

# Erratum to “The robust superreplication problem: a dynamic approach”

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The assertions of Proposition 3.7 in our paper [1] may fail to hold without an additional assumption. We express our gratitude to Sergey Smirnov who spotted the mistake and alerted us. We also want to draw the readers’ attention to Sergey Smirnov’s works [2, 4, 5] (and the references therein), in which an alternative pathwise approach to modelling financial markets is developed independently of our results [1].

In Appendix A in the proof of Proposition 3.7, all statements are correct apart from one sentence, in the middle of page 923, which claims that  $\tilde{H} \in \text{span}(f_{T-1}(\omega) - S_{T-1}(\omega))$ . In fact it may happen that

$$\begin{aligned} (\#) \quad \limsup_{N \rightarrow \infty} \text{span}(f_{T-1}(\tilde{\omega}^N) - S_{T-1}(\tilde{\omega}^N)) &\supsetneq \text{span}\left(\limsup_{N \rightarrow \infty} f_{T-1}(\tilde{\omega}^N) - S_{T-1}(\tilde{\omega}^N)\right) \\ &= \text{span}(f_{T-1}(\omega) - S_{T-1}(\omega)). \end{aligned}$$

If this is the case, then  $\omega \mapsto \pi_t(\xi)(\omega)$  is not necessarily continuous and (3.8) may fail. A counterexample is presented below. In conclusion, Proposition 3.7 should read as follows:

**Proposition 3.7.** *Suppose  $(\mathcal{P}_t)_{0 \leq t \leq T-1}$  is generated by compact-valued, uniformly continuous correspondences  $\{f_t\}_{0 \leq t \leq T-1}$  and that  $NA(\mathcal{P})$  holds. Suppose further that for any  $0 \leq t \leq T-1$ ,  $\Omega^t \ni \omega \mapsto \text{span}(f_t(\omega) - S_t(\omega))$  is upper semicontinuous for inclusion, i.e.,*

$$(*) \quad \limsup_{N \rightarrow \infty} \text{span}(f_t(\omega^N) - S_t(\omega^N)) \subseteq \text{span}(f_t(\omega) - S_t(\omega)),$$

for any  $\omega^N, \omega \in \Omega^t$  with  $|\omega^N - \omega| \rightarrow 0$  as  $N \rightarrow \infty$ , where the  $\limsup$  denotes the limit superior in Kuratowski sense.

Take any measure  $P = P_0 \otimes \cdots \otimes P_{T-1}$  such that

$$\text{supp}(P_t(\omega)) = f_t(\omega), \quad 0 \leq t \leq T-1, \quad \omega \in \Omega^t.$$

Then, for any continuous function  $\xi : \Omega^T \rightarrow \mathbb{R}$ , for all  $0 \leq t \leq T-1$  and  $\omega \in \Omega^t$ ,

$$\pi_t(\xi)(\omega) = \inf\{x \in \mathbb{R} \mid \exists H \in \mathbb{R}^d \text{ such that } x + H\Delta S_{t+1}(\omega, \cdot) \geq \pi_{t+1}(\xi)(\omega, \cdot) \text{ } P\text{-a.s.}\}.$$

and  $\omega \mapsto \pi_t(\xi)(\omega)$  is continuous.

**Remark.** Note that a sufficient condition for  $(*)$  is that  $\text{span}(f_t(\omega) - S_t(\omega)) = \mathbb{R}^d$  for all  $0 \leq t \leq T-1$  and  $\omega \in \Omega^t$ . Continuity of  $\omega \mapsto \pi_t(\xi)(\omega)$  then also follows from [2]. This condition is, in particular, satisfied in the setting of Proposition 3.8 in [1].

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**Remark.** Note that it is enough to assume that  $f_t$  are continuous. Indeed,  $f_0$  is only evaluated on  $s_0$  and  $f_t$  is evaluated on the range of values determined by  $(f_s)_{s=0,\dots,t-1}$  and  $s_0$ ,  $1 \leq t \leq T-1$ . Since a continuous image of a compact is compact, by iteration, this range is compact and hence  $f_t$  is continuous on it<sup>1</sup>. Likewise, the novel assumption (\*) is only required for  $\omega^N, \omega$  in the range of values determined by  $(f_s)_{s=0,\dots,t-1}$  and  $s_0$ .

**Remark.** In most, if not all, applications, the range of  $f_t$  will be a finite union of disjoint compact convex sets and then (\*) requires that the dimension of  $\text{span}(f_t(\omega) - S_t(\omega))$  is constant on each of these components.

Let us present briefly a counterexample, constructed and communicated to us by Sergey Smirnov [3], which highlights the necessity of the additional assumption. Take  $T = 2$  and  $d = 2$ , i.e., two risky assets. Fix some real number  $\omega^* > 0$ . Choose two bounded Lipschitz-continuous functions  $\gamma : \mathbb{R} \rightarrow \mathbb{R}$  and  $\beta : \mathbb{R} \rightarrow \mathbb{R}$  such that  $\beta(\omega^1) > 0$  and  $\gamma(\omega^1) > 0$  for  $\omega^1 < \omega^*$  and  $\beta(\omega^1) = \gamma(\omega^1) = 0$  for  $\omega^1 \geq \omega^*$ , while the left limit is

$$\gamma(\omega^1)/\beta(\omega^1) \rightarrow 0 \quad \text{as } \omega^1 \nearrow \omega^*.$$

Denote  $\beta^* = \sup\{\beta(\omega^1) : \omega^1 \in \mathbb{R}\}$  and choose real numbers  $\alpha, a^1, b^1, a^2$ , and  $b^2$  such that  $0 < \alpha < a^1 < \omega^* < b^1$  and  $\beta^* < a^2 < b^2$ . Suppose  $a^1 < s_0^1 < b^1$  and  $a^2 < s_0^2 < b^2$  and define  $f_0(\omega) = [a^1, b^1] \times [a^2, b^2] \subseteq \mathbb{R}^2$ . The compact-valued mapping  $f_1(\cdot)$  describing possible values of price increments is defined as follows. For any  $\omega_1 = (\omega^1, \omega^2) \in \mathbb{R}^2$ , the compact set  $f_1(\omega_1) \subseteq \mathbb{R}^2$  represents the triangle with vertices

$$\begin{aligned} y^{(1)}(\omega^1, \omega^2) &= \omega_1 + (-\alpha, -\beta(\omega^1)), & y^{(2)}(\omega^1, \omega^2) &= \omega_1 + (\alpha, -\beta(\omega^1)), \\ y^{(3)}(\omega^1, \omega^2) &= \omega_1 + (0, \gamma(\omega^1)), \end{aligned}$$

when  $\omega^1 < \omega^*$ , and the line segment with endpoints

$$y^{(1)}(\omega^1, \omega^2) = \omega_1 + (-\alpha, 0), \quad y^{(2)}(\omega^1, \omega^2) = \omega_1 + (\alpha, 0)$$

when  $\omega^1 \geq \omega^*$ . Thanks to the choice of real numbers  $\alpha, a^1, b^1, a^2$ , and  $b^2$  the possible prices of risky assets at time moment  $t = 1$  are positive. It is clear that  $\tilde{\omega}^N = (\omega^* - 1/N, (a^2 + b^2)/2)$ ,  $\omega = (\omega^*, (a^2 + b^2)/2)$  give the strict inclusion in (#).

More importantly, we show by hand that  $\omega_1 \mapsto \pi_1(\xi)(\omega_1)$  is not continuous for a simple choice of  $\xi$ . Take the convex function  $f : \mathbb{R}^2 \times \mathbb{R}^2 \rightarrow \mathbb{R}$  given by  $f(\omega_1, \tilde{\omega}^1, \tilde{\omega}^2) = |\tilde{\omega}^1|$  and set  $\xi(\omega_1, \omega_2) = f(\omega_1, \omega_2 - \omega_1)$ . Furthermore take the unique martingale measure  $Q^*(\omega_1)$  supported on the extreme points of  $f_1(\omega_1)$ , i.e., when  $\omega^1 < \omega^*$ ,  $Q^*(\omega_1, \cdot) = q_1(\omega_1)\delta_{y^{(1)}(\omega_1)}(\cdot) + q_2(\omega_1)\delta_{y^{(2)}(\omega_1)}(\cdot) + q_3(\omega_1)\delta_{y^{(3)}(\omega_1)}(\cdot)$ , where  $\delta_a$  the Dirac measure, concentrated at a single point  $a$  and

$$q_1(\omega_1) = q_2(\omega_1) = \frac{\gamma(\omega^1)}{2[\beta(\omega^1) + \gamma(\omega^1)]}, \quad q_3(\omega_1) = \frac{\beta(\omega^1)}{\beta(\omega^1) + \gamma(\omega^1)}.$$

This yields a lower bound for the superhedging price of

$$\int \xi(\omega_1, \omega_2) Q^*(\omega_1, d\omega_2) = \alpha \frac{\gamma(\omega_1)}{\beta(\omega_1) + \gamma(\omega_1)} + 0.$$

<sup>1</sup>This too was pointed out to us by Sergey Smirnov.

On the other hand, for  $\omega^1 < \omega^*$ , a superhedging strategy is given by

$$H_2(\omega_1) = \left( 0, -\frac{\alpha}{\beta(\omega_1) + \gamma(\omega_1)} \right),$$

yielding an upper bound on the superhedging price of

$$\alpha - \alpha \frac{\beta(\omega_1)}{\beta(\omega_1) + \gamma(\omega_1)} = \alpha \frac{\gamma(\omega_1)}{\beta(\omega_1) + \gamma(\omega_1)}.$$

As the upper and lower bounds agree, the superhedging price is given by  $\alpha\gamma(\omega_1)/(\beta(\omega_1) + \gamma(\omega_1))$ , which converges to 0 as  $\omega^1 \nearrow \omega^*$  due to the growth assumptions on  $\beta$  and  $\gamma$ . Lastly, for  $\omega^1 \geq \omega^*$ , a martingale measure  $Q^*(\omega_1)$  supported on the extreme points of  $f_1(\omega_1)$  is given by

$$Q^*(\omega_1, \cdot) = \frac{1}{2}\delta_{y^{(1)}(\omega_1)}(\cdot) + \frac{1}{2}\delta_{y^{(2)}(\omega_1)}(\cdot),$$

while  $H_2(\omega_1) = (0, 0)$  is a superhedging strategy. As above one now can check that the superhedging price is equal to  $\alpha \neq 0$ .

In conclusion  $\omega_1 \mapsto \pi_1(\xi)(\omega_1)$  is not continuous at, e.g.,  $\omega_1 = (\omega^*, (a^2 + b^2)/2)$ .

## REFERENCES

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