Asymptotically Locally Euclidean metrics with holonomy SU(m)

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1 Introduction

Let G be a finite subgroup of U(m) acting freely on $\mathbb{C}^m \setminus \{0\}$. Then $\mathbb{C}^m/G$ has an *isolated quotient singularity* at 0. Suppose (X, π) is a *resolution* of $\mathbb{C}^m/G$. Then X is a noncompact complex manifold modelled at infinity on $\mathbb{C}^m/G$.

In this paper we will study Kähler metrics g on X which are also *Asymptotically Locally Euclidean*, or *ALE* for short. This means that g approximates the Euclidean metric h on $\mathbb{C}^m/G$ by $g = h + O(r^{-2m})$, with appropriate decay in the derivatives of g. We are particularly interested in *Ricci-flat* ALE Kähler manifolds.

The *Calabi conjecture* [4] describes the possible Ricci curvatures of Kähler metrics on a fixed compact complex manifold M, in terms of the first Chern class $c_1(M)$ of M. It was proved by Yau [22] in 1976. The following theorem is a corollary of Yau's proof.

Theorem 1.1 *Let M be a compact complex manifold admitting Kähler metrics, with $c_1(M) = 0$. Then there is a unique Ricci-flat Kähler metric in each Kähler class on M.*

Our main results are Theorems 3.3 and 3.4. Theorem 3.3 is an analogue of Theorem 1.1 for ALE Kähler manifolds. It says that if X is a resolution of $\mathbb{C}^m/G$ with $c_1(X) = 0$, that is, a *crepant resolution*, then every Kähler class of ALE Kähler metrics on X contains a unique Ricci-flat Kähler metric.

Theorem 3.4 says that these metrics have holonomy SU(m). When $m = 2$ the metrics were constructed explicitly by Kronheimer and others.

Section 2 defines ALE metrics and ALE Kähler metrics, and §3 states the main results of the paper, postponing the proofs until §6, and gives some examples. Section 4 develops some analytical tools for ALE manifolds: Banach spaces of functions called weighted Hölder spaces, and elliptic regularity theory for the Laplacian Δ on them. In §5 we discuss k-forms and de Rham cohomology on ALE manifolds.

Section 6 states a version of the Calabi conjecture for ALE manifolds. Only a sketch of the proof is given; a complete proof, following Yau [22], will be given in the author's book [12, §8]. We apply this Calabi conjecture to prove Theorem 3.3, and then prove Theorem 3.4.

A number of other people have already written papers on noncompact versions of the Calabi conjecture, and I should at once admit that there is some overlap between their results and mine. In particular, Tian and Yau [20, 21] and independently Bando and Kobayashi [1, 2] prove the following result [21, Cor. 1.1], [2, Th. 1]:

Theorem 1.2 *Let X be a compact Kähler manifold with $c_1(X) > 0$, and D a smooth reduced divisor on X such that $c_1(X) = \alpha[D]$ for some $\alpha > 1$. Suppose D admits a Kähler-Einstein metric with positive scalar curvature. Then $X \setminus D$ has a complete Ricci-flat Kähler metric.*

Also Tian and Yau give estimates on the decay of the curvature of their Ricci-flat metric. With a certain amount of work, the existence of the metrics of Theorem 3.3 follows from the theorem above. But our estimates on the asymptotic behaviour of the metrics are stronger than those proved by Tian and Yau. For example, we show that the curvature is $O(r^{-2m-2})$ for large r, but Tian and Yau only show that it is $O(r^{-3})$, which is not good enough for the applications we have in mind.

In a sequel [13] we will extend the material of this paper to construct a class of Ricci-flat Kähler metrics on crepant resolutions of *non-isolated* singularities $\mathbb{C}^m/G$, which we will call *Quasi-ALE metrics.* These metrics are not covered by the work of Tian and Yau or Bando and Kobayashi.

The original motivation for this paper and [13] is that ALE and Quasi-ALE metrics with holonomy SU(2), SU(3), SU(4) and Sp(2) are essential ingredients in a new construction by the author of compact manifolds with the exceptional holonomy groups G_2 and Spin(7), which generalizes that of

[10, 11]. This construction will be described at length in the author's book [12], which also discusses the results of this paper and [13].

2 Asymptotically Locally Euclidean metrics

Suppose G is a finite subgroup of $\mathrm{SO}(n)$ that acts freely on $\mathbb{R}^n \setminus \{0\}$. Then $\mathbb{R}^n/G$ has an *isolated quotient singularity* at 0. Let h be the Euclidean metric on $\mathbb{R}^n$. Then h is preserved by G, as $G \subset \mathrm{SO}(n)$, and so h descends to $\mathbb{R}^n/G$. Let r be the radius function on $\mathbb{R}^n/G$, that is, $r(x)$ is the distance from 0 to x calculated using h. We will define a natural class of noncompact Riemannian manifolds (X, g) called *ALE manifolds*, that have one infinite end upon which the metric g asymptotically resembles the metric h on $\mathbb{R}^n/G$ for large r.

Definition 2.1 Let X be a noncompact manifold of dimension n, and g a Riemannian metric on X. We say that (X, g) is an *Asymptotically Locally Euclidean manifold* asymptotic to $\mathbb{R}^n/G$, or an *ALE manifold* for short, and we say that g is an *ALE metric*, if the following conditions hold.

There should exist a compact subset $S \subset X$ and a map $\pi : X \setminus S \to \mathbb{R}^n/G$ that is a diffeomorphism between $X \setminus S$ and the subset $\{z \in \mathbb{R}^n/G : r(z) > R\}$ for some fixed $R > 0$. Under this diffeomorphism, the push-forward metric $\pi_*(g)$ should satisfy

$$\nabla^k\big(\pi_*(g) - h\big) = O(r^{-n-k}) \quad \text{on } \{z \in \mathbb{R}^n/G : r(z) > R\}, \tag{1}$$

for all $k \geq 0$. Here ∇ is the Levi-Civita connection of h, and $T = O(r^{-j})$ if $|T| \leq K r^{-j}$ for some $K > 0$.

If $G = \{1\}$, so that (X, g) is asymptotic to $\mathbb{R}^n$, then we call (X, g) an *Asymptotically Euclidean manifold*, or *AE manifold*. We shall call the map $\pi : X \setminus S \to \mathbb{R}^n/G$ an *asymptotic coordinate system* for X. Equation (1) says that towards infinity the metric g on X (and its derivatives) must converge to the Euclidean metric on $\mathbb{R}^n/G$, with a given rate of decay. We will explain in §3 why we have chosen the powers r^{-n-k} here.

Definition 2.2 Let (X, g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$. We say that a smooth function $\rho : X \to [1, \infty)$ is a *radius function* on X if, given any asymptotic coordinate system $\pi : X \setminus S \to \mathbb{R}^n/G$, we have

$$\nabla^k\big(\pi_*(\rho) - r\big) = O(r^{1-n-k}) \quad \text{on } \{z \in \mathbb{R}^n/G : r(z) > R\}, \tag{2}$$

for all $k \geq 0$. This condition is independent of the choice of asymptotic coordinate system, and radius functions exist for every ALE manifold.

A radius function is a function ρ on X that approximates the function r on $\mathbb{R}^n/G$ near infinity. In doing analysis on ALE manifolds, we will find it useful to consider Hölder spaces of functions in which the norms are weighted by powers ρ^β of a radius function. Note that by definition $\rho \geq 1$, so we do not have to worry about small values of ρ.

Here is one way to think about ALE metrics. The manifold X is noncompact, but it can be compactified in a natural way by adding the boundary $\mathcal{S}^{n-1}/G$ at infinity. So we can instead regard X as a *compact manifold with boundary*. Then ALE metrics are metrics on X satisfying a certain natural boundary condition.

It is a general principle in differential geometry that most results about compact manifolds can also be extended to results about compact manifolds with boundary, provided the right boundary conditions are imposed in the problem. ALE manifolds are an example of this principle, because many results about compact Riemannian manifolds have natural analogues for ALE manifolds.

Next we define *ALE Kähler metrics*. Suppose G is a finite subgroup of $\mathrm{U}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$. Then $\mathbb{C}^m/G$ has an isolated quotient singularity at 0, and the standard Hermitian metric h on $\mathbb{C}^m$ descends to $\mathbb{C}^m/G$. Let r be the radius function on $\mathbb{C}^m/G$. Suppose (X,π) is a *resolution* of $\mathbb{C}^m/G$, that is, X is a normal nonsingular variety with a proper birational morphism $\pi : X \to \mathbb{C}^m/G$. Then we can consider metrics on X which are both Kähler, and ALE.

Definition 2.3 Let (X,π) be a resolution of $\mathbb{C}^m/G$, with complex structure J, and let g be a Kähler metric on X. We say that (X,J,g) is an *ALE Kähler manifold asymptotic to* $\mathbb{C}^m/G$, and that g is an *ALE Kähler metric*, if for some $R > 0$ we have

$$\nabla^k\bigl(\pi_*(g) - h\bigr) = O(r^{-2m-k}) \quad \text{on } \{z \in \mathbb{C}^m/G : r(z) > R\}, \tag{3}$$

for all $k \geq 0$. We say that a smooth function $\rho : X \to [1,\infty)$ is a *radius function* on X if $\rho = \pi^*(r)$ on the subset $\bigl\{x \in X : \pi^*(r) \geq 2\bigr\}$. A radius function exists for every ALE Kähler manifold.

Because X is a resolution of $\mathbb{C}^m/G$, it comes equipped with a resolving map $\pi : X \to \mathbb{C}^m/G$, which gives a natural asymptotic coordinate system for X. The consequence of using this preferred asymptotic coordinate system is that on an ALE Kähler manifold (X, J, g), both the metric g and the complex structure J are simultaneously asymptotic to the metric and complex structure on $\mathbb{C}^m/G$. We also use π to simplify the definition of radius function.

In dimension 2 one can also desingularize $\mathbb{C}^2/G$ by *deformation*. By adopting a slightly more general definition of ALE Kähler manifold we can include deformations and resolutions of deformations of $\mathbb{C}^2/G$, and most of our results also apply to them. This will be discussed in [12, §8.9]. However, by *Schlessinger's Rigidity Theorem* [19], if $m \geq 3$ then an isolated quotient singularity $\mathbb{C}^m/G$ admits no nontrivial deformations.

3 Ricci-flat ALE Kähler manifolds

We now state some results on Ricci-flat ALE Kähler manifolds, and give some examples. The proofs will be deferred until §6. A resolution (X, π) of $\mathbb{C}^m/G$ with $c_1(X) = 0$ is called a *crepant resolution*, as in Reid [17]. A great deal is known about the algebraic geometry of crepant resolutions, especially when $\dim X$ is 2 or 3. In particular, for $\mathbb{C}^m/G$ to admit a crepant resolution G must be a subgroup of $\mathrm{SU}(m)$, and when m is 2 or 3 a crepant resolution of $\mathbb{C}^m/G$ exists for every finite subgroup G of $\mathrm{SU}(m)$.

Our first proposition shows that Ricci-flat ALE Kähler metrics exist only on *crepant resolutions*. The proof is elementary, and we omit it.

Proposition 3.1 *Let G be a finite subgroup of $\mathrm{U}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$, let (X, π) be a resolution of $\mathbb{C}^m/G$, and suppose g is a Ricci-flat ALE Kähler metric on X. Then X is a crepant resolution of $\mathbb{C}^m/G$ and $G \subset \mathrm{SU}(m)$.*

Next we define *Kähler classes* and the *Kähler cone* for ALE manifolds.

Definition 3.2 Let (X, J, g) be an ALE Kähler manifold asymptotic to $\mathbb{C}^m/G$ for some $m > 1$, with Kähler form ω. Then ω defines a de Rham cohomology class $[\omega] \in H^2(X, \mathbb{R})$ called the *Kähler class* of g. Define the *Kähler cone* $\mathcal{K}$ of X to be the set of Kähler classes $[\omega] \in H^2(X, \mathbb{R})$ of ALE Kähler metrics on (X, J). It is not difficult to prove that $\mathcal{K}$ is an open convex cone in $H^2(X, \mathbb{R})$, which does not contain zero.

The following two theorems will be proved in §6.

Theorem 3.3 *Let G be a nontrivial finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$, and (X, π) a crepant resolution of $\mathbb{C}^m/G$. Then each Kähler class of ALE Kähler metrics on X contains a unique Ricci-flat ALE Kähler metric g. The Kähler form ω of g satisfies*

$$\pi_*(\omega) = \omega_0 + A\,\mathrm{dd}^c(r^{2-2m}) + \mathrm{dd}^c\chi \tag{4}$$

on the set $\{z \in \mathbb{C}^m/G : r(z) > R\}$, where $A < 0$ and $R > 0$ are constants, ω_0 is the Kähler form of the Euclidean metric on $\mathbb{C}^m/G$, r the radius function on $\mathbb{C}^m/G$, and χ a smooth function on $\{z \in \mathbb{C}^m/G : r(z) > R\}$ such that $\nabla^k\chi = O(r^{\gamma-k})$ for each $k \geq 0$ and $\gamma \in (1-2m, 2-2m)$.

Theorem 3.3 is the main result of this paper, and is an analogue of Theorem 1.1 for ALE Kähler manifolds. We use the notation that $\mathrm{d}^c f = i(\bar\partial - \partial)f$, when f is a differentiable function on a complex manifold. Then d^c is a real operator, and $\mathrm{dd}^c = 2i\partial\bar\partial$.

Note that because $A < 0$ in Theorem 3.3, the term $A\,\mathrm{dd}^c(r^{2-2m})$ in (4) is nonzero. Therefore $\pi_*(g) - h$ decays with order exactly $O(r^{-2m})$, and similarly $\nabla^k(\pi_*(g) - h)$ decays with order exactly $O(r^{-2m-k})$. Thus in Definition 2.1 the decay rates given in (1) are *sharp* for all Ricci-flat ALE Kähler metrics, and cannot be improved upon. This is why we chose the powers r^{-n-k} in our definition (1) of ALE metrics.

Theorem 3.4 *Let G be a nontrivial finite subgroup of $\mathrm{SU}(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$, let (X, π) be a crepant resolution of $\mathbb{C}^m/G$, and let g be a Ricci-flat ALE Kähler metric on X. Then g has holonomy $\mathrm{SU}(m)$.*

For an introduction to holonomy groups of Riemannian manifolds, and the connection between Ricci-flat Kähler metrics and holonomy $\mathrm{SU}(m)$, see Salamon [18].

3.1 Examples

ALE Kähler manifolds with holonomy SU(2) are already very well understood. Eguchi and Hanson [7] gave an explicit formula in coordinates for the metrics of ALE spaces with holonomy SU(2) asymptotic to $\mathbb{C}^2/\{\pm 1\}$, and this was generalized by Gibbons and Hawking [8] to explicit expressions for ALE spaces asymptotic to $\mathbb{C}^2/\mathbb{Z}_k$ for $k \geq 2$. More generally, Kronheimer

[14, 15] gave an explicit, algebraic construction of every ALE manifold with holonomy SU(2), using the hyperkähler quotient.

Thus, we can write down many explicit examples of ALE manifolds with holonomy SU(2). For $m \geq 3$, Calabi [5, p. 285] found an explicit ALE Kähler manifold with holonomy $\mathrm{SU}(m)$ asymptotic to $\mathbb{C}^m/\mathbb{Z}_m$, which we describe next. In the case $m = 2$, Calabi's example coincides with the Eguchi–Hanson metric.

Example 3.5 Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, let $\zeta = \mathrm{e}^{2\pi i/m}$, and let α act on $\mathbb{C}^m$ by $\alpha : (z_1, \ldots, z_m) \mapsto (\zeta z_1, \ldots, \zeta z_m)$. Then $\alpha^m = 1$, and the group $G = \langle\alpha\rangle$ generated by α is a subgroup of $\mathrm{SU}(m)$ isomorphic to $\mathbb{Z}_m$, which acts freely on $\mathbb{C}^m \setminus \{0\}$. Thus the quotient $\mathbb{C}^m/G$ has an isolated singular point at 0. Let (X, π) be the *blow-up* of $\mathbb{C}^m/G$ at 0, so that $\pi^{-1}(0) \cong \mathbb{CP}^{m-1}$. It is easy to show that X is in fact a *crepant resolution* of $\mathbb{C}^m/G$.

Let r be the radius function on $\mathbb{C}^m/G$, and define $f : \mathbb{C}^m/G \setminus \{0\} \to \mathbb{R}$ by

$$f = \sqrt[m]{r^{2m}+1} + \frac{1}{m}\sum_{j=0}^{m-1} \zeta^j \log\Bigl(\sqrt[m]{r^{2m}+1} - \zeta^j\Bigr). \tag{5}$$

To define the logarithm of the complex number $\sqrt[m]{r^{2m}+1} - \zeta^j$ we cut $\mathbb{C}$ along the negative real axis, and set $\log(R\mathrm{e}^{i\theta}) = \log R + i\theta$ for $R > 0$ and $\theta \in (-\pi, \pi)$. Then f is well-defined, and it is a smooth *real* function on $\mathbb{C}^m/G \setminus \{0\}$, despite its complex definition.

Define a (1,1)-form ω on $X \setminus \pi^{-1}(0)$ by $\omega = \mathrm{dd}^c\pi^*(f)$. It can be shown that ω extends to a smooth, closed, positive (1,1)-form on all of X. Let g be the Kähler metric on X with Kähler form ω. Then Calabi [5, §4] shows that g is complete and Ricci-flat, with $\mathrm{Hol}(g) = \mathrm{SU}(m)$. Equation (5) is derived from [5, eqn (4.14), p. 285]. Note also that the action of $\mathrm{U}(m)$ on $\mathbb{C}^m$ pushes down to $\mathbb{C}^m/G$ and lifts through π to X, and g is invariant under this action of $\mathrm{U}(m)$ on X.

This metric g on X is an *ALE Kähler metric*. To prove this, we show using (5) that

$$f = r^2 - \frac{1}{m(m-1)}\, r^{2-2m} + O(r^{-2m}) \quad \text{on } \mathbb{C}^m/G \setminus \{0\}, \text{ for large } r. \tag{6}$$

Now the Kähler form of the Euclidean metric on $\mathbb{C}^m/G$ is $\omega_0 = \mathrm{dd}^c(r^2)$. Hence

$$\pi_*(\omega) = \omega_0 - \frac{1}{m(m-1)}\,\mathrm{dd}^c(r^{2-2m}) + \mathrm{dd}^c\chi \quad \text{on } \mathbb{C}^m/G \setminus \{0\}, \tag{7}$$

where $\chi = f - r^2 + \frac{1}{m(m-1)} r^{2-2m}$. It is easy to show that $\nabla^k\chi = O(r^{-k-2m})$ on $\mathbb{C}^m/G \setminus \{0\}$ for large r, and it quickly follows that g is an ALE Kähler metric on X, by Definition 2.3. Also, g is one of the Ricci-flat ALE Kähler metrics of Theorem 3.3, and comparing (7) with (4) we see that $A = -\frac{1}{m(m-1)}$, which verifies that $A < 0$.

For $m \geq 3$, the metrics of Example 3.5 are the *only* explicit examples of ALE metrics with holonomy $\mathrm{SU}(m)$ that are known, at least to the author. It is possible to find these metrics explicitly because they have a large symmetry group $\mathrm{U}(m)$, whose orbits are of real codimension 1 in X. Because of this, the problem can be reduced to a nonlinear, second-order ODE in one real variable, which can then be explicitly solved.

It is a natural question whether we can find an explicit, algebraic form for any or all of the other ALE metrics with holonomy $\mathrm{SU}(m)$ for $m \geq 3$, that exist on crepant resolutions of $\mathbb{C}^m/G$ by Theorem 3.3. The author believes that general ALE metrics with holonomy $\mathrm{SU}(m)$ for $m \geq 3$ are essentially transcendental, nonalgebraic objects, and that one cannot write them down explicitly using simple functions. Furthermore, the author conjectures that for $m \geq 3$, the metrics of Example 3.5 are the only ALE metrics with holonomy $\mathrm{SU}(m)$ that can be written down explicitly in coordinates.

4 Analysis on ALE manifolds

Let (M, g) be a Riemannian manifold. Then the *Hölder spaces* $C^{k,\alpha}(M)$ are Banach spaces of functions on M, defined in Besse [3, p. 456-7]. When M is compact, elliptic operators such as the Laplacian Δ have very good regularity properties on Hölder spaces. Here is a typical elliptic regularity result, following from [3, Th. 27 & Th. 31, p. 463-4]. Theorems of this kind are essential tools in analytic problems such as the proof of the Calabi conjecture.

Theorem 4.1 *Let (M, g) be a compact Riemannian manifold, let $k \geq 0$ be an integer, and $\alpha \in (0, 1)$. Then for each $f \in C^{k,\alpha}(M)$ with $\int_M f\,\mathrm{d}V_g =$*

0 *there exists a unique* $u \in C^{k+2,\alpha}(M)$ *with* $\int_M u \, \mathrm{d}V_g = 0$ *and* $\Delta u = f$. *Moreover,* $\|u\|_{C^{k+2,\alpha}} \le C\|f\|_{C^{k,\alpha}}$ *for some* $C > 0$ *independent of* u *and* f.

However, if (X, g) is an ALE manifold then the results of Theorem 4.1 are false for X. This tells us that the $C^{k,\alpha}(X)$ are not good choices of Banach spaces of functions for studying elliptic operators on an ALE manifold. Instead, it turns out to be helpful to introduce *weighted Hölder spaces*, which we define next.

Definition 4.2 Let (X, g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$, and ρ a radius function on X. For $\beta \in \mathbb{R}$ and k a nonnegative integer, define $C^k_\beta(X)$ to be the space of continuous functions f on X with k continuous derivatives, such that $\rho^{j-\beta}\bigl|\nabla^j f\bigr|$ is bounded on X for $j = 0, \ldots, k$. Define the norm $\|\,.\,\|_{C^k_\beta}$ on $C^k_\beta(X)$ by

$$\|f\|_{C^k_\beta} = \sum_{j=0}^{k} \sup_X \bigl|\rho^{j-\beta}\nabla^j f\bigr|. \tag{8}$$

Let $\delta(g)$ be the injectivity radius of g, and write $d(x, y)$ for the distance between x, y in X. For T a tensor field on X and $\alpha, \gamma \in \mathbb{R}$, define

$$\bigl[T\bigr]_{\alpha,\gamma} = \sup_{\substack{x \ne y \in X \\ d(x,y) < \delta(g)}} \left[\min\bigl(\rho(x), \rho(y)\bigr)^{-\gamma} \cdot \frac{\bigl|T(x) - T(y)\bigr|}{d(x, y)^\alpha}\right]. \tag{9}$$

Here we interpret $|T(x) - T(y)|$ using parallel translation along the unique geodesic of length $d(x, y)$ joining x and y.

For $\beta \in \mathbb{R}$, k a nonnegative integer, and $\alpha \in (0, 1)$, define the *weighted Hölder space* $C^{k,\alpha}_\beta(X)$ to be the set of $f \in C^k_\beta(X)$ for which the norm

$$\bigl\|f\bigr\|_{C^{k,\alpha}_\beta} = \bigl\|f\bigr\|_{C^k_\beta} + \bigl[\nabla^k f\bigr]_{\alpha,\beta-k-\alpha} \tag{10}$$

is finite. Define $C^\infty_\beta(X)$ to be the intersection of the $C^k_\beta(X)$ for all $k \ge 0$. Both $C^k_\beta(X)$ and $C^{k,\alpha}_\beta(X)$ are Banach spaces, but $C^\infty_\beta(X)$ is not a Banach space.

This definition is taken from Lee and Parker [16, §9]. A function f in $C^k_\beta(X)$ or $C^{k,\alpha}_\beta(X)$ grows at most like ρ^β as $\rho \to \infty$, and so the index β should be interpreted as an *order of growth*. Similarly, the derivatives $\nabla^j f$

grow at most like $\rho^{\beta-j}$ for $j = 1, \ldots, k$. As vector spaces of functions $C^k_\beta(X)$ and $C^{k,\alpha}_\beta(X)$ are independent of the choice of radius function ρ. The norms on these spaces do depend on ρ, but not in a significant way, as all choices of ρ give equivalent norms.

There is also another useful class of Banach spaces on ALE manifolds, the *weighted Sobolev spaces* $L^q_{k,\beta}(X)$, which we will not define. They have similar analytic properties to the weighted Hölder spaces, and are described in [16, §9]. We have chosen to use weighted Hölder spaces instead, as they are often more convenient for nonlinear problems.

Next we discuss the analysis of the Laplacian Δ on ALE manifolds. Much work has been done on the behaviour of Δ on weighted Sobolev spaces and Hölder spaces on $\mathbb{R}^n$, and more generally on AE manifolds. A useful guide, with references, can be found in Lee and Parker [16, §9]. Most of these results apply immediately to ALE manifolds, with only very minor cosmetic changes to their proofs.

Proposition 4.3 *Let (X, g) be an ALE manifold of dimension n asymptotic to $\mathbb{R}^n/G$, let $\beta, \gamma \in \mathbb{R}$ satisfy $\beta + \gamma < 2 - n$, and suppose $u \in C^2_\beta(X)$ and $v \in C^2_\gamma(X)$. Then*

$$\int_X u\,\Delta v\,\mathrm{d}V_g = \int_X v\,\Delta u\,\mathrm{d}V_g. \tag{11}$$

Let ρ be a radius function on X. Then $\Delta(\rho^{2-n}) \in C^\infty_{-2n}(X)$ and

$$\int_X \Delta(\rho^{2-n})\,\mathrm{d}V_g = \frac{(n-2)\,\Omega_{n-1}}{|G|}, \tag{12}$$

where Ω_{n-1} is the volume of the unit sphere $\mathcal{S}^{n-1}$ in $\mathbb{R}^n$.

Proof. Let S_R be the subset $\{x \in X : \rho(x) \le R\}$ in X. Stokes' Theorem gives that

$$\int_{S_R} \bigl(u\Delta v - v\Delta u\bigr)\,\mathrm{d}V_g = \int_{\partial S_R} \bigl[(u\nabla v - v\nabla u)\cdot \mathbf{n}\bigr]\,\mathrm{d}V_g, \tag{13}$$

where $\mathbf{n}$ is the inward-pointing unit normal to ∂S_R. But for large R we have $\mathrm{vol}(\partial S_R) = O(R^{n-1})$ and $u\nabla v - v\nabla u = O(R^{\beta+\gamma-1})$ on ∂S_R, so that the r.h.s. of (13) is $O(R^{\beta+\gamma+n-2})$. Since $\beta + \gamma < 2 - n$ we see that the r.h.s. of (13) tends to zero as $R \to \infty$, and this proves (11).

The point about the power ρ^{2-n} is that $\Delta(r^{2-n}) = 0$ away from 0 in $\mathbb{R}^n/G$. Using the definitions of radius function and ALE manifold one can show that $\Delta(\rho^{2-n}) \in C^{\infty}_{-2n}(X)$, as we want. Using Stokes' Theorem again we find that

$$\int_{S_R} \Delta(\rho^{2-n})\,\mathrm{d}V_g = \int_{\partial S_R} \big[\nabla(\rho^{2-n}) \cdot \mathbf{n}\big]\,\mathrm{d}V_g. \tag{14}$$

But for large R we have $\nabla(\rho^{2-n})\cdot\mathbf{n} \approx (n\text{–}2)R^{1-n}$ and $\mathrm{vol}(S_R) \approx R^{n-1}\Omega_{n-1}/|G|$. Thus, letting $R \to \infty$ gives (12). □

Theorem 4.4 *Let $n > 2$ and $k \geq 0$ be integers and $\alpha \in (0,1)$, and let $\mathbb{R}^n$ have its Euclidean metric. Then*

(a) *Suppose $\beta \in (-n,-2)$. Then for each $f \in C^{k,\alpha}_{\beta}(\mathbb{R}^n)$ there is a unique $u \in C^{k+2,\alpha}_{\beta+2}(\mathbb{R}^n)$ with $\Delta u = f$.*

(b) *Suppose $\beta \in (-1-n,-n)$. Then for each $f \in C^{k,\alpha}_{\beta}(\mathbb{R}^n)$ there exists $u \in C^{k+2,\alpha}_{\beta+2}(\mathbb{R}^n)$ with $\Delta u = f$ if and only if $\int_{\mathbb{R}^n} f\,\mathrm{d}V = 0$, and u is then unique.*

In each case $\|u\|_{C^{k+2,\alpha}_{\beta+2}} \leq C\|f\|_{C^{k,\alpha}_{\beta}}$ for some $C > 0$ depending only on n, k, α and β.

Proof. This is an analogue for $\mathbb{R}^n$ of Theorem 4.1. If $u \in C^2_{\beta+2}(\mathbb{R}^n)$ for $\beta < -2$ and $\Delta u = f$, then by [9, §2.4] we have

$$u(y) = \frac{1}{(n-2)\Omega_{n-1}} \int_{x\in\mathbb{R}^n} |x-y|^{2-n} f(x)\mathrm{d}x, \tag{15}$$

where Ω_{n-1} is the volume of the unit sphere $\mathcal{S}^{n-1}$ in $\mathbb{R}^n$. This is *Green's representation* for u. Let ρ be a radius function on $\mathbb{R}^n$. Then $|f(x)| \leq \|f\|_{C^0_\beta}\rho(x)^\beta$, so (15) gives

$$|u(y)| \leq \frac{1}{(n-2)\Omega_{n-1}}\,\|f\|_{C^0_\beta} \cdot \int_{x\in\mathbb{R}^n} |x-y|^{2-n}\rho(x)^\beta \mathrm{d}x. \tag{16}$$

We split this into integrals over the three regions $|x| \leq \frac{1}{2}|y|$, $\frac{1}{2}|y| < |x| \leq 2|y|$ and $|x| > 2|y|$ in $\mathbb{R}^n$. Estimating the integral on each region separately we prove

$$\int_{x\in\mathbb{R}^n} |x-y|^{2-n}\rho(x)^\beta \mathrm{d}x \leq \begin{cases} C'\rho(y)^{\beta+2} & \text{for } \beta \in (-n,-2), \\ C'\rho(y)^{2-n} & \text{for } \beta < -n. \end{cases} \tag{17}$$

In case (a), if $\beta\in(-n,-2)$ then $|u(y)|\le C''\|f\|_{C^0_\beta}\rho(y)^{\beta+2}$ for some $C''>0$ depending only on n and β, and so $u\in C^0_{\beta+2}(\mathbb{R}^n)$ and $\|u\|_{C^0_{\beta+2}}\le C''\|f\|_{C^0_\beta}$.

One can extend this to show that $u\in C^{k+2,\alpha}_{\beta+2}(\mathbb{R}^n)$ and $\|u\|_{C^{k+2,\alpha}_{\beta+2}}\le C\|f\|_{C^{k,\alpha}_\beta}$ for some $C>0$ using the method of *Schauder estimates*, as in [9, §6]. The difficulty in doing this is to correctly include the powers of ρ involved in the weighted Hölder norm. To do this, for each $x\in\mathbb{R}^n$ we consider the ball $B_{\rho(x)/2}(x)$ of radius $\frac{1}{2}\rho(x)$ about x in $\mathbb{R}^n$.

On this ball we have $u=O\bigl(\rho(x)^{\beta+2}\bigr)$, $\nabla^j f=O\bigl(\rho(x)^{\beta-j}\bigr)$ for $j=0,\ldots,k$, and $[\nabla^k f]_\alpha=O\bigl(\rho(x)^{\beta-k-\alpha}\bigr)$. Using the Schauder interior estimates on the unit ball in $\mathbb{R}^n$ and rescaling distances by a factor $\frac{1}{2}\rho(x)$, we show that $\nabla^j u=O\bigl(\rho(x)^{\beta+2-j}\bigr)$ for $j=0,\ldots,k+2$ and $[\nabla^{k+2}u]_\alpha=O\bigl(\rho(x)^{\beta-k-\alpha}\bigr)$ on the interior of $B_{\rho(x)/2}(x)$. Thus $u\in C^{k+2,\alpha}_{\beta+2}(\mathbb{R}^n)$ and $\|u\|_{C^{k+2,\alpha}_{\beta+2}}\le C\|f\|_{C^{k,\alpha}_\beta}$, completing the proof of case (a).

Next we prove (b). Suppose $\beta\in(-1-n,-n)$, $u\in C^{k+2,\alpha}_{\beta+2}(\mathbb{R}^n)$ and $\Delta u=f$. Then

$$\int_{\mathbb{R}^n} f\,\mathrm{d}V=\int_{\mathbb{R}^n}1\Delta u\,\mathrm{d}V=\int_{\mathbb{R}^n}u\Delta(1)\,\mathrm{d}V=0 \tag{18}$$

by Proposition 4.3, since $u\in C^2_{\beta+2}(\mathbb{R}^n)$ and $1\in C^2_0(\mathbb{R}^n)$ and $\beta+2+0<2-n$. Thus, given $f\in C^{k,\alpha}_\beta(\mathbb{R}^n)$, there can only exist $u\in C^{k+2,\alpha}_{\beta+2}(\mathbb{R}^n)$ with $\Delta u=f$ if $\int_{\mathbb{R}^n}f\,\mathrm{d}V=0$. So suppose that $\int_{\mathbb{R}^n}f\,\mathrm{d}V=0$, and define u by

$$u(y)=\frac{1}{(n-2)\Omega_{n-1}}\int_{x\in\mathbb{R}^n}\Bigl[|x-y|^{2-n}-\rho(y)^{2-n}\Bigr]f(x)\mathrm{d}x. \tag{19}$$

Since $\int_{\mathbb{R}^n}f\,\mathrm{d}V=0$ the term involving $\rho(y)^{2-n}$ in this integral vanishes, so the equation reduces to (15) and thus $\Delta u=f$. From (19) we see that

$$|u(y)|\le\frac{1}{(n-2)\Omega_{n-1}}\,\|f\|_{C^0_\beta}\cdot\int_{x\in\mathbb{R}^n}\Bigl||x-y|^{2-n}-\rho(y)^{2-n}\Bigr|\rho(x)^\beta\mathrm{d}x,$$

and estimating as before shows that $|u(y)|\le C\|f\|_{C^0_\beta}\rho(y)^{\beta+2}$ when $\beta\in(-1-n,-n)$. Thus $u\in C^0_{\beta+2}(\mathbb{R}^n)$ and $\|u\|_{C^0_{\beta+2}}\le C\|f\|_{C^0_\beta}$. The rest of case (b) follows as above. □

Now we extend Theorem 4.4 to ALE manifolds.

Theorem 4.5 *Suppose (X,g) is an ALE manifold asymptotic to $\mathbb{R}^n/G$ for $n>2$, and ρ a radius function on X. Let $k\ge 0$ be an integer and $\alpha\in(0,1)$. Then*

(a) *Let* $\beta \in (-n, -2)$. *Then there exists* $C > 0$ *such that for each* $f \in C^{k,\alpha}_\beta(X)$ *there is a unique* $u \in C^{k+2,\alpha}_{\beta+2}(X)$ *with* $\Delta u = f$, *which satisfies* $\|u\|_{C^{k+2,\alpha}_{\beta+2}} \le C\|f\|_{C^{k,\alpha}_\beta}$.

(b) *Let* $\beta \in (-1-n, -n)$. *Then there exist* $C_1, C_2 > 0$ *such that for each* $f \in C^{k,\alpha}_\beta(X)$ *there is a unique* $u \in C^{k+2,\alpha}_{2-n}(X)$ *with* $\Delta u = f$. *Moreover* $u = A\rho^{2-n} + v$, *where*

$$A = \frac{|G|}{(n-2)\,\Omega_{n-1}} \cdot \int_X f \,\mathrm{d}V_g \tag{20}$$

and $v \in C^{k+2,\alpha}_{\beta+2}(X)$ *satisfy* $|A| \le C_1\|f\|_{C^0_\beta}$ *and* $\|v\|_{C^{k+2,\alpha}_{\beta+2}} \le C_2\|f\|_{C^{k,\alpha}_\beta}$. *Here* Ω_{n-1} *is the volume of the unit sphere* $\mathcal{S}^{n-1}$ *in* $\mathbb{R}^n$.

Proof. The theory of weighted Hölder spaces on AE manifolds and the Laplacian is developed by Chaljub-Simon and Choquet-Bruhat [6], who restrict their attention to the case $n = 3$. In particular, they prove part (a) of the Theorem for the case $n = 3$, $k = 0$ and $G = \{1\}$, [6, p. 15-16]. Their proof uses a result equivalent to part (a) of Theorem 4.4 in the case $n = 3$ and $k = 0$. By using Theorem 4.4 together with the methods of [6] one can show that Theorem 4.4 applies not only to $\mathbb{R}^n$ with its Euclidean metric, but also to any ALE manifold (X, g) asymptotic to $\mathbb{R}^n/G$. This proves case (a) of of the Theorem immediately.

For case (b), let $f \in C^{k,\alpha}_\beta(X)$, and define A by (20). Then by equation (12) we have $\int_X \bigl[f - \Delta(A\rho^{2-n})\bigr]\mathrm{d}V_g = 0$. Also $\Delta(\rho^{2-n}) \in C^\infty_{-2n}(X)$ by Proposition 4.3, and so $f - \Delta(A\rho^{2-n})$ lies in $C^{k,\alpha}_\beta(X)$ and has integral zero on X. Since $|f| \le \|f\|_{C^0_\beta}\rho^\beta$ we have $|A| \le C_1\|f\|_{C^0_\beta}$ for $C_1 = \int_X \rho^\beta \,\mathrm{d}V_g$, as we have to prove.

Applying case (b) of Theorem 4.4 for X to $f - \Delta(A\rho^{2-n})$, we see that there is a unique $v \in C^{k+2,\alpha}_{\beta+2}(X)$ with $\Delta v = f - \Delta(A\rho^{2-n})$, which satisfies

$$\|v\|_{C^{k+2,\alpha}_{\beta+2}} \le C\bigl(\|f\|_{C^{k,\alpha}_\beta} + |A| \cdot \|\Delta(\rho^{2-n})\|_{C^{k,\alpha}_\beta}\bigr). \tag{21}$$

Defining $u = A\rho^{2-n} + v$ gives $\Delta u = f$ as we want. Clearly $u \in C^{k+2,\alpha}_{2-n}(X)$, and the inequality $\|v\|_{C^{k+2,\alpha}_{\beta+2}} \le C_2\|f\|_{C^{k,\alpha}_\beta}$ then follows from (21) and the estimate on $|A|$ above. □

5 Exterior forms and de Rham cohomology

Let (X,g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$. Let $H^*(X,\mathbb{R})$ be the de Rham cohomology of X, and $H^*_c(X,\mathbb{R})$ the de Rham cohomology of X *with compact support*. That is,

$$H^k_c(X,\mathbb{R}) = \frac{\bigl\{\eta : \eta \text{ is a smooth, closed, compactly-supported } k\text{-form on } X\bigr\}}{\bigl\{\mathrm{d}\zeta : \zeta \text{ is a smooth, compactly-supported } (k-1)\text{-form on } X\bigr\}}.$$

Both $H^k(X,\mathbb{R})$ and $H^k_c(X,\mathbb{R})$ are finite-dimensional vector spaces. Let us regard X as a compact manifold with boundary $\mathcal{S}^{n-1}/G$. Using the long exact sequence

$$\ldots \to H^k_c(X,\mathbb{R}) \to H^k(X,\mathbb{R}) \to H^k(\mathcal{S}^{n-1}/G,\mathbb{R}) \to H^{k+1}_c(X,\mathbb{R}) \to \ldots,$$

the de Rham cohomology of $\mathcal{S}^{n-1}/G$, and the fact that $H^k_c(X,\mathbb{R}) \cong \bigl[H^{n-k}(X,\mathbb{R})\bigr]^*$ by Poincaré duality for manifolds with boundary, one can show that

$$\begin{gathered} H^0(X,\mathbb{R}) = \mathbb{R}, \quad H^0_c(X,\mathbb{R}) = 0, \quad H^n(X,\mathbb{R}) = 0, \quad H^n_c(X,\mathbb{R}) = \mathbb{R}, \quad \text{and} \\ H^k(X,\mathbb{R}) \cong H^k_c(X,\mathbb{R}) \cong \bigl[H^{n-k}(X,\mathbb{R})\bigr]^* \cong \bigl[H^{n-k}_c(X,\mathbb{R})\bigr]^* \quad \text{for } 0<k<n. \end{gathered}$$

Now the material on weighted Hölder spaces of functions in §4 generalizes naturally to weighted Hölder spaces of k-forms on ALE manifolds (X,g), so we may define the spaces $C^{l,\alpha}_\beta(\Lambda^k T^*X)$ and $C^\infty_\beta(\Lambda^k T^*X)$ in the obvious way. Similarly, the results of §4 on the Laplacian Δ on functions generalize to results on the Laplacian $\Delta = \mathrm{d}\mathrm{d}^* + \mathrm{d}^*\mathrm{d}$ on k-forms.

These tools can be used to generalize the ideas of Hodge theory to ALE manifolds. In particular, one can prove the following result.

Theorem 5.1 *Let (X,g) be an ALE manifold asymptotic to $\mathbb{R}^n/G$ for $n > 2$, and define*

$$\mathcal{H}^k = \bigl\{\eta \in C^\infty_{1-n}(\Lambda^k T^*X) : \mathrm{d}\eta = \mathrm{d}^*\eta = 0\bigr\}.$$

Then $\mathcal{H}^0 = \mathcal{H}^n = 0$, and the map $\mathcal{H}^k \to H^k(X,\mathbb{R})$ given by $\eta \mapsto [\eta]$ induces natural isomorphisms $\mathcal{H}^k \cong H^k(X,\mathbb{R}) \cong H^k_c(X,\mathbb{R})$ for $0 < k < n$. The Hodge star gives an isomorphism $* : \mathcal{H}^k \to \mathcal{H}^{n-k}$. Suppose $1-n \le \beta < -n/2$. Then

$$C^\infty_\beta(\Lambda^k T^*X) = \mathcal{H}^k \oplus \mathrm{d}\Bigl[C^\infty_{\beta+1}(\Lambda^{k-1}T^*X)\Bigr] \oplus \mathrm{d}^*\Bigl[C^\infty_{\beta+1}(\Lambda^{k+1}T^*X)\Bigr],$$

where the summands are L^2-orthogonal.

This is an analogue of the Hodge Decomposition Theorem and Hodge's Theorem. For the rest of the section we shall restrict our attention to ALE Kähler manifolds. If (X, J, g) is an ALE Kähler manifold then we can define the weighted Hölder spaces of (p, q)-forms $C^{l,\alpha}_{\beta}(\Lambda^{p,q}X)$ on X in the obvious way. The Laplacian Δ acts on these spaces by

$$\Delta : C^{l+2,\alpha}_{\beta+2}(\Lambda^{p,q}X) \to C^{l,\alpha}_{\beta}(\Lambda^{p,q}X). \tag{22}$$

They have very similar analytic properties to the weighted Hölder spaces of functions on an ALE manifold discussed in §4.

We can use facts about the Laplacian on weighted Hölder spaces of (p, q)-forms to develop an analogue for ALE Kähler manifolds of Hodge theory for compact Kähler manifolds.

Theorem 5.2 *Let (X, J, g) be an ALE Kähler manifold asymptotic to $\mathbb{C}^m/G$. Define*

$$\mathcal{H}^{p,q} = \bigl\{\eta \in C^{\infty}_{1-2m}(\Lambda^{p,q}X) : \mathrm{d}\eta = \mathrm{d}^*\eta = 0\bigr\}. \tag{23}$$

Then $\mathcal{H}^{p,q}$ is finite-dimensional, and the map $\mathcal{H}^{p,q} \to H^{p+q}(X, \mathbb{C})$ defined by $\eta \mapsto [\eta]$ is injective. Define $H^{p,q}(X)$ to be the image of this map. Then

$$H^k(X, \mathbb{C}) = \bigoplus_{j=0}^{k} H^{j,k-j}(X) \qquad \text{for } 0 < k < 2m. \tag{24}$$

In fact, if X is a crepant resolution of $\mathbb{C}^m/G$ then $H^{p,q}(X) = 0$ for $p \neq q$.

Theorem 5.3 *Let (X, J, g) be an ALE Kähler manifold, where X is a resolution of $\mathbb{C}^m/G$. Then $H^{2,0}(X) = H^{0,2}(X) = 0$, and each element of $H^{1,1}(X)$ is represented by a closed, compactly-supported $(1, 1)$-form on X.*

Here is a sketch of the proof of this theorem. Since X is a resolution of $\mathbb{C}^m/G$, it can be shown that the homology group $H_{2m-2}(X, \mathbb{C})$ is generated by the homology classes of the exceptional divisors of the resolution. But $H_{2m-2}(X, \mathbb{C}) \cong H^2_c(X, \mathbb{C})$. Thus $H^2_c(X, \mathbb{C})$ is generated by cohomology classes dual to the homology classes $[D]$ of exceptional divisors D in X. If U is any open neighbourhood of D in X, then we can find a closed $(1, 1)$-form supported in U representing the cohomology class dual to $[D]$. Therefore $H^2_c(X,\mathbb{C})$ is generated by cohomology classes represented by closed, compactly-supported $(1,1)$-forms. It easily follows that $H^{2,0}(X) = H^{0,2}(X) = 0$, and the proof is finished.

Next we prove a version of the Global dd^c-Lemma for ALE Kähler manifolds.

Theorem 5.4 *Let (X,J,g) be an ALE Kähler manifold asymptotic to $\mathbb{C}^m/G$ for some $m>1$, and let $\beta<-m$. Suppose that $\eta\in C^\infty_\beta(\Lambda^{1,1}_{\mathbb{R}}X)$ is a closed real $(1,1)$-form and $[\eta]=0$ in $H^2(X,\mathbb{R})$. Then there exists a unique real function $u\in C^\infty_{\beta+2}(X)$ with $\eta=\mathrm{dd}^c u$.*

Proof. Let ω be the Kähler form of g. Then if u is a smooth function on X we have

$$\mathrm{dd}^c u\wedge\omega^{m-1} = -\tfrac{1}{m}\,\Delta u\,\omega^m. \tag{25}$$

Also, if ζ is a real (1,1)-form on X and $\zeta\wedge\omega^{m-1}=0$ it can be shown that

$$\zeta\wedge\omega^{m-2} = -\tfrac{1}{2}(m-2)!\,*\zeta \quad\text{and}\quad \zeta\wedge\zeta\wedge\omega^{m-2} = -\tfrac{1}{2}(m-2)!\,|\zeta|^2\mathrm{d}V_g, \tag{26}$$

where $*$ is the Hodge star and $\mathrm{d}V_g$ the volume form of g. Equations (25) and (26) hold on any Kähler manifold of dimension m.

Define a function f on X by $\eta\wedge\omega^{m-1}=-\frac{1}{m}f\,\omega^m$. Since $\eta\in C^\infty_\beta(\Lambda^{1,1}_{\mathbb{R}}X)$, it follows that $f\in C^\infty_\beta(X)$. Now suppose for simplicity that $-2m<\beta<-m$. Then by part (a) of Theorem 4.5 there exists a unique function $u\in C^\infty_{\beta+2}(X)$ with $\Delta u=f$. Set $\zeta=\eta-\mathrm{dd}^c u$, which is an exact 2-form in $C^\infty_\beta(\Lambda^{1,1}_{\mathbb{R}}X)$. As $\beta<-m$ we can use the last part of Theorem 5.1 to prove that $\zeta=\mathrm{d}\theta$, for some $\theta\in C^\infty_{\beta+1}(T^*X)$.

By (25) we have $\zeta\wedge\omega^{m-1}=-\frac{1}{m}(f-\Delta u)\,\omega^m=0$, so (26) gives

$$\mathrm{d}\big[\theta\wedge\zeta\wedge\omega^{m-2}\big] = \zeta\wedge\zeta\wedge\omega^{m-2} = -\tfrac{1}{2}(m-2)!\,|\zeta|^2\mathrm{d}V_g. \tag{27}$$

Let ρ be a radius function on X, and define $S_R=\big\{x\in X:\rho(x)\le R\big\}$ for $R>1$. Integrating (27) over S_R and using Stokes' Theorem gives that

$$-\tfrac{1}{2}(m-2)!\cdot\int_{S_R}|\zeta|^2\mathrm{d}V_g = \int_{\partial S_R}\theta\wedge\zeta\wedge\omega^{m-2}. \tag{28}$$

But for large R we have $\theta = O(R^{\beta+1})$, $\zeta = O(R^{\beta})$ and $\omega = O(1)$ on ∂S_R, and $\mathrm{vol}(\partial S_R) = O(R^{2m-1})$. Thus the r.h.s. of (28) is $O(R^{2\beta+2m})$. As $\beta < -m$, taking the limit as $R \to \infty$ shows that $\int_X |\zeta|^2 \mathrm{d}V_g = 0$, and so $\zeta = 0$ on X. Thus $\eta = \mathrm{dd}^c u$, as we have to prove.

We have proved the theorem assuming that $-2m < \beta < -m$, but we wish to prove it for all $\beta < -m$. If $\beta \le 2m$ and $\eta \in C^\infty_\beta(\Lambda^{1,1}_{\mathbb{R}} X)$ then $\eta \in C^\infty_\gamma(\Lambda^{1,1}_{\mathbb{R}} X)$ for any γ with $-2m < \gamma < -m$, and so from above we have $\eta = \mathrm{dd}^c u$ for some unique u in $C^\infty_{\gamma+2}(X)$. However, if $u \in C^\infty_{\gamma+2}(X)$ and $\mathrm{dd}^c u \in C^\infty_\beta(\Lambda^{1,1}_{\mathbb{R}} X)$, one can show that $u \in C^\infty_{\beta+2}(X)$ as we want. This is because $\mathrm{dd}^c u$ is a stronger derivative of u than Δu is, and contains more information. □

Finally, we show we can modify any ALE Kähler metric to be flat outside a compact set.

Proposition 5.5 *Let $\mathbb{C}^m/G$ have an isolated singularity at 0 for some $m > 1$, let (X, π) be a resolution of $\mathbb{C}^m/G$ that admits ALE Kähler metrics, and let ρ be a radius function on X. Then in each Kähler class there exists an ALE Kähler metric $\hat{g}$ on X such that $\hat{g} = \pi^*(h)$ on the subset $\{x \in X : \rho(x) > R\}$, where h is the Hermitian metric on $\mathbb{C}^m/G$ and $R > 0$ is a constant.*

Proof. Let g be an ALE Kähler metric on X, with Kähler form ω. By Theorems 5.2 and 5.3 there exists a closed, compactly-supported, real (1,1)-form θ on X with $[\theta] = [\omega]$ in $H^2(X, \mathbb{R})$. Define $\eta = \omega - \mathrm{dd}^c(\rho^2) - \theta$. Then η is an exact real (1,1)-form on X. Now the Kähler form of h on $\mathbb{C}^m/G$ is $\omega_0 = \mathrm{dd}^c(r^2)$. So from the definition of ALE Kähler metric we see that $\omega - \mathrm{dd}^c(\rho^2) \in C^\infty_{-2m}(\Lambda^{1,1}_{\mathbb{R}} X)$, and therefore $\eta \in C^\infty_{-2m}(\Lambda^{1,1}_{\mathbb{R}} X)$ as θ has compact support. Thus by Theorem 5.4 there is a unique real function $u \in C^\infty_{2-2m}(X)$ with $\eta = \mathrm{dd}^c u$, and we have $\omega = \theta + \mathrm{dd}^c(\rho^2) + \mathrm{dd}^c u$.

Let $\mu : \mathbb{R} \to [0, 1]$ be a smooth function with $\mu(t) = 1$ for $t \le -1$ and $\mu(t) = 0$ for $t \ge 0$. For each $R > 0$ define a closed (1,1)-form ω_R by

$$\omega_R = \theta + \mathrm{dd}^c(\rho^2) + \mathrm{dd}^c\big[\mu(\rho - R) \cdot u\big]. \tag{29}$$

Then $\omega_R = \omega$ wherever $\rho < R - 1$, and $\omega_R = \mathrm{dd}^c(\rho^2)$ wherever $\rho > R$ and outside the support of θ. It is easy to show that ω_R is a positive (1,1)-form for large R, which therefore defines a Kähler metric g_R on X. Define $\hat{g}$ to be g_R for some R sufficiently large that ω_R is positive, $\rho \le R$ on the support of

θ and $R \geq 2$. Then $\hat{g}$ is an ALE Kähler metric in the Kähler class of g, and where $\rho > R$ we have $\hat{g} = \pi^*(h)$, since the Kähler form of $\hat{g}$ is $\mathrm{dd}^c(\rho^2)$, the Kähler form of h is $\mathrm{dd}^c(r^2)$, and $\rho = \pi^*(r)$ as $\rho > R \geq 2$. $\square$

6 The Calabi conjecture for ALE manifolds

We can now state the following version of the Calabi conjecture for ALE Kähler manifolds.

The Calabi conjecture for ALE manifolds *Suppose that (X, J, g) is an ALE Kähler manifold of dimension m asymptotic to $\mathbb{C}^m/G$ for some $m > 1$, with Kähler form ω, and that ρ is a radius function on X. Then*

(a) *Let $\beta \in (-2m, -2)$. Then for each $f \in C^\infty_\beta(X)$ there is a unique $\phi \in C^\infty_{\beta+2}(X)$ such that $\omega + \mathrm{dd}^c\phi$ is a positive $(1,1)$-form and $(\omega + \mathrm{dd}^c\phi)^m = \mathrm{e}^f\omega^m$ on X.*

(b) *Let $\beta \in (-1-2m, -2m)$. Then for each $f \in C^\infty_\beta(X)$ there is a unique $\phi \in C^\infty_{2-2m}(X)$ such that $\omega + \mathrm{dd}^c\phi$ is a positive $(1,1)$-form and $(\omega + \mathrm{dd}^c\phi)^m = \mathrm{e}^f\omega^m$ on X. Moreover we can write $\phi = A\rho^{2-2m} + \psi$, where $\psi \in C^\infty_{\beta+2}(X)$ and*

$$A = \frac{|G|}{(m-1)\Omega_{2m-1}} \cdot \int_X (1 - \mathrm{e}^f)\mathrm{d}V_g. \tag{30}$$

Here Ω_{2m-1} is the volume of the unit sphere $\mathcal{S}^{2m-1}$ in $\mathbb{C}^m$.

It is easy to rewrite this in terms of the existence of ALE Kähler metrics with prescribed Ricci curvature, as in the original Calabi conjecture. The two cases (a) $\beta \in (-2m, -2)$ and (b) $\beta \in (-1-2m, -2m)$ come from Theorem 4.5. By combining the method of Yau's proof [22] of the Calabi conjecture with the ideas of §4 on analysis on ALE manifolds, we can prove the Calabi conjecture for ALE manifolds.

The conjecture will be proved in [12, §8.5–§8.6], and we give only a sketch of the proof of part (a) here. We use the *continuity method*. Suppose $\beta \in (-2m, -2)$. Fix $f \in C^{3,\alpha}_\beta(X)$, and define S to be the set of all $t \in [0,1]$ for which there exists $\phi \in C^{5,\alpha}_{\beta+2}(X)$ such that $\omega + \mathrm{dd}^c\phi$ is a positive (1,1)-form and $(\omega + \mathrm{dd}^c\phi)^m = \mathrm{e}^{tf}\omega^m$ on X.

Clearly $0 \in S$, taking $\phi = 0$. We prove that S is both *open* and *closed* in $[0,1]$. Thus $S = [0,1]$ as $[0,1]$ is connected, so $1 \in S$, and there exists $\phi \in C^{5,\alpha}_{\beta+2}(X)$ with $\omega + \mathrm{dd}^c\phi$ positive and $(\omega + \mathrm{dd}^c\phi)^m = \mathrm{e}^f\omega^m$ on X. We then use Theorem 4.5 to show that if $f \in C^\infty_\beta(X)$ then $\phi \in C^\infty_{\beta+2}(X)$, and this completes the proof.

To prove that S is open, we fix $t \in S$ and show that S contains a small neighbourhood of t by considering the *linearization* of the equation at t. This linearization turns out to involve the Laplacian of the metric with Kähler form $\omega + \mathrm{dd}^c\phi$, and part (a) of Theorem 4.5 gives us what we need.

To prove that S is closed, we take a sequence $\{t_j\}_{j=0}^\infty$ in S such that $t_j \to t \in [0,1]$ as $j \to \infty$. Let $\{\phi_j\}_{j=0}^\infty$ be the sequence of solutions to $(\omega + \mathrm{dd}^c\phi_j)^m = \mathrm{e}^{t_j f}\omega^m$. Then ϕ_j converges to some $\phi \in C^{5,\alpha}_{\beta+2}(X)$ as $j \to \infty$ with $(\omega + \mathrm{dd}^c\phi)^m = \mathrm{e}^{tf}\omega^m$, and thus $t \in S$. Therefore S contains its limit points, and is closed.

The difficult part in showing S closed is finding an *a priori estimate* for ϕ_j in $C^{5,\alpha}_{\beta+2}(X)$. To do this we first follow Yau's proof to get an a priori estimate in $C^{5,\alpha}(X)$. Then we use a 'weighted' version of Yau's method to estimate ϕ_j in $C^0_\delta(X)$ for some small $\delta < 0$. This can be improved to $C^{5,\alpha}_\delta(X)$, and then to $C^{5,\alpha}_{\beta+2}(X)$ by a kind of induction, decreasing δ step by step until $\delta = \beta + 2$.

This concludes our treatment of the Calabi conjecture for ALE manifolds, and we are now ready to prove Theorems 3.3 and 3.4.

6.1 The proof of Theorem 3.3

Let X be a crepant resolution of $\mathbb{C}^m/G$, where G acts freely on $\mathbb{C}^m \setminus \{0\}$. By Proposition 5.5, in each Kähler class of ALE Kähler metrics on X we can choose a metric $\hat{g}$ with $\hat{g} = \pi^*(h)$ wherever $\rho > R \geq 2$, where h is the Euclidean metric on $\mathbb{C}^m/G$. Let $\hat{\omega}$ be the Kähler form and η the Ricci form of $\hat{g}$. Then η is closed and $[\eta] = 2\pi\, c_1(X)$ in $H^2(X,\mathbb{R})$. But $c_1(X) = 0$ as X is a crepant resolution, so $[\eta] = 0$ in $H^2(X,\mathbb{R})$. Also, $\eta = 0$ wherever $\rho > R$, since there $\hat{g} = \pi^*(h)$ and h is flat.

Thus η is a closed, compactly-supported $(1,1)$-form on X with $[\eta] = 0$ in $H^2(X,\mathbb{R})$, and by Theorem 5.4 there exists a unique function $f \in C^\infty_\beta(X)$ for each $\beta < 0$ with $\eta = \frac{1}{2}\mathrm{dd}^c f$. In fact $f = 0$ wherever $\rho > R$, so f is compactly supported. The Calabi conjecture for ALE manifolds holds by [12, §8.5–§8.6]. Part (b) of the conjecture shows that there exists a unique function $\phi = A\rho^{2-2m} + \psi$ where A is given by (30) and $\psi \in C^\infty_{\beta+2}(X)$ for

$\beta \in (-1-2m, -2m)$, such that $\omega = \hat{\omega} + \mathrm{dd}^c\phi$ is a positive (1,1)-form and $\omega^m = \mathrm{e}^f \hat{\omega}^m$.

Let g be the Kähler metric on X with Kähler form ω. Then since the Ricci form of $\hat{g}$ is $\frac{1}{2}\mathrm{dd}^c f$ it follows by standard properties of the Ricci form that g has Ricci form zero, and is Ricci-flat. On $\{z \in \mathbb{C}^m/G : r(z) > R\}$ we have $\pi_*(\hat{g}) = h$, so that $\pi_*(\hat{\omega}) = \omega_0$, and $\pi_*(\rho) = r$. Thus defining $\chi = \pi_*(\psi)$ gives (4). Since $\psi \in C^\infty_{\beta+2}(X)$ for $\beta \in (-1-2m, -2m)$, putting $\gamma = \beta + 2$ we see that $\nabla^k \chi = O(r^{\gamma-k})$ for $k = 0, 1, 2, \dots$ and $\gamma \in (1-2m, 2-2m)$, as we have to prove.

From (4) we see that $\nabla^k(\pi_*(g) - h) = O(r^{-2m-k})$ for $k \geq 0$, and thus g is an ALE Kähler metric by Definition 2.3. Also, g is unique in its Kähler class of ALE metrics because ϕ is unique. It only remains to prove that $A < 0$. We can do this by giving an explicit expression for A. Let ζ be the unique element of $\mathcal{H}^{1,1}$ with $[\zeta] = [\omega]$. Then a calculation shows that

$$A = -\frac{|G|}{2m(m-1)^2\Omega_{2m-1}} \int_X |\zeta|^2 \mathrm{d}V_g, \tag{31}$$

where Ω_{2m-1} is the volume of the unit sphere $\mathcal{S}^{2m-1}$ in $\mathbb{C}^m$. Now $[\omega] \neq 0$ as this is outside the Kähler cone, so $\zeta \neq 0$ and A is negative. This completes the proof of Theorem 3.3.

6.2 The proof of Theorem 3.4

First we show that $\mathbb{C}^m/G$ has no crepant resolutions when $m > 2$ and $G \subset \mathrm{Sp}(m/2)$.

Proposition 6.1 *Suppose that $m > 2$ is even and that G is a nontrivial finite subgroup of $\mathrm{Sp}(m/2)$ which acts freely on $\mathbb{C}^m \setminus \{0\}$. Then $\mathbb{C}^m/G$ is a terminal singularity, and admits no crepant resolutions.*

Proof. Let $\gamma \neq 1$ in G. Then there are coordinates $(z^1, \dots, z^m)$ on $\mathbb{C}^m$ in which γ acts by

$$\left(z^1, \dots, z^m\right) \xmapsto{\gamma} \left(\mathrm{e}^{2\pi i a_1} z^1, \dots, \mathrm{e}^{2\pi i a_m} z^m\right). \tag{32}$$

As γ acts freely on $\mathbb{C}^m \setminus \{0\}$ we can take $a_j \in (0,1)$ for $j = 1, \dots, m$. Since $G \subset \mathrm{Sp}(m/2)$ we know that γ preserves a complex symplectic form on $\mathbb{C}^m$, and we can choose $(z^1, \dots, z^m)$ so that this form is $\mathrm{d}z^1 \wedge \mathrm{d}z^2 + \cdots +$

$dz^{m-1} \wedge dz^m$. Thus (32) gives $e^{2\pi i a_{2j-1}} e^{2\pi i a_{2j}} = 1$ for $j = 1, \ldots, m/2$. But as $a_{2j-1}, a_{2j} \in (0,1)$ this implies that $a_{2j-1} + a_{2j} = 1$ for $j = 1, \ldots, m/2$.

Therefore $a_1 + \cdots + a_m = m/2 > 1$ for all $\gamma \neq 1$ in G. So by Reid [17, §4] it follows that $\mathbb{C}^m/G$ is a *terminal singularity*, as defined in [17, p. 347]. Terminal singularities are essentially singularities which have no crepant partial resolutions. To be more precise, a crepant resolution of a terminal singularity has no exceptional divisors. Thus, if X is a crepant resolution of $\mathbb{C}^m/G$ then $b_{2m-2}(X) = 0$. By Poincaré duality for manifolds with boundary we see that $b_2(X) = 0$, which is a contradiction, as X must contain a complex curve. So $\mathbb{C}^m/G$ has no crepant resolutions. □

We now prove Theorem 3.4. Let X be a crepant resolution of $\mathbb{C}^m/G$, where G is nontrivial and acts freely on $\mathbb{C}^m \setminus \{0\}$, and let g be a Ricci-flat ALE Kähler metric on X. As X is simply-connected, by general facts about holonomy groups we know that $\mathrm{Hol}(g)$ is a connected Lie subgroup of $\mathrm{SU}(m)$. Since g is Ricci-flat it is nonsymmetric. Also (X, g) is not a Riemannian product, because it is asymptotic to $\mathbb{C}^m/G$, which is not a product. Thus g is irreducible.

Therefore we may apply Berger's classification of Riemannian holonomy groups [18, §10]. The only two possibilities are $\mathrm{Hol}(g) = \mathrm{SU}(m)$ or $\mathrm{Hol}(g) = \mathrm{Sp}(m/2)$. When $m = 2$ the two groups coincide, so suppose $m > 2$. The holonomy of the Euclidean metric h on $\mathbb{C}^m/G$ is $G \subset \mathrm{SU}(m)$. Since g is asymptotic to h one can show that $G \subset \mathrm{Hol}(g) \subseteq \mathrm{SU}(m)$. Hence, if $\mathrm{Hol}(g) = \mathrm{Sp}(m/2)$ then $G \subset \mathrm{Sp}(m/2)$. But Proposition 6.1 then shows that $\mathbb{C}^m/G$ admits no crepant resolutions, a contradiction. So $\mathrm{Hol}(g) \neq \mathrm{Sp}(m/2)$, and thus $\mathrm{Hol}(g) = \mathrm{SU}(m)$, which completes the proof.

The essential point in this proof is that there do not exist ALE manifolds with holonomy $\mathrm{Sp}(m/2)$ for $m > 2$. One can also show this using Schlessinger's Rigidity Theorem [19], and properties of hyperkähler manifolds.

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