

A REMARK ON POTENTIALLY SEMI-STABLE REPRESENTATIONS OF HODGE-TATE TYPE (0,1)

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1. INTRODUCTION

The purpose of this note is to complement *part* of a theorem from the remarkable paper of Fontaine and Mazur on geometric Galois representations [5].

Fix a prime p , and let K be a finite extension of the p -adic numbers \mathbb{Q}_p . Fix an algebraic closure \bar{K} of K and let G be the Galois group of \bar{K} over K . Fontaine's theory [3] classifies various types of representations $\rho : G \rightarrow \text{Aut}(V)$ on finite-dimensional \mathbb{Q}_p -vector spaces V , and we refer to op. cit. for terminology.

The theorem (C2. (ii) \Leftrightarrow (iii)) in question from [5] says the following:

If $p \geq 5$ and (V, ρ) is a two-dimensional irreducible Hodge-Tate representation of Hodge-Tate type $(0,1)$, then ρ is potentially semi-stable if and only if it is potentially crystalline.

Of course, one should emphasize that the rest of the theorem gives much more detail, namely, a complete list of possibilities, and the theorem to follow is by no means a substitute for the refined statements. However, it might be worth remarking that at least this part admits an entirely simple proof in greater generality. We note also that [5] C2. (i) \Leftrightarrow (ii), the equivalence between crystalline representations of Hodge-Tate type $(0,1)$ and Barsotti-Tate representations, has been proved for $p \neq 2$ and arbitrary dimension by [2], [6] in the small ramification case and [1] in general.

We are grateful to the referee for suggesting improvements.

2. MAIN THEOREM

The main result of this note is the following.

Theorem 2.1. *Let (V, ρ) be a irreducible finite-dimensional Hodge-Tate representation of G of Hodge-Tate type $(0,1)$. Then V is potentially semi-stable if and only if V is potentially crystalline.*

Remark 2.2. Note that there is no restriction on p or the dimension.

Remark 2.3. It is not necessary to restrict oneself to the case of finite residue field. It suffices to work with perfect residue fields. The proof given below goes through without any difficulty.

Proof. Of course we need only prove the ‘only if’ part. So assume ρ is potentially semi-stable and let $H \subset G$ be a normal subgroup of finite index such that $\rho|_H$ is semi-stable. We claim that the H -representation has a non-zero crystalline subrepresentation. To see this let $M = D_{st}(V)$ be the associated filtered (ϕ, N) -module. Since M is weakly-admissible, the slopes of ϕ are all in the interval $[0, 1]$. The equality

$$p\phi N = N\phi$$

implies that N lowers the slopes by 1. Suppose there is a non-trivial slope-zero part, M_0 . Then M_0 is killed by N (in particular, it is stabilized by N) and hence has the structure of a sub-module. Now $t_N(M_0) = 0$ while $t_H(M_0) \leq t_N(M_0)$ from the weak admissibility of M . So we necessarily have $t_H(M_0) = 0 = t_N(M_0)$. Thus, M_0 is a weakly-admissible submodule of M on which $N = 0$, so it corresponds to a crystalline subrepresentation of V . Otherwise, the slopes of M are in $(0, 1]$ and thus, N must kill all of M and V is therefore crystalline.

Let $W \subset V$ be the maximal H -subrepresentation of V which is crystalline. This exists because the category of crystalline representations as a sub-category of H -representations is closed under direct sums and quotient objects, and hence, under taking sums of subspaces. The point is that W is actually G -stable, implying that $W = V$, and hence, that V is potentially crystalline. We see this as follows: Let $g \in G$ and consider the subspace $g^{-1}W \subset V$. Since H is a normal subgroup, $g^{-1}W$ is an H -subspace.

Claim: $g^{-1}W$ is H -crystalline.

For any representation $\pi : H \rightarrow \text{Aut}(U)$, denote by U_g the representation given by the H -action $\pi \circ c_g$ on the same vector space U , where $c_g(h) = ghg^{-1}$. Then as an H -representation, $g^{-1}W$ is isomorphic to W_g .

Thus, we need to consider the representation $W_g \otimes B_{\text{cris}}$. But since B_{cris} as an H -module is the restriction of a G -module, $(B_{\text{cris}})_g \simeq B_{\text{cris}}$. Therefore,

$$W_g \otimes B_{\text{cris}} \simeq W_g \otimes (B_{\text{cris}})_g \simeq (W \otimes B_{\text{cris}})_g$$

Since the images of the original and the twisted actions in the automorphism group of $W \otimes B_{\text{cris}}$ are the same, we get

$$[(W \otimes B)_g]^H = (W \otimes B)^H$$

and therefore, W_g is H -crystalline. Since W is the maximal H -crystalline subrepresentation of V , we must have $g^{-1}W \subset W$. So W is G -stable. \square

Remark 2.4. The idea of using kernel of N also occurs in Corollary 5.3.4 of [1].

3. AN EXAMPLE

An important ingredient in the proof is the fact that a semi-stable representation of Hodge-Tate type $(0,1)$ is either crystalline or has a *crystalline filtration*, i.e., a filtration whose associated graded objects are crystalline.

That this cannot be a general phenomenon was remarked by Jannsen in [7], where he considers a \mathbb{Q}_5 representation of dimension 2 associated to a modular form of weight 4 and level 5. Here we expand on a construction of [5] to illustrate that the conditions of the theorem cannot be relaxed in any dimension. In fact, the examples will be *strongly irreducible*, that is, irreducible even when restricted to a finite index subgroup, and a fortiori cannot have a crystalline filtration even after such a restriction.

The Fontaine-Mazur construction goes as follows: Let $M = \langle e_1, e_2 \rangle$ be a two dimensional \mathbb{Q}_p vector space spanned by vectors e_1, e_2 . Let $s \geq 3$ be an odd integer and let $b \in \mathbb{Q}_p$ be a p -adic integer such that $v_p(b) = (s-1)/2$. We define a filtration on M by $\text{Fil}^i M = M$ for $i \leq 0$, and $\text{Fil}^i M = \text{Fil}^s M = \langle e_1 \rangle$ for $1 \leq i \leq s$; and $\text{Fil}^i M = 0$ for $i > s$. The operator ϕ is defined by $\phi(e_1) = pbe_1$ and $\phi(e_2) = be_2$; $N(e_1) = e_2$. We see right away that $t_H(M) = s$ and $t_N(M) = 1 + (s-1)/2 + (s-1)/2 = s$. Now, the only (ϕ, N) -invariant subspace of M is $\langle e_2 \rangle$, to which the filtration restricts trivially. Thus, $t_N(\langle e_2 \rangle) = (s-1)/2 > t_H(\langle e_2 \rangle) = 0$ and therefore, M is weakly admissible, while it does not admit any weakly admissible submodules. Thus, $V = V_{st}(M)$ is an irreducible semi-stable representation for $G = \text{Gal}(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ [4]. Since $N \neq 0$, V is not crystalline, so V is a two-dimensional example of a representation which does not admit a crystalline filtration. But exactly the same argument shows that the filtered (ϕ, N) -module $M \otimes K$ has no weakly admissible submodules for any extension K of \mathbb{Q}_p (because of N , extending the coefficients does not allow us to pick up any other invariant lines). Extending the coefficients of M corresponds to restricting to a subgroup of G , therefore, we see that V is even irreducible when restricted to finite index subgroups of G .

We can say more by investigating the endomorphisms of M . Suppose f is such an endomorphism. By checking the conditions of commuting with ϕ and N , we see that f must be a scalar $a \in \mathbb{Q}_p$. That is, $\text{End}(M) = \mathbb{Q}_p$. In fact, we can see that $\text{End}(M \otimes K) = \mathbb{Q}_p$ for any extension K of \mathbb{Q}_p : Write $f(e_1) = xe_1 + ye_2$ and $f(e_2) = ze_1 + we_2$ for $x, y, z, w \in K$. The condition of commuting with N gives us $z = 0$ and $x = w$. $f(\phi(e_2)) = \phi(f(e_2))$ says that $bw = b\sigma(w) \Rightarrow w = \sigma(w)$, where $\sigma : K \rightarrow K$ is the Frobenius (lift) of K , so we get $w \in \mathbb{Q}_p$. Similarly, $f(\phi(e_1)) = \phi(f(e_1))$ gives us $bpwe_1 + bpye_2 = bpwe_1 + \sigma(y)be_2$ so that $\sigma(y) = py$. Since σ cannot change the valuation of non-zero elements, this implies $y = 0$. Note that the above computation does not require the filtration at all. By the equivalence of categories given by V_{st} and D_{st} , we see that for any finite index subgroup H of G , $\text{End}_{\mathbb{Q}_p[H]}(V) = \mathbb{Q}_p$. Since we already know that V is $\mathbb{Q}_p[H]$ -simple (this *does* require the filtration), this implies that $\text{Image}(\mathbb{Q}_p[H]) = \text{End}_{\mathbb{Q}_p}(V)$ by Wedderburn's theorem. In particular, H must act irreducibly on $\text{Sym}^n(V)$ for all $n > 0$. This way, we obtain a G -representation for all dimensions $n \geq 2$ which cannot admit a crystalline filtration for any finite index subgroup.

Remark 3.1. The proof of also shows that for the semi-stable Galois module V constructed above, the image of G in $GL(V)$ is open. This is a consequence of the following facts: 1) by the above calculation V remains irreducible on restriction to any open subgroup of G and its endomorphism ring is \mathbb{Q}_p . 2) By a standard result the image of Galois for any Hodge-Tate representation is open in its Zariski closure.

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