g(\varphi, t)$  writes:

$$\partial_t g = \mu \partial_\varphi J^g, \quad \text{with } J^g(\varphi, t) = 2\beta \tan \varphi g(\varphi, t) + \partial_\varphi g(\varphi, t), \quad \text{and } \beta := \alpha - \frac{1}{2}, \quad (7)$$

where the probability flux must now verify  $J^g(\pm\pi/2, t) = 0$  at all times. Note however that this is only exact in the limit  $\tan(\varphi) \ll \frac{N}{2\alpha}$ , as discussed in appendix B. Setting again  $J^g = 0$  everywhere, one finds the normalised stationary solution:

$$g_0(\varphi) = \frac{\Gamma(\alpha + \frac{1}{2})}{\sqrt{\pi}\Gamma(\alpha)} (\cos \varphi)^{2\alpha-1}. \quad (8)$$

The advantage of this formulation in  $\varphi$  is that, in contrast with the former, the second order derivative term  $\partial_{\varphi\varphi}$  in equation (7) only depends on  $\varphi$  through  $g(\varphi, t)$ , as diffusion is no longer position-dependent. Standard techniques for the resolution of Fokker–Planck equations, see e.g. [14], motivate the introduction of a function  $\Psi$  such that:

$$g(\varphi, t) := \sqrt{g_0(\varphi)} \Psi(\varphi, t), \quad (9)$$

and  $\Psi(\varphi, t) \rightarrow \sqrt{g_0(\varphi)}$  when  $t \rightarrow \infty$ .

Combining equations (7) and (9) one obtains a Schrödinger-like equation of the form [22]:

$$-\frac{1}{\mu} \partial_t \Psi = \mathbf{H} \Psi, \quad (10)$$

where the Hamiltonian  $\mathbf{H}$  is defined as:

$$\mathbf{H} := -\partial_{\varphi\varphi} + V(\varphi), \quad V(\varphi) := -\beta + \beta(\beta-1)\tan^2 \varphi, \quad (11)$$

and with boundary conditions given by:

$$[\cos^\beta \varphi (\beta \tan \varphi \Psi(\varphi, t) + \partial_\varphi \Psi(\varphi, t))]_{\varphi=\pm\pi/2} = 0. \quad (12)$$

We have left the  $\mu$  parameter out of the Hamiltonian  $\mathbf{H}$  in order to ease the comparison to the canonical form presented in [23, 24]. The  $\tan^2$  term in equation (10) is known as the Pöschl–Teller potential [23], and appears in a similar context of social dynamics within a version of the voter model in reference [25]. The potential we have here was fully solved in the case  $\beta > 0$  with boundary conditions  $\Psi(\pm\pi/2, t) = 0$  in [24]. To be applicable to our framework, we shall verify that their solutions also satisfy equation (12) in the general case  $\beta > -1/2$ . The Hamiltonian  $\mathbf{H}$  is Hermitian (contrarily to the Fokker–Planck operator) and has a discrete set of orthogonal eigenfunctions and eigenvalues, given by:

$$\mathbf{H} \Psi_n = \mathcal{E}_n \Psi_n, \quad (13)$$

where, splitting into even ( $n = 2k$ ) and odd ( $n = 2k + 1$ ) states:

$$\mathcal{E}_n = n(2\alpha + n - 1), \quad (14a)$$

$$\Psi_{2k}(\varphi) = A_{2k}(\beta) {}_2F_1 \left( -k, \beta + k; \beta + \frac{1}{2}, \cos^2 \varphi \right) \cos^\beta \varphi, \quad (14b)$$

$$\Psi_{2k+1}(\varphi) = A_{2k+1}(\beta) {}_2F_1 \left( -k, \beta + k + 1; \beta + \frac{1}{2}, \cos^2 \varphi \right) \sin \varphi \cos^\beta \varphi, \quad (14c)$$

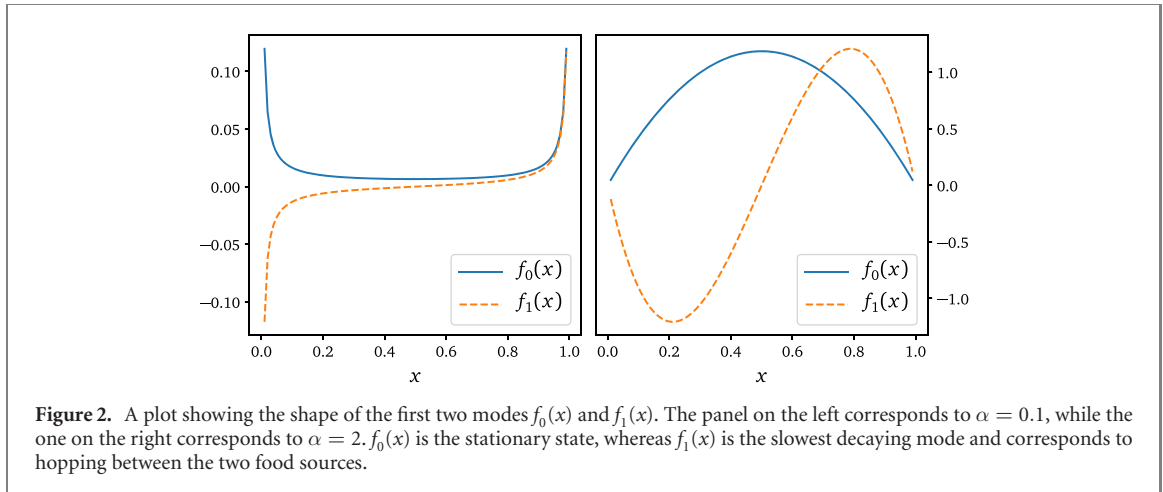
with  ${}_2F_1$  the ordinary hypergeometric function<sup>7</sup>. The coefficients  $A_n$  are set such as to ensure normalisation,  $\int_{[-\pi/2, \pi/2]} \Psi_n \Psi_m = \delta_{n,m}$ , and can be expressed as integrals of hypergeometric functions. Note that the parity of  $n$  also defines the parity of the function  $\Psi_n$  with respect to the  $y$ -axis. One can then easily check that for all  $n$  (both even and odd):

$$\cos^\beta \varphi [\beta \tan \varphi \Psi_n(\varphi) + \Psi'_n(\varphi)] \underset{\varphi \rightarrow \pm\pi/2}{\sim} \left( \frac{\pi}{2} \mp \varphi \right)^{1+2\beta}, \quad (15)$$

which, since  $\beta > -1/2$ , ensure that the boundary conditions given by equation (12) are satisfied. Noting that  $\Psi_0 = \sqrt{g_0}$ , the general solution of equation (10) then reads:

$$\Psi(\varphi, t) = \lambda_0 \sqrt{g_0(\varphi)} + \sum_{n>1} \lambda_n \Psi_n(\varphi) e^{-\mu \mathcal{E}_n t}, \quad (16)$$

<sup>7</sup> Here, the function  ${}_2F_1$  takes the form of a polynomial:  ${}_2F_1(-k, a; b, u) = \sum_{\ell=0}^k \binom{k}{\ell} (-1)^\ell \frac{\Gamma(a+\ell)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+\ell)} u^\ell$  for any integer  $k$ .



with  $\lambda_n$  given by the projections of the initial conditions on each mode  $n$ , namely  $\lambda_n = \int_{-\pi/2}^{\pi/2} d\varphi \Psi_n(\varphi) \Psi(\varphi, 0)$ .

Back to the physical variable  $x$ , the initial condition  $f(x, 0) = \delta(x - x_0)$  becomes  $g(\varphi, 0) = \delta(\varphi - \varphi_0)$  with  $\varphi_0 = \arcsin(2x_0 - 1)$ . Further using equation (9), it is easy to see that the initial condition in turn translates into  $\Psi(\varphi, 0) = \delta(\varphi - \varphi_0) / \sqrt{g_0(\varphi)}$ . The full solution for  $g(\varphi, t)$  follows:

$$g(\varphi, t) = g_0(\varphi) + \sum_{n>1} e^{-\mu \mathcal{E}_n t} \Psi_n(\varphi_0) \sqrt{g_0(\varphi)} \Psi_n(\varphi), \quad (17)$$

with the orthogonality between  $\Psi_0 = \sqrt{g_0}$  and  $\Psi_n$  for  $n > 1$  ensuring that  $\int_{-\pi/2}^{\pi/2} d\varphi g(\varphi, t) = \int_{-\pi/2}^{\pi/2} d\varphi g_0(\varphi) = 1$ , or equivalently for  $f(x, t)$ :

$$f(x, t) = f_0(x) + \sum_{n>1} e^{-\mu \mathcal{E}_n t} \Psi_n(\varphi_0) f_n(x), \quad (18)$$

with:

$$f_n(x) = \frac{\sqrt{g_0(\varphi(x))} \Psi_n(\varphi(x))}{2\sqrt{x(1-x)}}, \quad (19)$$

[see appendix (D2) for an explicit expression]. Equation (18) is the central result of the present communication.

## 5. Relaxation towards the stationary state

With the full dynamical solution of equation (18) at hand, one can see how long a system initially prepared at an initial value  $x_0 \approx 0$ , for example, takes to explore the whole space. In other words, one can ask how much time  $\tau$  is required to reach, say,  $x(\tau) \approx 1$  with a reasonable probability.

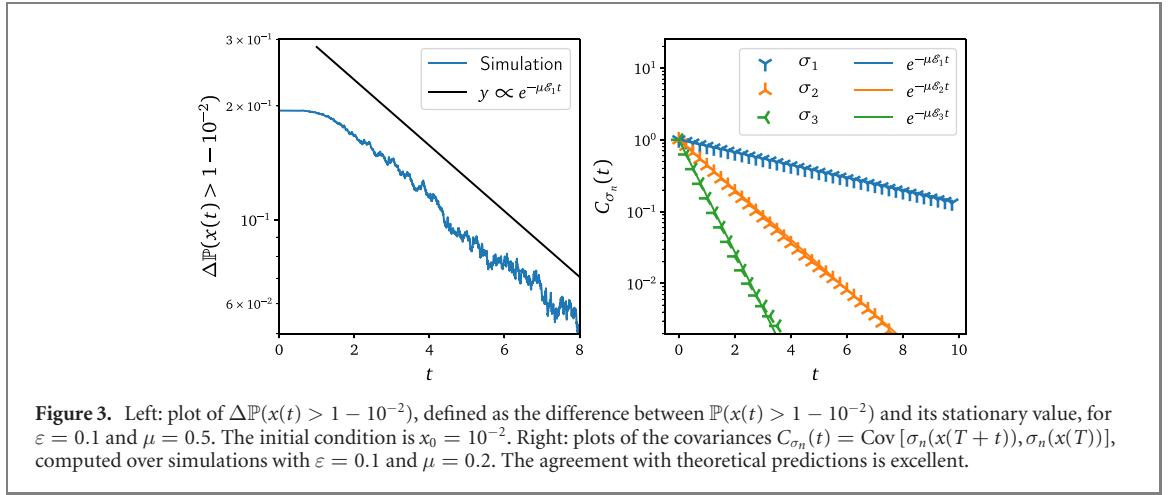
Since the stationary distribution  $f_0$  has weight on the whole interval  $[0; 1]$ , this time  $\tau$  is none other than the relaxation time (or ergodic time)  $\tau_R$  required to converge to stationarity. Owing to the form of equation (18) this convergence is asymptotically exponential, with the slowest mode given by  $n = 1$ . Hence, we find:

$$\tau_R := \frac{1}{\mu \mathcal{E}_1} \equiv \frac{1}{2\varepsilon}. \quad (20)$$

Perhaps surprisingly, this relaxation time depends only on the spontaneous switching rate  $\varepsilon$ , but not on the recruitment intensity  $\mu$ . Since  $n = 1$  corresponds to the slowest mode of the system, it also governs the collective ‘switch time’ between the two food sources,  $A$  and  $B$ —see figure 2.

We have checked our prediction for the switching time numerically by running trajectories starting at  $x_0 = \Delta x \ll 1$  and computing the probability  $\mathbb{P}(x(t) > 1 - \Delta x)$ . This quantity should converge to  $\int_{[1-\Delta x; 1]} f_0$  at an exponential rate  $\propto e^{-\mu \mathcal{E}_1 t}$ , which is in perfect agreement with our simulations, see figure 3.

Similarly, given an initial condition  $x_0 = 1/2$  where the ants are initially distributed evenly between the two sources, one may ask how long it takes for all the ants to ‘decide’ on concentrating on one of them. Since this condition is equivalent to  $\varphi_0 = 0$ , and since  $\Psi_1$  is an odd function of  $\varphi$ , it follows that  $\Psi_1(\varphi_0) = 0$  in this case. The convergence to the stationary distribution is then controlled by the second mode, with a much shorter



relaxation time given by:

$$\tau'_R := \frac{1}{\mu \mathcal{E}_2} \equiv \frac{1}{4\varepsilon + 2\mu}. \quad (21)$$

Directly applying tools from stochastic calculus on equation (4), one can obtain the following correlation functions (see appendix E):

$$\text{Cov}[\sigma_n(x(T+t)), \sigma_n(x(T))] \propto e^{-\mu \mathcal{E}_n t}, \quad (22)$$

where  $\sigma_n(x)$  are polynomials of degree  $n$  that allow one to ‘diagonalize’ the evolution of the correlations:

$$\begin{aligned} \sigma_1(x) &= x, \\ \sigma_2(x) &= x(1-x), \\ \sigma_3(x) &= (2x-1) \left[ \left(1 + \frac{2\alpha}{3}\right) (2x-1)^2 - 1 \right]. \end{aligned} \quad (23)$$

See appendix E for further details and figure 3 for a comparison with numerical results.

This result actually hides a deeper interpretation of the different modes  $f_n$ . In the case described above, one can surmise that the dynamics of the moments  $\mathbb{E}[x]$ ,  $\mathbb{E}[x^2]$  and  $\mathbb{E}[x^3]$  are determined exclusively by the modes  $f_1, f_2$  and  $f_3$ . In fact, focusing on any moment  $\mathbb{E}[x^m]$ , it is possible to prove that:

$$\forall n > m, \quad B_{n,m} = \int_0^1 dx f_n(x) x^m = 0, \quad (24)$$

as well as for all values  $n$  that do not have the same parity as  $m$ . This implies in fact that the dynamics of the moments  $\mathbb{E}[x^m]$  are fully described by the modes  $(f_1, \dots, f_m)$ , with only even values of  $n$  contributing to even moments  $m$  and vice versa. For example, for  $m = 3$  with the initial condition  $x(0) = x_0$  we can compute:

$$\mathbb{E}[x^3(t)] = B_{1,3} \Psi_1(\varphi_0) e^{-2\varepsilon t} + B_{3,3} \Psi_3(\varphi_0) e^{-3(2\varepsilon + 2\mu)t}, \quad (25)$$

where the exact expression of  $B_{n,m}$  is given in appendix (D3), equations (D20) and (D21). Mind that  $B_{0,m}$  is the stationary value of moment  $\mathbb{E}[x^m(t)]$  for all moments.

## 6. Conclusion

In this work, we have shown how that the approach to equilibrium in Kirman’s ants model can be fully characterized in terms of the spectrum of relaxation times, itself computable as the eigenvalues of a Schrödinger equation with a Pöschl–Teller ( $\tan^2$ ) potential. The different techniques we have presented can also be used for the exploration of a wide array of models, see for example reference [26] for an application of the Fokker–Planck equation to a different version of the Moran model, and reference [27], where a similar mapping of a stochastic evolution to a Schrödinger-like equation was used to discuss the dynamics of wealth inequality.

Among other interesting properties, we have found that in the bimodal phase where ants visit mostly one food site at a time, the switch time between the two sources only depends on the ‘spontaneous conversion’ rate  $\varepsilon$  and *not* on the recruitment rate  $\mu$ . This means that a single ant deciding on its own to explore an



alternative food source can trigger an ‘avalanche’ where the whole colony follows suit. More complicated correlation functions can be computed exactly, and involve higher and higher eigenvalues and eigenfunctions of the Schrödinger operator.

The possibility to solve exactly the dynamics of Kirman’s model is of course intellectually satisfying. It is also important in view of the number of possible applications of such a model, recalled in the introduction, and which has reappeared recently in the context of self-fulfilling prophecies in a simple economic model [28] and in the empirical study of the dynamics of fishers seeking to exploit fishing zones with finite resources [29]. Our analytical approach, while similar to the techniques used in [25], can also be easily generalized to other models of genetic population or social dynamics, such as the general setting discussed in [17], as the change of variable we introduce always leads to a Schrödinger equation with a trigonometric potential provided the drift is linear in  $x$ . These equations may then be solved using known analytical tools [30].

A number of natural extensions to this model can be imagined. For example, the original version of Kirman’s model does not take into account the heterogeneity in encounter probabilities induced by the topology of the social network; but one can easily (at least numerically!) modulate the probability of encounters according to their distance along such a network, for example restricting recruitments to nearest neighbours only. Similarly, one can imagine extending the model to a setting where the ants have not two but  $M > 2$  possibly heterogeneous choices for where they get their food. Such explorations, which we leave for later work, would naturally enrich the scope of this extremely simple but extremely rich model.

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## Data Availability

The data that support the findings of this study are available upon reasonable request from the authors.

## Appendix A. Derivation of the Fokker–Planck equation and stationary solution

We define the continuous distribution  $f(x, t)$  as:

$$f(x, t) = \lim_{N \rightarrow \infty} \sum_{k=0}^N \delta\left(x - \frac{k}{N}\right) \mathbf{P}(k, t), \quad (\text{A1})$$

which amounts to replacing  $\frac{k}{N}$  by  $x$  in equations (1) and (2). In this case, and to leading order in  $\frac{1}{N}$ , the term e.g.  $W(k+1 \rightarrow k)\mathbf{P}(k+1, t)$  reads:

$$\left(1 - \left(x + \frac{1}{N}\right)\right) \left(\varepsilon + \frac{\mu}{N} \left(x + \frac{1}{N}\right)\right) f\left(x + \frac{1}{N}, t\right). \quad (\text{A2})$$

We proceed similarly for all terms in the right-hand side of equation (1), and Taylor-expand the left-hand side to leading order in the time variable, to obtain:

$$\begin{aligned} \partial_t f(x, t) = & \frac{\varepsilon}{\Delta} \left[ (x + \Delta) f(x + \Delta, t) - x f(x, t) - (1 - x) f(x, t) + (1 - (x - \Delta)) f(x - \Delta, t) \right] \\ & + \frac{\mu}{\Delta^2} \left[ (x + \Delta) (1 - (x + \Delta)) f(x + \Delta, t) + (x - \Delta) (1 - (x - \Delta)) f(x - \Delta, t) - 2x(1 - x) f(x, t) \right], \end{aligned} \quad (\text{A3})$$

where  $\Delta = \frac{1}{N}$  for simplicity. We next Taylor-expand the right-hand side terms, such as e.g.  $(x + \Delta) f(x + \Delta, t) \approx x f(x, t) + \Delta \partial_x [x f(x, t)] + \mathcal{O}(\Delta^2)$ , to order  $\Delta$  for the terms with prefactor  $\varepsilon/\Delta$  and to order  $\Delta^2$  for the terms with prefactor  $\mu/\Delta^2$ . Gathering everything, we obtain the Fokker–Planck equation:

$$\partial_t f(x, t) = -\varepsilon \partial_x [(1 - 2x) f(x, t)] + \mu \partial_{xx} [x(1 - x) f(x, t)], \quad (\text{A4})$$

the same as given in equation (3). This equation can be written as  $\partial_t f(x, t) = \partial_x J^f(x, t)$ , where  $J^f$  is the probability flux, a function such that  $J^f(x) \Delta$  corresponds to the probability mass flowing from  $x + \Delta$  to  $x$ . To ensure



the conservation of probability in  $[0; 1]$ , we impose  $J^f = 0$  at the boundaries, meaning that no probability mass comes in or goes out during the dynamic evolution of the process.

In other words, writing  $I_f(t) = \int_0^1 dx f(x, t)$ , direct integration of equation (A4) leads to  $\dot{I}_f(t) = J^f(1, t) - J^f(0, t) = 0$ , ensuring that  $I_f(t) = 1$  at all times. Keeping the next term of order  $\Delta$  only slightly alters the equation:

$$\partial_t f(x, t) = -\varepsilon \partial_x [(1 - 2x)f(x, t)] + \partial_{xx} [(\mu x(1 - x) + \varepsilon \Delta)f(x, t)]. \quad (\text{A5})$$

Recalling now that a Fokker–Planck equation of the form

$$\partial_t p(y, t) = -\partial_y [a(y, t)p(y, t)] + \partial_{yy} [b(y, t)p(y, t)] \quad (\text{A6})$$

corresponds to the Ito stochastic differential equation

$$\dot{y} = a(y, t) + \sqrt{b(y, t)}\eta(t) \quad (\text{A7})$$

where  $\eta$  is a brownian white noise, one readily recovers equation (4). Physically, the 0-flux boundary condition corresponds to a reflecting boundary condition: a ‘wall’ that prevents  $x$  from getting out of  $[0; 1]$ .

### A.1. Determining the stationary solution

Looking for a stationary solution, one sets the right-hand side of equation (A4) to 0, leading to

$$\frac{f'_0(x)}{f_0(x)} = (\alpha - 1) \frac{1 - 2x}{x(1 - x)} \quad \text{with } \alpha := \frac{\varepsilon}{\mu} \quad (\text{A8})$$

which, after direct integration, yields  $f_0(x) \propto (x(1 - x))^{\alpha-1}$ . Integrating for  $x \in [0; 1]$  allows one to find the normalisation constant in terms of the Beta function, or equivalently as a ratio of Gamma functions, to get equation (5).

## Appendix B. Change of variables under an SDE

Obtaining equation (7) and understanding the rationale behind the change of variables of equation (6) is easier by starting from equation (4).

A change of variables  $x \rightarrow \varphi(x)$  leads to a new stochastic differential equation for  $\varphi$ , which after applying the Ito rule for differentiation reads

$$\frac{d\varphi(x)}{dt} = \varepsilon(1 - 2x)\varphi'(x) + \mu x(1 - x)\varphi''(x) + \sqrt{2\mu x(1 - x)}\varphi'(x)\eta(t), \quad (\text{B1})$$

which is still difficult to interpret because of the dependence on  $x$  of the term in front of the white noise  $\eta$ .

Picking however  $\varphi'(x) = \frac{1}{\sqrt{x(1-x)}}$  amounts to  $\varphi(x) = \arcsin(2x - 1)$  and rids us of this dependence. Computing the derivatives  $\varphi' = 2/\cos \varphi$  and  $\varphi'' = -4 \tan \varphi / \cos^2 \varphi$  and replacing in equation (B1):

$$\dot{\varphi} = -(2\varepsilon - \mu) \tan \varphi + \sqrt{2\mu}\eta(t), \quad (\text{B2})$$

which because of the equivalence between stochastic differential equations and Fokker–Planck equations discussed in appendix (A4) leads to equation (7). As before, imposing the reflecting boundary conditions  $J^g(\pm\pi/2, t) = 0$  ensures conservation of probability.

If we keep instead the term of order  $\Delta$  given in equation (A5), we get a Langevin equation

$$\dot{x} = \varepsilon(1 - 2x) + \sqrt{2\mu x(1 - x) + 2\varepsilon\Delta}\eta(t), \quad (\text{B3})$$

which leads to the change of variables

$$\phi = \arctan \left( \frac{2x - 1}{2\sqrt{x(1 - x) + \alpha\Delta}} \right), \quad (\text{B4})$$

where now  $|\phi| \leq \arctan(1/\sqrt{2\alpha\Delta}) \approx \frac{\pi}{2} - 2\sqrt{\alpha\Delta}$ , and naturally one can check that the definition of  $\phi$  corresponds to  $\varphi$  as  $\Delta \rightarrow 0$ , with  $\phi \approx \varphi - 2\alpha\Delta \tan(\varphi)$  to leading order in  $\Delta$ . The analysis in the limit  $N \rightarrow \infty$  therefore holds only in the limit  $\tan(\varphi) \ll \frac{N}{2\alpha}$ .

This new variable actually verifies the very same SDE, equation (B1), but with a different boundary.

## Appendix C. Schrödinger from Fokker–Planck

The following is a common ‘trick’ to transform a non-hermitian dynamic evolution coming from a Fokker–Planck equation with drift into a hermitian evolution determined by a Schrödinger equation. We start from a generic Fokker–Planck equation such as the one defined in equation (A6), but with constant  $b(y, t) = 1$  and time-independent drift, which we represent with the derivative of some function  $A$ ,  $a(y, t) = -A'(y)$ . The resulting Fokker–Planck equation reads

$$\partial_t p(y, t) = \partial_y [A'(y)p(y, t)] + \partial_{yy} p(y, t) \quad (C1)$$

and has a stationary solution that can be written as a Boltzmann distribution  $p_0(y) = e^{-A(y)}/Z$ , where  $Z$  is a constant ensuring normalisation.

We next introduce a function  $\Psi$  verifying  $p(y, t) = e^{-A(y)/2}/\sqrt{Z}\Psi(y, t)$ . We can compute derivatives to find

$$\begin{aligned} \partial_y [A'(y)p(y, t)] &= e^{-A(y)/2}/\sqrt{Z} \left[ \left( A''(y) - \frac{A'(y)^2}{2} \right) \Psi(y, t) + A'(y)\partial_y \Psi(y, t) \right] \\ \partial_{yy} p(y, t) &= e^{-A(y)/2}/\sqrt{Z} \left[ -\frac{1}{2} \left( A''(y) - \frac{A'(y)^2}{2} \right) \Psi(y, t) - A'(y)\partial_y \Psi(y, t) + \partial_{yy} \Psi(y, t) \right]. \end{aligned} \quad (C2)$$

Adding these terms and simplifying, we find the following Schrödinger’s equation for  $\Psi$ :

$$-\partial_t \Psi(y, t) = \mathbf{H}\Psi, \quad (C3)$$

where the Hamiltonian is here defined as

$$\mathbf{H} = -\partial_{yy} + V(y), \quad V(y) = -\frac{1}{2} \left( A''(y) - \frac{A'(y)^2}{2} \right). \quad (C4)$$

Equation (10) simply uses this substitution, with  $\int d\varphi \tan \varphi = \log \cos \varphi$  playing the role of  $A(y)$  (up to a multiplicative constant).

## Appendix D. Properties of the solution

We take the solutions in equation (14) as those given in [24]. We first check that they satisfy the boundary condition.

### D.1. Checking the boundary condition

We recall that

$${}_2F_1(-k, a; b, u) = \sum_{\ell=0}^k \binom{k}{\ell} (-1)^\ell \frac{\Gamma(a+\ell)}{\Gamma(a)} \frac{\Gamma(b)}{\Gamma(b+\ell)} u^\ell. \quad (D1)$$

In this case, direct differentiation in equation (14) for e.g. even modes  $n = 2k$  in the limit  $\varphi \rightarrow \pm \frac{\pi}{2}$  leads to

$$\begin{aligned} \frac{d}{d\varphi} \left( {}_2F_1 \left( -k, \beta + k; \beta + \frac{1}{2}, \cos^2 \varphi \right) \right) &= 2 \sin \varphi \cos \varphi \frac{k(\beta + k)}{\beta + 1/2} + \mathcal{O}(\cos \varphi) \\ &\approx \pm 2 \left( \frac{\pi}{2} \mp \varphi \right) \frac{k(\beta + k)}{\beta + 1/2}. \end{aligned} \quad (D2)$$

With this one can directly compute, with  ${}_2F_1(-k, \beta + k; \beta + \frac{1}{2}, 1) := c_1$  and for  $\varphi \rightarrow \pm \frac{\pi}{2}$ :

$$\begin{aligned} \beta \tan \varphi \Psi_{2k}(\varphi) + \Psi'_{2k}(\varphi) &\approx \beta \tan \varphi \cos^\beta \varphi c_1 - \beta \tan \varphi \cos^\beta \varphi c_1 \pm 2 \cos^\beta \varphi \left( \frac{\pi}{2} \mp \varphi \right) \frac{k(\beta + k)}{\beta + 1/2} \\ &\approx \pm 2 \frac{k(\beta + k)}{\beta + 1/2} \left( \frac{\pi}{2} \mp \varphi \right)^{1+\beta}, \end{aligned} \quad (D3)$$

which after multiplication with  $\cos^\beta \varphi \approx (\frac{\pi}{2} \mp \varphi)^\beta$  proves equation (15) for  $n = 2k$ . The proof for odd  $n = 2k + 1$  is strictly equivalent. It therefore follows that the solutions of [24], although found initially for vanishing boundary conditions, also satisfy the boundary condition given in equation (12).

## D.2. Explicit expressions

In this section we discuss the explicit expressions of the functions  $f_n$  and the constants  $A_n$ .

The constants  $A_n$  are set so that  $\int_{-\pi/2}^{\pi/2} \Psi_n \Psi_m = \delta_{n,m}$ , and therefore implies, in terms of the variable  $\alpha$ ,

$$A_{2k}(\alpha) = \left( \int_{-\pi/2}^{\pi/2} d\varphi \cos^{2\alpha-1} \varphi {}_2F_1 \left( -k, \alpha + k - \frac{1}{2}; \alpha, \cos^2 \varphi \right)^2 \right)^{-1/2} \quad (D4)$$

$$A_{2k+1}(\alpha) = \left( \int_{-\pi/2}^{\pi/2} d\varphi \cos^{2\alpha-1} \varphi \sin^2 \varphi {}_2F_1 \left( -k, \alpha + k + \frac{1}{2}; \alpha, \cos^2 \varphi \right)^2 \right)^{-1/2}.$$

To substitute and find the expressions of  $f_n(x)$ , we recall that

$$\sin \varphi = 2x - 1, \quad \cos \varphi = 2\sqrt{x(1-x)} \quad (D5)$$

and get, using equation (14) and replacing into  $f_n(x) = \frac{\sqrt{g_0(\varphi(x))} \Psi_n(\varphi(x))}{2\sqrt{x(1-x)}}$ , the explicit expression

$$f_{2k}(x) = A_{2k}(\alpha) \sqrt{\frac{\Gamma(\alpha + 1/2)}{\sqrt{\pi}\Gamma(\alpha)}} {}_2F_1 \left( -k, \alpha + k - \frac{1}{2}; \alpha, 4x(1-x) \right) (4x(1-x))^{\alpha-1} \quad (D6)$$

$$f_{2k+1}(x) = A_{2k+1}(\alpha) \sqrt{\frac{\Gamma(\alpha + 1/2)}{\sqrt{\pi}\Gamma(\alpha)}} {}_2F_1 \left( -k, \alpha + k + \frac{1}{2}; \alpha, 4x(1-x) \right) (4x(1-x))^{\alpha-1} (2x-1). \quad (D7)$$

## D.3. Computing the moments of the distribution

To understand the dynamics of the moments of the distribution

$$\mathbb{E}[x^m(t)] = \int_0^1 dx f(x, t) x^m \quad (D8)$$

it is necessary to understand the behaviour of  $B_{n,m} = \int_0^1 dx f_n(x) x^m$ . Owing to the parity of  $f_n(x)$  with respect to  $x = 1/2$  it is clear that for even moments  $m = 2p$  only even modes  $n = 2k$  will be non zero, and vice versa for odd moments and modes.

We therefore develop the computation of even moments only, as the extension to odd moments is direct. We wish to evaluate the integral  $\int_0^1 dx f_{2k}(x) x^{2p} = 2 \int_0^{1/2} dx f_{2k}(x) x^{2p}$ , after changing variables as  $t = 4x(1-x)$ , it is clear that this integral is proportional to

$$I_{2p,2m} = \int_0^1 dt {}_2F_1 \left( -k, \alpha + k - \frac{1}{2}; \alpha, t \right) t^{\alpha-1} (1-t)^{p-1/2}. \quad (D9)$$

After expanding the hypergeometric function and integrating explicitly, we find

$$I_{2k,2p} = \frac{\Gamma(\alpha)\Gamma(1/2+p)}{\Gamma(\beta+k)} \sum_{l=0}^k \binom{k}{l} (-1)^l \frac{\Gamma(\beta+k+l)}{\Gamma(\beta+1+l+p)} = \frac{\Gamma(\alpha)\Gamma(1/2+p)}{\Gamma(\beta+k)} S_{2k,2p} \quad (D10)$$

requiring then the explicit computation of the sum  $S_{2k,2p}$ .

Mind that for  $m = 2p + 1$  the only modes that contribute are  $n = 2k + 1$ , and the equivalent of the previous integral is

$$I_{2k+1,2p+1} = \frac{\Gamma(\alpha)\Gamma(3/2+p)}{\Gamma(\beta+k+1)} \sum_{l=0}^k \binom{k}{l} (-1)^l \frac{\Gamma(\beta+1+k+l)}{\Gamma(\beta+2+l+p)} = \frac{\Gamma(\alpha)\Gamma(3/2+p)}{\Gamma(\beta+k+1)} S_{2k+1,2p+1} \quad (D11)$$

We discuss this for  $k \geq 1$  in two situations,  $k > m$  and  $k \leq m$ .

*D.3.1. First case:  $k > m$ .* We can then write the sum  $S_{2k,2p}$  as

$$\sum_{l=0}^k \binom{k}{l} (-1)^l \prod_{i=p+1}^{k-1} (\beta + l + i), \quad (D12)$$

which, written as such, leads us to introduce the function

$$P(X) = \sum_{l=0}^k \binom{k}{l} (-1)^l X^{\beta+l+k-1} = X^{\beta+k-1} (1-X)^k. \quad (D13)$$

Applying the generalized Leibniz rule to compute the  $k - p - 1$ th derivative of this function, we obtain directly that  $S_{2k,2p} = P^{(k-p-1)}(1) = 0$  in this case. A similar calculation can be done for  $S_{2k+1,2p+1}$ , and it follows therefore that

$$\int_0^1 dx f_n(x) x^m = 0 \quad \text{for } n > m. \quad (\text{D14})$$

*D.3.2. Second case:  $k \leq m$ .* In this case, we now write the sum as

$$\sum_{l=0}^k \binom{k}{l} (-1)^l \prod_{i=k}^p \frac{1}{\beta + l + i}, \quad (\text{D15})$$

which can instead be seen as the result of successive integrations on the function defined in equation (D13).

To compute it, we define the functions  $B_0(t; a, b) = t^{a-1}(1-t)^{b-1}$  and  $B_{n+1}(t; a, b) = \int_0^t du B_n(u; a, b)$ , with  $B_1$  corresponding to the standard incomplete Beta function. With this definition, the sum reads

$$S_{2k,2p} = \int_0^1 du B_{p-k}(u; \beta + k + 1, k + 1), \quad (\text{D16})$$

while on the other hand successive integration by parts gives

$$\begin{aligned} B_n(1; a, b) &= \left[ \sum_{j=0}^{n-1} (-1)^{j+1} \frac{(t-1)^{j+1}}{\Gamma(j+2)} B_{n-j}(u; a, b) \right]_0^1 + (-1)^n \int_0^1 du \frac{(t-1)^n}{\Gamma(n+1)} B_0(u; a, b) \\ &= \frac{B_1(n+a, b)}{\Gamma(n+1)}. \end{aligned} \quad (\text{D17})$$

Finally, gathering everything we get

$$S_{2k,2p} = \frac{\Gamma(p+\beta+1)\Gamma(k+1)}{\Gamma(p+\beta+k+2)\Gamma(p-k+1)}, \quad (\text{D18})$$

while replacing  $\beta \rightarrow \beta + 1$  gives the similar expression

$$S_{2k+1,2p+1} = \frac{\Gamma(p+\beta+2)\Gamma(k+1)}{\Gamma(p+\beta+k+3)\Gamma(p-k+1)}. \quad (\text{D19})$$

The final result follows,

$$B_{n,m} = \int_0^1 dx f_n(x) x^m = A_n(\alpha) \sqrt{\frac{\Gamma(\alpha+1/2)}{\sqrt{\pi}\Gamma(\alpha)}} I_{n,m} \mathbf{1}(n \leq m) \quad (\text{D20})$$

with

$$\begin{aligned} I_{2k,2p+1} &= 0 \\ I_{2k,2p} &= \frac{\Gamma(\alpha)\Gamma(1/2+p)\Gamma(p+\beta+1)\Gamma(k+1)}{\Gamma(\beta+k)\Gamma(p+\beta+k+2)\Gamma(p-k+1)} \\ I_{2k+1,2p+1} &= \frac{\Gamma(\alpha)\Gamma(3/2+p)\Gamma(p+\beta+2)\Gamma(k+1)}{\Gamma(\beta+k+1)\Gamma(p+\beta+k+3)\Gamma(p-k+1)}, \end{aligned} \quad (\text{D21})$$

allowing then for explicit computation of the dynamics of  $\mathbb{E}[x^m(t)]$ .

## Appendix E. Stochastic calculus techniques

In this appendix, we shall directly integrate stochastic differential equations describing the model to obtain information on the covariances of moments  $x^n(t)$ . We begin by looking at the covariance  $\text{Cov}(x(t+T), x(T))$ .

A direct integration of equation (4) leads to

$$x(t+T) = x(T) + \varepsilon t - 2\varepsilon \int_T^{t+T} ds x(s) + \int_T^{t+T} ds \sqrt{2\mu x(s)(1-x(s))} \eta(s). \quad (\text{E1})$$

Taking now the covariance with  $x(t)$  and using linearity,

$$\text{Cov}(x(t+T), x(T)) = \text{Cov}(x(T), x(T)) - 2\varepsilon \int_T^{t+T} ds \text{Cov}(x(s), x(T)) + \int_T^{t+T} ds \mathbb{E} \left[ \sqrt{2\mu x(s)(1-x(s))} x(T) \eta(s) \right] \quad (\text{E2})$$

with the last integral being equal to 0, as

$$\mathbb{E} \left[ \sqrt{2\mu x(s)(1-x(s))} x(T) \eta(s) \right] = \mathbb{E} \left[ \sqrt{2\mu x(s)(1-x(s))} x(T) \right] \mathbb{E} [\eta(s)] = 0. \quad (\text{E3})$$

Taking finally the derivative with respect to  $t$  and solving the resulting differential equation we find

$$\begin{aligned} \frac{d}{ds} \text{Cov}(x(T+s), x(T)) &= -2\varepsilon \text{Cov}(x(T+s), x(T)) \\ \text{Cov}(x(t+T), x(T)) &\propto e^{-2\varepsilon t}. \end{aligned} \quad (\text{E4})$$

Similarly, one can derive the stochastic differential equation followed by  $\sigma_2(x) = x(1-x)$  using the differentiation rule exemplified in equation (B1), namely

$$\frac{d[x(1-x)]}{dt} = \varepsilon - (4\varepsilon + 2\mu)x(1-x) + \sqrt{2\mu x(1-x)}(1-2x)\eta(t), \quad (\text{E5})$$

and as before, we can take the covariance  $\text{Cov}(\sigma_2(x(t+T)), \sigma_2(x(T)))$ , differentiate with respect to  $t$  and find that it satisfies a differential equation, which after integrating reads

$$\text{Cov}(\sigma_2(x(t+T)), \sigma_2(x(T))) \propto e^{-(4\varepsilon+2\mu)t}. \quad (\text{E6})$$

This method can be extended to computing  $C_{n,k}(t+T, T) = \text{Cov}(x(t+T)^n, x(T)^k)$ . Applying Ito calculus as before, one can show that these functions satisfy the following ODE system:

$$\frac{d}{ds} [C_{n,k}(T+s, T)] = -\mu \mathcal{E}_n C_{n,k}(T+s, T) + \mu n(n-1+\alpha) C_{n-2,k}(T+s, T). \quad (\text{E7})$$

Owing to its triangular structure, it can be diagonalized iteratively to find functions  $\sigma_n$ , such that  $\sigma_n(x)$  is a polynomial of degree  $n$  and that the covariances  $C_{\sigma_n}(T+s, T) = \text{Cov}[\sigma_n(x(T+s)), \sigma_n(x(T))]$  satisfy

$$\frac{d}{ds} C_{\sigma_n}(T+s, T) = -\mu \mathcal{E}_n C_{\sigma_n}(T+s, T). \quad (\text{E8})$$

Knowing that  $\sigma_1(x) = x$  and  $\sigma_2(x) = x(1-x)$ , it is possible to find the third combination  $\sigma_3(x) = (2x-1)[(1+\frac{2\alpha}{3})(2x-1)^2-1]$ . Integrating the equations in equation (E8), one finds then that

$$C_{\sigma_n}(t+T, T) \propto e^{-\mu \mathcal{E}_n t}. \quad (\text{E9})$$

These results can also be obtained directly from the eigenvalues and eigenfunctions of the Schrödinger problem.

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