K-STABILITY OF CONTINUOUS $C(X)$-ALGEBRAS

APURVA SETH, PRAHLAD VAIDYANATHAN

ABSTRACT. A C*-algebra is said to be K-stable if its nonstable K-groups are naturally isomorphic to the usual K-theory groups. We study continuous $C(X)$-algebras, each of whose fibers are K-stable. We show that such an algebra is itself K-stable under the assumption that the underlying space X is compact, metrizable, and of finite covering dimension.

Nonstable K-theory is the study of the homotopy groups of the unitary group of a C*-algebra. The study of these groups was initiated by Rieffel [13], who showed that, for an irrational rotation algebra A, the inclusion map from A to $M_n(A)$ induces an isomorphism between the corresponding homotopy groups. In other words, the nonstable K-groups are naturally isomorphic to the usual K-theory groups of the algebra.

The theory was further explored by Thomsen [16], who used the notion of quasi-unitaries to profitably extend nonstable K-theory to encompass non-unital C*-algebras. Furthermore, he showed that this forms a homology theory, which allowed him to explicitly calculate these groups for certain C*-algebras. In particular, he showed that certain infinite dimensional C*-algebras (including the Cuntz algebras and simple infinite dimensional AF-algebras) satisfy the property enjoyed by the irrational rotation algebra mentioned above; a property he termed *K-stability.*

Since then, it has been proved (See Section 1.1) that a variety of interesting simple C*-algebras are K-stable. The goal of this paper is to enlarge this class of C*-algebras to include non-simple C*-algebras.

By the Dauns-Hoffmann theorem (See [11] or [14]), any non-simple C*-algebra may be represented as the section algebra of an upper semi-continuous C*-bundle over a compact space. If we assume that the underlying space X is Hausdorff, then such an algebra carries a non-degenerate, central action of $C(X)$, and is called a $C(X)$-algebra. An interesting sub-class of $C(X)$-algebras are ones that come equipped with a natural continuity condition. These algebras, called *continuous $C(X)$-algebras*, are particularly tractable as one can often take phenomena that occur at each fiber and propagate them to understand local behaviour of the algebra. Using a compactness

2010 *Mathematics Subject Classification.* Primary 46L85; Secondary 46L80.
Key words and phrases. Nonstable K-theory, C*-algebras.

argument, one may even be able to understand global behaviour. It is this idea that we employ in this paper to prove our main theorem.

Theorem A. *Let X be a compact metric space of finite covering dimension, and let A be a continuous $C(X)$-algebra. If each fiber of A is K-stable, then A is K-stable.*

1. Preliminaries

1.1. **Nonstable K-theory.** We begin by reviewing the work of Thomsen of constructing the nonstable K-groups associated to a C*-algebra. For the proofs of all the facts mentioned below, the reader may refer to [16].

Let A be a C^*-algebra (not necessarily unital). Define an associative composition $\cdot$ on A by

$$a \cdot b = a + b - ab \tag{1}$$

Henceforth, if $a, b \in A$, then ab will denote the usual multiplication in the algebra, and should not be confused with $a \cdot b$, which will denote the above composition.

An element $a \in A$ is said to be quasi-invertible if there exists $b \in A$ such that $a \cdot b = b \cdot a = 0$, and we write $\widehat{\mathcal{GL}}(A)$ for the set of all quasi-invertible elements in A. An element $u \in A$ is said to be a quasi-unitary if $u \cdot u^* = u^* \cdot u = 0$, and we write $\widehat{\mathcal{U}}(A)$ for the set of all quasi-unitary elements in A.

If B is a unital C^*-algebra, we write $GL(B)$ for the group of invertibles in B and $U(B)$ for the group of unitaries in B. Let A^+ denote the unitization of A. It follows by [16, Lemma 1.2] that an element $a \in A$ is quasi-invertible if and only if $1 - a \in GL(A^+)$; and similarly, $u \in A$ is a quasi-unitary if and only if $(1 - u) \in U(A^+)$. Therefore, $\widehat{\mathcal{GL}}(A)$ is open in A, $\widehat{\mathcal{U}}(A)$ is closed in A, and they both form topological groups. Furthermore, the map $r : \widehat{\mathcal{GL}}(A) \to \widehat{\mathcal{U}}(A)$ given by

$$r(a) := 1 - (1 - a)((1 - a^*)(1 - a))^{-1/2}$$

is a strong deformation retract, and hence a homotopy equivalence.

For elements $u, v \in \widehat{\mathcal{U}}(A)$, we write $u \sim v$ if there is a continuous function $f : [0, 1] \to \widehat{\mathcal{U}}(A)$ such that $f(0) = u$ and $f(1) = v$. We write $\widehat{\mathcal{U}}_0(A)$ for the set of $u \in \widehat{\mathcal{U}}(A)$ such that $u \sim 0$. The next result, which we will use repeatedly throughout the paper, follows from [16, Theorem 1.9] and [3, Theorem 4.8].

Theorem 1.1. *If $\varphi : A \to B$ is a surjective $*$-homomorphism between two C^*-algebras, then the induced map $\varphi : \widehat{\mathcal{U}}_0(A) \to \widehat{\mathcal{U}}_0(B)$ is a Serre fibration.*

For a C*-algebra A, the suspension of A is defined to be $SA := C_0(\mathbb{R}) \otimes A$. For $n > 1$, we set $S^nA := S(S^{n-1}A)$. We then have

Lemma 1.2. [16, Lemma 2.3] *For any C*-algebra A, $\pi_n(\widehat{\mathcal{U}}(A)) \cong \pi_0(\widehat{\mathcal{U}}(S^nA))$.*

Definition 1.3. The nonstable K-groups of a C*-algebra A are defined as

$$k_n(A) := \pi_{n+1}(\widehat{\mathcal{U}}(A)), \qquad \text{for } n = -1, 0, \ldots$$

By Lemma 1.2 and Bott periodicity, it follows [16, Proposition 2.6] that, if A is a stable C*-algebra, then $k_n(A) \cong K_n(A)$, where $K_n(A)$ denotes the usual K-theory groups of A. Motivated by this, Thomsen defines the notion of K-stability of a C*-algebra.

Definition 1.4. Let A be a C^*-algebra and $m \geq 2$. Define $\iota_m : M_{m-1}(A) \to M_m(A)$ by

$$a \mapsto \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$$

We say that A is K-stable if $(\iota_m)_* : k_n(M_{m-1}(A)) \to k_n(M_m(A))$ is an isomorphism for all $n = -1, 0, 1, 2 \ldots$ and all $m = 2, 3, 4 \ldots$.

Remark 1.5. The following C*-algebras are known to be K-stable:

- If $\mathcal{Z}$ denotes the Jiang-Su algebra, then $A \otimes \mathcal{Z}$ is K-stable for any C*-algebra A [7]. In particular, every separable, approximately divisible C*-algebra is K-stable [17].
- Every irrational rotation algebra is K-stable [13].
- If $\mathcal{O}_n$ denotes the Cuntz algebra, then $A \otimes \mathcal{O}_n$ is K-stable for any C*-algebra A [16].
- If A is an infinite dimensional simple AF-algebra, then $A \otimes B$ is K-stable for any C*-algebra B [16] .
- If A is a purely infinite, simple C*-algebra, and p any non-zero projection of A, then pAp is K-stable [18].

Note that some of the examples mentioned above may be subsumed into the first example as they are known to absorb $\mathcal{Z}$ tensorially. However, it is worth mentioning that the original proofs of K-stability for these algebras does not use $\mathcal{Z}$-stability. Furthermore, in the case where the algebras are $\mathcal{Z}$-stable, Theorem A may be deduced from [6, Theorem 4.6].

We conclude this section with an important observation about K-stable C*-algebras.

Lemma 1.6. *If A is K-stable, then for any $m \geq 2, \iota_m(\widehat{\mathcal{U}}(M_{m-1}(A))$ is a strong deformation retract of $\widehat{\mathcal{U}}(M_m(A))$.*

Proof. Note that $\widehat{\mathcal{GL}}(A)$ is an absolute neighbourhood retract [12, Theorem 5], and therefore, the pair $(\widehat{\mathcal{GL}}(M_m(A)), \iota_m(\widehat{\mathcal{GL}}(M_{m-1}(A))))$ has the homotopy extension property with respect to all spaces [12, Theorem 7]. If A is K-stable, then $\iota_m : \widehat{\mathcal{GL}}(M_{m-1}(A)) \to \widehat{\mathcal{GL}}(M_m(A))$ is a weak homotopy equivalence. However, since $\widehat{\mathcal{GL}}(A)$ is an open subset of a normed linear space,

$\widehat{\mathcal{GL}}(A)$ has the homotopy type of a CW-complex [10, Chapter IV, Corollary 5.5]. By Whitehead's theorem, it follows that ι_m is a homotopy equivalence, so $\iota_m(\widehat{\mathcal{GL}}(M_{m-1}(A))$ is a strong deformation retract of $\widehat{\mathcal{GL}}(M_m(A))$ by [5, Theorem 0.20]. Since the retractions $r : \widehat{\mathcal{GL}}(M_k(A)) \to \widehat{\mathcal{U}}(M_k(A))$ commute with the inclusion map ι_k, we conclude that $\iota_m(\widehat{\mathcal{U}}(M_{m-1}(A))$ is a strong deformation retract of $\widehat{\mathcal{U}}(M_m(A))$. □

1.2. $C(X)$**-algebras.** Let A be a C*-algebra, and X a compact Hausdorff space. We say that A is a $C(X)$-algebra [8, Definition 1.5] if there is a unital $*$-homomorphism $\theta : C(X) \to ZM(A)$, where $ZM(A)$ denotes the center of the multiplier algebra of A.

If $Y \subset X$ is closed, the set $C_0(X,Y)$ of functions in $C(X)$ that vanish on Y is a closed ideal of $C(X)$. By the Cohen factorization theorem [1, Theorem 4.6.4], $C_0(X,Y)A$ is a closed, two-sided ideal of A. The quotient of A by this ideal is denoted by $A(Y)$, and we write $\pi_Y : A \to A(Y)$ for the quotient map (also referred to as the restriction map). If $Z \subset Y$ is a closed subset of Y, we write $\pi_Z^Y : A(Y) \to A(Z)$ for the natural restriction map, so that $\pi_Z = \pi_Z^Y \circ \pi_Y$. If $Y = \{x\}$ is a singleton, we write $A(x)$ for $A(\{x\})$ and π_x for $\pi_{\{x\}}$. The algebra $A(x)$ is called the fiber of A at x. For $a \in A$, write $a(x)$ for $\pi_x(a)$. For each $a \in A$, we have a map $\Gamma_a : X \to \mathbb{R}$ given by $x \mapsto \|a(x)\|$. We say that A is a continuous $C(X)$-algebra if Γ_a is continuous for each $a \in A$.

If A is a continuous $C(X)$-algebra, we will often have reason to consider other $C(X)$-algebras obtained from A. At that time, the following result of Kirchberg and Wasserman will be useful.

Theorem 1.7. [9, Remark 2.6] *Let X be a compact Hausdorff space, and let A be a continuous $C(X)$-algebra. If B is a nuclear C*-algebra, then $A \otimes B$ is a continuous $C(X)$-algebra whose fiber at a point $x \in X$ is $A(x) \otimes B$.*

Finally, one fact that plays a crucial role in our investigation is that a $C(X)$-algebra may be patched together from quotients in the following way: Let B, C, and D be C*-algebras, and $\delta : B \to D$ and $\gamma : C \to D$ be $*$-homomorphisms. We define the pullback of this system to be

$$A = B \oplus_D C := \{(b,c) \in B \oplus C : \delta(b) = \gamma(c)\}$$

This is describe by a diagram

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\delta} \\ C & \xrightarrow{\gamma} & D \end{array}$$

where $\phi(b,c) = b$ and $\psi(b,c) = c$.

Lemma 1.8. [2, Lemma 2.4] *Let X be a compact Hausdorff space and Y and Z be two closed subsets of X such that $X = Y \cup Z$. If A is a $C(X)$-algebra, then A is isomorphic to the pullback*

$$\begin{array}{ccc} A & \xrightarrow{\pi_Y} & A(Y) \\ {\scriptstyle \pi_Z}\downarrow & & \downarrow{\scriptstyle \pi^Y_{Y\cap Z}} \\ A(Z) & \xrightarrow{\pi^Z_{Y\cap Z}} & A(Y\cap Z) \end{array}$$

1.3. **Notational Conventions.** We fix some notational conventions we will use repeatedly: If A is a C*-algebra, we write $\iota_A : A \to M_2(A)$ for the natural inclusion map. When there is no ambiguity, we write ι for this map. Moreover, if $\varphi : A \to B$ is a $*$-homomorphism between two C*-algebras, then the induced map from $\widehat{\mathcal{U}}(A)$ to $\widehat{\mathcal{U}}(B)$ is also denoted by φ.

If A is a continuous $C(X)$-algebra, then $M_2(A)$ is also a continuous $C(X)$-algebra with fibers $M_2(A(x))$ by Theorem 1.7. We will often consider both simultaneously, so we fix the following convention: If $Y \subset X$ is a closed set, we denote the restriction map by $\eta_Y : M_2(A) \to M_2(A(Y))$, and write $\iota_Y : A(Y) \to M_2(A(Y))$ for the natural inclusion map. If $Y = X$, we simply write ι for ι_X. Note that $\eta_Y \circ \iota = \iota_Y \circ \pi_Y$. Once again, if $Y = \{x\}$, we simply write ι_x for $\iota_{\{x\}}$.

Finally, suppose f and g are two continuous paths in a topological space Y. If $f(1) = g(0)$, we write $f \bullet g$ for the concatenation of the two paths. If f and g agree at end-points, we write $f \sim_h g$ if there is a path homotopy between them. Furthermore, we write $\overline{f}$ for the path $\overline{f}(t) := f(1-t)$. If $Y = \widehat{\mathcal{U}}(A)$ for some C*-algebra A, we write $f \cdot g$ for the path $t \mapsto f(t) \cdot g(t)$, and we write f^* for the path $t \mapsto f(t)^*$.

2. Main Results

Lemma 2.1. *Let A be a K-stable C*-algebra and X a locally compact Hausdorff space, then $C_0(X) \otimes A$ is K-stable.*

Proof. Suppose first that X is compact. For simplicity of notation, we write B for $C(X) \otimes A$. By Lemma 1.6, for each $m \geq 2$, there is a retraction $r : \widehat{\mathcal{U}}(M_m(A)) \to \widehat{\mathcal{U}}(M_{m-1}(A))$ and a homotopy $F : [0,1] \times \widehat{\mathcal{U}}(M_m(A)) \to \widehat{\mathcal{U}}(M_m(A))$ such that $F(0, v) = v$ and $F(1, v) = r(v)$ for all $v \in \widehat{\mathcal{U}}(M_m(A))$. Clearly, we may identify $\widehat{\mathcal{U}}(M_m(B))$ with $C(X, \widehat{\mathcal{U}}(M_m(A))$, where the latter denotes the space of continuous functions from X to $\widehat{\mathcal{U}}(M_m(A))$, equipped with the uniform topology (which coincides with the compact-open topology). Therefore, we may define $\widetilde{r} : \widehat{\mathcal{U}}(M_m(B)) \to \widehat{\mathcal{U}}(M_{m-1}(B))$ by

$$\widetilde{r}(f)(x) := r(f(x))$$

and $\widetilde{F} : [0,1] \times \widehat{\mathcal{U}}(M_m(B)) \to \widehat{\mathcal{U}}(M_m(B))$ by

$$\widetilde{F}(s,f)(x) := F(s, f(x))$$

It is easy to see that $\widetilde{r}$ is a retraction, and that $\widetilde{F}$ is continuous, and implements a homotopy between $\iota_{M_{m-1}(B)} \circ \widetilde{r}$ and $\text{id}_{\widehat{\mathcal{U}}(M_m(B))}$. Thus, B is K-stable.

If X is not compact, let X^+ denote its one-point compactification. We now have a short exact sequence $0 \to C_0(X) \otimes A \to C(X^+) \otimes A \to A \to 0$, which induces a long exact sequence of k-groups by [16, Theorem 2.5]. By the first part of the argument, $C(X^+) \otimes A$ is K-stable, so the result follows from the five lemma. □

Let A be a C*-algebra, and $c \in A$ a self-adjoint element. We define

$$\Lambda(c) := -\sum_{n=1}^{\infty} \frac{(ic)^n}{n!}$$

Observe that, in $A^+, \Lambda(c) = 1 - \exp(ic)$, so that $\Lambda(c) \sim 0$ in $\widehat{\mathcal{U}}(A)$ via the path $t \mapsto \Lambda(tc)$. The next lemma is implicit in [16, Lemma 1.7], but we spell it out since its proof is crucial to us.

Lemma 2.2. *Let $a, b \in \widehat{\mathcal{U}}(A)$ such that $\|a-b\| < 2$, then $a \sim b$ in $\widehat{\mathcal{U}}(A)$*

Proof. Consider A as an ideal in A^+. If $\|a-b\| < 2$, then $d := a \cdot b^*$ satisfies

$$\|d\| = \|1 - ((1-a)(1-b^*))\| = \|((1-b)-(1-a))(1-b^*)\| \leq \|a-b\| < 2$$

Hence, $(1-d)$ is a unitary in A^+, whose spectrum does not contain 1. Therefore, there is a continuous function $g : S^1 \to \mathbb{R}$ such that $g(1) = 0$ and $\exp(ig(x)) = x$ for all $x \in \sigma(1-d)$. Define $c := g(1-d)$, then c is self-adjoint and $a \cdot b^* = \Lambda(c)$. Thus $a = \Lambda(c) \cdot b \sim b$ in $\widehat{\mathcal{U}}(A)$. □

The next lemma is a variant of [15, Exercise 2.8] for quasi-unitaries.

Lemma 2.3. *Given $\epsilon > 0$, there is a $\delta > 0$ with the following property: If A is a C*-algebra and $a \in A$ such that $\|a \cdot a^*\| < \delta$ and $\|a^* \cdot a\| < \delta$, then there is a quasi-unitary $u \in \widehat{\mathcal{U}}(A)$ such that $\|a - u\| < \epsilon$.*

Proof. Fix $0 < \delta < \min\{\epsilon, 1\}$, and consider A as an ideal in A^+. If $b := (1-a)$, the hypothesis implies that $\|1 - bb^*\| = \|a \cdot a^*\| < \delta < 1$, and $\|1 - b^*b\| < 1$ as well. Hence, bb^* and b^*b are both invertible, and therefore b and b^* are also invertible. Thus, $|b| = (b^*b)^{1/2} \in GL(A^+)$, and $v := b|b|^{-1} \in U(A^+)$. Furthermore, $\sigma(b^*b) \subset (1-\delta, 1+\delta)$, so $\sigma(|b|) \subset ((1-\delta)^{1/2}, (1+\delta)^{1/2}) \subset (1-\delta, 1+\delta)$. Hence, $\||b| - 1\| < \delta$, so that $u := 1 - v \in \widehat{\mathcal{U}}(A)$, and

$$\|u - a\| = \|v - b\| = \|v(1 - |b|)\| \leq \|1 - |b|\| < \delta < \epsilon$$

as required. □

Lemma 2.4. *Let X be a compact, Hausdorff space and let A be a continuous $C(X)$-algebra. Let $a \in \widehat{\mathcal{U}}(A)$ and $x \in X$ such that $a(x) \sim 0$ in $\widehat{\mathcal{U}}(A(x))$. Then there is a closed neighbourhood Y of x such that $\pi_Y(a) \sim 0$ in $\widehat{\mathcal{U}}(A(Y))$.*

Proof. Since $\pi_x : \widehat{\mathcal{U}}_0(A) \to \widehat{\mathcal{U}}_0(A(x))$ is a fibration, there is a path $H : [0,1] \to \widehat{\mathcal{U}}(A)$ such that $H(0) = 0$ and $H(1)(x) = a(x)$. Let $b := H(1)$, then since A is a continuous $C(X)$-algebra, there is a closed neighbourhood Y of x such that $\|\pi_Y(a-b)\| < 2$. It follows by Lemma 2.3 that $\pi_Y(a) \sim \pi_Y(b)$ in $\widehat{\mathcal{U}}(A(Y))$. Furthermore, the path $G : [0,1] \to \widehat{\mathcal{U}}(A(Y))$ is of the form

$$G(t) = \pi_Y(b) \cdot \Lambda(t\pi_Y(c))$$

for some self-adjoint element $c \in A$ (since every self-adjoint element in $A(Y)$ lifts to a self-adjoint element in A). Since $a(x) = b(x)$, the proof of Lemma 2.2 in fact ensures that we may choose c such that $c(x) = 0$. Therefore, $\pi_x \circ G(t) = a(x)$ for all $t \in [0,1]$. Concatenating the paths $\pi_Y \circ H$ and G, we obtain a path connecting 0 to $\pi_Y(a)$ in $\widehat{\mathcal{U}}(A(Y))$. □

Our proof of Theorem A is by induction on the covering dimension of the underlying space. The next theorem is the base case, and it holds even if the space is not metrizable.

Theorem 2.5. *Let X be a compact Hausdorff space of zero covering dimension, and let A be a continuous $C(X)$-algebra. If each fiber of A is K-stable, then so is A.*

Proof. If A is a continuous $C(X)$-algebra, then so is every suspension of A. Furthermore, $(S^nA)(x) \cong S^n(A(x))$ by Theorem 1.7, and $S^n(A(x))$ is K-stable by Lemma 2.1. Hence, by Lemma 1.2, it suffices to show that, for each $m \geq 2$, the map

$$(\iota_m)_* : \pi_0(M_{m-1}(A)) \to \pi_0(M_m(A))$$

is an isomorphism. However, by Theorem 1.7, each $M_n(A)$ is also a continuous $C(X)$-algebra, with fibers $M_n(A(x))$, which is K-stable if $A(x)$ is K-stable. Therefore, suffices to show that the map

$$\iota_* : \pi_0(\widehat{\mathcal{U}}(A)) \to \pi_0(\widehat{\mathcal{U}}(M_n(A))$$

is an isomorphism for each $n \geq 2$. For simplicity of notation, we fix $n = 2$.

We first consider injectivity. Suppose $a \in \widehat{\mathcal{U}}(A)$ such that $\iota(a) \sim 0$ in $\widehat{\mathcal{U}}(M_2(A))$. Then, for any $x \in X$, $\iota_x(a(x)) \sim 0$ in $\widehat{\mathcal{U}}(M_2(A(x))$. Since $A(x)$ is K-stable, $a(x) \sim 0$ in $\widehat{\mathcal{U}}(A(x))$. By Lemma 2.4, there is a closed neighbourhood Y_x of x such that $\pi_{Y_x}(a) \sim 0$ in $\widehat{\mathcal{U}}(A(Y_x))$. Since X is compact and zero dimensional, we obtain disjoint open sets $\{Y_{x_1}, Y_{x_2}, \ldots, Y_{x_n}\}$ which cover X. Then by Lemma 1.8,

$$A \cong A(Y_{x_1}) \oplus A(Y_{x_2}) \oplus \ldots \oplus A(Y_{x_n})$$

via the map $b \mapsto (\pi_{Y_{x_1}}(b), \pi_{Y_{x_2}}(b), \ldots, \pi_{Y_{x_n}}(b))$. Since $\pi_{Y_{x_i}}(a) \sim 0$ for each $1 \leq i \leq n$, it follows that $a \sim 0$, so ι_* is injective.

For surjectivity, choose $u \in \widehat{\mathcal{U}}(M_2(A))$, and we wish to construct a quasi-unitary $\omega \in \widehat{\mathcal{U}}(A)$ such that $u \sim \iota(\omega)$. To this end, fix $x \in X$. Then

$u(x) \in \widehat{\mathcal{U}}(M_2(A(x))$. Since $A(x)$ is K-stable, there exists $f_x \in \widehat{\mathcal{U}}(A(x))$ and a path $g_x : [0,1] \to \widehat{\mathcal{U}}(M_2(A(x))$ such that $g_x(0) = u(x)$ and $g_x(1) = \iota_x(f_x)$. Choose $e_x \in A$ such that $e_x(x) = f_x$ (Note that e_x may not be a quasi-unitary).

Since the map $\eta_x : \widehat{\mathcal{U}}_0(M_2(A)) \to \widehat{\mathcal{U}}_0(M_2(A(x)))$ is a fibration, g_x lifts to a path $G_x : [0,1] \to \widehat{\mathcal{U}}(M_2(A))$ such that $G_x(0) = u$. Let $b_x := G_x(1)$, and so that $b_x(x) = \iota_x(e_x(x))$. Choose $\delta > 0$ so that conclusion of Lemma 2.3 holds for $\epsilon = 1$. Since A is a continuous $C(X)$-algebra, there is a closed neighbourhood Y_x of x such that

$$\|\eta_{Y_x}(b_x) - \eta_{Y_x}(\iota(e_x))\| < 1, \|\pi_{Y_x}(e_x^* \cdot e_x)\| < \delta, \text{ and} \|\pi_{Y_x}(e_x \cdot e_x^*)\| < \delta$$

By Lemma 2.3, there is a quasi-unitary $d_x \in \widehat{\mathcal{U}}(A(Y_x))$ such that $\|d_x - \pi_{Y_x}(e_x)\| < 1$, so that $\|\iota_{Y_x}(d_x) - \eta_{Y_x}(b_x)\| < 2$. By Lemma 2.2, $\iota_{Y_x}(d_x) \sim \eta_{Y_x}(b_x)$ in $\widehat{\mathcal{U}}(A(Y_x))$. Hence, $\iota_{Y_x}(d_x) \sim \eta_{Y_x}(u)$. As before, since X is compact and zero-dimensional, we may choose disjoint, open sets $\{Y_{x_1}, Y_{x_2}, \ldots, Y_{x_n}\}$ so that

$$A \cong A(Y_{x_1}) \oplus A(Y_{x_2}) \oplus \ldots \oplus A(Y_{x_n})$$

via the map $a \mapsto (\pi_{Y_{x_1}}(a), \pi_{Y_{x_2}}(a), \ldots, \pi_{Y_{x_n}}(a))$. Similarly,

$$M_2(A) \cong M_2(A(Y_{x_1})) \oplus M_2(A(Y_{x_2})) \oplus \ldots \oplus M_2(A(Y_{x_n}))$$

via the map $b \mapsto (\eta_{Y_{x_1}}(b), \eta_{Y_{x_2}}(b), \ldots, \eta_{Y_{x_n}}(b))$. Therefore, there exists $\omega \in \widehat{\mathcal{U}}(A)$ such that $\pi_{Y_{x_i}}(\omega) = d_{x_i}$ for all $1 \leq i \leq n$. Furthermore, for each $1 \leq i \leq n, \eta_{Y_{x_i}}(\iota(\omega)) = \iota_{Y_{x_i}}(d_{x_i}) \sim \eta_{Y_{x_i}}(u)$ in $\widehat{\mathcal{U}}(A(Y_{x_i}))$, so that $\iota(\omega) \sim u$ in $\widehat{\mathcal{U}}(A)$, as required. □

The next two lemmas help us to extend the above argument to higher dimensional spaces.

Lemma 2.6. *Let X be a compact Hausdorff space, and A be a continuous $C(X)$-algebra. Let $f_i : [0,1] \to \widehat{\mathcal{U}}(A), i = 1,2$ be two paths, and $x \in X$ be a point such that $\pi_x \circ f_1 = \pi_x \circ f_2$.*

1. *There is a closed neighbourhood Y of x and a homotopy $H : [0,1] \times [0,1] \to \widehat{\mathcal{U}}(A(Y))$ such that $H(0) = \pi_Y \circ f_1$ and $H(1) = \pi_Y \circ f_2$.*
2. *If, in addition, $f_1(0) = f_2(0)$ and $f_1(1) = f_2(1)$, then the homotopy in part (1) may be chosen to be a path homotopy.*

Proof. For the first part, note that $C[0,1] \otimes A$ is itself a continuous $C(X)$-algebra by Theorem 1.7. Hence, there is a closed neighbourhood Y of x such that $\|\pi_Y \circ f_1 - \pi_Y \circ f_2\| < 2$. The result now follows from Lemma 2.2.

For the second part, an examination of Lemma 2.2 shows that the homotopy is implemented by a path $H : [0,1] \times [0,1] \to \widehat{\mathcal{U}}(A(Y))$ given by

$$H(s,t) = \pi_Y(f_2(t)) \cdot \Lambda(sh(t))$$

where $h(t) = g(1 - \pi_Y(f_1)(t) \cdot \pi_Y(f_2)(t)^*)$ for an appropriate branch $g : S^1 \to \mathbb{R}$ of the log function such that $g(1) = 0$. Since f_1 and f_2 agree at end-points, it follows that $h(0) = h(1) = 0$. Hence, H is a path homotopy. $\square$

Lemma 2.7. *Let X be a compact Hausdorff space, A be a continuous $C(X)$-algebra, and $x \in X$ be a point such that $A(x)$ is K-stable. Let $a \in \widehat{\mathcal{U}}(A)$ be a quasi-unitary and $F : [0,1] \to \widehat{\mathcal{U}}(M_2(A))$ be a path such that*

$$F(0) = 0 \text{ and } F(1) = \iota(a)$$

Then, there is a closed neighbourhood Y of x and a path $L_Y : [0,1] \to \widehat{\mathcal{U}}(A(Y))$ such that

$$L_Y(0) = 0, \quad L_Y(1) = \pi_Y(a)$$

and $\iota_Y \circ L_Y$ is path homotopic to $\eta_Y \circ F$ in $\widehat{\mathcal{U}}(M_2(A(Y)))$.

Proof. Let $\iota_x : \widehat{\mathcal{U}}(A(x)) \to \widehat{\mathcal{U}}(M_2(A(x))$ be the natural inclusion map. Since $A(x)$ is K-stable, Lemma 1.6 implies that there is a continuous function $r_x : \widehat{\mathcal{U}}(M_2(A(x))) \to \widehat{\mathcal{U}}(A(x))$ such that $\iota_x \circ r_x \sim \mathrm{id}_{\widehat{\mathcal{U}}(M_2(A(x)))}$. Furthermore, since r_x is a retract, the function $F' := r_x \circ \eta_x \circ F$ is a path in $\widehat{\mathcal{U}}(A(x))$ such that $F'(0) = 0$ and $F'(1) = a(x)$. Consider the commutative diagram

$$\begin{array}{ccc} \widehat{\mathcal{U}}(A) & \xrightarrow{\iota_A} & \widehat{\mathcal{U}}(M_2(A)) \\ \downarrow{\scriptstyle \pi_x} & & \downarrow{\scriptstyle \eta_x} \\ \widehat{\mathcal{U}}(A(x)) & \underset{r_x}{\overset{\iota_x}{\rightleftarrows}} & \widehat{\mathcal{U}}(M_2(A(x)) \end{array}$$

The map $\pi_x : \widehat{\mathcal{U}}_0(A) \to \widehat{\mathcal{U}}_0(A(x))$ is a fibration, so F' lifts to a path $G_x : [0,1] \to \widehat{\mathcal{U}}(A)$ such that $G_x(0) = 0$. Set $b_x := G_x(1)$, then $\iota_A \circ G_x : [0,1] \to \widehat{\mathcal{U}}(M_2(A))$ is a path such that $\iota_A \circ G_x(0) = 0$ and $\iota_A \circ G_x(1) = \iota_A(b_x)$. Furthermore,

$$\eta_x \circ \iota_A \circ G_x = \iota_x \circ \pi_x \circ G_x = \iota_x \circ F' = \iota_x \circ r_x \circ \eta_x \circ F$$

Since $\iota_x(\widehat{\mathcal{U}}(A(x))$ is a strong deformation retract of $\widehat{\mathcal{U}}(M_2(A(x)))$, it follows that there is a path homotopy $\widetilde{H} : [0,1] \times [0,1] \to \widehat{\mathcal{U}}(M_2(A(x)))$ such that, for all $s, t \in [0,1]$,

$$\widetilde{H}(0,t) = \eta_x \circ F(t), \widetilde{H}(1,t) = \eta_x \circ \iota_A \circ G_x(t), \widetilde{H}(s,0) = 0, \widetilde{H}(s,1) = \iota_x(a(x))$$

The map $\eta_x : \widehat{\mathcal{U}}_0(M_2(A)) \to \widehat{\mathcal{U}}_0(M_2(A(x))$ is a fibration, and $\eta_x \circ F$ has a lift, so $\widetilde{H}$ lifts to a homotopy $H_1 : [0,1] \times [0,1] \to \widehat{\mathcal{U}}(M_2(A))$ such that $H_1(0,t) = F(t)$, and $\eta_x \circ H_1 = \widetilde{H}$. Similarly, $\eta_x \circ \iota_A \circ G_x$ lifts to $\iota_A \circ G_x$, so $\widetilde{H}$ lifts to a homotopy $H_2 : [0,1] \times [0,1] \to \widehat{\mathcal{U}}(M_2(A))$ such that $H_2(1,t) = \iota_A \circ G_x(t)$, and $\eta_x \circ H_2 = \widetilde{H}$. Note that

$$\eta_x \circ F(t) = \eta_x \circ H_2(0,t) \text{ and } \eta_x \circ H_1(1,t) = \eta_x \circ \iota_A \circ G_x(t)$$

Therefore, by Lemma 2.4 applied to the continuous $C(X)$-algebra $C[0,1]\otimes A$, there is a closed neighbourhood Y' of x and a homotopy $H : [0,1]\times[0,1] \to \widehat{\mathcal{U}}(M_2(A(Y')))$ such that

$$H(0,t) = \eta_{Y'}\circ F(t), H(1,t) = \eta_{Y'}\circ\iota_A\circ G_x(t), \text{ and } \eta_x^{Y'}\circ H(s,1) = \iota_x(a(x))$$

and a path $G_{Y'} : [0,1] \to \widehat{\mathcal{U}}(A(Y'))$ such that $G_{Y'}(0) = \pi_{Y'}(b_x), G_{Y'}(1) = \pi_{Y'}(a)$, and $\pi_x^{Y'}\circ G_{Y'}$ is a constant path $a(x)$ in $\widehat{\mathcal{U}}(A(x))$ (the last statement follows from the proof of Lemma 2.4). Furthermore, $f : [0,1] \to \widehat{\mathcal{U}}(M_2(A(Y')))$ given by $f(s) := H(1-s,1)$ is a path such that $f(0) = \iota_{Y'}\circ\pi_{Y'}(b_x)$ and $f(1) = \iota_{Y'}\circ\pi_{Y'}(a)$. Also,

$$\eta_x^{Y'}\circ f(s) = \iota_x(a(x)) = \eta_x^{Y'}\circ\iota_{Y'}\circ G_{Y'}(s) \quad \forall s\in[0,1]$$

Thus, by Lemma 2.6 applied to f and $\iota_{Y'}\circ G_{Y'}$, there is a closed neighbourhood Y of x such that $Y\subset Y'$ and $\eta_Y^{Y'}\circ\iota_{Y'}\circ G_{Y'} \sim_h \eta_Y^{Y'}\circ f$ in $\widehat{\mathcal{U}}(M_2(A(Y)))$. But $\eta_Y\circ F \sim_h (\eta_Y\circ\iota_A\circ G_x)\bullet(\eta_Y^{Y'}\circ f)$, so $\eta_Y\circ F\sim_h \iota_Y\circ L_Y$ where $L_Y : [0,1]\to\widehat{\mathcal{U}}(A(Y))$ is given by

$$L_Y := (\pi_Y\circ G_x)\bullet(\pi_Y^{Y'}\circ G_{Y'})$$

and L_Y satisfies the required conditions. □

Remark 2.8. We are now in a position to prove Theorem A, but first, we need one important fact, which allows us to use induction: If X is a finite dimensional compact metric space, then covering dimension agrees with the small inductive dimension [4, Theorem 1.7.7]. Therefore, by [4, Theorem 1.1.6], X has an open cover $\mathcal{B}$ such that, for each $U\in\mathcal{B}$,

$$\dim(\partial U) \leq \dim(X) - 1$$

Now suppose $\{U_1, U_2, \ldots, U_m\}$ is an open cover of X such that $\dim(\partial U_i)\leq \dim(X)-1$ for $1\leq i\leq m$, we define sets $\{V_i : 1\leq i\leq m\}$ inductively by

$$V_1 := \overline{U_1}, \text{ and } V_k := \overline{U_k\setminus\left(\bigcup_{i<k} U_i\right)} \text{ for } k>1$$

and subsets $\{W_j : 1\leq j\leq m-1\}$ by

$$W_j := \left(\bigcup_{i=1}^{j} V_i\right)\cap V_{j+1}$$

It is easy to see that $W_j\subset\bigcup_{i=1}^j \partial U_i$, so by [4, Theorem 1.5.3], $\dim(W_j)\leq \dim(X)-1$ for all $1\leq j\leq m-1$

Proof of Theorem A. Let X be a compact metric space of finite covering dimension, and let A be a continuous $C(X)$-algebra, each of whose fibers are K-stable. We wish to show that A is K-stable. As in the proof of Theorem 2.5, it suffices to show that the map

$$\iota_* : \pi_0(\widehat{\mathcal{U}}(A)) \to \pi_0(\widehat{\mathcal{U}}(M_2(A)))$$

is bijective. The proof is by induction on $\dim(X)$, so by Theorem 2.5, we assume that $\dim(X) \geq 1$, and that $A(Y)$ is K-stable for any closed subset Y of X such that $\dim(Y) \leq \dim(X) - 1$.

For injectivity, suppose $a \in \widehat{\mathcal{U}}(A)$ such that $\iota(a) \sim 0$ in $\widehat{\mathcal{U}}(M_2(A))$. Let $F : [0,1] \to \widehat{\mathcal{U}}(M_2(A))$ be a path such that $F(0) = \iota(a)$ and $F(1) = 0$. For $x \in X$, by Lemma 2.7, there is a closed neighbourhood Y_x of x and a path $L_{Y_x} : [0,1] \to \widehat{\mathcal{U}}(A(Y_x))$ such that

$$L_{Y_x}(0) = \pi_{Y_x}(a), \quad L_{Y_x}(1) = 0$$

and $\iota_{Y_x} \circ L_{Y_x}$ is path homotopic to $\pi_{Y_x} \circ F$ in $\widehat{\mathcal{U}}(A(Y_x))$. By Remark 2.8, we may choose Y_x to be the closure of a basic open set U_x such that $\dim(\partial U_x) \leq \dim(X) - 1$. Since X is compact, we may choose a finite subcover $\{U_1, U_2, \ldots, U_m\}$. Now define $\{V_1, V_2, \ldots, V_m\}$ and $\{W_1, W_2, \ldots, W_{m-1}\}$ as in Remark 2.8. We observe that each V_i is a closed set such that $\pi_{V_i}(a) \sim 0$ in $\widehat{\mathcal{U}}(A(V_i))$ since $V_i \subset \overline{U_i}$ for all $1 \leq i \leq m$.

Note that $W_1 = V_1 \cap V_2$, and $\dim(W_1) \leq \dim(X) - 1$. By induction hypothesis, $A(W_1)$ is K-stable. Let $H_i : [0,1] \to \widehat{\mathcal{U}}(A(V_i)), i = 1,2$ be paths such that $H_i(0) = \pi_{V_i}(a), H_i(1) = 0$, and $\iota_{V_i} \circ H_i \sim_h \eta_{V_i} \circ F$. Let $S : [0,1] \to \widehat{\mathcal{U}}(A(W_1))$ be the path

$$S(t) := \pi_{W_1}^{V_1}(H_1(t)) \cdot \pi_{W_1}^{V_2}(H_2(t))^*$$

Note that $S(0) = S(1) = 0$, so S is a loop, and

$$\iota_{W_1} \circ S = (\eta_{W_1}^{V_1} \circ \iota_{V_1} \circ H_1) \cdot (\eta_{W_1}^{V_2} \circ \iota_{V_2} \circ H_2^*) \sim_h \eta_{W_1} \circ F \cdot (\eta_{W_1} \circ F)^* = 0$$

Hence, $\iota_{W_1} \circ S$ is null-homotopic. Since the map $(\iota_{W_1})_* : \pi_1(\widehat{\mathcal{U}}(A(W_1))) \to \pi_1(\widehat{\mathcal{U}}(M_2(A(W_1))))$ is injective, it follows that S is null-homotopic in $\widehat{\mathcal{U}}(A(W_1))$. Since the map $\pi_{W_1}^{V_2} : \widehat{\mathcal{U}}_0(A(V_2)) \to \widehat{\mathcal{U}}_0(A(W_1))$ is a fibration, it follows that S has a lift $f : [0,1] \to \widehat{\mathcal{U}}(A(V_2))$ such that $f(0) = f(1) = 0$ and $\pi_{W_1}^{V_2} \circ f = S$. Define $H_2' : [0,1] \to \widehat{\mathcal{U}}(A(V_2))$ be defined by

$$H_2'(t) := f(t) \cdot H_2(t)$$

Then $H_2'(0) = \pi_{V_2}(a), H_2'(1) = 0$, and

$$\pi_{W_1}^{V_1} \circ H_1 = \pi_{W_1}^{V_2} \circ H_2'$$

By Lemma 1.8, $A(V_1 \cup V_2)$ is a pullback

$$\begin{array}{ccc} A(V_1 \cup V_2) & \longrightarrow & A(V_1) \\ \downarrow & & \downarrow \\ A(V_2) & \longrightarrow & A(W_1) \end{array}$$

so we obtain a path $H : [0,1] \to \widehat{\mathcal{U}}(A(V_1 \cup V_2))$ such that $\pi_{V_1}^{V_1\cup V_2} \circ H = H_1$ and $\pi_{V_2}^{V_1\cup V_2} \circ H = H_2'$. In particular, $\pi_{V_1}^{V_1\cup V_2} \circ H(0) = \pi_{V_1}(a)$ and $\pi_{V_2}^{V_1\cup V_2} \circ H(0) = \pi_{V_2}(a)$ so $H(0) = \pi_{V_1\cup V_2}(a)$. Similarly, $H(1) = 0$, so that $\pi_{V_1\cup V_2}(a) \sim 0$ in $\widehat{\mathcal{U}}(A(V_1 \cup V_2))$.

Now observe that $W_2 = (V_1 \cup V_2) \cap V_3$, and $\dim(W_2) \leq \dim(X) - 1$. Replacing V_1 by $V_1 \cup V_2$, and V_2 by V_3 in the earlier argument, we may repeat the earlier procedure. By induction on the number of elements in the finite subcover, we see that $a \sim 0$ in $\widehat{\mathcal{U}}(A)$. This completes the proof of injectivity of ι_*.

Now consider the surjectivity of ι_*: Fix $u \in \widehat{\mathcal{U}}(M_2(A))$, and we wish to show that there is a quasi-unitary $\omega \in \widehat{\mathcal{U}}(A)$ such that $u \sim \iota(\omega)$. So fix $x \in X$. Then by K-stability of $A(x)$, there exists $f_x \in \widehat{\mathcal{U}}(A(x))$ such that $\eta_x(u) \sim \iota_x(f_x)$. As in the proof of Theorem 2.5, there is a closed neighbourhood Y_x of x and a quasi-unitary $d_x \in \widehat{\mathcal{U}}(A(Y_x))$ such that

$$\eta_{Y_x}(u) \sim \iota_{Y_x}(d_x)$$

As in the first part of the proof, we may reduce to the case where $X = V_1 \cup V_2$, and there are quasi-unitaries $d_{V_1} \in \widehat{\mathcal{U}}(A(V_1)), d_{V_2} \in \widehat{\mathcal{U}}(A(V_2))$ such that

$$\eta_{V_i}(u) \sim \iota_{V_i}(d_{V_i}) \text{ in } \widehat{\mathcal{U}}(M_2(A(V_i)), i = 1, 2$$

and if $W := V_1 \cap V_2$, then

$$\dim(W) \leq \dim(X) - 1$$

Fix paths $H_i : [0,1] \to \widehat{\mathcal{U}}(M_2(A(V_i))$ such that $H_1(0) = \iota_{V_1}(d_{V_1})$ and $H_1(1) = \eta_{V_1}(u)$, $H_2(0) = \eta_{V_2}(u)$ and $H_2(1) = \iota_{V_2}(d_{V_2})$ and consider the path $F : [0,1] \to \widehat{\mathcal{U}}(M_2(A(W))$ given by

$$F := (\eta_W^{V_1} \circ H_1) \bullet (\eta_W^{V_2} \circ H_2)$$

Then $F(0) = \iota_W \circ \pi_W^{V_1}(d_{V_1})$ and $F(1) = \iota_W \circ \pi_W^{V_2}(d_{V_2})$. By induction, $A(W)$ is K-stable, so by Lemma 1.6, there is a retraction $r_W : \widehat{\mathcal{U}}(M_2(A(W)) \to \widehat{\mathcal{U}}(A(W))$. Define $H := r_W \circ F$, then $H : [0,1] \to \widehat{\mathcal{U}}(A(W))$ is a path such that

$$H(0) = \pi_W^{V_1}(d_{V_1}), \text{ and } H(1) = \pi_W^{V_2}(d_{V_2})$$

The map $\pi_W^{V_2} : \widehat{\mathcal{U}}_0(A(V_2)) \to \widehat{\mathcal{U}}_0(A(W))$ is a fibration, so there is a path $H' : [0,1] \to \widehat{\mathcal{U}}(A(V_2))$ such that

$$H'(1) = d_{V_2}, \text{ and } \pi_W^{V_2} \circ H' = H$$

Define $e_{V_2} := H'(0)$ so that

$$\pi_W^{V_2}(e_{V_2}) = \pi_W^{V_1}(d_{V_1})$$

By Lemma 1.8, $A = A(V_1 \cup V_2)$ is a pullback

$$\begin{array}{ccc} A & \xrightarrow{\pi_{V_1}} & A(V_1) \\ {\scriptstyle \pi_{V_2}}\downarrow & & \downarrow{\scriptstyle \pi_W^{V_1}} \\ A(V_2) & \xrightarrow{\pi_W^{V_2}} & A(W) \end{array}$$

so that $\omega := (d_{V_1}, e_{V_2})$ defines a quasi-unitary in A. We claim that $\iota(\omega) \sim u$ in $\widehat{\mathcal{U}}(A)$. To this end, define $H_3 : [0,1] \to \widehat{\mathcal{U}}(M_2(A(V_2))$ by

$$H_3 := H_2 \bullet (\iota_{V_2} \circ \overline{H'})$$

then $H_3(0) = \eta_{V_2}(u)$ and $H_3(1) = \iota_{V_2}(e_{V_2}) = \iota_{V_2}(\pi_{V_2}(\omega)) = \eta_{V_2}(\iota_{V_2}(\omega))$. So if $S : [0,1] \to \widehat{\mathcal{U}}(M_2(A(W))$ is given by

$$S(t) := \eta_W^{V_2}(H_3(t)) \cdot \eta_W^{V_1}(H_1(1-t))^*$$

Then S is a path with $S(0) = \eta_W(u) \cdot \eta_W(u)^* = 0$ and $S(1) = \iota_W^{V_2}(e_{V_2}) \cdot \eta_W^{V_1}(\iota_{V_1}(d_{V_1})) = 0$. Furthermore,

$$\begin{aligned} \eta_W^{V_2} \circ H_3 &= (\eta_W^{V_2} \circ H_2) \bullet (\eta_W^{V_2} \circ \iota_{V_2} \circ \overline{H'}) = (\eta_W^{V_2} \circ H_2) \bullet (\iota_W \circ \pi_W^{V_2} \circ \overline{H'}) \\ &= (\eta_W^{V_2} \circ H_2) \bullet (\iota_W \circ \overline{H}) = (\eta_W^{V_2} \circ H_2) \bullet (\iota_W \circ r_W \circ \overline{F}) \\ &\sim_h (\eta_W^{V_2} \circ H_2) \bullet \overline{F} \end{aligned}$$

Hence,

$$\begin{aligned} S &\sim_h [(\eta_W^{V_2} \circ H_2) \bullet \overline{F}] \cdot (\eta_W^{V_1} \circ \overline{H_1})^* \\ &= [(\eta_W^{V_2} \circ H_2) \bullet (\eta_W^{V_2} \circ \overline{H_2}) \bullet (\eta_W^{V_1} \circ \overline{H_1})] \cdot (\eta_W^{V_1} \circ \overline{H_1})^* \\ &\sim_h (\eta_W^{V_1} \circ \overline{H_1}) \cdot (\eta_W^{V_1} \circ \overline{H_1})^* = 0 \end{aligned}$$

Hence, S is null-homotopic. Once again, the map $\eta_W^{V_1} : \widehat{\mathcal{U}}_0(M_2(A(V_1)) \to \widehat{\mathcal{U}}_0(M_2(A(W))$ is a fibration, so there is a loop $f : [0,1] \to \widehat{\mathcal{U}}(M_2(A(V_1))$ such that $f(0) = f(1) = 0$ and $\eta_W^{V_1} \circ f = S$. Define $H_4 : [0,1] \to \widehat{\mathcal{U}}(M_2(A(V_1))$ by

$$H_4(t) := f(t) \cdot H_1(1-t)$$

Then $H_4(0) = \eta_{V_1}(u), H_4(1) = \iota_{V_1}(d_{V_1}) = \eta_{V_1}(\iota(\omega))$. Finally, by construction

$$\eta_W^{V_1} \circ H_4 = \eta_W^{V_2} \circ H_3$$

Therefore, the pair (H_3, H_4) defines a path in $\widehat{\mathcal{U}}(M_2(A))$ connecting u to $\iota(\omega)$. This concludes the proof that ι_* is surjective. ∎

Acknowledgements. The first named author is supported by UGC Junior Research Fellowship No. 1229 and the second named author was supported by SERB Grant YSS/2015/001060.

References

[1] Nathanial P. Brown and Narutaka Ozawa. *C^*-algebras and finite-dimensional approximations*, volume 88 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2008.

[2] Marius Dadarlat. Continuous fields of C^*-algebras over finite dimensional spaces. *Advances in Mathematics*, 222(5):1850–1881, 2009.

[3] Albrecht Dold. Partitions of unity in the theory of fibrations. *Annals of Mathematics. Second Series*, 78:223–255, 1963.

[4] Ryszard Engelking. *Dimension theory.* North-Holland Publishing Co., Amsterdam-Oxford-New York; PWN—Polish Scientific Publishers, Warsaw, 1978. Translated from the Polish and revised by the author, North-Holland Mathematical Library, 19.

[5] Allen Hatcher. *Algebraic topology.* Cambridge University Press, Cambridge, 2002.

[6] Ilan Hirshberg, Mikael Rø rdam, and Wilhelm Winter. $C_0(X)$-algebras, stability and strongly self-absorbing C^*-algebras. *Mathematische Annalen*, 339(3):695–732, 2007.

[7] Xinhui Jiang. Nonstable k-theory for $\mathcal{Z}$-stable c*-algebras. *arXiv preprint math/9707228*, 1997.

[8] G. G. Kasparov. Equivariant KK-theory and the Novikov conjecture. *Inventiones Mathematicae*, 91(1):147–201, 1988.

[9] Eberhard Kirchberg and Simon Wassermann. Operations on continuous bundles of C^*-algebras. *Mathematische Annalen*, 303(4):677–697, 1995.

[10] Albert T. Lundell and Stephen Weingram. *The topology of CW complexes.* The University Series in Higher Mathematics. Van Nostrand Reinhold Co., New York, 1969.

[11] May Nilsen. C^*-bundles and $C_0(X)$-algebras. *Indiana University Mathematics Journal*, 45(2):463–477, 1996.

[12] Richard S. Palais. Homotopy theory of infinite dimensional manifolds. *Topology. An International Journal of Mathematics*, 5:1–16, 1966.

[13] Marc A. Rieffel. The homotopy groups of the unitary groups of noncommutative tori. *Journal of Operator Theory*, 17(2):237–254, 1987.

[14] Marc A. Rieffel. Continuous fields of C^*-algebras coming from group cocycles and actions. *Mathematische Annalen*, 283(4):631–643, 1989.

[15] M. Rø rdam, F. Larsen, and N. Laustsen. *An introduction to K-theory for C^*-algebras*, volume 49 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2000.

[16] Klaus Thomsen. Nonstable K-theory for operator algebras. *K-Theory. An Interdisciplinary Journal for the Development, Application, and Influence of K-Theory in the Mathematical Sciences*, 4(3):245–267, 1991.

[17] Andrew S. Toms and Wilhelm Winter. $\mathcal{Z}$-stable ASH algebras. *Canadian Journal of Mathematics. Journal Canadien de Mathématiques*, 60(3):703–720, 2008.

[18] Shuang Zhang. Matricial structure and homotopy type of simple C^*-algebras with real rank zero. *Journal of Operator Theory*, 26(2):283–312, 1991.

Department of Mathematics, Indian Institute of Science Education and Research Bhopal, Bhopal ByPass Road, Bhauri, Bhopal 462066, Madhya Pradesh. India.

Email address: `apurva17@iiserb.ac.in, prahlad@iiserb.ac.in`