Finiteness Properties and CAT(0) groups

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For my Family.

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Abstract

In this thesis, we explore several areas of geometric topology. We first prove that all groups G which fit into a short exact sequence $F_2 \hookrightarrow G \twoheadrightarrow \mathbb{Z}$, act properly, freely, cellularly and cocompactly on CAT(0) square complexes. This shows, among other things, that their cubical dimension is equal to their geometric dimension.

In the second part, we consider finiteness properties of subgroups of CAT(0) groups. We construct two infinite families of finitely presented subgroups of hyperbolic groups, that are not themselves hyperbolic. We also construct the first examples of CAT(0) groups that do not contain $\mathbb{Z}^3$ subgroups but have subgroups of type FP_2 that are not finitely presented.

We give examples of groups that are of type F_{n-1} not of type F_n and contain no free abelian subgroups of rank $> \lceil \frac{n}{3} \rceil$.

In the final part of the thesis we examine stable diffeomorphisms of smooth 4-manifolds and place an upper bound on the number of $S^2 \times S^2$ summands required to gain a stable diffeomorphism.

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Chapter 1

Introduction

In this thesis we will look at three key areas of geometric topology.

Firstly, we will look at the class of groups of the form $F_2 \rtimes_\phi \mathbb{Z}$ where F_2 is the free group on two generators. We will prove the following.

Theorem A. *If G is of the form $F_2 \rtimes_\phi \mathbb{Z}$, then G is the fundamental group of a compact non-positively curved square complex.*

If a hyperbolic group acts properly and cocompactly on a CAT(0) cube complex, then this group is virtually special [1] and this has some strong group theoretic consequences, such as being residually finite and linear over $\mathbb{Z}$. However, no automorphism of F_2 produces a hyperbolic mapping torus since every automorphism fixes the conjugacy class of $[a, b]^{\pm 1}$. Non-hyperbolic groups acting properly and cocompactly on CAT(0) cube complexes need no longer have such nice properties as hyperbolic ones; for instance they can be simple [15].

Nevertheless, there are many advantages to having a group act on a CAT(0) cube complex. For instance, abelian subgroups are quasi-isometrically embedded, and such groups are biautomatic [22, 29] yielding a deterministic solution to the word problem in quadratic time [20].

Groups which act on CAT(0) square complexes have the further nice property that all of their finitely presented subgroups also act properly and cocompactly on CAT(0) square complexes. This is proved using a tower argument see [12, II.5.27] and the fact that a subcomplex of a non-positively curved square complex is itself a non-positively curved square complex (which in general fails in higher dimensions).

Theorem A also shows that for all F_2-by-$\mathbb{Z}$ groups Γ, the geometric dimension of Γ is equal to its CAT(0) dimension, namely 2.

In the second part of this thesis we will focus on developing some hyperbolisation techniques and studying finiteness properties of subgroups of CAT(0) groups. Recall the following definitions.

Definition 1.0.1. A group is of *type F_n*, if there is a classifying space for the group with finite n-skeleton.

Definition 1.0.2. A group G is of *type FP_n* if there is a partial resolution of the trivial $\mathbb{Z}G$-module $\mathbb{Z}$,

$$P_n \to \cdots \to P_1 \longrightarrow P_0 \longrightarrow \mathbb{Z} \longrightarrow 0$$

where each P_i is a finitely generated projective $\mathbb{Z}G$-module.

It is a well known fact that if a group is of type FP_n and is finitely presented then it is of type F_n. Many interesting examples of groups with exotic finiteness properties were given by Bestvina and Brady [7]. They gave, for each n, examples of groups which are of type F_n but not of type F_{n+1}, as well as the first examples of groups that are of type FP_n but not of type F_n. These examples are subgroups of right angled Artin groups. Similar constructions can be found in [17, 28]. The examples in [7] which are type F_n but not type F_{n+1} are subgroups of Artin groups that contain abelian groups of rank $n+1$. It is of interest to seek examples where this abelian subgroup rank is smaller. Brady [8] shows that in case $n = 2$ there are subhyperbolic examples; so the rank is reduced from 3 to 1. A similar construction was given by Lodha [27]. We construct infinite families of such examples by proceeding as follows.

A non-empty connected graph is *2-full* if the vertices can be partitioned into 2 sets such that the end points of each edge are in distinct sets in this partition. If the graph were simplicial this would be equivalent to bipartite, however, we will reserve the use of the word "bipartite" for simplicial graphs.

Theorem B. *Let Γ_1, Γ_2 and Γ_3 be 2-full graphs where every vertex has valence at least 4. There is a branched cover of $\Gamma_1 \times \Gamma_2 \times \Gamma_3$ with hyperbolic fundamental group which contains a finitely presented subgroup that is not hyperbolic.*

This theorem extends work of Brady [8]. Brady considered the special case where Γ_i are all cage graphs with four edges. Here, we study what happens when these conditions are relaxed and we make considerable use of Brady's methods.

A graph Γ is *sizeable* if it satisfies the following:

- Γ is bipartite on two sets A and B,
- Γ contains no cycles of length 4,
- there exist partitions $A = A^+ \sqcup A^-$ and $B = B^+ \sqcup B^-$, such that $\Gamma(A^s \sqcup B^t)$ is connected for all $s, t \in \{-, +\}$.

We can now state our second theorem giving an infinite family of hyperbolic groups with finitely presented subgroups:

Theorem C. *For $i = 1, 2$ and 3, let Γ_i be a sizeable graph with vertex set $A_i \sqcup B_i$. Let K_{ij} be the complete bipartite graph on A_i and B_j. Let X be the full cubical subcomplex of $K_{13} \times K_{21} \times K_{32}$ spanned by vertices $(v_1, v_2, v_3) \in K_{13} \times K_{21} \times K_{32}$ such that one of the following holds,*

- *$v_i \in A_i$ for all i,*
- *$v_i \in B_{i-1}$ for all i,*
- *$v_1 \in A_1, v_2 \in B_1$ and $[v_1, v_2]$ is an edge of Γ_1,*
- *$v_2 \in A_2, v_3 \in B_2$ and $[v_2, v_3]$ is an edge of Γ_2,*
- *$v_3 \in A_3, v_1 \in B_3$ and $[v_3, v_1]$ is an edge of Γ_3.*

Then X has hyperbolic fundamental group containing a finitely presented subgroup that is not hyperbolic.

In this case the theorem is inspired by work of Lodha [27], in which he considers the special case where Γ_i are all equal. Note also that our sizeable graphs include considerably more general examples than those considered by Lodha. This is possible in light of Theorem 2.4.9.

In both cases we compute the Euler characteristic of the ambient hyperbolic group. This shows that there are infinite families of hyperbolic groups which contain subgroups of type F_2 not F_3. These calculations also show that the groups defined in [8] and [27] are not isomorphic. It would be of interest to show that these groups are not commensurable, however, this work is ongoing.

Each of the methods presented has it's advantages. The first allows the interesting extension we will see in Theorem D′. On the other hand, there is some amount of loss of precision in taking branched coverings. The second method is more concrete in nature since we have an explicit description of the cube complex X. It also allowed for some form of extension to the key theorems of Chapter 8.

In all of the examples discussed above, if a group has a subgroup which is of type FP_2 but not finitely presented then the ambient group has a subgroup which is free abelian of rank 3. We use the following theorem to find CAT(0) groups which contain subgroups that are of type FP_2 but not of type F_2, but do not contain free abelian groups of rank greater than 2.

Theorem D. *Given a tripartite flag complex, there is a branched cover of a classifying space for the associated RAAG that does not contain any flats of dimension* > 2.

Theorem D′. *There exists a non positively curved space* X *such that* $\pi_1(X)$ *contains no subgroups isomorphic to* $\mathbb{Z}^3$*, which contains a subgroup of type* FP_2 *not* F_2.

We hope that a refinement of this construction will, in future, yield examples of hyperbolic groups with subgroups of type FP_2 that are not finitely presented.

Using the above constructions and some insights into Bestvina-Brady Morse theory we can reduce the lowest known rank of free abelian subgroups in groups of type F_{n-1} not of type F_n. This gives a partial answer to a question of Brady [6], who asked whether there exist groups of type F_{n-1} not of type F_n which do not contain $\mathbb{Z}^2$; he notes that the current bound is $\mathbb{Z}^{n-2}$. We improve on this bound considerably.

Theorem E. *There exist groups that have type* F_{n-1} *but not type* F_n *which contain no free abelian subgroups of rank greater than* $\lceil \frac{n}{3} \rceil$.

Finally, we show the failure of certain methods of hyperbolisation in dimensions greater than 3; this shows that our first construction of hyperbolic groups with finitely presented non-hyperbolic subgroups (Theorem B) cannot be generalised to give examples of type F_n but not type F_{n+1} for all n.

Theorem F. *If X is a cube complex of dimension $n > 3$, and there is a cubical subcomplex isometric to T^n, then no non-positively curved branched cover of X is hyperbolic.*

In the final section of this thesis we take a brief trip into the world of smooth 4-manifolds. Smooth 4-manifolds have a rich structure when studied up to diffeomorphism; however when one allows "stabilisation" by taking the connected sum with finitely many copies of $S^2 \times S^2$, this world collapses and there are no exotic structures. This was proved by Wall [33].

Given two homeomorphic 4-manifolds with associated handle decompositions our theorem gives an explicit bound on the number of summands needed for stabilisation. At the time of writing we are not aware of any other literature dealing with this question. However, it is conjectured that the bound should be 1. We denote the number of summands required by k_{Wall}. We denote the intersection form on $H_2(M)$ by Q_M.

Theorem G. *If M and N are homeomorphic simply connected, smooth, closed 4-manifolds and Q_M is indefinite or of rank at most 8, then*

$$k_{\text{Wall}}(M,N) \leq \max_{J=M,N} \left\{ h_1^J + h_3^J + 1 \right\}$$

where h_i^J is the number of i-handles in a handle decomposition of the manifold J.

If Q_M is definite of rank greater than 8, then

$$k_{\text{Wall}}(M,N) \leq \max_{J=M,N} \left\{ h_1^J + h_3^J + 1, 2 \right\}.$$

Chapter 2

Preliminaries

We recall preliminaries that will be required throughout.

2.1 Polyhedral and cube complexes

We will be studying CAT(0) spaces, which are spaces satisfying an inequality saying that geodesic triangles are no fatter than their comparison triangles in Euclidean space. Here is the formal definition.

Definition 2.1.1. Let X be a geodesic metric space. Let Δ a geodesic triangle in X. Let $\overline{\Delta}$ be a comparison triangle in $\mathbb{E}^2$. Then Δ satisfies the *CAT(0) inequality* if for all $x, y \in \Delta$ and all comparison points $\bar{x}, \bar{y} \in \overline{\Delta}$,

$$d(x, y) \leq d(\bar{x}, \bar{y}).$$

X is *CAT(0)* if every geodesic triangle satisfies the CAT(0) inequality.

Definition 2.1.2. A metric space is *non-positively curved* if its metric is locally CAT(0).

We will assume that the reader is familiar with the basics of CAT(0) geometry; for details see [12].

Our constructions involving CAT(0) geometry will use piecewise Euclidean ($P\mathbb{E}$) complexes. We restrict to the special cases of 2-dimensional complexes or cube complexes. In both cases the general theory has major simplifications; for full details see [12].

A piecewise Euclidean complex is obtained by taking the disjoint union of polyhedral cells in $\mathbb{E}^n$, each of which is the convex hull of a finite set of points, and then identifying certain faces by isometries. We put the resulting path metric on this complex coming from the piecewise Euclidean metric on each cell. We call such a complex a cube complex if each cell is isometric to $[0,1]^n$ for some n. The precise definition is given in [12, I.7.37] as follows.

Definition 2.1.3. A *convex polyhedral cell* C is the convex hull of a finite set of points $P \subset \mathbb{E}^n$, the *dimension* of C is the dimension of the smallest m-plane containing it. The *interior* of C is the interior of C as a subset of this plane.

Let H be a hyperplane in $\mathbb{E}^n$. If C lies in one of the closed half-spaces bounded by H and $H \cap C \neq \emptyset$ then $F = H \cap C$ is called a *face* of C; if $F \neq C$ then it is called a *proper face* of C. The *dimension of a face* F is the dimension of the smallest m-plane containing F. The *interior* of F is the interior of F in this plane. The 0-dimensional faces of C are called its *vertices*. The *support* of $x \in C$, denoted $\operatorname{supp}(x)$ is the unique face containing x in its interior.

We are now ready to define piecewise Euclidean ($P\mathbb{E}$) complexes.

Definition 2.1.4. Let $(C_\lambda : \lambda \in \Lambda)$ be a family of convex polyhedral cells and let $X = \bigcup_{\lambda\in\Lambda}(C_\lambda \times \{\lambda\})$ denote their disjoint union. Let $\sim$ be an equivalence relation on X and let $K = X/\sim$. Let $p\colon X \to K$ be the natural projection and define the *characteristic map* $\chi_\lambda\colon C_\lambda \to K$ by $\chi_\lambda(x) = p(x,\lambda)$.

K is called a *piecewise Euclidean complex* if:

1. for every $\lambda \in \Lambda$, the restriction of χ_λ to the interior of each face of C_λ is injective;

2. for all $\lambda_1, \lambda_2 \in \Lambda$ and $x_1 \in C_{\lambda_1}, x_2 \in C_{\lambda_2}$, if $\chi_{\lambda_1}(x_1) = \chi_{\lambda_2}(x_2)$ then there is an isometry $h\colon \operatorname{supp}(x_1) \to \operatorname{supp}(x_2)$ such that $\chi_{\lambda_1}(y) = \chi_{\lambda_2}(h(y))$ for all $y \in \operatorname{supp}(x_1)$.

Definition 2.1.5. A *cube complex* X is a piecewise Euclidean complex where all of the cells are isometric to $[0,1]^n$ for some n.

Using the combinatorial structure on piecewise Euclidean complexes we can define links of points.

Definition 2.1.6. The *link of a point* x in a piecewise Euclidean complex K, denoted $\mathrm{Lk}(x, K)$, is the space of directions at x endowed with the quotient metric associated to the projection $\coprod \mathrm{Lk}(x_\lambda, C_\lambda) \to \mathrm{Lk}(x, K)$ induced by the map $\coprod C_\lambda \to K$.

The *link of a cell* C in a piecewise Euclidean complex K, denoted $\mathrm{Lk}(C, K)$, is the link of any interior point of C.

The above definition of link of a cell is somewhat non-standard. The normal definition considers only directions perpendicular to the cell in question, we will denote this link $\overline{\mathrm{Lk}}(C, K)$. These definitions differ by taking a join with a sphere of the appropriate definition.

In a finite piecewise Euclidean complex K, $\mathrm{Lk}(x, K)$ can be thought of as a rescaled version of the ε-sphere about x, for sufficiently small ε. We can define a natural cell complex structure on $\mathrm{Lk}(x, X)$. An n-cell in this structure comes from the intersection with an $(n + 1)$-cell of K.

Given a metric space (Y, d), let $d_\pi(y, y') = \min\{\pi, d(y, y')\}$ be the *truncated metric* on Y.

Definition 2.1.7. The *spherical join* $X * Y$ of two metric spaces X and Y is, as a set, $[0, \pi/2] \times X \times Y$ modulo the equivalence relation which identifies (θ, x, y) to (θ', x', y') whenever $(\theta = \theta' = 0$ and $x = x')$ or $(\theta = \theta' = \pi/2$ and $y = y')$. We define a metric on this by requiring that the distance between the points $z = (\theta, x, y)$ and $z' = (\theta', x', y')$ be at most π and satisfy the formula

$$\cos\big(d(z, z)\big) = \cos\theta \cos\theta' \cos\big(d_\pi(x, x')\big) + \sin\theta \sin\theta' \sin\big(d_\pi(y, y')\big).$$

The *simplicial join* $X * Y$ of two simplicial complexes is the simplicial complex with vertex set $X^{(0)} \sqcup Y^{(0)}$, such that the set $[z_1, \ldots, z_n]$ spans a simplex if and only if the sets $Z_X = \big\{z_i : z_i \in X^{(0)}\big\}$ and $Z_Y = \big\{z_i : z_i \in Y^{(0)}\big\}$ span simplices in X and Y respectively.

The ambiguity in the above definition vanishes in light of the fact that if X and Y are metric simplicial complexes, then the spherical join can be given a simplicial structure making it simplicially isomorphic to the simplicial join.

The following insight of Gromov allows us to check whether a $P\mathbb{E}$ complex is non-positively curved.

Lemma 2.1.8 (Gromov [24], Bridson [10])**.** *A finite $P\mathbb{E}$ complex is non-positively curved if and only if the link of each vertex is a CAT(1) space.*

In the cases mentioned above this reduces to the following lemmata.

For 2-dimensional $P\mathbb{E}$ complexes the link of a vertex is a graph and we have the following:

Lemma 2.1.9 (Bridson-Haefliger [12], II.5.6)**.** *A graph is CAT(1) if and only if there are no circuits of length $< 2\pi$.*

In a cube complex the link of each vertex is a spherical complex built from all-right spherical simplices (see [12, II.5.17]). Gromov realised that an all-right spherical complex is CAT(1) if and only if it is flag.

Definition 2.1.10. A complex L is a *flag complex* if it is simplicial and every set $\{v_1, \dots, v_n\}$ of pairwise adjacent vertices spans a simplex.

Remark 2.1.11. Flag complexes are completely determined by their 1-skeleta.

Thus we arrive at the following combinatorial condition for cube complexes.

Lemma 2.1.12 (Gromov [24])**.** *A finite cube complex is non-positively curved if and only if the link of every vertex is a flag complex.*

We will be looking at various simplicial complexes throughout and will require some terminology.

Definition 2.1.13. Given a simplicial complex L and a subset of the vertices S, the *full subcomplex spanned by S* is defined to be the simplicial complex with vertex set S and there is a simplex $[s_0, \dots, s_n]$ if $[s_0, \dots, s_n]$ is a simplex of L.

A subcomplex M of L is a *full subcomplex* if it is isomorphic to the full subcomplex spanned by $V(M)$.

The *link of a cell σ* in a simplicial complex L, denoted $\mathrm{Lk}(\sigma, L)$, is defined to be the full subcomplex spanned by those vertices which are adjacent to all vertices of σ.

Later, we will be looking to limit the rank of free abelian subgroups. The following theorem shows that we can look at the maximal rank of an isometrically embedded copy of $\mathbb{E}^n$ instead.

Theorem 2.1.14 (Bridson-Haefliger [12], II.7)**.** *Let X be a compact non-positively curved cube complex, $\widetilde{X}$ its universal cover and $G = \pi_1(X)$. If $H = \mathbb{Z}^n$ is a subgroup of G, then there is an isometrically embedded copy of $i\colon \mathbb{E}^n \hookrightarrow \widetilde{X}$. Moreover, the quotient $i(\mathbb{E}^n)/H$ is an n-torus.*

Definition 2.1.15. A geodesic metric space X is *δ-hyperbolic* if given x, y, z in X and geodesics $[x, y], [y, z], [x, z]$. We have $[x, z] \subset N_\delta([x, y] \cup [y, z])$.

A geodesic metric space X is *hyperbolic* if it is δ-hyperbolic for some δ.

A group is *hyperbolic* if it acts properly and cocompactly on a hyperbolic space.

When we wish to prove that a non-positively curved space has hyperbolic fundamental group we will use the following.

Theorem 2.1.16 (Gromov [24], Bridson [11])**.** *Let X be a compact non-positively curved space. The universal cover $\widetilde{X}$ is not hyperbolic if and only if there is an isometric embedding $\mathbb{E}^2 \to \widetilde{X}$.*

2.1.1 CAT(0) cones

We will consider flats in CAT(0) cube complexes. For every point $x \in X$ on the flat, there is a geodesic loop of length 2π in $\mathrm{Lk}(x, X)$. In many cases the link of such a point will be a spherical join, and it will be natural to consider the cone on this link and examine flat planes in this cone.

Definition 2.1.17. Given a metric space Y the *CAT(0) cone* $X = C_0(Y)$ over Y is defined as follows. As a set X is a quotient of $[0, \infty) \times Y$ by the equivalence relation given by $(t, y) \sim (t', y')$ if $(t = t' = 0)$ or $(t = t' > 0$ and $y = y')$. The equivalence class of (t, y) is denoted ty and the class of $(0, y)$ is denoted 0.

The distance between two point $x = ty$ and $x' = t'y'$ in X is defined by

$$d(x, x')^2 = t^2 + t'^2 - 2tt' \cos\big(d_\pi(y, y')\big).$$

This formula defines a metric on X see [12, I.5.9].

The key theorem we will require addresses the join of two metric spaces.

Theorem 2.1.18 (Bridson-Haefliger [12], I.5.15)**.** *For any metric spaces Y_1 and Y_2 there is a natural isometry of $C_0(Y_1 * Y_2)$ onto $C_0(Y_1) \times C_0(Y_2)$.*

Thus we can consider the projection onto each factor. These projections do not increase distances and after reparameterisation map geodesics to geodesics.

Two geodesic lines $c, c' \colon \mathbb{R} \to X$ are said to be *asymptotic* if there exists a $k \in \mathbb{R}$ such that $d\big(c(t), c'(t)\big) \leq k$ for all $t \in \mathbb{R}$. Asymptotic rays in CAT(0) spaces behave very nicely.

Theorem 2.1.19 (Bridson-Haefliger [12], II.2.13)**.** *Let X be a CAT(0) space and c, c' be asymptotic geodesic lines. Then the convex hull of $c(\mathbb{R}) \cup c'(\mathbb{R})$ is isometric to a flat strip*

$$\mathbb{R} \times [0, D] \subset \mathbb{E}^2$$

.

2.2 Right angled Artin groups

Right angled Artin groups have been at the centre of a lot of study, particularly because of their interesting subgroup structure [1,7,25,34]. In particular, their subgroups have interesting connections with finiteness properties of groups, as shown in [7]. For completeness, we recall the basic theory of right angled Artin groups.

Definition 2.2.1. Given a flag complex Γ, we define the associated *right angled Artin group (RAAG)* A_Γ, as the group,

$$A_\Gamma = \left\langle \Gamma^{(0)} \;\middle|\; [v, w] = 1 \text{ if } [v, w] \in \Gamma^{(1)} \right\rangle.$$

These groups have non-positively curved cube complexes as classifying spaces; these are unions of tori:

Definition 2.2.2. Given a flag complex Γ the *Salvetti complex* S_Γ is defined as follows.

For each vertex v_i in Γ, let $S^1_{v_i} = S^1$ be a copy of the circle cubulated with 1 vertex. For each simplex $\sigma = [v_0, \ldots, v_n]$ of Γ there is an associated torus $T_\sigma = S^1_{v_0} \times \cdots \times S^1_{v_n}$. If $\tau < \sigma$, then there is a natural inclusion $T_\tau \hookrightarrow T_\sigma$. Now define

$$S_\Gamma = \left. \coprod_{\sigma < \Gamma} T_\sigma \middle/ \sim \right.$$

where the equivalence is generated by the inclusions $T_\tau \hookrightarrow T_\sigma$.

There is a map $S_\Gamma \to (S^1)^{\Gamma^{(0)}}$ sending each circle in S_Γ to the respective circle in $(S^1)^{\Gamma^{(0)}}$ and extending linearly over cubes; this map is an inclusion of cubical complexes.

We need to prove that S_Γ has the correct fundamental group and is a non-positively curved cube complex. Proofs of the following can be found in [7].

Lemma 2.2.3. $\pi_1(S_\Gamma) = A_\Gamma$.

Lemma 2.2.4. S_Γ *is non-positively curved.*

2.2.1 A new classifying space

We would like to construct a new classifying space, which will also be a non-positively curved cube complex and will be more amenable to taking branched covers. We will build such a classifying space in the case that the flag complex satisfies the following condition.

Definition 2.2.5. A simplicial complex Γ has an *n-partite structure* if $\Gamma \subset V_1 * \cdots * V_n$. In the case $n = 2$ we will say *bipartite* and in the case $n = 3$, *tripartite*.

Any finite simplicial complex can be given an n-partite structure as a subcomplex of a simplex.

We define a cube complex K_Γ for an n-partite complex Γ as follows.

Let $V_i = \{v_1^i, \ldots, v_{m_i}^i\}$ and $I = \{1, \ldots, n\}$. For each vertex v_k^i, let $S^1_{v_k^i}$ be a copy of S^1 cubulated with two vertices labelled 0 and 1, and two edges $e_{v_k^i}$ and $e_{v_0^i}$.

Definition 2.2.6. Given $J \in \mathcal{P}(I)$, $J = \{i_1, \ldots, i_l\}$, we say that σ is a *J-simplex* if it is of the form $[v_{k_1}^{i_1}, \ldots, v_{k_l}^{i_l}]$.

Remark 2.2.7. Every simplex is a J simplex for some unique (possibly empty) J.

For each J-simplex $\sigma = [v_{k_1}^{i_1}, \ldots, v_{k_l}^{i_l}]$ in Γ we associate the following space.

$$T_\sigma = \left(\prod_{v \in \sigma} S^1_v \right) \times \left(\prod_{i \notin J} e_{v_0^i} \right).$$

This is the product of a torus and a cube.

Let $\mathcal{T}_J = \{T_\sigma : \sigma \text{ is a } J\text{-simplex}\}$ and $\mathcal{T} = \coprod_{J \in \mathcal{P}(I)} \mathcal{T}_J$. Given simplices $\tau < \sigma$ there is a natural inclusion $T_\tau \hookrightarrow T_\sigma$.

$$K_\Gamma = \coprod_{T \in \mathcal{T}} T \Big/ \sim .$$

Where once again the equivalence relation is generated via the inclusions $T_\tau \to T_\sigma$. We need to prove the two key lemmata: the fundamental group of K_Γ is A_Γ, and K_Γ is non-positively curved.

Lemma 2.2.8. $\pi_1(K_\Gamma) = A_\Gamma$.

Proof. Apply the Seifert-van Kampen Theorem repeatedly. □

Lemma 2.2.9. *K_Γ is non-positively curved.*

Proof. There are 2^n vertices in K_Γ. Given two vertices v, w there is a cellular isomorphism $K_\Gamma \to K_\Gamma$ which sends v to w. Thus we only need to check the link of one vertex; we will check the link of $\mathbf{0} = (0, \ldots, 0)$. Let $L = \mathrm{Lk}(\mathbf{0}, K_\Gamma)$. Let $\overline{V_i} = V_i \cup \{v_0^i\}$.

There is a vertex in L for each edge at $\mathbf{0}$, so $L^{(0)} = \coprod \overline{V_i}$. Two distinct vertices $v_{k_1}^{i_1}$ and $v_{k_2}^{i_2}$ are connected if one of the following conditions holds:

1. $k_1 = 0, k_2 = 0$,
2. $k_1 = 0, i_1 \neq i_2$,
3. $i_1 \neq i_2, k_2 = 0$,
4. $[v_{k_1}^{i_1}, v_{k_2}^{i_2}]$ is an edge of Γ.

These edges come from the following subcomplexes.

1. $T_\emptyset$,
2. $T_{v_{k_1}^{i_1}}$,
3. $T_{v_{k_2}^{i_2}}$,
4. $T_{[v_{k_1}^{i_1}, v_{k_2}^{i_2}]}$.

This tells us that $L \subset \overline{V_1} * \cdots * \overline{V_n}$. We now want to prove that L is in fact a flag complex. Given a set $W = \{v^{i_1}_{k_1}, \dots v^{i_l}_{k_l}\}$ of pairwise adjacent vertices in L we want to show that $[v^{i_1}_{k_1}, \dots v^{i_l}_{k_l}]$ is also in L. We split W into two sets $W_0 = \left\{v^{i_j}_{k_j} : k_j = 0\right\}$ and $W_1 = W \setminus W_0 = \{w_1, \dots, w_a\}$. Since the vertices in W_1 are pairwise adjacent there is a simplex $\sigma \subset \Gamma$ spanning them. We see that the subcomplex T_σ fills the required simplex. $\square$

It should be noted the natural projections are injective on each closed cube and not just on the interior.

Remark 2.2.10. The complex K_Γ requires a choice of n-partite structure. Given two n-partite structures the complexes are homotopy equivalent but are not isomorphic as cube complexes. This is shown in Figure 2.1, together with some examples of the construction.

Remark 2.2.11. Let Θ_i be a "cage graph" with two vertices that has an edge for each element of $\overline{V_i}$, then the complex K_Γ constructed above embeds in $\prod_{i=1}^n \Theta_i$.

This fact will come in useful later, as will the fact that in the case $n = 3$ there is an embedded copy of $\Theta_1 \amalg \Theta_2 \amalg \Theta_3$ in K_Γ.

2.2.2 Finiteness properties of subgroups of RAAGs

We will require one theorem on the finiteness properties of subgroups of RAAGs from [17]. A homomorphism $f : A_\Gamma \to \mathbb{Z}$ can be defined by putting an integer label on each vertex and defining $f(v)$ by its label.

Definition 2.2.12. Let $f \colon A_\Gamma \to \mathbb{Z}$ be a homomorphism. We denote $\Gamma^\dagger < \Gamma$ the full subcomplex spanned by those vertices with label 0. Let L^* be the full subcomplex spanned by vertices not in $\Gamma^\dagger$.

Theorem 2.2.13 (Bux-Gonzalez [17], Theorem A)**.** *Let $f \colon A_\Gamma \to \mathbb{Z}$ be a homomorphism. Then the following are equivalent:*

- *The kernel of f is of type FP_n, respectively F_2.*
- *For every, possibly empty, dead simplex $\sigma < \Gamma^\dagger$ the complex $\Gamma^* \cap \mathrm{Lk}(\sigma, \Gamma)$ is homologically $(n - \dim(\sigma) - 2)$-connected, respectively L^* is simply connected.*

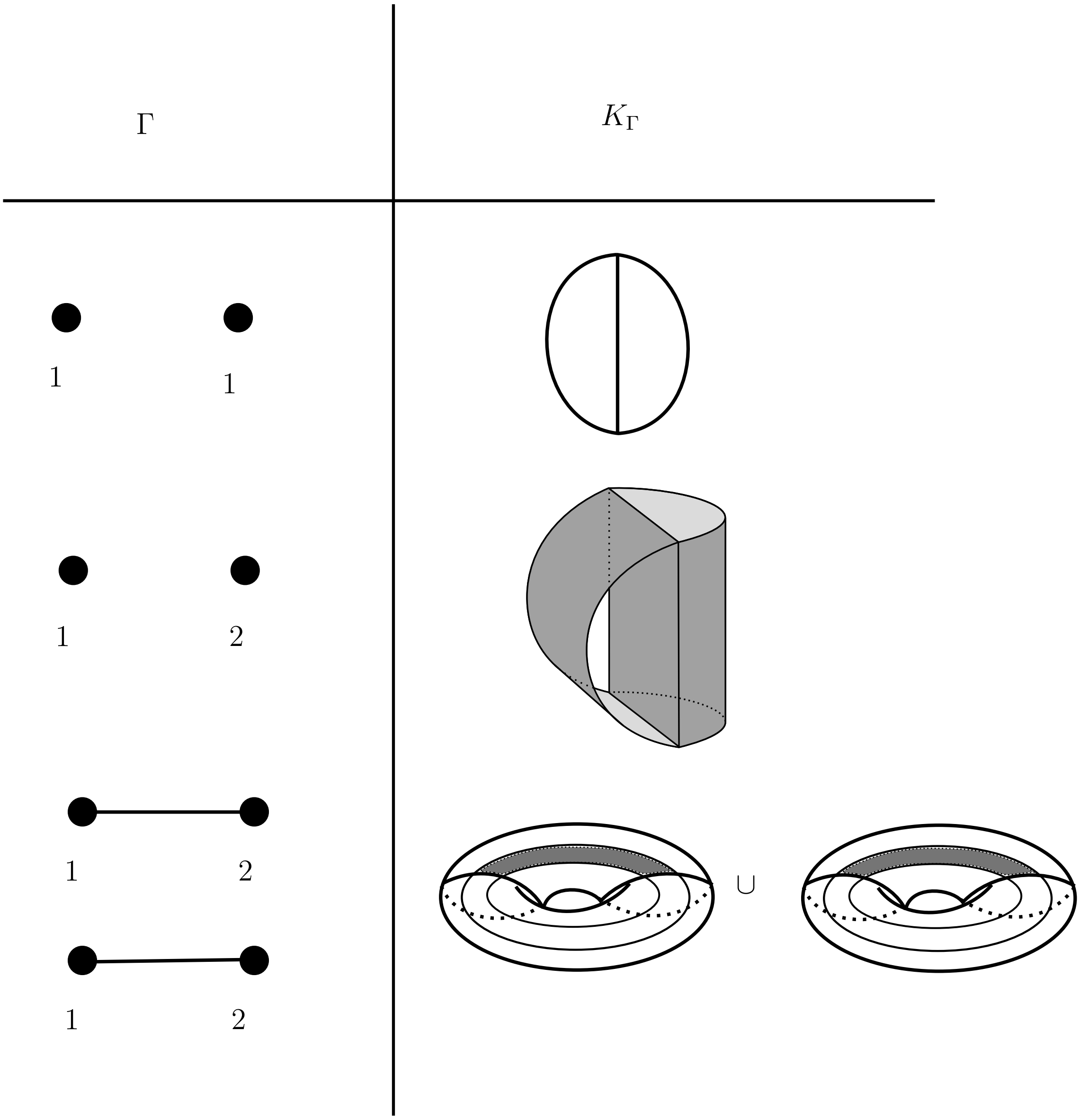


Figure 2.1: Some examples of K_Γ. The labels on vertices exhibit the n-partite structure.

2.3 Branched covers of cube complexes

We will take branched covers of cube complexes to get rid of high dimensional flats. The techniques we will use were developed by Brady in [8]. The idea is to branch over an appropriate subset which intersects all the high-dimensional flats.

Definition 2.3.1. Let X be a non-positively curved cube complex. We say that $Y \subset X$ is a *branching locus* if it satisfies the following conditions:

1. Y is a locally convex cubical subcomplex,
2. $\mathrm{Lk}(c, X) \smallsetminus Y$ is connected and non-empty for all cubes c in Y.

The first condition is required to prove that non-positive curvature is preserved when taking branched covers. The second is a reformulation of the classical requirement that the branching locus has codimension 2 in the theory of branched covers of manifolds; it ensures that the trivial branched covering of X is X.

Definition 2.3.2. A *branched cover* $\widehat{X}$ of X over the branching locus Y is the result of the following process.

1. Take a finite covering $\overline{X \smallsetminus Y}$ of $X \smallsetminus Y$.
2. Lift the piecewise Euclidean metric locally and consider the induced path metric on $\overline{X \smallsetminus Y}$.
3. Take the metric completion $\widehat{X}$ of $\overline{X \smallsetminus Y}$.

We require some key results from [8] which allow us to conclude that this process is natural and that the resulting complex is still a non-positively curved cube complex.

Lemma 2.3.3 (Brady [8], Lemma 5.3)**.** *There is a natural length preserving surjection* $\widehat{X} \to X$ *and* $\widehat{X}$ *is a piecewise Euclidean cube complex.*

Lemma 2.3.4 (Brady [8], Lemma 5.5)**.** *If* Y *is a finite graph, then* $\widehat{X}$ *is non-positively curved.*

2.4 Bestvina-Brady Morse theory

While Bestvina-Brady Morse theory is defined in the more general setting of affine cell complexes, in this instance we shall only need it for non-positively curved cube complexes.

For the remainder of this section, let X be a CAT(0) cube complex and let G be a group which acts freely, cellularly, properly and cocompactly on X. Let $\phi\colon G \to \mathbb{Z}$ be a homomorphism and let $\mathbb{Z}$ act on $\mathbb{R}$ by translations.

Recall that χ_c is the characteristic map of the cube c.

Definition 2.4.1. We say that a function $f\colon X \to \mathbb{R}$ is a *ϕ-equivariant Morse function* if it satisfies the following 3 conditions.

- For every cube $c \subset X$ of dimension n, the map $f\chi_c\colon [0,1]^n \to \mathbb{R}$ extends to an affine map $\mathbb{R}^n \to \mathbb{R}$ and $f\chi_c\colon [0,1]^n \to \mathbb{R}$ is constant if and only if $n = 0$.
- The image of the 0-skeleton of X is discrete in $\mathbb{R}$.
- f is ϕ-equivariant, i.e. $f(g \cdot x) = \phi(g) \cdot f(x)$.

We will consider the level sets of our function, which we will denote as follows.

Definition 2.4.2. For a non-empty closed subset $I \subset \mathbb{R}$ we denote by X_I the preimage of I. We also use X_t to denote the preimage of $t \in \mathbb{R}$.

The kernel H of ϕ acts on the cube complex X in a manner preserving each level set X_I. Moreover, it acts properly and cocompactly on the level sets. We will use the topological properties of the level sets to gain information about the finiteness properties of the group. We will need to examine how they vary as we pass to larger level sets.

Theorem 2.4.3 (Bestvina-Brady, [7], Lemma 2.3)**.** *If $I \subset I' \subset \mathbb{R}$ are connected and $X_{I'} \smallsetminus X_I$ contains no vertices of X, then the inclusion $X_I \hookrightarrow X_{I'}$ is a homotopy equivalence.*

If $X_{I'} \smallsetminus X_I$ contains vertices of X, then the topological properties of $X_{I'}$ can be very different from those of X_I. This difference is encoded in the ascending and descending links.

Definition 2.4.4. The *ascending link* of a vertex is

$Lk_{\uparrow}(v, X) = \cup\{\mathrm{Lk}(w, c) \mid \chi_c(w) = v$ and w is a minimum of $f\chi_c\} \subset \mathrm{Lk}(v, X)$

The *descending link* of a vertex is

$\mathrm{Lk}_{\downarrow}(v, X) = \cup\{\mathrm{Lk}(w, c) \mid \chi_c(w) = v$ and w is a maximum of $f\chi_c\} \subset \mathrm{Lk}(v, X)$

Theorem 2.4.5 (Bestvina-Brady, [7], Lemma 2.5)**.** *Let f be a Morse function. Suppose that $I \subset I' \subset \mathbb{R}$ are connected, closed and $\min I = \min I'$ ($\max I = \max I'$), and that $I' \smallsetminus I$ contains only one point r of $f\big(X^{(0)}\big)$. Then $X_{I'}$ is homotopy equivalent to the space obtained from X_I by coning off the descending (resp. ascending) links of v for each $v \in f^{-1}(r)$.*

We can now deduce a lot about the topology of the level and sub-level sets. We know how they change as we pass to larger intervals and so we have the following.

Corollary 2.4.6 (Bestvina-Brady, [7], Corollary 2.6)**.** *Let I, I' be as above.*

1. *If each ascending and descending link is homologically $(n-1)$-connected, then the inclusion $X_I \hookrightarrow X_{I'}$ induces an isomorphism on H_i for $i \leq n-1$ and is surjective for $i = n$.*
2. *If the ascending and descending links are connected, then the inclusion $X_I \hookrightarrow X_{I'}$ induces a surjection on π_1.*
3. *If the ascending and descending links are simply connected, then the inclusion $X_I \hookrightarrow X_{I'}$ induces an isomorphism on π_1.*

Knowing that the direct limit of this system is a contractible space allows us to compute the finiteness properties of the kernel of ϕ.

Theorem 2.4.7 (Bestvina-Brady, [7], Theorem 4.1)**.** *Let $f\colon X \to \mathbb{R}$ be a ϕ-equivariant Morse function and let $H = \ker(\phi)$. If all ascending and descending links are simply connected, then H is finitely presented (i.e. is of type F_2).*

We would also like to have conditions which will allow us to deduce that H does not satisfy certain other finiteness properties. A well known result in this direction is:

Proposition 2.4.8 (Brown, [14], p. 193)**.** *Let H be a group acting freely, properly, cellularly and cocompactly on a cell complex X. Assume further that $\widetilde{H}_i(X,\mathbb{Z}) = 0$ for $0 \leq i \leq n-1$ and that $\widetilde{H}_n(X,\mathbb{Z})$ is not finitely generated as a $\mathbb{Z}H$-module. Then H is of type FP_n but not FP_{n+1}.*

In [8], the above result was used to prove that a certain group is not of type FP_3. However in our theorems not all the links will satisfy the assumptions of [8, Theorem 4.7] and we require the following.

Theorem 2.4.9. *Let $f\colon X \to \mathbb{R}$ be a ϕ-equivariant Morse function and let $H = \ker(\phi)$. Suppose that for all vertices v the reduced homology of $\mathrm{Lk}_\uparrow(v)$ and $\mathrm{Lk}_\downarrow(v)$ is 0 in dimensions $0,\ldots,n-1$ and $n+1$. Further assume that there is a vertex v' such that $\widetilde{H}_n\big(\mathrm{Lk}_\uparrow(v')\big) \neq 0$ or $\widetilde{H}_n\big(\mathrm{Lk}_\downarrow(v')\big) \neq 0$ (possibly both). Then H is of type FP_n but not of type FP_{n+1}.*

Proof. From Corollary 2.4.6 we know that, $\widetilde{H}_i\big(X_{[t-N,t+N]}\big) = \widetilde{H}_i(X_t)$ for $0 \leq i \leq n-1$ and all $N \in \mathbb{N}$. Since homology commutes with direct limits, we can pass from X_t to X and deduce that all these homology groups are trivial. We now show that $\widetilde{H}_n(X_t)$ is not finitely generated as a $\mathbb{Z}H$-module.

Suppose that we have a finite set of n-cycles $z_1,\ldots,z_l$ that generate $\widetilde{H}_n(X_t)$ as a $\mathbb{Z}H$-module. There exists an N such that each z_i bounds and $(n+1)$-cycle in $X_{[t-N,t+N]}$. So the inclusion induced map $\widetilde{H}_n(X_t) \to \widetilde{H}_n(X_{[t-N,t+N]})$ is zero but by Corollary 2.4.6 it is also onto, so $\widetilde{H}_n(X_{[t-N,t+N]}) = 0$.

Theorem 2.4.5 implies that for every closed interval J containing $[t-N,t+N]$, X_J can be obtained (up to homotopy) from $X_{[t-N,t+N]}$ by coning off the ascending and descending links of vertices v such that $f(v) \in \big(J \smallsetminus [t-N,t+N]\big)$. Since all these links have 0 homology in dimension $n+1$ we can see from the Mayer-Vietoris sequence that the inclusion induced map

$$\widetilde{H}_{n+1}(X_{J'}) \to \widetilde{H}_{n+1}(X_J)$$

is injective for all $[t-N,t+N] \subseteq J' \subseteq J$.

We will assume that the vertex v' satisfies $\widetilde{H}_n\big(\mathrm{Lk}_\uparrow(v')\big) \neq 0$. The case with descending links is analogous. Using the fact that the short exact sequence

$$0 \to H \to G \to \mathbb{Z} \to 0$$

splits, we get a $\mathbb{Z}$ action on X which gives us an action on the collection of level sets with $r \in \mathbb{Z}$ by sending X_t to X_{t+r}. Translating by this action, we can assume that the vertex v' is in $X_{[t-N-s,t+N]} \smallsetminus X_{[t-N,t+N]}$ for some large s. Let L be the union of all the ascending links coned off in the process of going from $X_{[t-N,t+N]}$ to $X_{[t-N-s,t+N]}$.

Once again looking at the Mayer-Vietoris sequence we will see that

$$\widetilde{H}_{n+1}\big(X_{[t-N-s,t+N]}\big) \to \widetilde{H}_n(L)$$

is a surjective map. Since the latter is non-zero by assumption, the former must also be non-zero. This implies that $\widetilde{H}_{n+1}(X)$ is non-zero since homology commutes with direct limits; but X is a contractible space giving a contradiction.

Therefore $\widetilde{H}_n(X_t)$ must be infinitely generated and so by Proposition 2.4.8 the result follows. □

Finally, we prove a key theorem regarding ascending and descending links.

Theorem 2.4.10. *For $i = 1, 2$ let X_i be a CAT(0) cube complex, G_i a group acting freely, properly and cocompactly on X_i, $\phi_i \colon G_i \to \mathbb{Z}$ a surjective homomorphism and $f_i \colon X_i \to \mathbb{R}$ a ϕ_i-equivariant Morse function. Then, there is a Morse function $f \colon X_1 \times X_2 \to \mathbb{R}$ such that $\operatorname{Lk}_{\uparrow}\big((v_1, v_2), X_1 \times X_2\big) = \operatorname{Lk}_{\uparrow}(v_1, X_1) * \operatorname{Lk}_{\uparrow}(v_2, X_2)$.*

A similar statement is true for descending links. The proof really relies on the following key lemma.

Lemma 2.4.11. *Let X be a CAT(0) cube complex with Morse function f. Then $\operatorname{Lk}_{\uparrow}(v, X)$ is the full subcomplex of $\operatorname{Lk}(v, X)$ spanned by the vertices in $\operatorname{Lk}_{\uparrow}(v, X)$.*

At first sight this does not appear to be an improvement, however, it allows for simple calculation of the ascending and descending links once the link of a vertex is known.

Proof. Let $v_1, \dots, v_n$ be n pairwise adjacent vertices in $\operatorname{Lk}_{\uparrow}(v, X)$, proving that the simplex $[v_1, \dots, v_n]$ is in $\operatorname{Lk}_{\uparrow}(v, X)$ will prove the claim.

Let e_i be the edge in X corresponding to v_i. We must prove that v is a minimum for f restricted to $c = e_1 \times \cdots \times e_n$. We note that since f extends to an affine map, c is foliated by level sets $f^{-1}(t)$ for $t \in \mathbb{R}$. Each of these level sets corresponds to a linear subspace of dimension $n-1$ not containing any subcubes of dimension > 0. When intersected with the cube c there are exactly two subspaces where the intersection is a single vertex. One of these corresponds to the minimum of $f|_c$ and the other to the maximum of $f|_c$. One of these must be at the vertex mapping to v and this is the minimum. $\square$

Proof of Theorem 2.4.10. We define a Morse function $f\colon X_1 \times X_2 \to \mathbb{R}$ by $f(x_1, x_2) = f_1(x_1) + f_2(x_2)$. This satisfies all the conditions of being a Morse function and is ϕ-equivariant with respect to the map $\phi(g_1, g_2) = \phi_1(g_1) + \phi_2(g_2)$.

The link of a vertex (v_1, v_2) in $X_1 \times X_2$ is $\mathrm{Lk}(v_1, X_1) * \mathrm{Lk}(v_2, X_2)$. It is easy to see that the ascending link is the full subcomplex spanned by $\mathrm{Lk}_{\uparrow}(v_1, X_1) \sqcup \mathrm{Lk}_{\uparrow}(v_2, X_2) \subset \mathrm{Lk}(v_1, X_1) * \mathrm{Lk}(v_2, X_2)$, which is equal to $\mathrm{Lk}_{\uparrow}(v_1, X_1) * \mathrm{Lk}_{\uparrow}(v_2, X_2)$. $\square$

Part I

Cubical structures for free-by-cyclic groups

Chapter 3

CAT(0) structures for F_2-by-$\mathbb{Z}$ groups

The work in this chapter is published in a joint paper with Jack Button [16]

The groups G_ϕ that we shall be concerned with are mapping tori of $F_2 = F(x, y)$ by a single automorphism $\phi \in \mathrm{Aut}(F_2)$. These groups have presentations of the form

$$\langle x, y, t | txt^{-1} = \phi(x), tyt^{-1} = \phi(y)\rangle.$$

We shall prove the following theorem:

Theorem A. *Given $\phi \in Aut(F_2)$, the group G_ϕ acts freely, cellularly, properly and cocompactly on a CAT(0) square complex.*

It was previously known that all the groups of the form G_ϕ are CAT(0). This was originally proved by T. Brady in [9] where it is proved that the G_ϕ acts on a CAT(0) complex built from equilateral triangles. (Examples of equilateral triangle groups which do not act on any cube complex without a fixed point are constructed by Ballman and Świątkowski in [5].)

Another proof that G_ϕ is CAT(0) can be deduced from work of Piggott, Ruane and Walsh [30], where it is proved that $\mathrm{Aut}(F_2)$ is a CAT(0) group. To deduce the result from this we require a tower argument [12, II.5.27].

The current proof builds on unpublished work of Bridson and Lustig [13]. From this work we use the technique of folding used to get the building blocks in Figure 3.2. They used this in a similar way to gain CAT(0) structures for F_2-by-$\mathbb{Z}$ groups.

We will start by considering the case of automorphisms which are of finite order.

Proposition 3.0.1. *If $\phi \in \mathrm{Aut}(F_n)$ has order q in $\mathrm{Out}(F_n)$ then G_ϕ is the fundamental group of a non-positively curved 2-complex. Furthermore, this is finitely covered by $\Gamma \times S^1$ where Γ is a graph with fundamental group F_n.*

Proof. Every finite order automorphism ϕ of F_n can be realised as an isometry of a finite graph Γ (see [18, Theorem 2.1]). Let $X = \Gamma \times [0,1]/(x,0) \sim (\phi(x),1)$.

X is locally isometric to $\Lambda \times (-\epsilon, \epsilon)$ where Λ is a contractible subset of a graph. This product is CAT(0) and so X is non-positively curved.

There is a surjection $G_\phi \to \mathbb{Z}_q$. The cover corresponding to the kernel of this map will be isometric to $\Gamma \times S^1$. □

We now want to look at automorphisms of infinite order. We require the following lemmata to ensure that we account for the general case with our construction.

Lemma 3.0.2. *G_ϕ is defined up to isomorphism by $[\phi] \in \mathrm{Out}(F_2)$.*

Proof. If $\phi = ad_g\psi$, for $ad_g \in \mathrm{Inn}(F_2)$ the inner automorphism conjugation by g, then

$$\begin{aligned} G_\phi &\cong \langle x,y,t | txt^{-1} = \phi(x), tyt^{-1} = \phi(y)\rangle \\ &\cong \langle x,y,t | txt^{-1} = g\psi(x)g^{-1}, tyt^{-1} = g\psi(y)g^{-1}\rangle \\ &\cong \langle x,y,t,t' | t'xt'^{-1} = \psi(x), t'yt'^{-1} = \psi(y), t' = g^{-1}t\rangle \\ &\cong \langle x,y,t' | t'xt'^{-1} = \psi(x), t'yt'^{-1} = \psi(y)\rangle \cong G_\psi. \end{aligned}$$ □

Lemma 3.0.3. *G_ϕ is defined up to isomorphism by the conjugacy class of $[\phi] \in \mathrm{Out}(F_2)$.*

Proof. Let $\psi = \xi^{-1}\phi\xi$ then the following map defines an isomorphism.

$$\begin{aligned} \Omega : G_\psi &\to G_\phi \\ g &\mapsto \xi(g) \\ t &\mapsto t. \end{aligned}$$ □

As such we will restrict to conjugacy in $\mathrm{Out}(F_2) = \mathrm{GL}_2(\mathbb{Z})$.

Theorem 3.0.4. *[9] The semigroup Σ generated by $-I, F, L$ and R contains a conjugate of every infinite order matrix in $\mathrm{GL}_2(\mathbb{Z})$, where*

$$F = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, R = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, L = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}.$$

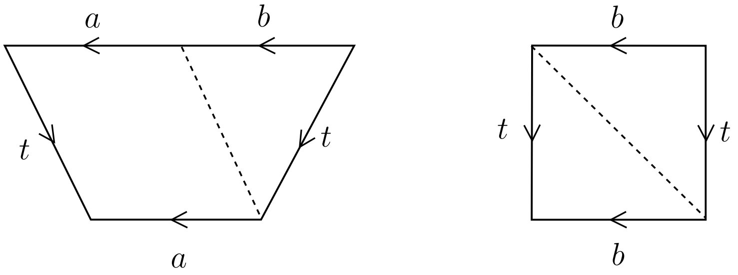


Figure 3.1: The 2-complex associated to λ.

3.1 Non-positively curved classifying spaces for F_2-by-$\mathbb{Z}$ groups

Theorem 3.1.1. *Given* ϕ *in the semigroup* $S < Aut(F_2)$ *generated by*

$$\begin{aligned} \lambda : a &\mapsto ba & \rho : a &\mapsto a \\ b &\mapsto b & b &\mapsto ab \\ \iota : a &\mapsto a^{-1} & \sigma : a &\mapsto b \\ b &\mapsto b^{-1} & b &\mapsto a, \end{aligned}$$

there is a non-positively curved space X_ϕ *such that* $\pi_1(X_\phi) = G_\phi$.

We shall see that from the above and the fact that the natural map $\mathrm{Out}(F_2) \to \mathrm{GL}_2(\mathbb{Z})$ is an isomorphism, that one obtains non-positively curved structures for all F_2-by-$\mathbb{Z}$ groups.

Proof. We start with the obvious 2-complex shown in Figure 3.1 for a mapping torus of the automorphism λ. This has a repeated corner which means it cannot support a metric of non-positive curvature.

To get rid of the repeated corner we cut our building blocks along the dotted lines identifying the triangles with the repeated corner, resulting in our basic building blocks shown in Figure 3.2.

By the above lemmata we can without loss of generality assume that our automorphism has the form $\phi = \eta_0 \dots \eta_{n-1}\theta$ where $\eta_i = \rho$ or λ and θ is one of the following

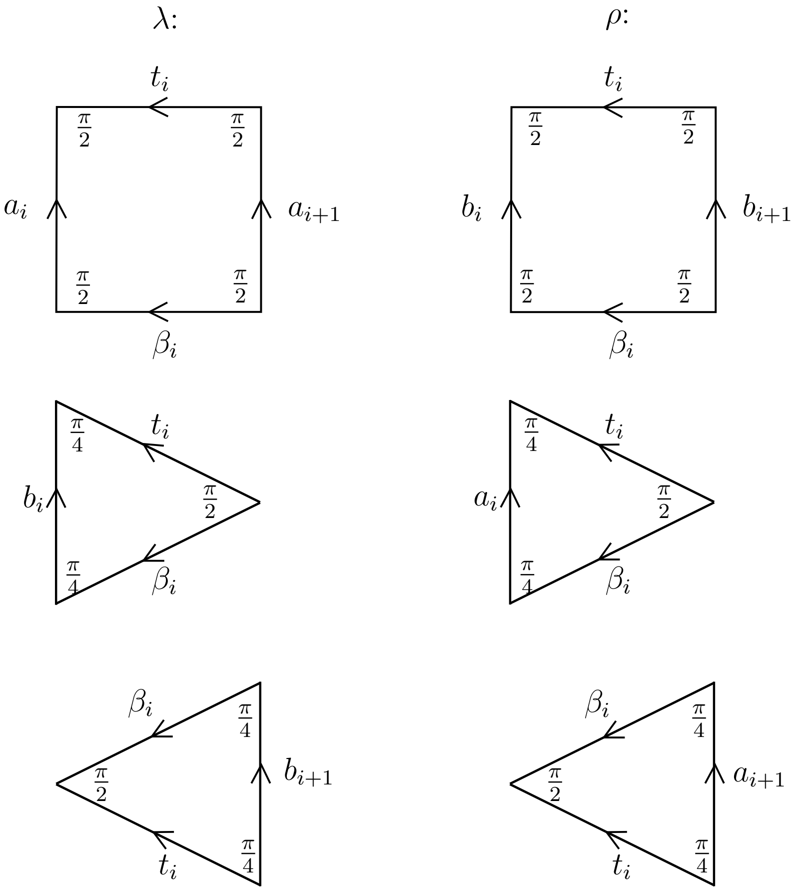


Figure 3.2: The basic building blocks for the construction with angles labelled.

automorphisms of order 2:

$$\begin{aligned} id &: (a,b) \mapsto (a,b) \\ \iota &: (a,b) \mapsto (a^{-1},b^{-1}) \\ \sigma &: (a,b) \mapsto (b,a) \\ \sigma \circ \iota &: (a,b) \mapsto (b^{-1},a^{-1}). \end{aligned}$$

We can assume that we only apply one of these and that we can do it at the end. This is because in $\mathrm{Aut}(F_2)$ the first and second give central elements and $\rho\sigma = \sigma\lambda$.

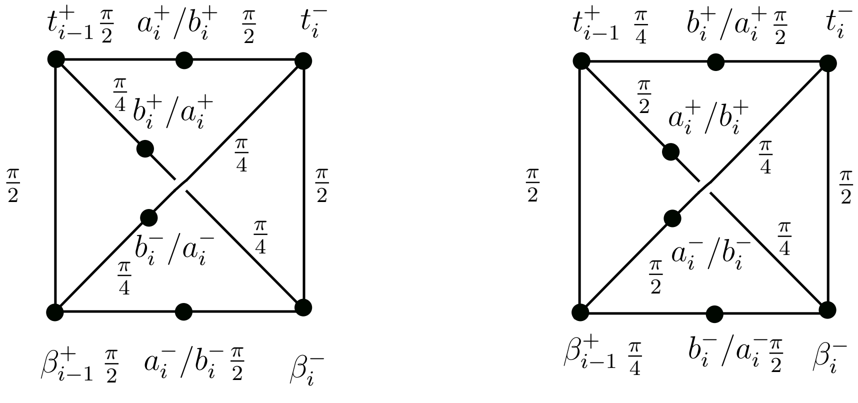
We can now glue these building blocks together to obtain a non-positively curved space X_ϕ with $\pi_1(X_\phi) = G_\phi$, for $\phi \in \Sigma$, so up to isomorphism we have all groups G_ψ. We do this in the following way: in Figure 3.2 let $i = 0$ so that we have the initial and terminal vertices $t_0^{\pm 1}$ at each end of t_0. On performing the given gluing we have that $t_0^{\pm 1}$ are not identified, but if we further stick these two vertices together by identifying a_0, b_0 with a_1, b_1, respectively, then the resulting 2-complex has fundamental group $F_2 \rtimes_\lambda \mathbb{Z}$ or $F_2 \rtimes_\rho \mathbb{Z}$. Now suppose that our automorphism $\phi = \eta_0 \ldots \eta_{n-1}\theta$, where θ is one of the four special finite order automorphisms above. For each i between 0 and $n-1$ take a copy C_i of the 2-complex associated to either λ or ρ in Figure 3.2; this contains the edge t_i. We glue C_i to C_{i+1} by the unique labelled isometry preserving orientation between the a_{i+1}, b_{i+1}, which means that the vertex t_i^+ is identified with t_{i+1}^-. Finally we glue C_{n-1} back to C_0 by identifying a_n, b_n with a_0, b_0 via an isometry realising the map θ so that t_{n-1}^+ becomes equal to t_0^-. We must now check that the resulting complex satisfies the link condition.

We say a vertex is at *time* i if it is the vertex where t_{i-1} and t_i meet.

We start with the case of vertices of time not equal to 0. These will be where our complexes C_{i-1} and C_i are glued together by the unique labelled orientation preserving isometry between the a_i, b_i.

The link of such a vertex is shown in Figure 3.3. We want to assign angles so that there are no circuits of length less than 2π in the link.

If we assign angles as in Figure 3.2, then we get two types of link as shown in Figure 3.3. Figure 3.3 i) corresponds to the automorphisms at the i-th stage both being ρ or both being λ. Figure 3.3 ii) corresponds to when one automorphism is λ and one is ρ.



i) *ii)*

Figure 3.3: The possible links of a vertex.

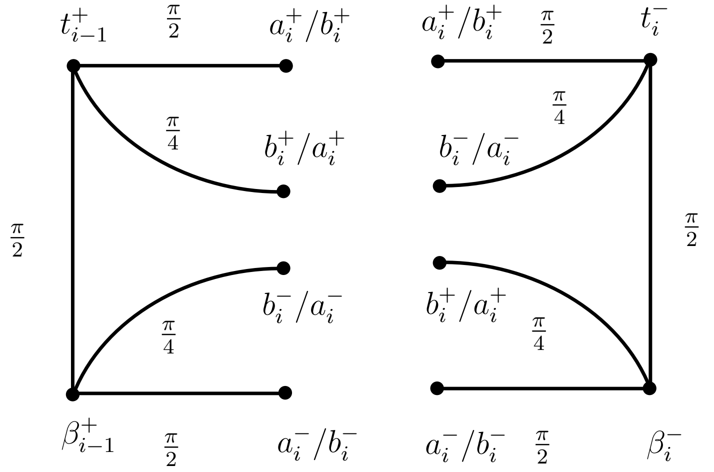
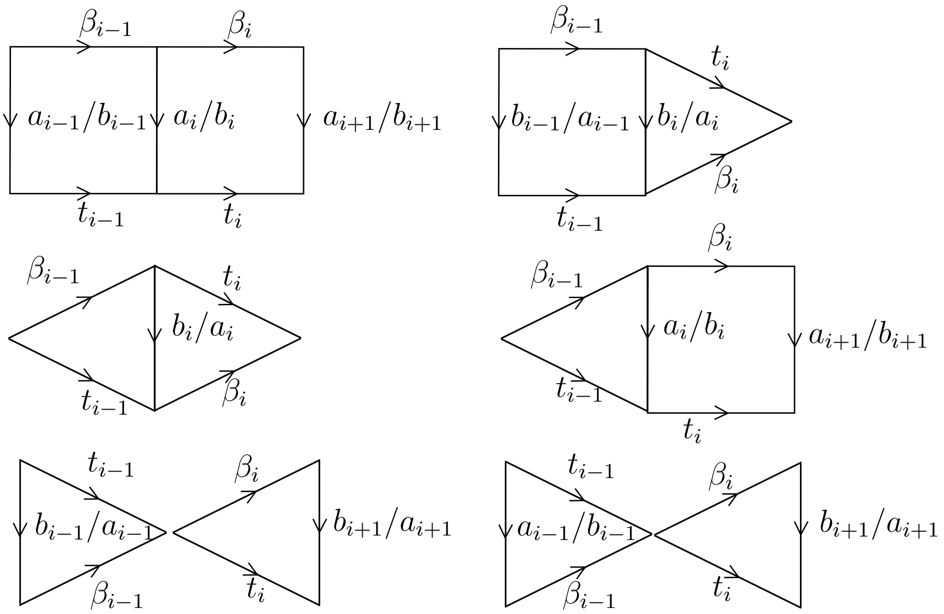


Figure 3.4: The two halves of a link.

We can see in either case that the link has no circuits of length less than 2π.

We now look at the case of the vertex at time 0. This will have to take into account the map θ. We can consider the link as being split into 2 halves as shown in Figure 3.4. The finite order maps defined earlier give vertex identifications. There are 16 possible links we may get in this way, corresponding to which automorphisms meet and to which of the 4 finite order automorphisms we have. All of these possibilities give a link which is homeomorphic to the 1 skeleton of a tetrahedron with the set of angles depicted in Figure 3.3 in which it can be seen that there are no circuits of length less than 2π. This completes the proof. □



i) ii)

Figure 3.5: The 2 possible cases of automorphisms meeting at a vertex.

3.2 Square complexes

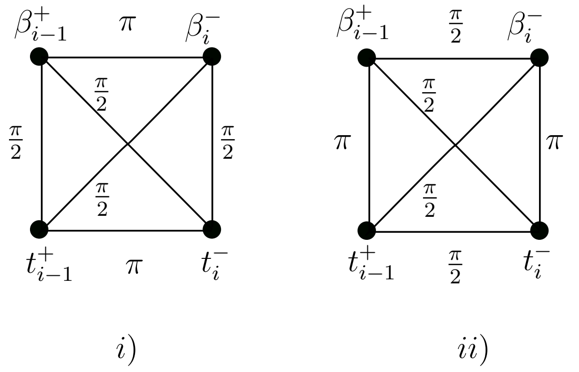
In this section we will prove that with a more careful assignment of angles in the construction of X_ϕ we can ensure that the complexes above can be made into square complexes.

Theorem 3.2.1. *X_ϕ is homotopy equivalent to a non-positively curved square complex, for all ϕ above.*

Proof. We start by splitting the automorphism ϕ into one of three types depending on its decomposition in the semigroup described earlier:

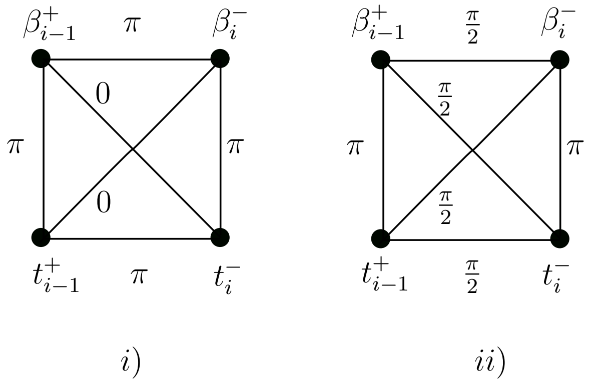
1. $\phi = \rho^n$ or λ^n
2. $\phi = (\rho\lambda)^n(\rho\theta)^\epsilon$ or $(\lambda\rho)^n(\lambda\theta)^\epsilon$ where $\theta \in \{\psi_3, \psi_4\}$ and $\epsilon \in \{0, 1\}$
3. all other automorphisms.

We will assign angles to our building blocks based on which meet at time i. In Figure 3.5 we have depicted the two cases of our building blocks meeting at a vertex.



i) *ii)*

Figure 3.6: The links for automorphisms in cases 1 and 2.



i) *ii)*

Figure 3.7: The links for automorphisms in case 3.

In case 1 each time our building blocks meet each other, it will be of the type depicted in Figure 3.5 i). In this case we keep the angle assignment we had before and make the edges labelled a_j or b_j on the vertical sides of the two rectangles of length 2 and the other edges of length 1. We then make the edge between the 2 triangles of length $\sqrt{2}$ and subdivide the rectangles into squares of edge length 1, thus replacing the two triangles with a new building block which is the square formed by gluing them together. The link of the original vertices in this complex are depicted in Figure 3.6 i), where again the edges corresponding to a_i and b_i have been suppressed as they have valence 2.

In case 2, each time our building blocks meet each other, it will be of the type depicted in Figure 3.5 ii). In this case we collapse the triangles to lines by letting the a_i/b_i edge have length 2 and the edges labelled β_i, t_i have length 1. Subdivide the resulting rectangles into 2 squares of side length 1. The link of the original vertices in this complex are depicted in Figure 3.6 ii), where the edges corresponding to a_i and b_i have been suppressed as they have valence 2.

In case 3 we will have a mix of both Figures 3.5 i) and ii) and we collapse all the triangles to lines as we did for case 2. If we have two triangles meeting at an edge as in Figure 3.5 i), this will produce a degenerate 2-cell, namely a segment of length 2. This could affect the overall topology of complex if there were a cylinder of such squares. However, this will not happen as a cylinder of degenerate squares corresponds to every automorphism meeting as in Figure 3.5 i). This only occurs if the automorphism is as in case 1.

The links of vertices in these complexes are depicted in Figure 3.7, where the edges corresponding to a_i and b_i have been suppressed as they have valence 2. The case of edges of 0 length are the degenerate squares where in fact the two edges become identified.

In all the cases we see that the resulting complex will be a non-positively curved square complex. This completes the proof of theorem A. $\square$

Part II

Finiteness properties of subgroups of CAT(0) groups

Chapter 4

The branched covering method

4.1 Hyperbolisation in dimension 2

This section will be a warm up to our key theorems which are all related to dimension 3. The lower dimensional case carries a lot of the ideas that will be used later.

Throughout this section Γ will be a bipartite graph; say $\Gamma \subset V_1 * V_2$. Let $V_i = \{v_1^i, \ldots, v_{m_i}^i\}$. Let Θ_i be the graph with two vertices, labelled 0 and 1, and $|V_i| + 1$ edges, labelled $v_0^i, v_1^i, \ldots, v_{m_i}^i$, all directed from 0 to 1. We noted in 2.2.11 that the complex K_Γ constructed in 2.2.6 is a subcomplex of $\Theta_1 \times \Theta_2$.

Theorem 4.1.1. *Let Γ be a bipartite graph, let A_Γ be the associated RAAG and let K_Γ be the classifying space constructed in 2.2.6. Then there is a branched cover R_Γ of K_Γ which has hyperbolic fundamental group.*

There are in fact many branched covers delivering this result. During the course of the proof we will pick a prime p and as this varies different hyperbolic branched covers are obtained.

Proof. The branching locus will be one of the 4 vertices of K_Γ. Given two vertices v and w there is a homeomorphism of K_Γ which sends v to w. Therefore, it does not matter which vertex we pick to be our branching locus. We will choose the vertex $\mathbf{0} = (0, 0)$.

$K_\Gamma \smallsetminus \{\mathbf{0}\}$ deformation retracts onto the graph $\Theta_1 \vee \Theta_2$. This can be seen as follows. We start with a torus cubulated as in Figure 4.1. From each torus we remove the center vertex. The complement deformation retracts onto the graph depicted in Figure 4.1. We now identify the edges via their labels, which will result in $\Theta_1 \vee \Theta_2$.

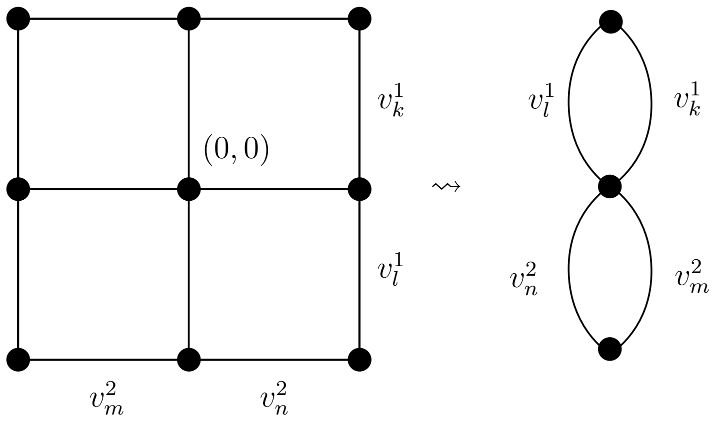


Figure 4.1: Depiction of the deformation retraction $T^2 \setminus \{(0,0)\} \to S^1 \vee S^1$.

This argument shows that $\pi_1(K_\Gamma \setminus \{\mathbf{0}\})$ is a free group on $|V_1| + |V_2|$ generators. We will denote these generators $a_i = v_{i-1}^1\overline{v_i^1}$ for $i \in \{1, \dots, |V_1|\}$ and $b_j = v_{j-1}^2\overline{v_j^2}$ for $j \in \{1, \dots, |V_2|\}$.

From the deformation described we get a map $\mathrm{Lk}(\mathbf{0}, K_\Gamma) \to \Theta_1 \vee \Theta_2$. Each loop of length 4 in $\mathrm{Lk}(\mathbf{0}, K_\Gamma)$ corresponds to a torus as in Figure 4.1. Under the deformation retraction this gets sent to a loop in $\Theta_1 \vee \Theta_2$ corresponding to the commutator $[v_k^1\overline{v_l^1}, v_m^2\overline{v_n^2}] = [\prod_{i=k+1}^{k+l} a_i, \prod_{j=m+1}^{m+n} b_j]$.

Let $p > 1 + \max(|V_1|, |V_2|)$ be a prime and S_p be the symmetric group on p letters. Let λ be a p-cycle in S_p and μ an element which conjugates λ to λ^l, where l is a generator of $\mathbb{Z}_p^\times$. We define our cover using the map,

$$\begin{aligned} \rho : \pi_1(\Theta_1 \vee \Theta_2) &\to S_p, \\ a_i &\mapsto \lambda, \\ b_j &\mapsto \mu, \end{aligned}$$

taking the cover corresponding to the stabiliser of 1 in S_p.

Note that the commutator $[\lambda^m, \mu^n] = \lambda^{(l^n-1)m}$ is a p-cycle. This means that the loops of length 4 in the link have connected preimage in the cover. We take the completion R_Γ of the resulting complex; there is a natural map $b : R_\Gamma \to K_\Gamma$. The link of the vertex which maps to $\mathbf{0}$ will contain no cycles of length < 5.

We now prove that the resulting complex has hyperbolic universal cover. We know that $\widetilde{R}_\Gamma$ is a non-positively curved cube complex. So to prove that it is hyperbolic we just have to show that there are no isometric embeddings $\mathbb{E}^2 \to \tilde{R}_\Gamma$. If there is such an embedding, then it will contain at least one square 2-cell. However, each square contains one vertex which is a lift of **0** and in the link of this vertex there are no loops of length 2π. However, were the flat plane to be isometrically embedded there would be a loop of length 2π in the link of every vertex on the plane. This contradiction completes the proof. $\square$

We use this theorem along with Morse theoretic ideas to find more examples of hyperbolic groups with finitely generated subgroups that are not finitely presentable.

Proposition 4.1.2. *Let* Γ *be a complete bipartite graph on sets* $A = \{a_1, a_2, a_3\}$ *and* $B = \{b_1, b_2, b_3\}$*, fix* p *satisfying the above hypotheses and let* $b : R_\Gamma \to K_\Gamma$ *be the branched cover constructed above. Then* $\pi_1(R_\Gamma)$ *has a finitely generated subgroup which is not finitely presentable.*

Proof. In this case $K_\Gamma = \Theta_1 \times \Theta_2$ where Θ_i is as above and has 4 edges. Put an orientation on each edge of Θ_i such that there are 2 edges oriented towards each vertex and 2 edges oriented away from each vertex. Cubulating S^1 with one vertex and one oriented edge. Define maps $h_i : \Theta_i \to S^1$ on each edge by their orientation. Define $h : K_\Gamma \to S^1$ by $h(x_1, x_2) = h_1(x_1) + h_2(x_2)$. Precomposing with the branched covering map b and lifting to universal covers, we obtain a Morse function $f : \tilde{R}_\Gamma \to \mathbb{R}$, which is $(hb)_*$-equivariant.

The ascending and descending links of this Morse functions are the preimages, under b, of the ascending and descending links of h. For the Morse function h the ascending and descending links are joins of the ascending and descending links for h_i. These will be copies of S^0, so the ascending and descending links will be copies of S^1.

There are now two possibilities, if we look at a vertex in R_Γ which does not map to **0** the ascending and descending links will remain unchanged and will still be copies of S^1.

If the vertex in question maps to **0** we will study the ascending link the other case being identical. The ascending link of **0** is a loop of length 4. Taking a branched cover cause this loop of length 4 to lengthen but in the preimage it will still be a copy of S^1. It follows that the kernel of $(hb)_*$ is finitely generated but not finitely presentable by 2.4.9. $\square$

4.2 Hyperbolisation of products of 3 graphs

In this section we are going to take branched covers of products of graphs and prove that the result has hyperbolic fundamental group, with a finitely presented subgroup that is not of type F_3. We will do this in the following way.

1. Start with three 2-full graphs Γ_1, Γ_2 and Γ_3 (Definition 4.2.1).
2. Find a locally isometric copy of $L = \Gamma_1 \sqcup \Gamma_2 \sqcup \Gamma_3$ in $K = \Gamma_1 \times \Gamma_2 \times \Gamma_3$.
3. Take a cover of $K \setminus L$ and complete to get a branched cover X.
4. Define a Morse function on the universal cover $\tilde{X}$ of X and examine the ascending and descending links of this function; from this we will conclude that there is a finitely presented subgroup that is not of type F_3.
5. Finally, prove that $\tilde{X}$ is a hyperbolic space.

Definition 4.2.1. A connected graph Γ is *2-full* if the vertices of Γ can be divided into 2 sets A and B such that every edge has one end point in A and the other in B.

For simplicial graphs, this is equivalent to being bipartite. However, we allow multiple edges in our graphs and will reserve the use of "bipartite" for simplicial graphs.

We will prove the following.

Theorem B. *Let Γ_1, Γ_2 and Γ_3 be three 2-full graphs such that the valence of any vertex is at least 4. Then there is a finite branched cover X of $\Gamma_1 \times \Gamma_2 \times \Gamma_3$ such that there are no isometrically embedded flat planes in $\tilde{X}$. Furthermore, there is a finitely presented subgroup of $\pi_1(X)$ that is not of type F_3.*

Proof. Let the vertices of Γ_i be divided into two sets A_i, B_i as in the definition of 2-full and let $K = \Gamma_1 \times \Gamma_2 \times \Gamma_3$.

Our branching locus will be

$$L = (\Gamma_1 \times A_2 \times B_3) \sqcup (B_1 \times \Gamma_2 \times A_3) \sqcup (A_1 \times B_2 \times \Gamma_3).$$

This is a locally convex subcomplex of K. We now check that $\mathrm{Lk}(c, K) \setminus L$ is connected and non-empty for all cubes in L. We will do the case of a cube on

$\Gamma_1 \times A_2 \times B_3$, the other cases are identical. If c is the edge $[0,1] \times \{(a,b)\}$, then $\mathrm{Lk}(c,K) = S^0 * \mathrm{Lk}(a,\Gamma_2) * \mathrm{Lk}(b,\Gamma_3)$. We are removing the copy of S^0 therefore, we deformation retract onto $\mathrm{Lk}(a,\Gamma_2) * \mathrm{Lk}(b,\Gamma_3)$. If c is the vertex (v_1,a,b), then $\mathrm{Lk}(c,K) = \mathrm{Lk}(v_1,\Gamma_1) * \mathrm{Lk}(a,\Gamma_2) * \mathrm{Lk}(b,\Gamma_3)$, we are removing the copy of $\mathrm{Lk}(v_1,\Gamma_1)$. This deformation retracts onto a join of two non-empty sets. Therefore, $\mathrm{Lk}(c,K) \smallsetminus L$ is connected and non-empty for all cubes in L.

We now consider the complex $K \smallsetminus L$. There are three surjective projections:

$$\begin{aligned} p_1 &: K \smallsetminus L \to \Gamma_2 \times \Gamma_3 \smallsetminus (A_2 \times B_3), \\ p_2 &: K \smallsetminus L \to \Gamma_3 \times \Gamma_1 \smallsetminus (A_3 \times B_1), \\ p_3 &: K \smallsetminus L \to \Gamma_1 \times \Gamma_2 \smallsetminus (A_1 \times B_2), \end{aligned}$$

which are the restrictions of the projection maps $K \to \Gamma_i \times \Gamma_j$.

Proposition 4.2.2. *Each of the complexes $\Gamma_i \times \Gamma_j \smallsetminus (A_i \times B_j)$ deformation retracts onto a graph.*

Proof. We start at a removed vertex and push out radially in adjacent squares. This allows us to remove the interior of each cube adjacent to a vertex in $A_i \times B_j$. By the definition of 2-full, all 2-cells are retracted by this process leaving us with a graph $\mathcal{G}_{ij} := (\Gamma_i \times A_j) \cup (B_i \times \Gamma_j) \subset \Gamma_i \times \Gamma_j$. $\square$

Let $q_{ij} > \max\{\deg(v) : v \in \Gamma_i \sqcup \Gamma_j\}$ be a prime number, $\alpha \in S_{q_{ij}}$ a q_{ij}-cycle and $\beta \in S_{q_{ij}}$ an element such that $\beta\alpha\beta^{-1} = \alpha^l$, where l is a generator of $\mathbb{Z}_{q_{ij}}^\times$.

A homomorphism $\pi_1(\mathcal{G}_{ij}) \to S_{q_{ij}}$ can be defined by labelling each edge of $\mathcal{G}_{ij}$ with an element of $S_{q_{ij}}$, with the condition that if e has label g, then $\bar{e}$ has label g^{-1}.

We start by labelling the edges of Γ_i and Γ_j, then extend this to a labelling of $\mathcal{G}_{ij}$ in an obvious way.

For each edge of Γ_i we label with an element of $\{\alpha, \ldots, \alpha^p\}$, such that at each vertex the edges oriented towards it are labelled by different powers of α. We similarly label the edges of Γ_j with powers of β such that at each vertex the edges oriented towards it are labelled by different powers of β.

This defines a homomorphism $\pi_1(\Gamma_i \times \Gamma_j \smallsetminus (A_i \times B_j)) \to S_{q_{ij}}$. A loop of length 4 in the link of a removed vertex comes from a diagram as in Figure 4.2. This loop then deformation retracts onto a loop of length 8 in $\mathcal{G}_{ij}$. This loop is labelled by a

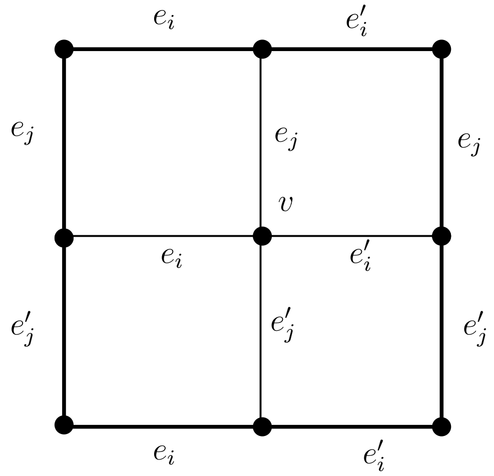


Figure 4.2: The loops e_i and e_i' are in Γ_i and e_j and e_j' are in Γ_j. The vertex labelled v is in $A_i \times B_j$.

commutator of the form $[\beta^{-m}, \alpha^{-n}] = \beta^m \alpha^n \beta^{-m} \alpha^{-n} = \alpha^{n(l^m-1)}$. When $n < p$ and $m < p-1$, this is a non-trivial power of α. These conditions will be satisfied by the choice of p, hence this commutator is a q_{ij}-cycle.

We now have three maps:

$$\begin{aligned}(\rho_1 \circ p_1)_* &: \pi_1(K \smallsetminus L) \to S_{q_{23}},\\ (\rho_2 \circ p_2)_* &: \pi_1(K \smallsetminus L) \to S_{q_{31}},\\ (\rho_3 \circ p_3)_* &: \pi_1(K \smallsetminus L) \to S_{q_{12}}.\end{aligned}$$

Let $q = q_{12}q_{23}q_{31}$. We combine the above representations to get a representation $\rho : \pi_1(K \smallsetminus L) \to S_q$. We then take the cover of $K \smallsetminus L$ corresponding to the stabiliser of 1 in S_q and complete to get the desired branched cover X. By Lemma 2.3.4, this cover will be non-positively curved.

In this part of the proof we consider links of vertices and a Morse function.

The vertices of X can be split into 2 types:

1. Vertices that do not map to L.

2. Vertices that map to L.

Given a vertex v of type 1, let $w = (v_1, v_2, v_3)$ be the vertex in K to which v maps. Since w is disjoint from the branching locus, a small neighbourhood lifts to X. The link of v is therefore isomorphic to the link of w, that is,

$$\mathrm{Lk}(v, X) = \mathrm{Lk}(v_1, \Gamma_1) * \mathrm{Lk}(v_2, \Gamma_2) * \mathrm{Lk}(v_3, \Gamma_3).$$

We now examine the link of a vertex v of type 2. Let $w = (v_1, v_2, v_3)$ be the vertex in K to which v maps. We will do the case where $v_2 \in A_2$ and $v_3 \in B_3$; the other cases can be treated similarly.

Recall that, $\mathrm{Lk}((v_1, v_2, v_3), K) = \mathrm{Lk}(v_1, \Gamma_1) * \mathrm{Lk}(v_2, \Gamma_2) * \mathrm{Lk}(v_3, \Gamma_3)$. Further recall that during the branching process we are removing the set corresponding to the vertices in $\mathrm{Lk}(v_1, \Gamma_1)$. We have a map on fundamental groups

$$\pi_1(\mathrm{Lk}(w, K) \setminus \mathrm{Lk}(v_1, \Gamma_1)) \to \pi_1(K \setminus L).$$

We consider the image of $\pi_1(\mathrm{Lk}(w, K) \setminus \mathrm{Lk}(v_1, \Gamma_1))$ under the maps p_{i*}.

$\mathrm{Lk}(w, K) \setminus \mathrm{Lk}(v_1, \Gamma_1)$ is mapped by p_1 to the link of a removed vertex in

$$\Gamma_2 \times \Gamma_3 \setminus (A_2 \times B_3).$$

The image is thus isomorphic to $\mathrm{Lk}(v_2, \Gamma_2) * \mathrm{Lk}(v_3, \Gamma_3)$. The space $(A * B) \setminus B$ is homotopy equivalent to A. This homotopy equivalence comes from the projection of

$$(C_0(A) \times C_0(B)) \setminus (\{0\} \times C_0(B)) \to C_0(A).$$

Where $C_0(X)$ is the CAT(0) cone from Definition 2.1.17. The map $\mathrm{Lk}(w, K) \setminus \mathrm{Lk}(v_1, \Gamma_1) \to \mathrm{Lk}(v_2, \Gamma_2) * \mathrm{Lk}(v_3, \Gamma_3)$ coming from the projection is a homotopy equivalence. Under the maps p_2 and p_3, $\mathrm{Lk}(w, K) \setminus \mathrm{Lk}(v_1, \Gamma_1)$ is sent to a union of contractible sets. We are now reduced to considering the image of $\pi_1(\mathrm{Lk}(w, K) \setminus \mathrm{Lk}(v_1, \Gamma_1))$ under the map ρ_1 to $S_{q_{23}}$. We picked the maps such that loops of length 4 in $\mathrm{Lk}(v_2, \Gamma_2) * \mathrm{Lk}(v_3, \Gamma_3)$ are sent to q_{23}-cycles under the homomorphism ρ_1.

We can cover $\mathrm{Lk}(v_2, \Gamma_2) * \mathrm{Lk}(v_3, \Gamma_3)$ with loops of length 4, such that each loop has non-empty intersection with the union of previous loops. Every loop of length 4 has connected preimage in the cover. In the sequence each will have non-empty intersection with the union of previous loops. Thus the corresponding cover will be a connected graph with no loops of length 4, as the covering map is length non-increasing and any loop of length 4 has preimage a loop of length > 4. We will denote this graph Λ_v. The link of v is the join of Λ_v with the discrete set $\mathrm{Lk}(v_1, \Gamma_1)$.

We put an orientation on each edge of Γ_i; we orient the edges such that at each vertex there are at least two incoming edges and at least two outgoing edges. We cubulate S^1 with 1 vertex and 1 edge, putting an orientation on this edge. Define a map $f_i : \Gamma_i \to S^1$ by mapping each edge to the edge of S^1 by the given orientation and mapping all vertices to the vertex of S^1. These assignments define a map

$$f : \Gamma_1 \times \Gamma_2 \times \Gamma_3 \to S^1 = \mathbb{R}/\mathbb{Z},$$
$$f(x, y, z) = f_1(x) + f_2(y) + f_3(z),$$

which lifts to an f_*-equivariant Morse function $\tilde{f} : \tilde{K} \to \mathbb{R}$.

Precompose with b to get a map $h = f \circ b : X \to S^1$. Lift this to universal covers to get a Morse function $\tilde{h} : \tilde{X} \to \mathbb{R}$, which is h_*-equivariant. We will examine the ascending and descending links of $\tilde{h}$. We concentrate on the case of the ascending link; the arguments are the same for the descending link.

Note that $\tilde{h} = \tilde{f} \circ \tilde{b}$ where $\tilde{b} : \tilde{X} \to \tilde{K}$ is the lift of b to universal covers. We also consider the derivative map $\tilde{b}_{\mathrm{Lk}(v)} : \mathrm{Lk}(v, \tilde{X}) \to \mathrm{Lk}(\tilde{b}(v), \tilde{K})$. Given a vertex v of $\tilde{X}$ the ascending link is the preimage of the ascending link of $\tilde{b}(v)$. The ascending link of $\tilde{b}(v)$ is $U_1 * U_2 * U_3$ where U_i is a discrete set whose size is the number of edges oriented away from $\tilde{b}(v)$ in $\tilde{\Gamma}_i$.

If v is a type 1 vertex, then the ascending link of v is isomorphic to the ascending link of $\tilde{b}(v)$, which is $U_1 * U_2 * U_3$. This is simply connected but has non-zero second homology as $|U_i| > 1$ it has the homotopy type of a wedge of 2-spheres.

Let v be a vertex of type 2; without loss of generality, assume that it lies on a lift of $\Gamma_1 \times A_2 \times B_3$. The ascending link of v will be the preimage of $U_1 * U_2 * U_3$ in $U_1 * \Lambda_v$. Since U_i contains at least 2 points, the graph $U_2 * U_3$ is connected but not simply connected. We cover $U_2 * U_3$ with loops of length 4 such that successive loops have non-empty intersection with the union of the previous loops. Since each of these loops has connected preimage in Λ_v, the intersection of a loop with the union of the previous loops will remain non-empty. Therefore, the cover $\mathcal{U}$ of $U_2 * U_3$ in Λ_v will be connected. The ascending link will be $\mathcal{U} * U_1$ which will be simply connected as the join of a connected set and a non-empty set.

The kernel of f_* is finitely presented by Theorem 2.4.7. The ascending link of a vertex which is of type 1 has the form $U_1 * U_2 * U_3$, which although simply connected

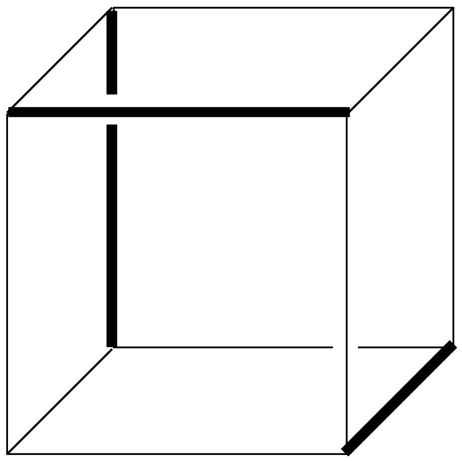


Figure 4.3: Intersection pattern of $\tilde{L}$ on a cube in $\tilde{X}$.

has non-zero second homology. Applying Theorem 2.4.9, we see that the kernel will not be of type FP_3.

To complete the proof of Theorem B we must prove that $\tilde{X}$ is hyperbolic.

Definition 4.2.3. Given a CAT(0) cube complex X and an isometric embedding $i : \mathbb{E}^2 \to X$. We say that a subset D of X intersects $\mathbb{E}^2$ *transversally* at a point p if there is an $\epsilon > 0$ such that $N_\epsilon(p) \cap D \cap \mathbb{E}^2 = \{p\}$.

Let $\tilde{L}$ be the preimage of L in $\tilde{X}$. We will prove that the cube complex $\tilde{X}$ is hyperbolic by contradiction. By Theorem 2.1.16 it is enough to show that there are no isometrically embedded copies of $\mathbb{E}^2$ in $\tilde{X}$. Suppose that $i : \mathbb{E}^2 \to \tilde{X}$ is such an embedding. First, we show that $i(\mathbb{E}^2)$ has a transverse intersection point with $\tilde{L}$. Around such an intersection point we will see that the angle sum in $i(\mathbb{E}^2)$ is forced to be at least 3π, contradicting the assumption that i is an isometric embedding.

Each 3-cube of $\tilde{X}$ intersects $\tilde{L}$ in 3 edges in the pattern depicted in Figure 4.3.

Because i is an isometric embedding, it cannot be contained in the 1-skeleton of $\tilde{X}$, so there is a 3-cube c where the intersection is 2-dimensional.

There are two cases to consider:

1. $i(\mathbb{E}^2) \cap \tilde{L} \cap c = \emptyset$
2. $i(\mathbb{E}^2) \cap \tilde{L} \cap c \neq \emptyset$

In case 1, $F = i(\mathbb{E}^2)$ must have intersected c in an edge adjacent to a vertex v of c which does not map to L. We can develop $\mathbb{E}^2$ in a cubical neighbourhood of this

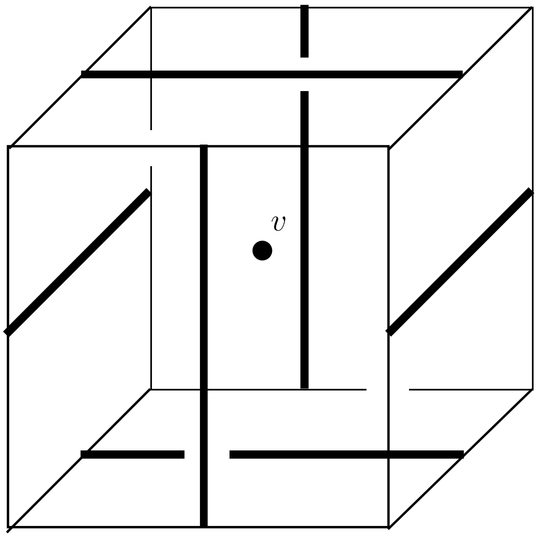


Figure 4.4: Cubes around a vertex not mapping to L.

vertex. The link of v in this neighbourhood will produce part of an octahedron. We have depicted this in Figure 4.4. We can consider this cube as a subset of $\mathbb{E}^3$, where v sits at the origin and the edge with non-empty intersection is one of the coordinate axes. Developing the plane into these cubes we see that we will intersect L in one of the adjacent cubes from Figure 4.4.

We have now reduced to case 2. We have shown that there is a cube c and an edge $e \subset \tilde{L} \cap c$ such that $F \cap e \neq \emptyset$. Either this intersection point is a single point in which case we will see that it is a transverse intersection. If this is not the case then the edge e is entirely contained in F. In this case we can see from the intersection pattern in Figure 4.3 that $i(\mathbb{E}^2)$ intersects one of the other edges of $\tilde{L}$ in c in a single point.

If the intersection point is in the interior of an edge, then it is a transverse intersection point. This can be checked by using a ball whose radius is less than the minimum distance to an endpoint of e.

In the case where the intersection point x is the end point of an edge, we will prove that it is a transverse point of intersection by contradiction.

If x is not a transverse point of intersection, then there are points of $\mathbb{E}^2 \cap \tilde{L}$ arbitrarily close to x. Since i is an isometric embedding, if two points of an edge are in $i(\mathbb{E}^2)$, then the whole edge is contained in $i(\mathbb{E}^2)$. Therefore, $i(\mathbb{E}^2)$ will contain both a 2-dimensional cross section of the cube c and an edge e' of $\tilde{L}$ in another cube that is adjacent to c. Since the link of x is the join of a graph and a discrete set and the vertex corresponding to e' is contained in the discrete set, it follows that there is a cube c' containing e' such that c and c' share a 2-dimensional face. Consider the union of these two cubes as a subset of $\mathbb{E}^3$. The plane defined by the polygon in c does not contain the edge in $\tilde{L}$, so i cannot have been an isometric embedding.

We now have a transverse intersection point with $\tilde{L}$. To see that the flat plane is not isometrically embedded we split into 2 cases.

Firstly, we will study the case where the transverse intersection point x is in the interior of an edge e. The plane will intersect several cubes in various polygonal subsets each contributing an angle. Since we assumed that the plane was isometrically embedded these angles must sum to 2π. Two such polygons joined along an edge, correspond to two cubes meeting along a face containing e. These polygons make a contribution of π. To get an angle sum of 2π we would therefore have to have exactly four cubes meeting around e. But the link of e in $\tilde{X}$ contains no loops of length less than 6.

If the transverse intersection point x is the endpoint of the edge e we proceed similarly. We will assume that the component of $\tilde{L}$ with the transverse intersection is a lift of Γ_1. An angle sum of π will correspond to the intersection of a half plane with 4 unit cubes in $\mathbb{E}^3$. We can define a map $\mathrm{Lk}(x, \tilde{X}) \smallsetminus \mathrm{Lk}(x, \tilde{L}) \to \Lambda_x$. This map is the corresponding map on links from the projection of a cube to a face. We see that an angle sum of π corresponds to a segment of length 2 in Λ_x, so an angle sum of 2π corresponds to a circuit of length 4. However, by choice of the branched covering, Λ_x has no loops of length < 6.

Therefore, there cannot be an isometrically embedded flat plane, and so $\tilde{X}$ is a hyperbolic CAT(0) cube complex. □

4.2.1 Euler characteristics

We now have an infinite family of spaces, each of which has hyperbolic fundamental group. We will now calculate the Euler characteristic of these groups to show that we have an infinite subfamily of non-isomorphic hyperbolic groups. We would like to improve this result to show that the groups are not commensurable. This work is ongoing.

Proposition 4.2.4. *Let* Γ_1, Γ_2 *and* Γ_3 *be 2-full graphs, let* $V_i = A_i \sqcup B_i$ *be the vertex set of* Γ_i *and let* E_i *be the edge set of* Γ_i*. We use the shorthand* $|A_i| = a_i, |B_i| = b_i, |V_i| = v_i$ *and* $|E_i| = e_i$*. Then the* $q_{12}q_{23}q_{31}$*- fold branched cover* X *of* $K = \Gamma_1 \times \Gamma_2 \times \Gamma_3$ *has Euler characteristic*

$$\begin{aligned} q_{12}q_{23}q_{31}(&v_1v_2v_3 + a_1b_2e_3 + b_1e_2a_3 + e_1a_2b_3 + e_1e_2v_3 + e_1v_2e_3 + v_1e_2e_3 \\ &- v_1a_2b_3 - b_1v_2a_3 - a_1b_2v_3 - e_1v_1v_2 - v_1e_2v_3 - v_1v_2e_3 - e_1e_2e_3) \\ &- q_{12}q_{31}(e_1a_2b_3 - v_1a_2b_3) - q_{12}q_{23}(b_1e_2a_3 - b_1v_2a_3) \\ &- q_{23}q_{31}(a_1b_2e_3 - a_1b_2v_3) \end{aligned}$$

Proof. To prove this we count the number of cells of each dimension in X. We do this by examining the cells in the cover of $K \smallsetminus L$ and how many cells we add when we complete to obtain X.

There are no 2 or 3-cells in L, therefore each such cell in K lifts to the cover. Thus we have

$$q_{12}q_{23}q_{31}(e_1e_2v_3 + e_1v_2e_3 + v_1e_2e_3)$$

2-cells and

$$q_{12}q_{23}q_{31}(e_1e_2e_3)$$

3-cells in X.

We are removing the

$$a_1b_2e_3 + b_1e_2a_3 + e_1a_2b_3$$

1-cells of L from K, so in the cover of $K \smallsetminus L$ we have

$$q_{12}q_{23}q_{31}(e_1v_1v_2 + v_1e_2v_3 + v_1v_2e_3 - a_1b_2e_3 - b_1e_2a_3 - e_1a_2b_3)$$

1-cells before completing.

Similarly, we are removing the

$$v_1a_2b_3 + b_1v_2a_3 + a_1b_2v_3$$

0-cells of L from K, so in the cover of $K \smallsetminus L$ we have

$$q_{12}q_{23}q_{31}(v_1v_2v_3 - v_1a_2b_3 - b_1v_2a_3 - a_1b_2v_3)$$

0-cells.

We now account for the cells added on completing the cover. We look at the completion of a component of Γ_1. The link of a lift of an edge e in Γ_1 will be a q_{23} fold cover of the link of e. The lift of Γ_1 will thus be a $q_{12}q_{31}$-fold cover of Γ_1. This cover will have $q_{12}q_{31}(v_1a_2b_3)$ 0-cells and $q_{12}q_{31}(e_1a_2b_3)$ 1-cells. Repeating this count for the graphs Γ_2 and Γ_3 gives the desired result. □

A simple example of a 2-full graph Γ satisfying the hypothesis of Theorem B is obtained by taking a prime $p > 3$ and $p-1$ copies of $[0, 1]$, then identifying all the 0 endpoint and all the 1 endpoints. This is a "cage graph" with 2 vertices and $p-1$ edges. The complex X constructed as above by taking $\Gamma_1 = \Gamma_2 = \Gamma_3 = \Gamma$ and $q_{12} = q_{13} = q_{23} = p$ has Euler characteristic $p^2(9 + 15p - 24p^2 + 9p^3 - p^4)$. Thus we obtain an infinite family of hyperbolic groups with finitely presented subgroups not of type F_3.

4.3 Almost hyperbolisation in dimension 3

Notation 4.3.1. Given a tripartite complex L, let L_{ij} be the full subcomplex spanned by the vertices of $V_i \cup V_j$.

The main theorem of this section is the following.

Theorem D. *Let Γ be a tripartite flag complex, A_Γ the associated RAAG and K_Γ the classifying space constructed in Section 2.2.1. Then there exists a branched cover X of K_Γ, such that $\pi_1(X)$ contains no subgroups isomorphic to $\mathbb{Z}^3$.*

Proof. Our branching locus will be

$$B = (\Theta_1 \times \{0\} \times \{1\}) \cup (\{1\} \times \Theta_2 \times \{0\}) \cup (\{0\} \times \{1\} \times \Theta_3),$$

where Θ_i is the graph with 2 vertices, 0 and 1, and $|V_i| + 1$ edges, each directed from 0 to 1. We have three maps

$$\begin{aligned} p_1 &: K_\Gamma \setminus B \to K_{\Gamma_{12}} \setminus \{(0, 1)\}, \\ p_2 &: K_\Gamma \setminus B \to K_{\Gamma_{23}} \setminus \{(0, 1)\}, \\ p_3 &: K_\Gamma \setminus B \to K_{\Gamma_{31}} \setminus \{(0, 1)\}, \end{aligned}$$

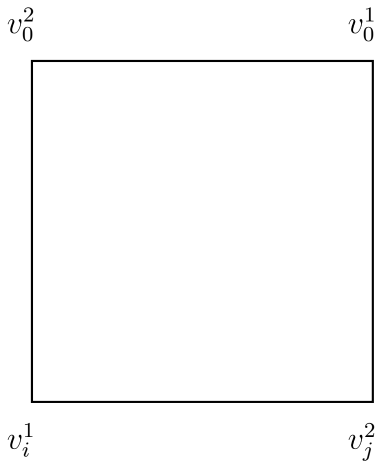


Figure 4.5: A loop corresponding to the commutator $[v_i^1, v_j^2]$.

which are the restrictions of the projections $\Theta_1 \times \Theta_2 \times \Theta_3 \to \Theta_i \times \Theta_j$. The maps from Section 4.1 give us three maps

$$\begin{aligned}
\pi_1(K_{\Gamma_{12}} \smallsetminus \{(0,1)\}) &\to S_{q_{12}} \\
\pi_1(K_{\Gamma_{23}} \smallsetminus \{(0,1)\}) &\to S_{q_{23}} \\
\pi_1(K_{\Gamma_{31}} \smallsetminus \{(0,1)\}) &\to S_{q_{31}}.
\end{aligned}$$

Here, q_{ij} are the primes picked in the process of taking a branched cover in Theorem 4.1.1. Let $q = q_{12}q_{23}q_{31}$. We can combine these permutation representations with the projection maps above to get a map $\pi_1(K_\Gamma \smallsetminus B) \to S_q$, which defines a q-fold cover of $K_\Gamma \smallsetminus B$ by taking the subgroup corresponding to the stabiliser of 1 in S_q. We complete this cover to get our branched cover X.

Let $\{i, j, k\} = \{1, 2, 3\}$. The maps p_1, p_2, p_3 retractions to see this consider the natural map

$$K_{\Gamma_{ij}} \hookrightarrow K_{\Gamma_{ij}} \times \left\{ \frac{1}{2} \right\} \hookrightarrow K_{\Gamma_{ij}} \times e_{v_0^k} \hookrightarrow K_\Gamma.$$

It follows that $\pi_1(K_\Gamma \smallsetminus B) \to \pi_1(K_{\Gamma_{ij}} \smallsetminus \{(0,1)\})$ is surjective. We will now consider what the link of a vertex in the branched cover is. We will restrict our attention to a vertex mapping to $(0,1,0)$ all the other cases are similar.

We will consider the image of the link of $(0,1,0)$ under the three maps p_1, p_2, p_3. In the image of the map p_1, this link is sent surjectively onto the link of $(0,1)$ in $K_{\Gamma_{12}}$. By Section 4.1 we know that $K_{\Gamma_{12}} \smallsetminus \{(0,1)\}$ deformation retracts onto the graph $\Theta_1 \vee \Theta_2$. Loops of length 4 are sent to commutators of the form $[v_i^1\overline{v}_j^1, v_k^2\overline{v}_l^2]$. In the map $\pi_1(K_{\Gamma_{12}} \smallsetminus \{(0,1)\}) \to S_q$ this commutator is sent to a q_{12}-cycle.

We must now consider the image under the maps p_2, p_3. These maps send the link to a disjoint union of contractible subsets, so the maps $\pi_1(K_\Gamma \smallsetminus B) \to S_{q_{23}}$ and $\pi_1(K_\Gamma \smallsetminus B) \to S_{q_{31}}$ send the image of the fundamental group of the link to the identity.

From this we can see that, in the cover of $K_\Gamma \smallsetminus B$ corresponding to the stabiliser of 1 in S_q, the preimage of one of the loops of length 4 depicted in Figure 4.5 will have $q_{23}q_{31}$ components, and each component is a loop of length $4q_{12}$. We will now prove that there are no isometrically embedded planes of dimension > 2. This, combined with Theorem 2.1.14, will complete the proof of the theorem.

Since the resulting cube complex has cubes of dimension at most 3, we can see that the dimension of an isometrically embedded flat plane is at most 3. If such a copy E of $\mathbb{E}^3$ were isometrically embedded in $\tilde{X}$, it would contain at least one cube and would in fact be a cubical embedding of the flat. For each vertex contained in the flat, the link would contain a subcomplex isomorphic to an octahedron. Let $\tilde{B}$ be the lift of the branching locus B. We can see how this intersects each cube in $\tilde{X}$ by Figure 4.3. As such any 3-flat will intersect a vertex on $\tilde{B}$.

Let x be a vertex on $\tilde{B} \cap E$. Then $\mathrm{Lk}(x, \tilde{X})$ has a tripartite structure. If there is an octahedron in this complex it has a tripartite structure of the form $S^0 * S^0 * S^0$. One of the copies of S^0 will be contained in the vertices corresponding to $\tilde{B}$. The other 4 vertices form a loop of length 4 in the bipartite graph defined by the edges not in $\tilde{B}$. However, we constructed the branched cover so that this graph has no cycles of length < 6. □

4.3.1 Subgroups of type FP_2 that cannot be finitely presented

We will now apply our branched covering technique with carefully chosen flag complexes Γ to prove the following.

Theorem D′. *There exists a non positively curved space X such that $\pi_1(X)$ contains no subgroups isomorphic to $\mathbb{Z}^3$, which contains a subgroup of type FP_2 not F_2.*

We will do this using the following steps.

1. Start with a connected 2-dimensional tripartite flag complex L such that $\pi_1(L)$ is a perfect group with the the link of every vertex connected and not a point. Take an auxiliary complex $\Gamma = \mathcal{R}(L)$. Then build the complex K_Γ in Section 2.2.1.
2. Define a function $f : K_\Gamma \to S^1$ which lifts to a Morse function on universal covers. Examine the ascending and descending links of this Morse function.
3. Take a branched covering of K_Γ as in Section 4.2 to get a complex X with an associated Morse function. Examine the ascending and descending links of this Morse function. This will show that the kernel of f_* is of type FP_2.
4. Prove that the kernel of f_* is not finitely presented.

4.3.1.1 The complex $\mathcal{R}(L)$

The construction of this complex is required, the key point of this complex is that it satisfies Proposition 4.3.6. This is required to make sure that the fundamental group of the links is not changed in the branching process.

Firstly, we prove that none of the assumptions from the first step are restrictive. We can realise any finitely presented group as the fundamental group of a finite connected 2-dimensional simplicial complex. It is also well known that the barycentric subdivision of a 2-dimensional simplicial complex is flag and we can put an obvious tripartite structure labelling vertices by the dimension of the corresponding cell. So we can realise any group as the fundamental group of a connected 2-dimensional tripartite flag complex.

Given a connected 2-dimensional tripartite flag complex there is a homotopy equivalent complex of the same form such that the link of every vertex is connected and not a point.

To see this, first note that if there is a vertex where the link is just a single point, then we can contract this edge without changing the homotopy type of the complex. Next, note that if there is a vertex x with disconnected link, then we perform the following procedure: pick two vertices v, w in two different components of the link, add an extra vertex y and connect it to v, w and x while also adding two triangles $[v, y, x]$ and $[u, y, x]$. The result is that we have reduced the number of components of the link at x, without adding any extra components to the link of v or w, and the link of y is connected. We have also not changed the homotopy type, since we have added a contractible space glued along a contractible subspace. Repeating this procedure, we can make sure that the link of every vertex is connected.

Definition 4.3.2. For a simplicial complex L the *octahedralisation* $S(L)$ is defined as follows. For each vertex v of L, let $S^0_v = \{v^+, v^-\}$ be a copy of S^0. For every simplex σ of L take $S_\sigma = *_{v\in\sigma} S^0_v$. If $\tau < \sigma$, then there is a natural map $S_\tau \to S_\sigma$. $S(L) = \cup_{\sigma<L} S_\sigma/\sim$. Where the equivalence is generated by the inclusions $S_\tau \to S_\sigma$.

The complex $S(L)$ can also be seen as the link of the vertex in the Salvetti complex for the RAAG defined by L. If L has an n-partite structure then there is a natural n-partite structure on $S(L)$. Let L be contained in $V_1 * \cdots * V_n$. Then $S(L)$ is contained in $S(V_1) * \cdots * S(V_n)$.

Remark 4.3.3. The map defined by $S^0_v \to \{v\}$ extends to a retraction of $S(L)$ to L (not a deformation retraction).

In particular, if $\pi_1(L) \neq 0$, then $\pi_1(S(L)) \neq 0$.

It is proved in [7] that if L is a flag complex, then $S(L)$ is a flag complex.

Lemma 4.3.4. *Assume L is a connected simplicial complex and* $\mathrm{Lk}(v, L)$ *is connected and not equal to a point for all vertices $v \in L$. If $H_1(L) = 0$, then $H_1(S(L)) = 0$.*

Proof. Let L^+ be the full subcomplex of $S(L)$ spanned by the set $\{v^+ : v \in L\}$, and let L^- be defined similarly. Let N^+ be the interior of the star of L^+ and N^- the interior of the star of L^-. It is clear that $S(L)$ is contained in $N^- \cup N^+$. We can also see that $N^+ \cap N^-$ is the union of the open simplices which are contained in neither L^+ nor L^-. Here we are using the fact that N^+ is homotopy equivalent to L and similarly N^- is homotopy equivalent to L.

By considering the Mayer-Vietoris sequence for N^- and N^+ we see that $H_1(S(L))$ is isomorphic to $\tilde{H}_0(N^+ \cap N^-)$, so we must prove that $N^+ \cap N^-$ is connected.

Let x, y be points of $N^+ \cap N^-$. We can always connect x and y to open edges contained in $N^+ \cap N^-$; label these edges e_x and e_y. Let v_x be the end point of e_x in L^+ and w_x the end point in L^-; define vertices v_y and w_y similarly. Let $\mathfrak{v} = (v_x = v_0, v_1, \ldots, v_n = v_y)$ be a sequence of vertices corresponding to a geodesic in $L^{(1)}$ from v_x to v_y.

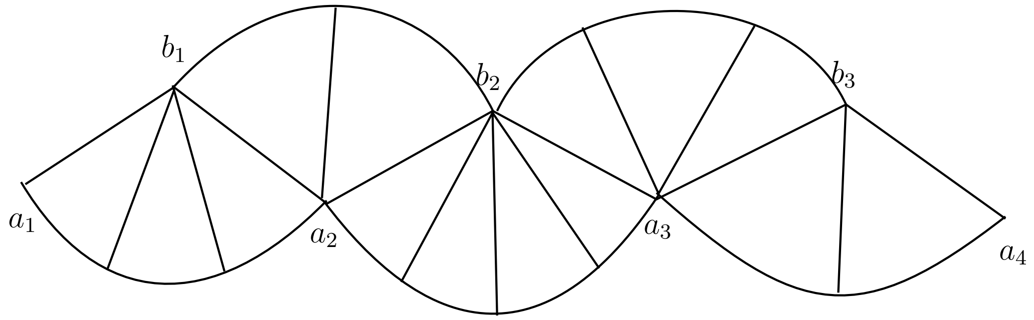


Figure 4.6: The key idea from Lemma 4.3.4.

The vertices w_x and w_y have corresponding vertices in L^+, these are adjacent to v_x and v_y respectively. We split into 4 cases by whether these vertices are on the geodesic $\mathfrak{v}$. We will define a path P in each of these four cases.

1. If $w_x = v_1$ and $w_y = v_{n-1}$, then let $P = v_0, v_1, \ldots, v_n$.
2. If $w_x = v_1$ and $w_y \neq v_{n-1}$, then let $P = v_0, v_1, \ldots, v_n, w_y$.
3. If $w_x \neq v_1$ and $w_y = v_{n-1}$, then let $P = w_x, v_0, v_1, \ldots, v_n$.
4. If $w_x \neq v_1$ and $w_y \neq v_{n-1}$, then let $P = w_x, v_0, v_1, \ldots, v_n, w_x$.

We will now describe how to get a sequence of edges $e_x = e_0, \ldots, e_m = e_y$ such that e_i, e_{i+1} are on a 2-simplex in L. The corresponding sequence of edges in $N^+ \cap N^-$ will give a path from e_x to e_y. Thus completing the proof of the lemma.

A path in the link of a vertex v can be viewed as a sequence of edges in L such that adjacent edges are on a 2-simplex.

We will relabel the path P to $a_1, b_1, \ldots$. Let A_i be a path in the link of b_i from a_i to a_{i+1}. Let B_i be a path in the link of a_i from b_i to b_{i+1}. This can be done since the link of every vertex is connected. The sequence of edges $A_1, B_2, A_2, \ldots$ defines the sequence of edges we require. The key idea is encapsulated in Figure 4.6, where the curved arcs correspond to the paths A_i, B_i. □

We can now define the complex $\mathcal{R}(L)$.

1. Let L be a 2-dimensional tripartite flag complex with $\pi_1(L)$ perfect, such that the link of every vertex in L is connected and not a point. Label the sets of vertices from the tripartite structure V_0, V_1, V_2.

2. Construct $S(L)$ as above. This is a 2-dimensional tripartite flag complex with $\pi_1(L)$ perfect.
3. Add 3 extra vertices v^0, v^1, v^2 which are of type $0, 1, 2$ respectively. Connect v^i to all the vertices not of type i. Define $\mathcal{R}(L)$ to be the flag completion of the resulting complex.

Take a simplicial complex L as above and let $\Gamma = \mathcal{R}(L)$. We can now construct the cubical complex K_Γ from Section 2.2.1.

4.3.1.2 The Morse function f

As noted in Remark 2.2.11, we can view K_Γ as a subcomplex of $\Theta_1 \times \Theta_2 \times \Theta_3$, where Θ_i is a graph that has two vertices $0, 1$ and has edges labelled by the vertices of Γ as well as one extra edge labelled v_0^i, such that each edge runs from 0 to 1. We define a Morse function on the product by putting an orientation on each edge of Θ_i as follows: if it is an edge corresponding to a vertex of $S(L) \subset \Gamma$ we orient it towards 1, while for the vertices v^i and v_0^i we orient towards 0. Now put an orientation on S^1 and map each graph by its orientation; we then extend linearly across cubes. Restricting this map to K_Γ we get a map g, and by lifting to universal covers we get a Morse function $f : \tilde{K}_\Gamma \to \mathbb{R}$ which is g_*-equivariant.

Let the vertices of type i in $S(L) \subset \Gamma$ be the set V_i^+ and let $V_i^- = \{v^i, v_0^i\} =: S_i^0$.

The ascending and descending links of this Morse function are given in Table 4.1.

Notation 4.3.5. Given a simplicial complex Λ and a subset S of the vertices of Λ, $\Lambda(S)$ denotes the full subcomplex of Λ spanned by the vertices in S.

Proposition 4.3.6. *Given a complex of the form* $V_i^+ * V_j^- * S_k^0$ *or* $\Gamma(V_i^+ \cup V_j^+) * S_k^0$ *there is an ordering* $v_1, v_2, \dots, v_q$ *on the set* V_j^s *such that* $St(v_m) \cap (\cup_{l<m} St(v_l))$ *is connected and* $\pi_1(St(v_m) \cap (\cup_{l<m} St(v_l))) \neq 0$ *for* $1 < m \leq q$*.*

Proof. In the case $V_j^s = V_j^- = S_j^0$ we have 2 vertices v_1 and v_2 and $St(v_1) \cap St(v_2) = S_k^0 * V_i^+$, which is connected but not simply connected. So either ordering will do.

In the case $V_j^s = V_j^+$, let $V_j = \{w_1, \dots, w_n\}$ and $V_j^+ = \{w_1^+, w_1^-, w_2^+, \dots, w_n^-\}$.

Vertex	$\mathrm{Lk}_{\uparrow}$	$\mathrm{Lk}_{\downarrow}$
$(0,0,0)$	$\Gamma(V_1^+ \cup V_2^+ \cup V_3^+) = S(L)$	$S^2 = S_1^0 * S_2^0 * S_3^0$
$(0,0,1)$	$\Gamma(V_1^+ \cup V_2^+) * S_3^0$	$V_3^+ * S_1^0 * S_2^0$
$(0,1,0)$	$\Gamma(V_1^+ \cup V_3^+) * S_2^0$	$V_2^+ * S_1^0 * S_3^0$
$(0,1,1)$	$V_1^+ * S_2^0 * S_3^0$	$\Gamma(V_2^+ \cup V_3^+) * S_1^0$
$(1,0,0)$	$\Gamma(V_2^+ \cup V_3^+) * S_1^0$	$V_1^+ * S_2^0 * S_3^0$
$(1,0,1)$	$V_2^+ * S_1^0 * S_3^0$	$\Gamma(V_1^+ \cup V_3^+) * S_2^0$
$(1,1,0)$	$V_3^+ * S_1^0 * S_2^0$	$\Gamma(V_1^+ \cup V_2^+) * S_3^0$
$(1,1,1)$	$S^2 = S_1^0 * S_2^0 * S_3^0$	$\Gamma(V_1^+ \cup V_2^+ \cup V_3^+) = S(L)$

Table 4.1: The ascending and descending links of the Morse function $f : K_\Gamma \to S^1$.

The subgraph $Z = L(V_i \cup V_j)$ is connected since L is connected and the link of every vertex is connected and not a point. We can thus assume that $St(w_l, Z) \cap (\cup_{m<l} St(w_m)) \neq \emptyset$.

Noting that $St(v^s, S(L)) = C(S(\mathrm{Lk}(v, L)))$. We can see that $St(v^s, S(L)) \cap St(w^t, S(L)) = S(St(v, L) \cap St(w, L))$. Thus ordering $V_j^+ = \{w_1^+, w_1^-, w_2^+, \dots, w_n^-\}$ we can see that $V_i^+ \cap St(v_l, S(L)) \cap (\cup_{m<l} St(v_m, S(L)))$ contains at least two points.

Noting that

$$St(v_l, S(L)) \cap (\cup_{m<l} St(v_m, S(L))) = (V_i^+ \cap St(v_l, S(L)) \cap (\cup_{m<l} St(v_m, S(L)))) * S_k^0$$

we can see that this is connected but not simply connected. □

Remark 4.3.7. In the above proof we are gluing contractible complexes along connected complexes. This shows that all the complexes in the statement of Proposition 4.3.7 are simply connected.

Remark 4.3.8. At each stage in the above proof $St(v_l, S(L)) \cap (\cup_{m<l} St(v_m, S(L)))$ could be covered by cycles of length 4, since it is the join of a discrete set and a copy of S^0.

4.3.1.3 Almost hyperbolisation and the Morse function $h : X \to S^1$

We shall now use the almost hyperbolisation technique from Section 4.3 to get a branched cover X of K_Γ, recall that there is a natural length preserving map $b \colon X \to K_\Gamma$. We define a Morse function $h = g \circ b : X \to S^1$. In what follows G is the fundamental group of X which does not contain any copies of $\mathbb{Z}^3$. In what follows, $H := \ker(h_* : G \to \mathbb{Z})$.

It is worth noting that in the almost hyperbolisation procedure we ensure that loops of length 4 in the link of a vertex have connected preimage.

4.3.1.4 Ascending and descending links of h

We distinguish between 4 types of vertices in X and label them as follows:

Vertices of type A are those which map to $(0,0,0)$ or $(1,1,1)$.

Vertices of type B are those which map to $(0,0,1)$ or $(1,0,1)$.

Vertices of type C are those which map to $(1,0,0)$ or $(1,1,0)$.

Vertices of type D are those which map to $(0,1,0)$ or $(0,1,1)$.

For a vertex in X the ascending (descending) link is the preimage of the ascending (descending) link of the corresponding vertex in K_Γ.

Type A vertices are disjoint from the branching locus, so a small neighbourhood of each lifts to X and the ascending and descending links are isomorphic to those of the corresponding vertex of K_Γ. We claim that for vertices *not* of type A the ascending and descending links are simply connected. We will prove this in the case of a vertex of type B and the ascending link; the other cases are similar.

A vertex x of type B is on a lift of Θ_1. We may assume that x maps to $(0,0,1)$. Now, $\mathrm{Lk}_\uparrow(0,0,1) = \Gamma(V_1^+ \cup V_2^+) * S_3^0$. Let us consider the preimage of $\mathrm{Lk}_\uparrow(0,0,1)$ in $\mathrm{Lk}(x,X)$. To envisage this, we start by removing the vertices in V_1 and taking the covering of the remaining space coming from the derivative map b_* of b, then add back the vertices of V_1. By Remark 4.3.7 we can cover $\Gamma(V_1^+ \cup V_2^+) * S_3^0$ by $St(v)$ as v runs over vertices in V_1^+. We noted in Remark 4.3.8 that we can construct this cover in such a way that each $St(v_m) \cap (\cup_{l<m} St(v_l))$ is connected and covered by loops of length 4 with non-empty intersection. Specifically, if S_4^1 and $S_{4'}^1$ are 2 loops of length 4 in $St(v_m) \cap (\cup_{l<m} St(v_l))$, then $S_4^1 \cap S_{4'}^1 \supset S_3^0$.

In the procedure of passing to the branched covering $b : X \to K_\Gamma$. Associated to each vertex of X there is a derivative map $b_{\mathrm{Lk}(v)} : \mathrm{Lk}(v,X) \to \mathrm{Lk}(b(v), K_\Gamma)$. Each of the 4 cycles above has connected preimage under the map $b_{\mathrm{Lk}(v)}$, therefore the preimage of $St(v_m) \cap (\cup_{l<m} St(v_l))$ is connected. Upon taking the completion, we replace the vertices in V_1^+; this corresponds to coning off the lifts of their links. Thus we see $\mathrm{Lk}_\uparrow(x)$ is made from a sequence of contractible spaces glued along connected subspaces, and so is simply connected.

4.3.1.5 Proof that the kernel is not finitely presented

To prove that H is not finitely presented we need the following lemma.

Lemma 4.3.9. *$\pi_1(X_I) \neq 0$ for all compact intervals $I \subset \mathbb{R}$*

Proof. For the purposes of this proof let $Y = \widetilde{K_\Gamma}$. In our case $\Gamma^\dagger = S^2$ and $\Gamma^* = \mathcal{R}(L)$. By Theorem 2.2.13 we know that the kernel of g_* is not finitely presented. Since the kernel of g_* acts cocompactly on the level set Y_I we can see that Y_I is not simply connected.

Let $\gamma : S^1 \to Y_I$ be a non-trivial loop. Then there is a larger interval J' such that γ is trivial in $Y_{J'}$. Let $J' = [a, b]$. We can assume that γ is non-trivial in $J = [a+\epsilon, b]$ or $J = [a, b-\epsilon]$ for all $\epsilon > 0$; assume the latter. There is a sequence of vertices $v_1, v_2, \dots$ such that $Y_{J'} = Y_J \cup C(\mathrm{Lk}_\uparrow(v_i))$. There is an integer m such that γ is trivial in $Y_J \cup_{i\leq m} C(\mathrm{Lk}_\uparrow(v_i))$ but not trivial in $Y_J \cup_{i<m} C(\mathrm{Lk}_\uparrow(v_i)) = \widehat{Y_J}$.

Since adding $C(\mathrm{Lk}_\uparrow(x_m))$ changes the fundamental group (γ becomes trivial), we can find a non-trivial loop $\hat{\gamma}$ in $\widehat{Y_J}$ which is also contained in $\mathrm{Lk}_\uparrow(x_m)$. We also know that it is contained in Y_J since $\mathrm{Lk}_\uparrow(v_i) \cap \mathrm{Lk}_\uparrow(v_j) = \emptyset$ whenever $i \neq j$, as the restriction of an affine map to a cube can only have one maximum and one minimum or it is constant on a subcube.

Since adding $C(\mathrm{Lk}_\uparrow(x_m))$ changes π_1, we see that $\pi_1(\mathrm{Lk}_\uparrow(x_m)) \neq 0$, so x_m is a vertex mapping to $(0, 0, 0)$ and $\hat{\gamma}$ bounds a disc in Y which does not intersect $\tilde{L}$. It follows that this disc lifts to $\tilde{X}$ under the branched covering $\tilde{b}$ coming from the following commutative diagram:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{b}"] \arrow[d] & Y \arrow[d] \\
X \arrow[r, "b"] & K_\Gamma
\end{tikzcd}$$

The boundary of this lifted disc is in X_J; call this loop μ. If μ bounded a disc in X_J, then we could map this to Y_J via $\tilde{b}$, but this would imply that $\tilde{b}(\mu) = \hat{\gamma}$ bounds a disc in Y_J, which it does not. Thus μ is non-trivial in $\pi_1(X_J) \neq 1$. Since $\pi_1(X_I) \to \pi_1(X_J)$ is surjective we deduce by Theorem 2.4.7 that $\pi_1(X_I) \neq 1$. $\square$

Recall that H is the kernel of g_*.

Theorem 4.3.10. *H is not finitely presented.*

Proof. Assume that H is finitely presented.

H acts cocompactly on X_I so we can add finitely many 2-cells to the quotient to gain fundamental group H. Taking a universal cover of the space obtained in this way we arrive at X_I with finitely many H-orbits of 2-cells attached, which is simply connected. In other words, there are finitely many H-orbits of loops which generate $\pi_1(X_I)$.

The direct limit of all the X_I is the space X which is CAT(0) and in particular contractible. We can pass to a larger interval such that the H-orbits of loops which generate $\pi_1(X_I)$ are trivial. In other words, the map $\pi_1(X_I) \to \pi_1(X_J)$ is trivial. But it is also surjective by Theorem 2.4.7 and because we have assumed that all the ascending and descending links are connected. Thus $\pi_1(X_J)$ is trivial, however we know this not to be the case by Lemma 4.3.9. □

Chapter 5

The subcomplex method

The second method we will present extends Lodha's work [27].

Definition 5.0.1. A graph Γ is *sizeable* if it satisfies the following conditions:

- Γ is bipartite on two sets A and B,
- Γ contains no loops of length 4,
- there exist partitions $A = A^+ \sqcup A^-$ and $B = B^+ \sqcup B^-$, such that $\Gamma(A^s \sqcup B^t)$ is connected for all $s, t \in \{-, +\}$.

Theorem C. *For $i = 1, 2$ and 3, let Γ_i be a sizeable graph with vertex set $A_i \sqcup B_i$. Let K_{ij} be the complete bipartite graph on A_i and B_j. Let X be the full cubical subcomplex of $K_{13} \times K_{21} \times K_{32}$ spanned by vertices $(v_1, v_2, v_3) \in K_{13} \times K_{21} \times K_{32}$ such that one of the following holds,*

- *$v_i \in A_i$ for all i,*
- *$v_i \in B_{i-1}$ for all i,*
- *$v_1 \in A_1, v_2 \in B_1$ and $[v_1, v_2]$ is an edge of Γ_1,*
- *$v_2 \in A_2, v_3 \in B_2$ and $[v_2, v_3]$ is an edge of Γ_2,*
- *$v_3 \in A_3, v_1 \in B_3$ and $[v_3, v_1]$ is an edge of Γ_3.*

Then X has hyperbolic fundamental group containing a finitely presented subgroup that is not hyperbolic.

We will give the proof of this theorem in several stages.

1. Terminology relating to the complex X.
2. Define a Morse function $\tilde{X} \to \mathbb{R}$, where $\tilde{X}$ is the universal cover of X.
3. Examine the ascending and descending links of the Morse function to see that $\pi_1(X)$ has a finitely presented subgroup which is not of type F_3.
4. Prove that $\tilde{X}$ is hyperbolic.

5.1 The complex X

Proof. By hypotheses, a vertex (v_1, v_2, v_3) is in X if one of the following conditions holds,

- $v_i \in A_i$ for all i,
- $v_i \in B_{i-1}$ for all i,
- $v_1 \in A_1, v_2 \in B_1$ and $[v_1, v_2]$ is an edge of Γ_1,
- $v_2 \in A_2, v_3 \in B_2$ and $[v_2, v_3]$ is an edge of Γ_2,
- $v_3 \in A_3, v_1 \in B_3$ and $[v_3, v_1]$ is an edge of Γ_3.

We say that a vertex is of type 1 if it satisfies either of the first two conditions and of type 2 otherwise. We include a cube in X if all the vertices defining it are in X.

Putting an orientation on each edge of K_{ij} by orienting it from A_i^s to B_j^t if $s = t$ and orienting it towards A_i^s otherwise. Cubulate S^1 with one vertex and one oriented edge. We define a map $h_i : K_{ij} \to S^1$ by mapping open edges homeomorphically to the open edge of S^1 respecting orientation, and extend this to a map

$$h : K_{13} \times K_{21} \times K_{32} \to S^1,$$
$$h(x, y, z) := h_1(x) + h_2(y) + h_3(z).$$

Restricting to X, we get a map $f : X \to S^1$. Lifting to universal covers we get a Morse function $\tilde{f} : \tilde{X} \to \mathbb{R}$ which is f_*-equivariant.

5.2 The ascending and descending links of f

We first examine the links of vertices and prove that X is a non-positively curved cube complex. We will then move on to looking at the ascending and descending links.

Notation 5.2.1. For a vertex v in Γ_i, let N_v be the set of vertices adjacent to v in Γ_i.

5.2.1 A type 1 vertex

Let $v = (v_1, v_2, v_3)$ be a vertex of type 1. We will consider the case where $v_i \in A_i$, the other case (where $v_i \in B_{i-1}$) being similar. Consider adjacent vertices in $K_{13} \times K_{21} \times K_{32}$ that are of the form $(v'_1, v_2, v_3), (v_1, v'_2, v_3)$ and (v_1, v_2, v'_3) where $v'_i \in B_{i-1}$. These vertices are in X under the following conditions:

- (v'_1, v_2, v_3) is in the complex if $[v'_1, v_3]$ is an edge of Γ_3.
- (v_1, v'_2, v_3) is in the complex if $[v'_2, v_1]$ is an edge of Γ_1.
- (v_1, v_2, v'_3) is in the complex if $[v'_3, v_2]$ is an edge of Γ_2.

Thus $Lk(v, X)^{(0)} = N_{v_1} \sqcup N_{v_2} \sqcup N_{v_3}$.

We now look at which edges will be in $Lk(v, X)$. There is an edge between the vertices corresponding to (v_1, v_2, v'_3) and (v_1, v'_2, v_3) if the vertex (v_1, v'_2, v'_3) is in the complex, since this will mean that the edges $[(v_1, v_2, v_3), (v_1, v'_2, v_3)]$ and $[(v_1, v_2, v_3), (v_1, v_2, v'_3)]$ are adjacent on a square. The vertex (v_1, v'_2, v'_3) is in the complex if $[v_1, v'_2]$ is an edge of Γ_1 which is the case above. Thus $Lk(v, X)^{(1)} = (N_{v_1} * N_{v_2} * N_{v_3})^{(1)}$. To understand the 2-skeleton we check which cubes are in X. This corresponds to checking vertices of the form (v'_1, v'_2, v'_3), we can see that $v'_i \in B_{i-1}$ so this vertex is always in the complex. We conclude that $Lk(v, X) = N_{v_1} * N_{v_2} * N_{v_3}$.

We now examine the ascending and descending links of v. We will examine the case of the ascending link, the descending link being similar.

$Lk_{\uparrow}(v, X)$ will consist of the full subcomplex of $Lk(v, X)$ corresponding to edges in the graphs K_{ij} oriented away from A_i. This is the join of three sets Q_{v_1}, Q_{v_2} and Q_{v_3} where $Q_{v_i} \subset N_{v_i}$ is the subset consisting of those edges in K_{ij} oriented away from A_i.

The ascending and descending links will all be simply connected as the join of three discrete sets. For a vertex $v_i \in \Gamma_i$, let $Q_{v_i} \subset N_{v_i}$ be the subset consisting of those edges in K_{ij} oriented away from v_i and $O_{v_i} \subset N_{v_i}$ be those edges oriented towards v_i. There is a vertex where at least one of Q_{v_i} and O_{v_i} contains at least 2 points. If this is not the case then every vertex has exactly one edge oriented towards it and one away. Since the subgraphs $\Gamma(A^s \sqcup B^t)$ are connected they would have to be segments meaning that Γ is a copy of $S^0 * S^0$ which contains a loop of length 4. Thus there is a vertex v such that at least one of $Lk_{\uparrow}(v, X)$ or $Lk_{\downarrow}(v, X)$ has non-zero second homology.

5.2.2 A type 2 vertex

Now examine the link of a type 2 vertex. We will look at a vertex $v = (v_1, v_2, v_3)$, where $v_1 \in A_1, v_2 \in B_1, v_3 \in A_3$ and $[v_1, v_2]$ is an edge of Γ_1. All other cases are similar.

We start by considering adjacent vertices which will correspond to the 0-skeleton of $Lk(v, X)$. We can see that the possible adjacent vertices are of the form $(v_1', v_2, v_3), (v_1, v_2', v_3)$ or (v_1, v_2, v_3'). All vertices of the form (v_1, v_2', v_3) are in X as these are type 1 vertices. Also vertices of the form (v_1, v_2, v_3') are in X as $[v_1, v_2]$ is an edge of Γ_1. Vertices of the form (v_1', v_2, v_3) are in the complex if $[v_1', v_3]$ is an edge of Γ_3. Therefore, $Lk(v, X)^{(0)} = A_2 \sqcup B_2 \sqcup N_{v_3}$.

We now consider which squares are in the complex X which will enable us to compute $Lk(v, X)^{(1)}$. We will do the three cases individually.

1. There is an edge between vertices corresponding to (v_1, v_2', v_3) and (v_1, v_2, v_3') if the vertex (v_1, v_2', v_3') is in the cube complex. We can see that $v_1 \in A_1, v_2' \in A_2, v_3' \in B_2$, which means this vertex is in the cube complex if $[v_2', v_3']$ is an edge of Γ_2.

2. There is an edge between vertices corresponding to (v_1', v_2, v_3) and (v_1, v_2, v_3') if the vertex (v_1', v_2, v_3') is in the cube complex. We can see that $v_1' \in B_3, v_2 \in B_1, v_3' \in B_2$, therefore this is a type 1 vertex and is always in the cube complex.

3. Similarly, there is an edge between vertices corresponding to (v_1', v_2, v_3) and (v_1, v_2', v_3) if the vertex (v_1', v_2', v_3) is in the complex. We can see that $v_1' \in$

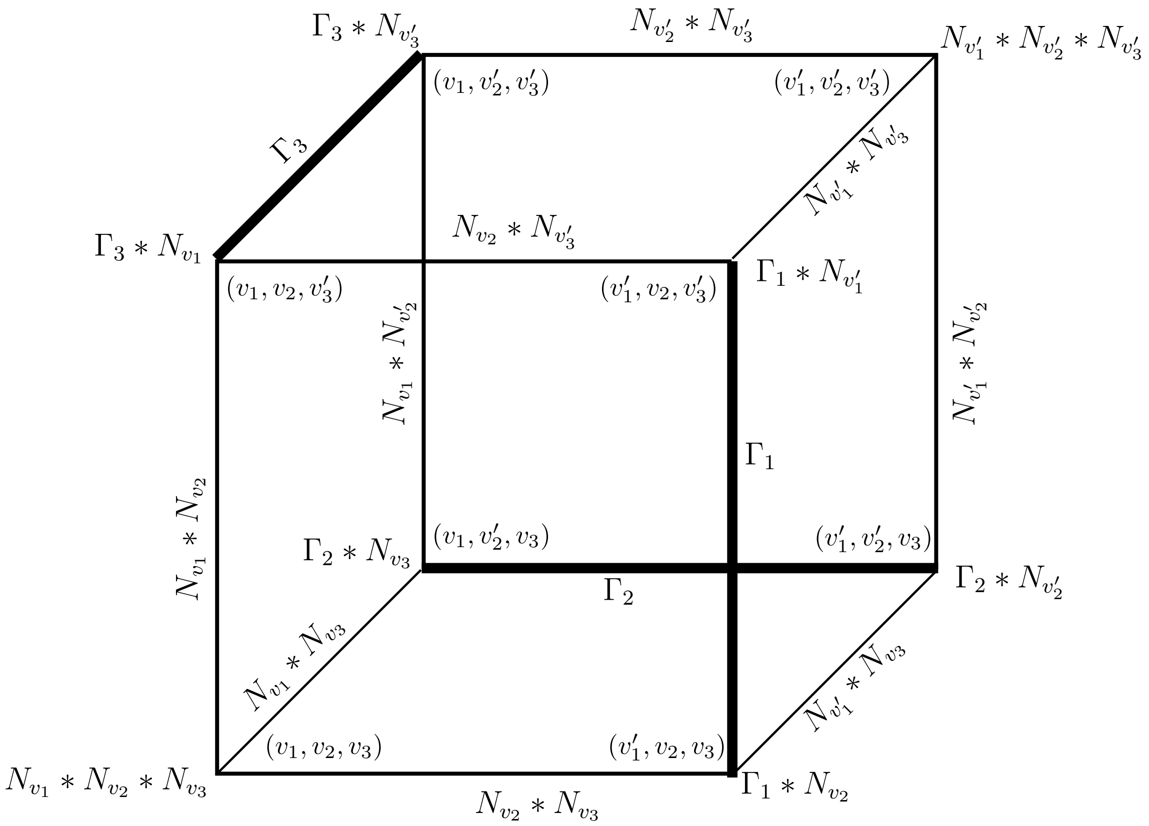


Figure 5.1: The link of each vertex and each edge of X. $v_i \in A_i$ and $v'_i \in B_{i-1}$. The bold edges are Γ-edges.

$B_3, v'_2 \in A_2, v_3 \in A_3$. So the vertex (v'_1, v'_2, v_3) is in the complex if $[v'_1, v_3]$ is an edge of Γ_3, which is the case if (v'_1, v_2, v_3) is in the complex.

Putting all of this together the 1-skeleton of $Lk(v, X)$ is the 1-skeleton of $\Gamma_2 * N_{v_3}$.

To ensure the complex is flag we must consider the cubes in X. Three vertices $(v'_1, v_2, v_3), (v_1, v'_2, v_3), (v_1, v_2, v'_3)$ which are pairwise adjacent in the link span a triangle if (v'_1, v'_2, v'_3) is in X. We can see that $v'_1 \in B_3, v'_2 \in A_2, v'_3 \in B_2$ and this is in the complex as long as $[v'_2, v'_3]$ is an edge of Γ_2, which is always the case, as $(v_1, v'_2, v_3), (v_1, v_2, v'_3)$ are adjacent in the link. This shows that $Lk(v, X) = \Gamma_2 * N_{v_3}$.

The ascending link of v is the full subcomplex spanned by those edges which are oriented away from v this is the join of Q_{v_3} with the full subgraph of Γ_2 spanned by $A^s \cup B^t$ for some $s, t \in \{-, +\}$. This is simply connected since Γ_2 is sizeable. For convenience of the reader we have depicted the link of each vertex in Figure 5.1.

5.3 Proof that $\tilde{X}$ is hyperbolic

To prove that $\tilde{X}$ is hyperbolic we use a similar argument to that of Section 4.2.

Recall, that a subset D of X intersects $\mathbb{E}^2$ transversally at a point p if there is an $\epsilon > 0$ such that $N_\epsilon(p) \cap D \cap \mathbb{E}^2 = \{p\}$.

In each 3-cube there are six type 2 vertices and two type 1 vertices. Between vertices of type 2 there are two types of edge, which can be determined by their link which is either Γ_i or a complete bipartite graph. This is shown in Figure 5.1. We will be concerned with those edges which have link Γ_i, which we will refer to as Γ-*edges*. Let C be the union of all Γ-edges. In a 3-cube the Γ-edges have the same arrangement as the edges of $\tilde{L}$ from Figure 4.3. We will prove that there cannot be an isometrically embedded flat plane. Firstly we will show that any flat plane would have a transverse intersection with C and we will see that around such a transverse point the angle sum must be at least 3π.

Assume that $i : \mathbb{E}^2 \to \tilde{X}$ is an isometric embedding. It is not contained in the 1-skeleton so there is a cube c where the intersection is 2 dimensional. If $i(\mathbb{E}^2) \cap c \cap C = \emptyset$, then we have intersected c in a neighbourhood of a type 1 vertex. We can develop this plane into cubes meeting at this type 1 vertex. If we develop the flat plane we see that it will intersect a Γ-edge in one of these cubes. This situation is shown in Figure 4.4 where the bold edges are Γ-edges.

We have reduced to the case $i(\mathbb{E}^2) \cap c \cap C \neq \emptyset$. From the intersection pattern pictured in Figure 4.3 the plane must intersect a Γ-edge in 1 point.

If the intersection point is in the interior of an edge, then we can see that it is a transverse intersection by taking a ball of radius less than the distance to either end point x.

If the intersection point with the Γ-edge is a vertex v, then we must prove that it intersects all Γ-edges at this vertex transversally. Since i was an isometric embedding, if we do not intersect all Γ-edges at v transversally, then there is a Γ-edge e' entirely contained in $i(\mathbb{E}^2)$. However, there is a cube e sharing a face with c in which this edge e' is contained. Thus $i(\mathbb{E}^2)$ will contain a polygonal subset in c and a Γ-edge e' in e. Considering $c \cup e$ as a subset of $\mathbb{E}^3$ we can see that the Γ-edge is not contained in the plane defined by the polygon in c. Therefore, the plane could not have been isometrically embedded.

We conclude that there is a transverse intersection point with C. Examining the angle sum around such a point will give the desired result.

Firstly, we will study the case where the intersection is in the interior of an edge e. The plane will intersect several cubes in various polygonal subsets each contributing an angle. Since we assumed that the plane was isometrically embedded these angles must sum to 2π. We can see that two such polygons joined along an edge corresponds to two cubes which share a face meeting along e, the angle contribution from these two polygons is π. To get an angle sum of 2π we must therefore have four cubes meeting around e but the link of e in X is Γ_i which contains no cycles of length less than 6.

If the transverse intersection point x is the endpoint of the edge e we will proceed similarly. We will assume that the link of e is Γ_1. An angle sum of π will correspond to the intersection of a half plane with 4 unit cubes in $\mathbb{E}^3$. We can define a map $Lk(x, \tilde{X}) \smallsetminus Lk(x, C) \to \Gamma_1$. This map is the corresponding map on links from the projection of a cube to a face. We see that an angle sum of π corresponds to a segment of length 2 in Γ_1, so an angle sum of 2π corresponds to a circuit of length 4. However, Γ_1 has no loops of length < 6.

Therefore, there are no isometrically embedded flat planes in $\tilde{X}$ and $\pi_1(X)$ is hyperbolic. $\square$

5.4 Euler characteristics

We once again have an infinite family of CAT(0) spaces, each having hyperbolic fundamental group. We prove that there are infinitely many non-isomorphic groups in this family by calculating the Euler characteristic of the spaces in terms of the sizeable graphs Γ_1, Γ_2 and Γ_3. We would like to improve this result to show that the groups are not commensurable. This work is ongoing.

Proposition 5.4.1. *Let* Γ_1, Γ_2 *and* Γ_3 *be sizeable graphs, such that* Γ_i *has vertices divided into* $A_i \sqcup B_i$ *as in Definition 5.0.1; also let* E_i *be the edge set of* Γ_i. *We will denote* $a_i = |A_i|, b_i = |B_i|$ *and* $e_i = |E_i|$. *The Euler characteristic of the cube complex*

$X = X(\Gamma_1, \Gamma_2, \Gamma_3)$ *is*

$$\begin{aligned} a_1a_2a_3 + b_1b_2b_3 + a_1e_2 + a_3e_1 + a_2e_3 + b_2e_1 + b_3e_2 + b_1e_3 \\ + e_1e_2a_3 + e_1a_2e_3 + a_1e_2e_3 + e_1e_2b_3 + e_1b_2e_3 + b_1e_2e_3 \\ - a_1a_2e_3 - a_1e_2a_3 - e_1a_2a_3 - b_1b_2e_3 - b_1e_2b_3 - e_1b_2b_3 \\ - e_1e_2 - e_2e_3 - e_1e_3 - a_1e_2b_3 - e_1b_2a_3 - b_1a_2e_3 - e_1e_2e_3. \end{aligned}$$

Proof. We examine the links of vertices. Vertices, edges and 2-cells in $Lk(v, X)$ correspond to edges, squares and cubes adjacent to v in X. Since each edge is adjacent to two vertices, each square is adjacent to four vertices and each cube is adjacent to eight vertices. The Euler characteristic can be computed by knowing the number of 0, 1 and 2-cells in the link of each vertex.

We start by examining the 8 types of vertices and their links. A vertex v in X is of the form (v_1, v_2, v_3); the 8 types correspond to v_i being in A_i or B_{i-1}. We will use N_{v_i} to denote the neighbours of v_i in the sizeable graph Γ_i or Γ_{i-1} and $n_i = N_{v_i}$.

(v_1, v_2, v_3)	Link of vertex	0-cells in link	1-cells in link	2-cells in link
$A_1 \times A_2 \times A_3$	$N_{v_1} * N_{v_2} * N_{v_3}$	$n_1 + n_2 + n_3$	$n_1n_2 + n_1n_3 + n_2n_3$	$n_1n_2n_3$
$A_1 \times A_2 \times B_2$	$\Gamma_3 * N_{v_1}$	$a_3 + b_3 + n_1$	$e_3 + n_1(a_3 + b_3)$	e_3n_1
$A_1 \times B_1 \times A_3$	$\Gamma_2 * N_{v_3}$	$a_2 + b_2 + n_3$	$e_2 + n_3(a_2 + b_2)$	e_2n_3
$A_1 \times B_1 \times B_2$	$\Gamma_3 * N_{v_3}$	$a_3 + b_3 + n_3$	$e_3 + n_3(a_3 + b_3)$	e_3n_3
$B_3 \times A_2 \times A_3$	$\Gamma_1 * N_{v_2}$	$a_1 + b_1 + n_2$	$e_1 + n_2(a_1 + b_1)$	e_1n_2
$B_3 \times A_2 \times B_2$	$\Gamma_1 * N_{v_1}$	$a_1 + b_1 + n_1$	$e_1 + n_1(a_1 + b_1)$	e_1n_1
$B_3 \times B_1 \times A_3$	$\Gamma_2 * N_{v_2}$	$a_2 + b_2 + n_2$	$e_2 + n_2(a_2 + b_2)$	e_2n_2
$B_3 \times B_1 \times B_2$	$N_{v_1} * N_{v_2} * N_{v_3}$	$n_1 + n_2 + n_3$	$n_1n_2 + n_1n_3 + n_2n_3$	$n_1n_2n_3$

Summing over the appropriate sets gives the desired result. We go through the first 2 cases in detail.

The number of 0-cells in the links of all vertices which are in $A_1 \times A_2 \times A_3$ is

$$\sum_{v_1 \in A_1} \sum_{v_2 \in A_2} \sum_{v_3 \in A_3} (n_1 + n_2 + n_3) = (a_1a_2 \sum_{v_3 \in A_3} n_3) + (a_1a_3 \sum_{v_2 \in A_2} n_2) + (a_2a_3 \sum_{v_1 \in A_1} n_1),$$

since each edge of Γ_i has one end point in A_i we can see that this is equal to

$$a_1a_2e_3 + a_1e_2a_3 + e_1a_2a_3.$$

The number of 1-cells in the links of all vertices which are in $A_1 \times A_2 \times A_3$ is

$$\sum_{v_1 \in A_1} \sum_{v_2 \in A_2} \sum_{v_3 \in A_3} (n_1n_2 + n_1n_3 + n_2n_3),$$

once again we can expand this. Using the fact that Γ_i is bipartite, this is equal to

$$e_1e_2a_3 + e_1a_2e_3 + a_1e_2e_3.$$

Finally, the number of 2-cells in the links of all vertices which are in $A_1 \times A_2 \times A_3$ is

$$\sum_{v_1\in A_1}\sum_{v_2\in A_2}\sum_{v_3\in A_3} n_1n_2n_3 = e_1e_2e_3.$$

We get similar results for the vertices in $B_3 \times B_1 \times B_2$.

For vertices in $A_1 \times A_2 \times B_2$ we get the following. The number of 0-cells across all such vertices is

$$\sum_{v_1\in A_1}\sum_{[v_2,v_3]\in E_2} (n_1 + a_3 + b_3) = a_1e_2a_3 + a_1e_2b_3 + e_1e_2.$$

The number of 1-cells across all such vertices is

$$\sum_{v_1\in A_1}\sum_{[v_2,v_3]\in E_2} (e_3 + n_1(a_3 + b_3)) = a_1e_2e_3 + e_1e_2a_3 + e_1e_2b_3.$$

The number of 2-cells across all such vertices is

$$\sum_{v_1\in A_1}\sum_{[v_2,v_3]\in E_2} e_3n_1 = e_1e_2e_3.$$

Repeating this for the other vertices gives the desired result. □

Using the construction detailed in Proposition 5.5.1 with $|A^+| = |B^+| = 2$ and $|A^-| = |B^-| = 1$ we can create sizeable graphs where $a_i = 4p = b_i$ and $e_i = 16p$ for all primes $p \geq 5$. The Euler characteristic of X in these examples is $-64p^2(p+6)$ and as such we have an infinite family of non-isomorphic hyperbolic groups each having a finitely presented subgroup not of type F_3.

5.5 A small example of a sizeable graph

Finally we present both a process for constructing sizeable graphs and show how we can distinguish the complexes constructed in this section from those constructed previously.

A sizeable graph was constructed in [27]. We can construct many examples using the procedure for hyperbolising 2-dimensional right angled Artin groups detailed in Section 4.1.

Proposition 5.5.1. *Let A and B be sets with partitions $A = A^+ \sqcup A^-$ and $B = B^+ \sqcup B^-$ where A^-, B^- are non-empty and $|A^+|, |B^+| > 1$. Let Γ be the complete bipartite graph on A and B. Let A_Γ be the associated right angled Artin group, K_Γ the classifying space from 2.2.6 and R_Γ the branched covering constructed in Section 4.1. Let $v \in R_\Gamma$ be a vertex mapping to $(0,0)$. Then $Lk(v, R_\Gamma)$ is sizeable.*

Proof. The link of a vertex is a cover $\overline{\Lambda}$ of the graph Λ. Λ is the complete bipartite graph on sets $A' = A \cup \{a_0\}$ and $B' = B \cup \{b_0\}$. Define $A'^+ = A^+$ and $A'^- = A' \setminus A'^+$, defining B'^+ and B'^- similarly.

The graph $\overline{\Lambda}$ has the following properties. It is bipartite as it is the cover of a bipartite graph and it has no cycles of length 4 since the branching process was designed to remove these. To check the last property we let $A^+_{\overline{\Lambda}}$ be the set of vertices mapping to $A'^+ \subset \Lambda$ and $A^-_{\overline{\Lambda}}$ the complement of these in the bipartite structure. We define $B^+_{\overline{\Lambda}}$ and $B^-_{\overline{\Lambda}}$ similarly.

We must prove that $\overline{\Lambda}(A^s_{\overline{\Lambda}} \cup B^t_{\overline{\Lambda}})$ is connected. $\Lambda(A'^s, B'^t)$ can be covered by finitely many loops $c_1, c_2, \ldots$ of length 4 such that $c_{m+1} \cap \cup_{i=1}^m c_m \neq \emptyset$ for all m. When taking the branched cover each c_i has connected preimage and the intersection will still be non empty so the resulting union will be connected. $\square$

Lastly, we consider one specific construction of a sizeable graph. This allows us to distinguish the complexes constructed from those in [27] and [8].

Definition 5.5.2. We define Λ to be the graph defined as follows.

The vertices of Λ are split into 4 sets $A_\Lambda = \{a_1, \ldots, a_9\}, B_\Lambda = \{b_1, \ldots, b_9\}, C_\Lambda = \{c_1, \ldots c_{10}\}, D_\Lambda = \{d_1, \ldots, d_9\}$. The edges of Λ are depicted in Figure 5.2.

Proposition 5.5.3. *Λ is a sizeable graph.*

Proof. Λ is clearly bipartite on $A = A_\Lambda \cup D_\Lambda$ and $B = B_\Lambda \cup C_\Lambda$. We can also see from Figure 5.2 that $\Lambda(S_\Lambda \cup T_\Lambda)$ is connected for $S \in \{A, D\}$ and $T \in \{B, C\}$. It is in fact contractible.

Finally we check that there are no loops of length 4 in Λ. There are 5 possibilities for a loop of length 4. These are:

1. a_i, b_j, a_k, c_l

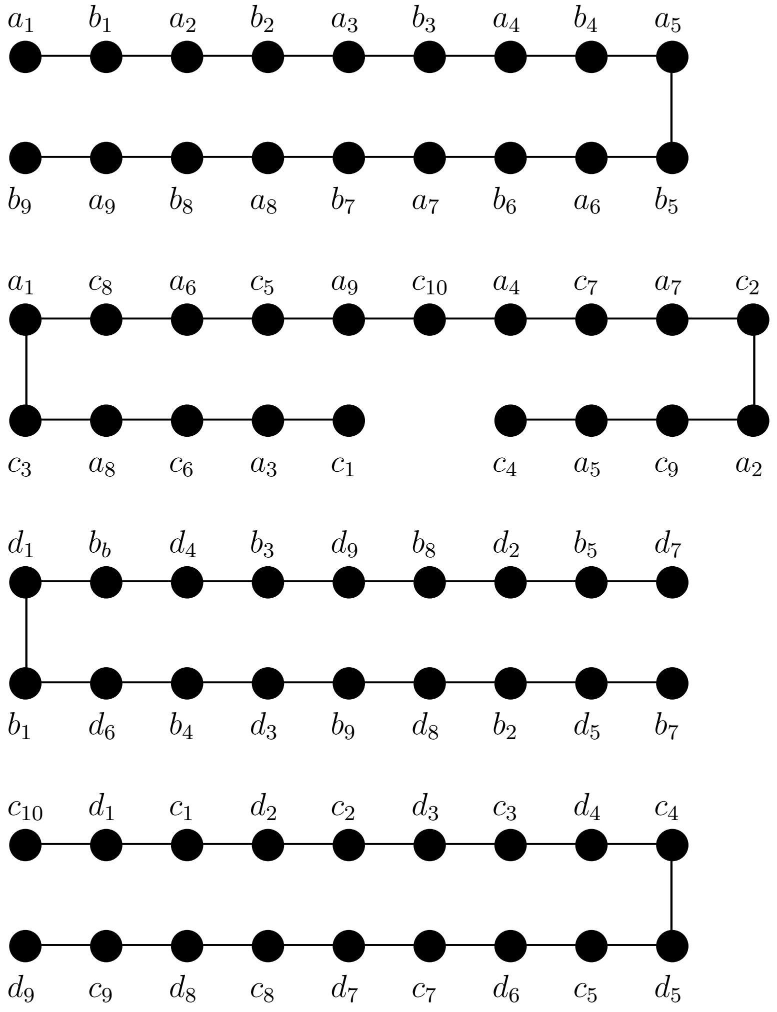


Figure 5.2: A description of the edges in the graph Λ.

2. a_i, b_j, d_k, b_l

3. a_i, c_j, d_k, c_l

4. d_i, b_j, d_k, c_l

5. a_i, b_j, d_k, c_l

In the first case we can see that a_i and a_k would be adjacent in both the (A, C) graph and the (A, B) graph. This can easily be verified to not be the case. The proof is similar for cases 2, 3 and 4.

The final case is more tricky. Consider the set $N_E(f_i) = \{e_j : [e_j, f_i] \in \Lambda\}$. This is the set of neighbours of f_i in E, where $E \in \{A, B, C, D\}$. We will examine the sets

$$N_1(a_i) = \bigcup_{b_j \in N_B(a_i)} N_D(b_j) \text{ and } N_2(a_i) = \bigcup_{c_k \in N_C(a_i)} N_D(c_k).$$

$N_1(a_i)$ is the set of all vertices in D that are at distance two from a_i such that the intermediate vertex is in B. N_2 is defined similarly. There exists a loop of length 4 if and only if $N_1(a_i) \cap N_2(a_i) \neq \emptyset$. One can check that this is not the case. We will do the example of a_5 to illustrate this point.

$N_B(a_5) = \{b_4, b_5\}$ and $N_1(a_5) = \{d_3, d_6, d_2, d_7\}$, similarly $N_C(a_5) = \{c_4, c_9\}$ and $N_2(a_5) = \{d_8, d_9, d_4, d_5\}$. These sets are disjoint and so there is no loop of length 4 containing a_5. We can repeat this for the other a_i to get the desired result. □

Remark 5.5.4. In this graph the subgraphs $\Lambda(S_\Lambda \cup T_\Lambda)$ are contractible for $S \in \{A, D\}$ and $T \in \{B, C\}$. When we create the cube complex X from Theorem C, the ascending and descending links come in two types. We have some which are contractible and those which are the join of three discrete sets each with 2 points.

The example of Brady [8] can be seen as a special case of Theorem B by taking a cage graph with 4 edges and in the proof taking $q_1 2 = q_2 3 = q_3 1 = 5$. The example of Lodha [27] is a special case of Theorem C, using the sizeable graph defined in [27]. In both the cases the ascending and descending links come in two varieties either a join of 3 discrete sets each with 2 points or the suspension of a loop of length at least 20. We can see that Remark 5.5.4 differentiates our complexes from those constructed in [8, 27].

Chapter 6

Groups of type F_{n-1} but not F_n

In [6], Question 8.5 Brady asks whether there exist groups of type F_n but not F_{n+1} which do not contain $\mathbb{Z}^2$; he notes that the known examples all contain $\mathbb{Z}^{n-1}$. While we are not able to find examples without $\mathbb{Z}^2$, we can drastically reduce the rank of a free abelian subgroup as shown by the following.

Theorem E. *For every positive integer n, there exists a group of type F_{n-1} but not F_n that contains no abelian subgroups of rank greater than $\lceil \frac{n}{3} \rceil$.*

Proof. For our general construction, we require 4-tuples (G_i, X_i, ϕ_i, f_i) for $i = 1, 2, 3$, where:

- G_i is a hyperbolic group,
- X_i is a CAT(0) cube complex with a free cocompact G_i action,
- $\phi_i : G_i \to \mathbb{Z}$ is a surjective homomorphism,
- f_i is a ϕ_i-equivariant Morse function.

We also require that all ascending and descending links of f_i are $(i-2)$-connected but at least one is not $(i-1)$-connected.

The existence of such a Morse function would show by Theorem 2.4.9 that G_i has a subgroup of type F_{i-1} not F_i, namely $\ker(\phi_i)$.

Let G_1 be a free group of rank 2, X_1 a 4-valent tree (the Cayley graph of G_i with respect to generators a and b), ϕ_1 the exponent sum homomorphism with respect to a and b, and f_1 the map that is linear on edges and whose restriction to the vertices of X_1 is ϕ_1.

Let (G_2, X_2, ϕ_2, f_2) be the group, classifying space, homomorphism and Morse function from Proposition 4.1.2.

Let (G_3, X_3, ϕ_3, f_3) be the group, classifying space, homomorphism and Morse function from Theorem B in the case where each vertex in Γ_j has valence 4 for all j. We could also use the examples of groups from [8, 27].

An important point to note is that for $i = 1, 2, 3$, the ascending and descending links for (G_i, X_i, ϕ_i, f_i) are isomorphic to S^{i-1}.

Now let n be an integer, let $l = \lfloor \frac{n}{3} \rfloor$, and let (G_l, X_l, ϕ_l, f_l) be the 4-tuple with

$$G_l = \prod_{i=1}^{l} G_3, \;\; X_l = \prod_{i=1}^{l} X_3,$$

$$\phi_l = \sum_{i=1}^{l} \phi_3, \;\; f_l = \sum_{i=1}^{l} f_3.$$

By Theorem 2.4.10 the ascending and descending links of f_l are $*_{i=1}^{l} S^2 = S^{3l-1}$ which is $(3l-2)$-connected but not $(3l-1)$-connected. If $n \equiv 0 \mod 3$, then $3l = n$, and we define $G = G_l$. If not, let m be the residue of $n \mod 3$ and consider the 4-tuple $(G = G_l \times G_m, X = X_l \times X_m, \phi = \phi_l + \phi_m, f = f_l + f_m)$. The ϕ-equivariant Morse function f on the cube complex X has ascending and descending links that are all copies of S^{n-1}. In all cases, $Ker(\phi)$ is of type F_{n-1} not F_n by Theorem 2.4.9.

The group G contains no free abelian subgroups of rank greater than $\lceil \frac{n}{3} \rceil$ since G_i as a hyperbolic group contains no copy of $\mathbb{Z}^2$. $\square$

Chapter 7

An obstruction to hyperbolisation in dimension ≥ 4

Our procedure for hyperbolising products of three graphs is special to dimension three. We will now show that in higher dimensions hyperbolisation via branched covers is not always possible.

Recall the process of taking a branched cover of X over a branching locus Y is the following,

1. Take a finite covering $\overline{X \smallsetminus Y}$ of $X \smallsetminus Y$.
2. Lift the piecewise Euclidean metric locally and consider the induced path metric on $\overline{X \smallsetminus Y}$.
3. Take the metric completion $\hat{X}$ of $\overline{X \smallsetminus Y}$.

Theorem F. *If X is a cube complex of dimension $n > 3$, and there is a cubical subcomplex isometric to T^n, then no non-positively curved branched cover of X is hyperbolic.*

Brady [8] states that Bestvina proved a similar result about T^5.

We will start by looking at the case of an n-torus and this will quickly imply the result. Let C be a branching locus for $X = T^n$, let $\overline{X \smallsetminus C}$ be the finite cover in the definition of branched cover and let Y be the completion of this finite cover.

Lemma 7.0.1. *Let e be an open cube contained in C of dim $\leq n-3$ and assume there does not exist a cube e' such that $\bar{e} \subsetneq e' \subsetneq C$. Let $C' = C \smallsetminus e$. Then there is a cover of $T^n \smallsetminus C'$ whose completion is isomorphically isometric to Y.*

Proof. Within C, e contained in no larger cube so each $x \in \mathring{e}$ has a neighbourhood $N(x)$ in T^n such that $N(x) \smallsetminus C$ is homeomorphic to $\mathbb{R}^n \smallsetminus \mathbb{R}^{\dim(e)}$. Then $N(x) \smallsetminus C$ is homotopy equivalent to a sphere of dimension at least 2, so is simply connected. This deleted neighbourhood lifts homeomorphically to the cover and when we complete we are just gluing back in the copy of $\mathbb{R}^{\dim(e)}$. This shows that in a neighbourhood of each point of e the branched covering map is an unramified cover. Thus we can remove e from the branching locus without affecting $\hat{X}$. $\square$

This lemma allows us to assume that the branching locus is a union of cubes of dimension $n-2$.

Lemma 7.0.2. *Suppose that $e = e^{n-2} \subset C$ has a free face e' (i.e. there is no other cube glued to e along e') and let $C' = C \smallsetminus (int(e) \cup e')$. Then there is a cover of $T^n \smallsetminus C'$ whose completion is isometrically isomorphic to Y.*

Proof. Let e' be a free face of e. Each point $x \in \mathring{e}'$ has a neighbourhood $N(x)$ such that $N(x) \smallsetminus C$ is homeomorphic to $\mathbb{R}^n \smallsetminus \mathbb{R}_+ \times \mathbb{R}^{n-4}$ which is simply connected. This deleted neighbourhood lifts to the cover, and when we take completions we are just gluing back in the copy of $\mathbb{R}_+ \times \mathbb{R}^{n-4}$. This shows that in a neighbourhood of the free face the branched covering map is an unramified covering.

Let x be an arbitrary point in the interior of e and let γ be a geodesic from x to the free face of e. The boundary ∂ of a regular neighbourhood of γ in T^n is homeomorphic to $S^{n-1} = S^{n-1}_{-} \cup (S^{n-2} \times [0,1]) \cup S^{n-1}_{+}$ where $S^{n-1}_{\pm}$ are hemispheres centered around the end points of γ. Then $(\partial \smallsetminus C) \simeq (S^{n-1} \smallsetminus S^{n-1}_{-} \cup (S^{n-2} \times [0,1])) \simeq \star$. A deleted neighbourhood of γ in $T^n \smallsetminus C$ lifts to the cover. So, again, the branched covering is unramified in a neighbourhood of γ, which shows that we can remove e from the branching locus. $\square$

Lemma 7.0.3. *A locally convex subcomplex of T^n of dimension $n-2$ without free faces is a disjoint union of copies of T^{n-2}.*

Proof. The locally convex subcomplex with no free faces is a closed non-positively curved manifold and defines a class in $H_{n-2}(T^n)$. The link of every vertex in T^n is a copy of S^{n-1} triangulated as $S^0 * \cdots * S^0$. After a possible cubical subdivision we can assume that the cubical neighbourhood of every vertex is $[-1,1]^n$. Each copy of

$[-1,1]$ in this product defines a generator of the fundamental group of T^n. In the subcomplex each vertex will have link isomorphic to $S^0 * \cdots * S^0 = S^{n-3}$ the $n-2$ copies of S^0 give $n-2$ generators showing that this submanifold has fundamental group $\mathbb{Z}^{n-2}$ and the manifold is an $(n-2)$-torus. $\square$

Proposition 7.0.4. *Let $b : Y \to T^n$ be a branched cover. If a component Δ of the branching locus is a copy of T^{n-2}, then the restriction of b to the preimage of Δ is a covering map.*

Proof. A deleted neighbourhood of Δ in T^n is homeomorphic to $T^{n-2} \times (D^2 \smallsetminus \{0\})$ any finite cover of this is again homeomorphic to $T^{n-2} \times (D^2 \smallsetminus \{0\})$ and its metric completion has the effect of adding the obvious $T^{n-2} \times \{0\}$. From this we see that the map on the preimage of Δ is an unramified covering map. $\square$

If we know that the cover is non-positively curved and the torus is locally convex, then the fundamental group of our cover will have a $\mathbb{Z}^2$ subgroup and will not be hyperbolic.

Proposition 7.0.5. *The preimage of C is locally convex and the cover is non-positively curved.*

Proof. Looking at a point in C we see a neighbourhood of the form $S^1 * D^{n-2}$ where S^1 has length 2π. The preimage of this in the cover will be several disjoint copies of $S^1 * D^{n-2}$ where the length of S^1 may now be greater than 2π. The copy of D^{n-2} is convex in $S^1 * D^{n-2}$ and so the embedding of the preimage of C is locally convex.

The link of a vertex will be copy of $S^1 * S^{n-3}$ which is a CAT(1) space, being the join of two CAT(1) spaces. Thus the cover is non-positively curved. $\square$

Theorem 7.0.6. *If X is an n-dimensional cube complex, $n > 3$, which contains an isometric copy of n-dimensional torus as a cubical subcomplex, then no non-positively curved branched cover of X is hyperbolic.*

Proof. If we look at a component of the preimage of T^n, then we will have a branched cover of T^n. This will contribute a $\mathbb{Z}^2$ subgroup to the fundamental group of the branched covering hence this is not hyperbolic. $\square$

In the case where X is the product of $n \geq 4$ graphs with no vertices of valence 1 every branched cover is CAT(0). Using the work proved earlier it is clear to see that we can assume that the branching locus L is a union of $n-2$ cubes. Let $\hat{L}$ be the preimage of L under the branching map b. Using the above we can also see that the map $b : \hat{L} \to L$ is a covering map.

Lemma 7.0.7 (Brady [8], Lemma 5.4)**.** *Let K be a non-postively curved cube complex, $L \subset K$ a branching locus, and $v \in L$ a vertex. Then $\mathrm{Lk}(v, L)$ is a full subcomplex of $\mathrm{Lk}(v, K)$.*

The following propostition tells us that this means the cover will be non-positively curved.

Proposition 7.0.8. *Let K be a non-positively curved cube complex, L be a branching locus. Let $\hat{L}$ be the preimage of L under the branched covering $b : \hat{K} \to K$. If the map $b : \hat{L} \to L$ is an unramified covering, then $\hat{K}$ is non-positively curved.*

Proof. The branched covering $\hat{K}$ is a cube complex and we must prove that the link of each vertex is a flag complex. Let v be a vertex of $\hat{K}$ as before we have the derivative map $b_{\mathrm{Lk}(v)} : \mathrm{Lk}(v, \hat{K}) \to \mathrm{Lk}(b(v), K)$. We must first show that the complex $\mathrm{Lk}(v, \hat{K})$ is simplicial. If not, there exist simplices $\sigma, \tau \subset \mathrm{Lk}(v, \hat{K})$ which are not equal but have the same boundary. Since $\mathrm{Lk}(b(v), K)$ is a simplicial complex we can see that $b_{\mathrm{Lk}(v)}(\sigma) = b_{\mathrm{Lk}(v)}(\tau)$.

A vertex w in σ defines an edge at v in $\hat{K}$. if $b(e) \not\subset L$ then the map $\overline{\mathrm{Lk}}(e, \hat{K}) \to \overline{\mathrm{Lk}}(b(e), K)$ is an isomorphism. Since $\mathrm{Lk}(e, \hat{K})$ can be identified with $\mathrm{Lk}(w, \mathrm{Lk}(v, \hat{K}))$. This implies that the map of links $\mathrm{Lk}(v, \hat{K}) \to \mathrm{Lk}(b(v), K)$ is locally injective at $w \in \mathrm{Lk}(v, \hat{K})$ contradciting $b_{\mathrm{Lk}(v)}(\sigma) = b_{\mathrm{Lk}(v)}(\tau)$.

This shows that all the vertices of σ is contained in $\hat{L}$. But the map $b : \hat{L} \to L$ is a covering map and hence locally injective once again contradicting the fact that $b_{\mathrm{Lk}(v)}(\sigma) = b_{\mathrm{Lk}(v)}(\tau)$.

Once it is simplicial we see that the simplex in K filling the 1-skeleton of $b_{\mathrm{Lk}(v)}(\sigma)$ lifts to fill the 1-skeleton of σ in $\mathrm{Lk}(v, \hat{K})$. $\square$

Chapter 8

More hyperbolic groups

One notable difficulty in creating groups that are of type F_n but not F_{n+1} as subgroups of hyperbolic groups is that the cohomological dimension of the ambient hyperbolic group is at least $n+1$. There are several methods of constructing hyperbolic groups with high dimension. However from the perspective of Morse theory, many of these are unsuitable for our purposes: they use systolic groups [4, 26], and any finitely presented subgroup of a systolic group is systolic [35] hence of type F_∞.

In what follows we construct hyperbolic groups which are the fundamental group of high-dimensional cube complexes. We anticipate in the future that these will produce examples of hyperbolic groups with subgroups with exotic finiteness properties.

An n-dimensional simplicial complex is *pure* if every point is contained in an n-simplex. A complex is 5-*large*, also known as "flag no square", if every 4-cycle has a chord see Figure 8.1.

With this required terminology we are ready to state our next construction.

Construction 8.0.1. *There is an algorithm that associates to each* 5*-large n-partite pure simplicial complex* Γ *and n-tuple of integers* $\mathbf{j} = (j_1, \dots, j_n)$ *with all* $j_i > 1$, *a non-positively curved cubed complex* $X_{\Gamma,\mathbf{j}}$.

Γ is n-partite, and is naturally contained in a join $A_1 * \cdots * A_n$. The cube complex $X = X_{\Gamma,\mathbf{j}}$ is a subcomplex of the product

$$P := \prod_{i=1}^{n} (A_i * B_i),$$

where B_i is a discrete set with j_i points.

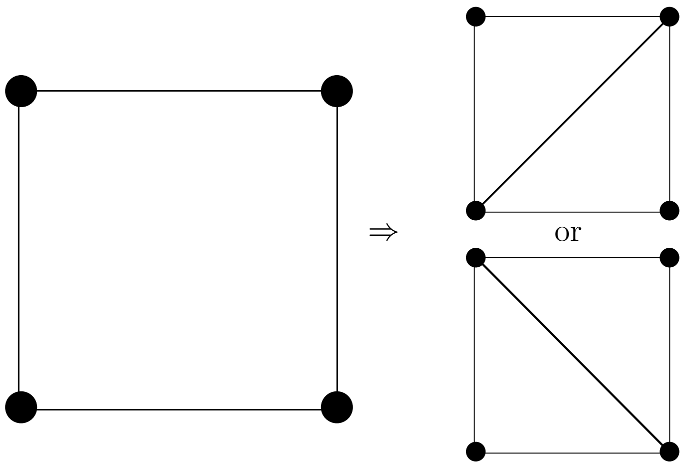


Figure 8.1: A depiction of the 5-large condition.

A vertex $v = (v_1, \ldots, v_n)$ of P is in the complex X if the set of vertices $\{v_i : v_i \in A_i\}$ spans a simplex of Γ. P has a natural cube complex structure as a product of graphs. We include a cube if all its defining vertices are in X.

Claim. *X is non-positively curved.*

Proof. Let us examine the link of a vertex $v = (v_1, \ldots, v_n)$. Let $W = \{v_i : v_i \in A_i\}$, $\Delta_W \subset \Gamma$ the simplex spanned by W and $I_v = \{i : v_i \notin B_i\}$. We will show that $Lk(v, X) = Lk(\Delta_W, \Gamma) *_{i \in I_v} B_i$. We first prove that these complexes are isomorphic at the level of 1-skeleta. We are then left to prove that $Lk(v, X)$ is flag; since the complex $Lk(\Delta_W, \Gamma) *_{i \in I_v} B_i$ is flag this will guarantee the required isomorphism.

We want to understand which vertices v' adjacent to v in P are contained in X. A vertex v' is adjacent to v in P if we can obtain v' by changing one coordinate of v. Thus a vertex in the link of v corresponds to changing one coordinate v_i of v.

There are two cases:

1. Switch $v_i \in A_i$ to $v'_i \in B_i$.

2. Switch $v_i \in B_i$ to $v'_i \in A_i$.

In the first case, the vertices in W' span a simplex $\Delta_{W'} < \Delta_W$ so the vertex v' is in X. In the second case, v' is in X only if $W \cup \{v'_i\}$ spans a simplex of Γ. $W \cup \{v'_i\}$ spans a simplex if and only if v'_i is adjacent to every vertex of W; that is, $v'_i \in Lk(\Delta_W, \Gamma)$. In conclusion, the 0-skeleton of $Lk(v, X)$ is $Lk(\Delta_W, \Gamma)^{(0)} \sqcup_{i \in I_v} B_i$.

Now that we have described the 0-skeleton, we will describe the 1-skeleton of the link. Two points of $Lk(v, X)$ are connected if there is a square between the corresponding edges in X. This results in checking whether changing both coordinates gives another vertex of X. As such we must examine the possibilities of changing two coordinates of v.

1. Change two A_i coordinates to two B_i coordinates.
2. Change one A_i coordinate and one B_j coordinate.
3. Change two B_i coordinates to two A_i coordinates.

In the first two cases the set W' spans a simplex in Γ. In the last case there is an edge if the two new a_i vertices are adjacent in Γ. This shows that the 1-skeleton of the link is isomorphic to $(Lk(\Delta_W, \Gamma) *_{i \in I_v} B_i)^{(1)}$.

We must now prove that $Lk(v, X)$ is flag. Given a set of pairwise adjacent vertices in the link $D = \{w_1, \dots w_k\}$, let $S = W \cup \{w_i \in D : w_i \in A_j\}$. The presence of the cube which fills this simplex in $Lk(v, X)$ corresponds to the existence in Γ of the simplex spanned by S. All the vertices in $S \setminus W$ are adjacent to all vertices in W by definition, and they are also pairwise adjacent, so the vertices of S are pairwise adjacent in Γ. Γ is a flag complex so S spans a simplex in Γ as required. $Lk(v, X)$ is flag and the complex is non-positively curved. $\square$

Theorem 8.0.2. *The fundamental group of $X = X_{\Gamma, \mathbf{j}}$ is hyperbolic.*

Proof. We are left to prove that the universal cover $\tilde{X}$ of X is hyperbolic, which by 2.1.16 requires us to prove that there are no flat planes in $\tilde{X}$.

We will start by assuming that there is an isometrically embedded flat plane and then show that there is a point on the flat plane where the angle sum is greater than 2π, providing a contradiction.

Let $i : \mathbb{E}^2 \to \tilde{X}$ be an isometric embedding of $\mathbb{E}^2$ into the universal cover of X. The flat plane cannot be contained in the 1-skeleton so there is a maximal dimensional cube such that the intersection is 2-dimensional. In such a cube there is a vertex with link $*_i B_i$.

A *generalised octahedron* is a join of r copies of S^0.

There is a generalised octahedron at this vertex into which this plane develops. The union of the cubes defining this generalised octahedron form the subdivision of a larger cube c. We will show that $E = i(\mathbb{E}^2)$ intersects the $(n-2)$-skeleton of c transversely in a point.

Take a point on E and take 2 non-parallel geodesics emanating from this point; these will eventually intersect the $(n-1)$-skeleton of c in two separate points. If these are in different cubes of the $(n-1)$-skeleton, then the sector in E spanned by the two geodesics has an intersection with the $(n-2)$-skeleton. If they are in the same cube then the geodesic between them is contained in E and extends to give an intersection of E with the $(n-2)$-skeleton.

Now that we have an intersection point with the $(n-2)$-skeleton of c, we must prove that there is a cube e of the $(n-2)$-skeleton which has a unique intersection point with E. If for some e this is not the case, then the cube e has two distinct points of intersection with E and the geodesic between them is contained in the plane. This geodesic extends to intersect another cube e' of dimension $< n-2$ in the $(n-2)$-skeleton of c. If we are not a unique intersection point with e' then we can repeat the process to intersect a cube in the boundary of e' which has dimension $< \dim(e')$, repeating if necessary we continue. This process will eventually terminate as the dimension of the intersecting cube decreases at each stage.

So let $e \subset c$ be a cube in the $(n-2)$-skeleton with $e \cap E = \{p\}$. In the case that the point p is in the interior of an $(n-2)$-cube then it is a transverse intersection. If this is not the case, then e can be chosen to have dimension $k < n-2$ and there are at least 3 cubes of dimension $k-1$ meeting around this lower dimensional cube. These all meet at angles of $\frac{\pi}{2}$ and so the plane can intersect at most two of these non transversely and therefore has a transverse intersection with any others. (A vertex is a transverse point of intersection.)

We will now show that around a transverse intersection point x the angle sum is greater than 2π giving us the desired contradiction. If x is a vertex of this larger cube then there are no geodesic loops of length 2π since the link of such a point is 5-large, so we are done. If the intersection is with a cube e of dimension > 0 in c then $L := Lk(x, X) = \Lambda * J_1 * \cdots * J_l$ where Λ is the link of a cell in Γ corresponding to e and J_i is a discrete set and we must study the possible loops of length 2π in L.

Given a geodesic loop of length 2π in L we can develop this into a flat plane E' in the CAT(0) cone $C_0(L)$ over L (see Definition 2.1.17). Since L is a join we know by Lemma 2.1.18 that $C_0(L) = C_0(\Lambda) \times C_0(J_1) \times \cdots \times C_0(J_l)$. Here, $C_0(J_i)$ is the union of several copies of $[0, \infty)$ identifying all the 0 points. Given a flat plane in this cone we can project to each factor; the image of this projection is either a point, a line or a plane. Since the point of intersection p is transverse the projection of E' to $C_0(\Lambda)$ must be either a line or a plane.

If the projection is a line, then there is a line in the plane E' perpendicular to $C_0(\Lambda)$. This would run parallel to the cube e, giving a contradiction to the transversality assumption.

We have now reduced to the case that the projection is a plane. Consider two parallel geodesics γ_1 and γ_2 in the plane E'. These project to geodesics in the projection of the plane E', which are still parallel as the projection is distance non-increasing. We now have two parallel geodesics in $C_0(\Lambda)$ so by Theorem 2.1.19 they bound a flat strip. This would give us a loop of length 2π in Λ, this cannot be the case as Λ is a 5-large flag complex. This completes the proof that the complex is hyperbolic. $\square$

We would like to use this construction to construct interesting examples of hyperbolic groups with exotic subgroups. To do this we would like to have a Morse function on this cube complex with nice ascending and descending links.

Theorem 8.0.3. *Let Γ be a pure 5-large complex and X be the cube complex constructed above. There is a Morse function $f : \tilde{X} \to \mathbb{R}$ for which the ascending and descending links are joins of full subcomplexes of Γ with discrete sets.*

Proof. We start by defining functions $A_i * B_i \to S^1$ we then extend this to the product in an obvious way. We define such a function by dividing the sets A_i and B_i into disjoint subsets A_i^+, A_i^- and B_i^+, B_i^-. We cubulate S^1 as $\{a_+, a_-\} * \{b_+, b_-\}$ and we send all the vertices in A_i^s to a_s and all the vertices in B_i^s to b_s. We put an orientation on S^1 so that an edge from a_s to b_t is oriented towards b_t if and only if $s = t$. We extend this to the product $\prod_i A_i * B_i$ and restrict to the complex X. Lifting to universal covers gives us our Morse function f.

We must now examine the ascending and descending links of this Morse function; we will examine the ascending link of a vertex, the computation for the descending link being similar. We will examine the vertex $v = (v_1, \dots, v_n)$. Let $W = \{v_i : v_i \in A_i\}$, Δ_W be the simplex spanned by W and $I_v = \{i : v_i \notin B_i\}$. We know that $Lk(v, X) = Lk(\Delta_W, \Gamma) *_{i \in I_v} B_i$. We will now consider which of the vertices of $Lk(v, X)$ are in the ascending link. By the orientation we put on S^1 $Lk_{\uparrow}^{(0)}(v, X)$ this will be $\cup L_i$ where:

$$L_i = \begin{cases} Lk(\Delta_W, \Gamma) \cap A_i^+ & \text{if } v_i \in B_i^-, \\ Lk(\Delta_W, \Gamma) \cap A_i^- & \text{if } v_i \in B_i^+, \\ B_i^+ & \text{if } v_i \in A_i^+, \\ B_i^- & \text{if } v_i \in A_i^-. \end{cases}$$

We know from Lemma 2.4.11 that $Lk_{\uparrow}(v, X)$ will be the full complex spanned by $Lk_{\uparrow}^{(0)}(v, X)$, and this is isomorphic to the simplicial join of a full subcomplex of Γ and several discrete sets B_i^s. □

Corollary 8.0.4. *Let Γ be a pure 5-large complex. Then there is a cube complex with hyperbolic fundamental group and a Morse function such that each of the ascending and descending links is either contractible or a full subcomplex of Γ.*

Proof. Take the cube complex constructed above with B_i a set with two elements and B_i^+ a set with one element. The ascending and descending links will either be full subcomplexes of Γ or be the cone over another space. □

8.0.1 Strategy for future

Unfortunately, at this time we have been unable to construct interesting examples of 5-large complexes which will give us exotic subgroups. We hope that in the near future we will both find interesting such examples and strengthen the above construction to allow for a weakening of the 5-large condition.

We are looking for 5-large complexes Γ on vertex sets $A_1, \dots, A_n$ such that the full subcomplex spanned by the vertices in $\Gamma(s_1, \dots, s_n) := \coprod_i A_i^{s_i}$ has certain homological properties for any choices of $s_i \in \{+, -\}$ (e.g. homology in only dimension k). Another way to try and find such a complex is to start with the complexes $\Gamma(s_1, \dots, s_n)$ and glue them together along the full subcomplexes $A_i^{s_i}$. We can do this in the case of a graph Γ: examples of such graphs include the sizeable graphs constructed earlier. We can however relax this and allow that Γ be n-partite rather than bipartite. The precise definition of such a graph is as follows.

Definition 8.0.5. A graph Γ is *n-sizeable* if it has an n-partite structure $\Gamma \subset A_1 * \cdots * A_n$ satisfying the following conditions:

- Γ is 5-large,
- and there exist partitions $A_i = A_i^+ \sqcup A_i^-$ such that $\Gamma(A_1^{s_1} \sqcup \cdots \sqcup A_n^{s_n})$ is connected for all $s_i \in \{-, +\}$.

Given n-sizeable graphs (e.g. the 2-sizeable graphs constructed earlier), we are able to construct more examples of hyperbolic groups with subgroups which are finitely generated but not finitely presented.

Corollary 8.0.6. *Given an n-sizeable graph Γ there is a cube complex X_Γ with hyperbolic fundamental group which contains a subgroup which is finitely generated but not finitely presentable.*

Proof. Taking the above construction with B_1 a set with 3 vertices and B_i a set with 2 vertices for $i \neq 1$. We obtain a cube complex with a Morse function that has connected ascending and descending links. One of the ascending links is not simply connected. From Theorem 2.4.7 the kernel is finitely generated but not finitely presentable. $\square$

Part III

A brief foray into 4-manifolds

Chapter 9

A crash course in intersection forms

For a closed oriented 4-manifold M the *intersection form* is a symmetric 2 form $Q_M\colon H_2(M;\mathbb{Z}) \times H_2(M;\mathbb{Z}) \to \mathbb{Z}$. Since each class in $H_2(M;\mathbb{Z})$ is represented by a surface, the image of this form can be seen as the class in $H_0(M;\mathbb{Z})$ that counts (with sign) the points of intersection between the representatives for the two homology classes.

Lemma 9.0.1 (Scorpan [31], p.118). *If $N = M_1 \# M_2$ is a connected sum of M_1 and M_2 of two 4-manifolds, then $Q_N = Q_{M_1} \oplus Q_{M_2}$.*

We will be concerned with simply connected 4-manifolds M. By the Hurewicz isomorphism; given a simply connected 4-manifold every class in $H_2(M;\mathbb{Z})$ can be represented by a map of S^2 into M. Using the fact that M is simply connected $H_2(M;\mathbb{Z})$ is a free $\mathbb{Z}$-module, so the intersection form can be described by a matrix.

One invariant of intersection forms that we will use is their definiteness: recall that Q_M is *positive definite* if for all non-zero $\alpha \in H_2(M;\mathbb{Z})$ we have $Q_M(\alpha,\alpha) > 0$ and *negative definite* if $Q_M(\alpha,\alpha) < 0$, for all $\alpha \neq 0$. If the form is neither positive or negative definite then it is termed indefinite.

Also recall, two 4-manifolds M, M' are *h-cobordant* if there is a 5-manifold W such that $\partial W = M \sqcup \overline{M'}$.

A theorem of Wall relates intersection forms to h-cobordisms.

Theorem 9.0.2 (Wall [32]). *If M_1 and M_2 are smooth, simply connected, closed 4-manifolds and Q_{M_1} is isomorphic to Q_{M_2} then M_1 is h-cobordant to M_2.*

Once we have our h-cobordism W we can arrange it so that there only 2 and 3-handles. After attaching all 2-handles, the upper boundary of the cobordism is a submanifold along which we can cut W. If we examine the topology of the resulting manifold we get the following corollary.

Corollary 9.0.3. *If M_1, M_2 are smooth, simply connected, 4-manifolds which are h-cobordant then for some k, $M_1 \#_k (S^2 \times S^2)$ is diffeomorphic to $M_2 \#_k (S^2 \times S^2)$.*

Notation 9.0.4. The minimal number of $S^2 \times S^2$ summands required above will be denoted $k_{\text{Wall}}(M, N)$.

By combining Theorem 9.0.2 with the following theorem of Freedman we can see that intersection forms are a very strong invariant of 4-manifolds.

Theorem 9.0.5 (Freedman [21])**.** *If W is a h-cobordism between simply connected 4-manifolds M and N then W is homeomorphic to $M \times [0, 1]$.*

Corollary 9.0.6. *If M and N are smooth simply connected 4-manifolds and Q_M, Q_N are isomorphic, then M and N are homeomorphic.*

We will require three more theorems on intersection forms, the first of which allows us to realise automorphisms of intersection forms via self diffeomorphims of certain 4-manifolds.

Theorem 9.0.7 (Wall [33])**.** *If Q_M is indefinite or has rank at most 8 then any automorphism of $Q_{M\#S^2\times S^2}$ can be realised by a self diffeomorphism of $M\#(S^2 \times S^2)$.*

The exclusion of large rank definite forms in Theorem 9.0.7 is less troublesome in the light of the following.

Theorem 9.0.8 (Donaldson [19])**.** *If M is a smooth, simply connected, closed 4-manifold with Q_M definite then M is homeomorphic to one of $S^4, m\mathbb{CP}^2$ or $m\overline{\mathbb{CP}}^2$.*

Finally, we would like to be able to read a description of the intersection form from a handlebody diagram; this can be done easily in the case where there are no 1 or 3-handles.

Definition 9.0.9. Given a framed link $L = (L_1, L_2, \ldots, L_n)$, the *linking matrix* A is the $n \times n$-matrix given by $A_{ij} = \text{Lk}(L_i, L_j)$ for $i \neq j$, and A_{ii} =framing of L_i, where $1 \leq i, j \leq n$.

Theorem 9.0.10 (Gompf-Stipsicz [23], §4.5)**.** *If we have a closed 4-manifold with no 1 or 3-handles then the matrix for the intersection form is equal to the linking matrix of the 2-handles.*

Chapter 10

A bound on stabilisation

Theorem G. *If M and N are homeomorphic simply connected, smooth, closed 4-manifolds and Q_M is indefinite or of rank at most 8 then,*

$$k_{\mathrm{Wall}}(M,N) \leq \max_{J=M,N}\{h_1^J + h_3^J + 1\}$$

where h_i^J is the number of i-handles in a handle decomposition of the manifold J.

If Q_M is definite of rank greater than 8 then,

$$k_{\mathrm{Wall}}(M,N) \leq \max_{J=M,N}\{h_1^J + h_3^J + 1, 2\}.$$

Proof. The proof will proceed in 2 steps:

1. Cancel all 1 and 3-handles using an $S^2 \times S^2$ for each one. The resulting manifold will have a handle decomposition with only 2-handles.
2. Theorem 9.0.10 describes the intersection form of this manifold. Adding another $S^2 \times S^2$ summand we can assume that this form is indefinite. We then apply Theorem 9.0.7.

Now we move onto the application of Kirby calculus and handle sliding to bounding the constant as above.

A presentation of the fundamental group of a manifold is determined by the 1- and 2-handles; each 1-handle corresponds to a generator and the (homotopy class of the) attaching map of a 2-handle gives a relator (cf. [23, p.149]).

The presentation for the fundamental group can be changed by adding cancelling handle pairs and sliding handles in the following ways.

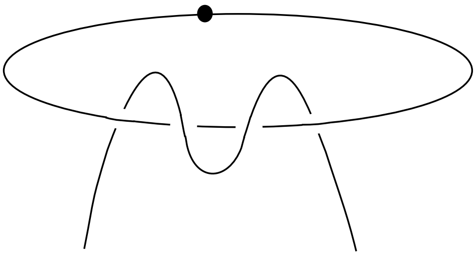


Figure 10.1: Conjugating a relator.

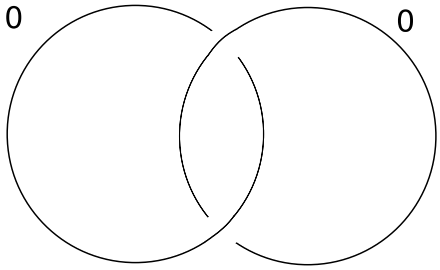


Figure 10.2: A Kirby diagram for $S^2 \times S^2$.

1. Adding a cancelling 1 and 2-handle pair (cf. [23, p.146]) corresponds to adding a generator and a relator killing it.

2. Adding a cancelling 2 and 3-handle pair (cf. [23, p.147]) corresponds to adding a trivial relator to the presentation.

3. Sliding the 2-handle corresponding to r_1 over the 2-handle corresponding to r_2 replaces r_1 with $r_1 r_2$.

4. Sliding the 1-handle corresponding to x_1 over the 1-handle corresponding to x_2 replaces every instance of x_1 with x_2 in the relations.

5. We can conjugate relators by generators by sliding 2-handles through 1-handles as in Figure 10.1.

We now proceed with Step 1 showing how to cancel any chosen 1-handle using an $S^2 \times S^2$ summand.

When taking the connected sum with $S^2 \times S^2$ we just add Figure 10.2 into the Kirby diagram for our manifold. We then slide the left-hand handle of this figure until it intersects our chosen 1-handle in 1 point, up to homotopy. If we wish to cancel this 1-handle we need to make sure that it intersects it in one point, up to isotopy. To do

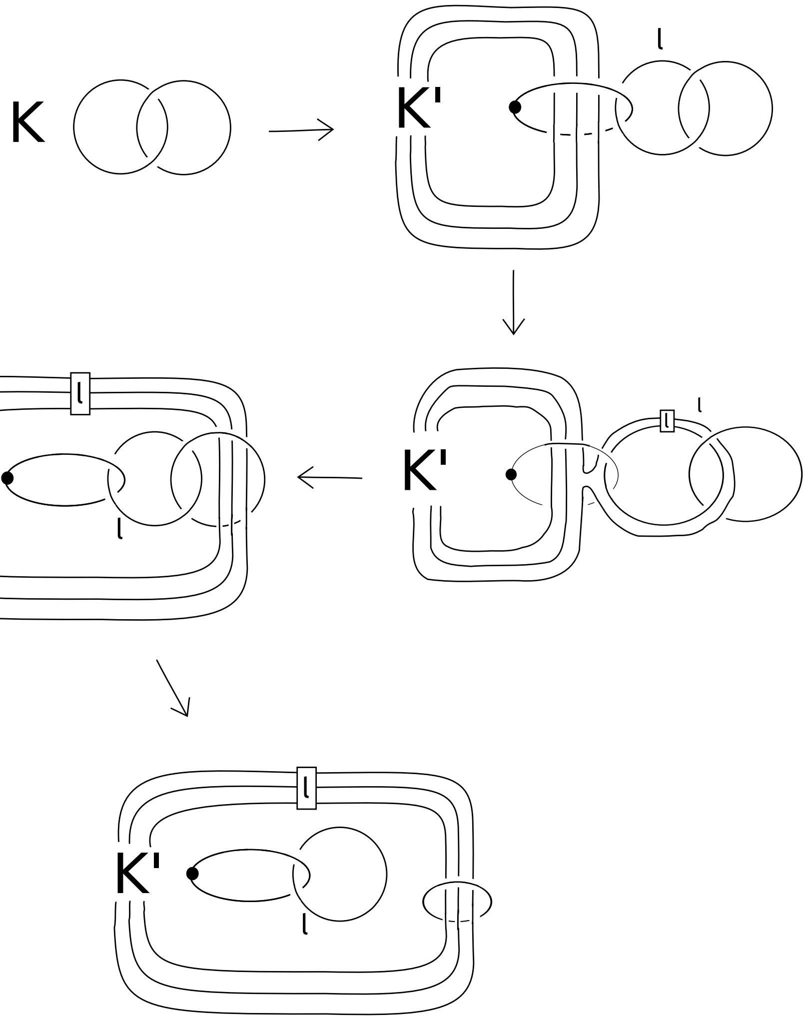


Figure 10.3: Sliding a copy of $S^2 \times S^2$ to cancel a 1-handle.

this we use the right-hand 2-handle in Figure 10.2 to make sure that the intersection is a single point, up to isotopy. The issue is that the left-hand circle could be knotted with itself to rectify this we can use the right-hand circle to switch any overcrossing or this not into an undercrossing. This will eventually turn the knot into the unknot and we can make the intersection a single point isotopically. We will then be in the situation depicted in the top right of Figure 10.3. Our chosen 1-handle is the dotted circle in the picture, the handle with framing l is the left-hand 2-handle and the far right-hand 2-handle is the right-hand 2-handle from Figure 10.2. We also show in Figure 10.3 how to rectify this so as to cancel the 1-handle.

We repeat the process from the preceding paragraph until there are no 1-handles in the Kirby diagram. We then consider a dual diagram where k-handles are switched with $(4-k)$-handles. This diagram will have only $0, 1, 2$ and 4-handles. We use the same technique to cancel the 1 handles in this new diagram. We are now left with a manifold which has no 1 or 3-handles. We can stabilise with one more $S^2 \times S^2$ summand then we can proceed with step 2.

We have seen that we require one summand to get rid of each 1-handle and each 3-handle and so we have shown $k_{\text{Wall}} \leq \max\limits_{J=M,N}\{h_1^J + h_3^J + 1\}$.

In the definite case, if there are no 1 or 3-handles, then we take a connected sum with $S^2 \times S^2$ first thus making the form indefinite. We are now left with a manifold which has no 1 or 3-handles and indefinite intersection form. We can stabilise with one more $S^2 \times S^2$ summand then we can proceed with step 2. □

This theorem can be made better in the case where the manifolds in question are the doubles of certain compact 4-manifolds.

Theorem 10.0.1. *If M and N are the doubles of smooth, compact, simply connected 4-manifolds with just 1 and 2-handles, then*

$$k_{\text{Wall}}(M, N) \leq \max_{J=M,N}\{h_1^J - 1\}.$$

Corollary 10.0.2. *If M is the double of a smooth, compact, contractible, 4-manifold with just 1 and 2-handles then $M \# k S^2 \times S^2 \cong k(S^2 \times S^2)$ where $k = h_1^J - 1$.*

Proof of Theorem 10.0.1. We proceed as before cancelling a chosen 1-handle with a copy of $S^2 \times S^2$. We will then be in the situation at the end of Figure 10.3, however, each 2-handle now has a dual 2-handle so we proceed as in Figure 10.4 to cancel a 3-handle.

Once we have cancelled all but one 1-handle and one 3-handle, we use the dual 2-handles to move all but two 2-handles away from this 1-handle. This will allow us to see our new 4-manifold as the connected sum of a Mazur manifold [2] and some 2-sphere bundles. It was proved in [2] that Mazur manifolds are diffeomorphic to S^4 and so we are left with a connected sum of 2-sphere bundles.

Once we have done this for both manifolds M and N we will be left with the same number of twisted and untwisted 2-sphere bundles in each case since they have the same intersection form. They will then be diffeomorphic.

This is how the bound of $\max\limits_{J=M,N}\{h_1^J - 1\}$ can be realised. □

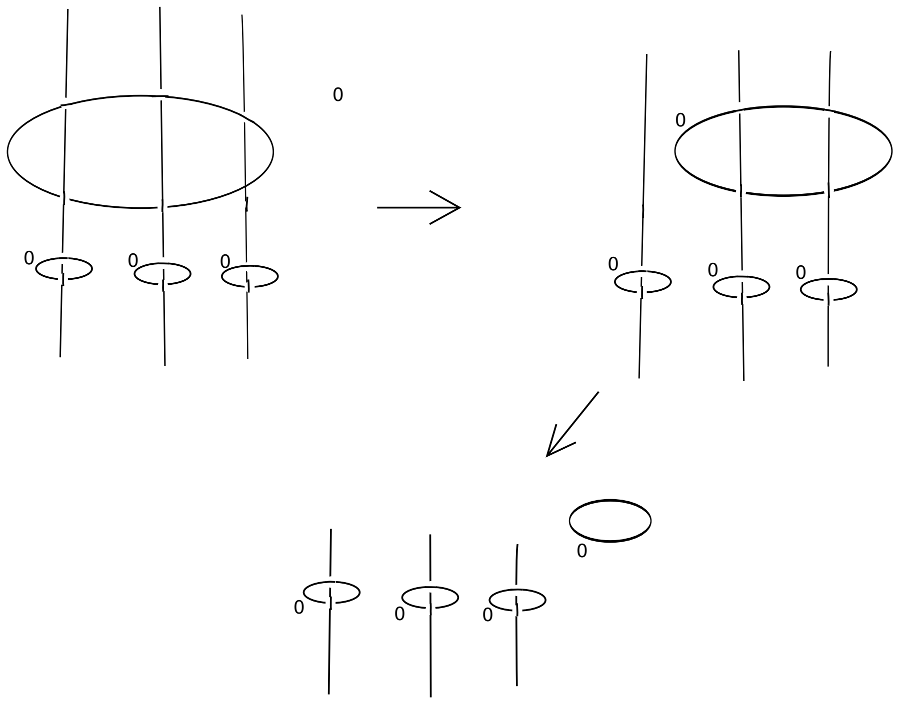


Figure 10.4: Sliding the extra 2-handle to cancel a 3-handle.

Question 1. *Can we improve on this bound? For instance can we reduce it to 1 stabilisation?*

There are relations between the work on doubles and a partial Andrews-Curtis conjecture.

Conjecture 10.0.3 (Andrews-Curtis [3])**.** *If we are given a balanced presentation of the trivial group*

$$\langle x_1, \ldots, x_n \mid r_1, \ldots, r_n \rangle$$

then we can transform this to the presentation

$$\langle x_1, \ldots, x_n \mid x_1, \ldots, x_n \rangle$$

through the following moves:

1. *Replace r_i by $r_i r_j$ for $i \neq j$,*
2. *replace r_i by $w^{-1} r_i w$ for some word w in the free group on $x_1, \ldots, x_n$.*

We could add two extra moves above:

1. Add a new generator a and a new relator a killing it.
2. Add a new relator equal to the empty word e.

The first of these extra moves does not seem to add to the conjecture. However, the second allows us to realise any Tietze transformation and so the conjecture becomes true. Ideally we would like to be able to minimise the use of this move.

Question 2. *If we allow this extra move can we restrict ourselves to a single use?*

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Appendices

Appendix A

Computing Presentations for Branched Covers

> This is like being in a house built by a child using nothing but a hatchet and a picture of a house.
>
> XKCD 1513: Code Quality

A.1 Theoretical Computations

The process of taking a branched cover has a drastic effect on the fundamental group. For instance, there are branched covers of T^3 which have hyperbolic fundamental groups in stark contrast to the free abelian fundamental groups of unramified covers of T^3.

In this section we are going to give a method for computing the fundamental group of a branched cover. The process, while not simple, can be effectively computed in certain cases. We have written a GAP program to compute the presentations of the fundamental group of certain branched covers constructed in this thesis.

Recall that by taking a branched covering of K over L we mean the following process.

1. Take a finite covering $\overline{K \setminus L}$ of $K \setminus L$.

2. Lift the piecewise Euclidean metric locally and consider the induced path metric on $\overline{K \setminus L}$.

3. Take the metric completion $\hat{X}$ of $\overline{K \setminus L}$.

Also recall there is a natural length-preserving map $b\colon \hat{K} \to K$.

There is a commutative diagram in the category of topological spaces for the branched covering of K over L:

$$\begin{tikzcd}
\overline{K \smallsetminus L} \arrow[r, hook] \arrow[d, "p"] & \hat{K} \arrow[d, "b"] \\
K \smallsetminus L \arrow[r, hook] & K
\end{tikzcd}$$

The first stage of computing the fundamental group is to compute the fundamental group of $K \smallsetminus L$. This is possibly the hardest stage and in general there is no concrete method. We have only done this in the special case below where some of the logistical problems already come to light.

Once this has been computed we have a prescribed map to a finite symmetric group S_m and we are looking at the stabiliser of a point in S_m. This is the fundamental group of $K \smallsetminus L$. We can use the Todd-Coxeter algorithm to compute a presentation of this finite index subgroup.

Finally we take a completion which corresponds to gluing back the lifts of the branching locus; this will add some extra relations. Thus, $\pi_1(\hat{K})$ is a quotient of $\pi_1(K \smallsetminus L)$. We examine this last step now.

By assumption K is a non-positively curved cube complex and L is a locally convex subcomplex satisfying the conditions of Definition 2.3.1. There is an $\epsilon > 0$ such that $N(L) = N_\epsilon(L)$ deformation retracts onto L and the map from

$$\pi_1\big(N(L) \smallsetminus L\big) \to \pi_1(L)$$

is surjective. Let $\overline{L}$ be the preimage of L in $\hat{K}$. We have the following commutative diagram:

$$\begin{tikzcd}
N(\overline{L}) \arrow[d] & N(\overline{L}) \cap \overline{K \smallsetminus L} \arrow[l] \arrow[r] \arrow[d] & \overline{K \smallsetminus L} \arrow[d] \\
N(L) & N(L) \cap (K \smallsetminus L) \arrow[l] \arrow[r] & K \smallsetminus L.
\end{tikzcd}$$

K is the pushout of the bottom row and $\hat{K}$ is the pushout of the top row. We are only left to compute the maps on fundamental groups.

We first compute the fundamental group of K by considering the relations added by taking the pushout of

$$N(L) \longleftarrow N(L) \cap K \smallsetminus L \longrightarrow K \smallsetminus L.$$

Since L is finite we are adding finitely many relations $r_1, \dots, r_n$.

$$\pi_1(K) = \pi_1(K \smallsetminus L)/\langle\langle r_1, \dots, r_n \rangle\rangle,$$

lifting these relations to $\overline{K \smallsetminus L}$ corresponds to taking the intersection of $\langle\langle r_1, \dots, r_n \rangle\rangle$ with $\pi_1(K \smallsetminus L)$. Since each relation comes from a disc in $N(L)$ and L is a branching locus we can pick a disc such that the point not in L are dense. Thus

$$\pi_1(\hat{K}) = \pi_1(\overline{K \smallsetminus L})/\langle\langle r_1, \dots, r_n \rangle\rangle \cap \pi_1(\overline{K \smallsetminus L}).$$

While this $\langle\langle r_1, \dots, r_n \rangle\rangle \cap \pi_1(\overline{K \smallsetminus L})$ is finitely normally generated it is not necessarily the case that it is finitely generated. We would like to be able to give a finite list of relations for this normal subgroup.

We can consider the group $\langle r_1, \dots, r_n \rangle < \pi_1(K \smallsetminus L)$. Note this is just a subgroup and not a normal subgroup. We have that

$$\langle r_1, \dots, r_n \rangle \cap \pi_1(\overline{K \smallsetminus L}) < \langle\langle r_1, \dots, r_n \rangle\rangle \cap \pi_1(\overline{K \smallsetminus L}).$$

This is unlikely to be all the relations. However each other relation in

$$\langle\langle r_1, \dots, r_n \rangle\rangle \cap \pi_1(\overline{K \smallsetminus L})$$

is a different lift of an element in $\langle\langle r_1, \dots, r_n \rangle\rangle$ which differs from an element of $\langle r_1, \dots, r_n \rangle$ by a path in $\overline{K \smallsetminus L}$. The path in $\overline{K \smallsetminus L}$ projects down to a loop in $K \smallsetminus L$. These loops correspond to coset representatives $l_1, \dots, l_m$ for $\pi_1(\overline{K \smallsetminus L})$ in $\pi_1(K \smallsetminus L)$.

The subgroup $\langle\langle r_1, \dots, r_n \rangle\rangle \cap \pi_1(\overline{K \smallsetminus L})$ is the subgroup normally generated by

$$\bigcup_{i=1}^{n} l_i^{-1} \langle r_1, \dots, r_n \rangle l_i \cap \pi_1(\overline{K \smallsetminus L}).$$

$l_i^{-1} \langle r_1, \dots, r_n \rangle l_i \cap \pi_1(\overline{K \smallsetminus L})$ is a finite index subgroup of $l_i^{-1} \langle r_1, \dots, r_n \rangle l_i$ so we can compute generators algorithmically. This gives us a way to describe a finite presentation of $\pi_1(\hat{K})$.

Remark A.1.1. The above process is an application of the Reidemeister-Schreier Theorem.

We implement this in one special case where the space K is the product of three cage graphs.

For $i = 1, 2, 3$ let Θ_i be a graph with 2 vertices $\{0, 1\}$ and $e_i > 3$ edges each directed from 0 to 1. Let $K = \Theta_1 \times \Theta_2 \times \Theta_3$. We will compute the fundamental group of a $q_1q_2q_3$-fold branched covering corresponding to the branching locus

$$L = (\Theta_1 \times \{0\} \times \{1\}) \sqcup (\{1\} \times \Theta_2 \times \{0\}) \sqcup (\{0\} \times \{1\} \times \Theta_3),$$

where q_1, q_2, q_3 are primes which satisfy

$$\begin{aligned} q_1 &> \max\{e_2, e_3\}, \\ q_2 &> \max\{e_1, e_3\}, \\ q_3 &> \max\{e_1, e_2\}. \end{aligned}$$

The particular branched covering we need is that arising in the proof of Theorem B.

In this case the space K is a union of copies of T^3 and the space $K \smallsetminus L$ is a union of copies of $T^3 \smallsetminus L$. We start by computing this fundamental group. For what follows let $\Theta = S^1$ be cubulated with 2 vertices 0 and 1.

Proposition A.1.2. *The fundamental group of*

$$X = \Theta^3 \smallsetminus \Big(\Theta \times \{0\} \times \{1\}\Big) \sqcup \Big(\{1\} \times \Theta \times \{0\}\Big) \sqcup \Big(\{0\} \times \{1\} \times \Theta\Big)$$

is isomorphic to

$$\big\langle a, b, c \mid [a, [c^{-1}, b]], [b, [a, c]]\big\rangle$$

At first this presentation seems surprising. There is an obvious three-fold symmetry of

$$X = \Theta^3 \smallsetminus \Big(\Theta \times \{0\} \times \{1\}\Big) \sqcup \Big(\{1\} \times \Theta \times \{0\}\Big) \sqcup \Big(\{0\} \times \{1\} \times \Theta\Big),$$

which does not appear in the presentation above. However, an application of the Hall-Witt identity shows that the third commutator relation is implied by these two, restoring the three-fold symmetry.

We will be using the commutator convention $[a, b] = a^{-1}b^{-1}ab$.

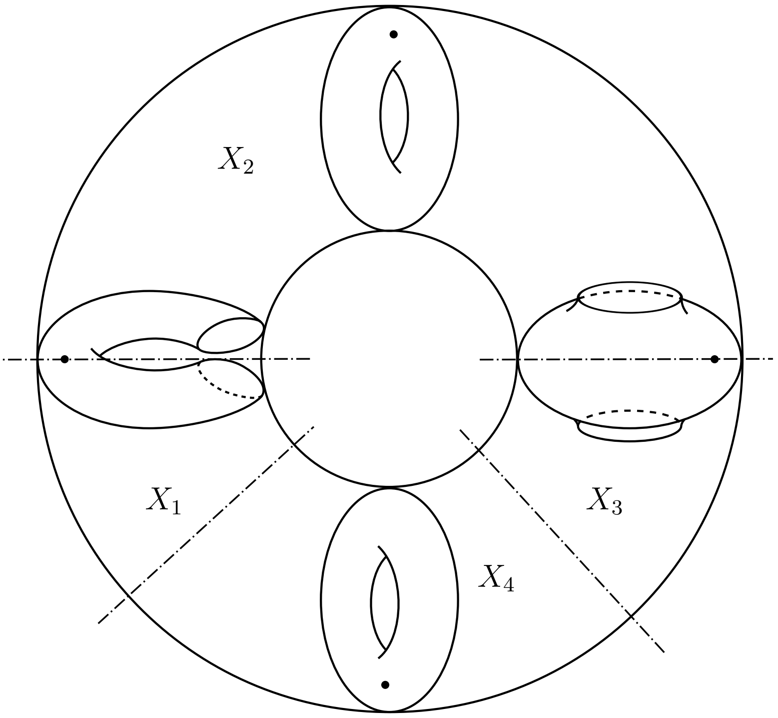

Figure A.1: A depiction of the space X from Proposition A.1.2.

Proof. The space obtained by removing one copy of Θ is homeomorphic to $\Sigma \times S^1$, where Σ is a punctured torus. We can thus view X as in Figure A.1. We cut this space along the 4 lines shown in Figure A.1. Each of the 4 pieces $X_1, \ldots, X_4$ has fundamental group F_2 we now need to compute the fundamental group of their union.

Let $Y_1 = X_1 \cup X_2$.

$$\pi_1(Y_1) = \langle \alpha_1, \alpha_2, \beta_1, \beta_2 \mid \alpha_1 = \beta_1, \alpha_2\alpha_1\alpha_2^{-1} = \beta_2\beta_1\beta_2^{-1} \rangle.$$

Let $Y_3 = Y_1 \cup X_3$.

$$\begin{aligned} \pi_1(Y_3) = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2 \mid \alpha_1 = \beta_1, \alpha_2\alpha_1\alpha_2^{-1} = \beta_2\beta_1\beta_2^{-1}, \\ \gamma_2 = \beta_2, \beta_1\beta_2\beta_1^{-1} = \gamma_1\gamma_2\gamma_1^{-1} \rangle. \end{aligned}$$

Finally, $X = Y_3 \cup X_4$ and

$$\begin{aligned} \pi_1(X) = \langle \alpha_1, \alpha_2, \beta_1, \beta_2, \gamma_1, \gamma_2, t \mid \alpha_1 = \beta_1, \alpha_2\alpha_1\alpha_2^{-1} = \beta_2\beta_1\beta_2^{-1}, \\ \gamma_2 = \beta_2, \beta_1\beta_2\beta_1^{-1} = \gamma_1\gamma_2\gamma_1^{-1}, \\ t\alpha_1 t^{-1} = \gamma_1, t\alpha_2 t^{-1} = \gamma_2 \rangle. \end{aligned}$$

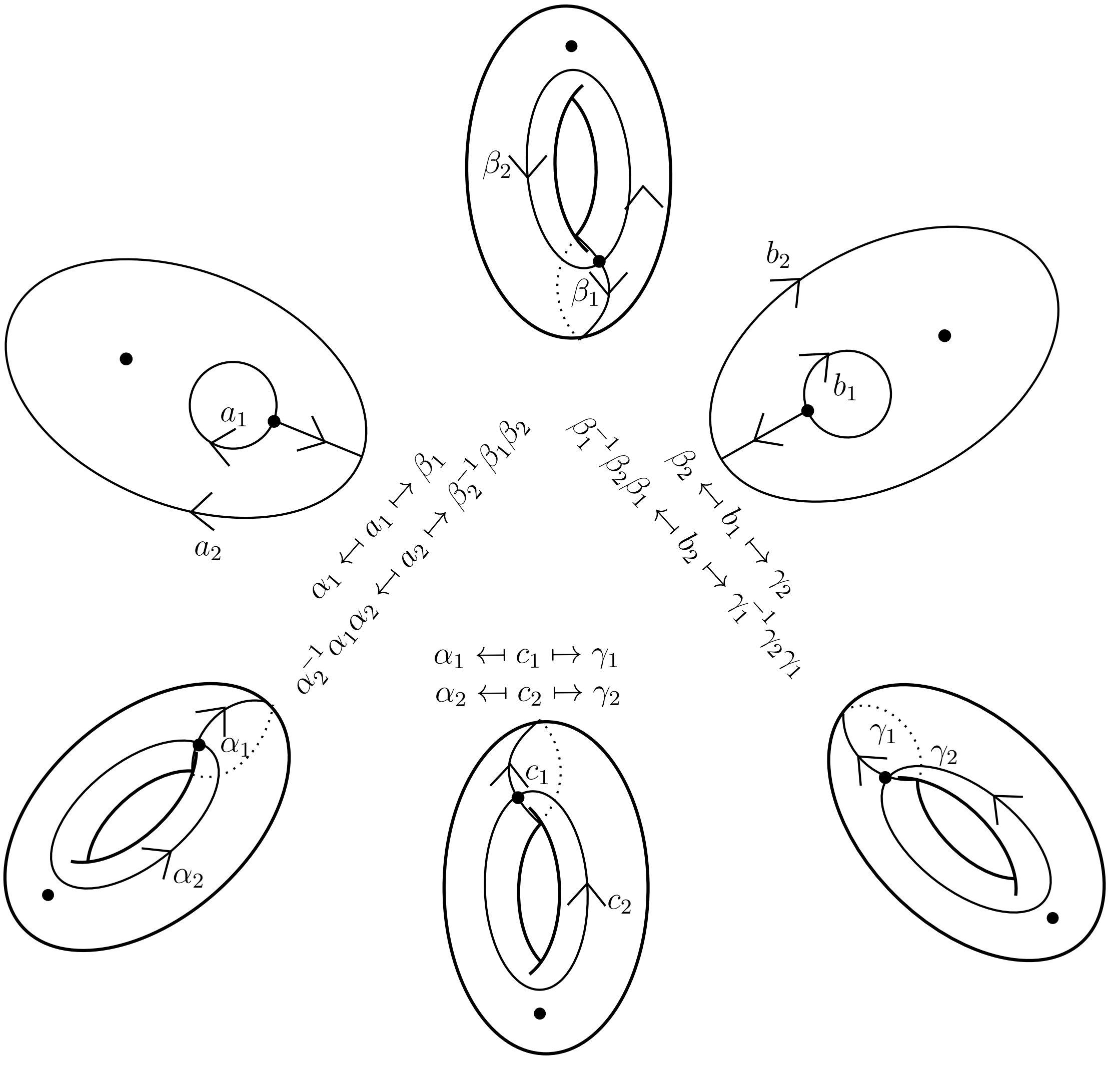

Figure A.2: Maps required for Proposition A.1.2.

The maps required to get make these computations are depicted in Figure A.2. We can then simplify this to

$$\left\langle \alpha_1, \alpha_2, t \mid [\alpha_2, [\alpha_1, t]], [\alpha_1, [t^{-1}, \alpha_2]] \right\rangle$$

□

For $i = 1, 2, 3$ let $d_i = e_i - 1$. We can thus see that the fundamental group of $K \smallsetminus L$ is:

$$\left\langle \begin{array}{l} a_1, \ldots, a_{d_1}, \\ b_1, \ldots, b_{d_2}, \\ c_1, \ldots, c_{d_3} \end{array} \;\middle|\; \begin{array}{l} [a_i, [c_k^{-1}, b_j]], [b_j, [a_i, c_k]] \\ 1 \le i \le d_1, 1 \le j \le d_2, 1 \le k \le d_3 \end{array} \right\rangle$$

The relations we add from taking the union with $N(L)$ corresponds to the commutators $[a_i, b_j], [b_j, c_k]$ and $[a_i, c_k]$ for $1 \leq i \leq d_1, 1 \leq j \leq d_2, 1 \leq k \leq d_3$. The computation is completed with some code in GAP which we have included below.

A.2 Practical GAP Code

This section should start with an apology; I didn't read that style guide yet.

```
#Function to find a generator of Zp X
genmodp := function(p)
        local a, i;
        a := 2;
        i := 2;
        repeat
                if RemInt(a^i, p)=1 then
                        a := a + 1;
                        i := 2;
                else
                        i := i + 1;
                fi;
        until i > ((p - 1) / 2) + 1;
        return a;
end;

#Function to produce p cycle.
cycle := function(p)
        local i, j, k;
        i := 2;
        j := (1, 2);
        k := (1, 2);
        repeat
                j := (i, i+1);
                k := j * k;
                i := i + 1;
        until
        i = p;
        return k;
end;

#Same as above with offset
cycleoff := function(p, q)
        local i, j, k;
        i := 2;
        j := (q + 1, q + 2);
```

```
        k := (q + 1, q + 2);
        repeat
                j := (q + i, q + i + 1);
                k := j * k;
                i := i + 1;
        until
        i = p;
        return k;
end;


#Function to find a permutation mapping a p cycle to a power
   of it
genconj := function(p)
        local a, m, b, t;
        m := genmodp(p);
        b := m + 1;
        t := (1, b);
        repeat
                a :=  RemInt(b + m, p);
                if b + m > p then
                        t := t * (b, b + m - p);
                        b := b + m - p;
                else
                        t := t * (b, b + m);
                        b := b + m;
                fi;
        until a = 1;
        return t;
end;

#Same as above with an offset
genconjoff := function(p, q)
        local a, m, b, t;
        m := genmodp(p);
        b := m + 1;
        t := (q + 1, b + q);
        repeat
                a :=  RemInt(b + m, p);
                if b + m > p then
                        t := t * (b + q, b + m - p + q);
                        b := b + m - p;
                else
                        t := t * (b + q, b + m + q);
                        b := b + m;
```

```
                fi;
        until a = 1;
        return t;
end;


#Commutator relations
comms := function(x, y, z)
        local i, x1, x2, x3, j, k, l, comms, group;
        group := FreeGroup(x + y + z);
        x1 := [];
        x3 := [];
        x2 := [];
        i := 1;
        if x + y + z <> Size(GeneratorsOfGroup(group)) then
           return 0;
        else
                repeat
                        if i < x + 1 then Add(x1,
                           GeneratorsOfGroup(group)[i]); i :=
                            i + 1;
                        elif i < x + y + 1 then Add(x2,
                           GeneratorsOfGroup(group)[i]); i :=
                            i + 1;
                        elif i< x + y + z + 1 then Add(x3,
                           GeneratorsOfGroup(group)[i]); i :=
                            i + 1;
                        else i := i + 1;
                        fi;
                until
                i = x + y + z + 1;
                fi;
                j := 1;
                k := 1;
                l := 1;
                comms := [];
                repeat
                        if k < y + 1 then Add(comms, Comm(x1[
                           j], x2[k])); k := k + 1;
                        elif l < z + 1 then Add(comms,Comm(x1
                           [j], x3[l])); l := l + 1;
                        elif j < x then j := j + 1; k := 1; l
                            := 1;
                        fi;
                until j + k + l > x + y + z + 1;
```

```
                j := 1;
                k := 1;
                l := 1;
                repeat
                        if k < y + 1 then Add(comms, Comm(x2[
                           k], x3[l])); k := k + 1;
                        elif l < z + 1 then l := l + 1; k :=
                           1;
                        fi;
                until l = z + 1;
        return comms;
end;

#Function to find first fixed point of a permutation
firstfixw1 := function(perm,p)
        local i, l;
        i := 1;
        repeat
                if i^perm = i then
                        return i;
                else i := i + 1;
                fi;
        until
        i = p + 1;
        return 0;
end;

#Image under homomorphism to sp1*p2*p3
imunderhomo := function(p1,p2,p3)
        local i, j, image, k;
        image := [];

        i := 1;
        j := ();
        repeat
                if \mod(i, p1) <> 0 then
                        j := (i, i + 1) * j;
                        i := i + 1;
                else
                        j := (i + 1, i + 2) * j;
                        i := i + 2;
                fi;
                until i = p1 * p2 * p3;
                Add(image, j);
        i := 1;
```

```
j := ();
repeat
        j := (i, i + 1) * j;
        i := i + p1;
until i > p1 * p2 * p3;
Add(image, j);
i := 1;
j := ();
k := ();
repeat
        i := firstfixw1(j, p1 * p2 * p3);
repeat
        if Order(k) <> p2 then j := (i, i + p1) * j;
           i := i + p1; k := (i, i + p1) * k;
        else k := (); i := i + p1;
        fi;
until i > p1 * p2 * p3;
until firstfixw1(j, p1 * p2 * p3) = 0;
Add(image, j);
i := 1;
j := ();
repeat
        if \mod(i, p1) <> 0 then j := (i, i + p1) * j
           ; i := i + 1;
else j := (i, i + p1) * j; i := i + p1 * p2 - p1 +1;
        fi;
until i + p1 > p1 * p2 * p3;
Add(image, j);
i := 1;
j := ();
k := ();
repeat
        i := firstfixw1(j, p1 * p2 * p3);
repeat
        if Order(k) <> p3 then j := (i, i + p1 * p2)
           * j; i := i + p1 * p2; k := (i, i + p1 *
           p2) * k;
else k := (); i := i + p1 * p2;
        fi;
until i > p1 * p2 * p3;
until firstfixw1(j, p1 * p2 * p3) = 0;
Add(image, j);
i := 1;
j := ();
repeat
```

```
                j := (i, i + p1 * p2) * j; i := i + 1;
        until i > p1 * p2;
        Add(image, j);
        return image;
end;

#Function defining the fundamental group of K minus L
group1 := function(a, b, c)
        local i, j, k, els, G, g;
        G := FreeGroup(a + b + c);
        g := GeneratorsOfGroup(G);
        i := 1;
        j := a + 1;
        k := a + b + 1;
        els := [];
        repeat
                j := a + 1;
                repeat
                        k := a + b + 1;
                        repeat
                                Add(els, g[i]^-1 * g[j]^-1 *
                                   g[k] * g[j] * g[k]^-1 * g[
                                   i] * g[k] * g[j]^-1 * g[k
                                   ]^-1 * g[j]);
                                Add(els, g[j]^-1 * g[k]^-1 *
                                   g[i]^-1 * g[k] * g[i] * g[
                                   j] * g[i]^-1 * g[k]^-1 * g
                                   [i] * g[k]);
                                k := k+1;
                        until k=a + b + c + 1;
                        j := j + 1;
                until j=a + b + 1;
                i := i + 1;
        until i=a + 1;
        return \/(G, els);
end;



#Homomorphism to sxsxs
homo1 := function(group, a, b, c, d, e, f)
        local s1, s2, s3, p1, p2, p3, i, image, homo;
        group := group1(a, b, c);
        s1 := SymmetricGroup(d);
```

```
        s2 := SymmetricGroup(e);
        s3 := SymmetricGroup(f);
        image := [];
        i := 1;
        repeat
                Add(image, cycle(d) * genconjoff(e, d)); i :=
                    i + 1;
        until i = a + 1;
        i := 1;
        repeat
                Add(image, cycleoff(e,d) * genconjoff(f,e+d))
                   ; i := i + 1;
        until i = b + 1;
        i := 1;
        repeat
                Add(image, cycleoff(f,e+d)*genconj(d)); i :=
                   i + 1;
        until i = c + 1;
        homo := GroupHomomorphismByImages(group,
           DirectProduct(s1, s2, s3), GeneratorsOfGroup(group
           ), image);
        return homo;
end;

#Function to produce hyperbolic group with f2 not f3 subgroup
    .
hypgroup := function(a, b, c, d, e, f)
        local f1, f2, f3, s1, s2, s3, gens, rels, relations,
           iso, group, stab, stabup, inters, sx, group, prim,
            rt, lrt, i, commrel, j, commlen, x, y, z, rels1,
           prim2;
        group := group1(a, b, c);
        s1 := SymmetricGroup(d);
        s2 := SymmetricGroup(e);
        s3 := SymmetricGroup(f);
        f1 := homo1(group, a, b, c, d, e, f);
        sx := DirectProduct(s1, s2, s3);
        f2 := GroupHomomorphismByImages(sx, SymmetricGroup(d*
           e*f), GeneratorsOfGroup(sx), imunderhomo(d, e, f))
           ;
        f3 := CompositionMapping(f2, f1);
        stab := Stabilizer(SymmetricGroup(d*e*f), 1);
        prim := PreImage(f3, stab);
        rt := RightTransversal(group, prim2);
        lrt := Length(rt);
```

```
        i := 1;
        j := 1;
        commrel := comms(a,b,c);
        commlen := Length(commrel);
        rels := [];
        repeat
                j := 1;
                repeat
                        Add(rels, rt[i]^-1 * commrel[j] * rt[i]);
                        j := j + 1;
                until j = commlen + 1;
                i := i + 1;
        until i = lrt + 1;
        rels1 := Subgroup(group, rels);
        inters := Intersection(PreImage(f3, stab), rels1);
        z := PresentationSubgroupMtc(group, prim);
        x := GeneratorsOfGroup(inters);
        y := 1;
        repeat
                AddRelator(z, x[y]);
                y := y + 1;
        until y=Length(x) + 1;
        relations := Image(iso,GeneratorsOfGroup(inters));
        group := \/(gens, relations);
        return group;
end;
```