

# Infinite loop spaces and nilpotent K–theory

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Using a construction derived from the descending central series of the free groups, we produce filtrations by infinite loop spaces of the classical infinite loop spaces  $BSU$ ,  $BU$ ,  $BSO$ ,  $BO$ ,  $BSp$ ,  $BGL_\infty(R)^+$  and  $Q_0(S^0)$ . We show that these infinite loop spaces are the zero spaces of nonunital  $E_\infty$ –ring spectra. We introduce the notion of  $q$ –nilpotent K–theory of a CW–complex  $X$  for any  $q \geq 2$ , which extends the notion of commutative K–theory defined by Adem and Gómez, and show that it is represented by  $\mathbb{Z} \times B(q, U)$ , where  $B(q, U)$  is the  $q^{\text{th}}$  term of the aforementioned filtration of  $BU$ .

For the proof we introduce an alternative way of associating an infinite loop space to a commutative  $\mathbb{I}$ –monoid and give criteria for when it can be identified with the plus construction on the associated limit space. Furthermore, we introduce the notion of a commutative  $\mathbb{I}$ –ring and show that they give rise to nonunital  $E_\infty$ –ring spectra.

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## 1 Introduction

Let  $G$  denote a locally compact, Hausdorff topological group such that  $1_G \in G$  is a nondegenerate base point. It is well known that we can obtain a model for the classifying space  $BG$  as the geometric realization of the classical bar construction  $B_*G$ . Now fix an integer  $q \geq 2$  and let  $\Gamma_n^q$  be the  $q^{\text{th}}$  stage of the descending central series of the free group on  $n$  letters  $F_n$ , with the convention  $\Gamma_n^1 = F_n$ . Consider the set of homomorphisms  $B_n(q, G) := \text{Hom}(F_n/\Gamma_n^q, G)$ . If  $e_1, \dots, e_n$  are generators of  $F_n$ , then evaluation on the classes corresponding to  $e_1, \dots, e_n$  provides a natural inclusion  $B_n(q, G) \subset G^n$ . Using this inclusion we can give  $B_n(q, G)$  the subspace topology. Therefore  $B_n(q, G)$  is precisely the space of ordered  $n$ –tuples in  $G$  generating a subgroup of  $G$  with nilpotence class less than  $q$ . For each fixed  $q \geq 2$  the collection  $\{B_n(q, G)\}_{n \geq 0}$  forms a simplicial space with face and degeneracy maps induced by those in the bar construction. The geometric realization of this simplicial space is

denoted by  $B(q, G)$ . These spaces were first introduced by Adem, Cohen and Torres Giese [1], where many of their basic properties were established. They give rise to a natural filtration of the classifying space

$$B(2, G) \subset B(3, G) \subset \cdots \subset B(q, G) \subset B(q+1, G) \subset \cdots \subset BG.$$

For  $q = 2$  we obtain  $B_{\text{com}}G := B(2, G)$ , which is constructed by assembling the different spaces of ordered commuting  $n$ -tuples in the group  $G$ . Adem and Gómez [2] showed that for Lie groups this space plays the role of a classifying space for commutativity. More generally  $B(q, G)$  is a classifying space for  $G$ -bundles of transitional nilpotency class less than  $q$ .

For the infinite unitary group  $U = \text{colim}_{n \rightarrow \infty} U(n)$ , it is well known that  $BU$  is the infinite loop space underlying a nonunital  $E_\infty$ -ring spectrum, namely the homotopy fiber of the Postnikov section  $ku \rightarrow H\mathbb{Z}$ . In other words,  $BU$  is a so-called nonunital  $E_\infty$ -ring space. A basic question is whether the above gives rise to a filtration of  $BU$  by nonunital  $E_\infty$ -ring spaces. The main purpose of this paper is to show that indeed this is the case, not only for  $U$  but also for other linear groups.

**Theorem 1.1** *The spaces  $B(q, \text{SU})$ ,  $B(q, U)$ ,  $B(q, \text{SO})$ ,  $B(q, O)$  and  $B(q, \text{Sp})$  provide a filtration by nonunital  $E_\infty$ -ring spaces of the classical infinite loop spaces  $B\text{SU}$ ,  $BU$ ,  $B\text{SO}$ ,  $BO$  and  $B\text{Sp}$ , respectively.*

The  $q$ -nilpotent K-theory of a space  $X$  is defined using isomorphism classes of bundles on  $X$  whose transition functions generate subgroups of nilpotence class less than  $q$ . We show that  $K_{q\text{-nil}}(X) \cong [X, \mathbb{Z} \times B(q, U)]$ , from which we obtain:

**Corollary 1.2**  *$K_{q\text{-nil}}(-)$  is the zeroth term of a generalized multiplicative cohomology theory.*

In particular we obtain a sequence of multiplicative cohomology theories

$$K_{\text{com}}(X) = K_{2\text{-nil}}(X) \rightarrow K_{3\text{-nil}}(X) \rightarrow \cdots \rightarrow K_{q\text{-nil}}(X) \rightarrow \cdots \rightarrow K(X).$$

We also show that  $B(q, U) \rightarrow BU$  splits as a map of infinite loop spaces, whence we see that topological K-theory is a direct summand in  $K_{q\text{-nil}}$ .

The infinite loop space structure on  $B(q, G)$  for  $G = U, \text{SU}, \text{SO}, O, \text{Sp}$  is obtained by using the machinery of commutative  $\mathbb{I}$ -monoids first introduced by Bökstedt and developed by Schlichtkrull [19], Sagave and Schlichtkrull [18] and Lind [9]. Here  $\mathbb{I}$  is the category of finite sets and injections. In addition to the usual construction, we associate an infinite loop space to a commutative  $\mathbb{I}$ -monoid by restricting the usual

homotopy colimit construction to the subcategory  $\mathbb{P}$  of finite sets and isomorphisms. This allows us to identify the homotopy type of the homotopy colimit under certain conditions. Another addition to infinite loop space theory is the introduction of the notion of a commutative  $\mathbb{I}$ -rig, which we show to give rise to a bipermutative category and hence an  $E_\infty$ -ring spectrum.

Our main examples above all arise from commutative  $\mathbb{I}$ -rigs where we can identify the infinite loop space as the plus construction of the associated limit space. A more complicated situation arises for  $Q_0(\mathbb{S}^0) \simeq B\Sigma_\infty^+$  and  $BGL_\infty(R)^+$ . Our methods give rise to natural sequences of  $E_\infty$ -ring spaces but the terms are not easy to describe.

The outline of this article is as follows. In Section 2 we use the machinery of commutative  $\mathbb{I}$ -monoids to produce two associated infinite loop spaces, one of which is a nonunital  $E_\infty$ -ring space when the  $\mathbb{I}$ -monoid is an  $\mathbb{I}$ -rig. In Section 3 we show that these are homotopy equivalent and identify them under suitable assumptions. Then in Section 4 we apply these results to prove Theorem 1.1 and show that the spaces  $B(q, U)$  for  $q \geq 2$  are infinite loop spaces and that  $BU$  splits off. Finally, in Section 5 we introduce the notion of  $q$ -nilpotent K-theory and show that it is represented by the infinite loop spaces  $\mathbb{Z} \times B(q, U)$ , answering the question raised for commutative K-theory in [2].

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## 2 Commutative $\mathbb{I}$ -monoids and infinite loop spaces

The standard construction of the infinite loop space structure on  $BU$  from the permutative category of complex vector spaces and their isomorphisms does not restrict to give an infinite loop space structure on  $B(q, U)$ . Instead we are going to use certain constructions on commutative  $\mathbb{I}$ -monoids. More precisely, we will give two constructions of permutative categories from commutative  $\mathbb{I}$ -monoids. For the case of interest the permutative categories are actually bipermutative and hence give rise to  $E_\infty$ -ring spectra. We start by setting up some notations and basic definitions following [19; 18; 9]. We will use [5] as a reference for bipermutative categories and the associated multiplicative infinite loop space machinery.

## 2.1 The category $\mathbb{I}$ and its subcategories $\mathbb{P}$ and $\mathbb{N}$

These three categories are skeletons of the category of finite sets and injections, the category of finite sets and isomorphisms, and the translation category associated to the monoid of natural numbers. We will use the following notation.

For every integer  $n \geq 0$ , let  $\mathbf{n}$  denote the set  $\{1, 2, \dots, n\}$ . When  $n = 0$  we use the convention  $\mathbf{0} := \emptyset$ . Let  $\mathbb{I}$  denote the category whose objects are the elements of the form  $\mathbf{n}$  for all integers  $n \geq 0$  with morphisms given by all injective maps. Note that in particular  $\mathbf{0}$  is an initial object in the category  $\mathbb{I}$  and  $\mathbb{I}$  is a symmetric monoidal category under the concatenation  $\mathbf{m} \sqcup \mathbf{n} := \{1, 2, \dots, m+n\}$  with the symmetry morphism given by the  $(m, n)$ -shuffle map

$$\tau_{\mathbf{m}, \mathbf{n}}: \mathbf{m} \sqcup \mathbf{n} \rightarrow \mathbf{n} \sqcup \mathbf{m}.$$

It is also symmetric monoidal under the Cartesian product

$$\mathbf{m} \times \mathbf{n} := \{1 = (1, 1), 2 = (1, 2), \dots, n+1 = (2, 1), \dots, mn = (m, n)\}$$

given by lexicographic ordering. By definition,  $\mathbf{0} \times \mathbf{n} = \mathbf{0} = \mathbf{n} \times \mathbf{0}$ . The associated symmetry morphism is given by a permutation

$$\tau_{\mathbf{m} \times \mathbf{n}}^{\times}: \mathbf{m} \times \mathbf{n} \rightarrow \mathbf{n} \times \mathbf{m}.$$

The latter monoidal product is distributive over the former. More precisely, left distributivity

$$\delta_{\mathbf{m}, \mathbf{n}, \mathbf{k}}^l: \mathbf{m} \times \mathbf{k} \sqcup \mathbf{n} \times \mathbf{k} \rightarrow (\mathbf{m} \sqcup \mathbf{n}) \times \mathbf{k}$$

is given by the identity and right distributivity is given by a permutation

$$\delta_{\mathbf{m}, \mathbf{n}, \mathbf{k}}^r: \mathbf{m} \times \mathbf{n} \sqcup \mathbf{m} \times \mathbf{k} \rightarrow \mathbf{m} \times (\mathbf{n} \sqcup \mathbf{k}).$$

These two structures make  $\mathbb{I}$  into a bipermutative category, as in [5, Definition 3.6].

The category  $\mathbb{I}$  has two natural subcategories. Let  $\mathbb{P}$  be the totally disconnected subcategory containing all objects and all isomorphisms  $\sigma: \mathbf{n} \rightarrow \mathbf{n}$  but no other morphisms, and let  $\mathbb{N}$  denote the connected subcategory containing all objects, their identities and only the canonical inclusions  $j: \mathbf{n} \rightarrow \mathbf{m}$ . While  $\mathbb{P}$  is a bipermutative subcategory,  $\mathbb{N}$  does not inherit any monoidal structure from  $\mathbb{I}$ .

## 2.2 Definitions of commutative $\mathbb{I}$ -monoids and $\mathbb{I}$ -rigs

An  $\mathbb{I}$ -space is a functor  $X: \mathbb{I} \rightarrow \text{Top}$ . Every morphism in  $\mathbb{I}$  can be factored as a composition of a canonical inclusion  $j: \mathbf{n} \hookrightarrow \mathbf{m}$  and a permutation  $\sigma: \mathbf{m} \rightarrow \mathbf{m}$ . Therefore an  $\mathbb{I}$ -space  $X: \mathbb{I} \rightarrow \text{Top}$  determines a sequence of spaces  $X(\mathbf{n})$  together

with an induced action of the symmetric group  $\Sigma_n$  for  $n \geq 0$ , and structural maps  $j_n: X(\mathbf{n}) \rightarrow X(\mathbf{n} + \mathbf{1})$  that are equivariant in the sense that  $j_n(\sigma \cdot x) = \sigma \cdot j_n(x)$  for every  $\sigma \in \Sigma_n$  and  $x \in X(\mathbf{n})$ . On the right-hand side we see  $\sigma$  as element in  $\Sigma_{n+1}$  via the canonical inclusion  $\Sigma_n \hookrightarrow \Sigma_{n+1}$ . Vice versa, given such a sequence of  $\Sigma_n$ -spaces  $X(\mathbf{n})$  and compatible structure maps  $j_n$ , they give rise to an  $\mathbb{I}$ -space if and only if for  $m \geq n$  and any two elements  $\sigma, \sigma' \in \Sigma_m$  which restrict to the same permutation of  $\mathbf{n}$  we have  $\sigma(x) = \sigma'(x)$  for all  $x \in j(X(\mathbf{n}))$ . We note that this condition is not satisfied by the sequence  $X(\mathbf{n}) = \Sigma_n$  with the left or right multiplication action, but *is* satisfied by the sequence  $X(\mathbf{n}) = \mathbf{n}$  with the natural permutation action since  $\mathbf{n} \cong \mathbb{I}(\mathbf{1}, \mathbf{n})$ .

We say that an  $\mathbb{I}$ -space is an  $\mathbb{I}$ -monoid if it comes equipped with a natural transformation

$$\mu_{\mathbf{m}, \mathbf{n}}: X(\mathbf{m}) \times X(\mathbf{n}) \rightarrow X(\mathbf{m} \sqcup \mathbf{n})$$

of functors defined on  $\mathbb{I} \times \mathbb{I}$  and a natural transformation

$$\eta_{\mathbf{n}}: * \rightarrow X(\mathbf{n})$$

from the constant  $\mathbb{I}$ -space  $*(\mathbf{n}) = *$  to  $X$  satisfying associativity and unit axioms for  $* \in X(\mathbf{0})$ . We say that  $X$  is a commutative  $\mathbb{I}$ -monoid if  $\mu$  is commutative, meaning that the diagram

$$\begin{array}{ccc} X(\mathbf{m}) \times X(\mathbf{n}) & \xrightarrow{\mu_{\mathbf{m}, \mathbf{n}}} & X(\mathbf{m} \sqcup \mathbf{n}) \\ \tau \downarrow & & \tau_{\mathbf{m}, \mathbf{n}} \downarrow \\ X(\mathbf{n}) \times X(\mathbf{m}) & \xrightarrow{\mu_{\mathbf{n}, \mathbf{m}}} & X(\mathbf{n} \sqcup \mathbf{m}) \end{array}$$

commutes, where  $\tau(x, y) = (y, x)$ .

An  $\mathbb{I}$ -rig is a commutative  $\mathbb{I}$ -monoid equipped with a natural transformation

$$\pi_{\mathbf{m}, \mathbf{n}}: X(\mathbf{m}) \times X(\mathbf{n}) \rightarrow X(\mathbf{m} \times \mathbf{n})$$

of functors defined on  $\mathbb{P} \times \mathbb{P}$  and an element  $1 \in X(\mathbf{1})$  satisfying associativity and unit axioms, as well as left distributivity, ie that the diagram

$$\begin{array}{ccc} (X(\mathbf{m}) \times X(\mathbf{n})) \times X(\mathbf{k}) & \xrightarrow{\pi_{\mathbf{m} \sqcup \mathbf{n}, \mathbf{k}} \circ (\mu_{\mathbf{m}, \mathbf{n}} \times 1)} & X((\mathbf{m} \sqcup \mathbf{n}) \times \mathbf{k}) \\ (1 \times \tau \times 1) \circ (1 \times 1 \times \Delta) \downarrow & & \uparrow \delta_{\mathbf{m}, \mathbf{n}, \mathbf{k}}^l \\ X(\mathbf{m}) \times X(\mathbf{k}) \times X(\mathbf{n}) \times X(\mathbf{k}) & \xrightarrow{\mu_{\mathbf{m} \times \mathbf{k}, \mathbf{n} \times \mathbf{k}} \circ (\pi_{\mathbf{m}, \mathbf{k}} \times \pi_{\mathbf{n}, \mathbf{k}})} & X(\mathbf{m} \times \mathbf{k} \sqcup \mathbf{n} \times \mathbf{k}) \end{array}$$

commutes, and right distributivity, which is given by an analogous commutative diagram. Here  $\Delta$  is the diagonal map. We emphasize that  $\pi$  is only required to be natural on the subcategory  $\mathbb{P} \times \mathbb{P}$  of  $\mathbb{I} \times \mathbb{I}$ .<sup>1</sup>

A commutative  $\mathbb{I}$ -rig is an  $\mathbb{I}$ -rig in which  $\pi$  is commutative in the sense that the diagram

$$\begin{array}{ccc} X(\mathbf{m}) \times X(\mathbf{n}) & \xrightarrow{\pi_{\mathbf{m},\mathbf{n}}} & X(\mathbf{m} \times \mathbf{n}) \\ \tau \downarrow & & \tau_{\mathbf{m},\mathbf{n}}^\times \downarrow \\ X(\mathbf{n}) \times X(\mathbf{m}) & \xrightarrow{\pi_{\mathbf{n},\mathbf{m}}} & X(\mathbf{n} \times \mathbf{m}) \end{array}$$

commutes. A natural transformation  $T$  between two  $\mathbb{I}$ -spaces  $X$  and  $Y$  defines a map of commutative  $\mathbb{I}$ -monoids ( $\mathbb{I}$ -rigs) if it commutes with  $\mu$  (and  $\pi$ ) in the sense that  $T \circ \mu_{\mathbf{m},\mathbf{n}} = \mu_{\mathbf{m},\mathbf{n}} \circ T \times T$  (and  $T \circ \pi_{\mathbf{m},\mathbf{n}} = \pi_{\mathbf{m},\mathbf{n}} \circ T \times T$ ). We have thus defined a category of  $\mathbb{I}$ -spaces, a category of commutative  $\mathbb{I}$ -monoids and a category of  $\mathbb{I}$ -rigs.

### 2.3 Associated (bi)permutative translation categories

We will use the following notation for translation categories. If  $Y: \mathcal{C} \rightarrow \mathbf{Top}$  is a functor from a category  $\mathcal{C}$  to the category of topological spaces, we let  $\mathcal{C} \ltimes Y$  denote the translation category on  $Y$ . The translation category, also known as the Grothendieck construction, is a topological category whose objects are pairs  $(c, x)$  consisting of an object  $c$  of  $\mathcal{C}$  and a point  $x \in Y(c)$ . A morphism in  $\mathcal{C} \ltimes Y$  from  $(c, x)$  to  $(c', x')$  is a morphism  $\alpha: c \rightarrow c'$  in  $\mathcal{C}$  satisfying the equation  $Y(\alpha)(x) = x'$ . For example, if  $\mathcal{C} = G$  is a group, thought of as a one object category, then the translation category  $G \ltimes Y$  is the action groupoid for the  $G$ -space  $Y$  and its classifying space is the homotopy orbit space  $B(G \ltimes Y) = EG \times_G Y$ . In general, the classifying space  $B(\mathcal{C} \ltimes Y)$  is homeomorphic to the homotopy colimit  $\mathrm{hocolim}_{\mathcal{C}} Y$  of  $Y$  over  $\mathcal{C}$  defined using the bar construction.

Suppose now that  $X$  is a commutative  $\mathbb{I}$ -monoid. Then the translation category  $\mathbb{I} \ltimes X$  is a permutative category, as we now explain. The monoidal structure  $\oplus$  is defined on objects  $(\mathbf{m}, x)$  and  $(\mathbf{n}, y)$  by

$$(\mathbf{m}, x) \oplus (\mathbf{n}, y) = (\mathbf{m} \sqcup \mathbf{n}, \mu_{\mathbf{m},\mathbf{n}}(x, y))$$

<sup>1</sup>In fact, we do not know of any nontrivial examples where  $\pi$  may be extended to a natural transformation of functors defined on  $\mathbb{I} \times \mathbb{I}$ . The examples of  $\mathbb{I}$ -rigs that we discuss in Section 2.5 do not satisfy this additional naturality condition. Indeed, as we will see in the following sections, an  $\mathbb{I}$ -rig that does satisfy this condition and has each level  $X(\mathbf{n})$  a connected space would give rise to a connected  $E_\infty$ -ring space  $\mathrm{hocolim}_{\mathbb{I}} X$ . An  $E_\infty$ -ring space whose multiplicative unit and additive unit lie in the same path component is contractible, so such examples would only give rise to trivial  $E_\infty$ -ring spectra.

and on morphisms  $\alpha: (\mathbf{m}, x) \rightarrow (\mathbf{m}', x')$  and  $\beta: (\mathbf{n}, y) \rightarrow (\mathbf{n}', y')$  by letting

$$\alpha \oplus \beta: (\mathbf{m}, x) \oplus (\mathbf{n}, y) \rightarrow (\mathbf{m}', x') \oplus (\mathbf{n}', y')$$

be determined by the morphism

$$\alpha \sqcup \beta: \mathbf{m} \sqcup \mathbf{n} \rightarrow \mathbf{m}' \sqcup \mathbf{n}'$$

in the category  $\mathbb{I}$ . Notice that  $X(\alpha \sqcup \beta)(\mu_{\mathbf{m}, \mathbf{n}}(x, y)) = \mu_{\mathbf{m}', \mathbf{n}'}(x', y')$  by the naturality of  $\mu$ , so that this is well-defined. The associativity and unit conditions for  $X$  imply that  $\mathbb{I} \ltimes X$  is a strict monoidal category with strict unit object  $(\mathbf{0}, *)$  determined by the unit  $\eta$  of the  $\mathbb{I}$ -monoid  $X$ . The commutativity of  $X$  implies that  $\mathbb{I} \ltimes X$  is a permutative category, see for example [5, Definition 3.1]. Note that the permutative structure on  $\mathbb{I} \ltimes X$  restricts to the subcategory  $\mathbb{P} \ltimes X$ .

Suppose now that  $X$  is a commutative  $\mathbb{I}$ -rig. Then by the same reasoning as above, there is another permutative category structure on  $\mathbb{P} \ltimes X$  with product  $\otimes$  induced by  $\pi$  and strict unit object  $(\mathbf{1}, 1)$ . The distributivity axioms for  $X$  translate to distributivity axioms for bipermutative categories [5, Definition 3.6].

Furthermore, a natural transformation  $T$  between two  $\mathbb{I}$ -spaces  $X$  and  $Y$  induces a functor  $\mathbb{I} \ltimes X \rightarrow \mathbb{I} \ltimes Y$ . If  $X$  and  $Y$  are commutative  $\mathbb{I}$ -monoids ( $\mathbb{I}$ -rigs) and  $T$  is a morphism of such then the induced functor of translation categories is a functor of (bi)permutative categories.

We have thus proved the following result:

**Proposition 2.1** *The assignment  $X \mapsto \mathbb{I} \ltimes X$  defines a functor from the category of commutative  $\mathbb{I}$ -monoids to the category of permutative categories, and the assignment  $X \mapsto \mathbb{P} \ltimes X$  defines a functor from the category of commutative  $\mathbb{I}$ -monoids ( $\mathbb{I}$ -rigs) to the category of (bi)permutative categories.*

## 2.4 Construction of two infinite loop spaces

Let  $X$  be a commutative  $\mathbb{I}$ -monoid. As explained in [12], the classifying space of a permutative category is an  $E_\infty$ -space structured by an action of the Barratt–Eccles operad. We have proved the next theorem.

**Theorem 2.2** *Suppose that  $X: \mathbb{I} \rightarrow \mathbf{Top}$  is a commutative  $\mathbb{I}$ -monoid. Then the homotopy colimit*

$$\mathrm{hocolim}_{\mathbb{I}} X = B(\mathbb{I} \ltimes X)$$

*is an  $E_\infty$ -space.*

Without further assumptions on  $X$ , this  $E_\infty$ -space need not be grouplike (ie the monoid  $\pi_0(\mathrm{hocolim}_{\mathbb{I}} X)$  need not be a group). However, we can always form the group completion  $\Omega B(\mathrm{hocolim}_{\mathbb{I}} X)$  to get the associated infinite loop space. Note that an algebra over the Barratt–Eccles operad has an underlying monoid structure that is always strictly associative (and homotopy commutative) so that the usual functorial construction of the classifying space for monoids built using the bar construction can be applied. We will always use this model for  $B$  in defining the group completion functor  $\Omega B(-)$ . The consistency results in [12] guarantee that the group completion  $\Omega B(\mathrm{hocolim}_{\mathbb{I}} X)$  defines an infinite loop space weakly equivalent to that obtained using any other delooping machine.

Schlichtkrull [19] defined a different infinite loop space associated to  $X$ , using the language of  $\Gamma$ -spaces. Schlichtkrull's construction is the same as May's construction [14] of a  $\Gamma$ -space applied to the permutative category  $\mathbb{I} \ltimes X$ . By the uniqueness result of [14], the infinite loop space  $\Omega B(\mathrm{hocolim}_{\mathbb{I}} X)$  is equivalent to that defined by Schlichtkrull.

We now give a different construction of an infinite loop space associated to  $X$ . To start note the decomposition of categories

$$\mathbb{P} \ltimes X = \bigsqcup_{n \geq 0} \Sigma_n \ltimes X(n),$$

where  $\Sigma_n$  is seen as a category with one object. Thus  $\mathbb{P} \ltimes X$  is a topological category with classifying space

$$M := \mathrm{hocolim}_{\mathbb{P}} X = B(\mathbb{P} \ltimes X) \simeq \bigsqcup_{n \geq 0} E\Sigma_n \times_{\Sigma_n} X(n).$$

As  $\mathbb{P} \ltimes X$  is a permutative category,  $M = B(\mathbb{P} \ltimes X)$  is an  $E_\infty$ -space and thus its group completion,  $\Omega BM$ , is an infinite loop space. The reduction maps  $X(n) \rightarrow *$  define a map of permutative categories  $\mathbb{P} \ltimes X \rightarrow \mathbb{P} \ltimes *$  and hence a map of infinite loop spaces

$$\rho^X: \Omega B(\mathrm{hocolim}_{\mathbb{P}} X) \rightarrow \Omega B(\mathrm{hocolim}_{\mathbb{P}} *).$$

In particular, the homotopy fiber  $\mathrm{hofib} \rho^X$  is naturally an infinite loop space.

When  $X$  is a commutative  $\mathbb{I}$ -rig, we process the associated bipermutative category  $\mathbb{P} \ltimes X$  using the machinery of Elmendorf and Mandell. To a bipermutative category  $\mathcal{C}$ , they functorially associate a commutative symmetric ring spectrum [5, Corollary 3.9 and Theorem 9.3.8]. By [5, Theorem 4.6] and the original work of Segal [22], its underlying infinite loop space is weak homotopy equivalent to  $\Omega BBC$ . By a theorem due to Schwede [21] and later refined by Mandell and May [10, Section 1], the

homotopy category of commutative symmetric ring spectra is equivalent to that of  $E_\infty$ -ring spectra. We write  $K\mathcal{C}$  for the  $E_\infty$ -ring spectrum associated to  $\mathcal{C}$  under this equivalence of homotopy categories. The underlying infinite loop space of an  $E_\infty$ -ring spectrum is an  $E_\infty$ -ring space, as defined in [13, Chapter VI], so we may functorially associate to each bipermutative category an  $E_\infty$ -ring space  $\Omega^\infty K\mathcal{C}$ . Moreover, by [9, Theorem 1.2], the space  $\Omega^\infty K\mathcal{C}$  is weak homotopy equivalent to the group completion  $\Omega BBC$ .

We now apply this machinery to the morphism  $\mathbb{P} \ltimes X \rightarrow \mathbb{P} \ltimes *$  of bipermutative categories. We obtain a map of  $E_\infty$ -ring spectra

$$K(\mathbb{P} \ltimes X) \rightarrow K(\mathbb{P} \ltimes *)$$

which is equivalent to  $\rho^X$  after applying  $\Omega^\infty$ . The homotopy fiber of a map of  $E_\infty$ -ring spectra is a nonunital  $E_\infty$ -ring spectrum. By a nonunital  $E_\infty$ -ring space, we mean the underlying infinite loop space of a nonunital  $E_\infty$ -ring spectrum. Since  $\Omega^\infty$  preserves homotopy fiber sequences, this means that the homotopy fiber of a map of  $E_\infty$ -ring spaces is a nonunital  $E_\infty$ -ring space. We have proved the next theorem.

**Theorem 2.3** *For any commutative  $\mathbb{I}$ -monoid  $X$  the homotopy fiber  $\text{hofib } \rho^X$  of*

$$\rho^X: \Omega B(\text{hocolim}_{\mathbb{P}} X) \rightarrow \Omega B(\text{hocolim}_{\mathbb{P}} *)$$

*is an infinite loop space. If furthermore  $X$  is a commutative  $\mathbb{I}$ -rig, then  $\text{hofib } \rho^X$  is a nonunital  $E_\infty$ -ring space.*

## 2.5 The main example

For any group  $G$ , conjugation by  $G$  or action by any other automorphism of  $G$  induces a well-defined action on  $B_n(q, G) = \text{Hom}(F_n / \Gamma_n^q, G)$  by postcomposition. The action is also compatible with the simplicial face and degeneracy maps in the bar construction and hence induces an action on  $B(q, G)$ .

For every  $q \geq 2$  we define an  $\mathbb{I}$ -space  $B(q, U(-))$  by setting  $\mathbf{n} \mapsto B(q, U(n))$  with morphisms induced by the natural inclusions and the action of  $\Sigma_n$  on  $B(q, U(n))$  given by conjugation through permutation matrices. Being induced by the natural action of  $\Sigma_n$  on  $\mathbf{n}$ , it can be checked that this compatible sequence defines indeed an  $\mathbb{I}$ -space.

We give  $B(q, U(-))$  the structure of an  $\mathbb{I}$ -monoid by defining the unit map  $\eta_{\mathbf{n}}: * \rightarrow B(q, U(n))$  to be the inclusion of the base-point and defining the monoid structure map

$$\mu_{\mathbf{n}, \mathbf{m}}: B(q, U(n)) \times B(q, U(m)) \rightarrow B(q, U(n+m))$$

to be induced by the block sum of matrices. To see that  $\mu_{\mathbf{n}, \mathbf{m}}$  is well-defined note that block sum defines a group homomorphism  $U(n) \times U(m) \rightarrow U(n+m)$ . When taking elements of the symmetric groups to permutation matrices, the disjoint union of sets corresponds to block sum of matrices. Thus  $\mu$  defines a natural transformation of functors defined on  $\mathbb{I} \times \mathbb{I}$ . One checks compatibility with  $\tau$  and hence  $B(q, U(-))$  is a commutative  $\mathbb{I}$ -monoid.

Next we note that tensor product of matrices induces a well-defined map

$$\pi_{\mathbf{n}, \mathbf{m}}: B(q, U(n)) \times B(q, U(m)) \rightarrow B(q, U(nm)).$$

To see this note that tensor product commutes with matrix multiplication and hence induces a homomorphism  $U(n) \times U(m) \rightarrow U(nm)$ . The map is equivariant for the symmetric group actions because the permutation matrix associated to the product of two permutations is the same as the tensor product of the corresponding permutation matrices. Hence  $\pi$  is a natural transformation of functors defined on the category  $\mathbb{P} \times \mathbb{P}$ . Note, however, that  $\pi$  is not natural for proper injections. The map  $\pi$  is compatible with  $\tau$  and the distributivity of block sum and tensor product of matrices induces distributivity maps for  $\mu$  and  $\pi$ . We have shown:

**Theorem 2.4**  $B(q, U(-))$  is a commutative  $\mathbb{I}$ -rig.

As a consequence, we may apply Theorems 2.2 and 2.3 to get a pair of infinite loop spaces, the latter of which carries a nonunital  $E_\infty$ -ring structure. In the next section, we will show that these two infinite loop spaces are equivalent.

### 3 Identifying and comparing the infinite loop spaces

Let  $X$  be a commutative  $\mathbb{I}$ -monoid. We will first identify  $\text{hofib } \rho^X$  under certain assumptions and then show it is homotopy equivalent as an infinite loop space to  $\text{hocolim}_{\mathbb{I}} X$ .

Consider the space

$$X_\infty := \text{hocolim}_{n \in \mathbb{N}} X(\mathbf{n}).$$

Note that  $X_\infty \simeq \text{colim}_{n \in \mathbb{N}} X(\mathbf{n})$  if the structural maps  $j_n: X(\mathbf{n}) \rightarrow X(\mathbf{n} + \mathbf{1})$  are cofibrations. In our applications this will always be the case. Let  $X_\infty^+$  denote Quillen's plus construction applied with respect to the maximal perfect subgroup of  $\pi_1(X_\infty)$  (which we take to be understood to be done in each component separately, if  $X_\infty$  is not connected). Also recall that a space  $Z$  is abelian if  $\pi_1(Z)$  is abelian and acts trivially on homotopy groups  $\pi_*(Z)$ . It is well known that  $H$ -spaces are abelian.

**Theorem 3.1** *Let  $X: \mathbb{I} \rightarrow \text{Top}$  be a commutative  $\mathbb{I}$ -monoid. Assume that*

- *the action of  $\Sigma_\infty$  on  $H_*(X_\infty)$  is trivial;*
- *the inclusions induce natural isomorphisms  $\pi_0(X(\mathbf{n})) \simeq \pi_0(X_\infty)$  of finitely generated abelian groups with multiplication compatible with the Pontrjagin product and in the center of the homology Pontrjagin ring;*
- *the commutator subgroup of  $\pi_1(X_\infty)$  is perfect (for each component) and  $X_\infty^+$  is abelian.*

*Then  $\text{hofib } \rho^X \simeq X_\infty^+$  and, in particular,  $X_\infty^+$  is an infinite loop space.*

**Proof** Let  $M = \text{hocolim}_{\mathbb{P}} X = B(\mathbb{P} \ltimes X)$  and  $m$  be the point corresponding to the base point in  $X(\mathbf{1})$  (in the identity component of  $\pi_0(X(\mathbf{1}))$ ). Then

$$\text{Tel}(M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} M \xrightarrow{\cdot m} \cdots) \simeq \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty).$$

As  $\mathbb{P} \ltimes X$  is a symmetric monoidal category, its classifying space  $M$  is a homotopy commutative topological monoid. The hypotheses imply that  $\pi_0(M)$  is in the center of  $H_*(M)$ . Hence  $H_*(M)[\pi_0(M)^{-1}]$  can be constructed by right fractions, so that we may apply the group completion theorem [15; 17]. Therefore there is a map

$$f: \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty) \rightarrow \Omega BM$$

which induces an isomorphism on homology with all systems of local coefficients on  $\Omega BM$ . Furthermore, the fundamental group (of each component) of  $E\Sigma_\infty \times_{\Sigma_\infty} X_\infty$  has a perfect commutator subgroup by [17], and  $f$  extends to a homology equivalence between abelian spaces

$$f^+: \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \rightarrow \Omega BM,$$

which is thus a homotopy equivalence. This shows, in particular, that the space  $\mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+$  is an infinite loop space as  $\Omega BM$  is the group completion of an  $E_\infty$ -space.

Consider now the fibration sequence

$$(1) \quad X_\infty \rightarrow E\Sigma_\infty \times_{\Sigma_\infty} X_\infty \xrightarrow{p} B\Sigma_\infty$$

and the associated map of plus constructions

$$p^+: \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \rightarrow \mathbb{Z} \times B\Sigma_\infty^+.$$

Since  $f^+$  is a homotopy equivalence and  $\Omega B(\text{hocolim}_{\mathbb{P}} *) \simeq \mathbb{Z} \times B\Sigma_\infty^+$ , we can identify the homotopy fiber of  $p^+$  with  $\text{hofib } \rho^X$ . By assumption the action of  $\Sigma_\infty$  on  $X_\infty$  is homologically trivial. We are also assuming that  $X_\infty^+$  is abelian and in particular

nilpotent. Under these conditions the fiber sequence (1) remains a fiber sequence after passing to plus constructions; see [4, Theorem 1.1]. Thus we have a homotopy fibration

$$X_\infty^+ \rightarrow \mathbb{Z} \times (E\Sigma_\infty \times_{\Sigma_\infty} X_\infty)^+ \rightarrow \mathbb{Z} \times B\Sigma_\infty^+.$$

This shows that the homotopy fiber of  $p^+$  is  $X_\infty^+$  and so  $X_\infty^+ \simeq \text{hofib } \rho^X$ .  $\square$

**Remark 3.2** For any commutative  $\mathbb{I}$ -monoid  $X$ , the multiplication on  $M_X := \bigsqcup_{n \geq 0} X(n)$  is commutative up to the action of the shuffle maps  $\tau_{\mathbf{m}, \mathbf{n}}$ . These are induced by the action of the symmetric group. So, assuming that these actions are trivial in homology, it follows that the Pontrjagin product is commutative on the level of homology. In particular  $\pi_0(M_X)$  is in the center of the Pontrjagin ring  $H_*(M_X)$ . Thus by the group completion theorem [15], the map

$$\mathbb{Z} \times X_\infty \rightarrow \Omega B(M_X)$$

is a homology isomorphism. In recent work, Gritschacher [7] has shown that without any further assumption, the commutator subgroup of  $\pi_1(X_\infty)$  is always perfect and that  $X_\infty^+$  is always an abelian space. In other words, the assumptions in Theorem 3.1 on  $\pi_1(X_\infty)$  and  $X_\infty^+$  are actually consequences.<sup>2</sup>

In contrast, the condition that the symmetric groups act homologically trivially is necessary. To see this consider the commutative  $\mathbb{I}$ -space  $X$  with  $X(n) := Z^n$  for some pointed connected space  $Z$ . Then, by the parametrized version of the Barratt–Priddy–Quillen theorem (see for example [12; 22]),

$$\Omega B(\text{hocolim}_{\mathbb{P}} X) \simeq Q(Z_+)$$

and thus  $\text{hofib } \rho^X \simeq \text{hofib } p^+ \simeq Q(Z)$  while  $X_\infty \simeq \text{hocolim}_n Z^n$ . Here  $Q = \Omega^\infty \Sigma^\infty$  and  $Z_+$  denotes the space  $Z$  with an additional base point.

We now turn to the question of comparing the infinite loop spaces  $\text{hofib } \rho^X$  and  $\text{hocolim}_{\mathbb{I}} X$ . Suppose that  $X$  is a commutative  $\mathbb{I}$ -monoid. Consider the following commutative diagram of strict functors between permutative categories:

$$\begin{array}{ccc} \mathbb{P} \ltimes X & \xrightarrow{\alpha_X} & \mathbb{I} \ltimes X \\ \rho^X \downarrow & & \downarrow \rho_1^X \\ \mathbb{P} \ltimes * & \xrightarrow{\alpha_*} & \mathbb{I} \ltimes * \end{array}$$

The horizontal maps are induced by the inclusion  $\mathbb{P} \rightarrow \mathbb{I}$ . In the above diagram  $*$  is the terminal commutative  $\mathbb{I}$ -monoid and the vertical maps  $\rho^X$  and  $\rho_1^X$  are induced by

<sup>2</sup>As we do not know whether  $M_X$  is homotopy commutative, the results of [17] cannot be applied directly to conclude that the induced map  $\mathbb{Z} \times X_\infty^+ \rightarrow \Omega B(M_X)$  is a homotopy equivalence.

the projection maps to a point. Passing to the level of classifying spaces and applying group completion we obtain a commutative diagram of infinite loop spaces:

$$(2) \quad \begin{array}{ccc} \Omega B(\mathrm{hocolim}_{\mathbb{P}} X) & \xrightarrow{\alpha_X} & \Omega B(\mathrm{hocolim}_{\mathbb{I}} X) \\ \rho^X \downarrow & & \downarrow \rho_1^X \\ \Omega B(\mathrm{hocolim}_{\mathbb{P}} *) & \xrightarrow{\alpha_*} & \Omega B(\mathrm{hocolim}_{\mathbb{I}} *) \simeq * \end{array}$$

Note that the empty set is an initial object for  $\mathbb{I}$  and hence  $\mathrm{hocolim}_{\mathbb{I}} * = B\mathbb{I} \simeq *$ .

The above diagram induces an infinite loop map between the homotopy fibers of the maps  $\rho^X$  and  $\rho_1^X$ . By definition the homotopy fiber on the left is the space  $\mathrm{hofib} \rho^X$ . Also, since  $\mathrm{hocolim}_{\mathbb{I}} *$  is contractible, the homotopy fiber on the right can be identified with  $\Omega B(\mathrm{hocolim}_{\mathbb{I}} X)$ . This shows that we have a map of infinite loop spaces

$$\mathrm{hofib} \rho^X \xrightarrow{g} \Omega B(\mathrm{hocolim}_{\mathbb{I}} X).$$

Note that  $\rho^X$  has a canonical splitting of permutative categories induced by the unit  $* \rightarrow X$  of the  $\mathbb{I}$ -monoid  $X$ . Thus it follows from the following theorem that  $g$  is a homotopy equivalence whenever the stated conditions on  $X$  are satisfied.

**Theorem 3.3** *Let  $X$  be a commutative  $\mathbb{I}$ -monoid such that all maps  $j: X(\mathbf{n}) \rightarrow X(\mathbf{m})$  induced by injections  $j: \mathbf{n} \rightarrow \mathbf{m}$  are monomorphisms. Furthermore, assume that, for all  $x \in X(\mathbf{n})$  and  $y \in X(\mathbf{m})$ , the sum  $\mu_{\mathbf{n}, \mathbf{m}}(x, y)$  is in the image of a map induced by a nonidentity order preserving injection if and only if  $x$  or  $y$  is. Then*

$$\alpha_X \times \rho^X: \Omega B(\mathrm{hocolim}_{\mathbb{P}} X) \rightarrow \Omega B(\mathrm{hocolim}_{\mathbb{I}} X) \times \Omega B(\mathrm{hocolim}_{\mathbb{P}} *)$$

*is a weak homotopy equivalence of infinite loop spaces which is natural for commutative  $\mathbb{I}$ -monoids.*

Notice that, when  $X$  is a commutative  $\mathbb{I}$ -rig, we may use the theorem to transfer the nonunital  $E_\infty$ -ring space structure on  $\mathrm{hofib} \rho^X$  along  $g$  to obtain a nonunital  $E_\infty$ -ring space structure on the group completion of  $\mathrm{hocolim}_{\mathbb{I}} X$ .

A version of the theorem was proved by Fiedorowicz and Ogle [6] in the setting of simplicial sets. This was revisited in Gritschacher [7, Section 4]. For convenience of the reader we sketch a streamlined argument following [7].

**Proof** Given  $x \in X(\mathbf{n})$  we can write it as  $x = j_x(\bar{x})$ , where  $\bar{x} \in X(\bar{\mathbf{n}})$ ,  $j_x: \bar{\mathbf{n}} \rightarrow \mathbf{n}$  is an order-preserving injection and  $\bar{\mathbf{n}}$  is minimal. We call  $x$  reduced if  $x = \bar{x}$ . Note that  $\bar{x}$  and  $j_x$  are uniquely determined. Denote by  $\bar{X}(\mathbf{n})$  the set of reduced elements in  $X(\mathbf{n})$ . The assignment  $\mathbf{n} \mapsto \bar{X}(\mathbf{n})$  defines a  $\mathbb{P}$ -diagram. By the assumption on  $\mu$  the commutative  $\mathbb{I}$ -monoid structure of  $X$  induces the structure of a permutative category on  $\mathbb{P} \ltimes \bar{X}$ .

Assume now that  $X$  is discrete. Then the assignment  $(\mathbf{n}, x) \mapsto (\bar{\mathbf{n}}, \bar{x})$  on objects extends to define a functor

$$R_X: \mathbb{I} \ltimes X \rightarrow \mathbb{P} \ltimes \bar{X}.$$

It has a right inverse given by the inclusion  $\iota_X: \mathbb{P} \ltimes \bar{X} \rightarrow \mathbb{I} \ltimes X$ . Furthermore, the maps  $j_x$  define a natural transformation from  $\iota_X \circ R_X$  to the identity on  $\mathbb{I} \ltimes X$ . Hence,  $R_X$  defines a homotopy deformation retract on classifying spaces. We also note that by our assumption on  $\mu$ , the functor  $R_X$  is a strict symmetric monoidal functor.

The inclusions  $\mathbb{P} \ltimes \bar{X} \rightarrow \mathbb{P} \ltimes X$  and  $\mathbb{P} \rightarrow \mathbb{P} \ltimes X$  combine via the monoidal product functor to a functor

$$T_X: (\mathbb{P} \ltimes \bar{X}) \times \mathbb{P} \rightarrow \mathbb{P} \ltimes X$$

that maps the object  $((\bar{\mathbf{n}}, \bar{x}), \mathbf{n})$  to  $(\bar{\mathbf{n}} + \mathbf{n}, j(\bar{x}))$ , where  $j$  is the canonical inclusion  $\bar{\mathbf{n}} \hookrightarrow \bar{\mathbf{n}} + \mathbf{n}$ . We claim this is a homotopy equivalence on classifying spaces. Indeed, an analysis of the effect of permutations on reduced points shows that the functor is bijective on automorphism groups of objects. As both source and target categories are groupoids and every isomorphism class of the target category has a representative in the image, this is an equivalence of categories. We note that  $T_X$  is not a strict monoidal functor (only up to conjugation by a block permutation). However, the left inverse functor  $(\mathbf{n}, x) \mapsto ((\bar{\mathbf{n}}, \bar{x}), \mathbf{n} - \bar{\mathbf{n}})$  does commute strictly with the monoidal structure. Hence, this defines a homotopy equivalence of monoids on classifying spaces, and induces a homotopy equivalence of group completions. Compare [6, Lemma 1.7].

Consider now the map of permutative categories

$$\alpha_X \times \rho^X: \mathbb{P} \ltimes X \rightarrow (\mathbb{I} \ltimes X) \times \mathbb{P}$$

and take the group completion of their classifying spaces

$$(3) \quad \alpha_X \times \rho^X: \Omega B(B(\mathbb{P} \ltimes X)) \rightarrow \Omega B(B(\mathbb{I} \ltimes X)) \times \Omega B(B\mathbb{P}).$$

We claim that this is a weak homotopy equivalence which is natural in commutative  $\mathbb{I}$ –monoids. To see this precompose with the map of group completed classifying spaces induced by  $T_X$  and postcompose with the map induced by  $R_X \times \text{Id}$ . The resulting composite is homotopic to the endofunctor of  $(\mathbb{P} \ltimes \bar{X}) \times \mathbb{P}$  given by

$$((\bar{\mathbf{n}}, \bar{x}), \mathbf{m}) \mapsto ((\bar{\mathbf{n}}, \bar{x}), \bar{\mathbf{n}} + \mathbf{m}).$$

This map is the identity on the first component and an equivalence on the second component because we are working with group-complete monoids.

Using the naturality of the weak homotopy equivalence in (3) and applying it to boundary and face maps allows us to extend it to  $\mathbb{I}$ –diagrams in simplicial sets. More

precisely, for any commutative  $\mathbb{I}$ -monoid  $X$  in simplicial sets that satisfies levelwise the condition on  $\mu$ , we have a map of simplicial permutative categories which is a weak homotopy equivalence on applying  $\Omega B(B(-))$  to each simplicial level, and hence a weak homotopy equivalence on total spaces:

$$\alpha_X \times \rho^X : |\mathbf{n} \mapsto \Omega B(B(\mathbb{P} \ltimes X(\mathbf{n})))| \simeq |\mathbf{n} \mapsto \Omega B(B(\mathbb{I} \ltimes X(\mathbf{n}))) \times \Omega B(B\mathbb{P})|.$$

As  $\Omega$  commutes with Cartesian product, and as  $|\mathbf{n} \mapsto \Omega Z(\mathbf{n})| \simeq \Omega |\mathbf{n} \mapsto Z(\mathbf{n})|$  whenever each  $Z(\mathbf{n})$  is connected (see [11, Theorem 12.3]), we also have

$$\alpha_X \times \rho^X : \Omega |\mathbf{n} \mapsto B(B(\mathbb{P} \ltimes X(\mathbf{n})))| \simeq \Omega |\mathbf{n} \mapsto B(B(\mathbb{I} \ltimes X(\mathbf{n}))) \times B(B\mathbb{P})|.$$

Furthermore, as realizations of multisimplicial sets can be taken in any order, we deduce that

$$\alpha_X \times \rho^X : \Omega B(B(\mathbb{P} \ltimes |\mathbf{n} \mapsto X(\mathbf{n})|)) \simeq \Omega B(B(\mathbb{I} \ltimes |\mathbf{n} \mapsto X(\mathbf{n})|)) \times \Omega B(B\mathbb{P}).$$

Compare [6, Lemma 1.8]. Finally, by replacing every space by its singular simplicial set, any  $\mathbb{I}$ -diagram  $X$  in topological spaces gives rise to an  $\mathbb{I}$ -diagram in simplicial sets, taking commutative  $\mathbb{I}$ -monoids to simplicial ones. Note that the conditions on  $\mu$  are pointwise conditions and are automatically satisfied by the singular  $p$ -simplices for each  $p$ . As a space is weakly homotopy equivalent to the realization of its singular simplicial set, the theorem follows.  $\square$

**Example 3.4** Consider the commutative  $\mathbb{I}$ -space  $X$  with  $X(\mathbf{n}) := Z^n$ , where  $Z$  is a well-pointed connected space. Note that in this case  $\Sigma_n$  does not act trivially on  $H_*(Z^n)$  and hence Theorem 3.1 does not apply. As before, by the parametrized version of the Barratt–Priddy–Quillen theorem,

$$\Omega B(\operatorname{hocolim}_{\mathbb{P}} X) \simeq Q(Z_+) \simeq Q(\mathbb{S}^0) \times Q(Z)$$

and hence  $\operatorname{hofib} \rho^X \simeq Q(Z)$ . Thus, by Theorem 3.3 we also have  $\operatorname{hocolim}_{\mathbb{I}} X \simeq Q(Z)$ , which is in agreement with a result of Schlichtkrull [20].

## 4 Constructing filtrations by infinite loop spaces

In this section we use the results obtained in the previous sections to produce filtrations of classical infinite loop spaces by sequences of infinite loop spaces arising from the descending central series of the free groups.

**Theorem 4.1** *The spaces  $B(q, U)$ ,  $B(q, \operatorname{SU})$ ,  $B(q, \operatorname{SO})$ ,  $B(q, O)$  and  $B(q, \operatorname{Sp})$  provide a filtration by nonunital  $E_\infty$ -ring spaces of the classical nonunital  $E_\infty$ -ring spaces  $BU$ ,  $B\operatorname{SU}$ ,  $B\operatorname{SO}$ ,  $BO$  and  $B\operatorname{Sp}$ , respectively.*

**Proof** Consider first the case of  $BU$ . Recall that the spaces  $B(q, U)$  provide a filtration of the space  $BU$

$$B(2, U) \subset B(3, U) \subset \cdots \subset B(q, U) \subset B(q+1, U) \subset \cdots \subset BU.$$

We will show that this filtration is a filtration by nonunital  $E_\infty$ -ring spaces. For this notice that by the main example in Section 2, each  $n \mapsto B(q, U(n))$  for  $q \geq 2$  is a commutative  $\mathbb{I}$ -rig. In what follows we are going to show that the conditions of Theorem 3.1 are satisfied, and hence  $B(q, U) \simeq \text{hofib } \rho^{B(q, U(-))}$  is a nonunital  $E_\infty$ -ring space by Theorem 2.3.

The conjugation action of  $\Sigma_n$  on  $B(q, U(n))$  is homologically trivial because this action factors through the conjugation action of  $U(n)$ . The conjugation action by any element in  $U(n)$  is trivial, up to homotopy, since the action of the identity matrix is trivial and  $U(n)$  is path-connected. This implies in particular that the action of  $\Sigma_\infty$  on  $B(q, U)$  is homologically trivial.

Note that  $B(q, U(n))$  and hence  $B(q, U)$  is path connected. Next, we argue that the space  $B(q, U)$  is an  $H$ -space under direct sum multiplication. To be more precise, consider the injection  $\mathbb{N} \sqcup \mathbb{N} \rightarrow \mathbb{N}$  defined by  $(1, 2, 3, 4, \dots) \cup (1', 2', 3', 4', \dots) \mapsto (1, 2, 1', 2', 3, 4, 3', 4', \dots)$ . It defines a map of vector spaces  $\mathbb{C}^\infty \times \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$  and hence a continuous homomorphism  $U \times U \rightarrow U$ . The image of  $U(n)$  in  $U$  under right or left multiplication by the identity matrix  $I$  differs from the image under the standard inclusion by conjugation of an even permutation. As such a permutation is in the path-component of the identity matrix, we see that the multiplication is unital up to homotopy.

$H$ -spaces have abelian fundamental group and hence Theorem 3.1 applies. We conclude that  $B(q, U) \simeq \text{hofib } \rho^{B(q, U(-))}$  for every  $q \geq 2$  and is a nonunital  $E_\infty$ -ring space by Theorem 2.3. The very same arguments can be used to prove analogous statements for the commutative  $\mathbb{I}$ -rig  $n \mapsto B(q, \text{SU}(n))$ , and  $n \mapsto B(q, \text{Sp}(n))$  for any  $q \geq 2$ .

In case of the commutative  $\mathbb{I}$ -rig  $n \mapsto B(q, \text{SO}(n))$  we note that  $\Sigma_n$  is not a subgroup of  $\text{SO}(n)$ . Nevertheless, the alternating group  $A_n$  is contained in  $\text{SO}(n)$  and by the same argument as above acts therefore trivially on the homology of  $B(q, \text{SO}(n))$ . Furthermore, when  $n$  is odd, any odd permutation is represented by a matrix with determinant equal to  $-1$ . Hence it can be path-connected to the diagonal matrix  $-I$  with constant entry  $-1$ . As  $-I$  is in the center of  $O(n)$  it acts trivially by conjugation on  $B(q, \text{SO}(n))$  and hence also on its homology. But then so does any odd permutation. This proves that when  $n$  is odd the action of  $\Sigma_n$  on  $B(q, \text{SO}(n))$  is homologically trivial. This in turn implies that the action of  $\Sigma_\infty$  on  $B(q, \text{SO})$  is

homologically trivial. We also have that  $B(q, \text{SO})$  is an  $H$ -space and hence abelian. Thus  $B(q, \text{SO}) \simeq \text{hofib } \rho^{B(q, \text{SO}(-))}$  for every  $q \geq 2$  and it is a nonunital  $E_\infty$ -ring space by Theorem 2.3. This line of argument can also be used to prove the analogous statement for the commutative  $\mathbb{I}$ -rig  $n \mapsto B(q, O(n))$ .  $\square$

As remarked in [1, Theorem 6.3], the natural map  $\Omega B(q, G) \rightarrow \Omega BG$  admits a splitting up to homotopy. It is given by a factorization of the usual homotopy equivalence  $G \rightarrow \Omega BG$ . Indeed we have that  $\Sigma G = F_1 B(q, G) = F_1 BG$ , where  $F_1$  denotes the first layer in the usual filtration of the geometric realization of these simplicial spaces. Hence, the adjoint of  $\Sigma G \rightarrow BG$  factors through  $\Omega B(q, G)$ . Note that this splitting does not in general admit a delooping; see [1, Section 6] for a counterexample. Nevertheless, we have the following theorem. Here  $E(q, G)$  denotes the pull-back of the universal  $G$ -bundle  $EG$  over  $BG$ . It is homotopy equivalent to the homotopy fiber of the inclusion  $B(q, G) \rightarrow BG$ .

**Theorem 4.2** *For all  $q \geq 2$ , and  $G = U, \text{SU}, \text{SO}, O$  and  $\text{Sp}$ , there is a homotopy split fibration of infinite loop spaces*

$$E(q, G) \rightarrow B(q, G) \rightarrow BG.$$

*In particular there is a splitting of spaces*

$$B(q, G) \simeq BG \times E(q, G).$$

*Both are natural in the entry  $q$ , meaning that both are compatible with the filtration maps.*

In order to prove the theorem, we will need to know the fundamental group of  $B(q, G)$  for the groups in question. We have the following general result:

**Lemma 4.3** *Let  $G$  be a topological group with a CW-structure. Assume  $\pi_0(G)$  is abelian and that the natural homomorphism  $G \rightarrow \pi_0(G)$  splits. Then, for all  $q \geq 2$ ,*

$$\pi_1(B(q, G)) = \pi_0(G).$$

**Proof** Consider  $\Sigma G = F_1 B(q, G) = F_1 BG$ . As the 1-skeleton of the realization of a (good) simplicial space is contained in the first filtration [11, Proposition 11.4], any map from  $S^1$  to  $B(q, G)$  will factor through  $\Sigma G$ . Hence the map  $\Sigma G \rightarrow B(q, G)$  is surjective on fundamental groups.

The fundamental group of a suspension  $\Sigma X$  for any space  $X$  has fundamental group the free group over the set  $\pi_0(X) - \{1\}$ ; hence we have

$$\pi_1(\Sigma G) = F(g \mid g \in \pi_0(G) - \{1\}).$$

The inclusion  $\Sigma G \rightarrow BG$  induces the surjective map of fundamental groups  $\pi_1(\Sigma G) \rightarrow \pi_0(G)$  which sends a generator  $g$  to the *element*  $g \in \pi_0(G)$  and, more generally, the word  $g_1 \bullet \cdots \bullet g_k$  to the product of the elements  $g_1 \cdots g_k$ . To see this geometrically, consider  $\pi_0(G)$  as a subgroup of  $G$ , and note that the 2-simplex  $(g, h)$  defines a homotopy from the 2-letter word  $g \bullet h$  to the product element  $gh$ .

We now note that, as  $\pi_0(G)$  is abelian, the 2-simplex  $(g, h)$  is contained in  $B_2(q, G)$  for  $q \geq 2$ . Hence all the above relations are already satisfied in  $\pi_1(B(q, G))$ . As the factorization  $\pi_1(\Sigma G) \rightarrow \pi_1(B(q, G)) \rightarrow \pi_1(BG)$  is surjective, the result follows.  $\square$

**Proof of Theorem 4.2** As  $EG_\infty \simeq *$ , for every  $q \geq 2$  we have a homotopy fibration sequence  $E(q, G_\infty) \rightarrow B(q, G_\infty) \rightarrow BG_\infty$ . As the map on the right is a map of infinite loop spaces, the homotopy fiber  $E(q, G_\infty)$  is an infinite loop space. It remains to show that it splits.

Let  $G_n$  denote one of the groups  $U(n)$ ,  $SU(n)$ ,  $SO(n)$ ,  $O(n)$  or  $Sp(n)$ , so that  $G_\infty = \operatorname{colim}_n G_n$  denotes the group  $U$ ,  $SU$ ,  $SO$ ,  $O$  or  $Sp$ , respectively. For each fixed  $q \geq 2$ , the assignment  $\mathbf{n} \mapsto \Omega B(q, G_n)$  defines a commutative  $\mathbb{I}$ -rig with  $\mu$  given by block sum and  $\pi$  given by tensor product of matrices. In the same way the assignment  $\mathbf{n} \mapsto \Omega BG_n$  also defines a commutative  $\mathbb{I}$ -rig and the inclusion map  $\Omega B(q, G_n) \rightarrow \Omega BG_n$  defines a morphism of commutative  $\mathbb{I}$ -rigs.

We claim that the commutative  $\mathbb{I}$ -rigs  $G_-$ ,  $\Omega B(q, G_-)$  and  $\Omega BG_-$  satisfy the hypotheses of Theorem 3.1. Indeed, except in the case  $G = O$ , the group  $G_n \simeq \Omega BG_n$  is path-connected for every  $n \geq 0$  and, as  $\pi_0(\Omega B(q, G_n)) \cong \pi_1(B(q, G_n))$  is trivial by Lemma 4.3,  $\Omega B(q, G_n)$  is also path-connected. When  $G = O$ ,

$$\pi_0(\Omega B(q, O(n))) = \pi_1(B(q, O(n))) = \mathbb{Z}/2\mathbb{Z}$$

for each  $n \geq 1$  by Lemma 4.3. The multiplication in  $\pi_0 \Omega B(q, O(n))$  is compatible with direct sum and stabilization. This checks the second condition in Theorem 3.1.

Except in the cases  $G = SO$  or  $G = O$ , the action of  $\Sigma_n$  is homologically trivial as conjugation by any element in the path component of the identity is trivial, up to homotopy, and  $G_n$  is path-connected. This implies that  $\Sigma_\infty$  acts homologically trivially on  $G_\infty$ ,  $\Omega B(q, G_\infty)$  and  $\Omega BG_\infty$ . The same conclusion can be obtained for  $G = SO$  or  $G = O$  using a similar argument as in the proof of Theorem 4.1. Hence the first condition from Theorem 3.1 holds.

To verify the third condition, observe that the commutator group of  $\pi_1(\Omega B(q, G_n)) \cong \pi_2(B(q, G_n))$  is trivial, as this group is abelian in all cases. Finally,  $\Omega B(q, G_\infty)$  is an abelian space since it is a loop space and hence in particular an  $H$ -space.

By Theorem 3.1 we thus have maps of  $E_\infty$ -spaces

$$G_\infty \rightarrow \Omega B(q, G_\infty) \rightarrow \Omega BG_\infty$$

whose composition is a homotopy equivalence. Taking classifying spaces is compatible with  $E_\infty$ -space structures and hence the above splitting deloops to give the splitting of the theorem.  $\square$

We have concentrated so far on compact groups such as  $O(n)$  and  $U(n)$ , although the methods clearly extend to other linear groups. Using some results by Pettet and Souto [16] and Bergeron [3] we can prove the following theorem:

**Theorem 4.4** *Suppose that  $G$  is the group of complex or real points in a reductive linear algebraic group (defined over  $\mathbb{R}$  in the real case). Let  $K \subset G$  be a maximal compact subgroup. Then the inclusion map  $i: B(q, K) \rightarrow B(q, G)$  is a homotopy equivalence for every  $q \geq 2$ .*

**Proof** By [3, Theorem I] it follows that the inclusion map  $i_n: B_n(q, K) \rightarrow B_n(q, G)$  is a homotopy equivalence for all  $q \geq 2$  and all  $n \geq 0$ . Thus the inclusion map induces a simplicial map  $i_*: B_*(q, K) \rightarrow B_*(q, G)$  that is a levelwise homotopy equivalence. Since  $G$  is assumed to be the group of complex or real points in a reductive linear algebraic group (defined over  $\mathbb{R}$  in the real case), we can identify  $G$  with a Zariski closed subgroup of  $\mathrm{SL}_N(\mathbb{C})$  for some  $N \geq 0$ . Also, for every  $n \geq 0$  we can see the space  $B_n(q, G)$  as an algebraic variety since it is defined in terms of iterated commutators of elements in  $G$  and such equations can be defined in terms of polynomial functions. Moreover, the subspace  $S_n^1(q, G) \subset B_n(q, G)$  consisting of all  $n$ -tuples in  $B_n(q, G)$  for which at least one of the coordinates is equal to  $1_G$  is an algebraic subvariety of  $B_n(q, G)$ . By the semialgebraic triangulation theorem (see [8, Section 1]) it follows that  $B_n(q, G)$  has the structure of a CW-complex in such a way that  $S_n^1(q, G)$  is a subcomplex. In particular, it follows that the pair  $(B_n(q, G), S_n^1(q, G))$  is a strong NDR pair. This proves that  $B_*(q, G)$  is a proper simplicial space. The same is true for  $B_*(q, K)$ . Using the gluing lemma — for example see [12, Theorem A.4] — we obtain the result of the theorem.  $\square$

Our tools can also be used to obtain a similar filtration for the infinite loop space defining algebraic K-theory for any discrete ring  $R$ . Indeed, suppose that  $R$  is a discrete ring with unit and let  $q \geq 2$ . Consider the commutative  $\mathbb{I}$ -rig  $B(q, \mathrm{GL}_-(R))$  defined by  $\mathbf{n} \mapsto B(q, \mathrm{GL}_n(R))$ . As before the morphisms are induced by the natural inclusions and the conjugation action of  $\Sigma_n$  on  $B(q, \mathrm{GL}_n(R))$ . The multiplication map

$$\mu_{\mathbf{n}, \mathbf{m}}: B(q, \mathrm{GL}_n(R)) \times B(q, \mathrm{GL}_m(R)) \rightarrow B(q, \mathrm{GL}_{n+m}(R))$$

is also given by the block sum and  $\pi$  by tensor product of matrices. Note that Theorem 3.3 applies to give

$$\mathrm{hocolim}_{\mathbb{I}} B(q, \mathrm{GL}_{-}(R)) \simeq \mathrm{hofib} \rho^{B(q, \mathrm{GL}_{-}(R))}.$$

By Theorem 2.3, this space has the structure of a nonunital  $E_{\infty}$ -ring space. This way we obtain a filtration of nonunital  $E_{\infty}$ -ring spaces:

$$\mathrm{hocolim}_{\mathbb{I}} B(2, \mathrm{GL}_{-}(R)) \subset \cdots \subset \mathrm{hocolim}_{\mathbb{I}} B(q, \mathrm{GL}_{-}(R)) \subset \cdots \subset \mathrm{hocolim}_{\mathbb{I}} B\mathrm{GL}_{-}(R).$$

As is well known, the conjugation action of  $\Sigma_n$  on  $B\mathrm{GL}_n(R)$  is homologically trivial. It follows from Theorems 3.1 and 3.3 that we have an equivalence

$$B\mathrm{GL}_{\infty}(R)^+ \simeq \mathrm{hofib} \rho^{B\mathrm{GL}_{-}(R)} \simeq \mathrm{hocolim}_{\mathbb{I}} B\mathrm{GL}_{-}(R).$$

Thus the above gives a filtration of nonunital  $E_{\infty}$ -ring spaces with final space weakly homotopy equivalent to the algebraic K-theory of  $R$ . However, unlike the case of  $B\mathrm{GL}_n(R)$ , we do not know whether the conjugation action of  $\Sigma_n$  on  $B(q, \mathrm{GL}_n(R))$  is homologically trivial, and we expect that the natural map

$$B(q, \mathrm{GL}_{\infty}(R)) \rightarrow \mathrm{hocolim}_{\mathbb{I}} B(q, \mathrm{GL}_{-}(R))$$

is not a homology isomorphism.

In a similar way we can obtain a filtration of  $Q(\mathbb{S}^0)$ . For this note that the conjugation action of  $\Sigma_n$  on  $B\Sigma_n$  is homologically trivial. Therefore, by the Barratt–Priddy–Quillen theorem, the level zero component of  $Q(\mathbb{S}^0)$  is equivalent to the homotopy colimit over  $\mathbb{I}$  of the classifying spaces of the symmetric groups:

$$Q_0(\mathbb{S}^0) \simeq (B\Sigma_{\infty})^+ \simeq \mathrm{hofib} \rho^{B\Sigma_{-}} \simeq \mathrm{hocolim}_{\mathbb{I}} B\Sigma_{-}.$$

Consider the commutative  $\mathbb{I}$ -rig  $B(q, \Sigma_{-})$  defined by  $n \mapsto B(q, \Sigma_n)$ . The structural maps are given by conjugation of  $\Sigma_n$  and inclusions in an analogous way as above. Then by Theorem 2.2 we have a filtration of nonunital  $E_{\infty}$ -ring spaces

$$\mathrm{hocolim}_{\mathbb{I}} B(2, \Sigma_{-}) \subset \cdots \subset \mathrm{hocolim}_{\mathbb{I}} B(q, \Sigma_{-}) \subset \cdots \subset \mathrm{hocolim}_{\mathbb{I}} B\Sigma_{-} \simeq Q_0(\mathbb{S}^0).$$

As in the case of  $B(q, \mathrm{GL}_n(R))$ , the conjugation action of  $\Sigma_n$  on  $B(q, \Sigma_n)$  may fail to be homologically trivial (for example this is the case for the conjugation action of  $\Sigma_3$  on  $B(2, \Sigma_3)$ ; see [1]). The conditions of Theorem 3.3 are satisfied but the homotopy types of the spaces  $\mathrm{hocolim}_{\mathbb{I}} B(q, \Sigma_{-}) \simeq \mathrm{hofib} \rho^{B(q, \Sigma_{-})}$  remain to be determined.

**Corollary 4.5** *The spaces*

$$\begin{aligned}\mathrm{hocolim}_{\mathbb{I}} B(q, \mathrm{GL}_-(R)) &\simeq \mathrm{hofib} \rho^{B(q, \mathrm{GL}_-(R))}, \\ \mathrm{hocolim}_{\mathbb{I}} B(q, \Sigma_-) &\simeq \mathrm{hofib} \rho^{B(q, \Sigma_-)}\end{aligned}$$

*provide filtrations of nonunitary  $E_\infty$ -ring spaces with final target the classical nonunitary  $E_\infty$ -ring spaces  $B\mathrm{GL}_\infty(R)^+$  and  $Q_0(\mathbb{S}^0)$ .*

## 5 Transitional nilpotence, bundles and K-theory

In this section we extend the notions of transitionally commutative bundles and commutative K-theory as defined in [2] to more general  $q$ -nilpotent notions for  $q \geq 2$ , reflecting the filtration induced by the descending central series of the free groups. We will show that these geometrically defined theories are represented by the infinite loop spaces  $\mathbb{Z} \times B(q, U)$ .

**Definition 5.1** For a CW-complex  $X$  a principal  $G$ -bundle  $\pi: E \rightarrow X$  is said to have *transitional nilpotency class* at most  $q$  if there exists an open cover  $\{U_i\}_{i \in I}$  of  $X$  such that the bundle  $\pi: E \rightarrow X$  is trivial over each  $U_i$  and for every  $x \in X$  the group generated by the collection  $\{\rho_{i,j}(x)\}_{i,j}$  is a group of nilpotency class at most  $q$ . Here  $\rho_{i,j}: U_i \cap U_j \rightarrow G$  denotes the transition functions, and  $i$  and  $j$  run through all indices in  $I$  for which  $x \in U_i \cap U_j$ . The minimum of all such numbers  $q$  is said to be transitional nilpotency class of  $\pi: E \rightarrow X$ .

The principal  $G$ -bundle  $p_q: E(q, G) \rightarrow B(q, G)$  is universal for all principal  $G$ -bundles with transitional nilpotency class less than  $q$ .

**Theorem 5.2** Assume that  $G$  is an algebraic subgroup of  $\mathrm{GL}_N(\mathbb{C})$  for some  $N \geq 0$ ,  $X$  is a finite CW-complex and that  $\pi: E \rightarrow X$  is a principal  $G$ -bundle over  $X$ . Then, for any  $q \geq 2$ , the classifying map  $f: X \rightarrow BG$  of  $\pi$  factors through  $B(q, G)$  (up to homotopy) if and only if  $\pi$  has transitional nilpotency class less than  $q$ .

**Proof** The case  $q = 2$  was treated in [2, Theorem 2.2] and in fact this theorem is true for any Lie group in this case. The proof goes through verbatim also for  $q > 2$  using the fact that when  $G$  is an algebraic subgroup of  $\mathrm{GL}_N(\mathbb{C})$ , then the simplicial space  $B_*(q, G)$  is proper, as was pointed out in the proof of Theorem 4.4.  $\square$

As  $[\Sigma X, BG] = [X, \Omega BG]$  and the canonical map  $\Omega B(q, G) \rightarrow \Omega BG$  always admits a splitting up to homotopy, any principal  $G$ -bundle on a suspension  $\Sigma X$  has transitional nilpotency class less than  $q$  for all  $q$ . However, the nilpotency structure is not unique in general, not even up to isomorphism in the sense of the following definition:

**Definition 5.3** Let  $\pi_0: E_0 \rightarrow X$  and  $\pi_1: E_1 \rightarrow X$  be two principal  $G$ -bundles with transitional nilpotency class less than  $q$ . We say that these bundles are  $q$ -transitionally isomorphic if there exists a principal  $G$ -bundle  $p: E \rightarrow X \times [0, 1]$  with transitional nilpotency class less than  $q$  such that  $\pi_0 = p|_{p^{-1}(X \times \{0\})}$  and  $\pi_1 = p|_{p^{-1}(X \times \{1\})}$ .

A complex vector bundle  $\pi: E \rightarrow X$  is said to have transitional nilpotency class less than  $q$  if the corresponding frame bundle, under a fixed Hermitian metric on  $E$ , has transitional nilpotency class less than  $q$ . Theorem 4.2 can then be interpreted to say that any vector bundle is stably of transitional nilpotency class less than  $q$  for all  $q \geq 2$ , and there is a functorial choice of such a structure. The set  $\text{Vect}_{q\text{-nil}}(X)$  of  $q$ -transitionally isomorphism classes of complex vector bundles over  $X$  with transitional nilpotency class less than  $q$  is a monoid under the direct sum of vector bundles. The  $q$ -nilpotent  $K$ -theory of  $X$  is defined as the associated Grothendieck group.

**Definition 5.4**  $K_{q\text{-nil}}(X) := \text{Gr}(\text{Vect}_{q\text{-nil}}(X))$ .

Tensor products induce a natural multiplication on  $K_{q\text{-nil}}(X)$  just as in classical  $K$ -theory.

**Theorem 5.5** For any finite CW-complex  $X$  there is a natural isomorphism of rings

$$K_{q\text{-nil}}(X) \cong [X, \mathbb{Z} \times B(q, U)].$$

Hence, it is the zeroth term of a multiplicative generalized cohomology theory.

**Proof** Let  $X$  be a finite CW-complex. By working one path-connected component at a time, we may assume without loss of generality that  $X$  is path-connected. By Theorem 5.2,

$$\text{Vect}_{q\text{-nil}}(X) = \left[ X, \bigsqcup_{n \geq 0} B(q, U(n)) \right]$$

as abelian monoids, where the addition is induced by direct sum of matrices on the right hand side. Any injection  $\mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$  induces a linear injection  $\mathbb{C}^\infty \times \mathbb{C}^\infty \rightarrow \mathbb{C}^\infty$ , which in turn induces an  $H$ -space product on  $\mathbb{Z} \times B(q, U)$ . The natural inclusions  $B(q, U(n)) \rightarrow B(q, U)$  define a map

$$\left[ X, \bigsqcup_{n \geq 0} B(q, U(n)) \right] \rightarrow [X, \mathbb{Z} \times B(q, U)].$$

As the symmetric groups act by homotopy equivalences on  $B(q, U)$ , we see that the above map is compatible with the product structure on both sets, ie it is a map of

monoids. By the universal property of the Grothendieck construction, this map factors through a unique map of abelian groups

$$K_{q\text{-nil}}(X) \rightarrow [X, \mathbb{Z} \times B(q, U)].$$

As  $X$  is compact, any map  $X \rightarrow B(q, U)$  factors through some  $B(q, U(n))$  for some large enough  $n$ . Hence the above map is surjective.

To prove that it is injective, suppose that the image of  $[A] - [B] \in K_{q\text{-nil}}(X)$  in  $[X, \mathbb{Z} \times B(q, U)]$  is zero. Let us write  $f_B: X \rightarrow B(q, U)$  for the image of a map representing  $B$  in the colimit  $B(q, U) = \text{colim}_{n \in \mathbb{N}} B(q, U(n))$ . Since  $B(q, U)$  is a grouplike  $H$ -space, the induced product on  $\text{Map}(X, B(q, U))$  is also a grouplike  $H$ -space structure. Let  $f_{B'}: X \rightarrow B(q, U)$  be a homotopy inverse for  $f_B$  under this product. Since  $X$  is compact, we may factor  $f_{B'}$  through a finite stage of the colimit and find a corresponding bundle  $B'$  over  $X$  with transitional nilpotency class less than  $q$  which is classified by the map  $f_{B'}$ . It follows that  $B \oplus B'$  is stably  $q$ -transitionally isomorphic to the trivial bundle  $\epsilon_k$  of rank  $k = \dim B + \dim B'$ . By our assumption, we see that the image of  $[A \oplus B'] - [\epsilon_k]$  in  $[X, \mathbb{Z} \times B(q, U)]$  is also zero. This means that  $A \oplus B'$  is stably  $q$ -transitionally isomorphic to a trivial bundle, say  $A \oplus B' \oplus \epsilon_t \cong \epsilon_{k+t}$ . We then have the relation

$$[A] - [B] = [A \oplus B' \oplus \epsilon_t] - [\epsilon_{k+t}] = 0$$

in  $K_{q\text{-nil}}(X)$ , which completes the proof.  $\square$

This answers the question raised in [2] for  $q = 2$ . Moreover, we have a sequence of cohomology theories and maps between them,

$$K_{\text{com}}(X) = K_{2\text{-nil}}(X) \rightarrow K_{3\text{-nil}}(X) \rightarrow \cdots \rightarrow K_{q\text{-nil}}(X) \rightarrow \cdots \rightarrow K(X).$$

By Theorem 4.2, topological K-theory splits off  $q$ -nilpotent K-theory for all  $q \geq 2$ . These theories are not well understood and would seem to warrant further attention. For example in [2] it was shown that  $K_{\text{com}}(\mathbb{S}^i) \cong K(\mathbb{S}^i)$  for  $0 \leq i \leq 3$ , but that  $K_{\text{com}}(\mathbb{S}^4) \neq K(\mathbb{S}^4)$ .

We leave it to the reader to formulate  $q$ -nilpotent versions of real and hermitian K-theory.

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