



# On the divisibility of characteristic classes of non-oriented surface bundles

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## ABSTRACT

In this note we introduce a construction which assigns to an arbitrary manifold bundle its fiberwise orientation covering. This is used to show that the zeta classes of non-oriented surface bundles are not divisible in the stable range.

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## 1. Introduction

The mapping class group  $\mathcal{N}_g$  of a non-orientable surface  $S_g$  of genus  $g$  (that is, the connected sum of  $g$  copies of  $\mathbb{RP}^2$ ) is defined to be

$$\mathcal{N}_g := \pi_0(\text{Diff}(S_g)),$$

the group of components of the diffeomorphism group of that surface. If  $g \geq 3$ , the components of  $\text{Diff}(S_g)$  are contractible [3], hence  $B\mathcal{N}_g \simeq B\text{Diff}(S_g)$ , and so the cohomology of  $B\mathcal{N}_g$  (or the group cohomology of  $\mathcal{N}_g$ ) can be interpreted as the ring of characteristic classes for  $S_g$ -bundles.

Wahl [10] has proved a homological stability theorem for these groups, which says that in degrees  $* \leq (g-3)/4$  the cohomology groups  $H^*(\mathcal{N}_g)$  are independent of the genus  $g$ . We call this range of degrees the *stable range*. Combining Wahl's result with that of Galatius, Madsen, Tillmann and Weiss [7], the stable rational cohomology of these groups can be identified: there are certain integrally defined characteristic classes  $\zeta_i$  in degrees  $4i$  (defined in Section 3) and the map

$$\mathbb{Q}[\zeta_1, \zeta_2, \zeta_3, \dots] \rightarrow H^*(\mathcal{N}_g; \mathbb{Q})$$

is an isomorphism in the stable range. In [9] the second author calculates these stable groups with coefficients in a finite field, and tabulates some low-dimensional integral groups.

The classes  $\zeta_i$  are analogues of the even Miller–Morita–Mumford classes, for non-oriented surface bundles. Galatius, Madsen and Tillmann [6] have studied the divisibility of the Miller–Morita–Mumford classes  $\kappa_i \in H^*(\Gamma_\infty; \mathbb{Z})$  in the free

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quotient of the integral cohomology of the stable mapping class group  $\Gamma_\infty$ . They find that the even classes are divisible by 2 and the odd classes are divisible by a denominator of a Bernoulli number. In [5] the first author studied the divisibility of the Miller–Morita–Mumford classes for surface bundles with spin structures, and it was shown that the divisibility increases by a certain power of 2 relative to the non-spin case. Continuing the study of divisibility of characteristic classes of surface bundles, we prove

**Theorem A.** *The universal zeta classes,  $\zeta_n \in H^{4n}(\mathcal{N}_g; \mathbb{Z})$ , are not divisible in the stable range. Indeed, they are not divisible in the free quotient  $H_{\text{free}}^{4n}(\mathcal{N}_g; \mathbb{Z})$  of  $H^{4n}(\mathcal{N}_g; \mathbb{Z})$  in this range.*

This gives the trend that extra structure on the vertical tangent bundle, such as an orientation or a spin structure, gives extra divisibility of characteristic classes of surface bundles.

## 2. Lifting diffeomorphisms to orientation coverings

In this section, we will construct a natural homomorphism from the diffeomorphism group  $\text{Diff}(M)$  of a smooth  $d$ -manifold to the group  $\text{Diff}^+(\tilde{M})$  of orientation-preserving diffeomorphisms of the orientation covering of  $M$ . This implies that any smooth fiber bundle  $p: E \rightarrow B$  admits a two-fold covering  $\pi: \tilde{E} \rightarrow E$ , such that  $p \circ \pi: \tilde{E} \rightarrow B$  is an oriented smooth fiber bundle and that the restriction of  $\pi: \tilde{E} \rightarrow E$  to a fiber of  $p$  is the orientation covering.

Let  $M$  be a smooth  $d$ -manifold,  $d > 0$ , and let  $\Lambda^d TM$  be the highest exterior power of the tangent bundle, which is a real line bundle. The total space of the orientation covering of  $M$  can be defined as the quotient

$$\tilde{M} := (\Lambda^d TM \setminus 0) / \mathbb{R}_{>0}. \quad (2.1)$$

The canonical map  $\pi: \tilde{M} \rightarrow M$  is a two-sheeted covering. The space  $\tilde{M}$  is a smooth oriented manifold with a preferred orientation. To see this, recall that an orientation of a  $d$ -dimensional real vector space  $V$  is a component of  $\Lambda^d V \setminus 0$ , or in other words, one of the two points of  $(\Lambda^d V \setminus 0) / \mathbb{R}_{>0}$ . Thus a point in  $x \in \tilde{M}$  is by definition an orientation of the tangent space  $T_{\pi(x)}M$ . The differential  $T_x\pi$  at  $x \in \tilde{M}$  is a linear isomorphism  $T_x\tilde{M} \rightarrow T_{\pi(x)}M$  so the orientation of  $T_{\pi(x)}M$  given by  $x$  gives us a preferred orientation of  $T_x\tilde{M}$ . Using local coordinates on  $M$ , it is easy to see that these orientations of the tangent spaces  $T_x\tilde{M}$  fit together continuously and define an orientation of  $\tilde{M}$ , the *preferred orientation*.

Moreover, this construction is natural: a diffeomorphism  $f: M \rightarrow N$  of smooth manifolds induces a diffeomorphism  $\tilde{f}: \tilde{M} \rightarrow \tilde{N}$  which covers  $f$ . It is easy to see that  $\tilde{f}$  is orientation-preserving provided  $\tilde{M}$  and  $\tilde{N}$  are endowed with the preferred orientations. If  $g: N \rightarrow P$  is another diffeomorphism, then  $\widetilde{g \circ f} = \tilde{g} \circ \tilde{f}$ . Also,  $\widetilde{\text{id}_M} = \text{id}_{\tilde{M}}$ . Finally, we did not use that  $f$  is a diffeomorphism, but only that the differential of  $f$  was nonsingular. It follows that the assignments  $M \mapsto \tilde{M}$  and  $f \mapsto \tilde{f}$  define a functor  $\mathcal{L}$  from the category  $\mathcal{X}_d$  of smooth  $d$ -manifolds and local diffeomorphisms to the category  $\mathcal{X}_d^+$  of oriented  $d$ -manifolds and orientation-preserving local diffeomorphisms. In particular, we defined a group homomorphism  $\mathcal{L}_M: \text{Diff}(M) \rightarrow \text{Diff}^+(\tilde{M})$ .

For a manifold  $M$ , we denote by  $\pi_M$  the covering map  $\tilde{M} \rightarrow M$  and by  $\iota_M: \tilde{M} \rightarrow \tilde{M}$  the unique nontrivial deck transformation. If  $f: M \rightarrow N$  is a (local) diffeomorphism, the following relations hold

$$\pi_N \circ \tilde{f} = f \circ \pi_M; \quad \tilde{f} \circ \iota_M = \iota_N \circ \tilde{f}. \quad (2.2)$$

The morphism spaces of the categories  $\mathcal{X}_d$  and  $\mathcal{X}_d^+$  have a natural topology, the weak  $C^\infty$ -topology, with respect to which the composition maps are continuous. Thus  $\mathcal{X}_d$  and  $\mathcal{X}_d^+$  are topological categories. Using local coordinates, it is easy to see that the functor  $\mathcal{L}$  is continuous. In particular, the homomorphism  $\mathcal{L}_M: \text{Diff}(M) \rightarrow \text{Diff}^+(\tilde{M})$  is continuous.

Let us now discuss smooth fiber bundles. Let  $p: E \rightarrow B$  be a smooth fiber bundle with fiber a  $d$ -dimensional smooth manifold  $M$  and structural group  $\text{Diff}(M)$  (with the weak  $C^\infty$ -topology). Consider the associated  $\text{Diff}(M)$ -principal bundle  $Q \rightarrow B$ , which has the property that  $Q \times_{\text{Diff}(M)} M \cong E$ . Via the homomorphism  $\mathcal{L}_M$ , the manifold  $\tilde{M}$  has a  $\text{Diff}(M)$ -action by orientation-preserving diffeomorphisms. Hence the fiber bundle

$$q: \tilde{E} := Q \times_{\text{Diff}(M)} \tilde{M} \rightarrow B$$

is an oriented smooth fiber bundle with fiber  $\tilde{M}$ . Because of (2.2), there is a twofold covering  $\pi_E: \tilde{E} \rightarrow E$ , such that  $q = p \circ \pi_E$ . Furthermore, there is a fiber-preserving and orientation-reversing involution  $\iota_E$  on  $\tilde{E}$ . We call  $\tilde{E}$  the *fiberwise orientation cover* of  $E$ . We summarize the results of this section.

**Theorem 2.1.** *The fiberwise orientation covering  $\pi_E: \tilde{E} \rightarrow E$  of a smooth fiber bundle  $p: E \rightarrow B$  is a two-sheeted covering whose restriction to any fiber  $E_b$  of  $p$  is the orientation covering of  $E_b$ . The composition  $q = p \circ \pi_E$  is an oriented fiber bundle. Furthermore,  $\tilde{E}$  and  $\pi_E$  are uniquely determined by these properties (up to orientation-preserving isomorphism).*

We conclude with a simple remark. All the constructions in this section make sense when the manifold  $M$  (or the fiber bundle  $E$ ) is orientable. If this is the case, then  $\tilde{M}$  is the disjoint sum of two copies of  $M$ . The choice of an orientation of  $M$  singles out a component of  $\tilde{M}$ .

### 3. Characteristic classes of surface bundles

In this section, we give a brief review of the theory of characteristic classes of surface bundles, both oriented and non-oriented. First we discuss the oriented case. Let  $\pi : E \rightarrow B$  be an oriented surface bundle and let  $T_\nu E$  be the vertical tangent bundle. It is an oriented 2-dimensional real vector bundle on  $E$  and thus it has an Euler class  $e(T_\nu(E)) \in H^2(E; \mathbb{Z})$ . We can consider  $T_\nu E$  also as a complex line bundle (there is a complex structure on it, which is unique up to isomorphism) and the Euler class agrees with the first Chern class. The Miller–Morita–Mumford classes are defined to be

$$\kappa_n(E) := \pi_!(e(T_\nu E)^{n+1}) \in H^{2n}(B; \mathbb{Z}),$$

where  $\pi_! : H^*(E; \mathbb{Z}) \rightarrow H^{*-2}(B; \mathbb{Z})$  is the umkehr, or cohomological fiber-integration, map. This definition cannot be generalized to the non-oriented case without further effort, because both the Euler class and the umkehr map only exist for oriented surface bundles.

The concept needed for a generalization is the Becker–Gottlieb transfer [1]. Let  $p : E \rightarrow B$  be a smooth fiber bundle with compact fibers diffeomorphic to  $F$  (not necessarily of dimension 2). The transfer is a stable map in the converse direction, more precisely, it is a map of suspension spectra

$$\mathrm{trf}_p : \Sigma^\infty B_+ \rightarrow \Sigma^\infty E_+.$$

Recall that the spectrum cohomology of the suspension spectrum of a space  $\Sigma^\infty X_+$  agrees with the usual cohomology of the space  $X$ . Thus we can form the map  $\mathrm{trf}_p^* \circ p^* : H^*(B; \mathbb{Z}) \rightarrow H^*(B; \mathbb{Z})$ , and for all  $x \in H^*(B; \mathbb{Z})$  we have

$$\mathrm{trf}_p^* \circ p^*(x) = \chi(F) \cdot x, \quad (3.1)$$

where  $\chi(F)$  denotes the Euler number of the fiber [1, Theorem 5.5]. Furthermore, if  $q : \tilde{E} \rightarrow E$  is another smooth fiber bundle with compact fibers, then  $p \circ q$  is also such a fiber bundle. In this situation the composition of the transfers is homotopic to the transfer of the composition (see [2, Eq. (2.3), p. 137]):

$$\mathrm{trf}_{p \circ q} \simeq \mathrm{trf}_q \circ \mathrm{trf}_p. \quad (3.2)$$

A diffeomorphism  $f : M \rightarrow N$  of manifolds can be considered as a fiber bundle whose fiber is a point. By (3.1),

$$\mathrm{trf}_f^* \circ f^* = \mathrm{id}_{H^*(N; \mathbb{Z})}, \quad f^* \circ \mathrm{trf}_f^* = \mathrm{id}_{H^*(M; \mathbb{Z})}. \quad (3.3)$$

In fact,  $\mathrm{trf}_f$  and  $\Sigma^\infty(f^{-1})$  are homotopic, but we do not need this fact. The transfer of an oriented fiber bundle  $p : E \rightarrow B$  is closely related to the umkehr map. For all  $x \in H^*(E; \mathbb{Z})$ , one has (see [1, Theorem 4.3])

$$\mathrm{trf}_p^*(x) = p_!(x \cup e(T_\nu E)). \quad (3.4)$$

The identity (3.4) implies that

$$\kappa_n(E) = \mathrm{trf}_p^*(e(T_\nu E)^n) \quad (3.5)$$

for the Miller–Morita–Mumford classes of an oriented surface bundle  $p : E \rightarrow B$ . Because of the identity  $p_1(L) = e(L)^2$  for the Pontrjagin class of a 2-dimensional oriented real vector bundle  $L$ , we see that

$$\kappa_{2n}(E) = \mathrm{trf}_p^*(p_1(T_\nu E)^n). \quad (3.6)$$

This can be generalized to the non-oriented case. Wahl defines [10, p. 3]

$$\zeta_i(E) := \mathrm{trf}_p^*(p_1(T_\nu E)^i) \in H^{4i}(B; \mathbb{Z}), \quad (3.7)$$

for a non-oriented surface bundle  $p : E \rightarrow B$ , where  $p_1(T_\nu E) \in H^4(E; \mathbb{Z})$  is the first Pontrjagin class of the vertical tangent bundle.

The spaces  $B\mathrm{Diff}^+(F_g)$  and  $B\mathrm{Diff}(S_g)$  carry universal oriented and non-orientable surface bundles of a fixed genus, so the above constructions define classes  $\kappa_i \in H^{2i}(B\mathrm{Diff}^+(F_g); \mathbb{Z})$  and  $\zeta_i \in H^{4i}(B\mathrm{Diff}(S_g); \mathbb{Z})$  which we call the *universal classes*, and omit the universal bundle from the notation.

Now we can state and prove the main result of this section.

**Theorem 3.1.** *Let  $p : E \rightarrow B$  be a non-oriented surface bundle with compact fibers and let  $c : \tilde{E} \rightarrow E$  be its fiberwise orientation covering. Denote  $q := p \circ c : \tilde{E} \rightarrow B$ . Then the following relations hold for all  $n \geq 0$ :*

- (1)  $\kappa_{2n}(\tilde{E}) = 2 \cdot \zeta_n(E)$ .
- (2)  $2 \cdot \kappa_{2n+1}(\tilde{E}) = 0$ .

**Proof.** For the identity (1), we compute

$$\begin{aligned}\kappa_{2n}(\tilde{E}) &= \text{tr}_q^*(p_1(T_V \tilde{E})^n) \\ &= \text{tr}_p^*(\text{tr}_c^*(c^*(p_1(T_V E)^n))) \\ &= \text{tr}_p^*(2 \cdot p_1(T_V E)^n) \\ &= 2 \cdot \zeta_n(E).\end{aligned}$$

The first equality is (3.6). Because  $c: \tilde{E} \rightarrow E$  is a smooth covering in every fiber,  $c^*(T_V E) \cong T_V(\tilde{E})$ , whence  $p_1(T_V \tilde{E}) = c^*(p_1(T_V E))$ . Together with (3.2), this fact implies the second equality. Because  $c$  is a double covering, the Euler number of its fiber is 2. Thus  $\text{tr}_c^* \circ c^* = 2$ , which gives the third equality. The fourth equality is the definition.

For the proof of identity (2), we use the orientation-reversing involution  $\iota$  on  $\tilde{E}$ . By (3.3),  $\text{tr}_\iota^* = (\iota^*)^{-1} = \iota^*$ . Because  $c \circ \iota = c$ , it follows that  $\text{tr}_c^* = \text{tr}_c^* \circ \text{tr}_\iota^* = \text{tr}_c^* \circ \iota^*$ . Because  $\iota$  is an orientation-reversing fiberwise diffeomorphism, it induces an orientation-reversing vector bundle isomorphism  $d\iota: T_V \tilde{E} \rightarrow \iota^* T_V \tilde{E}$ . Thus  $e(T_V \tilde{E}) = -\iota^* e(T_V \tilde{E})$ . Thus

$$\begin{aligned}\kappa_{2n+1}(\tilde{E}) &= \text{tr}_p^*(\text{tr}_c^*(e(T_V \tilde{E})^{2n+1})) \\ &= \text{tr}_p^*(\text{tr}_c^*(\iota^*(e(T_V \tilde{E})^{2n+1}))) \\ &= (-1)^{2n+1} \text{tr}_p^*(\text{tr}_c^*(e(T_V \tilde{E})^{2n+1})) \\ &= -\kappa_{2n+1}(\tilde{E}). \quad \square\end{aligned}$$

**Remark 3.2.** An implication of this theorem is that for an oriented surface bundle  $E' \rightarrow B$ , the characteristic classes  $2 \cdot \kappa_{2n+1}(E')$  are obstructions to  $E'$  admitting an orientation-reversing, fixed-point free, fiberwise involution. Furthermore, for bundles which do admit such an involution, it gives an interpretation of  $\frac{1}{2}\kappa_{2n}(E')$  as the zeta classes of the associated quotient bundle of non-orientable surfaces.

#### 4. An example

In this section, we consider the example of a genus zero surface bundle. Let  $\gamma_3 \rightarrow BSO(3)$  be the universal 3-dimensional oriented Riemannian real vector bundle and let  $\mathbb{S}(\gamma_3) \rightarrow BSO(3)$  be its unit sphere bundle. It is known that this is the universal smooth oriented bundle with fiber  $S^2$ , but we do not need this fact. In [4, Proposition 5.2.4] the first author has computed that  $\kappa_{2n}(\mathbb{S}(\gamma_3)) = 2p_1^n$ . The bundle  $\mathbb{S}(\gamma_3)$  admits an orientation-reversing, fixed-point free involution on its fibers, namely the antipodal map  $-\text{id}$ . The quotient is  $\mathbb{P}(\gamma_3)$ , the  $\mathbb{RP}^2$ -bundle associated to  $\gamma_3$ . By Theorem 3.1, we have

$$2\zeta_n(\mathbb{P}(\gamma_3)) = 2p_1^n. \quad (4.1)$$

It is well known that the free quotient  $H_{\text{free}}^*(BSO(3); \mathbb{Z})$  is the polynomial algebra  $\mathbb{Z}[p_1]$ . In particular, the powers  $p_1^n$  are not divisible in the free quotient of  $H^*(BSO(3); \mathbb{Z})$ . We have shown:

**Proposition 4.1.** *The class  $\zeta_n(\mathbb{P}(\gamma_3))$  is not divisible in  $H_{\text{free}}^*(BSO(3); \mathbb{Z})$ .*

#### 5. A review of the stable homotopy theory of surfaces and proof of Theorem A

In this section, we give a brief introduction to the modern homotopy theory of surface bundles developed by Galatius, Madsen, Tillmann and Weiss. A good survey can be found in [6, Sections 2 and 3],<sup>3</sup> and full proofs can be found in [7]. Let us first discuss the oriented case.

Consider the universal complex line bundle  $L \rightarrow BSO(2)$ . There does not exist a vector bundle  $V$  such that  $V \oplus L$  is trivial, but we can define an additive inverse  $L^\perp$  of  $L$  as a *stable vector bundle*. The *Madsen–Tillmann spectrum*  $\mathbf{MTSO}(2)$  is defined to be the Thom spectrum of  $L^\perp$ . For any oriented surface bundle  $E \rightarrow B$ , there exists a natural map

$$\alpha_E: B \rightarrow \Omega_0^\infty \mathbf{MTSO}(2)$$

into the unit component of the infinite loop space of the Madsen–Tillmann spectrum. In particular, it can be defined for the universal oriented surface bundle with fibers a surface  $F_g$  of genus  $g$ , to obtain a universal map

$$\alpha_g: B \text{Diff}^+(F_g) \rightarrow \Omega_0^\infty \mathbf{MTSO}(2).$$

For each  $n > 0$ , there is a cohomology class  $y_n \in H^{2n}(\Omega_0^\infty \mathbf{MTSO}(2); \mathbb{Z})$  such that for any surface bundle as above

$$\alpha_E^*(y_n) = \kappa_n(E). \quad (5.1)$$

<sup>3</sup> Note that this paper uses a different notation: they denote  $\mathbf{MTSO}(2)$  by  $\mathbb{CP}_-^\infty$ .

The rational cohomology of  $\Omega_0^\infty \mathbf{MTSO}(2)$  is isomorphic to the polynomial ring  $\mathbb{Q}[y_1, y_2, \dots]$ . The main result of [7], originally due to Madsen and Weiss [8], implies that the map  $\alpha_g$  induces an isomorphism on homology groups in the stable range, that is,

$$H_k(\alpha_g) : H_k(B \operatorname{Diff}^+(F_g); \mathbb{Z}) \rightarrow H_k(\Omega_0^\infty \mathbf{MTSO}(2); \mathbb{Z})$$

is an isomorphism as long as  $g \geq 2k + 2$ .

Similar results hold in the non-oriented case, and are detailed in [10, Section 6]. The Madsen–Tillmann spectrum is replaced by  $\mathbf{MTO}(2)$ , which is the Thom spectrum of the stable inverse of the universal 2-dimensional real vector bundle over  $BO(2)$ . There is an analogue of the map  $\alpha_E$  for any non-oriented surface bundle  $E \rightarrow B$ , and there are classes  $x_n \in H^{4n}(\Omega_0^\infty \mathbf{MTO}(2); \mathbb{Z})$  for  $n > 0$ , such that  $\alpha_E^*(x_n) = \zeta_n(E)$ . The rational cohomology ring of  $\Omega_0^\infty \mathbf{MTO}(2)$  is isomorphic to the polynomial ring  $\mathbb{Q}[x_1, x_2, \dots]$ , in complete analogy to the oriented case.

The analogue of the Madsen–Weiss theorem is also true in the non-oriented case, by [10] and [7]. More precisely

$$H_k(\alpha_g; \mathbb{Z}) : H_k(B \operatorname{Diff}(S_g); \mathbb{Z}) \rightarrow H_k(\Omega_0^\infty \mathbf{MTO}(2); \mathbb{Z}) \quad (5.2)$$

is an isomorphism as long as  $4k + 3 \leq g$ , and similarly in cohomology.

**Proof of Theorem A.** This is now straightforward. We assume that the universal class  $\zeta_n$  is divisible in the stable range. Under the isomorphism (5.2),  $\zeta_n$  corresponds to the class  $x_n \in H^{4n}(\Omega_0^\infty \mathbf{MTO}(2); \mathbb{Z})$ , which must also be divisible. We have seen in Proposition 4.1 that the image of  $x_n \in H^{4n}(\Omega_0^\infty \mathbf{MTO}(2); \mathbb{Z})$  under the map  $\alpha_{\mathbb{P}(\gamma_3)} : BSO(3) \rightarrow \Omega_0^\infty \mathbf{MTO}(2)$  is  $p_1^n$  in the free quotient and so not divisible. This is a contradiction.  $\square$

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