# Special Lagrangian submanifolds with isolated conical singularities. III. Desingularization, the unobstructed case

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## 1 Introduction

*Special Lagrangian m-folds (SL m-folds)* are a distinguished class of real $m$-dimensional minimal submanifolds which may be defined in $\mathbb{C}^m$, or in *Calabi–Yau m-folds*, or more generally in *almost Calabi–Yau m-folds* (compact Kähler $m$-folds with trivial canonical bundle). We write an almost Calabi–Yau $m$-fold as $M$ or $(M, J, \omega, \Omega)$, where the manifold $M$ has complex structure $J$, Kähler form $\omega$ and holomorphic volume form $\Omega$.

This is the third in a series of five papers [10, 11, 12, 13] studying SL $m$-folds with *isolated conical singularities*. That is, we consider an SL $m$-fold $X$ in an almost Calabi–Yau $m$-fold $M$ for $m > 2$ with singularities at $x_1, \ldots, x_n$ in $M$, such that for some special Lagrangian cones $C_i$ in $T_{x_i}M \cong \mathbb{C}^m$ with $C_i \setminus \{0\}$ nonsingular, $X$ approaches $C_i$ near $x_i$ in an asymptotic $C^1$ sense. Readers are advised to begin with the final paper [13], which surveys the series, and applies the results to prove some conjectures.

The first paper [10] laid the foundations for the series, and studied the *regularity* of SL $m$-folds with conical singularities near their singular points. The second paper [11] discussed the *deformation theory* of compact SL $m$-folds $X$ with conical singularities in an almost Calabi–Yau $m$-fold $M$.

This paper and the sequel [12] study *desingularizations* of compact SL $m$-folds $X$ with conical singularities. That is, we construct a family of compact, *nonsingular* SL $m$-folds $\tilde{N}^t$ in $M$ for $t \in (0, \epsilon]$ such that $\tilde{N}^t \to X$ as $t \to 0$, in the sense of currents.

Having a good understanding of the singularities of special Lagrangian submanifolds will be essential in clarifying the Strominger–Yau–Zaslow conjecture on the Mirror Symmetry of Calabi–Yau 3-folds [20], and also in resolving conjectures made by the author [7] on defining new invariants of Calabi–Yau 3-folds by counting special Lagrangian homology 3-spheres with weights. The series aims to develop such an understanding for simple singularities of SL $m$-folds.

Here is the basic idea of the paper. Let $X$ be a compact SL $m$-fold with conical singularities $x_1, \ldots, x_n$ in an almost Calabi–Yau $m$-fold $(M, J, \omega, \Omega)$.

Choose an isomorphism $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ for $i = 1, \ldots, n$. Then there is a unique *SL cone* $C_i$ in $\mathbb{C}^m$ with $X$ asymptotic to $\upsilon_i(C_i)$ at $x_i$.

Let $L_i$ be an *Asymptotically Conical SL m-fold* (*AC SL m-fold*) in $\mathbb{C}^m$, asymptotic to $C_i$ at infinity. As $C_i$ is a cone it is invariant under dilations, so $tC_i = C_i$ for all $t > 0$. Thus $tL_i = \{t\,\mathbf{x} : \mathbf{x} \in L_i\}$ is also an AC SL $m$-fold asymptotic to $C_i$ for $t > 0$. We explicitly construct a 1-parameter family of compact, nonsingular *Lagrangian* $m$-folds $N^t$ in $(M, \omega)$ for $t \in (0, \delta)$ by gluing $tL_i$ into $X$ at $x_i$, using a partition of unity.

When $t$ is small, $N^t$ is close to being special Lagrangian (its phase is nearly constant), but also close to being singular (it has large curvature and small injectivity radius). We prove using analysis that for small $\epsilon \in (0, \delta)$ we can deform $N^t$ to a *special* Lagrangian $m$-fold $\tilde{N}^t$ in $M$ for all $t \in (0, \epsilon]$, using a small Hamiltonian deformation. The proof involves a delicate balancing act, showing that the advantage of being close to special Lagrangian outweighs the disadvantage of being nearly singular.

Here are some of the issues involved in doing this in full generality:

(i) To ensure $N^t$ and $\tilde{N}^t$ are connected, we suppose $X$ is connected. But $X' = X \setminus \{x_1, \ldots, x_n\}$, the nonsingular part of $X$, may not be connected. If it is not then the Laplacian $\Delta$ on $N^t$ has small positive eigenvalues, of size $O(t^{m-2})$. These cause analytic problems in constructing $\tilde{N}^t$.

(ii) Let $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$. Then $\Sigma_i$ is a compact $(m-1)$-manifold, and $L_i$ effectively has boundary $\Sigma_i$ at infinity. There are natural *cohomological invariants* $Y(L_i) \in H^1(\Sigma_i, \mathbb{R})$ and $Z(L_i) \in H^{m-1}(\Sigma_i, \mathbb{R})$. It turns out that there are *topological obstructions* to the existence of $N^t$ or $\tilde{N}^t$, involving the $Y(L_i)$ and $Z(L_i)$.

(iii) Let $\big\{(M, J^s, \omega^s, \Omega^s) : s \in \mathcal{F}\big\}$ be a smooth family of almost Calabi–Yau $m$-folds for $0 \in \mathcal{F} \subset \mathbb{R}^d$ with $(M, J^0, \omega^0, \Omega^0) = (M, J, \omega, \Omega)$. Then we can consider special Lagrangian desingularizations $\tilde{N}^{s,t}$ of $X$ not just in $(M, J, \omega, \Omega)$ but in $(M, J^s, \omega^s, \Omega^s)$ for small $s \in \mathcal{F}$. To do this introduces new analytic problems, and new topological obstructions involving the cohomology classes $[\omega^s]$ and $[\operatorname{Im}\Omega^s]$.

Rather than tackling these questions all at once, we prove our first main result in §6 assuming that $X'$ is connected, that $Y(L_i) = 0$ and $L_i$ converges quickly to $C_i$ at infinity in $\mathbb{C}^m$, and working in a single almost Calabi–Yau $m$-fold $(M, J, \omega, \Omega)$ rather than a family. This simplifies (i)–(iii) above.

Section 7 extends this to the case when $X'$ is not connected, as in (i), but still supposing $Y(L_i) = 0$ and $L_i$ converges quickly to $C_i$. The sequel [12] deals with issues (ii) and (iii), allowing $Y(L_i) \neq 0$ and $L_i$ to converge more slowly to $C_i$, and working in a family of almost Calabi–Yau $m$-folds $(M, J^s, \omega^s, \Omega^s)$.

We begin in §2 with an introduction to special Lagrangian geometry. Sections 3 and 4 discuss SL $m$-folds with conical singularities and Asymptotically Conical SL $m$-folds respectively, recalling results we will need from [10].

Given a compact Lagrangian $m$-fold $N$ in an almost Calabi–Yau $m$-fold $(M, J, \omega, \Omega)$ which is close to being special Lagrangian, §5 uses analysis to construct an SL $m$-fold $\tilde{N}$ as a small Hamiltonian deformation of $N$. This existence result, Theorem 5.3 below, can probably be used elsewhere. In each of §6 and §7 we construct a family of Lagrangian $m$-folds $N^t$ in $(M, J, \omega, \Omega)$, and apply Theorem 5.3 to show that $N^t$ can be deformed to an SL $m$-fold $\tilde{N}^t$ for small $t$.

For simplicity we generally take all submanifolds to be *embedded*. However, all our results generalize immediately to *immersed* submanifolds, with only cosmetic changes.

We conclude by discussing similar work by other authors. Salur [18, 19] considers a nonsingular, connected, immersed SL 3-fold $N$ in a Calabi–Yau 3-fold with a codimension two self-intersection along an $\mathcal{S}^1$, and constructs new SL 3-folds by smoothing along the $\mathcal{S}^1$.

Butscher [3] studies SL $m$-folds $N$ in $\mathbb{C}^m$ with boundary in a symplectic submanifold $W^{2m-2} \subset \mathbb{C}^m$. Given two such SL $m$-folds $N_1, N_2$ intersecting transversely at $x$ and satisfying an angle criterion, he constructs a 1-parameter family of connect sum SL $m$-folds $N_1 \#_x N_2$ in $\mathbb{C}^m$, with boundary, by gluing in an explicit AC SL $m$-fold $L$ in $\mathbb{C}^m$ due to Lawlor [14], diffeomorphic to $\mathcal{S}^{m-1} \times \mathbb{R}$ and asymptotic to the union of two SL planes $\mathbb{R}^m$ in $\mathbb{C}^m$.

Closest to the present paper is Lee [6]. She considers a compact, connected, immersed SL $m$-fold $N$ in a Calabi–Yau $m$-fold $M$ with transverse self-intersection points $x_1, \ldots, x_n$ satisfying an angle criterion. She shows that $N$ can be desingularized by gluing in one of Lawlor's AC SL $m$-folds $L_i$ at $x_i$ for $i = 1, \ldots, n$, to get a family of compact, embedded SL $m$-folds in $M$. Her result follows from Theorem 6.13 below.

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## 2 Special Lagrangian geometry

We introduce special Lagrangian submanifolds (SL $m$-folds) in two different geometric contexts. First we define SL $m$-folds in $\mathbb{C}^m$. Then we discuss SL $m$-folds in *almost Calabi–Yau $m$-folds*, compact Kähler manifolds equipped with a holomorphic volume form, which generalize Calabi–Yau manifolds. Some references for this section are Harvey and Lawson [4] and the author [9]. We begin by defining *calibrations* and *calibrated submanifolds*, following [4].

**Definition 2.1** Let $(M, g)$ be a Riemannian manifold. An *oriented tangent $k$-plane* $V$ on $M$ is a vector subspace $V$ of some tangent space $T_xM$ to $M$ with $\dim V = k$, equipped with an orientation. If $V$ is an oriented tangent $k$-plane on $M$ then $g|_V$ is a Euclidean metric on $V$, so combining $g|_V$ with the orientation on $V$ gives a natural *volume form* $\mathrm{vol}_V$ on $V$, which is a $k$-form on $V$.

Now let $\varphi$ be a closed $k$-form on $M$. We say that $\varphi$ is a *calibration* on $M$ if for every oriented $k$-plane $V$ on $M$ we have $\varphi|_V \leqslant \mathrm{vol}_V$. Here $\varphi|_V = \alpha \cdot \mathrm{vol}_V$

for some $\alpha \in \mathbb{R}$, and $\varphi|_V \leqslant \mathrm{vol}_V$ if $\alpha \leqslant 1$. Let $N$ be an oriented submanifold of $M$ with dimension $k$. Then each tangent space $T_xN$ for $x \in N$ is an oriented tangent $k$-plane. We say that $N$ is a *calibrated submanifold* if $\varphi|_{T_xN} = \mathrm{vol}_{T_xN}$ for all $x \in N$.

It is easy to show that calibrated submanifolds are automatically *minimal submanifolds* [4, Th. II.4.2]. Here is the definition of special Lagrangian submanifolds in $\mathbb{C}^m$, taken from [4, §III].

**Definition 2.2** Let $\mathbb{C}^m$ have complex coordinates $(z_1, \ldots, z_m)$, and define a metric $g'$, a real 2-form $\omega'$ and a complex $m$-form $\Omega'$ on $\mathbb{C}^m$ by

$$\begin{gathered} g' = |\mathrm{d}z_1|^2 + \cdots + |\mathrm{d}z_m|^2, \quad \omega' = \tfrac{i}{2}(\mathrm{d}z_1 \wedge \mathrm{d}\bar{z}_1 + \cdots + \mathrm{d}z_m \wedge \mathrm{d}\bar{z}_m), \\ \text{and} \quad \Omega' = \mathrm{d}z_1 \wedge \cdots \wedge \mathrm{d}z_m. \end{gathered} \tag{1}$$

Then $\mathrm{Re}\,\Omega'$ and $\mathrm{Im}\,\Omega'$ are real $m$-forms on $\mathbb{C}^m$. Let $L$ be an oriented real submanifold of $\mathbb{C}^m$ of real dimension $m$. We say that $L$ is a *special Lagrangian submanifold* of $\mathbb{C}^m$, or *SL m-fold* for short, if $L$ is calibrated with respect to $\mathrm{Re}\,\Omega'$, in the sense of Definition 2.1.

Harvey and Lawson [4, Cor. III.1.11] give the following alternative characterization of special Lagrangian submanifolds:

**Proposition 2.3** *Let $L$ be a real $m$-dimensional submanifold of $\mathbb{C}^m$. Then $L$ admits an orientation making it into an SL submanifold of $\mathbb{C}^m$ if and only if* $\omega'|_L \equiv 0$ *and* $\mathrm{Im}\,\Omega'|_L \equiv 0$.

An $m$-dimensional submanifold $L$ in $\mathbb{C}^m$ is called *Lagrangian* if $\omega'|_L \equiv 0$. Thus special Lagrangian submanifolds are Lagrangian submanifolds satisfying the extra condition that $\mathrm{Im}\,\Omega'|_L \equiv 0$, which is how they get their name. We shall define special Lagrangian submanifolds not just in Calabi–Yau manifolds, but in the much larger class of *almost Calabi–Yau manifolds*.

**Definition 2.4** Let $m \geqslant 2$. An *almost Calabi–Yau m-fold* is a quadruple $(M, J, \omega, \Omega)$ such that $(M, J)$ is a compact $m$-dimensional complex manifold, $\omega$ is the Kähler form of a Kähler metric $g$ on $M$, and $\Omega$ is a non-vanishing holomorphic $(m, 0)$-form on $M$.

We call $(M, J, \omega, \Omega)$ a *Calabi–Yau m-fold* if in addition $\omega$ and $\Omega$ satisfy

$$\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m \Omega \wedge \bar{\Omega}. \tag{2}$$

Then for each $x \in M$ there exists an isomorphism $T_xM \cong \mathbb{C}^m$ that identifies $g_x, \omega_x$ and $\Omega_x$ with the flat versions $g', \omega', \Omega'$ on $\mathbb{C}^m$ in (1). Furthermore, $g$ is Ricci-flat and its holonomy group is a subgroup of $\mathrm{SU}(m)$.

This is not the usual definition of a Calabi–Yau manifold, but is essentially equivalent to it.

**Definition 2.5** Let $(M, J, \omega, \Omega)$ be an almost Calabi–Yau $m$-fold, and $N$ a real $m$-dimensional submanifold of $M$. We call $N$ a *special Lagrangian submanifold*, or *SL m-fold* for short, if $\omega|_N \equiv \mathrm{Im}\,\Omega|_N \equiv 0$. It easily follows that $\mathrm{Re}\,\Omega|_N$ is a nonvanishing $m$-form on $N$. Thus $N$ is orientable, with a unique orientation in which $\mathrm{Re}\,\Omega|_N$ is positive.

Again, this is not the usual definition of SL $m$-fold, but is essentially equivalent to it. Suppose $(M, J, \omega, \Omega)$ is an almost Calabi–Yau $m$-fold, with metric $g$. Let $\psi : M \to (0, \infty)$ be the unique smooth function such that

$$\psi^{2m}\omega^m/m! = (-1)^{m(m-1)/2}(i/2)^m\Omega \wedge \bar{\Omega}, \tag{3}$$

and define $\tilde{g}$ to be the conformally equivalent metric $\psi^2 g$ on $M$. Then $\mathrm{Re}\,\Omega$ is a *calibration* on the Riemannian manifold $(M, \tilde{g})$, and SL $m$-folds $N$ in $(M, J, \omega, \Omega)$ are calibrated with respect to it, so that they are minimal with respect to $\tilde{g}$.

If $M$ is a Calabi–Yau $m$-fold then $\psi \equiv 1$ by (2), so $\tilde{g} = g$, and an $m$-submanifold $N$ in $M$ is special Lagrangian if and only if it is calibrated w.r.t. $\mathrm{Re}\,\Omega$ on $(M, g)$, as in Definition 2.2. This recovers the usual definition of special Lagrangian $m$-folds in Calabi–Yau $m$-folds.

# 3 SL $m$-folds with conical singularities

The preceding papers [10, 11] studied SL $m$-folds $X$ with *conical singularities* in an almost Calabi–Yau $m$-fold $(M, J, \omega, \Omega)$. We now recall the definitions and results from [10] that we will need later. For brevity we keep explanations to a minimum, and readers are referred to [10] for further details.

## 3.1 Preliminaries on special Lagrangian cones

Following [10, §2.1] we give definitions and results on *special Lagrangian cones*.

**Definition 3.1** A (singular) SL $m$-fold $C$ in $\mathbb{C}^m$ is called a *cone* if $C = tC$ for all $t > 0$, where $tC = \{t\,\mathbf{x} : \mathbf{x} \in C\}$. Let $C$ be a closed SL cone in $\mathbb{C}^m$ with an isolated singularity at 0. Then $\Sigma = C \cap \mathcal{S}^{2m-1}$ is a compact, nonsingular $(m-1)$-submanifold of $\mathcal{S}^{2m-1}$, not necessarily connected. Let $g_\Sigma$ be the restriction of $g'$ to $\Sigma$, where $g'$ is as in (1).

Set $C' = C \setminus \{0\}$. Define $\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then $\iota$ has image $C'$. By an abuse of notation, *identify* $C'$ with $\Sigma \times (0, \infty)$ using $\iota$. The *cone metric* on $C' \cong \Sigma \times (0, \infty)$ is $g' = \iota^*(g') = \mathrm{d}r^2 + r^2 g_\Sigma$.

For $\alpha \in \mathbb{R}$, we say that a function $u : C' \to \mathbb{R}$ is *homogeneous of order* $\alpha$ if $u \circ t \equiv t^\alpha u$ for all $t > 0$. Equivalently, $u$ is homogeneous of order $\alpha$ if $u(\sigma, r) \equiv r^\alpha v(\sigma)$ for some function $v : \Sigma \to \mathbb{R}$.

In [10, Lem. 2.3] we study *homogeneous harmonic functions* on $C'$.

**Lemma 3.2** *In the situation of Definition 3.1, let $u(\sigma, r) \equiv r^\alpha v(\sigma)$ be a homogeneous function of order $\alpha$ on $C' = \Sigma \times (0, \infty)$, for $v \in C^2(\Sigma)$. Then*

$$\Delta u(\sigma, r) = r^{\alpha-2}\bigl(\Delta_\Sigma v - \alpha(\alpha + m - 2)v\bigr),$$

*where $\Delta$, $\Delta_\Sigma$ are the Laplacians on $(C', g')$ and $(\Sigma, g_\Sigma)$. Hence, $u$ is harmonic on $C'$ if and only if $v$ is an eigenfunction of $\Delta_\Sigma$ with eigenvalue $\alpha(\alpha+m-2)$.*

Following [10, Def. 2.5], we define:

**Definition 3.3** In the situation of Definition 3.1, suppose $m > 2$ and define

$$\mathcal{D}_\Sigma = \bigl\{\alpha \in \mathbb{R} : \alpha(\alpha+m-2) \text{ is an eigenvalue of } \Delta_\Sigma\bigr\}. \tag{4}$$

Then $\mathcal{D}_\Sigma$ is a countable, discrete subset of $\mathbb{R}$. By Lemma 3.2, an equivalent definition is that $\mathcal{D}_\Sigma$ is the set of $\alpha \in \mathbb{R}$ for which there exists a nonzero homogeneous harmonic function $u$ of order $\alpha$ on $C'$.

Define $m_\Sigma : \mathcal{D}_\Sigma \to \mathbb{N}$ by taking $m_\Sigma(\alpha)$ to be the multiplicity of the eigenvalue $\alpha(\alpha+m-2)$ of $\Delta_\Sigma$, or equivalently the dimension of the vector space of homogeneous harmonic functions $u$ of order $\alpha$ on $C'$. Define $N_\Sigma : \mathbb{R} \to \mathbb{Z}$ by

$$N_\Sigma(\delta) = -\sum_{\alpha \in \mathcal{D}_\Sigma \cap (\delta,0)} m_\Sigma(\alpha) \;\text{ if } \delta < 0, \text{ and } \; N_\Sigma(\delta) = \sum_{\alpha \in \mathcal{D}_\Sigma \cap [0,\delta]} m_\Sigma(\alpha) \;\text{ if } \delta \geqslant 0.$$

Then $N_\Sigma$ is monotone increasing and upper semicontinuous, and is discontinuous exactly on $\mathcal{D}_\Sigma$, increasing by $m_\Sigma(\alpha)$ at each $\alpha \in \mathcal{D}_\Sigma$. As the eigenvalues of $\Delta_\Sigma$ are nonnegative, we see that $\mathcal{D}_\Sigma \cap (2-m, 0) = \emptyset$ and $N_\Sigma \equiv 0$ on $(2-m, 0)$.

## 3.2 The definition of SL $m$-folds with conical singularities

Now we can define *conical singularities* of SL $m$-folds, following [10, Def. 3.6].

**Definition 3.4** Let $(M, J, \omega, \Omega)$ be an almost Calabi–Yau $m$-fold for $m > 2$, and define $\psi : M \to (0, \infty)$ as in (3). Suppose $X$ is a compact singular SL $m$-fold in $M$ with singularities at distinct points $x_1, \ldots, x_n \in X$, and no other singularities.

Fix isomorphisms $\upsilon_i : \mathbb{C}^m \to T_{x_i}M$ for $i = 1, \ldots, n$ such that $\upsilon_i^*(\omega) = \omega'$ and $\upsilon_i^*(\Omega) = \psi(x_i)^m \Omega'$, where $\omega', \Omega'$ are as in (1). Let $C_1, \ldots, C_n$ be SL cones in $\mathbb{C}^m$ with isolated singularities at 0. For $i = 1, \ldots, n$ let $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$, and let $\mu_i \in (2,3)$ with $(2, \mu_i] \cap \mathcal{D}_{\Sigma_i} = \emptyset$, where $\mathcal{D}_{\Sigma_i}$ is defined in (4). Then we say that $X$ has a *conical singularity* at $x_i$, with *rate* $\mu_i$ and *cone* $C_i$ for $i = 1, \ldots, n$, if the following holds.

By Darboux' Theorem [15, Th. 3.15] there exist embeddings $\Upsilon_i : B_R \to M$ for $i = 1, \ldots, n$ satisfying $\Upsilon_i(0) = x_i$, $\mathrm{d}\Upsilon_i|_0 = \upsilon_i$ and $\Upsilon_i^*(\omega) = \omega'$, where $B_R$ is the open ball of radius $R$ about 0 in $\mathbb{C}^m$ for some small $R > 0$. Define $\iota_i : \Sigma_i \times (0, R) \to B_R$ by $\iota_i(\sigma, r) = r\sigma$ for $i = 1, \ldots, n$.

Define $X' = X \setminus \{x_1, \ldots, x_n\}$. Then there should exist a compact subset $K \subset X'$ such that $X' \setminus K$ is a union of open sets $S_1, \ldots, S_n$ with $S_i \subset \Upsilon_i(B_R)$, whose closures $\bar{S}_1, \ldots, \bar{S}_n$ are disjoint in $X$. For $i = 1, \ldots, n$ and some $R' \in (0, R]$ there should exist a smooth $\phi_i : \Sigma_i \times (0, R') \to B_R$ such that $\Upsilon_i \circ \phi_i : \Sigma_i \times (0, R') \to M$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i$, and

$$\bigl|\nabla^k(\phi_i - \iota_i)\bigr| = O(r^{\mu_i - 1 - k}) \quad \text{as } r \to 0 \text{ for } k = 0, 1. \tag{5}$$

Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (0, R')$.

The reasoning behind this definition was discussed in [10, §3.3]. We suppose $m > 2$ for two reasons. Firstly, the only SL cones in $\mathbb{C}^2$ are finite unions of SL planes $\mathbb{R}^2$ in $\mathbb{C}^2$ intersecting only at 0. Thus any SL 2-fold with conical singularities is actually *nonsingular* as an immersed 2-fold. Secondly, $m = 2$ is a special case in the analysis of [10, §2], and it is simpler to exclude it. Therefore we will suppose $m > 2$ throughout the paper.

### 3.3 Lagrangian Neighbourhood Theorems and regularity

We recall some symplectic geometry, which can be found in McDuff and Salamon [15]. Let $N$ be a real $m$-manifold. Then its tangent bundle $T^*N$ has a *canonical symplectic form* $\hat\omega$, defined as follows. Let $(x_1, \ldots, x_m)$ be local coordinates on $N$. Extend them to local coordinates $(x_1, \ldots, x_m, y_1, \ldots, y_m)$ on $T^*N$ such that $(x_1, \ldots, y_m)$ represents the 1-form $y_1 \mathrm{d}x_1 + \cdots + y_m \mathrm{d}x_m$ in $T^*_{(x_1,\ldots,x_m)}N$. Then $\hat\omega = \mathrm{d}x_1 \wedge \mathrm{d}y_1 + \cdots + \mathrm{d}x_m \wedge \mathrm{d}y_m$.

Identify $N$ with the zero section in $T^*N$. Then $N$ is a *Lagrangian submanifold* of $T^*N$. The *Lagrangian Neighbourhood Theorem* [15, Th. 3.33] shows that any compact Lagrangian submanifold $N$ in a symplectic manifold looks locally like the zero section in $T^*N$.

**Theorem 3.5** *Let $(M, \omega)$ be a symplectic manifold and $N \subset M$ a compact Lagrangian submanifold. Then there exists an open tubular neighbourhood $U$ of the zero section $N$ in $T^*N$, and an embedding $\Phi : U \to M$ with $\Phi|_N = \mathrm{id} : N \to N$ and $\Phi^*(\omega) = \hat\omega$, where $\hat\omega$ is the canonical symplectic structure on $T^*N$.*

In [10, §4] we extend Theorem 3.5 to situations involving conical singularities, first to *SL cones*, [10, Th. 4.3].

**Theorem 3.6** *Let $C$ be an SL cone in $\mathbb{C}^m$ with isolated singularity at $0$, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Define $\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$, with image $C \setminus \{0\}$. For $\sigma \in \Sigma$, $\tau \in T^*_\sigma\Sigma$, $r \in (0, \infty)$ and $u \in \mathbb{R}$, let $(\sigma, r, \tau, u)$ represent the point $\tau + u\,\mathrm{d}r$ in $T^*_{(\sigma,r)}\bigl(\Sigma \times (0, \infty)\bigr)$. Identify $\Sigma \times (0, \infty)$ with the zero section $\tau = u = 0$ in $T^*\bigl(\Sigma \times (0, \infty)\bigr)$. Define an action of $(0, \infty)$ on $T^*\bigl(\Sigma \times (0, \infty)\bigr)$ by*

$$t : (\sigma, r, \tau, u) \longmapsto (\sigma, tr, t^2\tau, tu) \quad \textit{for } t \in (0, \infty), \tag{6}$$

*so that $t^*(\hat\omega) = t^2\hat\omega$, for $\hat\omega$ the canonical symplectic structure on $T^*\bigl(\Sigma \times (0, \infty)\bigr)$.*

*Then there exists an open neighbourhood $U_C$ of $\Sigma \times (0, \infty)$ in $T^*\bigl(\Sigma \times (0, \infty)\bigr)$ invariant under (6) given by*

$$U_C = \bigl\{(\sigma, r, \tau, u) \in T^*\bigl(\Sigma \times (0, \infty)\bigr) : \bigl|(\tau, u)\bigr| < 2\zeta r\bigr\} \quad \textit{for some } \zeta > 0,$$

*where $|\,.\,|$ is calculated using the cone metric $\iota^*(g')$ on $\Sigma \times (0, \infty)$, and an embedding $\Phi_C : U_C \to \mathbb{C}^m$ with $\Phi_C|_{\Sigma \times (0,\infty)} = \iota$, $\Phi_C^*(\omega') = \hat\omega$ and $\Phi_C \circ t = t\,\Phi_C$ for all $t > 0$, where $t$ acts on $U_C$ as in (6) and on $\mathbb{C}^m$ by multiplication.*

In [10, Th. 4.4] we construct a particular choice of $\phi_i$ in Definition 3.4.

**Theorem 3.7** *Let $(M,J,\omega,\Omega)$, $\psi, X, n, x_i, \upsilon_i, C_i, \Sigma_i, \mu_i, R, \Upsilon_i$ and $\iota_i$ be as in Definition 3.4. Theorem 3.6 gives $\zeta>0$, neighbourhoods $U_{C_i}$ of $\Sigma_i\times(0,\infty)$ in $T^*\bigl(\Sigma_i\times(0,\infty)\bigr)$ and embeddings $\Phi_{C_i}:U_{C_i}\rightarrow\mathbb{C}^m$ for $i=1,\ldots,n$.*

*Then for sufficiently small $R'\in(0,R]$ there exist unique closed 1-forms $\eta_i$ on $\Sigma_i\times(0,R')$ for $i=1,\ldots,n$ written $\eta_i(\sigma,r)=\eta_i^1(\sigma,r)+\eta_i^2(\sigma,r)\mathrm{d}r$ for $\eta_i^1(\sigma,r)\in T^*_\sigma\Sigma_i$ and $\eta_i^2(\sigma,r)\in\mathbb{R}$, and satisfying $\vert\eta_i(\sigma,r)\vert<\zeta r$ and $\vert\nabla^k\eta_i\vert=O(r^{\mu_i-1-k})$ as $r\rightarrow 0$ for $k=0,1$, computing $\nabla,\vert\,.\,\vert$ using the cone metric $\iota_i^*(g')$, such that the following holds.*

*Define $\phi_i:\Sigma_i\times(0,R')\rightarrow B_R$ by $\phi_i(\sigma,r)=\Phi_{C_i}\bigl(\sigma,r,\eta_i^1(\sigma,r),\eta_i^2(\sigma,r)\bigr)$. Then $\Upsilon_i\circ\phi_i:\Sigma_i\times(0,R')\rightarrow M$ is a diffeomorphism $\Sigma_i\times(0,R')\rightarrow S_i$ for open sets $S_1,\ldots,S_n$ in $X'$ with $\bar S_1,\ldots,\bar S_n$ disjoint, and $K=X'\setminus(S_1\cup\cdots\cup S_n)$ is compact. Also $\phi_i$ satisfies (5), so that $R',\phi_i,S_i,K$ satisfy Definition 3.4.*

In [10, §5] we study the asymptotic behaviour of the maps $\phi_i$ of Theorem 3.7, using the elliptic regularity of the special Lagrangian condition. Combining [10, Th. 5.1], [10, Lem. 4.5] and [10, Th. 5.5] proves:

**Theorem 3.8** *In the situation of Theorem 3.7 we have $\eta_i=\mathrm{d}A_i$ for $i=1,\ldots,n$, where $A_i:\Sigma_i\times(0,R')\rightarrow\mathbb{R}$ is given by $A_i(\sigma,r)=\int_0^r\eta_i^2(\sigma,s)\mathrm{d}s$. Suppose $\mu_i'\in(2,3)$ with $(2,\mu_i']\cap\mathcal{D}_{\Sigma_i}=\emptyset$ for $i=1,\ldots,n$. Then*

$$\begin{aligned}&\bigl\vert\nabla^k(\phi_i-\iota_i)\bigr\vert=O(r^{\mu_i'-1-k}),\quad \bigl\vert\nabla^k\eta_i\bigr\vert=O(r^{\mu_i'-1-k})\quad\text{and}\\&\bigl\vert\nabla^k A_i\bigr\vert=O(r^{\mu_i'-k})\quad\text{as } r\rightarrow 0 \text{ for all } k\geqslant 0 \text{ and } i=1,\ldots,n.\end{aligned}\tag{7}$$

*Hence $X$ has conical singularities at $x_i$ with cone $C_i$ and rate $\mu_i'$, for all possible rates $\mu_i'$ allowed by Definition 3.4. Therefore, the definition of conical singularities is essentially independent of the choice of rate $\mu_i$.*

Finally we extend Theorem 3.5 to SL $m$-folds with conical singularities [10, Th. 4.6], in a way compatible with Theorems 3.6 and 3.7.

**Theorem 3.9** *Suppose $(M,J,\omega,\Omega)$ is an almost Calabi–Yau $m$-fold and $X$ a compact SL $m$-fold in $M$ with conical singularities at $x_1,\ldots,x_n$. Let the notation $\psi,\upsilon_i,C_i,\Sigma_i,\mu_i,R,\Upsilon_i$ and $\iota_i$ be as in Definition 3.4, and let $\zeta,U_{C_i},\Phi_{C_i},R'$, $\eta_i,\eta_i^1,\eta_i^2,\phi_i,S_i$ and $K$ be as in Theorem 3.7.*

*Then making $R'$ smaller if necessary, there exists an open tubular neighbourhood $U_{X'}\subset T^*X'$ of the zero section $X'$ in $T^*X'$, such that under $\mathrm{d}(\Upsilon_i\circ\phi_i):T^*\bigl(\Sigma_i\times(0,R')\bigr)\rightarrow T^*X'$ for $i=1,\ldots,n$ we have*

$$\bigl(\mathrm{d}(\Upsilon_i\circ\phi_i)\bigr)^*(U_{X'})=\bigl\{(\sigma,r,\tau,u)\in T^*\bigl(\Sigma_i\times(0,R')\bigr):\bigl\vert(\tau,u)\bigr\vert<\zeta r\bigr\},\tag{8}$$

*and there exists an embedding $\Phi_{X'}:U_{X'}\rightarrow M$ with $\Phi_{X'}\vert_{X'}=\mathrm{id}:X'\rightarrow X'$ and $\Phi_{X'}^*(\omega)=\hat\omega$, where $\hat\omega$ is the canonical symplectic structure on $T^*X'$, such that*

$$\Phi_{X'}\circ\mathrm{d}(\Upsilon_i\circ\phi_i)(\sigma,r,\tau,u)\equiv\Upsilon_i\circ\Phi_{C_i}\bigl(\sigma,r,\tau+\eta_i^1(\sigma,r),u+\eta_i^2(\sigma,r)\bigr)\tag{9}$$

*for all $i=1,\ldots,n$ and $(\sigma,r,\tau,u)\in T^*\bigl(\Sigma_i\times(0,R')\bigr)$ with $\bigl\vert(\tau,u)\bigr\vert<\zeta r$. Here $\vert(\tau,u)\vert$ is computed using the cone metric $\iota_i^*(g')$ on $\Sigma_i\times(0,R')$.*

# 4 Asymptotically Conical SL $m$-folds

Let $C$ be an SL cone in $\mathbb{C}^m$ with an isolated singularity at 0. Section 3 considered SL $m$-folds with conical singularities, which are asymptotic to $C$ at 0. We now discuss *Asymptotically Conical* SL $m$-folds $L$ in $\mathbb{C}^m$, which are asymptotic to $C$ at infinity. Here is the definition.

**Definition 4.1** Let $C$ be an SL cone in $\mathbb{C}^m$ with isolated singularity at 0 for $m > 2$, as in Definition 3.1, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$, so that $\Sigma$ is a compact, nonsingular $(m-1)$-manifold, not necessarily connected. Let $g_\Sigma$ be the metric on $\Sigma$ induced by the metric $g'$ on $\mathbb{C}^m$ in (1), and $r$ the radius function on $\mathbb{C}^m$. Define $\iota : \Sigma \times (0, \infty) \to \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Then the image of $\iota$ is $C \setminus \{0\}$, and $\iota^*(g') = r^2 g_\Sigma + \mathrm{d}r^2$ is the cone metric on $C \setminus \{0\}$.

Let $L$ be a closed, nonsingular SL $m$-fold in $\mathbb{C}^m$ and $\lambda < 2$. We call $L$ *Asymptotically Conical (AC)* with *rate* $\lambda$ and *cone* $C$ if there exists a compact subset $K \subset L$ and a diffeomorphism $\varphi : \Sigma \times (T, \infty) \to L \setminus K$ for some $T > 0$, such that

$$\left|\nabla^k(\varphi - \iota)\right| = O(r^{\lambda-1-k}) \quad \text{as } r \to \infty \text{ for } k = 0, 1. \tag{10}$$

Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota^*(g')$ on $\Sigma \times (T, \infty)$.

This is very similar to Definition 3.4, and in fact there are strong parallels between the theories of SL $m$-folds with conical singularities and of AC SL $m$-folds. We recall some results from [10, §7], including versions of the material in §3.3. We continue to assume $m > 2$ throughout.

## 4.1 Cohomological invariants of AC SL $m$-folds

When $Y$ is a manifold, write $H^k(Y, \mathbb{R})$ for the $k^{\text{th}}$ *de Rham cohomology group* of $Y$, and $H_k(Y, \mathbb{R})$ for the $k^{\text{th}}$ *real singular homology group* of $Y$, defined using smooth simplices. Then the pairing between homology and cohomology is defined at the chain level by integrating $k$-forms over $k$-simplices. We can also define *relative* homology and cohomology groups in the usual way. The *Betti numbers* of $Y$ are $b^k(Y) = \dim H^k(Y, \mathbb{R})$.

Let $L$ be an AC SL $m$-fold in $\mathbb{C}^m$ with cone $C$, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. As $\Sigma$ is in effect the boundary of $L$, there is a natural map $H^k(L, \mathbb{R}) \to H^k(\Sigma, \mathbb{R})$. Following [10, Def. 7.2] we define *cohomological invariants* $Y(L), Z(L)$ of $L$.

**Definition 4.2** Let $L$ be an AC SL $m$-fold in $\mathbb{C}^m$ with cone $C$, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. As $\omega', \operatorname{Im}\Omega'$ in (1) are closed forms with $\omega'|_L \equiv \operatorname{Im}\Omega'|_L = 0$, they define classes in the relative de Rham cohomology groups $H^k(\mathbb{C}^m; L, \mathbb{R})$ for $k = 2, m$. For $k > 1$ we have the exact sequence

$$0 = H^{k-1}(\mathbb{C}^m, \mathbb{R}) \to H^{k-1}(L, \mathbb{R}) \xrightarrow{\cong} H^k(\mathbb{C}^m; L, \mathbb{R}) \to H^k(\mathbb{C}^m, \mathbb{R}) = 0.$$

Define $Y(L) \in H^1(\Sigma, \mathbb{R})$ to be the image of $[\omega']$ in $H^2(\mathbb{C}^m; L, \mathbb{R}) \cong H^1(L, \mathbb{R})$ under $H^1(L, \mathbb{R}) \to H^1(\Sigma, R)$, and $Z(L) \in H^{m-1}(\Sigma, \mathbb{R})$ to be the image of $[\operatorname{Im}\Omega']$ in $H^m(\mathbb{C}^m; L, \mathbb{R}) \cong H^{m-1}(L, \mathbb{R})$ under $H^{m-1}(L, \mathbb{R}) \to H^{m-1}(\Sigma, R)$.

Here are some conditions for $Y(L)$ or $Z(L)$ to be zero, [10, Prop. 7.3].

**Proposition 4.3** *Let $L$ be an AC SL $m$-fold in $\mathbb{C}^m$ with cone $C$ and rate $\lambda$, and let $\Sigma = C \cap \mathcal{S}^{2m-1}$. If $\lambda < 0$ or $b^1(L) = 0$ then $Y(L) = 0$. If $\lambda < 2 - m$ or $b^0(\Sigma) = 1$ then $Z(L) = 0$.*

In this paper we will consider only AC SL $m$-folds $L_i$ with rates $\lambda_i < 0$. These all have $Y(L_i) = 0$ by the proposition. Because of this we shall avoid some tricky issues of global symplectic topology in defining Lagrangian $m$-folds $N^t$ by gluing $tL_i$ in at a singular point $x_i$ of an SL $m$-fold $X$ with conical singularities, so §6 and §7 are simplified. The case $Y(L_i) \neq 0$ will be considered in the sequel [12]. Here is a (trivial) lemma on *dilations* of AC SL $m$-folds.

**Lemma 4.4** *Let $L$ be an AC SL $m$-fold in $\mathbb{C}^m$ with rate $\lambda$ and cone $C$, and let $t > 0$. Then $tL = \{t\,\mathbf{x} : \mathbf{x} \in L\}$ is also an AC SL $m$-fold in $\mathbb{C}^m$ with rate $\lambda$ and cone $C$, satisfying $Y(tL) = t^2 Y(L)$ and $Z(tL) = t^m Z(L)$.*

## 4.2 Lagrangian Neighbourhood Theorems and regularity

Next we generalize §3.3 to AC SL $m$-folds. Here is the analogue of Theorem 3.7, proved in [10, Th. 7.4].

**Theorem 4.5** *Let $C$ be an SL cone in $\mathbb{C}^m$ with isolated singularity at $0$, and set $\Sigma = C \cap \mathcal{S}^{2m-1}$. Define $\iota : \Sigma \times (0,\infty) \rightarrow \mathbb{C}^m$ by $\iota(\sigma, r) = r\sigma$. Let $\zeta$, $U_C \subset T^*\bigl(\Sigma \times (0,\infty)\bigr)$ and $\Phi_C : U_C \rightarrow \mathbb{C}^m$ be as in Theorem 3.6.*

*Suppose $L$ is an AC SL $m$-fold in $\mathbb{C}^m$ with cone $C$ and rate $\lambda < 2$. Then there exists a compact $K \subset L$ and a diffeomorphism $\varphi : \Sigma \times (T,\infty) \rightarrow L \setminus K$ for some $T > 0$ satisfying (10), and a closed $1$-form $\chi$ on $\Sigma \times (T,\infty)$ written $\chi(\sigma,r) = \chi^1(\sigma,r) + \chi^2(\sigma,r)\mathrm{d}r$ for $\chi^1(\sigma,r) \in T^*_\sigma\Sigma$ and $\chi^2(\sigma,r) \in \mathbb{R}$, satisfying*

$$\begin{aligned} &\bigl|\chi(\sigma,r)\bigr| < \zeta r, \quad \varphi(\sigma,r) \equiv \Phi_C\bigl(\sigma, r, \chi^1(\sigma,r), \chi^2(\sigma,r)\bigr) \\ &\text{and} \quad \bigl|\nabla^k\chi\bigr| = O(r^{\lambda-1-k}) \quad \text{as } r \rightarrow \infty \text{ for } k = 0, 1, \end{aligned} \tag{11}$$

*computing $\nabla, |\,.\,|$ using the cone metric $\iota^*(g')$.*

Now suppose that the rate $\lambda$ of $L$ satisfies $\lambda < 0$. Then $Y(L) = 0$ by Proposition 4.3, and the results of [10, §7.3] simplify. Combining [10, Prop. 7.6], [10, Th. 7.7] and [10, Th. 7.11] gives an analogue of Theorem 3.8.

**Theorem 4.6** *In the situation of Theorem 4.5, suppose $\lambda < 0$. Then $\chi = \mathrm{d}E$, where $E \in C^\infty\bigl(\Sigma \times (T,\infty)\bigr)$ is given by $E(\sigma,r) = -\int_r^\infty \chi^2(\sigma,s)\mathrm{d}s$. If either $\lambda = \lambda'$, or $\lambda' \in (2-m, 0)$, or $\lambda, \lambda'$ lie in the same connected component of $\mathbb{R} \setminus \mathcal{D}_\Sigma$, then $L$ is an AC SL $m$-fold with rate $\lambda'$ and*

$$\begin{aligned} &\bigl|\nabla^k(\varphi - \iota)\bigr| = O(r^{\lambda'-1-k}), \quad \bigl|\nabla^k\chi\bigr| = O(r^{\lambda'-1-k}) \quad \text{and} \\ &\bigl|\nabla^k E\bigr| = O(r^{\lambda'-k}) \quad \text{as } r \rightarrow \infty \text{ for all } k \geqslant 0. \end{aligned} \tag{12}$$

*Here $\nabla, |\,.\,|$ are computed using the cone metric $\iota^*(g')$ on $\Sigma \times (T,\infty)$.*

In particular, this shows that any AC SL $m$-fold $L$ with rate $\lambda < 0$ is *also* an AC SL $m$-fold with rate $\lambda'$ for $\lambda' \in \bigl(2-m, \frac{1}{2}(2-m)\bigr)$. This will be important in §6.2, where we need to assume that $\lambda < \frac{1}{2}(2-m)$ to make an error term sufficiently small. Here [10, Th. 7.5] is the analogue of Theorem 3.9.

**Theorem 4.7** *Suppose $L$ is an AC SL $m$-fold in $\mathbb{C}^m$ with cone $C$. Let $\Sigma, \iota, \zeta, U_C, \Phi_C, K, T, \varphi, \chi, \chi^1, \chi^2$ be as in Theorem 4.5. Then making $T, K$ larger if necessary, there exists an open tubular neighbourhood $U_L \subset T^*L$ of the zero section $L$ in $T^*L$, such that under $\mathrm{d}\varphi : T^*\bigl(\Sigma \times (T,\infty)\bigr) \to T^*L$ we have*

$$(\mathrm{d}\varphi)^*(U_L) = \bigl\{(\sigma, r, \tau, u) \in T^*\bigl(\Sigma \times (T,\infty)\bigr) : \bigl|(\tau, u)\bigr| < \zeta r\bigr\}, \qquad (13)$$

*and there exists an embedding $\Phi_L : U_L \to \mathbb{C}^m$ with $\Phi_L|_L = \mathrm{id} : L \to L$ and $\Phi_L^*(\omega') = \hat\omega$, where $\hat\omega$ is the canonical symplectic structure on $T^*L$, such that*

$$\Phi_L \circ \mathrm{d}\varphi(\sigma, r, \tau, u) \equiv \Phi_C\bigl(\sigma, r, \tau + \chi^1(\sigma, r), u + \chi^2(\sigma, r)\bigr) \qquad (14)$$

*for all $(\sigma, r, \tau, u) \in T^*\bigl(\Sigma \times (T,\infty)\bigr)$ with $|(\tau, u)| < \zeta r$, computing $|\,.\,|$ using $\iota^*(g')$.*

In [10, Th. 7.10] we study the *bounded harmonic functions* on $L$.

**Theorem 4.8** *Suppose $L$ is an AC SL $m$-fold in $\mathbb{C}^m$, with cone $C$. Let $\Sigma, T$ and $\varphi$ be as in Theorem 4.5. Let $l = b^0(\Sigma)$, and $\Sigma^1, \ldots, \Sigma^l$ be the connected components of $\Sigma$. Let $V$ be the vector space of bounded harmonic functions on $L$. Then $\dim V = l$, and for each $\mathbf{c} = (c^1, \ldots, c^l) \in \mathbb{R}^l$ there exists a unique $v^{\mathbf{c}} \in V$ such that for all $j = 1, \ldots, l$, $k \geqslant 0$ and $\beta \in (2-m, 0)$ we have*

$$\nabla^k\bigl(\varphi^*(v^{\mathbf{c}}) - c^j\,\bigr) = O\bigl(|\mathbf{c}| r^{\beta - k}\bigr) \quad \text{on } \Sigma^j \times (T,\infty) \text{ as } r \to \infty.$$

*Note also that $V = \{v^{\mathbf{c}} : \mathbf{c} \in \mathbb{R}^l\}$ and $v^{(1,\ldots,1)} \equiv 1$.*

# 5 An analytic existence result for SL $m$-folds

We shall now use analysis to prove that under certain conditions a compact, nonsingular Lagrangian $m$-fold $N$ in an almost Calabi–Yau $m$-fold $M$ which is approximately special Lagrangian can be deformed to a nearby special Lagrangian $m$-fold $\tilde N$ in $M$. We begin in §5.1 with some background material from analysis. The main result, Theorem 5.3, is stated in §5.2, and proved in §5.3–§5.5.

Theorem 5.3 and its proof are based on results by the author [8, Th. 11.6.1 & Th. 13.6.1], which are used to construct compact 7- and 8-manifolds $M$ with holonomy $G_2$ and Spin(7) by deforming a $G_2$- or Spin(7)-structure with small torsion on $M$. The geometry is rather different, but the underlying conception and structure of the proof is the same.

In each of §6 and §7 we will construct a family of compact, nonsingular Lagrangian $m$-folds $N^t$ in $M$ for $t \in (0, \delta)$ by gluing AC SL $m$-folds $L_1, \ldots, L_n$

in at the singular points $x_1, \ldots, x_n$ of a compact SL $m$-fold $X$ in $M$ with conical singularities. We then apply Theorem 5.3 to show that $N^t$ can be deformed to a nearby compact, nonsingular SL $m$-fold $\tilde{N}^t$ in $M$ for small $t$.

The proof of Theorem 5.3 is long and technical, and some readers may prefer to skip over it. The rest of the paper will use only the statement of Theorem 5.3, and not refer to its proof in §5.3–§5.5.

## 5.1 Banach spaces of functions

Let $(N, g)$ be a Riemannian manifold. To establish notation, we shall define various Banach spaces of functions on $N$. Some references for these spaces are Aubin [1] and Gilbarg and Trudinger [2]. For each integer $k \geqslant 0$, define $C^k(N)$ to be the vector space of continuous, bounded functions $f$ on $N$ that have $k$ continuous, bounded derivatives, and define the norm $\|.\|_{C^k}$ on $C^k(N)$ by $\|f\|_{C^k} = \sum_{j=0}^k \sup_N \left|\nabla^j f\right|$, where $\nabla$ is the Levi-Civita connection. Then $C^k(N)$ is a Banach space. Let $C^\infty(N) = \bigcap_{k\geqslant 0} C^k(N)$.

For $k \geqslant 0$ and $\alpha \in (0,1)$, define the *Hölder space* $C^{k,\alpha}(N)$ to be the subset of $f \in C^k(N)$ for which

$$[\nabla^k f]_\alpha = \sup_{\substack{x\neq y\in N \\ d(x,y)<\delta(g)}} \frac{\left|\nabla^k f(x) - \nabla^k f(y)\right|}{d(x,y)^\alpha}$$

is finite. Here $d(x,y)$ is the geodesic distance between $x$ and $y$ and $\delta(g) > 0$ the *injectivity radius*. Note that $\nabla^k f(x)$ and $\nabla^k f(y)$ lie in different vector spaces $\otimes^k T_x^* N$, $\otimes^k T_y^* N$ when $k > 0$, but we identify them by parallel translation using $\nabla$ along the unique geodesic $\gamma$ of length $d(x,y)$ joining $x$ and $y$. The *Hölder norm* is $\|f\|_{C^{k,\alpha}} = \|f\|_{C^k} + [\nabla^k f]_\alpha$.

For $q \geqslant 1$, define the *Lebesgue space* $L^q(N)$ to be the set of locally integrable functions $f$ on $N$ for which the norm

$$\|f\|_{L^q} = \left(\int_N |f|^q \,\mathrm{d}V_g\right)^{1/q}$$

is finite. Here $\mathrm{d}V_g$ is the volume form of $g$. Suppose $r, s, t \geqslant 1$ with $1/r = 1/s + 1/t$. If $\phi \in L^s(N)$ and $\psi \in L^t(N)$ then $\phi\psi \in L^r(N)$, and $\|\phi\psi\|_{L^r} \leqslant \|\phi\|_{L^s}\|\psi\|_{L^t}$; this is *Hölder's inequality*.

Let $q \geqslant 1$ and $k \geqslant 0$ be an integer. Define the *Sobolev space* $L^q_k(N)$ to be the set of $f \in L^q(N)$ such that $f$ is $k$ times weakly differentiable and $|\nabla^j f| \in L^q(N)$ for $j \leqslant k$. Define the *Sobolev norm* on $L^q_k(N)$ to be

$$\|f\|_{L^q_k} = \left(\sum_{j=0}^k \int_N |\nabla^j f|^q \,\mathrm{d}V_g\right)^{1/q}.$$

Then $L^q_k(N)$ is a Banach space, and $L^2_k(N)$ a Hilbert space.

The *Sobolev Embedding Theorem* [1, Th. 2.30] gives inclusions between the spaces $L^q_k(N)$ and $C^{l,\alpha}(N)$.

**Theorem 5.1** *Suppose $(N,g)$ is a compact Riemannian $n$-manifold, $k \geqslant l \geqslant 0$ are integers, $\alpha \in (0,1)$ and $q, r \geqslant 1$. If $\frac{1}{q} \leqslant \frac{1}{r} + \frac{k-l}{n}$, then $L^q_k(N)$ is continuously embedded in $L^r_l(N)$ by inclusion. If $\frac{1}{q} \leqslant \frac{k-l-\alpha}{n}$, then $L^q_k(N)$ is continuously embedded in $C^{l,\alpha}(N)$ by inclusion.*

## 5.2 Statement of the result

The following definition sets up the notation we shall use.

**Definition 5.2** Let $(M, J, \omega, \Omega)$ be an almost Calabi–Yau $m$-fold, with metric $g$. Let $N$ be a compact, oriented, immersed, Lagrangian $m$-submanifold in $M$, with immersion $\iota : N \to M$, so that $\iota^*(\omega) \equiv 0$. Define $h = \iota^*(g)$, so that $(N,h)$ is a Riemannian manifold. Let $\mathrm{d}V$ be the volume form on $N$ induced by the metric $h$ and orientation.

Let $\psi : M \to (0,\infty)$ be the smooth function given in (3). Then $\Omega|_N$ is a complex $m$-form on $N$, and using (3) and the Lagrangian condition we find that $\bigl|\Omega|_N\bigr| = \psi^m$, calculating $|\,.\,|$ using $h$ on $N$. Therefore we may write

$$\Omega|_N = \psi^m e^{i\theta}\,\mathrm{d}V \quad \text{on } N, \tag{15}$$

for some phase function $e^{i\theta}$ on $N$. Suppose that $\cos\theta \geqslant \frac{1}{2}$ on $N$. Then we can choose $\theta$ to be a smooth function $\theta : N \to (-\frac{\pi}{3}, \frac{\pi}{3})$. Suppose that $[\iota^*(\mathrm{Im}\,\Omega)] = 0$ in $H^m(N,\mathbb{R})$. Then $\int_N \psi^m \sin\theta\,\mathrm{d}V = 0$, by (15).

Suppose we are given a finite-dimensional vector subspace $W \subset C^\infty(N)$ with $1 \in W$. Define $\pi_W : L^2(N) \to W$ to be the projection onto $W$ using the $L^2$-inner product.

For $r > 0$, define $\mathcal{B}_r \subset T^*N$ to be the bundle of 1-forms $\alpha$ on $N$ with $|\alpha| < r$. Regard $\mathcal{B}_r$ as a noncompact $2m$-manifold with natural projection $\pi : \mathcal{B}_r \to N$, whose fibre at $x \in N$ is the ball of radius $r$ about 0 in $T^*_x N$. We will sometimes identify $N$ with the zero section of $\mathcal{B}_r$, and write $N \subset \mathcal{B}_r$.

At each $y \in \mathcal{B}_r$ with $\pi(y) = x \in N$, the Levi-Civita connection $\nabla$ of $h$ on $T^*N$ defines a splitting $T_y\mathcal{B}_r = H \oplus V$ into horizontal and vertical subspaces $H, V$, with $H \cong T_x N$ and $V \cong T^*_x N$. Write $\hat\omega$ for the natural symplectic structure on $\mathcal{B}_r \subset T^*N$, defined using $T\mathcal{B}_r \cong H \oplus V$ and $H \cong V^*$. Define a natural Riemannian metric $\hat h$ on $\mathcal{B}_r$ such that the subbundles $H, V$ are orthogonal, and $\hat h|_H = \pi^*(h)$, $\hat h|_V = \pi^*(h^{-1})$.

Let $\hat\nabla$ be the connection on $T\mathcal{B}_r \cong H \oplus V$ given by the lift of the Levi-Civita connection $\nabla$ of $h$ on $N$ in the horizontal directions $H$, and by partial differentiation in the vertical directions $V$, which is well-defined as $T\mathcal{B}_r$ is naturally trivial along each fibre. Then $\hat\nabla$ preserves $\hat h, \hat\omega$ and the splitting $T\mathcal{B}_r \cong H \oplus V$. It is *not* torsion-free in general, but has torsion $T(\hat\nabla)$ depending linearly on the Riemann curvature $R(h)$.

As $N$ is a Lagrangian submanifold of $M$, by Theorem 3.5 the symplectic manifold $(M,\omega)$ is locally isomorphic near $N$ to $T^*N$ with its canonical symplectic structure. That is, for some small $r > 0$ there exists an immersion $\Phi : \mathcal{B}_r \to M$ such that $\Phi^*(\omega) = \hat\omega$ and $\Phi|_N = \iota$. Define an $m$-form $\beta$ on $\mathcal{B}_r$ by $\beta = \Phi^*(\mathrm{Im}\,\Omega)$.

If $\alpha \in C^\infty(T^*N)$ with $|\alpha| < r$, write $\Gamma(\alpha)$ for the *graph* of $\alpha$ in $\mathcal{B}_r$. Then $\Phi_*(\Gamma(\alpha))$ is a compact, immersed submanifold in $M$ diffeomorphic to $N$.

With this notation, we can state our main result.

**Theorem 5.3** *Let $\kappa > 1$ and $A_1, \ldots, A_8 > 0$ be real, and $m \geqslant 3$ an integer. Then there exist $\epsilon, K > 0$ depending only on $\kappa, A_1, \ldots, A_8$ and $m$ such that the following holds.*

*Suppose $0 < t \leqslant \epsilon$ and Definition 5.2 holds with $r = A_1 t$, and*

(i) $\|\psi^m \sin\theta\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa+m/2}$, $\|\psi^m \sin\theta\|_{C^0} \leqslant A_2 t^{\kappa-1}$, $\|\mathrm{d}(\psi^m \sin\theta)\|_{L^{2m}} \leqslant A_2 t^{\kappa-3/2}$ *and* $\|\pi_W(\psi^m \sin\theta)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$.

(ii) $\psi \geqslant A_3$ *on* $N$.

(iii) $\|\hat\nabla^k \beta\|_{C^0} \leqslant A_4 t^{-k}$ *for* $k = 0, 1, 2$ *and* $3$.

(iv) *The injectivity radius* $\delta(h)$ *satisfies* $\delta(h) \geqslant A_5 t$.

(v) *The Riemann curvature* $R(h)$ *satisfies* $\|R(h)\|_{C^0} \leqslant A_6 t^{-2}$.

(vi) *If* $v \in L^2_1(N)$ *with* $\pi_W(v) = 0$, *then* $v \in L^{2m/(m-2)}(N)$ *by Theorem 5.1, and* $\|v\|_{L^{2m/(m-2)}} \leqslant A_7 \|\mathrm{d}v\|_{L^2}$.

(vii) *For all* $w \in W$ *we have* $\|\mathrm{d}^*\mathrm{d}w\|_{L^{2m/(m+2)}} \leqslant \frac{1}{2} A_7^{-1} \|\mathrm{d}w\|_{L^2}$. *For all* $w \in W$ *with* $\int_N w \,\mathrm{d}V = 0$ *we have* $\|w\|_{C^0} \leqslant A_8 t^{1-m/2} \|\mathrm{d}w\|_{L^2}$.

*Here norms are computed using the metric $h$ on $N$ in* (i), (v), (vi) *and* (vii), *and the metric $\hat h$ on $\mathcal{B}_{A_1 t}$ in* (iii). *Then there exists $f \in C^\infty(N)$ with $\int_N f \,\mathrm{d}V = 0$, such that $\|\mathrm{d}f\|_{C^0} \leqslant K t^\kappa < A_1 t$ and $\tilde N = \Phi_*\bigl(\Gamma(\mathrm{d}f)\bigr)$ is an immersed special Lagrangian $m$-fold in $(M, J, \omega, \Omega)$.*

The theorem will be proved in §5.3–§5.5. In the rest of the section we work in the situation of Theorem 5.3, so we suppose $M, J, \omega, \Omega$ and $N$ are given, we use the notation of Definition 5.2, and we suppose that $\kappa > 1, A_1, \ldots, A_8 > 0$ and $t > 0$ are given such that parts (i)–(vii) of Theorem 5.3 hold.

## 5.3 Special Lagrangian submanifolds close to $N$

We begin the proof by studying the conditions for a submanifold $\tilde N$ of $M$ close to $N$ to be special Lagrangian. We write $\tilde N$ as $\Phi\bigl(\Gamma(\alpha)\bigr)$, where $\alpha$ is a small 1-form on $N$ and $\Gamma(\alpha)$ its graph in $\mathcal{B}_{A_1 t} \subset T^*N$.

**Lemma 5.4** *In the situation above, let $\alpha \in C^\infty(T^*N)$ be a smooth 1-form with $\|\alpha\|_{C^0} < A_1 t$, and $\Gamma(\alpha)$ the graph of $\alpha$ in $\mathcal{B}_{A_1 t}$. Then $\tilde N = \Phi\bigl(\Gamma(\alpha)\bigr)$ is a special Lagrangian $m$-fold in $M$ if and only if $\mathrm{d}\alpha = 0$ and $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = 0$.*

*Proof.* Note that $\pi : \Gamma(\alpha) \to N$ is a diffeomorphism and $\Phi : \Gamma(\alpha) \to M$ an immersion. By Definition 2.5, $\tilde{N}$ is an SL $m$-fold in $M$ if and only if $\omega|_{\tilde{N}} \equiv \operatorname{Im}\Omega|_{\tilde{N}} \equiv 0$. Pulling back by $\Phi$, this holds if and only if $\hat{\omega}|_{\Gamma(\alpha)} \equiv \beta|_{\Gamma(\alpha)} \equiv 0$, since $\Phi^*(\omega) = \hat{\omega}$ and $\Phi^*(\operatorname{Im}\Omega) = \beta$.

Pushing forward by $\pi : \Gamma(\alpha) \to N$, we see that $\tilde{N}$ is special Lagrangian if and only if $\pi_*\bigl(\hat{\omega}|_{\Gamma(\alpha)}\bigr) \equiv \pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) \equiv 0$. But as $\mathcal{B}_{A_1 t} \subset T^*N$ and $\hat{\omega}$ is the natural symplectic structure on $T^*N$ we have $\pi_*\bigl(\hat{\omega}|_{\Gamma(\alpha)}\bigr) = -\mathrm{d}\alpha$ by a well-known piece of symplectic geometry, and the lemma follows. □

We rewrite the condition $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = 0$ in terms of a function $F$.

**Definition 5.5** Define $\mathcal{A} = \bigl\{\alpha \in C^\infty(T^*N) : \|\alpha\|_{C^0} < A_1 t\bigr\}$, and define $F : \mathcal{A} \to C^\infty(N)$ by $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr) = F(\alpha)\,\mathrm{d}V$. Then Lemma 5.4 shows that if $\alpha \in \mathcal{A}$ then $\Phi\bigl(\Gamma(\alpha)\bigr)$ is special Lagrangian if and only if $\mathrm{d}\alpha = F(\alpha) = 0$.

The value of $F(\alpha)$ at $x \in N$ depends on the tangent space $T_y\Gamma(\alpha)$, where $y \in \Gamma(\alpha)$ with $\pi(y) = x$. But $T_y\Gamma(\alpha)$ depends on both $\alpha|_x$ and $\nabla\alpha|_x$. Hence $F(\alpha)$ depends pointwise on both $\alpha$ and $\nabla\alpha$, rather than just $\alpha$. Therefore we may write

$$\begin{aligned} &F(\alpha)[x] = F'\bigl(x, \alpha(x), \nabla\alpha(x)\bigr) \quad \text{for all } x \in N, \text{ where} \\ &F' : \bigl\{(x,\gamma,\delta) : x \in N,\ \ \gamma \in T_x^*N,\ \ |\gamma| < A_1 t,\ \ \delta \in \otimes^2 T_x^*N\bigr\} \to \mathbb{R} \end{aligned} \tag{16}$$

is a smooth, nonlinear function. Note that $F$ maps between infinite-dimensional spaces $\mathcal{A} \to C^\infty(N)$, but $F'$ maps between finite-dimensional spaces.

For fixed $x \in N$ the variables $\gamma, \delta$ in the domain of $F'$ lie in vector spaces $T_x^*N, \otimes^2 T_x^*N$. Thus we may take partial derivatives in these directions (without using a connection), with values in the dual spaces $T_xN, \otimes^2 T_xN$. Write $\partial_1, \partial_2$ for the partial derivatives in the $\gamma, \delta$ directions respectively. Then we have

$$\begin{gathered} \partial_1 F'(x,\gamma,\delta) \in T_xN, \quad \partial_2 F'(x,\gamma,\delta) \in \otimes^2 T_xN, \quad \partial_1^2 F'(x,\gamma,\delta) \in S^2(T_xN), \\ \partial_1\partial_2 F'(x,\gamma,\delta) \in \otimes^3 T_xN \quad \text{and} \quad \partial_2^2 F'(x,\gamma,\delta) \in S^2(\otimes^2 T_xN). \end{gathered}$$

We compute the expansion of $F$ up to first order in $\alpha$.

**Proposition 5.6** *This function $F$ may be written*

$$F(\alpha) = \psi^m \sin\theta - \mathrm{d}^*\bigl(\psi^m \cos\theta\,\alpha\bigr) + Q(\alpha), \tag{17}$$

*where $Q : \mathcal{A} \to C^\infty(N)$ is smooth with $Q(\alpha) = O\bigl(|\alpha|^2 + |\nabla\alpha|^2\bigr)$ for small $\alpha$.*

*Proof.* It is easy to see that $F$ depends smoothly on $\alpha$. Therefore by Taylor's theorem we may expand $F$ about $\alpha = 0$ up to second order, and get an equation with the general form of (17), with $Q$ smooth. Since $F(\alpha)$ depends pointwise on $\alpha, \nabla\alpha$, the second-order remainder term $Q(\alpha)$ is of the form $O\bigl(|\alpha|^2 + |\nabla\alpha|^2\bigr)$, and the estimate valid when $|\alpha|, |\nabla\alpha|$ are small, that is, when $\alpha$ is small in $C^1$.

So to prove (17) we need to compute $F(0)$ and $\mathrm{d}F(0)$ and show they coincide with the first two terms on the right hand side of (17). When $\alpha=0$ we have $\Phi_*\bigl(\Gamma(0)\bigr)=N$ in $M$, and therefore $\pi_*\bigl(\beta|_{\Gamma(0)}\bigr)=\operatorname{Im}\Omega|_N=\psi^m\sin\theta\,\mathrm{d}V$ by (15). Thus $F(0)=\psi^m\sin\theta$ by Definition 5.5, giving the zeroth order term in (17).

Next we compute the first order term in $\alpha$. Let $v$ be the vector field on $T^*N$ with $v\cdot\hat\omega=-\pi^*(\alpha)$. Then $v$ is tangent to the fibres of $\pi:T^*N\rightarrow N$, and $\exp(v)$ maps $T^*N\rightarrow T^*N$ taking $\beta\mapsto\alpha+\beta$ for 1-forms $\beta$ on $N$. Identifying $N$ with the zero section of $T^*N$, the image $\exp(v)[N]$ of $N$ under $\exp(v)$ is $\Gamma(\alpha)\subset\mathcal{B}_{A_1t}\subset T^*N$. More generally, $\exp(sv)[N]=\Gamma(s\alpha)$ for $s\in[0,1]$.

Therefore $F(s\alpha)\,\mathrm{d}V=\exp(sv)^*(\beta)$ for $s\in[0,1]$. Differentiating gives

$$\begin{aligned}\mathrm{d}F(0)[\alpha]\,\mathrm{d}V&=\frac{\mathrm{d}}{\mathrm{d}s}\bigl(F(s\alpha)\bigr)\Big|_{s=0}\mathrm{d}V=\frac{\mathrm{d}}{\mathrm{d}s}\bigl(\exp(sv)^*(\beta)\bigr)\Big|_{s=0}\\&=\bigl(\mathcal{L}_v\beta\bigr)\Big|_N=\bigl(\mathrm{d}(v\cdot\beta)+v\cdot(\mathrm{d}\beta)\bigr)\Big|_N=\mathrm{d}\bigl((v\cdot\beta)|_N\bigr),\end{aligned}\tag{18}$$

where $\mathcal{L}_v$ is the Lie derivative, '$\cdot$' contracts together vector fields and forms in the usual way, and we have used the fact that $\mathrm{d}\beta=0$ since $\Omega$ is closed and $\beta=\Phi^*(\operatorname{Im}\Omega)$.

Fix $x\in N$. We may choose local real coordinates $(x_1,\ldots,x_m,y_1,\ldots,y_m)$ on $M$ near $x$ such that at $x$ we have

$$g=\sum_{j=1}^m(\mathrm{d}x_j^2+\mathrm{d}y_j^2),\quad T_xN=\Bigl\langle\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_m}\Bigr\rangle,\quad \mathrm{d}V=\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_m\big|_{T_xN},$$
$$\omega=\sum_{j=1}^m\mathrm{d}x_j\wedge\mathrm{d}y_j\quad\text{and}\quad\Omega=\psi^m\mathrm{e}^{i\theta}(\mathrm{d}x_1+i\mathrm{d}y_1)\wedge\cdots\wedge(\mathrm{d}x_m+i\mathrm{d}y_m).$$

Identify $T_xM$ with $T_x\mathcal{B}_{A_1t}$ using $\mathrm{d}\Phi$, so that $\hat\omega=\omega$ and $\beta=\operatorname{Im}\Omega$.

Write $\alpha=\sum_{j=1}^m a_j\,\mathrm{d}x_j$ at $x$ for $a_j\in\mathbb{R}$. Then $v=\sum_{j=1}^m a_j\frac{\partial}{\partial y_j}$ at $x$ as $v\cdot\omega=-\alpha$. Calculation with the above expression for $\Omega$ then shows that

$$\begin{aligned}(v\cdot\beta)|_N&=\psi^m\cos\theta\sum_{j=1}^m(-1)^{j-1}a_j\,\mathrm{d}x_1\wedge\cdots\wedge\mathrm{d}x_{j-1}\wedge\mathrm{d}x_{j+1}\wedge\cdots\wedge\mathrm{d}x_m\\&=\psi^m\cos\theta\,*_{T_xN}\alpha\qquad\text{at }x,\end{aligned}$$

where $*_{T_xN}$ is the Hodge star on $T_xN$, computed using the explicit expressions for $g$ and $\mathrm{d}V$ at $x$. Since $*\mathrm{d}V=1$ and $*\mathrm{d}*=-\mathrm{d}^*$ on 1-forms, equation (18) gives

$$\mathrm{d}F(0)[\alpha]\,\mathrm{d}V=\mathrm{d}(\psi^m\cos\theta\,*\alpha)=\bigl(*\mathrm{d}*(\psi^m\cos\theta\,\alpha)\bigr)\,\mathrm{d}V=\bigl(-\mathrm{d}^*(\psi^m\cos\theta\,\alpha)\bigr)\,\mathrm{d}V.$$

This shows that $\mathrm{d}F(0):\alpha\mapsto-\mathrm{d}^*(\psi^m\cos\theta\,\alpha)$, which yields the first order term in (17), and completes the proof. □

Here are some properties of $Q$.

**Lemma 5.7** *This function $Q$ satisfies $Q(0)=\mathrm{d}Q(0)=0$ and $\int_N Q(\alpha)\,\mathrm{d}V=0$ for all $\alpha\in\mathcal{A}$, and $\Phi\bigl(\Gamma(\alpha)\bigr)$ is special Lagrangian if and only if*

$$\mathrm{d}\alpha=0 \quad \textit{and} \quad \mathrm{d}^*\bigl(\psi^m\cos\theta\,\alpha\bigr)=\psi^m\sin\theta+Q(\alpha). \tag{19}$$

*Proof.* Proposition 5.6 gives $Q(\alpha)=O\bigl(|\alpha|^2+|\nabla\alpha|^2\bigr)$, which implies that $Q(0)=\mathrm{d}Q(0)=0$. By definition $\pi_*\bigl(\beta|_{\Gamma(\alpha)}\bigr)=F(\alpha)\,\mathrm{d}V$ for $\alpha\in\mathcal{A}$, so

$$\int_N F(\alpha)\,\mathrm{d}V=\int_{\Gamma(\alpha)}\beta=\int_{\Gamma(0)}\beta=\int_N \iota^*(\operatorname{Im}\Omega)=0,$$

as $\beta$ is closed, $\Gamma(\alpha)$ and $\Gamma(0)$ are homologous, and $[\iota^*(\operatorname{Im}\Omega)]=0$ in $H^m(N,\mathbb{R})$ by Definition 5.2. Now $F(0)=\psi^m\sin\theta$ by (17), so $\int_N\psi^m\sin\theta\,\mathrm{d}V=0$, and $\int_N \mathrm{d}^*\bigl(\psi^m\cos\theta\,\alpha\bigr)\,\mathrm{d}V=0$ by integration by parts. Therefore multiplying (17) by $\mathrm{d}V$ and integrating over $N$ gives $\int_N Q(\alpha)\,\mathrm{d}V=0$. Finally $\Phi\bigl(\Gamma(\alpha)\bigr)$ is special Lagrangian if and only if $\mathrm{d}\alpha=F(\alpha)=0$ by Definition 5.5, and by (17) this is equivalent to (19). □

The notation $Q(\alpha)$ was chosen because $Q$ is *approximately quadratic* for small $\alpha$. The following estimates of $Q$ are modelled on the fact that if $q$ is a homogeneous quadratic polynomial on $\mathbb{R}^n$ then $\bigl|q(\mathbf{x})-q(\mathbf{y})\bigr|\leqslant C|\mathbf{x}-\mathbf{y}|\bigl(|\mathbf{x}|+|\mathbf{y}|\bigr)$ for some $C\geqslant 0$ and all $\mathbf{x},\mathbf{y}\in\mathbb{R}^n$.

**Proposition 5.8** *There exist $C_1,\ldots,C_4>0$ depending only on $A_1,A_4,A_6,m$ such that $C_1<A_1$ and if $\alpha,\beta\in\mathcal{A}$ with $\|\alpha\|_{C^0},\|\beta\|_{C^0}\leqslant C_1t$ and $\|\nabla\alpha\|_{C^0}$, $\|\nabla\beta\|_{C^0}\leqslant C_2$ then*

$$\begin{aligned}\bigl|Q(\alpha)-Q(\beta)\bigr|\leqslant C_3\bigl(t^{-1}|\alpha-\beta|+|\nabla\alpha-\nabla\beta|\bigr)\cdot&\\ \bigl(t^{-1}|\alpha|+t^{-1}|\beta|+|\nabla\alpha|+|\nabla\beta|\bigr)&\qquad\textit{and}\end{aligned}\tag{20}$$

$$\begin{aligned}\bigl|\mathrm{d}\bigl(Q(\alpha)-Q(\beta)\bigr)\bigr|\leqslant C_4\Bigl(&t^{-3}|\alpha-\beta|\bigl(|\alpha|+|\beta|\bigr)+t^{-2}|\alpha-\beta|\bigl(|\nabla\alpha|+|\nabla\beta|\bigr)\\ &+t^{-1}|\alpha-\beta|\bigl(|\nabla^2\alpha|+|\nabla^2\beta|\bigr)+t^{-2}|\nabla\alpha-\nabla\beta|\bigl(|\alpha|+|\beta|\bigr)\\ &+t^{-1}|\nabla\alpha-\nabla\beta|\bigl(|\nabla\alpha|+|\nabla\beta|\bigr)+|\nabla\alpha-\nabla\beta|\bigl(|\nabla^2\alpha|+|\nabla^2\beta|\bigr)\\ &+t^{-1}|\nabla^2\alpha-\nabla^2\beta|\bigl(|\alpha|+|\beta|\bigr)+|\nabla^2\alpha-\nabla^2\beta|\bigl(|\nabla\alpha|+|\nabla\beta|\bigr)\Bigr).\end{aligned}\tag{21}$$

*Proof.* Let $\alpha,\beta\in\mathcal{A}$, fix $x\in N$, and define a real function $P$ on the triangle $\bigl\{(r,s):0\leqslant r\leqslant s\leqslant 1\bigr\}$ by $P(r,s)=Q\bigl(r(\alpha-\beta)+s\beta\bigr)[x]$. This is well-defined as $\mathcal{A}$ is convex and contains 0, so $r(\alpha-\beta)+s\beta\in\mathcal{A}$ when $0\leqslant r\leqslant s\leqslant 1$. Then

$$\bigl(Q(\alpha)-Q(\beta)\bigr)[x]=P(1,1)-P(0,1)=\int_0^1\frac{\partial P}{\partial r}(u,1)\mathrm{d}u.$$

Lemma 5.7 gives $\mathrm{d}Q(0)=0$, so $\frac{\partial P}{\partial r}(0,0)=0$. Therefore

$$\frac{\partial P}{\partial r}(u,1)=\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{\partial P}{\partial r}(us,s)\Big)\mathrm{d}s=\int_0^1\Big[u\frac{\partial^2 P}{\partial r^2}(us,s)+\frac{\partial^2 P}{\partial r\partial s}(us,s)\Big]\mathrm{d}s.$$

Substituting this into the previous equation and changing variables to $r=us$ and $s$, we obtain

$$\big(Q(\alpha)-Q(\beta)\big)[x]=\int_0^1\int_0^s\Big[\frac{r}{s^2}\,\frac{\partial^2 P}{\partial r^2}(r,s)+\frac{1}{s}\,\frac{\partial^2 P}{\partial r\partial s}(r,s)\Big]\mathrm{d}r\,\mathrm{d}s. \tag{22}$$

By the definitions of $P,Q$ and $F'$ we have

$$\begin{aligned}P(r,s)=&F'\big(x,r(\alpha(x)-\beta(x))+s\beta(x),r(\nabla\alpha(x)-\nabla\beta(x))+s\nabla\beta(x)\big)\\&-(\psi^m\sin\theta)[x]+r\,\mathrm{d}^*\big(\psi^m\cos\theta\,(\alpha-\beta)\big)[x]+s\,\mathrm{d}^*\big(\psi^m\cos\theta\,\beta\big)[x].\end{aligned}$$

Taking second derivatives, the last line drops out to give

$$\begin{aligned}\frac{\partial^2 P}{\partial r^2}(r,s)=&\otimes^2(\alpha-\beta)\cdot\partial_1^2F'+\otimes^2(\nabla\alpha-\nabla\beta)\cdot\partial_2^2F'\\&+2(\alpha-\beta)\otimes(\nabla\alpha-\nabla\beta)\cdot\partial_1\partial_2F'\qquad\text{and}\\\frac{\partial^2 P}{\partial r\partial s}(r,s)=&(\alpha-\beta)\otimes\beta\cdot\partial_1^2F'+(\nabla\alpha-\nabla\beta)\otimes\nabla\beta\cdot\partial_2^2F'\\&+\big((\alpha-\beta)\otimes\nabla\beta+\beta\otimes(\nabla\alpha-\nabla\beta)\big)\cdot\partial_1\partial_2F',\end{aligned}$$

evaluating $\partial_j\partial_kF'$ at $\big(x,r(\alpha-\beta)+s\beta,r(\nabla\alpha-\nabla\beta)+s\nabla\beta\big)$ and $\alpha,\beta,\nabla\alpha,\nabla\beta$ at $x$.

Substituting these two equations into (22) and taking mods gives

$$\begin{aligned}\big|Q(\alpha)-Q(\beta)\big|[x]\leqslant&\int_0^1\int_0^s\Big[\big(rs^{-2}|\alpha-\beta|^2+s^{-1}|\alpha-\beta|\,|\beta|\big)\big|\partial_1^2F'\big|\\&+\big(2rs^{-2}|\alpha-\beta||\nabla\alpha-\nabla\beta|+s^{-1}|\alpha-\beta||\nabla\beta|+s^{-1}|\beta||\nabla\alpha-\nabla\beta|\big)\big|\partial_1\partial_2F'\big|\\&+\big(rs^{-2}|\nabla\alpha-\nabla\beta|^2+s^{-1}|\nabla\alpha-\nabla\beta|\,|\nabla\beta|\big)\big|\partial_2^2F'\big|\ \Big]\mathrm{d}r\,\mathrm{d}s.\end{aligned}\tag{23}$$

Here $\alpha,\beta,\nabla\alpha,\nabla\beta$ are independent of $r,s$ and so $|\alpha-\beta|,\ldots,|\nabla\beta|$ are constants, but $\partial_j\partial_kF'$ is evaluated at $\big(x,r(\alpha-\beta)+s\beta,r(\nabla\alpha-\nabla\beta)+s\nabla\beta\big)$, so $\big|\partial_j\partial_kF'\big|$ is a function of $r,s$.

Let us interpret $F'(x,\gamma,\delta)$ in terms of $\hat\beta$. Regard $(x,\gamma)$ as a point in $\mathcal{B}_{A_1t}\subset T^*N$, with $\gamma\in T_x^*N$. Then $T_{(x,\gamma)}\mathcal{B}_{A_1t}\cong T_xN\oplus T_x^*N$ as in Definition 5.2. Using $\delta\in\otimes^2T_x^*N$ we define a map $I_\delta:T_xN\rightarrow T_xN\oplus T_x^*N=T_{(x,\gamma)}\mathcal{B}_{A_1t}$ by $v\mapsto(v,\delta\cdot v)$. Then $F'(x,\gamma,\delta)\,\mathrm{d}V|_x$ is the pullback to $T_xN$ under $I_\delta$ of the restriction of $\hat\beta$ to $T_{(x,\gamma)}\mathcal{B}_{A_1t}$.

Because of this, estimates on the derivatives of $\hat\beta$ imply estimates on the derivatives of $F'$. In particular, as $\|\hat\nabla^k\beta\|_{C^0}\leqslant A_4t^{-k}$ for $k=0,1,2$ by part (iii) of Theorem 5.3 we can show that there exist $C_1,C_2,C>0$ depending only on $A_4,m$ such that

$$\big|\partial_1^2F'\big|\leqslant Ct^{-2},\quad\big|\partial_1\partial_2F'\big|\leqslant Ct^{-1}\quad\text{and}\quad\big|\partial_2^2F'\big|\leqslant C\quad\text{at }(x,\gamma,\delta),\tag{24}$$

provided $|\gamma| \leqslant C_1 t$ and $|\delta| \leqslant C_2$. Here the power of $t$ is determined by the number of derivatives $\partial_1$. This is because changing $\delta$ does not affect the point $(x, \gamma)$ in $\mathcal{B}_{A_1 t}$, so $\partial_2$ does not involve differentiating $\hat{\beta}$ on $\mathcal{B}_{A_1 t}$. Note that $\hat{\nabla}, \partial_1$ are the same in the fibre directions, both given by partial differentiation.

Substituting (24) into (23) and integrating we prove (20), for some $C_3 > 0$ depending only on $A_4, m$. Equation (21) can be proved by a similar but rather more complicated argument, which we leave to the reader. The extra derivative on $Q$ means that we also use the inequalities $\|R(h)\|_{C^0} \leqslant A_6 t^{-2}$ and $\|\hat{\nabla}^3 \beta\|_{C^0} \leqslant A_4 t^{-3}$ in Theorem 5.3. □

## 5.4 Some analytic estimates on $N$

Section 5.3 studied the geometry of $M$ near $N$. We now give some estimates on $N$ itself, Propositions 5.11 and 5.13 below, depending only on the Riemannian manifold $(N, h)$. The proofs are based on that of Theorem G1 in the author's book [8, §11.7].

These estimates are all proved by considering small balls in $N$ of radius $O(t)$, and comparing them with balls of the same radius in $\mathbb{R}^m$. We begin by showing that the metric $h$ on balls of radius $O(t)$ in $N$ is close to the Euclidean metric $g_0$ on $\mathbb{R}^m$ in the $L^{2m}_2$ norm.

**Proposition 5.9** *Let $D_1 > 0$ be smaller than a positive bound depending on $m$. Then there exist $D_2, D_3, D_4 > 0$ depending only on $A_5, A_6, m$ and $D_1$ such that the following holds. Let $B_2, B_3$ be the balls of radii $2, 3$ about $0$ in $\mathbb{R}^m$, and $g_0$ the Euclidean metric on $B_3$. Set $r = D_2 t$. Then for each $x \in N$ we have $D_3 t^m \leqslant \mathrm{vol}\bigl(B_r(x)\bigr) \leqslant \mathrm{vol}\bigl(B_{4r}(x)\bigr) \leqslant D_4 t^m$, where $B_r(x)$ is the geodesic ball of radius $r$ about $x$, and there is a smooth, injective map $\Psi_x : B_3 \to N$ satisfying $\bigl\| r^{-2} \Psi_x^*(h) - g_0 \bigr\|_{L^{2m}_2} \leqslant D_1$ and $B_r(x) \subset \Psi_x(B_2) \subset \Psi_x(B_3) \subset B_{4r}(x)$.*

*Proof.* For simplicity, first suppose that $t = 1$. We require systems of coordinates on open balls in $N$, in which the metric $h$ appears close to the Euclidean metric $g_0$ in the $L^{2m}_2$ norm. These are provided by Jost and Karcher's theory of *harmonic coordinates* [5]. Jost and Karcher show that if the injectivity radius is bounded below and the sectional curvature is bounded above, then there exist coordinate systems on all balls of a given radius, in which the $C^{1,\alpha}$ norm of the metric is bounded in terms of $\alpha$ for each $\alpha \in (0, 1)$.

The $C^{1,\alpha}$ norm is not quite strong enough for our purposes, but fortunately Jost and Karcher's results can be improved to the $L^p_2$ norm, for $p > m/2$. This was done mainly by Anderson, and is described in Petersen [17, §4–§5]. From [17, Th. 5.1, p. 185] we deduce that since $\delta(h) \geqslant A_5$ and $\|R(h)\|_{C^0} \leqslant A_6$ (as $t = 1$), for $D_2 > 0$ depending only on $A_5, A_6, m$ and $D_1$, there exists a coordinate system $\Psi_x$ about $x$ for each $x \in N$, which we may write as a map $\Psi_x : B_3 \to N$ with $\Psi_x(0) = x$, such that $\|D_2^{-2} \Psi_x^*(h) - g_0\|_{L^{2m}_2} \leqslant D_1$, as we have to prove.

Now the radius and volume of balls are controlled by the $C^0$ norm of the metric on the balls, which is controlled by the $L^{2m}_2$ norm by Theorem 5.1. Thus if $D_1$ is small enough in terms of $m$, the balls $\Psi_x(B_2)$, $\Psi_x(B_3)$ in $N$ must have volume and radius close to those of the balls of radius $2D_2$ and $3D_2$ in $\mathbb{R}^m$.

By making $D_1$ and $D_2$ smaller if necessary, we can ensure that $D_3 \leqslant \mathrm{vol}(B_{D_2}(x))$ and $\mathrm{vol}(B_{4D_2}(x)) \leqslant D_4$ for some $D_3, D_4 > 0$ depending only on $A_5, A_6, m$ and $D_1$, and that $B_{D_1}(x) \subset \Psi_x(B_2)$ and $\Psi_x(B_3) \subset B_{4D_1}(x)$, for all $x \in N$. This completes the proof when $t = 1$. To prove the proposition for general $t > 0$, apply the case $t = 1$ to the rescaled metric $t^{-2}h$. □

By the Sobolev Embedding Theorem, Theorem 5.1, $L^{2m}_1$ embeds in $C^0$. Using this we may prove the following result on balls in $\mathbb{R}^m$, following [1, Lem. 2.22]. It is easy to modify the proof to get a bound involving $\|u\|_{L^2}$ rather than $\|u\|_{L^{2m}}$.

**Lemma 5.10** *Let $B_2, B_3$ be the balls of radii $2, 3$ about $0$ in $\mathbb{R}^m$. Then there exist $D_5, D_6 > 0$ depending only on $m$ such that if $u \in C^1(B_3)$ and $v \in L^{2m}_1(B_3)$ then $\|u|_{B_2}\|_{C^0} \leqslant D_5\bigl(\|\mathrm{d}u\|_{C^0} + \|u\|_{L^2}\bigr)$ and $\|v|_{B_2}\|_{C^0} \leqslant D_6\bigl(\|\mathrm{d}v\|_{L^{2m}} + \|v\|_{L^2}\bigr)$.*

We can now prove a *Sobolev embedding result* for 1-forms on $N$.

**Proposition 5.11** *There exist $C_5, C_6 > 0$ depending only on $A_5, A_6$ and $m$ such that if $\alpha \in L^{2m}_2(T^*N)$ then $\alpha \in C^1(T^*N)$ and*

$$\|\alpha\|_{C^0} \leqslant C_5\bigl(t\|\nabla\alpha\|_{C^0} + t^{-m/2}\|\alpha\|_{L^2}\bigr), \quad \textit{and} \tag{25}$$

$$\|\nabla\alpha\|_{C^0} \leqslant C_6\bigl(t^{1/2}\|\nabla^2\alpha\|_{L^{2m}} + t^{-m/2}\|\nabla\alpha\|_{L^2}\bigr). \tag{26}$$

*Proof.* Let $D_1 > 0$ be sufficiently small in terms of $m$, and let $x \in N$. Then Proposition 5.9 gives $D_2, D_3, D_4 > 0$ and $\Psi_x : B_3 \to N$. Define $u \in L^{2m}_2(B_3)$ and $v \in L^{2m}_1(B_3)$ by $u = \Psi_x^*\bigl(|\alpha|_h\bigr)$ and $v = \Psi_x^*\bigl(|\nabla\alpha|_h\bigr)$, where $|\,.\,|_h$ is taken using the metric $h$.

Lemma 5.10 then applies to $u$ and $v$, as $u \in C^1(B_3)$ by Theorem 5.1. The norms in Lemma 5.10 are calculated w.r.t. $g_0$ on $B_3$. But $\|r^{-2}\Psi_x^*(h) - g_0\|_{L^{2m}_2} \leqslant D_1$ by Proposition 5.9, so if $D_1$ is small then the metrics $r^{-2}\Psi_x^*(h)$ and $g_0$ are close in $C^0$. Hence we can increase $D_5, D_6$ to $D_5', D_6'$ depending only on $D_1, m$ such that $\|u|_{B_2}\|_{C^0} \leqslant D_5'\bigl(\|\mathrm{d}u\|_{C^0} + \|u\|_{L^2}\bigr)$ and $\|v|_{B_2}\|_{C^0} \leqslant D_6'\bigl(\|\mathrm{d}v\|_{L^{2m}} + \|v\|_{L^2}\bigr)$, where now all norms are taken w.r.t. the metric $r^{-2}\Psi_x^*(h)$ on $B_3$.

Pushing these forward via $\Psi_x$ we deduce that

$$\bigl\||\alpha|\big|_{\Psi_x(B_2)}\bigr\|_{C^0} \leqslant D_5'\bigl(r\bigl\|\mathrm{d}|\alpha|\big|_{\Psi_x(B_3)}\bigr\|_{C^0} + r^{-m/2}\bigl\||\alpha|\big|_{\Psi_x(B_3)}\bigr\|_{L^2}\bigr) \text{ and}$$

$$\bigl\||\nabla\alpha|\big|_{\Psi_x(B_2)}\bigr\|_{C^0} \leqslant D_6'\bigl(r^{1/2}\bigl\|\,\mathrm{d}|\nabla\alpha|\big|_{\Psi_x(B_3)}\bigr\|_{L^2} + r^{-m/2}\bigl\||\alpha|\big|_{\Psi_x(B_3)}\bigr\|_{L^2}\bigr),$$

where all mods and norms are taken w.r.t. $h$, and the powers of $r$ compensate for the change from $r^{-2}h$ to $h$. Substituting $r = D_2 t$, noting that $\bigl|\mathrm{d}|\alpha|\bigr| \leqslant |\nabla\alpha|$ and $\bigl|\mathrm{d}|\nabla\alpha|\bigr| \leqslant |\nabla^2\alpha|$, and taking the supremum of these two inequalities over all $x \in N$, we quickly prove (25) and (26) with $C_5 = D_5' \max(D_2, D_2^{-m/2})$ and $C_6 = D_6' \max(D_2^{1/2}, D_2^{-m/2})$. □

Next we prove some interior elliptic regularity estimates on $B_2, B_3$.

**Proposition 5.12** *Suppose $E_1, E_2 > 0$. Then there exist $E_3, E_4 > 0$ depending only on $E_1, E_2$ and $m$ such that the following holds.*

*Let $B_2, B_3$ be the balls of radii $2, 3$ about $0$ in $\mathbb{R}^m$, and suppose $a^{ij} \in L_2^{2m}(B_3)$ for $i, j = 1, \ldots, m$ and $b^i \in L_1^{2m}(B_3)$ for $i = 1, \ldots, m$ such that*

$$\begin{aligned} E_1 \sum_{i=1}^{m} \xi_i^2 \leqslant - \sum_{i,j=1}^{m} a^{ij} \xi_i \xi_j \quad \text{on } B_3 \text{ for all } (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m, \\ \text{and} \quad \|a^{ij}\|_{L_2^{2m}}, \|b^i\|_{L_1^{2m}} \leqslant E_2 \quad \text{for all } i, j = 1, \ldots, m. \end{aligned} \tag{27}$$

*Then whenever $\sigma \in L_3^{2m}(B_3)$ and $\tau \in L_1^{2m}(B_3)$ with*

$$\sum_{i,j=1}^{m} a^{ij} \frac{\partial^2 \sigma}{\partial x_i \partial x_j} + \sum_{i=1}^{m} b^i \frac{\partial \sigma}{\partial x_i} = \tau, \tag{28}$$

*we have*

$$\big\| \nabla^2 \sigma|_{B_2} \big\|_{L^2} \leqslant E_3 \big( \|\mathrm{d}\sigma\|_{L^2} + \|\tau\|_{L^2} \big), \quad \text{and} \tag{29}$$

$$\big\| \nabla^3 \sigma|_{B_2} \big\|_{L^{2m}} \leqslant E_4 \big( \|\mathrm{d}\sigma\|_{L^2} + \|\tau\|_{L_1^{2m}} \big). \tag{30}$$

*Proof.* Aubin [1, Cor. 4.3] shows that if $(M, g)$ is a compact Riemannian manifold and $\varphi \in L_1^2(M)$ with $\int_M \varphi \, \mathrm{d}V = 0$ then $\|\varphi\|_{L^2} \leqslant C \|\mathrm{d}\varphi\|_{L^2}$ for $C > 0$ depending on $(M, g)$. Using the technique of 'doubling' we see this also holds for compact Riemannian manifolds with boundary. Therefore if $\varphi \in L_1^2(B_3)$ with $\int_{B_3} \varphi \, \mathrm{d}V_{g_0} = 0$ then $\|\varphi\|_{L^2} \leqslant E_5 \|\mathrm{d}\varphi\|_{L^2}$ for $E_5 > 0$ depending only on $m$. Since (28)–(30) are unchanged by adding a constant to $\sigma$, we may assume that $\int_{B_3} \sigma \, \mathrm{d}V_{g_0} = 0$, and thus we have $\|\sigma\|_{L^2} \leqslant E_5 \|\mathrm{d}\sigma\|_{L^2}$.

Equation (28) is a *second-order linear elliptic equation*, with *coefficients* $a^{ij}, b^i$. As $L_1^{2m} \hookrightarrow C^{0,1/2}$ by Theorem 5.1, the bounds (27) imply a $C^{0,1/2}$ bound on the coefficients of (28), and also show that (28) is *uniformly elliptic*. By the interior elliptic regularity estimates of Gilbarg and Trudinger [2, Th. 9.11, p. 235] there exists $E_6 > 0$ depending only on $E_1, E_2$ and $m$ such that $\|\sigma|_{B_2}\|_{L_2^2} \leqslant E_6 \big( \|\sigma\|_{L^2} + \|\tau\|_{L^2} \big)$. Combining this with $\|\nabla^2 \sigma|_{B_2}\|_{L^2} \leqslant \|\sigma|_{B_2}\|_{L_2^2}$ and $\|\sigma\|_{L^2} \leqslant E_5 \|\mathrm{d}\sigma\|_{L^2}$ we deduce (29), with $E_3 = E_6 \max(E_5, 1)$.

We would like to apply the same approach to prove (30). However, there is a problem: to get an $L_3^{2m}$ estimate of $\sigma$ we need a $C^{1,1/2}$ bound on the coefficients of (28), which we do not have. So we instead rewrite (28) as

$$\sum_{i,j=1}^{m} a^{ij} \frac{\partial^2 \sigma}{\partial x_i \partial x_j} = \tau - \sum_{i=1}^{m} b^i \frac{\partial \sigma}{\partial x_i} = \tilde{\tau}. \tag{31}$$

Now the l.h.s. is a uniformly elliptic operator with coefficients $a^{ij}$, which are bounded in $C^{1,1/2}$ by $\|a^{ij}\|_{L_2^{2m}} \leqslant E_2$ and the Sobolev embedding $L_2^{2m} \hookrightarrow C^{1,1/2}$.

Let $B_{5/2}$ be the ball of radius 5/2 about 0 in $\mathbb{R}^m$. Applying [2, Th. 9.19, p. 243] to (31) on $B_{5/2}$ we get a bound of the form

$$\big\|\sigma|_{B_2}\big\|_{L^{2m}_3} \leqslant E_7\big(\big\|\sigma|_{B_{5/2}}\big\|_{L^2} + \|\tilde\tau|_{B_{5/2}}\|_{L^{2m}_1}\big), \tag{32}$$

for $E_7 > 0$ depending on $E_1, E_2$ and $m$. Now as $L^{2m}_1 \hookrightarrow C^0$ by Theorem 5.1 we can show that multiplication is a *continuous* map $L^{2m}_1(B_{5/2}) \times L^{2m}_1(B_{5/2}) \to L^{2m}_1(B_{5/2})$, and so there exists $C'' > 0$ depending only on $m$ with

$$\|\varphi\,\eta\|_{L^{2m}_1} \leqslant C''\|\varphi\|_{L^{2m}_1}\|\eta\|_{L^{2m}_1} \quad \text{for all } \varphi,\eta \in L^{2m}_1(B_{5/2}). \tag{33}$$

But $\|b^i\|_{L^{2m}_1} \leqslant E_2$, and by the method used to prove (29) we can bound $\|\sigma|_{B_{5/2}}\|_{L^{2m}_2}$, and hence $\|\frac{\partial\sigma}{\partial x_j}|_{B_{5/2}}\|_{L^{2m}_1}$, in terms of $\|\sigma\|_{L^2}$ and $\|\tau\|_{L^{2m}}$. Therefore using (31) and (33) we can bound $\|\tilde\tau|_{B_{5/2}}\|_{L^{2m}_1}$ in terms of $\|\sigma\|_{L^2}$ and $\|\tau\|_{L^{2m}_1}$. Combining this with (32) and $\|\sigma\|_{L^2} \leqslant E_5\|\mathrm{d}\sigma\|_{L^2}$, we prove (30) for some $E_4 > 0$ depending only on $E_1, E_2$ and $m$. □

Finally we prove an elliptic regularity result for $u \mapsto \mathrm{d}^*\big(\psi^m\cos\theta\,\mathrm{d}u\big)$ on $N$.

**Proposition 5.13** *There exist $C_7, C_8 > 0$ depending only on $A_3, A_4, A_5, A_6$ and $m$ such that if $u \in L^{2m}_3(N)$, $v \in L^{2m}_1(N)$ with $\mathrm{d}^*\big(\psi^m\cos\theta\,\mathrm{d}u\big) = v$ then*

$$\|\nabla^2 u\|_{L^2} \leqslant C_7\big(t^{-1}\|\mathrm{d}u\|_{L^2} + \|v\|_{L^2}\big), \quad \textit{and} \tag{34}$$

$$\|\nabla^3 u\|_{L^{2m}} \leqslant C_8\big(t^{-(m+3)/2}\|\mathrm{d}u\|_{L^2} + t^{-1}\|v\|_{L^{2m}} + \|\mathrm{d}v\|_{L^{2m}}\big). \tag{35}$$

*Proof.* Let $x \in N$, and define $\sigma = \Psi_x^*(u) \in L^{2m}_3(B_3)$ and $\tau = r^2\Psi_x^*(v) \in L^{2m}_1(B_3)$. Then pulling the equation $\mathrm{d}^*\big(\psi^m\cos\theta\,\mathrm{d}u\big) = v$ back using $\Psi_x$ and rewriting it in the standard coordinates $(x_1,\ldots,x_m)$ on $B_3$, calculation shows that (28) holds, where

$$\begin{gathered} a^{ij} = -\Psi_x^*(\psi^m\cos\theta)\big[\big(r^{-2}\Psi_x^*(h)\big)^{-1}\big]^{ij} \quad \text{and} \\ b^i = \sum_{j=1}^m\Big(\frac{\partial a^{ij}}{\partial x_j} + \frac{a^{ij}}{2}\frac{\partial}{\partial x_j}\big(\log\det\big[r^{-2}\Psi_x^*(h)\big]_{kl}\big)\Big). \end{gathered} \tag{36}$$

Here $(r^{-2}\Psi_x^*(h))^{-1}$ denotes the inverse in $S^2TB_3$ of the metric $r^{-2}\Psi_x^*(h)$, and $[\ldots]^{ij}$, $[\ldots]_{kl}$ is the index notation for tensors, and $[r^{-2}\Psi_x^*(h)]_{kl}$ is regarded as an $m \times m$ matrix, so that we can take its determinant.

Now by Proposition 5.9, if $D_1$ is small then $r^{-2}\Psi_x^*(h)$ is $L^{2m}_2$ close to $g_0$, and hence $C^1$ close as $L^{2m}_2 \hookrightarrow C^1$ by Theorem 5.1. Therefore $(r^{-2}\Psi_x^*(h))^{-1}$ is $L^{2m}_2$ and $C^1$ close to $g_0^{-1}$, and thus $[(r^{-2}\Psi_x^*(h))^{-1}]^{ij}$ is $L^{2m}_2$ and $C^1$ close to $\delta^i_j$.

Since the component of $\beta|_N$ in $\Lambda^{m-1}H^*\otimes V^*$ is $\psi^m\cos\theta$, the inequality $\|\beta\|_{C^0} \leqslant A_4$ in Theorem 5.3 implies that $\psi^m\cos\theta \leqslant A_4$. Using $\cos\theta \geqslant \frac{1}{2}$ from Definition 5.2 and $\psi \geqslant A_3 > 0$ by Theorem 5.3 then gives $0 < \frac{1}{2}A_3^m \leqslant$

$\Psi_x^*(\psi^m \cos\theta) \leqslant A_4$ on $B_3$. Combining this with the fact that $[(r^{-2}\Psi_x^*(h))^{-1}]^{ij}$ is $C^0$ close to $\delta^i_j$, we can find $E_1 > 0$ depending only on $D_1, A_3, A_4$ and $m$ such that the first equation of (27) holds.

Again, the inequality $\|\hat\nabla^k\beta\|_{C^0} \leqslant A_4 t^{-k}$ for $k = 0, 1, 2$ in Theorem 5.3 implies that $\left|\nabla^k(\psi^m \cos\theta)|_N\right|_h \leqslant A_4 t^{-k}$ for $k = 0, 1, 2$ on $N$. As $r = D_2 t$, this implies that $\left|\nabla^k(\psi^m \cos\theta)|_N\right|_{r^{-2}h} \leqslant A_4 D_2^k$ for $k = 0, 1, 2$ on $N$, taking $|\,.\,|$ using the metric $r^{-2}h$ rather than $h$. Pulling back to $B_3$ using $\Psi_x$ and using the fact that $r^{-2}\Psi_x^*(h)$ is $C^1$ close to $g_0$, we can find $E_8 > 0$ depending on $A_4, D_1, D_2$ and $m$ such that $\left|\partial^k \Psi_x^*(\psi^m \cos\theta)\right|_{g_0} \leqslant E_8$ on $B_3$ for $k = 0, 1, 2$.

Combining this with (36) and $\|r^{-2}\Psi_x^*(h) - g_0\|_{L^{2m}_2} \leqslant D_1$ we can find $E_2 > 0$ depending only on $A_3, A_4, D_1, D_2$ and $m$ such that the second equation of (27) holds. Therefore Proposition 5.12 gives $E_3, E_4 > 0$ such that (29) and (30) hold. Here $\nabla$ and all norms are taken w.r.t. $g_0$ on $B_3$. But as $\|r^{-2}\Psi_x^*(h) - g_0\|_{L^{2m}_2} \leqslant D_1$ we can increase $E_3, E_4$ to $E_3', E_4'$ depending only on $E_3, E_4, D_1$ and $m$ such that (29) and (30) hold with $\nabla$ and all norms taken w.r.t. $r^{-2}\Psi_x^*(h)$ on $B_3$.

Pushing these inequalities forward with $\Psi_x$ and remembering that $\sigma = \Psi_x^*(u)$ and $\tau = r^2\Psi_x^*(v)$ gives

$$\left\|\nabla^2 u|_{\Psi_x(B_2)}\right\|_{L^2} \leqslant E_3'\bigl(r^{-1}\|\,\mathrm{d}u|_{\Psi_x(B_3)}\bigr\|_{L^2} + \bigl\|v|_{\Psi_x(B_3)}\bigr\|_{L^2}\bigr) \quad \text{and} \tag{37}$$

$$\begin{aligned}\left\|\nabla^3 u|_{\Psi_x(B_2)}\right\|_{L^{2m}} \leqslant E_4'\bigl(&r^{-(m+3)/2}\bigl\|\,\mathrm{d}u|_{\Psi_x(B_3)}\bigr\|_{L^2} \\ &+ r^{-1}\bigl\|v|_{\Psi_x(B_3)}\bigr\|_{L^{2m}} + \bigl\|\,\mathrm{d}v|_{\Psi_x(B_3)}\bigr\|_{L^{2m}}\bigr).\end{aligned} \tag{38}$$

Here $\nabla$ and all norms are taken w.r.t. $h$, and the powers of $r$ compensate for the change of metrics from $r^{-2}h$ to $h$, and the $r^2$ factor in $\tau = r^2\Psi_x^*(v)$.

Raising (37) and (38) to the powers 2 and $2m$ respectively, we deduce

$$\begin{aligned}\int_{\Psi_x(B_2)} |\nabla^2 u|^2 \mathrm{d}V &\leqslant 2(E_3')^2 \int_{\Psi_x(B_3)} \bigl(r^{-2}|\mathrm{d}u|^2 + |v|^2\bigr)\mathrm{d}V \quad \text{and} \\ \int_{\Psi_x(B_2)} |\nabla^3 u|^{2m} &\leqslant 3^{2m-1}(E_4')^{2m} r^{-m(m+3)} \bigl\|\,\mathrm{d}u|_{\Psi_x(B_3)}\bigr\|_{L^2}^{2m-2} \int_{\Psi_x(B_3)} |\mathrm{d}u|^2 \mathrm{d}V \\ &+ 3^{2m-1}(E_4')^{2m} \int_{\Psi_x(B_3)} \bigl(r^{-2m}|v|^{2m} + |\mathrm{d}v|^{2m}\bigr)\mathrm{d}V,\end{aligned}$$

since $(a+b)^2 \leqslant 2(a^2+b^2)$ and $(a+b+c)^{2m} \leqslant 3^{2m-1}(a^{2m}+b^{2m}+c^{2m})$. As $B_r(x) \subset \Psi_x(B_2) \subset \Psi_x(B_3) \subset B_{4r}(x)$ by Proposition 5.9 and $\|\mathrm{d}u|_{\Psi_x(B_3)}\|_{L^2} \leqslant \|\mathrm{d}u\|_{L^2}$, this gives

$$\int_{B_r(x)} |\nabla^2 u|^2 \mathrm{d}V \leqslant 2(E_3')^2 \int_{B_{4r}(x)} \bigl(r^{-2}|\mathrm{d}u|^2 + |v|^2\bigr)\mathrm{d}V \quad \text{and} \tag{39}$$

$$\begin{aligned}\int_{B_r(x)} |\nabla^3 u|^{2m} &\leqslant 3^{2m-1}(E_4')^{2m} r^{-m(m+3)} \|\mathrm{d}u\|_{L^2}^{2m-2} \int_{B_{4r}(x)} |\mathrm{d}u|^2 \mathrm{d}V \\ &+ 3^{2m-1}(E_4')^{2m} \int_{B_{4r}(x)} \bigl(r^{-2m}|v|^{2m} + |\mathrm{d}v|^{2m}\bigr)\mathrm{d}V.\end{aligned} \tag{40}$$

Integrate (39) and (40) over $x \in N$. The left hand side of (39) gives

$$\begin{aligned}\int_{x\in N}\int_{y\in B_r(x)}|\nabla^2 u|^2(y)\mathrm{d}V_y\mathrm{d}V_x &= \int_{y\in N}\int_{x\in B_r(y)}|\nabla^2 u|^2(y)\mathrm{d}V_x\mathrm{d}V_y\\ &= \int_{y\in N}\mathrm{vol}\bigl(B_r(y)\bigr)|\nabla^2 u|^2(y)\mathrm{d}V_y,\end{aligned}$$

exchanging the order of integration of $x, y$ and noting that $y \in B_r(x)$ if and only if $x \in B_r(y)$. Using $D_3 t^m \leqslant \mathrm{vol}\bigl(B_r(y)\bigr) \leqslant \mathrm{vol}\bigl(B_{4r}(y)\bigr) \leqslant D_4 t^m$ from Proposition 5.9, we get

$$\begin{aligned}\int_N D_3 t^m|\nabla^2 u|^2\mathrm{d}V &\leqslant 2(E_3')^2\int_N D_4 t^m\bigl(r^{-2}|\mathrm{d}u|^2+|v|^2\bigr)\mathrm{d}V \quad\text{and}\\ \int_N D_3 t^m|\nabla^3 u|^{2m} &\leqslant 3^{2m-1}(E_4')^{2m}r^{-m(m+3)}\|\mathrm{d}u\|_{L^2}^{2m-2}\int_N D_4 t^m|\mathrm{d}u|^2\mathrm{d}V\\ &\quad+3^{2m-1}(E_4')^{2m}\int_N D_4 t^m\bigl(r^{-2m}|v|^{2m}+|\mathrm{d}v|^{2m}\bigr)\mathrm{d}V,\end{aligned}$$

or equivalently, dividing by $D_3 t^m$ and substituting $r = D_2 t$,

$$\|\nabla^2 u\|_{L^2}^2 \leqslant 2D_3^{-1}D_4(E_3')^2\bigl(D_2^{-2}t^{-2}\|\mathrm{d}u\|_{L^2}^2+\|v\|_{L^2}^2\bigr)\quad\text{and} \tag{41}$$

$$\begin{aligned}\|\nabla^3 u\|_{L^{2m}}^{2m} &\leqslant 3^{2m-1}D_3^{-1}D_4(E_4')^{2m}D_2^{-m(m+3)}t^{-m(m+3)}\|\mathrm{d}u\|_{L^2}^{2m}\\ &\quad+3^{2m-1}D_3^{-1}D_4(E_4')^{2m}\bigl(D_2^{-2m}t^{-2m}\|v\|_{L^{2m}}^{2m}+\|\mathrm{d}v\|_{L^{2m}}^{2m}\bigr).\end{aligned} \tag{42}$$

Raising (41), (42) to the powers $\frac{1}{2}$, $\frac{1}{2m}$ and using $(a+b)^{1/2} \leqslant a^{1/2}+b^{1/2}$ and $(a+b+c)^{1/2m} \leqslant a^{1/2m}+b^{1/2m}+c^{1/2m}$ for $a, b, c \geqslant 0$ yields (34) and (35) with

$$\begin{aligned}C_7 &= 2^{1/2}D_3^{-1/2}D_4^{1/2}E_3'\max(D_2^{-1},1)\quad\text{and}\\ C_8 &= 3^{1-1/2m}D_3^{-1/2m}D_4^{1/2m}E_4'\max(D_2^{-(m+3)/2},D_2^{-1},1).\end{aligned}$$

These depend only on $A_3, A_4, A_5, A_6$ and $m$, and the proof is complete. □

### 5.5 The proof of Theorem 5.3

We shall now prove Theorem 5.3. Let $f \in C^\infty(N)$, and apply Lemma 5.7 to the 1-form $\alpha = \mathrm{d}f$. As $\mathrm{d}\alpha = 0$ automatically, the lemma shows that if $\|\mathrm{d}f\|_{C^0} < A_1 t$ then $\Phi\bigl(\Gamma(\mathrm{d}f)\bigr)$ is special Lagrangian if and only if

$$\mathrm{d}^*\bigl(\psi^m\cos\theta\,\mathrm{d}f\bigr) = \psi^m\sin\theta + Q(\mathrm{d}f). \tag{43}$$

The idea of the proof is to construct by induction a sequence $(f_n)_{n=0}^\infty$ in $C^\infty(N)$ satisfying $\|\mathrm{d}f_n\|_{C^0} < A_1 t$ and

$$\mathrm{d}^*\bigl(\psi^m\cos\theta\,\mathrm{d}f_n\bigr) = \psi^m\sin\theta + Q(\mathrm{d}f_{n-1}) \tag{44}$$

for $n \geqslant 1$. Then we prove using a priori estimates that the $f_n$ converge in $L^{2m}_3(N)$ to $f$ which satisfies (43), and finally we show that $f \in C^\infty(N)$ by elliptic regularity.

We start with two lemmas. The first follows from Aubin [1, Th. 4.7], and will give existence for $f_n$ in (44) by induction.

**Lemma 5.14** *For each $v \in C^\infty(N)$ with $\int_N v\,\mathrm{d}V = 0$ there exists a unique $u \in C^\infty(N)$ with $\int_N u\,\mathrm{d}V = 0$ and $\mathrm{d}^*\bigl(\psi^m \cos\theta\,\mathrm{d}u\bigr) = v$.*

**Lemma 5.15** *Let $v \in C^\infty(N)$ with $\pi_W(v) = 0$ and $w \in W$. Then $\|\mathrm{d}v\|_{L^2} + \|\mathrm{d}w\|_{L^2} \leqslant 2\|\mathrm{d}v + \mathrm{d}w\|_{L^2}$.*

*Proof.* Using integration by parts and Hölder's inequality gives

$$\begin{aligned}\|\mathrm{d}v + \mathrm{d}w\|^2_{L^2} &= \|\mathrm{d}v\|^2_{L^2} + \|\mathrm{d}w\|^2_{L^2} + 2\langle \mathrm{d}v, \mathrm{d}w\rangle = \|\mathrm{d}v\|^2_{L^2} + \|\mathrm{d}w\|^2_{L^2} + 2\langle v, \mathrm{d}^*\mathrm{d}w\rangle \\ &\geqslant \|\mathrm{d}v\|^2_{L^2} + \|\mathrm{d}w\|^2_{L^2} - 2\|v\|_{L^{2m/(m-2)}}\|\mathrm{d}^*\mathrm{d}w\|_{L^{2m/(m+2)}} \\ &\geqslant \|\mathrm{d}v\|^2_{L^2} + \|\mathrm{d}w\|^2_{L^2} - 2A_7\|\mathrm{d}v\|_{L^2}\cdot \tfrac{1}{2}A_7^{-1}\|\mathrm{d}w\|_{L^2} \\ &= \tfrac{1}{4}\bigl(\|\mathrm{d}v\|_{L^2} + \|\mathrm{d}w\|_{L^2}\bigr)^2 + \tfrac{3}{4}\bigl(\|\mathrm{d}v\|_{L^2} - \|\mathrm{d}w\|_{L^2}\bigr)^2,\end{aligned}$$

by parts (vi), (vii) of Theorem 5.3. The lemma follows. □

The following proposition constructs the sequence $(f_n)_{n=0}^\infty$ and proves the a priori estimates we need. At various points in its proof we shall need $t$ to be smaller than some positive constant defined in terms of $\kappa$ and $A_1, \ldots, A_8$. As a shorthand we will simply say that this holds as $t \leqslant \epsilon$, and suppose without remark that $\epsilon > 0$ has been chosen so that the relevant restriction holds.

The constants $C_1, \ldots, C_8 > 0$ appearing in the proposition and proof are those of Propositions 5.8, 5.11 and 5.13. Note that as $C_1, \ldots, C_8$ depend only on $A_1, \ldots, A_7$ and $m$, it is all right for $\epsilon, K, F_2, \ldots, F_5$ to depend on them.

**Proposition 5.16** *There exist $\epsilon, K, F_1, \ldots, F_4 > 0$ depending only on $m, \kappa$ and $A_1, \ldots, A_8$, such that if $0 < t \leqslant \epsilon$ then there is a unique sequence $(f_n)_{n=0}^\infty$ in $C^\infty(N)$ with $f_0 = 0$ satisfying (44) and $\int_N f_n\,\mathrm{d}V = 0$ for all $n \geqslant 1$ and*

| | |
|---|---|
| (A) $\|\mathrm{d}f_n\|_{L^2} \leqslant F_1 t^{\kappa+m/2}$, | (a) $\|\mathrm{d}f_n - \mathrm{d}f_{n-1}\|_{L^2} \leqslant F_1 2^{-n} t^{\kappa+m/2}$, |
| (B) $\|\mathrm{d}f_n\|_{C^0} \leqslant K t^\kappa \leqslant C_1 t$, | (b) $\|\mathrm{d}f_n - \mathrm{d}f_{n-1}\|_{C^0} \leqslant K 2^{-n} t^\kappa$, |
| (C) $\|\nabla^2 f_n\|_{L^2} \leqslant F_2 t^{\kappa+m/2-1}$, | (c) $\|\nabla^2 f_n - \nabla^2 f_{n-1}\|_{L^2} \leqslant F_2 2^{-n} t^{\kappa+m/2-1}$, |
| (D) $\|\nabla^2 f_n\|_{C^0} \leqslant F_3 t^{\kappa-1} \leqslant C_2$, | (d) $\|\nabla^2 f_n - \nabla^2 f_{n-1}\|_{C^0} \leqslant F_3 2^{-n} t^{\kappa-1}$, |
| (E) $\|\nabla^3 f_n\|_{L^{2m}} \leqslant F_4 t^{\kappa-3/2}$, | (e) $\|\nabla^3 f_n - \nabla^3 f_{n-1}\|_{L^{2m}} \leqslant F_4 2^{-n} t^{\kappa-3/2}$. |

*Proof.* First note that as $f_0 = 0$ we have $f_k = \sum_{n=1}^k (f_n - f_{n-1})$. Suppose that (a) holds for $n = 1, \ldots, k$. Then we have

$$\|\mathrm{d}f_k\|_{L^2} \leqslant \sum_{n=1}^k \|\mathrm{d}f_n - \mathrm{d}f_{n-1}\|_{L^2} \leqslant F_1 t^{\kappa+m/2} \sum_{n=1}^k 2^{-n} \leqslant F_1 t^{\kappa+m/2}.$$

Therefore (a) for $n=1,\ldots,k$ implies (A) for $n=k$. In the same way, (b)–(e) for $n=1,\ldots,k$ imply (B)–(E) for $n=k$. The extra inequalities $Kt^\kappa\leqslant C_1t$ and $F_3t^{\kappa-1}\leqslant C_2$ in (B), (D) hold as $\kappa>1$ and $t\leqslant\epsilon$.

Next suppose that $f_1,\ldots,f_n$ exist and (a), (c), (e) hold for $n$, for some $F_1,F_2,F_4>0$ depending only on $m,\kappa,A_1,\ldots,A_8$. We shall prove (b), (d) for $n$. Apply Proposition 5.11 to $\alpha=\mathrm{d}f_n-\mathrm{d}f_{n-1}$. Equation (26) yields

$$\begin{aligned}\|\nabla^2f_n-\nabla^2f_{n-1}\|_{C^0}&\leqslant C_6\bigl(t^{1/2}\|\nabla^3f_n-\nabla^3f_{n-1}\|_{L^{2m}}+t^{-m/2}\|\nabla^2f_n-\nabla^2f_{n-1}\|_{L^2}\bigr)\\&\leqslant C_6\bigl(t^{1/2}F_42^{-n}t^{\kappa-3/2}+t^{-m/2}F_22^{-n}t^{\kappa+m/2-1}\bigr)=F_32^{-n}t^{\kappa-1}\end{aligned}$$

for $n$ by parts (c), (e), where $F_3=C_6(F_4+F_2)$. This proves (d). Similarly, (25) and parts (a), (d) prove part (b), with $K=C_5(F_3+F_1)$.

Therefore, to complete the proof we only need to show that a unique sequence $(f_n)_{n=0}^\infty$ exists and satisfies (44), $\int_Nf_n\,\mathrm{d}V=0$ and parts (a), (c) and (e) for all $n\geqslant1$. We will do this by induction on $n$. The first step is the case $n=1$. As $f_0=0$ and $Q(0)=0$ by Lemma 5.7, equation (44) gives

$$\mathrm{d}^*\bigl(\psi^m\cos\theta\,\mathrm{d}f_1\bigr)=\psi^m\sin\theta.\tag{45}$$

As in the proof of Lemma 5.7 $\int_N\psi^m\sin\theta\,\mathrm{d}V=0$, so Lemma 5.14 shows there exists a unique $f_1\in C^\infty(N)$ satisfying (45) and $\int_Nf_1\,\mathrm{d}V=0$.

Let $w=\pi_W(f_1)$. Multiplying (45) by $f_1$ and integrating over $N$ yields

$$\begin{aligned}\tfrac12A_3^m\|\mathrm{d}f_1\|_{L^2}^2&\leqslant\int_N\psi^m\cos\theta|\mathrm{d}f_1|^2\mathrm{d}V=\int_Nf_1\psi^m\sin\theta\,\mathrm{d}V\\&=\int_N(f_1-w)\psi^m\sin\theta\,\mathrm{d}V+\int_Nw\,\pi_W(\psi^m\sin\theta)\,\mathrm{d}V\\&\leqslant\|f_1-w\|_{L^{2m/(m-2)}}\|\psi^m\sin\theta\|_{L^{2m/(m+2)}}+\|w\|_{C^0}\bigl\|\pi_W(\psi^m\sin\theta)\bigr\|_{L^1}\\&\leqslant A_7\|\mathrm{d}f_1-\mathrm{d}w\|_{L^2}\cdot A_2t^{\kappa+m/2}+A_8t^{1-m/2}\|\mathrm{d}w\|_{L^2}\cdot A_2t^{\kappa+m-1}\\&\leqslant A_7\cdot2\|\mathrm{d}f_1\|_{L^2}\cdot A_2t^{\kappa+m/2}+A_8t^{1-m/2}\cdot2\|\mathrm{d}f_1\|_{L^2}\cdot A_2t^{\kappa+m-1}.\end{aligned}$$

Here the first line uses part (ii) of Theorem 5.3 and $\cos\theta\geqslant\frac12$ from Definition 5.2, the second the fact that $\langle w,\psi^m\sin\theta\rangle=\langle w,\pi_W(\psi^m\sin\theta)\rangle$ as $w\in W$, the third Hölder's inequality, the fourth parts (i), (vi) and (vii) of Theorem 5.3, and the fifth Lemma 5.15 with $v=f_1-w$. Therefore $\|\mathrm{d}f_1\|_{L^2}\leqslant4A_2A_3^{-m}(A_7+A_8)t^{\kappa+m/2}$, which proves part (a) for $n=1$ with $F_1=8A_2A_3^{-m}(A_7+A_8)$.

Now apply Proposition 5.13 with $u=f_1$ and $v=\psi^m\sin\theta$ by (45), to get

$$\begin{aligned}\|\nabla^2f_1\|_{L^2}&\leqslant C_7\bigl(t^{-1}\|\mathrm{d}f_1\|_{L^2}+\|\psi^m\sin\theta\|_{L^2}\bigr)\\&\leqslant C_7\bigl(t^{-1}\tfrac12F_1t^{\kappa+m/2}+A_2t^{\kappa+m/2-1}\bigr)=\tfrac12F_2t^{\kappa+m/2-1}\quad\text{and}\\\|\nabla^3f_1\|_{L^{2m}}&\leqslant C_8\bigl(t^{-(m+3)/2}\|\mathrm{d}f_1\|_{L^2}+t^{-1}\|\psi^m\sin\theta\|_{L^{2m}}+\|\mathrm{d}\psi^m\sin\theta\|_{L^{2m}}\bigr)\\&\leqslant C_8\bigl(t^{-(m+3)/2}\tfrac12F_1t^{\kappa+m/2}+t^{-1}A_2t^{\kappa-1/2}+A_2t^{\kappa-3/2}\bigr)=\tfrac12F_4t^{\kappa-3/2},\end{aligned}$$

where $F_2=C_7(F_1+2A_2)$ and $F_4=C_8(F_1+4A_2)$. Here we have used (a) when $n=1$ and part (i) of Theorem 5.3, noting that $\|\psi^m\sin\theta\|_{L^{2m/(m+2)}}\leqslant$

$A_2 t^{\kappa+m/2}$ and $\|\psi^m \sin\theta\|_{C^0} \leqslant A_2 t^{\kappa-1}$ imply that $\|\psi^m \sin\theta\|_{L^2} \leqslant A_2 t^{\kappa+m/2-1}$ and $\|\psi^m \sin\theta\|_{L^{2m}} \leqslant A_2 t^{\kappa-1/2}$ by interpolation. This proves (c) and (e) for $n = 1$, completing the first step.

For the inductive step, suppose by induction that $k \geqslant 1$ and that $f_1, \ldots, f_k$ exist and satisfy (44), $\int_N f_n \, \mathrm{d}V = 0$, (A)–(E) and (a)–(e) for $n = 1, \ldots, k$. We shall show that there exists a unique $f_{k+1}$ satisfying (44), $\int_N f_n \, \mathrm{d}V = 0$ and parts (a), (c) and (e) for $n = k + 1$.

By (B) we have $\|\mathrm{d}f_k\|_{C^0} \leqslant C_1 t$, and $C_1 < A_1$ by Proposition 5.8, so $\|\mathrm{d}f_k\|_{C^0} < A_1 t$, and $Q(\mathrm{d}f_k)$ is well-defined. Also $\int_N Q(\mathrm{d}f_k) \, \mathrm{d}V = 0$ by Lemma 5.7, and $\int_N \psi^m \sin\theta \, \mathrm{d}V = 0$ as above. Therefore by Lemma 5.14 there exists a unique $f_{k+1} \in C^\infty(N)$ satisfying (44) for $n = k + 1$ and $\int_N f_{k+1} \, \mathrm{d}V = 0$.

Let $u = f_{k+1} - f_k$. Subtracting (44) for $n = k + 1$, $n = k$ gives

$$\mathrm{d}^*\big(\psi^m \cos\theta \, \mathrm{d}u\big) = \mathrm{d}^*\big(\psi^m \cos\theta \, (\mathrm{d}f_{k+1} - \mathrm{d}f_k)\big) = Q(\mathrm{d}f_k) - Q(\mathrm{d}f_{k-1}) = v, \tag{46}$$

say. We shall estimate some norms of $v$.

**Lemma 5.17** *There exist $G_1, G_2 > 0$ depending only on $C_3, C_4, K, F_1, \ldots, F_4$ and $m$ such that $\|v\|_{L^p} \leqslant G_1 2^{-k} t^{2\kappa+m/p-2}$ for $p \geqslant 1$, $\|v\|_{C^0} \leqslant G_1 2^{-k} t^{2\kappa-2}$ and $\|\mathrm{d}v\|_{L^{2m}} \leqslant G_2 2^{-k} t^{2\kappa-5/2}$.*

*Proof.* Observe that (B) and (D) for $n = k, k-1$ give

$$\|\mathrm{d}f_k\|_{C^0}, \|\mathrm{d}f_{k-1}\|_{C^0} \leqslant C_1 t \quad \text{and} \quad \|\nabla^2 f_k\|_{C^0}, \|\nabla^2 f_{k-1}\|_{C^0} \leqslant C_2.$$

So we may apply Proposition 5.8 with $\alpha = \mathrm{d}f_k$ and $\beta = \mathrm{d}f_{k-1}$, and (20) gives

$$\begin{aligned} |v| \leqslant C_3 \big(t^{-1}|\mathrm{d}f_k - \mathrm{d}f_{k-1}| + |\nabla^2 f_k - \nabla^2 f_{k-1}|\big) \cdot \\ \big(t^{-1}|\mathrm{d}f_k| + t^{-1}|\mathrm{d}f_{k-1}| + |\nabla^2 f_k| + |\nabla^2 f_{k-1}|\big). \end{aligned} \tag{47}$$

Integrating (47) over $N$ and using parts (A), (C), (a), (c) for $n = k, k-1$ shows that $\|v\|_{L^1} \leqslant 4C_3(F_1 + F_2)^2 2^{-k} t^{2\kappa+m-2}$. Taking the supremum of (47) over $N$ and using parts (B), (D), (b), (d) for $n = k, k-1$ gives $\|v\|_{C^0} \leqslant 4C_3(K + F_3)^2 2^{-k} t^{2\kappa-2}$. Define $G_1 = 4C_3 \max\big((F_1 + F_2)^2, (K + F_3)^2\big)$. Then $\|v\|_{L^1} \leqslant G_1 2^{-k} t^{2\kappa+m-2}$ and $\|v\|_{C^0} \leqslant G_1 2^{-k} t^{2\kappa-2}$, so $\|v\|_{L^p} \leqslant G_1 2^{-k} t^{2\kappa+m/p-2}$ for $p \geqslant 1$ by interpolation. A similar but longer proof using (21) and (A)–(E), (a)–(e) for $n = k, k-1$ gives $G_2$ with $\|\mathrm{d}v\|_{L^{2m}} \leqslant G_2 2^{-k} t^{2\kappa-5/2}$. □

Let $w = \pi_W(u)$. Multiplying (46) by $u$ and integrating over $N$ yields

$$\begin{aligned} \tfrac{1}{2} A_3^m \|\mathrm{d}u\|_{L^2}^2 &\leqslant \int_N \psi^m \cos\theta |\mathrm{d}u|^2 \mathrm{d}V = \int_N uv \, \mathrm{d}V = \int_N (u - w)v \, \mathrm{d}V + \int_N wv \, \mathrm{d}V \\ &\leqslant \|u - w\|_{L^{2m/(m-2)}} \|v\|_{L^{2m/(m+2)}} + \|w\|_{C^0} \|v\|_{L^1} \\ &\leqslant A_7 \|\mathrm{d}u - \mathrm{d}w\|_{L^2} \cdot G_1 2^{-k} t^{2\kappa+m/2-1} + A_8 t^{1-m/2} \|\mathrm{d}w\|_{L^2} \cdot G_1 2^{-k} t^{2\kappa+m-2} \\ &\leqslant A_7 \cdot 2\|\mathrm{d}u\|_{L^2} \cdot G_1 2^{-k} t^{2\kappa+m/2-1} + A_8 t^{1-m/2} \cdot 2\|\mathrm{d}u\|_{L^2} \cdot G_1 2^{-k} t^{2\kappa+m-2}. \end{aligned}$$

Here the first line uses part (ii) of Theorem 5.3 and $\cos\theta \geqslant \frac{1}{2}$ from Definition 5.2, the second Hölder's inequality, the third parts (vi) and (vii) of Theorem 5.3 and Lemma 5.17, and the fourth Lemma 5.15 with $u-w$ in place of $v$.

Cancelling shows that $\|\mathrm{d}u\|_{L^2} \leqslant 4G_1A_3^{-m}(A_7+A_8)2^{-k}t^{2\kappa+m/2-1}$. As $u = f_{k+1}-f_k$ and $F_1 = 8A_2A_3^{-m}(A_7+A_8)$, this implies part (a) with $n=k+1$ if $t^{\kappa-1}G_1 \leqslant A_2$, which is true as $\kappa>1$ and $t\leqslant\epsilon$. Now apply Proposition 5.13 to (46), so that (34), part (a) with $n=k+1$ and Lemma 5.17 give

$$\begin{aligned}\|\nabla^2 f_{k+1}-\nabla^2 f_k\|_{L^2} &\leqslant C_7\bigl(t^{-1}\|\mathrm{d}f_{k+1}-\mathrm{d}f_k\|_{L^2}+\|v\|_{L^2}\bigr)\\ &\leqslant C_7\bigl(t^{-1}F_1 2^{-k-1}t^{\kappa+m/2}+G_1 2^{-k}t^{2\kappa+m/2-2}\bigr).\end{aligned}$$

Since $F_2 = C_7(F_1+2A_2)$, this implies part (c) with $n=k+1$ if $G_1t^{\kappa-1}\leqslant A_2$, which is true as $\kappa>1$ and $t\leqslant\epsilon$.

In the same way, from (35), part (a) with $n=k+1$ and Lemma 5.17 we get

$$\begin{aligned}\|\nabla^3 f_{k+1}-\nabla^3 f_k\|_{L^{2m}} &\leqslant C_8\bigl(t^{-(m+3)/2}\|\mathrm{d}f_{k+1}-\mathrm{d}f_k\|_{L^2}+t^{-1}\|v\|_{L^{2m}}+\|\mathrm{d}v\|_{L^{2m}}\bigr)\\ &\leqslant C_8\bigl(t^{-(m+3)/2}F_1 2^{-k-1}t^{\kappa+m/2}+t^{-1}G_1 2^{-k}t^{2\kappa-3/2}+G_2 2^{-k}t^{2\kappa-5/2}\bigr).\end{aligned}$$

As $F_4 = C_8(F_1+4A_2)$, this implies (e) with $n=k+1$ if $(G_1+G_2)t^{\kappa-1}\leqslant 2A_2$, which holds as $\kappa>1$ and $t\leqslant\epsilon$. This completes the inductive step, and the proof. □

We can now finish the proof of Theorem 5.3. If $u\in C^1(N)$ and $\int_N u\,\mathrm{d}V=0$ then $u(x)=0$ for some $x\in N$. If $y\in N$, then $x$ and $y$ are joined by a smooth path $\gamma$ in $N$ of length no more than $\mathrm{diam}(N)$, the diameter of $N$. Integrating along $\gamma$ shows that $\bigl|u(y)\bigr|\leqslant \mathrm{diam}(N)\|\mathrm{d}u\|_{C^0}$. Hence $\|u\|_{C^0}\leqslant\mathrm{diam}(N)\|\mathrm{d}u\|_{C^0}$.

Applying this to part (b) of Proposition 5.16 gives

$$\|f_n-f_{n-1}\|_{C^0}\leqslant K\,\mathrm{diam}(N)2^{-n}t^{\kappa}\quad\text{for } n\geqslant 1.$$

Combining this with parts (b) and (d) shows that $(f_n)_{n=0}^{\infty}$ is a *Cauchy sequence* in the Banach space $C^2(N)$. Let $f$ be the limit of $(f_n)_{n=0}^{\infty}$ in $C^2(N)$.

Part (B) gives $\|\mathrm{d}f_n\|_{C^0}\leqslant Kt^{\kappa}\leqslant C_1t<A_1t$, as $C_1<A_1$ by Proposition 5.8. Taking the limit as $n\to\infty$ gives $\|\mathrm{d}f\|_{C^0}\leqslant Kt^{\kappa}<A_1t$. Therefore $Q(\mathrm{d}f)$ is well-defined. Since $Q(\alpha)$ depends pointwise on $\alpha,\nabla\alpha$ and $f_n\to f$ in $C^2(N)$ it follows that $Q(\mathrm{d}f_n)\to Q(\mathrm{d}f)$ in $C^0(N)$ as $n\to\infty$. So taking the limit $n\to\infty$ in (44) we see that (43) holds for $f$.

By Definition 5.5 and Proposition 5.6, equation (43) is equivalent to

$$F'\bigl(x,\mathrm{d}f(x),\nabla^2 f(x)\bigr)\equiv 0. \tag{48}$$

This is a *second-order nonlinear elliptic equation* on $f$. Note that $F'$ is not linear in $\nabla^2 f(x)$, so that (48) is not quasilinear, and that $F'$ is a smooth function of its arguments. Now Aubin [1, Th. 3.56] gives a regularity result for $C^2$ solutions of such equations, which implies that $f\in C^{\infty}(N)$, as we wish.

We have constructed $\epsilon,K>0$ in Proposition 5.16 depending only on $m,\kappa$ and $A_1,\ldots,A_8$, and $f\in C^{\infty}(N)$ with $\|\mathrm{d}f\|_{C^0}\leqslant Kt^{\kappa}<A_1t$ satisfying (43). Taking the limit $n\to\infty$ in $\int_N f_n\,\mathrm{d}V=0$ yields $\int_N f\,\mathrm{d}V=0$. Finally, (43) and Lemma 5.7 show that $\Phi_*\bigl(\Gamma(\mathrm{d}f)\bigr)$ is an immersed special Lagrangian $m$-fold in $(M,J,\omega,\Omega)$. This concludes the proof of Theorem 5.3.

# 6 Desingularization: the simplest case

Let $(M,J,\omega,\Omega)$ be an almost Calabi–Yau $m$-fold, $X$ be a compact SL $m$-fold in $M$ with conical singularities at $x_1,\ldots,x_n$ with cones $C_i$, and $L_1,\ldots,L_n$ be AC SL $m$-folds in $\mathbb{C}^m$, where $L_i$ has cone $C_i$ and rate $\lambda_i$ for $i=1,\ldots,n$. The goal of the rest of the paper is to *desingularize* $X$ by 'gluing' $L_1,\ldots,L_n$ in at $x_1,\ldots,x_n$, to produce a family of compact, nonsingular SL $m$-folds $\tilde{N}^t$ for $t\in(0,\epsilon]$, which converge to $X$ as $t\to 0$ in an appropriate sense.

Very briefly, we do this by first shrinking $L_i$ by a small factor $t>0$ and gluing $tL_i$ into $X$ at $x_i$ to make a family of compact Lagrangian $m$-folds $N^t$ for $t\in(0,\delta)$, and then applying Theorem 5.3 to show that $N^t$ can be deformed to a nearby SL $m$-fold $\tilde{N}^t$ when $0<t\leqslant\epsilon<\delta$. Now to do this in full generality is rather complex. Therefore we begin in this section with the easiest case, in which $\lambda_i<0$ for all $i$ and $X'=X\setminus\{x_1,\ldots,x_n\}$ is connected.

As explained in §1, this simplifies the problem, avoiding issues of small eigenvalues and obstructions to the existence of $N^t$ from global symplectic topology. Section 7 will extend the results to the case when $X'$ is not connected. The sequel [12] will study the case when $\lambda_i=0$ and $Y(L_i)\neq 0$, and extend the results to *families* of almost Calabi–Yau $m$-folds $(M,J^s,\omega^s,\Omega^s)$ for $s\in\mathcal{F}$.

## 6.1 Setting up the problem

We shall consider the following situation.

**Definition 6.1** Let $(M,J,\omega,\Omega)$ be an almost Calabi–Yau $m$-fold with metric $g$, and define $\psi:M\to(0,\infty)$ as in (3). Let $X$ be a compact SL $m$-fold in $M$ with conical singularities at $x_1,\ldots,x_n$ with identifications $\upsilon_1,\ldots,\upsilon_n$, cones $C_1,\ldots,C_n$ and rates $\mu_1,\ldots,\mu_n\in(2,3)$, as in Definition 3.4. Define $\Sigma_i=C_i\cap\mathcal{S}^{2m-1}$ for $i=1,\ldots,n$. Let $L_1,\ldots,L_n$ be AC SL $m$-folds in $\mathbb{C}^m$, where $L_i$ has cone $C_i$ and rate $\lambda_i$ for $i=1,\ldots,n$, as in Definition 4.1. Suppose that $\lambda_i<\frac{1}{2}(2-m)$ for $i=1,\ldots,n$.

We use the following notation:

- Let $R,B_R,X'$ and $\iota_i,\Upsilon_i$ for $i=1,\ldots,n$ be as in Definition 3.4.
- Let $\zeta$ and $U_{C_i},\Phi_{C_i}$ for $i=1,\ldots,n$ be as in Theorem 3.6.
- Let $R',K$ and $\phi_i,\eta_i,\eta_i^1,\eta_i^2,S_i$ for $i=1,\ldots,n$ be as in Theorem 3.7.
- Let $U_{X'},\Phi_{X'}$ be as in Theorem 3.9.
- Let $A_i$ be as in Theorem 3.8 for $i=1,\ldots,n$, so that $\eta_i=\mathrm{d}A_i$.
- Apply Theorem 4.5 to $L_i$ with $\zeta,U_{C_i},\Phi_{C_i}$ as above, for $i=1,\ldots,n$. Let $T>0$ be as in the theorem, the same for all $i$. Let the subset $K_i\subset L_i$, the diffeomorphism $\varphi_i:\Sigma_i\times(T,\infty)\to L_i\setminus K_i$ and the 1-form $\chi_i$ on $\Sigma_i\times(T,\infty)$ with components $\chi_i^1,\chi_i^2$ be as in Theorem 4.5.
- Let $U_{L_i},\Phi_{L_i}$ be as in Theorem 4.7 for $i=1,\ldots,n$.
- Let $E_i\in C^\infty\bigl(\Sigma_i\times(T,\infty)\bigr)$ be as in Theorem 4.6 for $i=1,\ldots,n$.

In §6.4–§6.5 we will also suppose that $X'$ is *connected*, and in §6.5 we will relax the condition $\lambda_i < \frac{1}{2}(2-m)$ to $\lambda_i < 0$ using Theorem 4.6. With this notation, we define a family of Lagrangian $m$-folds $N^t$ in $(M,\omega)$ for $t \in (0,\delta)$.

**Definition 6.2** In the situation of Definition 6.1, choose a smooth, increasing function $F:(0,\infty)\rightarrow[0,1]$ with $F(r)\equiv 0$ for $r\in(0,1)$ and $F(r)\equiv 1$ for $r>2$. Write $F'$ for $\mathrm{d}F/\mathrm{d}r$. Let $t>0$ act as a *dilation* $\mathbf{x}\mapsto t\,\mathbf{x}$ on $\mathbb{C}^m$. Write $tK_i, tL_i$ for the images of $K_i, L_i$ under $t$, so that $tK_i=\{t\,\mathbf{x}:\mathbf{x}\in K_i\}$, and so on. Let $\tau\in(0,1)$ satisfy

$$\max_{i=1,\ldots,n}\Bigl(\frac{m+2}{2\mu_i+m-2}\Bigr)<\tau<1, \tag{49}$$

which is possible as $\mu_i>2$ implies $(m+2)/(2\mu_i+m-2)<1$.

For $i=1,\ldots,n$ and small enough $t>0$, define $P_i^t=\Upsilon_i(tK_i)$. This is well-defined if $tK_i\subset B_R\subset\mathbb{C}^m$, and is a compact submanifold of $M$ with boundary, diffeomorphic to $K_i$. As $K_i$ is Lagrangian in $(\mathbb{C}^m,\omega')$ and $\Upsilon_i^*(\omega)=\omega'$, we see that $P_i^t$ is *Lagrangian* in $(M,\omega)$.

For $i=1,\ldots,n$ and $t>0$ with $tT<t^\tau<2t^\tau<R'$, define a 1-form $\xi_i^t$ on $\Sigma_i\times(tT,R')$ by

$$\begin{aligned}\xi_i^t(\sigma,r)&=\mathrm{d}\bigl[F(t^{-\tau}r)A_i(\sigma,r)+t^2(1-F(t^{-\tau}r))E_i(\sigma,t^{-1}r)\bigr]\\&=F(t^{-\tau}r)\eta_i(\sigma,r)+t^{-\tau}F'(t^{-\tau}r)A_i(\sigma,r)\mathrm{d}r\\&\quad+t^2(1-F(t^{-\tau}r))\chi_i(\sigma,t^{-1}r)-t^{2-\tau}F'(t^{-\tau}r)E_i(\sigma,t^{-1}r)\mathrm{d}r.\end{aligned} \tag{50}$$

Let $\xi_i^{1,t},\xi_i^{2,t}$ be the components of $\xi_i^t$ in $T^*\Sigma$ and $\mathbb{R}$, as for $\eta_i,\chi_i$ in Theorems 3.7 and 4.5. Note that when $r\geqslant 2t^\tau$ we have $F(t^{-\tau}r)\equiv 1$ so that $\xi_i^t(\sigma,r)=\eta_i(\sigma,r)$, and when $r\leqslant t^\tau$ we have $F(t^{-\tau}r)\equiv 0$, so that $\xi_i^t(\sigma,r)=t^2\chi_i(\sigma,t^{-1}r)$. Thus $\xi_i^t$ is an exact 1-form which interpolates between $\eta_i(\sigma,r)$ near $r=R'$ and $t^2\chi_i(\sigma,t^{-1}r)$ near $r=tT$.

Choose $\delta\in(0,1]$ with $\delta T\leqslant\delta^\tau<2\delta^\tau\leqslant R'$ and $\delta K_i\subset B_R\subset\mathbb{C}^m$ and

$$\bigl|\xi_i^t(\sigma,r)\bigr|<\zeta r\quad\text{on }\Sigma_i\times(tT,R')\text{ for all }i=1,\ldots,n\text{ and }t\in(0,\delta). \tag{51}$$

Here $t\in(0,\delta)$ and $\delta T\leqslant\delta^\tau<2\delta^\tau\leqslant R'$ imply that $tT<t^\tau<2t^\tau<R'$, so $\xi_i^t$ exists. As $|\eta_i(\sigma,r)|<\zeta r$ in Theorem 3.7 and $\xi_i^t\equiv\eta_i$ when $r\geqslant 2t^\tau$, equation (51) holds automatically on $\Sigma_i\times[2t^\tau,R')$. Similarly, as $\bigl|\chi_i(\sigma,r)\bigr|<\zeta r$ in Theorem 4.5 and $\xi_i^t(\sigma,r)=t^2\chi_i(\sigma,t^{-1}r)$ when $r\leqslant t^\tau$, equation (51) holds on $\Sigma_i\times(tT,t^\tau]$. We can show using (7), (12) and (50) that (51) holds on $\Sigma_i\times(t^\tau,2t^\tau)$ for small enough $t>0$, so $\delta$ exists.

For $i=1,\ldots,n$ and $t\in(0,\delta)$, define $\Xi_i^t:\Sigma_i\times(tT,R')\rightarrow M$ by

$$\Xi_i^t(\sigma,r)=\Upsilon_i\circ\Phi_{C_i}\bigl(\sigma,r,\xi_i^{1,t}(\sigma,r),\xi_i^{2,t}(\sigma,r)\bigr). \tag{52}$$

Then $\Phi_{C_i}\bigl(\ldots\bigr)$ is well-defined as $\bigl|\xi_i^t(\sigma,r)\bigr|<\zeta r$ by (51), and making $R'$ smaller if necessary we can ensure that $\Phi_{C_i}\bigl(\ldots\bigr)$ lies in $B_R$, so $\Xi_i^t$ is well-defined, and is an embedding as $\Upsilon_i,\Phi_{C_i}$ are.

Define $Q_i^t=\Xi_i^t\bigl(\Sigma_i\times(tT,R')\bigr)$ for $i=1,\ldots,n$ and $t\in(0,\delta)$. As $\Upsilon_i^*(\omega)=\omega'$, $\Phi_{C_i}^*(\omega')=\hat\omega$ and $\xi_i^t$ is a closed 1-form we see that $(\Xi_i^t)^*(\omega)\equiv 0$. Thus $Q_i^t$ is

*Lagrangian* in $(M,\omega)$, and is a noncompact embedded submanifold diffeomorphic to $\Sigma_i\times(tT,R')$. For $t\in(0,\delta)$, define $N^t$ to be the disjoint union of $K$, $P_1^t,\ldots,P_n^t$ and $Q_1^t,\ldots,Q_n^t$, where $K\subset X'$ is as above.

Then $N^t$ is *Lagrangian* in $(M,\omega)$, as $K,P_i^t$ and $Q_i^t$ are. We claim that $N^t$ is a compact, smooth submanifold of $M$ *without boundary*. That is, the boundary $\partial P_i^t\cong\Sigma_i$ joins smoothly onto $Q_i^t\cong\Sigma_i\times(tT,R')$ at the $\Sigma_i\times\{tT\}$ end, and the boundary $\partial K$ is the disjoint union of pieces $\Sigma_i$ for $i=1,\ldots,n$ which join smoothly onto $Q_i^t\cong\Sigma_i\times(tT,R')$ at the $\Sigma_i\times\{R'\}$ end.

To see this, note that $\xi_i^t(\sigma,r)=t^2\chi_i(\sigma,t^{-1}r)$ on $\Sigma_i\times(tT,t^\tau]$, and so

$$\begin{aligned}\Xi_i^t(\sigma,r)&=\Upsilon_i\circ\Phi_{C_i}\bigl(\sigma,r,t^2\chi_i^1(\sigma,t^{-1}r),t\chi_i^2(\sigma,t^{-1}r)\bigr)\\&=\Upsilon_i\bigl(t\,\Phi_{C_i}\bigl(\sigma,t^{-1}r,\chi_i^1(\sigma,t^{-1}r),\chi_i^2(\sigma,t^{-1}r)\bigr)\bigr)=\Upsilon_i\bigl(t\,\varphi_i(\sigma,t^{-1}r)\bigr)\end{aligned}$$

on $\Sigma_i\times(tT,t^\tau]$, using (11) and the dilation equivariance of $\Phi_{C_i}$ in Theorem 3.6. Thus the end $\Xi_i^t\bigl(\Sigma_i\times(tT,t^\tau]\bigr)$ of $Q_i^t$ is $\Upsilon_i\bigl(t\,\varphi_i(\Sigma_i\times(T,t^{\tau-1}])\bigr)\subset\Upsilon_i(tL_i)$, and as $\varphi_i\bigl(\Sigma_i\times(T,t^{\tau-1}]\bigr)\subset L_i$ joins smoothly onto $K_i\subset L_i$ we see that $\Xi_i^t\bigl(\Sigma_i\times(tT,t^\tau]\bigr)\subset Q_i^t$ joins smoothly onto $P_i^t=\Upsilon_i(tK_i)$.

In the same way, as $\xi_i^t\equiv\eta_i$ on $\Sigma_i\times[2t^\tau,R')$ Theorem 3.7 gives

$$\Xi_i^t(\sigma,r)=\Upsilon_i\circ\Phi_{C_i}\bigl(\sigma,r,\eta_i^1(\sigma,r),\eta_i^2(\sigma,r)\bigr)=\Upsilon_i\circ\phi_i(\sigma,r)\ \text{ on }\ \Sigma_i\times[2t^\tau,R'),$$

so $\Xi_i^t\bigl(\Sigma_i\times[2t^\tau,R')\bigr)\subset Q_i^t$ is $\Upsilon_i\circ\phi_i(\Sigma_i\times[2t^\tau,R'))\bigr)\subset S_i\subset X'$, which joins smoothly onto $K$. Therefore $N^t$ is compact and smooth without boundary.

Here $N^t$ depends smoothly on $t$, and converges to the singular SL $m$-fold $X$ in $M$ as $t\to 0$, in the sense of currents in Geometric Measure Theory. Also $N^t$ is equal to $X'$ in $K$ and the annuli $\Xi_i^t\bigl(\Sigma_i\times[2t^\tau,R')\bigr)\subset Q_i^t$, and equal to $\Upsilon_i(tL_i)$ on $P_i^t=\Upsilon_i(tK_i)$ and the annuli $\Xi_i^t\bigl(\Sigma_i\times(tT,t^\tau]\bigr)\subset Q_i^t$. In between, on the annuli $\Xi_i^t\bigl(\Sigma_i\times(t^\tau,2t^\tau)\bigr)\subset Q_i^t$, $N^t$ interpolates smoothly between $X'$ and $\Upsilon_i(tL_i)$ as a Lagrangian submanifold.

At several points later on we shall make $\delta>0$ smaller if necessary to ensure that something works for all $t\in(0,\delta)$. This is for simplicity, to avoid introducing a series of further constants $\delta',\delta'',\ldots$ in $(0,\delta)$.

## 6.2 Estimating $\operatorname{Im}\Omega|_{N^t}$

We now prove estimates for $\operatorname{Im}\Omega|_{N^t}$, to use in part (i) of Theorem 5.3. Here is some more notation.

**Definition 6.3** In the situation of Definitions 6.1 and 6.2, let $h^t$ be the restriction of $g$ to $N^t$ for $t\in(0,\delta)$, so that $(N^t,h^t)$ is a compact Riemannian manifold. As $X',L_i$ are SL $m$-folds they are oriented, and $N^t$ is made by gluing $X',L_1,\ldots,L_n$ together in an orientation-preserving way, so $N^t$ is also oriented. Let $\mathrm{d}V^t$ be the volume form on $N^t$ induced by $h^t$ and this orientation. As in (15) we may write $\Omega|_{N^t}=\psi^m\mathrm{e}^{i\theta^t}\,\mathrm{d}V^t$ for some phase function $\mathrm{e}^{i\theta^t}$ on $N^t$. Write $\varepsilon^t=\psi^m\sin\theta^t$, so that $\operatorname{Im}\Omega|_{N^t}=\varepsilon^t\,\mathrm{d}V^t$ for $t\in(0,\delta)$.

We compute bounds for $\varepsilon^t$ at each point in $N^t$.

**Proposition 6.4** *In the situation above, making $\delta > 0$ smaller if necessary, there exists $C > 0$ such that for all $t \in (0, \delta)$ we have $\varepsilon^t = 0$ on $K$, the pull-back $(\Xi_i^t)^*(\varepsilon^t)$ of $\varepsilon^t$ on $Q_i^t$ satisfies*

$$\big|(\Xi_i^t)^*(\varepsilon^t)\big|(\sigma, r) \leqslant \begin{cases} Cr, & r \in (tT, t^\tau], \\ Ct^{\tau(\mu_i - 2)} + Ct^{(1-\tau)(2-\lambda_i)}, & r \in (t^\tau, 2t^\tau), \\ 0, & r \in [2t^\tau, R'), \end{cases} \tag{53}$$

$$\big|(\Xi_i^t)^*(\mathrm{d}\varepsilon^t)\big|(\sigma, r) \leqslant \begin{cases} C, & r \in (tT, t^\tau], \\ Ct^{\tau(\mu_i - 3)} + Ct^{(1-\tau)(2-\lambda_i)-\tau}, & r \in (t^\tau, 2t^\tau), \\ 0, & r \in [2t^\tau, R'), \end{cases} \tag{54}$$

$$\textit{and} \quad |\varepsilon^t| \leqslant Ct, \quad |\mathrm{d}\varepsilon^t| \leqslant C \quad \textit{on } P_i^t \textit{ for all } i = 1, \dots, n. \tag{55}$$

*Here $|\,.\,|$ is computed using the metrics $(\Xi_i^t)^*(h^t)$ in (54) and $h^t$ in (55).*

*Proof.* Since $N^t$ coincides with $X'$ in $K$ and $\Xi_i^t\big(\Sigma_i \times [2t^\tau, R')\big)$, and $\operatorname{Im}\Omega|_{X'}$ as $X'$ is special Lagrangian, we see that $\varepsilon^t \equiv 0$ on $K$ and $\Xi_i^t\big(\Sigma_i \times [2t^\tau, R')\big)$, giving the bottom lines of (53) and (54).

As $\Upsilon_i^*(\operatorname{Im}\Omega)$ is a smooth $m$-form on $B_R$ and $\Upsilon_i^*(\operatorname{Im}\Omega)|_0 = \upsilon_i^*(\operatorname{Im}\Omega) = \psi(x_i)^m \operatorname{Im}\Omega'$ by Definition 3.4, we see that $\Upsilon_i^*(\operatorname{Im}\Omega) = \psi(x_i)^m \operatorname{Im}\Omega' + O(r)$ on $B_R$, by Taylor's Theorem. Since $tL_i$ is special Lagrangian in $\mathbb{C}^m$ we have $\operatorname{Im}\Omega'|_{tL_i} \equiv 0$. Thus

$$\big|\Upsilon_i^*(\operatorname{Im}\Omega)|_{tL_i}\big| = O(r) \quad \text{on } tL_i \cap B_R, \tag{56}$$

computing $|\,.\,|$ using the metric $\Upsilon_i^*(g)$ on $B_R$, restricted to $tL_i$.

Now $N^t$ coincides with $\Upsilon_i(tL_i)$ on $P_i^t$ and $\Xi_i^t\big(\Sigma_i \times (tT, t^\tau]\big)$, so $\varepsilon^t \mathrm{d}V^t = \operatorname{Im}\Omega|_{\Upsilon_i(tL_i)}$ on these regions. As $h^t$ is the restriction of $g$ to $N^t$ we have $|\mathrm{d}V^t| = 1$, computing $|\,.\,|$ using $g$, so

$$\big|\Upsilon_i^*(\varepsilon^t)\big| = \big|\Upsilon_i^*(\operatorname{Im}\Omega)|_{tL_i}\big| \quad \text{on } t\big(K \cup \varphi_i(\Sigma_i \times (T, t^{\tau-1}]\big) \subset tL_i \cap B_R. \tag{57}$$

Combining (56) and (57) gives $\big|\varepsilon^t\big| = O\big((\Upsilon_i)_*(r)\big)$ on $P_i^t$ and $\Xi_i^t\big(\Sigma_i \times (tT, t^\tau]\big)$. As $(\Upsilon_i)_*(r) = O(t)$ on $P_i^t$, we see that

$$\big|(\Xi_i^t)^*(\varepsilon^t)\big|(\sigma, r) = O(r) \quad \text{for } r \in (tT, t^\tau], \text{ and} \quad |\varepsilon^t| = O(t) \quad \text{on } P_i^t. \tag{58}$$

A similar argument for the derivative $\mathrm{d}\varepsilon^t$ gives

$$\big|(\Xi_i^t)^*(\mathrm{d}\varepsilon^t)\big|(\sigma, r) = O(1) \quad \text{for } r \in (tT, t^\tau], \text{ and} \quad |\mathrm{d}\varepsilon^t| = O(1) \quad \text{on } P_i^t. \tag{59}$$

Next we estimate $\varepsilon^t$ and $\mathrm{d}\varepsilon^t$ on the annuli $\Xi_i^t\big(\Sigma_i \times (t^\tau, 2t^\tau)\big)$. First we bound $\xi_i^t$ and its derivatives on $\Sigma_i \times (t^\tau, 2t^\tau)$. From (7) and (12) we find that

$$\big|\nabla^k A_i(\sigma, r)\big| = O(t^{\tau(\mu_i - k)}) \quad \text{and} \quad \big|\nabla^k E_i(\sigma, t^{-1}r)\big| = O(t^{-\lambda_i + \tau(\lambda_i - k)})$$

for $r \in (t^\tau, 2t^\tau)$, computing $\nabla, |\,.\,|$ using the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (t^\tau, 2t^\tau)$. Substituting these into (50) gives

$$\bigl|\nabla^k \xi_i^t(\sigma, r)\bigr| = O(t^{\tau(\mu_i - 1 - k)}) + O(t^{2-\lambda_i+\tau(\lambda_i-1-k)}) \quad \text{for } r \in (t^\tau, 2t^\tau), \tag{60}$$

computing $\nabla, |\,.\,|$ using the cone metric $\iota_i^*(g')$.

Now $\Upsilon_i^*(\varepsilon^t)$ depends pointwise on $\xi_i^t$ and $\nabla\xi_i^t$, and when $\xi_i^t, \nabla\xi_i^t$ are small the dominant error terms in $\varepsilon^t$ are linear in $\nabla\xi_i^t$. Similarly, $\mathrm{d}\varepsilon^t$ depends pointwise on $\xi_i^t, \nabla\xi_i^t$ and $\nabla^2\xi_i^t$, and when $\xi_i^t, \nabla\xi_i^t$ are small the dominant error terms in $\mathrm{d}\varepsilon^t$ are linear in $\nabla^2\xi_i^t$. The metrics $\iota_i^*(g')$ and $(\Xi_i^t)^*(h^t)$ on $\Sigma_i \times (t^\tau, 2t^\tau)$ are equivalent uniformly in $t$, so (60) also holds with $\nabla, |\,.\,|$ computed using $(\Xi_i^t)^*(h^t)$.

Putting all this together we see that

$$\begin{aligned} \bigl|(\Xi_i^t)^*(\varepsilon^t)\bigr| &= O(t^{\tau(\mu_i-2)}) + O(t^{(1-\tau)(2-\lambda_i)}) && \text{and} \\ \bigl|(\Xi_i^t)^*(\mathrm{d}\varepsilon^t)\bigr| &= O(t^{\tau(\mu_i-3)}) + O(t^{(1-\tau)(2-\lambda_i)-\tau}) && \text{for } r \in (t^\tau, 2t^\tau), \end{aligned} \tag{61}$$

computing $|\,.\,|$ using $(\Xi_i^t)^*(h^t)$. Here we have used $\mu_i < 3$ and $\tau > 0$ to absorb error terms $O(t^\tau)$ and $O(1)$ respectively into the first term on the right hand side of each line of (61), from the same source as (58) and (59). Making $\delta > 0$ smaller if necessary, the rest of (53)–(55) now follow from (58), (59) and (61), for some $C > 0$ independent of $t$. □

Now we can estimate norms of $\varepsilon^t$ and $\mathrm{d}\varepsilon^t$, as in part (i) of Theorem 5.3.

**Proposition 6.5** *There exists $C' > 0$ such that for all $t \in (0, \delta)$ we have*

$$\|\varepsilon^t\|_{L^{2m/(m+2)}} \leqslant C' t^{\tau(1+m/2)} \sum_{i=1}^n \bigl(t^{\tau(\mu_i-2)} + t^{(1-\tau)(2-\lambda_i)}\bigr), \tag{62}$$

$$\|\varepsilon^t\|_{C^0} \leqslant C' \sum_{i=1}^n \bigl(t^{\tau(\mu_i-2)} + t^{(1-\tau)(2-\lambda_i)}\bigr), \tag{63}$$

$$\text{and} \quad \|\mathrm{d}\varepsilon^t\|_{L^{2m}} \leqslant C' t^{-\tau/2} \sum_{i=1}^n \bigl(t^{\tau(\mu_i-2)} + t^{(1-\tau)(2-\lambda_i)}\bigr), \tag{64}$$

*computing norms with respect to the metric $h^t$ on $N^t$.*

*Proof.* As $g'$ and $\Upsilon_i^*(g)$ are equivalent metrics on $B_R$, it is easy to see that there exist $D_1, D_2, D_3 > 0$ such that for all $t \in (0, \delta)$ we have

$$\begin{gathered} \mathrm{vol}\bigl(P_i^t\bigr) \leqslant D_1 t^m, \quad \mathrm{vol}\bigl(\Xi_i^t\bigl(\Sigma_i \times (t^\tau, 2t^\tau)\bigr)\bigr) \leqslant D_2 t^{m\tau} \\ \text{and} \quad (\Xi_i^t)^*(\mathrm{d}V^t) \leqslant D_3\, \mathrm{d}V_{g'} \quad \text{on } \Sigma_i \times (tT, t^\tau], \end{gathered} \tag{65}$$

where $\mathrm{d}V_{g'}$ is the volume form of the cone metric $\iota_i^*(g')$ on $\Sigma_i \times (tT, t^\tau]$.

Combining (53), (55) and (65) we find that

$$\begin{aligned}\int_{N^t}|\varepsilon^t|^{2m/(m+2)}\,\mathrm{d}V^t \leqslant D_1 t^m (Ct)^{2m/(m+2)}\\ + D_2 t^{m\tau}\sum_{i=1}^n \bigl(Ct^{\tau(\mu_i-2)}+Ct^{(1-\tau)(2-\lambda_i)}\bigr)^{2m/(m+2)}\\ + D_3\sum_{i=1}^n \mathrm{vol}(\Sigma_i)\int_{tT}^{t^\tau}(Cr)^{2m/(m+2)}r^{m-1}\mathrm{d}r.\end{aligned}$$

Raising this to the power $(m+2)/2m$ and manipulating, we can prove (62) for some $C'>0$ depending only on $C, D_1, D_2, D_3, m, n$ and $\mathrm{vol}(\Sigma_i)$. Equations (63) and (64) follow by similar arguments. □

Now for part (i) of Theorem 5.3 to hold, we want $\|\varepsilon^t\|_{L^{2m/(m+2)}}\leqslant A_2 t^{\kappa+m/2}$, $\|\varepsilon^t\|_{C^0}\leqslant A_2 t^{\kappa-1}$ and $\|\mathrm{d}\varepsilon^t\|_{L^{2m}}\leqslant A_2 t^{\kappa-3/2}$ for some $\kappa>1$. Clearly, as $t<1$ from (62)–(64) these hold with $A_2=2nC'$ provided for all $i=1,\ldots,n$ we have

$$\tau(1+m/2)+\tau(\mu_i-2)\geqslant\kappa+m/2,\quad \tau(1+m/2)+(1-\tau)(2-\lambda_i)\geqslant\kappa+m/2, \tag{66}$$
$$\tau(\mu_i-2)\geqslant\kappa-1,\qquad (1-\tau)(2-\lambda_i)\geqslant\kappa-1, \tag{67}$$
$$-\tau/2+\tau(\mu_i-2)\geqslant\kappa-3/2,\quad\text{and}\quad -\tau/2+(1-\tau)(2-\lambda_i)\geqslant\kappa-3/2. \tag{68}$$

Elementary calculations using $0<\tau<1$, $\mu_i>2$ and $\lambda_i<2$ show that the first equation of (66) admits a solution $\kappa>1$ provided $\tau>(2+m)/(2\mu_i-2+m)$, and the second equation of (66) admits a solution $\kappa>1$ provided $\lambda_i-1+m/2<0$, that is, provided $\lambda_i<\frac{1}{2}(2-m)$ as in Definition 6.1. Also (66) implies (67) and (68) as $\tau\leqslant 1$.

Therefore, in Definition 6.2 we choose $\tau\in(0,1)$ to satisfy (49). Then there exists $\kappa>1$ satisfying (66)–(68) for all $i=1,\ldots,n$, and we have proved:

**Theorem 6.6** *Making $\delta>0$ smaller if necessary, there exist $A_2>0$ and $\kappa>1$ such that the functions $\varepsilon^t=\psi^m\sin\theta^t$ on $N^t$ satisfy $\|\varepsilon^t\|_{L^{2m/(m+2)}}\leqslant A_2 t^{\kappa+m/2}$, $\|\varepsilon^t\|_{C^0}\leqslant A_2 t^{\kappa-1}$ and $\|\mathrm{d}\varepsilon^t\|_{L^{2m}}\leqslant A_2 t^{\kappa-3/2}$ for all $t\in(0,\delta)$, as in part* (i) *of Theorem 5.3.*

Here is why the condition $\lambda_i<\frac{1}{2}(2-m)$ is needed in Definition 6.1. In (13) the term $Ct^{(1-\tau)(2-\lambda_i)}$ when $r\in(t^\tau,2t^\tau)$ contributes $C't^{2-\lambda_i+\tau(\lambda_i-1+m/2)}$ to the bound for $\|\varepsilon^t\|_{L^{2m/(m+2)}}$ in (62). Now if $\lambda_i\geqslant\frac{1}{2}(2-m)$ then $2-\lambda_i+\tau(\lambda_i-1+m/2)\leqslant 1+m/2$ for all $\tau\in[0,1]$, so whatever value of $\tau$ we choose our estimate for $\|\varepsilon^t\|_{L^{2m/(m+2)}}$ will be at least as big as $O(t^{1+m/2})$.

But for Theorem 5.3 to work we need $\|\varepsilon^t\|_{L^{2m/(m+2)}}=O(t^{\kappa+m/2})$ for $\kappa>1$, that is, we need $\|\varepsilon^t\|_{L^{2m/(m+2)}}$ to be smaller than $O(t^{1+m/2})$. If $\lambda_i\geqslant\frac{1}{2}(2-m)$ then $L_i$ does not decay to the cone $C_i$ fast enough at infinity, so the errors we make in tapering $L_i$ off to $C_i$ are too great for the method of §5 to cope with.

The condition $\lambda_i<\frac{1}{2}(2-m)$ in Definition 6.1 will be relaxed to $\lambda_i<0$ in §6.5 using Theorem 4.6, so it is not as strong a restriction as it appears.

## 6.3 Lagrangian neighbourhoods and bounds on $R(h^t), \delta(h^t)$

Next we show that parts (ii)–(v) of Theorem 5.3 hold for $N^t$ when $t \in (0, \delta)$, with appropriate $A_1, A_3, \dots, A_6 > 0$ independent of $t$. We begin by gluing together the Lagrangian neighbourhoods $U_{X'}, \Phi_{X'}$ for $X'$ in Theorem 3.9 and $U_{L_i}, \Phi_{L_i}$ for $L_i$ in Theorem 4.7 to get a Lagrangian neighbourhood $U_{N^t}, \Phi_{N^t}$ for $N^t$, which we will use to define the $m$-form $\beta^t$ on $\mathcal{B}_{A_1 t} \subset T^*N^t$ in Definition 5.2, and so prove part (iii) of Theorem 5.3.

**Definition 6.7** Define an open neighbourhood $U_{N^t} \subset T^*N^t$ of the zero section $N^t$ in $T^*N^t$ and a smooth map $\Phi_{N^t} : U_{N^t} \to M$ as follows. Let $\pi : T^*N^t \to N^t$ be the natural projection. As $N^t$ is the disjoint union of $K$ and $P_i^t, Q_i^t$ for $i = 1, \dots, n$ we shall define $U_{N^t}$ and $\Phi_{N^t}$ separately over $K, P_i^t$ and $Q_i^t$.

Define $U_{N^t} \cap \pi^*(K)$ and $\Phi_{N^t}|_{U_{N^t} \cap \pi^*(K)}$ by

$$U_{N^t} \cap \pi^*(K) = U_{X'} \cap \pi^*(K) \quad \text{and} \quad \Phi_{N^t}|_{U_{N^t} \cap \pi^*(K)} = \Phi_{X'}|_{U_{X'} \cap \pi^*(K)}, \tag{69}$$

recalling that $K$ is part of $N^t$ and $X'$, so $U_{N^t} \cap \pi^*(K)$ and $U_{X'} \cap \pi^*(K)$ are both subsets of $T^*K$. For $i = 1, \dots, n$, define $U_{N^t} \cap \pi^*(P_i^t)$ and $\Phi_{N^t}|_{U_{N^t} \cap \pi^*(P_i^t)}$ by

$$\begin{aligned} U_{N^t} \cap \pi^*(P_i^t) &= \mathrm{d}(\Upsilon_i \circ t)\big(\{\alpha \in T^*K_i : t^{-2}\alpha \in U_{L_i}\}\big) \\ \text{and} \quad \Phi_{N^t} \circ \mathrm{d}(\Upsilon_i \circ t)(\alpha) &= \Upsilon_i \circ t \circ \Phi_{L_i}(t^{-2}\alpha). \end{aligned} \tag{70}$$

Here the diffeomorphism $\Upsilon_i \circ t : K_i \to P_i^t$ induces $\mathrm{d}(\Upsilon_i \circ t) : T^*K_i \to T^*P_i^t$, and $\alpha \mapsto t^{-2}\alpha$ is multiplication by $t^{-2}$ in the vector space fibres of $T^*K_i \to K_i$.

As in (8)–(9) and (13)–(14), define $U_{N^t} \cap \pi^*(Q_i^t)$ and $\Phi_{N^t}|_{U_{N^t} \cap \pi^*(Q_i^t)}$ by

$$(\mathrm{d}\Xi_i^t)^*(U_{N^t}) = \big\{(\sigma, r, \varsigma, u) \in T^*\big(\Sigma_i \times (tT, R')\big) : \big|(\varsigma, u)\big| < \zeta r\big\} \quad \text{and} \tag{71}$$

$$\Phi_{N^t} \circ \mathrm{d}\Xi_i^t(\sigma, r, \varsigma, u) \equiv \Upsilon_i \circ \Phi_{C_i}\big(\sigma, r, \varsigma + \xi_i^{1,t}(\sigma, r), u + \xi_i^{2,t}(\sigma, r)\big) \tag{72}$$

for all $(\sigma, r, \varsigma, u) \in T^*\big(\Sigma_i \times (tT, R')\big)$ with $\big|(\varsigma, u)\big| < \zeta r$, computing $\nabla, |\,.\,|$ using $\iota_i^*(g')$. Here $\Xi_i^t : \Sigma_i \times (tT, R') \to Q_i^t$ is a diffeomorphism, and induces an isomorphism $\mathrm{d}\Xi_i^t : T^*\big(\Sigma_i \times (tT, R')\big) \to T^*Q_i^t$.

Careful consideration shows that $U_{N^t}$ is well-defined, and $\Phi_{N^t}$ is well-defined in (70) and (72) for small $t$, so making $\delta > 0$ smaller if necessary $\Phi_{N^t}$ is well-defined for $t \in (0, \delta)$. Clearly $\Phi_{N^t}$ is smooth on each of $U_{N^t} \cap \pi^*(K)$, $U_{N^t} \cap \pi^*(P_i^t)$ and $U_{N^t} \cap \pi^*(Q_i^t)$, but we must still show that $\Phi_{N^t}$ is smooth over the joins between them.

As $\xi_i^t \equiv \eta_i$ on $\Sigma_i \times [2t^\tau, R')$, comparing (8)–(9) and (71)–(72) shows that over $\Xi_i^t\big(\Sigma_i \times [2t^\tau, R')\big) \subset N^t \cap X'$ we have $U_{N^t} = U_{X'}$ and $\Phi_{N^t} = \Phi_{X'}$. Therefore $U_{N^t}$ and $\Phi_{N^t}$ join smoothly over the $r = R'$ end of $Q_i^t$ and the corresponding end of $K$. Similarly, as $\xi_i^t(\sigma, r) = t^2 \chi_i(\sigma, t^{-1} r)$ when $r \leqslant t^\tau$, comparing (13)–(14) and (71)–(72) shows that $U_{N^t} = \mathrm{d}(\Upsilon_i \circ t)(t^2 U_{L_i})$ and $\Phi_{N^t} \circ \mathrm{d}(\Upsilon_i \circ t) = \Upsilon_i \circ t \circ \Phi_{L_i} \circ t^{-2}$ over $\Xi_i^t\big(\Sigma_i \times (tT, t^\tau]\big)$. So $U_{N^t}, \Phi_{N^t}$ join smoothly at the $r = tT$ end of $Q_i^t$ and $\partial P_i^t$, by (70).

Therefore $U_{N^t}$ is an open tubular neighbourhood of $N^t$ in $T^*N^t$, and $\Phi_{N^t} : U_{N^t} \to M$ is well-defined and smooth. We shall show that $\Phi_{N^t}^*(\omega) = \hat{\omega}$. On

$U_{N^t} \cap \pi^*(K)$ this follows from $\Phi_{X'}^*(\omega) = \hat\omega$. On $U_{N^t} \cap \pi^*(P_i^t)$ it follows from $\Upsilon_i^*(\omega) = \omega'$, $\Phi_{L_i}^*(\omega') = \hat\omega$, and the fact that the powers of $t$ in (70) cancel out in their effect on $\Phi_{N^t}^*(\omega)$. On $U_{N^t} \cap \pi^*(Q_i^t)$ it follows from $\Upsilon_i^*(\omega) = \omega'$, $\Phi_{C_i}^*(\omega') = \hat\omega$ and the fact that $\xi_i^t$ is closed.

Define an $m$-form $\beta^t$ on $U_{N^t}$ by $\beta^t = \Phi_{N^t}^*(\operatorname{Im}\Omega)$, as in Definition 5.2.

We now prove that parts (ii)–(v) of Theorem 5.3 hold for $N^t$ when $t \in (0,\delta)$.

**Theorem 6.8** *Making $\delta > 0$ smaller if necessary, there exist $A_1, A_3, \ldots, A_6 > 0$ such that for all $t \in (0,\delta)$, as in parts* (ii)–(v) *of Theorem 5.3 we have*

(ii) *$\psi \geqslant A_3$ on $N^t$.*

(iii) *The subset $\mathcal{B}_{A_1 t} \subset T^*N^t$ of Definition 5.2 lies in $U_{N^t}$, and $\|\hat\nabla^k \beta^t\|_{C^0} \leqslant A_4 t^{-k}$ on $\mathcal{B}_{A_1 t}$ for $k = 0, 1, 2$ and $3$.*

(iv) *The injectivity radius $\delta(h^t)$ satisfies $\delta(h^t) \geqslant A_5 t$.*

(v) *The Riemann curvature $R(h^t)$ satisfies $\|R(h^t)\|_{C^0} \leqslant A_6 t^{-2}$.*

*Here part* (iii) *uses the notation of Definition 5.2, and parts* (iv) *and* (v) *refer to the compact Riemannian manifold $(N^t, h^t)$.*

*Proof.* All of (ii)–(v) are in fact elementary, following from obvious facts about the behaviour of $N^t, h^t, U_{N^t}, \Phi_{N^t}$ for small $t$. Let $A_3 = \inf_M \psi$. Then $A_3 > 0$ as $M$ is compact and $\psi : M \to (0,\infty)$ is continuous, and $\psi \geqslant A_3$ on $N^t$ for all $t \in (0,\delta)$ as $N^t \subset M$. This proves (ii).

For part (iii), under $\Upsilon_i \circ t : K_i \to P_i^t$ we have $(\Upsilon_i \circ t)^*(U_{N^t}) = t^2 U_{L_i}$. Hence the radii of the fibres of $\pi : (\Upsilon_i \circ t)^*(U_{N^t}) \to K_i$ scale like $t^2$ in the metric $g'$ on $K_i$, and thus approximately like $t$ in the metric $(\Upsilon_i \circ t)^*(h^t) \approx t^2 g'$ on $K_i$. Therefore $U_{N^t}$ contains all the balls of radius $A_1 t$ in $T^*P_i^t$ for small $A_1 > 0$.

The fibre of $\pi : U_{N^t} \cap \pi^*(Q_i^t) \to Q_i^t$ over $\Xi_i^t(\sigma, r)$ for $(\sigma, r) \in \Sigma_i \times (tT, R')$ is a ball of radius $\zeta r$ about 0 in $T^*_{\Xi_i^t(\sigma,r)} Q_i^t$ w.r.t. the metric $(\Xi_i^t)_*\bigl(\iota_i^*(g')\bigr)$ on $Q_i^t$. As $h^t$ and $(\Xi_i^t)_*\bigl(\iota_i^*(g')\bigr)$ are equivalent metrics uniformly in $t$, we see that $U_{N^t}$ contains all the balls of radius $A_1 t$ in $T^*Q_i^t$ for small $A_1 > 0$. And $U_{N^t}$ and $h^t$ are independent of $t$ over $K$, so this is obviously true over $K$. Thus for small $A_1 > 0$ we have $\mathcal{B}_{A_1 t} \subset U_{N^t}$ for all $t \in (0,\delta)$.

To see that $\|\hat\nabla^k \beta^t\|_{C^0} \leqslant A_4 t^{-k}$ for $k = 0, \ldots, 3$ in (iii), that $\delta(h^t) \geqslant A_5 t$ in (iv) and $\|R(h^t)\|_{C^0} \leqslant A_6 t^{-2}$ in (v), consider the rôle of $t$ in defining $N^t, h^t, U_{N^t}, \Phi_{N^t}$ and $\beta^t$ in Definitions 6.2, 6.3 and 6.7. We make $N^t$ by shrinking $L_i$ by a factor $t > 0$ and gluing it into $X'$ using $\Upsilon_i$. Therefore for small $t > 0$ the metric $h^t$ on $P_i^t$, and on $Q_i^t$ near $P_i^t$, approximates the metric $t^2 g'$ on $L_i$, and the metric $h^t$ on $K$, and on $Q_i^t$ near $K$, is $g|_{X'}$ and independent of $t$.

Now under the homothety $g' \mapsto t^2 g'$ for $g'$ on $L_i$ we have $\delta(t^2 g') = t\,\delta(g')$ and $\|R(t^2 g')\|_{C^0} = t^{-2}\|R(g')\|_{C^0}$. Thus on and near $P_i^t$ we have $\delta(h^t) = O(t)$ and $\|R(h^t)\|_{C^0} = O(t^{-2})$, and it is easy to see that these are the dominant contributions to $\delta(h^t), \|R(h^t)\|_{C^0}$. (In particular, the derivatives of $F(t^{-\tau} r)$ on the annuli $\Sigma_i \times (t^\tau, 2t^\tau)$ in (50) only contribute terms $O(t^\tau), O(t^{-2\tau})$ respectively.)

Thus making $\delta$ smaller if necessary, there exist $A_5, A_6 > 0$ such that (iv), (v) hold for all $t \in (0, \delta)$.

In a similar way, on and near $P_i^t$ we can identify $U_{N^t}$ with $U_{L_i}$ by construction, and then for small $t \in (0, \delta)$ we have $\hat{h}^t \approx t^2 \hat{h}$ and $\beta^t \approx t^m \Phi_{L_i}^*(\operatorname{Im}\Omega')$, where $\hat{h}^t$ and $\hat{h}$ are the metrics constructed on $T^*N^t$ using $h^t$ and on $T^*L_i$ using $g'|_{L_i}$ in Definition 5.2. It then follows that $\|\hat{\nabla}^k \beta^t\|_{C^0} = O(t^{-k})$ on and near $P_i^t$ for small $t$ and all $k \geqslant 0$. This is the dominant contribution to $\|\hat{\nabla}^k \beta^t\|_{C^0}$ on $N^t$, so making $\delta$ smaller if necessary, part (iii) holds for some $A_4 > 0$. □

### 6.4 Sobolev embedding inequalities on $N^t$

We now prove that parts (vi) and (vii) of Theorem 5.3 hold in the simplest case that $X'$ is connected. The author learned the idea behind the next three results, and the reference [16], from Lee [6, §3].

Write $C^k_{\rm cs}(S)$ for the vector subspace of *compactly-supported* functions in $C^k(S)$. In [16], Michael and Simon prove a Sobolev inequality for submanifolds $S$ of $\mathbb{R}^l$, depending on their *mean curvature vector* $H$ in $\mathbb{R}^l$. Applying [16, Th. 2.1] with $h = |u|^{2(m-1)/(m-2)}$ and using Hölder's inequality, we easily prove:

**Theorem 6.9** *Let $S$ be an $m$-submanifold of $\mathbb{R}^l$ for $m > 2$, and $u \in C^1_{\rm cs}(S)$. Then $\|u\|_{L^{2m/(m-2)}} \leqslant D_1\bigl(\|{\rm d}u\|_{L^2} + \|u\,H\|_{L^2}\bigr)$, where $D_1 > 0$ depends only on $m$, and $H$ is the mean curvature of $S$ in $\mathbb{R}^l$.*

If $S$ is *minimal* in $\mathbb{R}^n$ then $H \equiv 0$, and $\|u\|_{L^{2m/(m-2)}} \leqslant D_1 \|{\rm d}u\|_{L^2}$. The SL $m$-folds $tL_i$ in $\mathbb{C}^m$ are automatically minimal, so we deduce:

**Corollary 6.10** *There exists $D_1 > 0$ such that $\|u\|_{L^{2m/(m-2)}} \leqslant D_1 \|{\rm d}u\|_{L^2}$ for all $t > 0$, $i = 1, \ldots, n$ and $u \in C^1_{\rm cs}(tL_i)$.*

Next we prove a similar inequality for $X'$, when it is connected.

**Proposition 6.11** *Suppose $X'$ is connected. Then there exists $D_2 > 0$ such that for all $v \in C^1_{\rm cs}(X')$ we have*

$$\|v\|_{L^{2m/(m-2)}} \leqslant D_2\bigl(\|{\rm d}v\|_{L^2} + \bigl|\textstyle\int_{X'} v \,{\rm d}V_g\bigr|\bigr). \tag{73}$$

*Proof.* By the Nash Embedding Theorem we can choose an isometric embedding of $(M, g)$ in some $\mathbb{R}^l$. Then $X'$ is also isometrically embedded in $\mathbb{R}^l$. The mean curvature of $X'$ in $M$ is zero, as $X'$ is minimal, and the mean curvature of $M$ in $\mathbb{R}^l$ is bounded, as $M$ is compact. Thus the mean curvature $H$ of $X'$ in $\mathbb{R}^l$ is bounded, say $|H| \leqslant D_3$. Applying Theorem 6.9 to $X'$ in $\mathbb{R}^l$ then gives

$$\|u\|_{L^{2m/(m-2)}} \leqslant D_1\bigl(\|{\rm d}u\|_{L^2} + D_3\|u\|_{L^2}\bigr) \quad \text{for all } u \in C^1_{\rm cs}(X'). \tag{74}$$

By studying the eigenvalues of $\Delta$ on $X'$, we show in [10, Th. 2.17] that if $X'$ is connected then for some $D_4 > 0$ and all $u \in C^2_{\rm cs}(X')$ with $\int_{X'} u \,{\rm d}V_g = 0$, we have $\|u\|_{L^2} \leqslant D_4\|\mathrm{d}u\|_{L^2} \leqslant D_4^2\|\Delta u\|_{L^2}$. As $C^2_{\rm cs}(X')$ is dense in $C^1_{\rm cs}(X')$, the first inequality $\|u\|_{L^2} \leqslant D_4\|\mathrm{d}u\|_{L^2}$ holds if $u \in C^1_{\rm cs}(X')$ with $\int_{X'} u\,\mathrm{d}V_g = 0$. Combining this with (74) and setting $D_5 = D_1(1+D_3D_4)$ proves that

$$\|u\|_{L^{2m/(m-2)}} \leqslant D_5\|\mathrm{d}u\|_{L^2} \quad \text{for all } u \in C^1_{\rm cs}(X') \text{ with } \textstyle\int_{X'} u\,\mathrm{d}V_g = 0. \tag{75}$$

Fix some $w \in C^1_{\rm cs}(X')$ with $\int_{X'} w\,\mathrm{d}V_g = 1$. For any $v \in C^1_{\rm cs}(X')$, define $u = v - w\int_{X'} v\,\mathrm{d}V_g$. Then $u \in C^1_{\rm cs}(X')$ with $\int_{X'} u\,\mathrm{d}V_g = 0$, so (75) gives

$$\begin{aligned}\|v\|_{L^{2m/(m-2)}} - \bigl|\textstyle\int_{X'} v\,\mathrm{d}V_g\bigr| \cdot \|w\|_{L^{2m/(m-2)}} &\leqslant \|u\|_{L^{2m/(m-2)}}\\ &\leqslant D_5\|\mathrm{d}u\|_{L^2} \leqslant D_5\bigl(\|\mathrm{d}v\|_{L^2} + \bigl|\textstyle\int_{X'} v\,\mathrm{d}V_g\bigr| \cdot \|\mathrm{d}w\|_{L^2}\bigr).\end{aligned}$$

Equation (73) then follows with $D_2 = \max\bigl(D_5, \|w\|_{L^{2m/(m-2)}} + D_5\|\mathrm{d}w\|_{L^2}\bigr)$. □

We now combine the inequalities on $L_i$ and $X'$ in Corollary 6.10 and Proposition 6.11 to prove an equality on $N^t$ for small $t$. The theorem implies parts (vi) and (vii) of Theorem 5.3 with $W = \langle 1\rangle$, as (vii) is trivial when $W = \langle 1\rangle$.

**Theorem 6.12** *Suppose $X'$ is connected. Making $\delta > 0$ smaller if necessary, there exists $A_7 > 0$ such that for all $t \in (0,\delta)$, if $v \in L^2_1(N^t)$ with $\int_{N^t} v\,\mathrm{d}V^t = 0$ then $v \in L^{2m/(m-2)}(N^t)$ and $\|v\|_{L^{2m/(m-2)}} \leqslant A_7\|\mathrm{d}v\|_{L^2}$.*

*Proof.* Choose $a, b \in \mathbb{R}$ with $0 < a < b < \tau$. Then for small $t > 0$ we have

$$2t^\tau < t^b < t^a < \min(1, R'). \tag{76}$$

Making $\delta > 0$ smaller if necessary, we suppose that (76) holds for all $t \in (0,\delta)$. Let $G : (0,\infty) \to [0,1]$ be a smooth, decreasing function with $G(s) = 1$ for $s \in (0,a]$ and $G(s) = 0$ for $s \in [b,\infty)$. Write $G'$ for $\mathrm{d}G/\mathrm{d}s$. For $t \in (0,\delta)$, define a function $F^t : N^t \to [0,1]$ by

$$F^t(x) = \begin{cases} 1, & x \in K,\\ G\bigl((\log r)/(\log t)\bigr), & x = \Xi^t_i(\sigma,r) \in Q^t_i, \quad i = 1,\ldots,n,\\ 0, & x \in P^t_i, \quad i = 1,\ldots,n.\end{cases} \tag{77}$$

Then $F^t$ is smooth, and $F^t \equiv 0$ on $P^t_i$ and $\Xi^t_i\bigl(\Sigma_i \times (tT, t^b]\bigr)$ for $i = 1,\ldots,n$, and $F^t \equiv 1$ on $K$ and $\Xi^t_i\bigl(\Sigma_i \times [t^a, R')\bigr)$ for $i = 1,\ldots,n$. It changes only on the annuli $\Xi^t_i\bigl(\Sigma_i \times (t^b, t^a)\bigr)$ for $i = 1,\ldots,n$, and there we have

$$(\Xi^t_i)^*(\mathrm{d}F^t) = (\log t)^{-1} \cdot G'\bigl((\log r)/(\log t)\bigr) r^{-1}\mathrm{d}r \quad \text{for } r \in (t^b, t^a). \tag{78}$$

Suppose now that $t \in (0,\delta)$ and $v \in C^1(N^t)$ with $\int_{N^t} v\,\mathrm{d}V^t = 0$. The main idea of the proof is to write $v = F^t v + (1-F^t)v$, where we treat $F^t v$ as a compactly-supported function on $X'$ and apply Proposition 6.11 to it, and we treat $(1-F^t)v$ as a sum over $i = 1,\ldots,n$ of compactly-supported functions on $tL_i$, and apply Corollary 6.10 to them.

The first is straightforward. As $2t^\tau < t^b$ by (76), the support of $F^t$ lies in the union of $K$ and $\Xi_i^t\bigl(\Sigma_i \times [2t^\tau, R')\bigr)$ for $i = 1, \ldots, n$, which is part of both $N^t$ and $X'$. Thus, extending $F^t v$ by zero to the rest of $X'$ we can regard it as an element of $C^1_{\rm cs}(X')$, and since $\int_{X'} F^t v \,{\rm d}V_g = \int_{N^t} F^t v \,{\rm d}V^t$ equation (73) gives

$$\begin{aligned}\bigl\|F^t v\bigr\|_{L^{2m/(m-2)}} &\leqslant D_2\bigl(\bigl\|\,{\rm d}(F^t v)\bigr\|_{L^2} + \bigl|\,\textstyle\int_{N^t} F^t v \,{\rm d}V^t\,\bigr|\bigr)\\ &\leqslant D_2\bigl(\|F^t {\rm d}v\|_{L^2} + \|v\|_{L^{2m/(m-2)}}\cdot(\|{\rm d}F^t\|_{L^m} + \|1-F^t\|_{L^{2m/(m+2)}})\bigr).\end{aligned} \tag{79}$$

Here all functions are on $N^t$ and all norms computed using $h^t$. This is valid because the metrics $g$ on $X'$ and $h^t$ on $N^t$ coincide on the support of $F^t$. We have also used Hölder's inequality upon $\|v\,{\rm d}F^t\|_{L^2}$, and

$$\bigl|\textstyle\int_{N^t} F^t v \,{\rm d}V^t\bigr| = \bigl|\textstyle\int_{N^t}(1-F^t)v \,{\rm d}V^t\bigr| \leqslant \|v\|_{L^{2m/(m-2)}}\cdot\|1-F^t\|_{L^{2m/(m+2)}},$$

since ${\rm d}V_g = {\rm d}V^t$ on the support of $F^t$ as $g = h^t$, and $\int_{N^t} v\,{\rm d}V^t = 0$.

For the second, identify the subset $tK_i \cup t\circ\varphi_i\bigl(\Sigma_i \times (T, t^{-1}R')\bigr)$ in $tL_i \subset \mathbb{C}^m$ with the subset $P_i^t \cup Q_i^t$ in $N^t \subset M$ for $i = 1, \ldots, n$ by

$$\begin{gathered} tK_i \ni tx \mapsto \Upsilon_i(tx) \in P_i^t \quad \text{and}\\ t\circ\varphi_i\bigl(\Sigma_i\times(T,t^{-1}R')\bigr) \ni t\circ\varphi_i(\sigma,r) \mapsto \Xi_i^t(\sigma,tr) \in Q_i^t.\end{gathered}$$

The restriction of $(1-F^t)v$ to $P_i^t \cup Q_i^t$ is compactly-supported, so using this identification we can regard $(1-F^t)v$ as an element of $C^1_{\rm cs}(tL_i)$, extended by zero outside $tK_i \cup t\circ\varphi_i\bigl(\Sigma_i\times(T,t^{-1}R')\bigr)$.

Under this identification between subsets of $tL_i$ and $N^t$, on the support $tK_i \cup t\circ\varphi_i\bigl(\Sigma_i\times(T,t^{a-1})\bigr)$ of $(1-F^t)v$ in $tL_i$, the metrics $g'$ on $tL_i$ and $h^t$ on $N^t$ are close when $t$ is small. Applying Corollary 6.10 to $(1-F^t)v$ on $tL_i$ gives $\|(1-F^t)v\|_{L^{2m/(m-2)}} \leqslant D_1\|{\rm d}((1-F^t)v)\|_{L^2}$, where norms are computed using $g'$ on $tL_i$. As $g', h^t$ are close for small $t$, increasing $D_1$ to $2D_1$ the same inequality holds with norms computed using $h^t$, for small $t$. So making $\delta > 0$ smaller if necessary, for all $t \in (0,\delta)$ and $i = 1, \ldots, n$ we have

$$\bigl\|(1-F^t)v|_{P_i^t\cup Q_i^t}\bigr\|_{L^{2m/(m-2)}} \leqslant 2D_1\bigl\|\,{\rm d}((1-F^t)v)|_{P_i^t\cup Q_i^t}\bigr\|_{L^2}. \tag{80}$$

Now $(1-F^t)v$ is supported on $\bigcup_{i=1}^n (P_i^t \cup Q_i^t)$. Therefore

$$\begin{gathered}\bigl\|(1-F^t)v\bigr\|_{L^{2m/(m-2)}} \leqslant \sum_{i=1}^n \bigl\|(1-F^t)v|_{P_i^t\cup Q_i^t}\bigr\|_{L^{2m/(m-2)}} \quad \text{and}\\ \sum_{i=1}^n \bigl\|\,{\rm d}((1-F^t)v)|_{P_i^t\cup Q_i^t}\bigr\|_{L^2} \leqslant \sqrt{n}\,\bigl\|\,{\rm d}((1-F^t)v)\bigr\|_{L^2},\end{gathered} \tag{81}$$

proving the second equation using $a_1 + \cdots + a_n \leqslant \sqrt{n}\,(a_1^2 + \cdots + a_n^2)^{1/2}$, which gives the factor $\sqrt{n}$. Equations (80) and (81) give

$$\begin{aligned}\bigl\|(1-F^t)v\bigr\|_{L^{2m/(m-2)}} &\leqslant 2\sqrt{n}\,D_1\bigl\|\,{\rm d}((1-F^t)v)\bigr\|_{L^2}\\ &\leqslant 2\sqrt{n}\,D_1\bigl(\bigl\|(1-F^t)\,{\rm d}v\bigr\|_{L^2} + \|v\|_{L^{2m/(m-2)}}\cdot\|{\rm d}F^t\|_{L^m}\bigr).\end{aligned} \tag{82}$$

Combining (79), (82) and $\|F^t \mathrm{d}v\|_{L^2}, \|(1-F^t)\mathrm{d}v\|_{L^2} \leqslant \|\mathrm{d}v\|_{L^2}$ then proves

$$\begin{aligned}\big[1-(D_2+2\sqrt{n}\,D_1)\|\mathrm{d}F^t\|_{L^m} - D_2\|1-F^t\|_{L^{2m/(m+2)}}\big]\cdot\|v\|_{L^{2m/(m-2)}}&\\ \leqslant (D_2+2\sqrt{n}\,D_1)\|\mathrm{d}v\|_{L^2}.&\end{aligned} \tag{83}$$

As the $L^m$ norm of $r^{-1}$ on $\Sigma_i \times (t^b, t^a)$ with the cone metric is $O\big(|\log t|^{1/m}\big)$, using (78) and $\mathrm{vol}\big(\mathrm{supp}(1-F^t)\big) = O\big(t^{ma}\big)$ we find that

$$\|\mathrm{d}F^t\|_{L^m} = O\big(|\log t|^{(1-m)/m}\big) \quad \text{and} \quad \|1-F^t\|_{L^{2m/(m+2)}} = O(t^{a(m+2)/2})$$

for small $t$. Thus making $\delta > 0$ smaller if necessary we can suppose that

$$\big[1-(D_2+2\sqrt{n}\,D_1)\|\mathrm{d}F^t\|_{L^m} - D_2\|1-F^t\|_{L^{2m/(m+2)}}\big] \geqslant \tfrac{1}{2} \quad \text{for all } t \in (0,\delta).$$

Setting $A_7 = 2(D_2 + 2\sqrt{n}\,D_1)$, we see from (83) that for all $v \in C^1(N^t)$ with $\int_{N^t} v\,\mathrm{d}V^t = 0$ we have $\|v\|_{L^{2m/(m-2)}} \leqslant A_7\|\mathrm{d}v\|_{L^2}$. Since $C^1(N^t)$ is dense in $L^2_1(N^t)$ and $L^2_1(N^t) \hookrightarrow L^{2m/(m-2)}(N^t)$ by the Sobolev Embedding Theorem, this completes the proof of Theorem 6.12. □

## 6.5 The main result, in the simplest case

We can now prove our main result on desingularizations of SL $m$-folds $X$ with conical singularities, in the simplest case. To make the statement of the theorem short and easily understood, we do not say much about the $\tilde{N}^t$. But the construction gives a much more precise and detailed description of the topology of $\tilde{N}^t$, and the topology and geometry of its embedding in $(M, J, \omega, \Omega)$, which we can read off from §5–§6 above if we need to.

**Theorem 6.13** *Suppose $(M, J, \omega, \Omega)$ is an almost Calabi–Yau $m$-fold and $X$ a compact SL $m$-fold in $M$ with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Let $L_1, \ldots, L_n$ be Asymptotically Conical SL $m$-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Suppose $\lambda_i < 0$ for $i = 1, \ldots, n$, and $X' = X \setminus \{x_1, \ldots, x_n\}$ is connected.*

*Then there exists $\epsilon > 0$ and a smooth family $\big\{\tilde{N}^t : t \in (0, \epsilon]\big\}$ of compact, nonsingular SL $m$-folds in $(M, J, \omega, \Omega)$, such that $\tilde{N}^t$ is constructed by gluing $tL_i$ into $X$ at $x_i$ for $i = 1, \ldots, n$. In the sense of currents, $\tilde{N}^t \to X$ as $t \to 0$.*

*Proof.* By assumption $L_i$ is an AC SL $m$-fold with rate $\lambda_i < 0$. Now Theorem 4.6 shows that if $L$ is an AC SL $m$-fold in $\mathbb{C}^m$ with rate $\lambda < 0$ then $L$ is *also* Asymptotically Conical with rate $\lambda'$ for all $\lambda' \in (2-m, 0)$. It follows that if $\lambda_i \geqslant \frac{1}{2}(2-m)$ then we can decrease the rate $\lambda_i$ of $L_i$ so that $\lambda_i \in \big(2-m, \frac{1}{2}(2-m)\big)$. Therefore we can suppose that $\lambda_i < \frac{1}{2}(2-m)$ for all $i = 1, \ldots, n$, as in Definition 6.1.

Let $\delta > 0$ and $N^t$ for $t \in (0, \delta)$ be as in Definition 6.2, and make $\delta$ smaller if necessary so that Theorems 6.6, 6.8 and 6.12 apply. For each $t \in (0, \delta)$, define a finite-dimensional vector subspace $W^t \subset C^\infty(N^t)$ by $W^t = \langle 1 \rangle$, the constant functions. This will be $W \subset C^\infty(N)$ in Definition 5.2. The natural projection $\pi_{W^t} : L^2(N^t) \to W^t$ is given by $\pi_{W^t}(v) = \mathrm{vol}(N^t)^{-1} \int_{N^t} v \,\mathrm{d}V^t$.

Let $\varepsilon^t = \psi^m \sin\theta^t \in C^\infty(N^t)$ for $t \in (0,\delta)$ be as in Definition 6.3. Then $\mathrm{Im}\,\Omega|_{N^t} = \psi^m \sin\theta^t \,\mathrm{d}V^t$. Thus

$$\int_{N^t} \psi^m \sin\theta^t \,\mathrm{d}V^t = \int_{N^t} \mathrm{Im}\,\Omega = [\mathrm{Im}\,\Omega]\cdot[N^t] = [\mathrm{Im}\,\Omega]\cdot[X] = 0,$$

where $[\mathrm{Im}\,\Omega] \in H^m(M,\mathbb{R})$ and $[N^t],[X] \in H_m(M,\mathbb{R})$, as $N^t, X$ are homologous in $M$ and $\mathrm{Im}\,\Omega|_{X'} \equiv 0$ as $X'$ is an SL $m$-fold. This implies that

$$\pi_{W^t}(\psi^m \sin\theta^t) = \mathrm{vol}(N^t)^{-1} \int_{N^t} \psi^m \sin\theta^t \,\mathrm{d}V^t = 0 \quad \text{for all } t \in (0,\delta). \tag{84}$$

Theorem 6.6 gives constants $\kappa > 1$ and $A_2 > 0$ such that

$$\begin{gathered} \|\psi^m \sin\theta^t\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa+m/2}, \quad \|\psi^m \sin\theta^t\|_{C^0} \leqslant A_2 t^{\kappa-1} \quad \text{and} \\ \|\mathrm{d}(\psi^m \sin\theta^t)\|_{L^{2m}} \leqslant A_2 t^{\kappa-3/2} \quad \text{for all } t \in (0,\delta). \end{gathered} \tag{85}$$

Equations (84) and (85) imply that part (i) of Theorem 5.3 holds for $N^t$ for all $t \in (0,\delta)$, replacing $N, W, \theta$ by $N^t, W^t, \theta^t$ respectively.

Let the Lagrangian neighbourhood $\Phi_{N^t} : U_{N^t} \to M$ and the $m$-form $\beta^t$ on $U_{N^t}$ be as in Definition 6.7. Then Theorem 6.8 gives constants $A_1, A_3, \ldots, A_6 > 0$ such that parts (ii)–(v) of Theorem 5.3 hold for $N^t$ for all $t \in (0,\delta)$, replacing $N, \beta, h$ by $N^t, \beta^t, h^t$ respectively.

As $\pi_{W^t}(v) = 0$ if and only if $\int_{N^t} v \,\mathrm{d}V^t = 0$, Theorem 6.12 gives $A_7 > 0$ such that if $v \in L^2_1(N^t)$ with $\pi_{W^t}(v) = 0$ then $v \in L^{2m/(m-2)}(N^t)$ and $\|v\|_{L^{2m/(m-2)}} \leqslant A_7 \|\mathrm{d}v\|_{L^2}$, for all $t \in (0,\delta)$. Thus part (vi) of Theorem 5.3 holds for $N^t$ for all $t \in (0,\delta)$, replacing $N, W$ by $N^t, W^t$ respectively. As $W^t = \langle 1 \rangle$ part (vii) of Theorem 5.3 is trivial for $N^t, W^t$, since $\mathrm{d}^*\mathrm{d}w = \mathrm{d}w = 0$ for all $w \in W^t$, and if $w \in W^t$ with $\int_{N^t} w \,\mathrm{d}V^t = 0$ then $w = 0$. Thus part (vii) holds for any $A_8 > 0$, and we take $A_8 = 1$.

We have not yet showed that the inequality $\cos\theta^t \geqslant \frac{1}{2}$ in Definition 5.2 holds. From parts (i) and (ii) of Theorem 5.3 we see that $|\sin\theta^t| \leqslant A_2 A_3^{-m} t^{\kappa-1}$ on $N^t$. Thus for small $t \in (0,\delta)$ we have $|\sin\theta^t| \leqslant \frac{\sqrt{3}}{2}$ as $\kappa > 1$, so that $|\cos\theta^t| \geqslant \frac{1}{2}$. But $\cos\theta^t$ is continuous, $N^t$ is connected, and $\cos\theta^t \equiv 1$ on $K$ as $K$ is special Lagrangian, so we must have $\cos\theta^t \geqslant \frac{1}{2}$ on $N^t$ for small $t \in (0,\delta)$.

We have constructed $\kappa > 1$ and $A_1, \ldots, A_8 > 0$ such that parts (i)–(vii) of Theorem 5.3 hold for $N^t$ in $M$ for all $t \in (0,\delta)$. Let $\epsilon, K > 0$ be as given in Theorem 5.3 depending on $\kappa, A_1, \ldots, A_8$ and $m$, and make $\epsilon > 0$ smaller if necessary to ensure that $\epsilon < \delta$ and $\cos\theta^t \geqslant \frac{1}{2}$ on $N^t$ for $t \in (0,\epsilon]$. Then Theorem 5.3 shows that for all $t \in (0,\epsilon]$ we can deform $N^t$ to a nearby compact, nonsingular SL $m$-fold $\tilde N^t$, given by $\tilde N^t = (\Phi_{N^t})_*\bigl(\Gamma(\mathrm{d}f^t)\bigr)$ for some $f^t \in C^\infty(N^t)$ with $\|\mathrm{d}f^t\|_{C^0} \leqslant K t^\kappa < A_1 t$.

Since $N^t$ and $\Phi_{N^t}$ depend smoothly on $t$, we see that $f^t$ is the locally unique solution of a nonlinear elliptic p.d.e. on $N^t$ depending smoothly on $t$. It quickly follows from general theory that $f^t$ depends smoothly on $t$, and so $\tilde{N}^t$ does. One can easily show from Definition 6.2 that $N^t \to X$ as currents as $t \to 0$. But the estimates on $\mathrm{d}f^t$ and $\nabla \mathrm{d}f^t$ in §5 imply that $\tilde{N}^t - N^t \to 0$ as currents as $t \to 0$, so $\tilde{N}^t \to X$ as $t \to 0$. This completes the proof of Theorem 6.13. $\square$

# 7 Desingularizing when $X'$ is not connected

We now generalize the material of §6 to the case when $X'$ is not connected. Suppose $X'$ has $q > 1$ connected components $X'_1, \ldots, X'_q$. Then Theorem 6.12 no longer holds. The basic reason for this is that the Laplacian $\Delta^t$ on $N^t$ has $q$ small eigenvalues $0 = \lambda^t_1, \ldots, \lambda^t_q$, with $\lambda^t_k = O(t^{m-2})$.

The corresponding eigenfunctions $1 = v^t_1, \ldots, v^t_k$ are approximately constant on the part of $N^t$ coming from $X'_k$ for each $k = 1, \ldots, q$, and change on the part of $N^t$ coming from $tL_i$ for $i = 1, \ldots, n$. If $q > 1$, calculation shows that the $v^t_k$ for $1 < k \leqslant q$ satisfy $\int_{N^t} v^t_k \mathrm{d}V^t = 0$ and $\|v^t_k\|_{L^{2m/(m-2)}} = O(t^{-(m-2)/2}) \cdot \|\mathrm{d}v^t_k\|_{L^2}$, so that Theorem 6.12 cannot hold.

To get round this, we define in §7.1 a vector subspace $W^t \subset C^\infty(N^t)$ which approximates $\langle v^t_1, \ldots, v^t_q \rangle$. This will be $W$ in Theorem 5.3. In §7.3 we show that if $v \in L^2_1(N^t)$ is $L^2$-orthogonal to $W^t$ then $\|v\|_{L^{2m/(m-2)}} \leqslant A_7 \|\mathrm{d}v\|_{L^2}$ for some $A_7 > 0$ independent of $t$, and this replaces Theorem 6.12.

There is also some more work to do. Part (i) of Theorem 5.3 requires that $\|\pi_{W^t}(\psi^m \sin\theta^t)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$. In §6 this was trivial, as there $W^t = \langle 1 \rangle$ and $\pi_{W^t}(\psi^m \sin\theta^t) \equiv 0$ for topological reasons. We shall show in §7.2 that here for $\pi_{W^t}(\psi^m \sin\theta^t)$ to be sufficiently small the invariants $Z(L_i)$ for $i = 1, \ldots, n$ must satisfy equation (86) below. The final results are given in §7.4.

## 7.1 Setting up the problem

We shall consider the following situation, the analogue of Definitions 6.1–6.3.

**Definition 7.1** We work in the situation of Definition 6.1. Thus, $(M, J, \omega, \Omega)$ is an almost Calabi–Yau $m$-fold for $m > 2$ with metric $g$, $\psi : M \to (0, \infty)$ satisfies (3), $X$ is a compact SL $m$-fold in $M$ with conical singularities at $x_1, \ldots, x_n$ with identifications $\upsilon_i$, cones $C_i$ and rates $\mu_i$, and $L_1, \ldots, L_n$ are AC SL $m$-folds in $\mathbb{C}^m$ with cones $C_i$ and rates $\lambda_i < 0$. We write $X' = X \setminus \{x_1, \ldots, x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$, and use the other notation of §6.

However, we do *not* assume that $X'$ is connected, as we did in §6.4. Set $q = b^0(X')$, so that $X'$ has $q$ connected components, and number them $X'_1, \ldots, X'_q$. For $i = 1, \ldots, n$ let $l_i = b^0(\Sigma_i)$, so that $\Sigma_i$ has $l_i$ connected components, and number them $\Sigma^1_i, \ldots, \Sigma^{l_i}_i$.

Now $\Upsilon_i \circ \varphi_i$ is a diffeomorphism $\Sigma_i \times (0, R') \to S_i \subset X'$. For each $j = 1, \ldots, l_i$, $\Upsilon_i \circ \varphi_i\bigl(\Sigma^j_i \times (0, R')\bigr)$ is a connected subset of $X'$, and so lies in exactly one of the $X'_k$ for $k = 1, \ldots, q$. Define numbers $k(i,j) = 1, \ldots, q$ for $i = 1, \ldots, n$

and $j = 1, \ldots, l_i$ by $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr) \subset X'_{k(i,j)}$. Suppose that

$$\sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i:\\ k(i,j)=k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j\,] = 0 \quad \text{for all } k = 1, \ldots, q. \tag{86}$$

Here $Z(L_i) \in H^{m-1}(\Sigma_i, \mathbb{R})$ is as in Definition 4.2, and $[\Sigma_i^j] \in H_{m-1}(\Sigma_i, \mathbb{Z})$, and '$\cdot$' is the contraction $H^{m-1}(\Sigma_i, \mathbb{R}) \times H_{m-1}(\Sigma_i, \mathbb{Z}) \to \mathbb{R}$. The reason for (86) will appear in Proposition 7.5 below.

Define Lagrangian $m$-folds $N^t$ for $t \in (0, \delta)$ as in Definition 6.2, but when we choose $\tau \in (0, 1)$ to satisfy (49) we also require that

$$\tfrac{m}{m+1} < \tau < 1. \tag{87}$$

Clearly this is possible. Suppose that the topology of $X$ and $L_i$ is such that the $N^t$ are *connected*. This requires that $X$ be connected, but not that $X'$ or the $L_i$ be connected. A sufficient (but not necessary) condition for the $N^t$ to be connected is that $X$ and the $L_i$ are connected.

Let $h^t = g|_{N^t}$, so that $(N^t, h^t)$ is a Riemannian manifold. As in Definition 6.3 the $N^t$ are oriented, and $\Omega|_{N^t} = \psi^m e^{i\theta^t} \mathrm{d}V^t$ for some phase function $e^{i\theta^t}$ on $N^t$, where $\mathrm{d}V^t$ is the volume form on $N^t$. We write $\varepsilon^t = \psi^m \sin\theta^t$, so that $\operatorname{Im}\Omega|_{N^t} = \varepsilon^t \,\mathrm{d}V^t$ for $t \in (0, \delta)$.

To apply Theorem 5.3 to $N^t$ we need a vector subspace $W^t \subset C^\infty(N^t)$, to be $W$ in Definition 5.2. In §6, where $X'$ was connected, we took $W^t = \langle 1 \rangle$. However, when $X'$ is not connected we must introduce a nontrivial $W^t$ with $\dim W^t = b^0(X')$ to repair the proof of Theorem 6.12. Here is the definition.

**Definition 7.2** We work in the situation of Definition 7.1. For $i = 1, \ldots, n$ apply Theorem 4.8 to the AC SL $m$-fold $L_i$ in $\mathbb{C}^m$, using the numbering $\Sigma_i^j$ chosen in Definition 7.1 for the connected components of $\Sigma_i$. This gives a vector space $V_i$ of bounded harmonic functions on $L_i$ with $\dim V_i = l_i$. For each $\mathbf{c}_i = (c_i^1, \ldots, c_i^{l_i}) \in \mathbb{R}^{l_i}$ there exists a unique $v_i^{\mathbf{c}_i} \in V_i$ with

$$\nabla^k\bigl(\varphi_i^*(v_i^{\mathbf{c}_i}) - c_i^j\,\bigr) = O\bigl(|\mathbf{c}_i| r^{\beta-k}\bigr) \quad \text{on } \Sigma_i^j \times (T, \infty) \text{ as } r \to \infty, \tag{88}$$

for all $i = 1, \ldots, n$, $j = 1, \ldots, l_i$, $k \geqslant 0$ and $\beta \in (2-m, 0)$.

We shall define a vector subspace $W^t \subset C^\infty(N^t)$ for $t \in (0, \delta)$, with an isomorphism $W^t \cong \mathbb{R}^q$. Fix $\mathbf{d} = (d_1, \ldots, d_q) \in \mathbb{R}^q$, and set $c_i^j = d_{k(i,j)}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$. Let $\mathbf{c}_i = (c_i^1, \ldots, c_i^{l_i})$. This defines vectors $\mathbf{c}_i \in \mathbb{R}^{l_i}$ for $i = 1, \ldots, n$, which depend linearly on $\mathbf{d}$. Hence we have harmonic functions $v_i^{\mathbf{c}_i} \in V_i \subset C^\infty(L_i)$, which also depend linearly on $\mathbf{d}$.

Let $F : (0, \infty) \to [0, 1]$ and $\tau \in (0, 1)$ be as in Definition 6.2. Make $\delta > 0$ smaller if necessary so that $tT < \frac{1}{2}t^\tau$ for all $t \in (0, \delta)$. For $t \in (0, \delta)$, define a function $w_{\mathbf{d}}^t \in C^\infty(N^t)$ as follows:

(i) The subset $K \subset N^t$ has $q$ connected components $K \cap X'_k$ for $k = 1, \ldots, q$. Define $w_{\mathbf{d}}^t \equiv d_k$ on $K \cap X'_k$ for $k = 1, \ldots, q$.

(ii) Define $w^t_{\mathbf{d}}$ on $P^t_i \subset N^t$ by $(\Upsilon_i \circ t \circ \varphi_i)^*(w^t_{\mathbf{d}}) \equiv v_i^{\mathbf{c}_i}$ on $K_i$ for $i = 1, \ldots, n$.

(iii) Define $w^t_{\mathbf{d}}$ on $Q^t_i \subset N^t$ by

$$(\Xi^t_i)^*(w^t_{\mathbf{d}})(\sigma, r) = \bigl(1 - F(2t^{-\tau}r)\bigr)\varphi_i^*(v_i^{\mathbf{c}_i})(\sigma, t^{-1}r) + F(2t^{-\tau}r)c_i^j \qquad (89)$$

on $\Sigma_i^j \times (tT, R')$, for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$.

It is easy to see that $w^t_{\mathbf{d}}$ is smooth over the joins between $P^t_i, Q^t_i$ and $K$, so $w^t_{\mathbf{d}} \in C^\infty(N^t)$. Also $w^t_{\mathbf{d}}$ is linear in $\mathbf{d}$, as $v_i^{\mathbf{c}_i}$ is. Thus $W^t = \{w^t_{\mathbf{d}} : \mathbf{d} \in \mathbb{R}^q\}$ is a vector subspace of $C^\infty(N^t)$ isomorphic to $\mathbb{R}^q$, for all $t \in (0, \delta)$.

If $\mathbf{d} = (1, \ldots, 1)$ then $c_i^j \equiv 1$, so $\mathbf{c}_i = (1, \ldots, 1)$ for $i = 1, \ldots, n$, and thus $v_i^{\mathbf{c}_i} \equiv 1$ for $i = 1, \ldots, n$ by Theorem 4.8. Therefore $w^t_{(1,\ldots,1)} \equiv 1$ by (i)–(iii) above, and $1 \in W^t$ for all $t \in (0, \delta)$. This corresponds to the condition $1 \in W$ in Definition 5.2. Define $\pi_{W^t} : L^2(N^t) \to W^t$ to be the projection onto $W^t$ using the $L^2$-inner product, as for $\pi_W$ in Definition 5.2.

In Definition 6.2 we defined the $N^t$ by gluing together $X'$ and $tL_i$ using $F(t^{-\tau}r)$ in (50). In contrast, equation (89) uses $F(2t^{-\tau}r)$. As $F$ increases from 0 to 1 on $[1, 2]$, this means that in Definition 6.2 we glued $X'$ and $tL_i$ together on the annuli $\Xi^t_i\bigl(\Sigma_i \times (t^\tau, 2t^\tau)\bigr)$, but in (89) we glue $v_i^{\mathbf{c}_i}$ and $c_i^j$ together on the annuli $\Xi^t_i\bigl(\Sigma_i^j \times (\frac{1}{2}t^\tau, t^\tau)\bigr)$. This will be important in §7.2, where the fact that $w^t_{\mathbf{d}}$ is *constant* on the annuli $\Xi^t_i\bigl(\Sigma_i \times (t^\tau, 2t^\tau)\bigr)$ will simplify the calculations.

The point of Definition 7.2 is that $w^t_{\mathbf{d}}$ is made by gluing together the constant function $d_k$ on $X'_k$ with the harmonic function $t_*(v_i^{\mathbf{c}_i})$ on $tL_i$. This is asymptotic to $d_{k(i,j)}$ on the $j^{\text{th}}$ end of $tL_i$, which is glued into $X'_{k(i,j)}$. Thus, for small $t$ the functions $d_{k(i,j)}$ and $t_*(v_i^{\mathbf{c}_i})$ are nearly equal in the annulus $\Xi^t_i\bigl(\Sigma_i^j \times (\frac{1}{2}t^\tau, t^\tau)\bigr)$ where they are glued together.

Thus we expect $w^t_{\mathbf{d}}$ to be *nearly harmonic* on $N^t$ for small $t$, that is, $\mathrm{d}^*\mathrm{d}w^t_{\mathbf{d}}$ is small compared to $w^t_{\mathbf{d}}$. The next proposition estimates $w^t_{\mathbf{d}}, \mathrm{d}w^t_{\mathbf{d}}$ and $\mathrm{d}^*\mathrm{d}w^t_{\mathbf{d}}$.

**Proposition 7.3** *In the situation of Definition 7.2, for all $t \in (0, \delta)$, $\mathbf{d} = (d_1, \ldots, d_q) \in \mathbb{R}^q$, $\beta \in (2 - m, 0)$ and $i = 1, \ldots, n$ we have*

$$|w^t_{\mathbf{d}}| \leqslant \max\bigl(|d_1|, \ldots, |d_q|\bigr) \quad \textit{on } N^t, \qquad (90)$$

$$\mathrm{d}w^t_{\mathbf{d}} = \mathrm{d}^*\mathrm{d}w^t_{\mathbf{d}} = 0 \quad \textit{on } K \textit{ and } \Xi^t_i\bigl(\Sigma_i \times [t^\tau, R')\bigr), \qquad (91)$$

$$\bigl|(\Xi^t_i)^*(\mathrm{d}w^t_{\mathbf{d}})\bigr|(\sigma, r) = O\bigl(|\mathbf{d}|t^{-\beta}r^{\beta-1}\bigr) \quad \textit{on } \Sigma_i \times (tT, t^\tau), \qquad (92)$$

$$\bigl|(\Xi^t_i)^*(\mathrm{d}^*\mathrm{d}w^t_{\mathbf{d}})\bigr|(\sigma, r) = O\bigl(|\mathbf{d}|t^{-\beta}r^{\beta-1}\bigr) \quad \textit{on } \Sigma_i \times (tT, \tfrac{1}{2}t^\tau], \qquad (93)$$

$$\bigl|(\Xi^t_i)^*(\mathrm{d}^*\mathrm{d}w^t_{\mathbf{d}})\bigr|(\sigma, r) = O\bigl(|\mathbf{d}|t^{-\beta+\tau(\beta-2)}\bigr) \quad \textit{on } \Sigma_i \times (\tfrac{1}{2}t^\tau, t^\tau), \qquad (94)$$

$$\textit{and} \quad |\mathrm{d}w^t_{\mathbf{d}}| = O\bigl(|\mathbf{d}|t^{-1}\bigr), \quad |\mathrm{d}^*\mathrm{d}w^t_{\mathbf{d}}| = O\bigl(|\mathbf{d}|t^{-1}\bigr) \quad \textit{on } P^t_i. \qquad (95)$$

*Here $\mathrm{d}^*$ and $|\,.\,|$ are computed using $h^t$ or $(\Xi^t_i)^*(h^t)$.*

*Proof.* Since $v_i^{\mathbf{c}_i}$ is harmonic it has no strict maxima or minima in $L_i$ by the maximum principle, and as it approaches $d_{k(i,j)}$ on the $j^{\text{th}}$ end we see that

$\left|v_i^{\mathbf{c}_i}\right| \leqslant \max\bigl(|d_1|,\ldots,|d_q|\bigr)$ on $L_i$. Equation (90) then follows from the definition of $w_{\mathbf{d}}^t$ above. Also $w_{\mathbf{d}}^t$ is constant on $K$ and $\Xi_i^t\bigl(\Sigma_i\times[t^\tau,R')\bigr)$, so (91) holds.

Let $h_i$ be the metric $g'|_{L_i}$ on $L_i$. Then we have

$$(\Upsilon_i\circ t)^*(h^t)=t^2h_i+O(t^3)\qquad\text{on } K_i\text{, and}\tag{96}$$

$$(\Upsilon_i\circ t\circ\varphi_i)^*(h^t)=t^2\varphi_i^*(h_i)+O(t^2r)\qquad\text{on } \Sigma_i\times(tT,t^\tau).\tag{97}$$

Therefore the metrics $h^t$ on $P_i^t$ and $\Xi_i^t\bigl(\Sigma_i\times(tT,t^\tau)\bigr)$ and $t^2h_i$ on the corresponding regions of $L_i$ are uniformly equivalent for small $t$. Using this, (88) and (89) we deduce (92) and the first equation of (95).

By the same method we also find that

$$|\nabla^2 w_{\mathbf{d}}^t|=O\bigl(|\mathbf{d}|t^{-2}\bigr)\qquad\text{on } P_i^t\text{, and}\tag{98}$$

$$\bigl|(\Xi_i^t)^*(\nabla^2 w_{\mathbf{d}}^t)\bigr|(\sigma,r)=O\bigl(|\mathbf{d}|t^{-\beta}r^{\beta-2}\bigr)\qquad\text{on } \Sigma_i\times(tT,t^\tau),\tag{99}$$

computing $\nabla$ and $|\,.\,|$ using $h^t$ or $(\Xi_i^t)^*(h^t)$. Equation (94) then follows from (99) with $r\in(\frac12 t^\tau,t^\tau)$, as $|\mathrm{d}^*\mathrm{d}w_{\mathbf{d}}^t|\leqslant|\nabla^2 w_{\mathbf{d}}^t|$.

Now from (96) and a similar equation for derivatives we find that

$$\mathrm{d}_{h^t}^*\mathrm{d}w_{\mathbf{d}}^t=\mathrm{d}_{t^2h_i}^*\mathrm{d}w_{\mathbf{d}}^t+O(t)\cdot|\nabla^2w_{\mathbf{d}}^t|+O(1)\cdot|\mathrm{d}w_{\mathbf{d}}^t|\quad\text{on } P_i^t,\tag{100}$$

where $\mathrm{d}_{h^t}^*,\mathrm{d}_{t^2h_i}^*$ are $\mathrm{d}^*$ computed using $h^t,t^2h_i$. But $w_{\mathbf{d}}^t=(\Upsilon_i\circ t)_*(v_i^{\mathbf{c}_i})$ on $P_i^t$, and $v_i^{\mathbf{c}_i}$ is harmonic w.r.t. $h_i$, and so w.r.t. $t^2h_i$. Hence $\mathrm{d}_{t^2h_i}^*\mathrm{d}w_{\mathbf{d}}^t=0$ on $P_i^t$, and combining (98), (100) and the first equation of (95) gives the second equation of (95). We prove (93) in the same way, using (92), (97) and (99). □

Another way to think about $W^t$ is that the Laplacian $\Delta^t=\mathrm{d}^*\mathrm{d}$ of $h^t$ on $N^t$ has $q$ *small eigenvalues* $\lambda_1^t,\ldots,\lambda_q^t$, counted with multiplicity and including 0, and $W^t$ approximates the sum of the corresponding eigenspaces of $\Delta^t$. From Proposition 7.3 one can show that

$$\|w_{\mathbf{d}}^t\|_{L^2}=O(|\mathbf{d}|)\quad\text{and}\quad\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2}=O(|\mathbf{d}|t^{(m-2)/2}),$$

so that $\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2}^2=O(t^{m-2})\cdot\|w_{\mathbf{d}}^t\|_{L^2}^2$. This implies that the dominant eigenvectors of $\Delta^t$ in $w_{\mathbf{d}}^t$ have eigenvalues $O(t^{m-2})$, so $\lambda_k^t=O(t^{m-2})$ as $t\to0$.

## 7.2 Part (i) of Theorem 5.3: estimating $\|\pi_{W^t}(\psi^m\sin\theta^t)\|_{L^1}$

We need to bound $\|\pi_{W^t}(\psi^m\sin\theta^t)\|_{L^1}$ to prove part (i) of Theorem 5.3 for $N^t,W^t$. Writing $\varepsilon^t=\psi^m\sin\theta^t$ as in Definition 6.3, we shall do this by first estimating $\int_{N^t}w_{\mathbf{d}}^t\varepsilon^t\,\mathrm{d}V^t$ for all $\mathbf{d}\in\mathbb{R}^q$. As $w_{\mathbf{d}}^t\equiv d_k$ on $K\cap X_k'$ and on $\Xi_i^t\bigl(\Sigma_i^j\times[t^\tau,R')\bigr)$ when $k(i,j)=k$, we see that

$$\begin{aligned}\int_{N^t}w_{\mathbf{d}}^t\varepsilon^t\,\mathrm{d}V^t=&\sum_{i=1}^n\biggl(\int_{P_i^t}w_{\mathbf{d}}^t\varepsilon^t\,\mathrm{d}V^t+\int_{\Xi_i^t(\Sigma_i\times(tT,t^\tau))}w_{\mathbf{d}}^t\varepsilon^t\,\mathrm{d}V^t\biggr)\\&+\sum_{k=1}^q d_k\biggl(\int_{K\cap X_k'}\varepsilon^t\,\mathrm{d}V^t+\sum_{\substack{1\leqslant i\leqslant n,\ 1\leqslant j\leqslant l_i:\\ k(i,j)=k}}\int_{\Xi_i^t(\Sigma_i^j\times[t^\tau,R'))}\varepsilon^t\,\mathrm{d}V^t\biggr).\end{aligned}\tag{101}$$

The next two propositions bound the bracketed terms on each line.

**Proposition 7.4** *For all $t \in (0,\delta)$, $\mathbf{d} \in \mathbb{R}^q$ and $i = 1,\dots,n$ we have*

$$\int_{P_i^t} w_{\mathbf{d}}^t \varepsilon^t \, \mathrm{d}V^t + \int_{\Xi_i^t(\Sigma_i \times (tT, t^\tau))} w_{\mathbf{d}}^t \varepsilon^t \, \mathrm{d}V^t = O\bigl(|\mathbf{d}| t^{(m+1)\tau}\bigr). \tag{102}$$

*Proof.* We have $|w_{\mathbf{d}}^t| \leqslant |\mathbf{d}|$ on $N^t$ by (90). On $P_i^t$ we have $|\varepsilon^t| \leqslant Ct$ by (55). As $\mathrm{vol}(P_i^t) = O(t^m)$, this implies that the first integral in (102) is $O\bigl(|\mathbf{d}| t^{m+1}\bigr)$. Similarly, on $\Sigma_i \times (tT, t^\tau)$ we have $\bigl|(\Xi_i^t)^*(\varepsilon^t)\bigr| \leqslant Cr \leqslant Ct^\tau$ by (53). As $\mathrm{vol}\bigl(\Xi_i^t(\Sigma_i \times (tT, t^\tau))\bigr) = O(t^{m\tau})$, this implies that the second integral in (102) is $O\bigl(|\mathbf{d}| t^{(m+1)\tau}\bigr)$. The proposition follows. □

**Proposition 7.5** *For all $t \in (0,\delta)$, $\mathbf{d} \in \mathbb{R}^q$ and $k = 1,\dots,q$ we have*

$$\begin{aligned} \int_{K \cap X_k'} \varepsilon^t \, \mathrm{d}V^t + \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j)=k}} \int_{\Xi_i^t(\Sigma_i^j \times [t^\tau, R'))} \varepsilon^t \, \mathrm{d}V^t = \\ - t^m \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j)=k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j\,] + O\bigl(t^{(m+1)\tau}\bigr), \end{aligned} \tag{103}$$

*where $Z(L_i) \in H^{m-1}(\Sigma_i, \mathbb{R})$ is as in §4.1, and $[\Sigma_i^j\,] \in H_{m-1}(\Sigma_i, \mathbb{Z})$.*

*Proof.* As $\varepsilon^t \mathrm{d}V^t = \mathrm{Im}\,\Omega|_{N^t}$, the left hand side of (103) is the integral of $\mathrm{Im}\,\Omega$ over the $m$-chain

$$Z_k = \bigl(K \cap X_k'\bigr) + \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j)=k}} \Xi_i^t\bigl(\Sigma_i^j \times [t^\tau, R')\bigr)$$

for $k = 1,\dots,q$, which is a closed subset of $N^t$, with boundary $(m-1)$-chain

$$\partial Z_k = - \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j)=k}} \Xi_i^t\bigl(\Sigma_i^j \times \{t^\tau\}\bigr). \tag{104}$$

For each $i = 1,\dots,n$ and $j = 1,\dots,l_i$, define an $m$-chain $A_i^j$ in $B_R$ to be the image of $\Sigma_i^j \times [0,1]$ under the map $\Sigma_i^j \times [0,1] \to B_R$ given by

$$(\sigma, r) \longmapsto r\Phi_{C_i}\bigl(\sigma, t^\tau, t^2\chi_i^1(\sigma, t^{\tau-1}), t^2\chi_i^2(\sigma, t^{\tau-1})\bigr).$$

As $\Upsilon_i \circ \Phi_{C_i}\bigl(\sigma, t^\tau, t^2\chi_i^1(\sigma, t^{\tau-1}), t^2\chi_i^2(\sigma, t^{\tau-1})\bigr) \equiv \Xi_i^t(\sigma, t^\tau)$ for $\sigma \in \Sigma_i$ by Definition 6.2, we see that

$$\partial\bigl(\Upsilon_i(A_i^j)\bigr) = \Xi_i^t\bigl(\Sigma_i^j \times \{t^\tau\}\bigr), \tag{105}$$

regarding $\Upsilon_i(A_i^j)$ as an $m$-chain in $M$.

Now define another $m$-chain $Z'_k$ for $k = 1, \ldots, q$ to be

$$Z'_k = \overline{X'_k} - \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j)=k}} \Upsilon_i(A_i^j).$$

As $\overline{X'_k}$ is an $m$-chain without boundary, we see from (104) and (105) that $\partial Z'_k = \partial Z_k$, and in fact it is easy to see that $Z'_k$ and $Z_k$ are homologous in $M$. Since $\operatorname{Im}\Omega$ is a closed $m$-form on $M$, this implies that $\int_{Z'_k} \operatorname{Im}\Omega = \int_{Z_k} \operatorname{Im}\Omega$. But $\operatorname{Im}\Omega|_{X'_k} \equiv 0$ as $X'_k$ is special Lagrangian. Hence we see that

$$\int_{Z_k} \operatorname{Im}\Omega = - \sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j)=k}} \int_{A_i^j} \Upsilon_i^*(\operatorname{Im}\Omega). \tag{106}$$

From Definition 3.4 we have $\upsilon_i^*(\Omega) = \psi(x_i)^m \Omega'$, where $\Omega'$ is as in (1). Thus as $\Upsilon_i^*(\Omega)$ is smooth on $B_R$, Taylor's theorem gives

$$\Upsilon_i^*(\operatorname{Im}\Omega) = \psi(x_i)^m \operatorname{Im}\Omega' + O(r) \quad \text{on } B_R. \tag{107}$$

Now $A_i^j$ is an $m$-chain in $B_R \subset \mathbb{C}^m$ with boundary in the AC SL $m$-fold $tL_i$, and $[\partial A_i^j] \in H_{m-1}(tL_i, \mathbb{R})$ is the image of $[\Sigma_i^j] \in H_{m-1}(\Sigma_i, \mathbb{R})$ under the map $H_{m-1}(\Sigma_i, \mathbb{R}) \to H_{m-1}(L_i, \mathbb{R})$ dual to the natural map $H^{m-1}(L_i, \mathbb{R}) \to H^{m-1}(\Sigma_i, \mathbb{R})$. It then follows easily from Definition 4.2 and Lemma 4.4 that

$$\int_{A_i^j} \operatorname{Im}\Omega' = Z(tL_i) \cdot [\Sigma_i^j] = t^m Z(L_i) \cdot [\Sigma_i^j]. \tag{108}$$

But as $r = O(t^\tau)$ on $A_i^j$ and $\operatorname{vol}(A_i^j) = O(t^{m\tau})$ we see from (107) that

$$\int_{A_i^j} \bigl( \Upsilon_i^*(\operatorname{Im}\Omega) - \psi(x_i)^m \operatorname{Im}\Omega' \bigr) = O(t^{(m+1)\tau}). \tag{109}$$

Equation (103) now follows from (106), (108), (109), and the fact that the left hand side of (103) is $\int_{Z_k} \operatorname{Im}\Omega$. This completes the proof. □

We can now explain the reason for the condition (86) in Definition 7.1. If (86) holds then the first term on the right hand side of (103) is zero, and therefore (101) and Propositions 7.4 and 7.5 show that $\int_{N^t} w_{\mathbf{d}}^t \varepsilon^t \, dV^t = O\bigl(|\mathbf{d}| t^{(m+1)\tau}\bigr)$. This in turn implies that $\|\pi_{W^t}(\varepsilon^t)\|_{L^1} = O(t^{(m+1)\tau})$.

Therefore $\|\pi_{W^t}(\psi^m \sin\theta^t)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$ for some $A_2 > 0$ and $\kappa > 1$ and all $t \in (0, \delta)$, as $\varepsilon^t = \psi^m \sin\theta^t$ and $\tau > \frac{m}{m+1}$ by (87). This is one of the conditions in part (i) of Theorem 5.3 for $N^t, W^t$. However, if (86) does not hold then $\|\pi_{W^t}(\psi^m \sin\theta^t)\|_{L^1} = O(t^m)$, and part (i) of Theorem 5.3 for $N^t, W^t$ does not hold for all $t \in (0, \delta)$ with $\kappa > 1$, so the construction fails.

Here is the analogue of Theorem 6.6.

**Theorem 7.6** *Making $\delta > 0$ smaller if necessary, there exist $A_2 > 0$ and $\kappa > 1$ such that $\varepsilon^t = \psi^m \sin\theta^t$ on $N^t$ satisfies $\|\varepsilon^t\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa+m/2}$, $\|\varepsilon^t\|_{C^0} \leqslant A_2 t^{\kappa-1}$, $\|\mathrm{d}\varepsilon^t\|_{L^{2m}} \leqslant A_2 t^{\kappa-3/2}$ and $\|\pi_{W^t}(\varepsilon^t)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$ for all $t \in (0,\delta)$, as in part* (i) *of Theorem 5.3.*

*Proof.* Theorem 6.6 shows that there exist $A_2 > 0$ and $\kappa > 1$ such that $\|\varepsilon^t\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa+m/2}$, $\|\varepsilon^t\|_{C^0} \leqslant A_2 t^{\kappa-1}$, and $\|\mathrm{d}\varepsilon^t\|_{L^{2m}} \leqslant A_2 t^{\kappa-3/2}$ for all $t \in (0,\delta)$. It remains to consider the condition $\|\pi_{W^t}(\varepsilon^t)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$.

Combining equations (86) and (101) and Propositions 7.4 and 7.5 gives

$$\int_{N^t} w^t_{\mathbf{d}} \varepsilon^t \,\mathrm{d}V^t = O\big(|\mathbf{d}| t^{(m+1)\tau}\big) \quad \text{for all } \mathbf{d} \in \mathbb{R}^q \text{ and } t \in (0,\delta). \tag{110}$$

One can show from Definition 7.2 that $\|w^t_{\mathbf{d}}\|_{L^2} \geqslant C|\mathbf{d}|$ for some $C > 0$ and all $\mathbf{d} \in \mathbb{R}^q$ and $t \in (0,\delta)$. This and (110) imply that $\|\pi_{W^t}(\varepsilon^t)\|_{L^2} = O(t^{(m+1)\tau})$. But $\|\pi_{W^t}(\varepsilon^t)\|_{L^1} \leqslant \mathrm{vol}(N^t)^{1/2} \|\pi_{W^t}(\varepsilon^t)\|_{L^2}$, and $\mathrm{vol}(N^t) = O(1)$. Therefore

$$\|\pi_{W^t}(\varepsilon^t)\|_{L^1} = O(t^{(m+1)\tau}) \quad \text{for all } t \in (0,\delta). \tag{111}$$

Make $\kappa > 1$ smaller if necessary so that $\kappa + m - 1 \leqslant (m+1)\tau$. This is possible as $(m+1)\tau > m$ by (87). Now make $A_2 > 0$ bigger and $\delta > 0$ smaller if necessary so that $\|\pi_{W^t}(\varepsilon^t)\|_{L^1} \leqslant A_2 t^{\kappa+m-1}$ for all $t \in (0,\delta)$. This is possible by (111), as $\kappa + m - 1 \leqslant (m+1)\tau$. The previous inequalities $\|\varepsilon^t\|_{L^{2m/(m+2)}} \leqslant A_2 t^{\kappa+m/2}$, $\|\varepsilon^t\|_{C^0} \leqslant A_2 t^{\kappa-1}$ and $\|\mathrm{d}\varepsilon^t\|_{L^{2m}} \leqslant A_2 t^{\kappa-3/2}$ for $t \in (0,\delta)$ still hold with the new $A_2, \kappa$, as we have increased $A_2$ and decreased $\kappa$, and $t < 1$. Thus there exist $A_2, \kappa$ satisfying the conditions of the theorem. □

## 7.3 Parts (vi) and (vii) of Theorem 5.3

We now explain how to modify the material of §6.4 to the case when $X'$ is not connected. Thus we prove that parts (vi) and (vii) of Theorem 5.3 hold for $N^t$ and $W^t$. Here is the analogue of Proposition 6.11.

**Proposition 7.7** *In the situation of §7.1, there exists $D_2 > 0$ such that for all $v \in C^1_{\mathrm{cs}}(X')$ we have*

$$\|v\|_{L^{2m/(m-2)}} \leqslant D_2\big(\|\mathrm{d}v\|_{L^2} + \textstyle\sum_{k=1}^q \big|\int_{X'_k} v \,\mathrm{d}V_g\big|\big). \tag{112}$$

*Proof.* Applying Proposition 6.11 to each connected component $X'_k$ of $X'$ gives constants $D_{2,k} > 0$ for $k = 1,\ldots,q$ such that

$$\big\|v|_{X'_k}\big\|_{L^{2m/(m-2)}} \leqslant D_{2,k}\big(\big\|\,\mathrm{d}v|_{X'_k}\big\|_{L^2} + \big|\textstyle\int_{X'_k} v \,\mathrm{d}V_g\big|\big). \tag{113}$$

Then summing (113) over $k = 1,\ldots,q$ and using

$$\big\|v\big\|_{L^{2m/(m-2)}} \leqslant \textstyle\sum_{k=1}^q \big\|v|_{X'_k}\big\|_{L^{2m/(m-2)}} \text{ and } \textstyle\sum_{k=1}^q \big\|\,\mathrm{d}v|_{X'_k}\big\|_{L^2} \leqslant q^{1/2}\big\|\,\mathrm{d}v\big\|_{L^2}$$

proves (112), with $D_2 = q^{1/2}\max\bigl(D_{2,1},\ldots,D_{2,q}\bigr)$. $\square$

Here is the analogue of Theorem 6.12. The condition $\int_{N^t} vw\,\mathrm{d}V^t = 0$ for all $w \in W^t$ is equivalent to $\pi_{W^t}(v) = 0$, so the theorem proves part (vi) of Theorem 5.3 for $N^t, W^t$, with $A_7$ independent of $t$.

**Theorem 7.8** *Making $\delta > 0$ smaller if necessary, there exists $A_7 > 0$ such that for all $t \in (0,\delta)$, if $v \in L^2_1(N^t)$ with $\int_{N^t} vw\,\mathrm{d}V^t = 0$ for all $w \in W^t$ then $v \in L^{2m/(m-2)}(N^t)$ and $\|v\|_{L^{2m/(m-2)}} \leqslant A_7\|\mathrm{d}v\|_{L^2}$.*

*Proof.* Let $a, b, G$ and $F^t$ be as in the proof of Theorem 6.12. Then $F^t : N^t \to [0,1]$ is smooth with $F^t \equiv 1$ on $K$, and $F^t \equiv 0$ on $P^t_i$ for $i = 1,\ldots,n$. The support of $F^t$ is a subset of $N^t \cap X'$, so we can also regard $F^t$ as a compactly-supported function on $X'$. For $k = 1,\ldots,q$, let $F^t_k$ be the smooth, compactly-supported function on $X'$ equal to $F^t$ on $X'_k$ and zero on $X'_{k'}$ for $k' \neq k$. Then $F^t_k$ is supported on $X'_k$, and $F^t = \sum_{k=1}^q F^t_k$. Moreover, extending $F^t_k$ by zero outside $N^t \cap X'$ we can also regard $F^t_k$ as a smooth, compactly-supported function on $N^t$, and $F^t = \sum_{k=1}^q F^t_k$ holds on $N^t$ as well.

Suppose now that $t \in (0,\delta)$ and $v \in C^1(N^t)$ with $\int_{N^t} vw\,\mathrm{d}V^t = 0$ for all $w \in W^t$. Then $F^t v$ is supported in $N^t \cap X'$, so we can regard it as a compactly-supported function on $X'$ and apply Proposition 7.7 to it. This gives

$$\begin{aligned}\|F^t v\|_{L^{2m/(m-2)}} &\leqslant D_2\bigl(\bigl\|\,\mathrm{d}(F^t v)\bigr\|_{L^2} + \textstyle\sum_{k=1}^q \bigl|\int_{X'_k} F^t v\,\mathrm{d}V_g\bigr|\bigr)\\ &= D_2\bigl(\bigl\|F^t\mathrm{d}v + v\,\mathrm{d}F^t\bigr\|_{L^2} + \textstyle\sum_{k=1}^q \bigl|\int_{N^t} F^t_k v\,\mathrm{d}V^t\bigr|\bigr).\end{aligned} \tag{114}$$

Here in the second line we use the fact that $F^t$ is supported in $N^t \cap X'$, so $h^t = g|_{X'}$ and $\mathrm{d}V_g = \mathrm{d}V^t$ in the support of $F^t$.

Let $e_1,\ldots,e_q$ be the usual basis of $\mathbb{R}^q$, so that $e_k = (\delta_{1k},\delta_{2k},\ldots,\delta_{qk})$ for $k = 1,\ldots,q$. As $\int_{N^t} vw^t_{e_k}\,\mathrm{d}V^t = 0$ by choice of $v$ we have

$$\bigl|\textstyle\int_{N^t} F^t_k v\,\mathrm{d}V^t\bigr| = \bigl|\int_{N^t}(F^t_k - w^t_{e_k})v\,\mathrm{d}V^t\bigr| \leqslant \bigl\|v\bigr\|_{L^{2m/(m-2)}}\cdot\bigl\|F^t_k - w^t_{e_k}\bigr\|_{L^{2m/(m+2)}}.$$

Substituting this into (114) and using Hölder's inequality we get

$$\begin{aligned}\|F^t v\|_{L^{2m/(m-2)}} \leqslant D_2\bigl(&\|F^t\mathrm{d}v\|_{L^2} + \|v\|_{L^{2m/(m-2)}}\cdot\|\mathrm{d}F^t\|_{L^m}\\ &+ \bigl\|v\bigr\|_{L^{2m/(m-2)}}\cdot\textstyle\sum_{k=1}^q\bigl\|F^t_k - w^t_{e_k}\bigr\|_{L^{2m/(m+2)}}\bigr).\end{aligned} \tag{115}$$

This is the analogue of (79).

Now by the definitions of $F^t_k$ and $w^t_{e_k}$ we have

$$F^t_k(x) = w^t_{e_k}(x) = \begin{cases} 1, & x \in K \cap X'_k,\\ 1, & x \in \Xi^t_i\bigl(\Sigma^j_i \times [t^a, R')\bigr) \text{ when } k(i,j) = k,\\ 0, & x \in K \cap X'_{k'} \text{ for } k' \neq k,\\ 0, & x \in \Xi^t_i\bigl(\Sigma^j_i \times [t^a, R')\bigr) \text{ for } k(i,j) \neq k.\end{cases}$$

Hence $F_k^t - w_{e_k}^t$ is *zero* on most of $N^t$. The support of $F_k^t - w_{e_k}^t$ is contained in the union of $P_i^t$ and $\Xi_i^t\big(\Sigma_i \times (tT, t^a)\big)$ over $i = 1, \ldots, n$, which has volume $O(t^{ma})$, and here $|F_k^t - w_{e_k}^t| \leqslant 1$ as $0 \leqslant F_k^t, w_{e_k}^t \leqslant 1$. Hence

$$\big\|F_k^t - w_{e_k}^t\big\|_{L^{2m/(m+2)}} = O(t^{a(m+2)/2}) \quad \text{for } k = 1, \ldots, q. \tag{116}$$

Following the proof of Theorem 6.12 without change, we prove (82). Using (115) instead of (79), in place of (83) we obtain

$$\begin{aligned}\big[1 - (D_2 + 2\sqrt{n}\, D_1)\|\mathrm{d}F^t\|_{L^m} - D_2 \textstyle\sum_{k=1}^q \big\|F_k^t - w_{e_k}^t\big\|_{L^{2m/(m+2)}}\big] \cdot \|v\|_{L^{2m/(m-2)}} \\ \leqslant (D_2 + 2\sqrt{n}\, D_1)\|\mathrm{d}v\|_{L^2}.\end{aligned}$$

Using (116) instead of $\|1 - F^t\|_{L^{2m/(m+2)}} = O(t^{a(m+2)/2})$, the rest of the proof follows that of Theorem 6.12. □

We now prove part (vii) of Theorem 5.3 for $N^t, W^t$, with $A_7$ as above.

**Theorem 7.9** *Making $\delta > 0$ smaller if necessary, for all $t \in (0, \delta)$ and $w \in W^t$ we have $\|\mathrm{d}^*\mathrm{d}w\|_{L^{2m/(m+2)}} \leqslant \frac{1}{2}A_7^{-1}\|\mathrm{d}w\|_{L^2}$, where $A_7 > 0$ is as in Theorem 7.8. Also there exists $A_8 > 0$ such that for all $t \in (0, \delta)$ and $w \in W^t$ with $\int_{N^t} w\,\mathrm{d}V^t = 0$ we have $\|w\|_{C^0} \leqslant A_8 t^{1-m/2}\|\mathrm{d}w\|_{L^2}$.*

*Proof.* Following the method of Proposition 6.5, using the estimates of Proposition 7.3 and taking $2 - m < \beta < \frac{1}{2}(2 - m)$ we find that for some $D_3 > 0$ and all $t \in (0, \delta)$ and $w_{\mathbf{d}}^t \in W^t$ we have

$$\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2} \leqslant D_3|\mathbf{d}|t^{(m-2)/2} \quad \text{and} \tag{117}$$

$$\|\mathrm{d}^*\mathrm{d}w_{\mathbf{d}}^t\|_{L^{2m/(m+2)}} \leqslant D_3|\mathbf{d}|t^{m/2} + D_3|\mathbf{d}|t^{2\beta(\tau-1)+\tau(m-2)}. \tag{118}$$

Also, as $w_{\mathbf{d}}^t = d_k$ on $K \cap X_k'$, from (90) we deduce that

$$\|w_{\mathbf{d}}^t\|_{C^0} = \max\big(|d_1|, \ldots, |d_q|\big) \leqslant |\mathbf{d}|. \tag{119}$$

However, (117)–(119) are not enough to prove the theorem, as (117) gives an *upper bound* for $\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2}$, but we actually need a *lower bound*. Now on $P_i^t$ and $\Xi_i^t\big(\Sigma_i \times (tT, \frac{1}{2}t^\tau]\big)$, by definition $w_{\mathbf{d}}^t$ coincides with $v_i^{\mathbf{c}_i}$ on the corresponding regions of $L_i$. Using (96) to compare the volume forms $\mathrm{d}V^t$ of $h^t$ on $N^t$ and $\mathrm{d}V_{h_i}$ of $h_i = g'|_{L_i}$ on $L_i$, from (91), (92) and (95) we can show that

$$\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2}^2 = t^{m-2}\sum_{i=1}^n \|\mathrm{d}v_i^{\mathbf{c}_i}\|_{L^2}^2 + O\big(|\mathbf{d}|^2 t^{m-1}\big) + O\big(|\mathbf{d}|^2 t^{\beta(\tau-1)+\tau(m-2)/2}\big), \tag{120}$$

where $\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2}$ is computed on $N^t$ using $h^t$, and $\|\mathrm{d}v_i^{\mathbf{c}_i}\|_{L^2}$ on $L_i$ using $h_i$.

Now $v_i^{\mathbf{c}_i}$ depends only on $\mathbf{d}$ and not on $t$, and in fact only on $d_{k(i,j)}$ for $j = 1, \ldots, l_i$. Also $\|\mathrm{d}v_i^{\mathbf{c}_i}\|_{L^2}$ depends only on the differences $d_{k(i,j)} - d_{k(i,j')}$ for $1 \leqslant j < j' \leqslant l_i$, as adding an overall constant to $d_k$ adds a constant to $v_i^{\mathbf{c}_i}$, and does not change $\mathrm{d}v_i^{\mathbf{c}_i}$.

For each $i = 1, \ldots, n$, define an equivalence relation $\sim$ on pairs $(i,j)$ for $j = 1, \ldots, l_i$ by $(i,j) \sim (i,j')$ if $\Sigma_i^j$ and $\Sigma_i^{j'}$ are ends at infinity of the same connected component of $L_i$. Then one can show that

$$\|\mathrm{d}v_i^{\mathbf{c}_i}\|_{L^2}^2 \geqslant C_i \sum_{\substack{1\leqslant j<j'\leqslant l_i:\\ (i,j)\sim(i,j')}} |d_{k(i,j)} - d_{k(i,j')}|^2, \tag{121}$$

for all $\mathbf{d} \in \mathbb{R}^q$ and $C_i > 0$ depending only on $L_i$ for $i = 1, \ldots, n$.

The point here is that if $d_{k(i,j)} = d_{k(i,j')}$ whenever $(i,j) \sim (i,j')$ then $v_i^{\mathbf{c}_i}$ is constant on each connected component of $L_i$, and $\mathrm{d}v_i^{\mathbf{c}_i} \equiv 0$, so both sides of (121) are zero. But otherwise $v_i^{\mathbf{c}_i}$ is not constant, and both sides of (121) are positive. Summing (121) over $i = 1, \ldots, n$, and remembering that the $N^t$ are connected by Definition 7.1, we can prove that

$$\sum_{i=1}^{n} \|\mathrm{d}v_i^{\mathbf{c}_i}\|_{L^2}^2 \geqslant 2D_4^2\bigl(|\mathbf{d}|^2 - \tfrac{1}{q}(d_1 + \ldots + d_q)^2\bigr), \tag{122}$$

for all $\mathbf{d} \in \mathbb{R}^q$, and some $D_4 > 0$ depending only on $C_1, \ldots, C_n$, the $k(i,j)$, and the equivalence relations $\sim$.

Combining (120) and (122) shows that for all $\mathbf{d} \in \mathbb{R}^q$ with $d_1 + \cdots + d_q = 0$ and $t \in (0, \delta)$ we have

$$\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2}^2 \geqslant 2D_4^2|\mathbf{d}|^2 t^{m-2} + O\bigl(|\mathbf{d}|^2 t^{m-1}\bigr) + O\bigl(|\mathbf{d}|^2 t^{\beta(\tau-1)+\tau(m-2)/2}\bigr).$$

As $\beta < \frac{1}{2}(2-m)$ and $\tau \in (0,1)$, both error terms are smaller than $|\mathbf{d}|^2 t^{m-2}$ for small $t$. Hence, making $\delta > 0$ smaller if necessary we see that when $t \in (0, \delta)$,

$$\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2} \geqslant D_4|\mathbf{d}|t^{(m-2)/2} \quad \text{for } \mathbf{d} \in \mathbb{R}^q \text{ with } d_1 + \cdots + d_q = 0. \tag{123}$$

Now make $\delta > 0$ smaller if necessary so that for all $t \in (0, \delta)$, we have

$$D_3|\mathbf{d}|t^{m/2} + D_3|\mathbf{d}|t^{2\beta(\tau-1)+\tau(m-2)} \leqslant \tfrac{1}{2}A_7^{-1}D_4|\mathbf{d}|t^{(m-2)/2}. \tag{124}$$

This is possible as both powers of $t$ on the left are greater than the power on the right. Combining (118), (123) and (124) shows that when $t \in (0, \delta)$,

$$\|\mathrm{d}^*\mathrm{d}w_{\mathbf{d}}^t\|_{L^{2m/(m+2)}} \leqslant \tfrac{1}{2}A_7^{-1}\|\mathrm{d}w_{\mathbf{d}}^t\|_{L^2} \text{ for } \mathbf{d}\in\mathbb{R}^q \text{ with } d_1+\cdots+d_q=0. \tag{125}$$

Now let $t \in (0, \delta)$ and $w \in W^t$. Then $w = w_{\mathbf{d}'}^t$ for some $\mathbf{d}' \in \mathbb{R}^q$. Let $c = \frac{1}{q}(d_1' + \ldots + d_q')$, let $d_k = d_k' - c$, and $\mathbf{d} = (d_1, \ldots, d_q)$. Then $d_1 + \ldots + d_q = 0$, and $w = w_{\mathbf{d}}^t + c$, since $w_{\mathbf{d}}^t$ depends linearly on $\mathbf{d}$ and $w_{(1,\ldots,1)}^t \equiv 1$. Thus $\mathrm{d}w = \mathrm{d}w_{\mathbf{d}}^t$, and (125) holds, giving $\|\mathrm{d}^*\mathrm{d}w\|_{L^{2m/(m+2)}} \leqslant \frac{1}{2}A_7^{-1}\|\mathrm{d}w\|_{L^2}$, as we have to prove.

Suppose also that $\int_{N^t} w \,\mathrm{d}V^t = 0$. Then by the proof of (119) we see that

$$\min(d_1', \ldots, d_q') = \min_{N^t} w \leqslant 0 \leqslant \max_{N^t} w = \max(d_1', \ldots, d_q').$$

Using this it is easy to show that

$$\|w\|_{C^0} = \max\bigl(|d'_1|, \ldots, |d'_q|\bigr) \leqslant 2 \max\bigl(|d_1|, \ldots, |d_q|\bigr) \leqslant 2|\mathbf{d}|. \tag{126}$$

Define $A_8 = 2D_4^{-1} > 0$. Then using (123), (126) and $\mathrm{d}w = \mathrm{d}w^t_{\mathbf{d}}$ we find that

$$\|w\|_{C^0} \leqslant 2|\mathbf{d}| = A_8 t^{1-m/2} \cdot D_4 |\mathbf{d}| t^{(m-2)/2} \leqslant A_8 t^{1-m/2} \|\mathrm{d}w\|_{L^2},$$

for all $t \in (0, \delta)$ and $w \in W^t$ with $\int_{N^t} w \, \mathrm{d}V^t = 0$. This completes the proof. $\square$

## 7.4 The main results, when $X'$ is not connected

We can now state our second main result on desingularizations of SL $m$-folds $X$ with conical singularities, this time allowing $X'$ not connected.

**Theorem 7.10** *Suppose $(M, J, \omega, \Omega)$ is an almost Calabi–Yau $m$-fold and $X$ a compact SL $m$-fold in $M$ with conical singularities at $x_1, \ldots, x_n$ and cones $C_1, \ldots, C_n$. Define $\psi : M \to (0, \infty)$ as in (3). Let $L_1, \ldots, L_n$ be Asymptotically Conical SL $m$-folds in $\mathbb{C}^m$ with cones $C_1, \ldots, C_n$ and rates $\lambda_1, \ldots, \lambda_n$. Suppose $\lambda_i < 0$ for $i = 1, \ldots, n$. Write $X' = X \setminus \{x_1, \ldots, x_n\}$ and $\Sigma_i = C_i \cap \mathcal{S}^{2m-1}$.*

*Set $q = b^0(X')$, and let $X'_1, \ldots, X'_q$ be the connected components of $X'$. For $i = 1, \ldots, n$ let $l_i = b^0(\Sigma_i)$, and let $\Sigma_i^1, \ldots, \Sigma_i^{l_i}$ be the connected components of $\Sigma_i$. Define $k(i,j) = 1, \ldots, q$ by $\Upsilon_i \circ \varphi_i\bigl(\Sigma_i^j \times (0, R')\bigr) \subset X'_{k(i,j)}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, l_i$. Suppose that*

$$\sum_{\substack{1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant l_i: \\ k(i,j) = k}} \psi(x_i)^m Z(L_i) \cdot [\Sigma_i^j] = 0 \quad \textit{for all } k = 1, \ldots, q. \tag{127}$$

*Suppose also that the compact $m$-manifold $N$ obtained by gluing $L_i$ into $X'$ at $x_i$ for $i = 1, \ldots, n$ is connected. A sufficient condition for this to hold is that $X$ and $L_i$ for $i = 1, \ldots, n$ are connected.*

*Then there exists $\epsilon > 0$ and a smooth family $\bigl\{\tilde{N}^t : t \in (0, \epsilon]\bigr\}$ of compact, nonsingular SL $m$-folds in $(M, J, \omega, \Omega)$ diffeomorphic to $N$, such that $\tilde{N}^t$ is constructed by gluing $tL_i$ into $X$ at $x_i$ for $i = 1, \ldots, n$. In the sense of currents in Geometric Measure Theory, $\tilde{N}^t \to X$ as $t \to 0$.*

The proof follows that of Theorem 6.13, but using Definition 7.1 instead of Definitions 6.1–6.3, and defining $W^t$ as in Definition 7.2 rather than $W^t = \langle 1 \rangle$. We use Theorem 7.6 instead of Theorem 6.6, and Theorems 7.8 and 7.9 instead of Theorem 6.12. Note that Theorem 6.8 still holds in this situation, as it does not assume that $X'$ is connected. The extra hypotheses (127) and that $N$ is connected come from Definition 7.1.

If $X'$ is connected, so that $q = 1$, then $k(i,j) \equiv 1$ and (127) becomes

$$\sum_{i=1}^n \psi(x_i)^m Z(L_i) \cdot \sum_{j=1}^{l_i} [\Sigma_i^j] = 0.$$

But $\sum_{j=1}^{l_i}[\Sigma_i^j] = [\Sigma_i]$, and $Z(L_i)\cdot[\Sigma_i]=0$ as $Z(L_i)$ is the image of a class in $H^{m-1}(L_i,\mathbb{R})$ by Definition 4.2, and $\Sigma_i$ is the boundary of $L_i$, so $[\Sigma_i]$ maps to zero in $H_{m-1}(L_i,\mathbb{R})$. Therefore (127) holds automatically when $X'$ is connected, and Theorem 7.10 reduces to Theorem 6.13 in this case.

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