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The Banach–Lie algebra of multiplication operators on a JBW*-triple ☆

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Abstract

The Banach–Lie algebra $\mathcal{L}(A)$ of multiplication operators on the JB*-triple A is introduced and it is shown that the hermitian part $\mathcal{L}(A)_h$ of $\mathcal{L}(A)$ is a unital GM-space the base of the dual cone in the dual GL-space $(\mathcal{L}(A)_h)^*$ of which is affine isomorphic and weak*-homeomorphic to the state space of $\mathcal{L}(A)$. In the case in which A is a JBW*-triple, it is shown that tripotents u and v in A are orthogonal if and only if the corresponding multiplication operators in the unital GM-space $\mathcal{L}(A)_h$ satisfy

$$0 \leqslant D(u,u) + D(v,v) \leqslant \mathrm{id}_A,$$

and that u is a pre-associate of v if and only if

$$D(u,u) \leqslant D(v,v).$$

Keywords: JB*-triple; JBW*-triple; Multiplication operator

1. Introduction

This paper presents a further investigation into the structure of JBW*-triples. The work of Kaup and Upmeier [24–26] and Vigué [39–42] shows how the holomorphic structure of the open

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unit ball in a complex Banach space A leads to the existence of a closed subspace A_s of A and a triple product $\{\ldots\}$ from $A \times A_s \times A$ to A. The purely algebraic properties of the triple product, namely the linearity and symmetry in the first and third variables, the conjugate linearity in the second variable and, most important, the existence of a Jordan triple identity, relate any complex Banach space to the Jordan triple systems studied by Koecher [27], Loos [28] and Meyberg [31]. When A_s exhausts A or, equivalently, when the open unit ball in A is a bounded symmetric domain, the complex Banach space A is said to be a JB*-triple. A JB*-triple that is the dual of a complex Banach space is said to be a JBW*-triple. Because of the very intimate nature of the relationship between their geometric and algebraic structure, considerable attention has been given to their properties in recent years. See, for example, [4–6,11,13,14,16,18–20,22,37,38].

Examples of JB*-triples are C*-algebras and JB*-algebras and, therefore, examples of JBW*-triples are W*-algebras and JBW*-algebras. A question often asked is when a certain property of a C*-algebra can be generalised to a JB*-triple. The main obstruction to such a programme stems from the lack of a global order structure for a JB*-triple. Many of the generalisations that have been proved depend upon developing new techniques that do not depend on order structure, some of which lead to the discovery of new properties of C*-algebras.

This paper is devoted to the study of a rather different order structure associated with JB*-triples. This structure is defined not on the JB*-triple A itself but within the complex unital Banach algebra $\mathcal{B}(A)$ of bounded linear operators from A to itself. Whilst this was touched upon in the formative work on JB*-triples mentioned above there appears not to have been a systematic investigation into this aspect of the theory of JB*-triples. A multiplication operator $D(a, b)$ corresponding to elements a and b in the JB*-triple is the element of $\mathcal{B}(A)$ defined, for an element c in A, by

$$D(a, b)c = \{a\, b\, c\}.$$

The main object of interest in this paper is the Banach–Lie subalgebra $\mathcal{L}(A)$ of $\mathcal{B}(A)$ which is the norm-closed linear span of the family of all multiplication operators and the identity operator on A. This possesses a natural involution and the self-adjoint part $\mathcal{L}(A)_h$ of $\mathcal{L}(A)$ which forms a linearly partially ordered real Banach space, which is, in fact, a unital GM-space. Moreover, it can be shown that the base of the dual cone in the dual GL-space $(\mathcal{L}(A)_h)^*$ consisting of elements of unit norm is affine isomorphic and weak*-homeomorphic to the state space of $\mathcal{L}(A)$.

The part played by idempotents in algebraic structures endowed with a binary product is played by tripotents in the theory of JB*-triples. A JBW*-triple A is particularly well-endowed with tripotents. Indeed, the linear span of the set $\mathcal{U}(A)$ of tripotents in the JBW*-triple is norm-dense in A. Furthermore, the set $\mathcal{U}(A)$ also possesses a natural partial ordering which has been much exploited in investigations into the properties of JBW*-triples. The main results of this paper connect this partial ordering with that in the ordered Banach space $\mathcal{L}(A)_h$. These show that tripotents u and v in A are orthogonal if and only if the corresponding multiplication operators in the unital GM-space $\mathcal{L}(A)_h$ satisfy

$$0 \leqslant D(u, u) + D(v, v) \leqslant \mathrm{id}_A,$$

and that u is a pre-associate of v if and only if

$$D(u, u) \leqslant D(v, v).$$

The paper is organised as follows. In Section 2, definitions and properties of unital GM- and GL-spaces are given, their importance in the study of subspaces of the real Banach space of hermitian elements in a complex unital Banach algebra is discussed, and the application of these results to the complex unital Banach algebras $\mathcal{B}(X)$ and $\mathcal{B}(X^*)$ of bounded linear operators on a complex Banach space X and on its dual X^* is explained. In Section 3, the basic definitions and properties of JB*- and JBW*-triples are given and the main object under discussion in this paper, the Banach–Lie algebra $\mathcal{L}(A)$ of multiplication operators on the JB*-triple A, is described. In addition, the results of Section 2 are applied to $\mathcal{L}(A)$ and it is shown how the state space of $\mathcal{L}(A)$ is related to the set of bounded sesquilinear functionals on A. The main results of the paper are contained in Section 4, after the definitions and properties involved in the study of weak*-closed inner ideals and tripotents in JBW*-triples are introduced.

2. Ordered Banach spaces and complex Banach algebras

In this section various preliminary results, some of them well known, are assembled. The first task is to review some results about partially ordered real Banach spaces.

Recall that a *unital GM-space* E is a real Banach space, linearly partially ordered by a norm-closed cone E_+, for which there exists an element 1 in E_+ such that the closed unit ball E_1 in E coincides with the order interval $(-1 + E_+) \cap (1 - E_+)$. In this case 1 is said to be the *order unit* in E [1,43].

Lemma 2.1. *Let E be a unital GM-space, with unit ball E_1, cone E_+, and order unit* 1*, and let F be a norm-closed subspace of E containing* 1*. Then, with respect to the restricted norm and cone $F \cap E_+$, the real Banach space F is a unital GM-space with order unit* 1.

Proof. Clearly $F \cap E_+$ is a norm-closed cone in F containing 1 and it only remains to show that the unit ball $F \cap E_1$ in F coincides with the set $(-1 + F \cap E_+) \cap (1 - F \cap E_+)$. Notice that if T is an element of $F \cap E_1$ then there exist elements T_1 and T_2 in E_+ such that

$$T = -1 + T_1 = 1 - T_2.$$

Since both 1 and T lie in F it follows that both T_1 and T_2 lie in F and

$$F \cap E_1 \subseteq (-1 + F \cap E_+) \cap (1 - F \cap E_+).$$

Conversely, if T lies in the set $(-1 + F \cap E_+) \cap (1 - F \cap E_+)$ then T lies in the set $F \cap (-1 + E_+) \cap (1 - E_+)$ which coincides with $F \cap E_1$, as required. □

Recall that a *GL-space* G is a real Banach space, linearly partially ordered by a norm-closed cone G_+, such that the norm is additive on G_+ and the closed unit ball G_1 in G coincides with the convex hull $\mathrm{conv}((G_+ \cap G_1) \cup (-G_+ \cap G_1))$ of the set $(G_+ \cap G_1) \cup (-G_+ \cap G_1)$. In this case the set K_G of elements of G_+ of norm one forms a base for the cone G_+ and

$$G_1 = \mathrm{conv}(K_G \cup -K_G).$$

Lemma 2.2. *Let E be a unital GM-space, with unit ball E_1, cone E_+, and order unit 1, let E^* be its Banach dual space, with unit ball E_1^*, and let*

$$E_+^* = \{\phi \in E^*: \phi(T) \geqslant 0 \ \forall T \in E_+\}.$$

Then, E_+^* is a weak-closed cone in E^* with respect to which the real Banach space E^* is a GL-space. Furthermore, the base K_{E^*} of the cone E_+^* is given by*

$$K_{E^*} = \{\phi \in E_+^*: \phi(1) = 1\} = \{\phi \in E_+^*: \|\phi\| = 1\}.$$

Proof. See [1, Proposition II.1.7]. □

Let $\mathcal{B}$ be a complex unital Banach algebra with unit $1_\mathcal{B}$ and let $\mathcal{B}^*$ be its Banach dual space. Then, the *state space* $S_\mathcal{B}$ of $\mathcal{B}$ consists of the set of elements ϕ in the unit ball $\mathcal{B}_1^*$ in $\mathcal{B}^*$ for which

$$\phi(1_\mathcal{B}) = 1.$$

For an element T in $\mathcal{B}$, the *numerical range* $V(\mathcal{B}, T)$ is defined by

$$V(\mathcal{B}, T) = \{\phi(T): T \in S_\mathcal{B}\}. \tag{2.1}$$

Observe that if $\mathcal{C}$ is a closed subspace of $\mathcal{B}$ containing $1_\mathcal{B}$ and T and the state space $S_\mathcal{C}$ and numerical range $V(\mathcal{C}, T)$ are similarly defined then

$$V(\mathcal{C}, T) = V(\mathcal{B}, T), \tag{2.2}$$

and, consequently, the numerical range of T is determined by the linear span of $1_\mathcal{B}$ and T. The *numerical radius* $\nu_\mathcal{B}(T)$ is defined by

$$\nu_\mathcal{B}(T) = \sup\{|\lambda|: \lambda \in V(\mathcal{B}, T)\}.$$

An element T of $\mathcal{B}$ is said to be *hermitian* if the numerical range $V(\mathcal{B}, T)$ is contained in $\mathbb{R}$ and is said to be *positive* if $V(\mathcal{B}, T)$ is contained in $\mathbb{R}_+$. Let $\mathcal{B}_h$ and $\mathcal{B}_{h+}$ denote the sets of hermitian and positive elements of $\mathcal{B}$, respectively. Observe that, for each element T in $\mathcal{B}_h$, the numerical range $V(\mathcal{B}, T)$ is the closed convex hull of the spectrum $\sigma_\mathcal{B}(T)$ of T, and, denoting the spectral radius of T by $r_\mathcal{B}(T)$, it follows that

$$\|T\| = \nu_\mathcal{B}(T) = r_\mathcal{B}(T). \tag{2.3}$$

For details of this result and those quoted above the reader is referred to [7,36].

The connection between the results quoted above and partially ordered real Banach spaces is described in the next lemma.

Lemma 2.3. *Let $\mathcal{B}$ be a complex unital Banach algebra with unit $1_\mathcal{B}$ and let $\mathcal{B}_h$ and $\mathcal{B}_{h+}$ be the sets of hermitian and positive elements of $\mathcal{B}$, respectively. Then, $\mathcal{B}_h$ is a norm-closed real subspace of $\mathcal{B}$, and $\mathcal{B}_{h+}$ forms a norm-closed cone in $\mathcal{B}_h$, containing $1_\mathcal{B}$, with respect to which $\mathcal{B}_h$ is a unital GM-space.*

Proof. That $\mathcal{B}_h$ forms a real Banach space with closed cone $\mathcal{B}_{h+}$ follows from [7, Lemma 2.6]. Moreover, since the numerical range $V(\mathcal{B}, 1_{\mathcal{B}})$ is the point $\{1\}$ in $\mathbb{R}_+$, the unit $1_{\mathcal{B}}$ lies in $\mathcal{B}_{h+}$. Observe that if the element T of $\mathcal{B}_h$ lies in the set $(-1_{\mathcal{B}} + \mathcal{B}_{h+}) \cap (1_{\mathcal{B}} - \mathcal{B}_{h+})$ then there exist elements T_1 and T_2 of $\mathcal{B}_{h+}$ such that

$$T = -1_{\mathcal{B}} + T_1 = 1_{\mathcal{B}} - T_2.$$

It follows that

$$\begin{aligned} V(\mathcal{B}, T) &= \{\phi(-1_{\mathcal{B}} + T_1)\colon \phi \in S_{\mathcal{B}}\} = \{-1 + \phi(T_1)\colon \phi \in S_{\mathcal{B}}\} \\ &= \{\phi(1_{\mathcal{B}} - T_2)\colon \phi \in S_{\mathcal{B}}\} = \{1 - \phi(T_2)\colon \phi \in S_{\mathcal{B}}\} \\ &\subseteq [-1, 1]. \end{aligned}$$

Hence, by (2.3),

$$\|T\| \leqslant 1.$$

Conversely, if T lies in the unit ball $\mathcal{B}_{h1}$ in $\mathcal{B}_h$, again using (2.3),

$$V(\mathcal{B}, T) \subseteq [-1, 1].$$

Hence, for all elements ϕ in $S_{\mathcal{B}}$,

$$\phi(1_{\mathcal{B}} - T) \geqslant 0, \qquad \phi(1_{\mathcal{B}} + T) \geqslant 0,$$

which implies that the elements $1_{\mathcal{B}} - T$ and $1_{\mathcal{B}} + T$ lie in $\mathcal{B}_{h+}$, and T lies in the set $(-1_{\mathcal{B}} + \mathcal{B}_{h+}) \cap (1_{\mathcal{B}} - \mathcal{B}_{h+})$, as required. □

The complex unital Banach algebra $\mathcal{B}$ may also be regarded as a Banach–Lie algebra relative to the Lie multiplication

$$(S, T) \mapsto [S, T] = ST - TS.$$

Recall that, by [7, Lemma 2.4], for elements S and T of $\mathcal{B}_h$, the element $\mathrm{i}[S, T]$ also lies in $\mathcal{B}_h$, and, therefore, the closed subspace $\mathcal{B}_j$ of $\mathcal{B}$, defined by

$$\mathcal{B}_j = \mathcal{B}_h + \mathrm{i}\mathcal{B}_h,$$

is a Banach–Lie subalgebra of $\mathcal{B}$ every element T of which has a unique representation

$$T = T_1 + \mathrm{i}T_2,$$

where T_1 and T_2 are elements of $\mathcal{B}_h$. For such an element T of $\mathcal{B}_j$ define

$$T^\dagger = T_1 - \mathrm{i}T_2.$$

Then, the mapping $T \mapsto T^\dagger$ is a norm-continuous involution on $\mathcal{B}_j$, the corresponding norm-continuous projections onto $\mathcal{B}_h$ being given by

$$T \mapsto \frac{1}{2}\big(T + T^\dagger\big), \qquad T \mapsto \frac{1}{2\mathrm{i}}\big(T - T^\dagger\big).$$

Let $\mathcal{L}$ be a norm-closed subspace of $\mathcal{B}_j$ which is closed under the involution $T \mapsto T^\dagger$ and contains $1_{\mathcal{B}}$. For each element ϕ in the Banach dual space $\mathcal{L}^*$ of $\mathcal{L}$ let $\phi^\dagger$ be the element of $\mathcal{L}^*$ defined, for an element T in $\mathcal{L}$, by

$$\phi^\dagger(T) = \overline{\phi\big(T^\dagger\big)}.$$

Then, the mapping $\phi \mapsto \phi^\dagger$ is a weak*-continuous involution on $\mathcal{L}^*$, the corresponding weak*-continuous projections onto the weak*-closed real subspace $(\mathcal{L}^*)_h$ of $\mathcal{L}^*$ being given by

$$\phi \mapsto \frac{1}{2}\big(\phi + \phi^\dagger\big), \qquad \phi \mapsto \frac{1}{2\mathrm{i}}\big(\phi - \phi^\dagger\big).$$

Observe that $(\mathcal{L}^*)_h$ consists of those elements of $\mathcal{L}^*$ that take real values on $\mathcal{L}_h$ and that

$$\mathcal{L}^* = (\mathcal{L}^*)_h \oplus \mathrm{i}(\mathcal{L}^*)_h.$$

Moreover, writing

$$\mathcal{L}_h = \mathcal{L} \cap \mathcal{B}_h, \qquad \mathcal{L}_{h+} = \mathcal{L} \cap \mathcal{B}_{h+},$$

it follows from Lemmas 2.1 and 2.3 that $\mathcal{L}_h$ is a unital GM-space, the dual $(\mathcal{L}_h)^*$ of which is a GL-space. In the next lemma the relationship that exists between the two real Banach spaces $(\mathcal{L}_h)^*$ and $(\mathcal{L}^*)_h$ is clarified.

Lemma 2.4. *Let $\mathcal{B}$ be a complex unital Banach algebra with unit $1_{\mathcal{B}}$, let $\mathcal{B}_j$ be the Banach–Lie subalgebra of $\mathcal{B}$ equal to $\mathcal{B}_h \oplus \mathrm{i}\mathcal{B}_h$, where $\mathcal{B}_h$ is the unital GM-space of hermitian elements of $\mathcal{B}$, let $T \mapsto T^\dagger$ be the norm-continuous involution on $\mathcal{B}_j$ defined, for elements T_1 and T_2 in $\mathcal{B}_h$ by*

$$(T_1 + \mathrm{i}T_2)^\dagger = T_1 - \mathrm{i}T_2,$$

let $\mathcal{L}$ be a norm-closed subspace of $\mathcal{B}$ containing $1_{\mathcal{B}}$ and closed under the involution, let $\mathcal{L}_h$ be the unital GM-space equal to $\mathcal{L} \cap \mathcal{B}_h$, with dual GL-space $(\mathcal{L}_h)^*$, let $\phi \mapsto \phi^\dagger$ be the adjoint weak-continuous involution on the Banach dual space $\mathcal{L}^*$, and let $(\mathcal{L}^*)_h$ be the weak*-closed real subspace of $\mathcal{L}^*$ consisting of elements ϕ for which ϕ and $\phi^\dagger$ coincide. For each element ϕ of $(\mathcal{L}^*)_h$, let $\hat{\phi}$ be the restriction of ϕ to $\mathcal{L}_h$. Then, the mapping $\phi \mapsto \hat{\phi}$ is a real linear weak*-continuous isometry from the real Banach space $(\mathcal{L}^*)_h$ onto the GL-space $(\mathcal{L}_h)^*$ mapping the state space $S_{\mathcal{L}}$ of $\mathcal{L}$ onto the base $K_{(\mathcal{L}_h)^*}$ of the cone $(\mathcal{L}_h)^*_+$ in $(\mathcal{L}_h)^*$.*

Proof. For each element ϕ in $(\mathcal{L}^*)_h$ and each element T in $\mathcal{L}_h$, $\phi(T)$ lies in $\mathbb{R}$ and,

$$\big|\phi(T)\big| \leqslant \|\phi\|\|T\|.$$

Hence, the restriction $\hat{\phi}$ of ϕ to $\mathcal{L}_h$ is a real linear functional on the unital GM-space $\mathcal{L}_h$ which lies in $(\mathcal{L}_h)^*$ and

$$\|\hat{\phi}\| \leqslant \|\phi\|. \tag{2.4}$$

It is clear that the mapping $\phi \mapsto \hat{\phi}$ is real linear from $(\mathcal{L}^*)_h$ to $(\mathcal{L}_h)^*$. If, for all elements S in $\mathcal{L}_h$,

$$\hat{\phi}(S) = 0$$

then, for all elements T in $\mathcal{L}$,

$$\phi(T) = \phi\left(\frac{1}{2}\left(T + T^{\dagger}\right)\right) + \mathrm{i}\phi\left(\frac{1}{2\mathrm{i}}\left(T - T^{\dagger}\right)\right) = 0,$$

and it follows that ϕ is equal to zero. Hence, the mapping $\phi \mapsto \hat{\phi}$ is an injection. For an element ψ of $(\mathcal{L}_h)^*$ define the mapping ϕ on $\mathcal{L}$, for each element T of $\mathcal{L}$, by

$$\phi(T) = \psi\left(\frac{1}{2}\left(T + T^{\dagger}\right)\right) + \mathrm{i}\psi\left(\frac{1}{2\mathrm{i}}\left(T - T^{\dagger}\right)\right).$$

Then, ϕ is clearly linear and, since the involution is norm-continuous,

$$\left|\phi(T)\right| \leqslant \frac{1}{2}\|\psi\|\left(\left\|T + T^{\dagger}\right\| + \left\|T - T^{\dagger}\right\|\right) \leqslant \|\psi\|(1+\gamma)\|T\|,$$

for some positive real number γ. It follows that ϕ lies in $\mathcal{L}^*$, and, since ϕ is real on $\mathcal{L}_h$, ϕ lies in $(\mathcal{L}^*)_h$. Clearly $\hat{\phi}$ and ψ coincide, from which it can be seen that the mapping $\phi \mapsto \hat{\phi}$ is a bijection from $(\mathcal{L}^*)_h$ onto $(\mathcal{L}_h)^*$.

Now, consider the pair $\mathcal{L}_h$ and $(\mathcal{L}^*)_h$ of real Banach spaces. If ϕ is an element of $(\mathcal{L}^*)_h$ such that, for all elements T in $\mathcal{L}_h$, $\phi(T)$ is equal to zero then, from above ϕ is equal to zero. Notice that, for each element ϕ of the state space $S_{\mathcal{L}}$ of $\mathcal{L}$ and each element T in $\mathcal{L}_h$, $\phi(T)$ is real and, hence, $S_{\mathcal{L}}$ is contained in $(\mathcal{L}^*)_h$. Therefore, if T is an element of $\mathcal{L}_h$ such that, for all elements ϕ in $(\mathcal{L}^*)_h$, $\phi(T)$ is equal to zero then, since $S_{\mathcal{L}}$ is contained in $(\mathcal{L}^*)_h$, by (2.1)–(2.3), the numerical range $V(\mathcal{B}, T)$ of T is equal to $\{0\}$ which implies that T is equal to zero. It follows that the real Banach spaces $\mathcal{L}_h$ and $(\mathcal{L}^*)_h$ form a dual pair in which the weak topology $\sigma(\mathcal{L}_h, (\mathcal{L}^*)_h)$ coincides with the weak topology $\sigma(\mathcal{L}, \mathcal{L}^*)$ restricted to $\mathcal{L}_h$ and the weak topology $\sigma((\mathcal{L}^*)_h, \mathcal{L}_h)$ coincides with the weak*-topology $\sigma(\mathcal{L}^*, \mathcal{L})$ restricted to $(\mathcal{L}^*)_h$. By definition, the mapping $\phi \mapsto \hat{\phi}$ is a homeomorphism from $(\mathcal{L}^*)_h$, endowed with the topology $\sigma((\mathcal{L}^*)_h, \mathcal{L}_h)$, onto $(\mathcal{L}_h)^*$, endowed with the weak*-topology.

Observe that, since the positive cone $\mathcal{L}_{h+}$ in the unital GM-space $\mathcal{L}_h$ is equal to the set of elements T of $\mathcal{L}$ for which the set $S_{\mathcal{L}}(T)$ is contained in $\mathbb{R}_+$, by duality it follows that the dual cone in $(\mathcal{L}^*)_h$ consists of the smallest weak*-closed cone in $(\mathcal{L}^*)_h$ containing $S_{\mathcal{L}}$. Notice that, since $S_{\mathcal{L}}$ is a weak*-compact convex set, a straightforward limit argument shows that $\mathbb{R}_+ S_{\mathcal{L}}$ is a weak*-closed cone and, hence, the smallest such cone containing $S_{\mathcal{L}}$. It follows by duality that

$$S_{\mathcal{L}} = \left\{\phi \in \mathcal{L}^*\colon \phi(\mathcal{L}_{h+}) \geqslant 0,\ \phi(1_{\mathcal{B}}) = 1\right\},$$

and, hence, that

$$\hat{S_{\mathcal{L}}} = \big\{\psi \in (\mathcal{L}_h)^*\colon\ \psi(\mathcal{L}_{h+}) \geqslant 0,\ \psi(1_{\mathcal{B}}) = 1\big\} = K_{(\mathcal{L}_h)^*}, \tag{2.5}$$

the base of the cone $(\mathcal{L}_h)^*_+$ in the GL-space $(\mathcal{L}_h)^*$. Notice that, for an element ψ in $(\mathcal{L}_h)^*$, there exist elements ψ_1 and ψ_2 in $K_{(\mathcal{L}_h)^*}$ and β in the real interval [0, 1] such that

$$\psi = \|\psi\|\big(\beta\psi_1 - (1-\beta)\psi_2\big),$$

and, hence, elements ϕ_1 and ϕ_2 in $S_{\mathcal{L}}$ such that

$$\psi = \|\psi\|\big(\beta\hat{\phi_1} - (1-\beta)\hat{\phi_2}\big) = \|\psi\|\big(\beta\phi_1 - (1-\beta)\phi_2\big)\hat{}.$$

It follows that ψ coincides with $\hat{\phi}$, where

$$\phi = \|\psi\|\big(\beta\phi_1 - (1-\beta)\phi_2\big).$$

Therefore,

$$\|\phi\| \leqslant \|\psi\|\big(\beta\|\phi_1\| + (1-\beta)\|\phi_2\|\big) = \|\hat{\phi}\|. \tag{2.6}$$

That the mapping $\phi \mapsto \hat{\phi}$ is an isometry follows from (2.4) and (2.6), and that the mapping is an order isomorphism follows from (2.5). □

In this paper, the complex unital Banach algebra $\mathcal{B}$ under consideration will be the set $\mathcal{B}(X)$ of bounded linear operators on a complex Banach space X, the unit being the identity operator id_X on X. The relationship that exists between the relevant properties of $\mathcal{B}(X)$ and $\mathcal{B}(X^*)$, where X^* is the Banach dual space of X is summarised in the result below.

Lemma 2.5. *Let X be a complex Banach space, with dual space X^*, let $\mathcal{B}(X)$ and $\mathcal{B}(X^*)$ be the complex unital Banach algebras of bounded linear operators on X and X^*, respectively, and let $T \mapsto T^*$ be the isometric linear anti-isomorphism from $\mathcal{B}(X)$ onto the sub-unital Banach algebra $\mathcal{B}(X)^*$ of weak*-continuous elements of $\mathcal{B}(X^*)$ obtained by taking adjoints. Then, the following results hold.*

(i) *The restriction of the mapping $T \mapsto T^*$ to the unital GM-space $\mathcal{B}(X)_h$ of hermitian elements of $\mathcal{B}(X)$ is an isometric real linear order isomorphism onto the unital GM-space $(\mathcal{B}(X)^*)_h$ of weak*-continuous hermitian elements of $\mathcal{B}(X^*)$.*

(ii) *The restriction of the mapping $T \mapsto T^*$ to the Banach–Lie algebra $\mathcal{B}(X)_j$, which is equal to $\mathcal{B}(X)_h \oplus \mathrm{i}\mathcal{B}(X)_h$, is an isometric linear Lie anti-isomorphism onto the sub-Banach–Lie algebra $(\mathcal{B}(X)_j)^*$ consisting of weak*-continuous elements of the Banach–Lie algebra $\mathcal{B}(X^*)_j$ such that, for each element T of $\mathcal{B}(X)_j$,*

$$\big(T^\dagger\big)^* = (T^*)^\dagger,$$

where $T \mapsto T^\dagger$ denotes the involution in $\mathcal{B}(X)$ and in $\mathcal{B}(X^*)$.

Proof. The result is an immediate consequence of standard properties of the adjoint mapping and [7, Theorem 9.4 and Corollary 9.6(iii)]. □

3. JB*-triples and JBW*-triples

A complex vector space A equipped with a triple product $(a, b, c) \mapsto \{a\,b\,c\}$ from $A \times A \times A$ to A which is symmetric and linear in the first and third variables, conjugate linear in the second variable and, for elements a, b, c and d in A, satisfies the identity

$$\big[D(a, b), D(c, d)\big] = D\big(\{a\,b\,c\}, d\big) - D\big(c, \{d\,a\,b\}\big), \tag{3.1}$$

where $[\,,\,]$ denotes the commutator, and D is the mapping from $A \times A$ to the algebra of linear operators on A defined by

$$D(a, b)c = \{a\,b\,c\},$$

is said to be a *Jordan*-triple*. A Jordan*-triple A for which the vanishing of $\{a\,a\,a\}$ implies that a itself vanishes is said to be *anisotropic*. For each element a in A, the conjugate linear mapping $Q(a)$ from A to itself is defined, for each element b in A, by

$$Q(a)b = \{a\,b\,a\}.$$

The mappings $D(a, b)$ and $Q(a)$ are said to be *multiplication operators* and *quadratic operators*, respectively. For details about the properties of Jordan*-triples the reader is referred to [28].

A Jordan*-triple A which is also a Banach space such that D is norm-continuous from $A \times A$ to the complex unital Banach algebra $\mathcal{B}(A)$ of bounded linear operators on A, and, for each element a in A, the multiplication operator $D(a, a)$ lies in the positive cone $\mathcal{B}(A)_{h+}$ in the unital GM-space $\mathcal{B}(A)_h$ of hermitian elements of $\mathcal{B}(A)$ and satisfies

$$\big\|D(a, a)\big\| = \|a\|^2, \tag{3.2}$$

is said to be a *JB*-triple*. It follows from the results of [19] that, in this case, for elements a, b and c in A,

$$\big\|\{a\,b\,c\}\big\| \leqslant \|a\|\,\|b\|\,\|c\|. \tag{3.3}$$

For details the reader is referred to [24,25,37] and [38]. Examples of JB*-triples are JB*-algebras, for the properties of which the reader is referred to [10,21,44], and [45], and, consequently, to C*-algebras [35].

The first main result shows how those of Section 2 apply to JB*-triples.

Theorem 3.1. *Let A be a JB*-triple and let $\mathcal{L}(A)$ be the norm-closed subspace of the complex unital Banach algebra $\mathcal{B}(A)$ of bounded linear operators on A which is the norm-closed complex linear span*

$$\mathcal{L}(A) = \overline{\mathrm{lin}_{\mathbb{C}}\big(\big\{D(a, b):\ a, b \in A\big\} \cup \{\mathrm{id}_A\}\big)}^{\,n}$$

of the union of the set of multiplication operators on A and the identity operator id_A on A. Then, the following results hold.

(i) *The norm-closed subspace* $\mathcal{L}(A)$ *is a sub-Banach–Lie algebra of the Banach–Lie algebra* $\mathcal{B}(A)_j$ *which is equal to* $\mathcal{B}(A)_h \oplus \mathrm{i}\mathcal{B}(A)_h$, *where* $\mathcal{B}(A)_h$ *is the unital GM-space of hermitian elements of* $\mathcal{B}(A)$, *and is closed under the involution* $T \mapsto T^\dagger$ *on* $\mathcal{B}(A)_j$.

(ii) *The real vector space*

$$\mathcal{L}(A)_h = \mathcal{L}(A) \cap \mathcal{B}(A)_h$$

is a unital GM-space with order unit id_A *such that*

$$\mathcal{L}(A)_h = \overline{\mathrm{lin}_{\mathbb{R}}\big(\big\{D(a,a)\colon\ a \in A\big\} \cup \{\mathrm{id}_A\}\big)}^{\,n}.$$

(iii) *The Banach–Lie algebra* $\mathcal{L}(A)$ *is given by*

$$\mathcal{L}(A) = \overline{\mathrm{lin}_{\mathbb{C}}\big(\big\{D(a,a)\colon\ a \in A\big\} \cup \{\mathrm{id}_A\}\big)}^{\,n}.$$

Proof. (i) Let $D(a,b)$ and $D(c,d)$ be multiplication operators on A and let α and β lie in $\mathbb{C}$. Then, using (3.1),

$$\big[D(a,b) + \alpha \mathrm{id}_A, D(c,d) + \beta \mathrm{id}_A\big] = \big[D(a,b), D(c,d)\big] = D\big(\{a\,b\,c\}, d\big) - D\big(c, \{d\,a\,b\}\big),$$

which lies in $\mathcal{L}(A)$. By linearity and norm-continuity of the triple product it follows that $\mathcal{L}(A)$ is a sub-Banach–Lie algebra of $\mathcal{B}(A)$. Observe that id_A lies in $\mathcal{B}(A)_h$ and, for all elements a and b in A,

$$\begin{aligned} D(a,b) = {} & \frac{1}{2}\big(D(a+b, a+b) - D(a,a) - D(b,b)\big) \\ & - \frac{\mathrm{i}}{2}\big(D(\mathrm{i}a+b, \mathrm{i}a+b) - D(a,a) - D(b,b)\big), \end{aligned} \tag{3.4}$$

which lies in $\mathcal{B}(A)_h \oplus \mathrm{i}\mathcal{B}(A)_h$, or, equivalently, in $\mathcal{B}(A)_j$. It follows that $\mathcal{L}(A)$ is a norm-closed Lie subalgebra of $\mathcal{B}(A)_j$, as required. Moreover, by (3.4), for elements a and b in A,

$$\begin{aligned} D(a,b)^\dagger &= \frac{1}{2}\big(D(a+b, a+b) - D(a,a) - D(b,b)\big) \\ &\quad + \frac{\mathrm{i}}{2}\big(D(\mathrm{i}a+b, \mathrm{i}a+b) - D(a,a) - D(b,b)\big) \\ &= \frac{1}{2}\big(\big(D(a,b) + D(b,a)\big) + \mathrm{i}\big(D(\mathrm{i}a,b) + D(b,\mathrm{i}a)\big)\big) \\ &= D(b,a). \end{aligned}$$

Since $(\mathrm{id}_A)^\dagger$ is equal to id_A, using the norm-continuity of the involution $T \mapsto T^\dagger$, it follows that $\mathcal{L}(A)$ is closed under the involution, as required.

(ii) That $\mathcal{L}_h$ is a unital GM-space follows from Lemma 2.1. From the definition of a JB*-triple it can be seen that the real vector space $\mathrm{lin}_{\mathbb{R}}(\{D(a,a)\colon\ a \in A\} \cup \{\mathrm{id}_A\})$ is contained in $\mathcal{L}(A)_h$, and, since $\mathcal{L}(A)_h$ is norm-closed, $\overline{\mathrm{lin}_{\mathbb{R}}(\{D(a,a)\colon\ a \in A\} \cup \{\mathrm{id}_A\})}^{\,n}$ is contained in $\mathcal{L}(A)_h$. Let T

be an element of $\mathcal{L}(A)_h$ and let ϵ be a positive real number. Then, there exists an element S in $\mathrm{lin}_{\mathbb{C}}(\{D(a,b)\colon\ a,b\in A\}\cup\{\mathrm{id}_A\})$ such that

$$\|S-T\|\leqslant\epsilon.$$

Suppose that

$$S=\sum_{j=1}^{n}\alpha_j D(a_j,b_j)+\beta\mathrm{id}_A.$$

Then,

$$S^{\dagger}=\sum_{j=1}^{n}\overline{\alpha_j}D(b_j,a_j)+\overline{\beta}\mathrm{id}_A,$$

and

$$\begin{aligned}\frac{1}{2}\big(S+S^{\dagger}\big)=&\sum_{j=1}^{n}\frac{1}{2}(\alpha_j+\overline{\alpha_j})\big(D(a_j+b_j,a_j+b_j)-D(a_j,a_j)-D(b_j,b_j)\big)\\&+\sum_{j=1}^{n}\frac{1}{2\mathrm{i}}(\alpha_j-\overline{\alpha_j})\big(D(\mathrm{i}a_j+b_j,\mathrm{i}a_j+b_j)-D(a_j,a_j)-D(b_j,b_j)\big)\\&+\frac{1}{2}(\beta+\overline{\beta})\mathrm{id}_A.\end{aligned}$$

Hence, the element $\frac{1}{2}(S+S^{\dagger})$ lies in $\mathrm{lin}_{\mathbb{R}}(\{D(a,a)\colon\ a\in A\}\cup\{\mathrm{id}_A\})$ and, using the norm-continuity of the involution, for some positive real number γ,

$$\left\|T-\frac{1}{2}\big(S+S^{\dagger}\big)\right\|\leqslant\frac{1}{2}\|T-S\|+\frac{1}{2}\big\|T^{\dagger}-S^{\dagger}\big\|<\frac{1}{2}(1+\gamma)\epsilon.$$

Hence, T lies in $\overline{\mathrm{lin}_{\mathbb{R}}(\{D(a,a)\colon\ a\in A\}\cup\{\mathrm{id}_A\})}^{n}$, as required.

(iii) This is immediate since

$$\mathcal{L}(A)=\mathcal{L}(A)_h\oplus\mathrm{i}\mathcal{L}(A)_h.\qquad\square$$

The Banach–Lie algebra $\mathcal{L}(A)$ is said to be the *Banach–Lie algebra of multiplication operators* on the JB*-triple A. The result below follows immediately from Lemma 2.4 and Theorem 3.1.

Corollary 3.2. *Under the conditions of Theorem* 3.1*, there exists a real linear weak*-continuous isometric order isomorphism from the self-adjoint part* $(\mathcal{L}(A)^*)_h$ *of the Banach dual space* $\mathcal{L}(A)^*$ *of* $\mathcal{L}(A)$ *onto the GL-space* $(\mathcal{L}(A)_h)^*$ *which maps the state space* $S_{\mathcal{L}(A)}$ *of* $\mathcal{L}(A)$ *onto the base* $K_{(\mathcal{L}(A)_h)^*}$ *of the cone* $(\mathcal{L}(A)_h)^*_+$.

A JB*-triple A which is the dual of a Banach space A_* is said to be a *JBW*-triple*. In this case the *predual* A_* of A is unique up to isometric linear isomorphism and, for elements a and b in A, the operators $D(a,b)$ and $Q(a)$ are weak*-continuous. Examples of JBW*-triples are JBW*-algebras, for the properties of which the reader is referred to [10,21,44], and [45], and, hence, to W*-algebras [35]. The second dual A^{**} of a JB*-triple A is a JBW*-triple. For details of these results the reader is referred to [4,5,8,9,18], and [23].

Theorem 3.1 and Corollary 3.2 do, of course, apply to JBW*-triples. Since, for elements a and b in the JBW*-triple A, the multiplication operator $D(a,b)$ is weak*-continuous, there exists an element $D(a,b)_*$ in the complex unital Banach algebra $\mathcal{B}(A_*)$ of bounded linear operators on the predual A_* of A with adjoint $D(a,b)$. Clearly, the operators $D(a,b)_*$ and $D(c,d)_*$ also satisfy the Jordan triple identity (3.1) and the result below follows immediately from this remark and Lemma 2.5.

Theorem 3.3. *Let A be a JBW*-triple with predual A_* and let $\mathcal{L}(A_*)$ be the norm-closed subspace of the complex unital Banach algebra of bounded linear operators on A_* which is the norm-closed linear span*

$$\mathcal{L}(A_*) = \overline{\operatorname{lin}_{\mathbb{C}}\big(\big\{D(a,b)_*\colon a,b \in A\big\} \cup \{\mathrm{id}_{A_*}\}\big)}^{n},$$

where $D(a,b)_*$ is defined, for each element x in A_* and c in A, by

$$D(a,b)_* x(c) = x\big(D(a,b)c\big) = x\big(\{a\,b\,c\}\big).$$

Then, the following results hold.

(i) *The norm-closed subspace $\mathcal{L}(A_*)$ is a sub-Banach–Lie algebra of the Banach–Lie algebra $\mathcal{B}(A_*)_j$ which is equal to $\mathcal{B}(A_*)_h \oplus \mathrm{i}\mathcal{B}(A_*)_h$, where $\mathcal{B}(A_*)_h$ is the unital GM-space of hermitian elements of $\mathcal{B}(A_*)$, and is closed under the involution $T \mapsto T^\dagger$ on $\mathcal{B}(A_*)_j$.*

(ii) *The real vector space*

$$\mathcal{L}(A_*)_h = \mathcal{L}(A_*) \cap \mathcal{B}(A_*)_h$$

is a unital GM-space with order unit id_{A_*} such that

$$\mathcal{L}(A_*)_h = \overline{\operatorname{lin}_{\mathbb{R}}\big(\big\{D(a,a)_*\colon a \in A\big\} \cup \{\mathrm{id}_{A_*}\}\big)}^{n}.$$

(iii) *The Banach–Lie algebra $\mathcal{L}(A_*)$ is given by*

$$\mathcal{L}(A_*) = \overline{\operatorname{lin}_{\mathbb{C}}\big(\big\{D(a,a)_*\colon a \in A\big\} \cup \{\mathrm{id}_{A_*}\}\big)}^{n}.$$

(iv) *The mapping $T \mapsto T^*$ is an isometric linear Lie anti-isomorphism from $\mathcal{L}(A_*)$ onto the Banach–Lie algebra $\mathcal{L}(A)$ of multiplication operators on A such that, for each element T in $\mathcal{L}(A_*)$,*

$$\big(T^\dagger\big)^* = (T^*)^\dagger,$$

the restriction of which to the unital GM-space $\mathcal{L}(A_*)_h$ is an isometric real linear order isomorphism onto the unital GM-space $\mathcal{L}(A)_h$ which is the hermitian part of $\mathcal{L}(A)$.

As a consequence of this result the study of the Banach–Lie algebra $\mathcal{L}(A)$ of multiplication operators on a JBW*-triple A is equivalent to the study of the Banach–Lie algebra $\mathcal{L}(A_*)$ of multiplication operators on its predual A_*. In particular, the GL-spaces $(\mathcal{L}(A)^*)_h$, $(\mathcal{L}(A)_h)^*$, $(\mathcal{L}(A_*)^*)_h$, and $(\mathcal{L}(A_*)_h)^*$ are all real linearly isometric and order isomorphic to each other.

Before embarking upon the proofs of the main results of the paper one further property of JB*-triples is required.

Lemma 3.4. *Let A be a JB*-triple, let ϕ be an element of the state space $S_{\mathcal{L}(A)}$ of the Banach–Lie algebra $\mathcal{L}(A)$ of multiplication operators on A and, for each pair a and b of elements of A, let*

$$\Omega_\phi(a,b)=\phi\big(D(a,b)\big),$$

where $D(a,b)$ is the multiplication operator corresponding to a and b. Then, the following results hold.

(i) *The mapping $\Omega_\phi : A\times A\to\mathbb{C}$ is a positive bounded sesquilinear functional on A.*
(ii) *The mapping $\|\cdot\|_{\Omega_\phi} : A\to\mathbb{R}_+$ defined, for an element a of A, by*

$$\|a\|_{\Omega_\phi}=\Omega_\phi(a,a)^{\frac{1}{2}}=\phi\big(D(a,a)\big)^{\frac{1}{2}}$$

is a pre-Hilbertian semi-norm on A.
(iii) *For all elements a in A,*

$$\|a\|_{\Omega_\phi}\leqslant\|a\|.$$

Proof. (i) Since ϕ is linear and the mapping $D : A\times A\to A$ is sesquilinear, it can be seen that Ω_ϕ is a sesquilinear functional on A. Moreover, since for all elements a in A, the multiplication operator $D(a,a)$ is positive, it follows that the sesquilinear functional Ω_ϕ is positive. Furthermore, for elements a and b in A, using (3.3),

$$\big|\Omega_\phi(a,b)\big|=\big|\phi\big(D(a,b)\big)\big|\leqslant\|\phi\|\big\|D(a,b)\big\|\leqslant\|a\|\|b\|,\tag{3.5}$$

and the sesquilinear functional Ω_ϕ is bounded.

(ii) This is an immediate consequence of (i).

(iii) This follows immediately from (3.5). □

4. Order theoretic properties of JBW*-triples

Before turning to the main results of the paper some more background material is summarised.

A subspace B of a JB*-triple A is said to be a *subtriple* if $\{B\,B\,B\}$ is contained in B. A subspace B is clearly a subtriple if and only if, for each element a in B, the element $\{a\,a\,a\}$ lies in B. Observe that every subtriple of a JB*-triple is an anisotropic Jordan*-triple. A subspace J of a JB*-triple A is said to be an *inner ideal* if $\{J\,A\,J\}$ is contained in J. Every norm-closed subtriple of a JB*-triple A is a JB*-triple [24]. It follows that a weak*-closed subtriple B of a JBW*-triple A is a JBW*-triple.

A pair a and b of elements in a JBW*-triple A is said to be *orthogonal*, when $D(a,b)$ is equal to zero. By [14, Lemma 3.1], it follows that orthogonality is a symmetric relation. For a subset B

of A, denote by $B^{\perp}$ the subset which consists of all elements in A which are orthogonal to all elements in B. The subset $B^{\perp}$ is said to be the *annihilator* of B. By [14, Lemma 3.2], $B^{\perp}$ is a weak*-closed inner ideal in A. Moreover, for subsets B, C of A, $B^{\perp} \cap B \subseteq \{0\}$, $B \subseteq B^{\perp\perp}$, $B \subseteq C$ implies that $C^{\perp} \subseteq B^{\perp}$, and $B^{\perp}$ and $B^{\perp\perp\perp}$ coincide.

An element u in a JBW*-triple A is said to be a *tripotent* if $\{u\,u\,u\}$ is equal to u. The set of tripotents in A is denoted by $\mathcal{U}(A)$. Observe that, by (3.3), a tripotent is either equal to 0 or is of norm one. For each tripotent u in the JBW*-triple A the weak*-continuous conjugate linear operator $Q(u)$ and the weak*-continuous linear operators, $P_j(u)$ are defined, for j equal to 0, 1, and 2, by

$$Q(u)a = \{u\,a\,u\}, \qquad P_2(u) = Q(u)^2,$$
$$P_1(u) = 2\big(D(u,u) - Q(u)^2\big), \qquad P_0(u) = \mathrm{id}_A - 2D(u,u) + Q(u)^2. \tag{4.1}$$

For j equal to 0, 1, and 2, the linear operators $P_j(u)$ are weak*-continuous projections onto the eigenspaces $A_j(u)$ of $D(u,u)$ corresponding to eigenvalues $j/2$. The corresponding decomposition

$$A = A_0(u) \oplus A_1(u) \oplus A_2(u),$$

or, equivalently,

$$\mathrm{id}_A = P_0(u) + P_1(u) + P_0(u)$$

is said to be the *Peirce decomposition* of A relative to u. It follows that

$$2D(u,u) = P_1(u) + 2P_2(u) = \mathrm{id}_A - P_0(u) + P_2(u). \tag{4.2}$$

For j, k, and l equal to 0, 1, and 2, $A_j(u)$ is a sub-JBW*-triple such that

$$\big\{A_j(u)\,A_k(u)\,A_l(u)\big\} \subseteq A_{j-k+l}(u) \tag{4.3}$$

when $j-k+l$ is equal to 0, 1 or 2, and

$$\big\{A_j(u)\,A_k(u)\,A_l(u)\big\} = \{0\}, \tag{4.4}$$

otherwise. Moreover,

$$\big\{A_2(u)\,A_0(u)\,A\big\} = \big\{A_0(u)\,A_2(u)\,A\big\} = \{0\} \tag{4.5}$$

and $A_0(u)$ and $A_2(u)$ are inner ideals in A. With respect to the product $(a,b) \to a \circ_u b = \{a\,u\,b\}$ and involution $a \to a^{\dagger_u} = \{u\,a\,u\}$, $A_2(u)$ is a JBW*-algebra with unit u. By weak*-continuity, for j equal to 0, 1, and 2, there exist contractive projections $P_j(u)_*$ on A_*, with ranges $A_j(u)_*$, which can be regarded as the preduals of the JBW*-triples $A_j(u)$, and A_* also enjoys the Peirce decomposition

$$A_* = A_0(u)_* \oplus A_1(u)_* \oplus A_2(u)_*.$$

A pair u and v of elements of $\mathcal{U}(A)$ is said to be *compatible* if, for j and k equal to 0, 1, and 2,

$$\big[P_j(u), P_k(v)\big] = 0,$$

or, equivalently, if

$$A = \bigoplus_{j,k=0}^{2} \big(A_j(u) \cap A_k(v)\big). \tag{4.6}$$

Two tripotents are compatible if one lies in any one of the Peirce spaces of the other. The reader is referred to [30] for details.

Observe that a pair u and v of elements of $\mathcal{U}(A)$ is orthogonal if v is contained in $A_0(u)$ or, equivalently, if u is contained in $A_0(v)$. For elements u and v of $\mathcal{U}(A)$ write $u \leqslant v$ if

$$\{u\, v\, u\} = u,$$

or, equivalently, if $v - u$ lies in $\mathcal{U}(A)$ and is orthogonal to u. This clearly defines a partial ordering on $\mathcal{U}(A)$. Let $\mathcal{U}(A)\tilde{}$ be the disjoint union of the set $\mathcal{U}(A)$ and a point set $\{\omega\}$ and define a relation on $\mathcal{U}(A)\tilde{}$ by writing $u \leqslant v$ if both u and v lie in $\mathcal{U}(A)$ and $u \leqslant v$ if u is an arbitrary element in $\mathcal{U}(A)$ and v is equal to ω. It is clear that this defines a partial ordering on $\mathcal{U}(A)\tilde{}$. Notice that if the supremum of a family (u_j) of elements of $\mathcal{U}(A)$ exists in $\mathcal{U}(A)$ then it is equal to its supremum in $\mathcal{U}(A)\tilde{}$.

For each subset L of the unit ball A_{*1} in A_* and each subset M of the unit ball A_1 in A, let

$$L' = \big\{a \in A_1\colon\ x(a) = 1\ \forall x \in L\big\}, \qquad M_{\prime} = \big\{x \in A_{*1}\colon\ x(a) = 1\ \forall a \in M\big\}. \tag{4.7}$$

Then, L is a norm-semi-exposed face of A_{*1} if and only if $(L')_{\prime}$ is equal to L and M is a weak*-semi-exposed face of A_1 if and only if $(M_{\prime})'$ is equal to M and the mappings $L \mapsto L'$ and $M \mapsto M_{\prime}$ are anti-order isomorphisms between the complete lattices of norm-semi-exposed faces of A_{*1} and weak*-closed semi-exposed faces of A_1 and are inverses of each other. For further details and the proof of the following result the reader is referred to [11] and [18].

Lemma 4.1. *Let A be a JBW*-triple with predual A_* and let A_1 and A_{*1} be the unit balls in A and A_*, respectively. Then, the following results hold.*

(i) *The mapping $u \mapsto \{u\}_{\prime}$, defined, for an element u of the partially ordered set $\mathcal{U}(A)$ of tripotents in A by (4.7) and for the element ω of $\mathcal{U}(A)\tilde{}$ by putting $\{\omega\}'$ equal to A_{*1}, is an order isomorphism from the partially ordered set $\mathcal{U}(A)\tilde{}$ onto the complete lattice of norm-closed faces of A_{*1}.*

(ii) *The mapping $u \mapsto \{u\}_{\prime}'$ is an anti-order isomorphism from the complete lattice $\mathcal{U}(A)\tilde{}$ onto the complete lattice of weak*-closed faces of A_1 such that, for an element u in $\mathcal{U}(A)$,*

$$\{u\}_{\prime}' = u + A_0(u)_1.$$

(iii) *An element u in $\mathcal{U}(A)$ is maximal if and only if $A_0(u)$ is equal to $\{0\}$.*

(iv) *For each element u in $\mathcal{U}(A)$ there exists a maximal tripotent w in $\mathcal{U}(A)$ for which $u \leqslant w$.*

Observe that, for each element a in the JBW*-triple A, there exists a smallest element r_a in $\mathcal{U}(A)$ for which a is a positive element in the JBW*-algebra $A_2(r_a)$. The tripotent r_a is said to be the *range tripotent* of a. For a detailed proof, see [15, Lemma 3.3]. The spectral theorem for JBW*-algebras can be used to show that a JBW*-triple coincides with the norm-closed linear span of the set $\mathcal{U}(A)$ of tripotents in A. This fact can be exploited to show that the Banach–Lie algebra $\mathcal{L}(A)$ of multiplication operators on the JBW*-triple A is also generated by the multiplication operators corresponding to tripotents.

Theorem 4.2. *Let A be a JBW*-triple, let $\mathcal{L}(A)$ be the Banach–Lie algebra of multiplication operators on A, let $\mathcal{L}(A)_h$ be the unital GM-space of hermitian elements of $\mathcal{L}(A)$, and let $\mathcal{U}(A)$ be the partially ordered set of tripotents in A. Then, the following results hold.*

(i) $\mathcal{L}(A)_h = \overline{\mathrm{lin}_{\mathbb{R}}(\{D(u,u)\colon\ u \in \mathcal{U}(A)\} \cup \{\mathrm{id}_A\})}^{n}$.
(ii) $\mathcal{L}(A) = \overline{\mathrm{lin}_{\mathbb{C}}(\{D(u,u)\colon\ u \in \mathcal{U}(A)\} \cup \{\mathrm{id}_A\})}^{n}$.

Proof. Let a be an element of A and let r_a be its range tripotent. Then a is a positive element in the JBW*-algebra $A_2(r_a)$ and it follows from [21, Proposition 4.2.3], that, given a positive real number ϵ, there exist pairwise orthogonal self-adjoint idempotents $u_1, u_2, \ldots, u_n$ in $A_2(r_a)$ and non-negative real numbers $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that

$$\left\| a - \sum_{j=1}^{n} \alpha_j u_j \right\| < \epsilon.$$

Observe that

$$D\left(\sum_{j=1}^{n} \alpha_j u_j, \sum_{k=1}^{n} \alpha_k u_k\right) = \sum_{j=1}^{n} \alpha_j^2 D(u_j, u_j),$$

and, hence, that

$$\begin{aligned}
\left\| D(a,a) - \sum_{j=1}^{n} \alpha_j^2 D(u_j, u_j) \right\| &\leqslant \left\| D\left(a - \sum_{j=1}^{n} \alpha_j u_j, a\right) \right\| \\
&\quad + \left\| D\left(\sum_{j=1}^{n} \alpha_j u_j, a - \sum_{j=1}^{n} \alpha_j u_j\right) \right\| \\
&\leqslant \left\| a - \sum_{j=1}^{n} \alpha_j u_j \right\| \|a\| + \left\| \sum_{j=1}^{n} \alpha_j u_j \right\| \left\| a - \sum_{j=1}^{n} \alpha_j u_j \right\| \\
&\leqslant \big(2\|a\| + \epsilon\big)\epsilon.
\end{aligned}$$

The result follows from Theorem 3.1. □

Using Lemma 4.1, it can be seen that, for each non-zero element x in A_*, there exists a smallest element e_x in $\mathcal{U}(A)$ for which

$$x(e_x) = \|x\|.$$

Furthermore,

$$P_2(e_x)_* x = x, \tag{4.8}$$

and the restriction of x to the JBW*-algebra $A_2(e_x)$ is a faithful positive weak*-continuous linear functional. Moreover, for α in $\mathbb{R}_+$ and non-zero and x in A_* non-zero, it is clear that

$$e_{\alpha x} = e_x,$$

and, using Lemma 4.1, it can be seen that the weak*-closed face

$$\{x/\|x\|\}' = e_x + A_0(e_x)_1. \tag{4.9}$$

When x is equal to zero, e_x is defined to be the zero tripotent. The tripotent e_x is said to be the *support tripotent* of x [18]. The proof of the following result is essentially given in the proof of [11, Theorem 4.4].

Lemma 4.3. *Let A be a JBW*-triple, let v be an element of the partially ordered set $\mathcal{U}(A)$ of tripotents in A, let $\{v\}_{\prime}$ be the corresponding norm-closed face of the unit ball A_{*1} in the predual A_* of A, and, for each element x in $\{v\}_{\prime}$, let e_x be the support tripotent of x. Then, the weak*-closed linear span $\overline{\mathrm{lin}_{\mathbb{C}}\{e_x \colon x \in \{v\}_{\prime}\}}^{w^*}$ of the set $\{e_x \colon x \in \{v\}_{\prime}\}$ coincides with the Peirce two-space $A_2(v)$.*

The next result, the second part of which owes much to [2], concerns the states of $\mathcal{L}(A)$ arising from elements of A_*.

Theorem 4.4. *Let A be a JBW*-triple with predual A_*, let $\mathcal{L}(A)$ be the Banach–Lie algebra of multiplication operators on A, let $\mathcal{L}(A)_h$ be the unital GM-space of hermitian elements of $\mathcal{L}(A)$, and let $(\mathcal{L}(A)^*)_h$ be the GL-space that is the self-adjoint part of the dual space $\mathcal{L}(A)^*$ of $\mathcal{L}(A)$. For each element x in A_*, let ϕ_x be the mapping from $\mathcal{L}(A)$ to $\mathbb{C}$ defined, for an element T in $\mathcal{L}(A)$, by*

$$\phi_x(T) = x(Te_x),$$

where e_x is the support tripotent of x. Then, the following results hold.

(i) *The mapping $x \mapsto \phi_x$ is a positive homogeneous isometry from A_* into the cone $(\mathcal{L}(A)^*)_{h+}$ in the GL-space $(\mathcal{L}(A)^*)_h$, mapping the set of elements in A_* of norm one into the state space $S_{\mathcal{L}(A)}$ of $\mathcal{L}(A)$.*
(ii) *For each non-zero element x in A_*, each element a in the weak*-closed face $\{x/\|x\|\}'$ of A_1 and each element T in $\mathcal{L}(A)$,*

$$\phi_x(T) = x(Ta).$$

Proof. (i) For x non-zero, the mapping ϕ_x is clearly a linear functional on $\mathcal{L}(A)$, and, for an element T in $\mathcal{L}(A)$,

$$\left|\phi_x(T)\right| = \left|x(Te_x)\right| \leqslant \|x\|\|T\|\|e_x\| = \|x\|\|T\|,$$

and ϕ_x is bounded with

$$\|\phi_x\| \leqslant \|x\|. \tag{4.10}$$

Moreover,

$$\phi_x(\mathrm{id}_A) = \left|x(e_x)\right| = \|x\|,$$

and, hence,

$$\|\phi_x\| \geqslant \|x\|. \tag{4.11}$$

It follows from (4.10), (4.11) that

$$\|\phi_x\| = \|x\| = \phi_x(\mathrm{id}_A), \tag{4.12}$$

and, hence, by Lemma 2.4, that ϕ_x lies in $(\mathcal{L}(A)^*)_{h+}$. Moreover, ϕ_0 is equal to zero and, hence, for each element x in A_* and α of $\mathbb{R}_+$,

$$\phi_{\alpha x} = \alpha\phi_x,$$

as required. Clearly, if x is an element of norm one in A_* then, by (4.12), ϕ_x lies in $S_{\mathcal{L}(A)}$.

(ii) The proof can be found in [2, Proposition 1.2]. □

It is now possible to relate the pre-Hilbertian semi-norms introduced in Lemma 3.4 to those introduced in [2] and subsequently used to define the strong*-topology on the JBW*-triple A. For details, the reader is referred to [3,33], and [34].

Corollary 4.5. *Under the conditions of Theorem* 4.4, *let x be an element of A_*, let $\Omega_{\phi_x} : A \times A \to \mathbb{C}$ be the positive bounded sesquilinear functional on A defined, for elements a and b in A, by*

$$\Omega_{\phi_x}(a, b) = \phi_x\bigl(D(a, b)\bigr) = x\bigl(\{a\,b\,e_x\}\bigr),$$

and let $\|\cdot\|_{\Omega_{\phi_x}} : A \to \mathbb{R}_+$ be the pre-Hilbertian semi-norm on A, defined, for an element a in A, by

$$\|a\|_{\Omega_{\phi_x}} = \phi_x\bigl(D(a, a)\bigr)^{\frac{1}{2}} = x\bigl(\{a\,a\,e_x\}\bigr)^{\frac{1}{2}}.$$

Then, the following results hold.

(i) $\{a \in A\colon \|a\|_{\Omega_{\phi_x}} = 0\} = A_0(e_x)$.

(ii) *For each pair a and b of elements of A,*

$$\Omega_{\phi_x}(a,b) = \Omega_{\phi_x}\big(P_2(e_x)a, P_2(e_x)b\big) + \Omega_{\phi_x}\big(P_1(e_x)a, P_1(e_x)b\big),$$

and

$$\|a\|^2_{\Omega_{\phi_x}} = \big\|P_2(e_x)a\big\|^2_{\Omega_{\phi_x}} + \big\|P_1(e_x)a\big\|^2_{\Omega_{\phi_x}}.$$

(iii) *The restriction of the semi-norm $a \mapsto \|a\|_{\Omega_{\phi_x}}$ to $A_2(e_x) \oplus A_1(e_x)$ is a norm with respect to which it forms a pre-Hilbert space.*

Proof. This is immediate from the results of [2]. □

It is now possible to turn to the main results of the paper which relate the order structure of the unital GM-space $\mathcal{L}(A)_h$ of hermitian elements of the Banach–Lie algebra of multiplication operators on the JBW*-triple A to the order structure of the partially ordered set $\mathcal{U}(A)$ of tripotents in A. Before stating the first theorem one further property of the states ϕ_x of $\mathcal{L}(A)$ which arise from elements of norm one in A_* is required.

Lemma 4.6. *Let A be a JBW*-triple, with predual A_*, let $\mathcal{L}(A)$ be the Banach–Lie algebra of multiplication operators on A, let $S_{\mathcal{L}(A)}$ be the state space of $\mathcal{L}(A)$, let x be an element of A_* of norm one, let ϕ_x be the corresponding element of $S_{\mathcal{L}(A)}$, and let u be an element of the partially ordered set $\mathcal{U}(A)$ of tripotents in A. Then,*

$$\phi_x\big(D(u,u)\big) = 1$$

if and only if x lies in the predual $A_2(u)_*$ of the JBW-algebra $A_2(u)$.*

Proof. Let x lie in $A_2(u)_*$. Then, by [16, Lemma 5.2], the support tripotent e_x of x lies in $A_2(u)$, which is the eigenspace of $D(u,u)$ corresponding to the eigenvalue 1. It follows that

$$\phi_x\big(D(u,u)\big) = x\big(D(u,u)e_x\big) = x(e_x) = \|x\| = 1,$$

as required.

Conversely, suppose that

$$1 = \phi_x\big(D(u,u)\big) = x\big(D(u,u)e_x\big).$$

Since, using (3.3),

$$\big\|D(u,u)e_x\big\| \leqslant \|u\|^2\|e_x\| = 1,$$

it follows that $D(u,u)e_x$ lies in the weak*-closed face $\{x\}'$ of the unit ball A_1 in A. Hence, by (4.9), there exists an element b_0 of the unit ball $A_0(e_x)_1$ in $A_0(e_x)$ such that

$$D(u,u)e_x = e_x + b_0. \tag{4.13}$$

For j equal to 0, 1, and 2, let u_j be equal to $P_j(e_x)u$. Then, by (4.13), using (4.3)–(4.5),

$$e_x + b_0 = \{u\,u\,e_x\} = \{(u_0+u_1+u_2)\,(u_0+u_1+u_2)\,e_x\}$$
$$= \{u_0\,u_1\,e_x\} + \{u_1\,u_1\,e_x\} + \{u_1\,u_2\,e_x\} + \{u_2\,u_2\,e_x\},$$

where the element $\{u_0\,u_1\,e_x\} + \{u_1\,u_2\,e_x\}$ lies in $A_1(e_x)$ and the element $\{u_1\,u_1\,e_x\} + \{u_2\,u_2\,e_x\}$ lies in $A_2(e_x)$. By linear independence, it follows that b_0 is equal to zero, which implies that

$$D(u,u)e_x = e_x.$$

It follows that e_x lies in $A_2(u)$ and, hence, that the tripotents e_x and u are compatible. Since $A_2(e_x)$ is contained in $A_2(u)$, it follows that

$$P_2(e_x)P_2(u) = P_2(u)P_2(e_x) = P_2(e_x). \tag{4.14}$$

Therefore, for each element a in A, using (4.8) and (4.14),

$$P_2(u)_*x(a) = x\big(P_2(u)a\big) = P_2(e_x)_*x\big(P_2(u)a\big) = x\big(P_2(e_x)P_2(u)a\big)$$
$$= x\big(P_2(e_x)a\big) = P_2(e_x)^*x(a) = x(a),$$

and

$$P_2(u)_*x = x,$$

which implies that x lies in $A_2(u)_*$, as required. □

It is now possible to state the first main result.

Theorem 4.7. *Let A be a JBW*-triple with predual A_*, let $\mathcal{L}(A)$ be the Banach–Lie algebra of multiplication operators on A, let $\mathcal{L}(A)_h$ be the unital GM-space of hermitian elements of $\mathcal{L}(A)$ with order unit* id_A*, and let u and v be elements of the partially ordered set $\mathcal{U}(A)$ of tripotents in A. Then, the following are equivalent*:

(i) *u and v are orthogonal*;
(ii) *in the unital GM-space $\mathcal{L}(A)_h$,*

$$0 \leqslant D(u,u) + D(v,v) \leqslant \mathrm{id}_A;$$

(iii) *the spectrum $\sigma_{\mathcal{B}(A)}(D(u,u)+D(v,v))$ in the complex unital Banach algebra $\mathcal{B}(A)$ of bounded linear operators on A is contained in the unit interval* [0, 1];
(iv) $\|D(u,u)+D(v,v)\| \leqslant 1$.

Proof. (i) ⇒ (ii). Since both $D(u,u)$ and $D(v,v)$ lie in the cone $\mathcal{L}(A)_{h+}$ in $\mathcal{L}(A)_h$, it follows that so also does $D(u,u)+D(v,v)$. Since u and v are orthogonal, it can be seen that $u+v$ lies in $\mathcal{U}(A)$ and, therefore, is of norm one. Let ϕ be an element of the state space $S_{\mathcal{L}(A)}$ of $\mathcal{L}(A)$. Then, by Lemma 3.4(iii),

$$\begin{aligned}0 &= 1 - \|u+v\|^2 \leqslant 1 - \|u+v\|^2_{\Omega_\phi} \\ &= 1 - \phi\big(D(u+v,u+v)\big) = 1 - \phi\big(D(u,u) + D(v,v)\big) \\ &= \phi\big(\mathrm{id}_A - \big(D(u,u) + D(v,v)\big)\big).\end{aligned}$$

Since, by Lemma 2.4, $\mathcal{L}(A)_{h+}$ and $\mathbb{R}_+ S_{\mathcal{L}(A)}$ are dual cones, it follows that $\mathrm{id}_A - (D(u,u) + D(v,v))$ lies in $\mathcal{L}(A)_{h+}$, as required.

(ii) ⇒ (iii). By [7, Lemma 1.2 and Theorem 1.6], and (2.1)–(2.2),

$$\begin{aligned}\sigma_{\mathcal{B}(A)}\big(D(u,u) + D(v,v)\big) &\subseteq V\big(\mathcal{B}(A), D(u,u) + D(v,v)\big) \\ &= V\big(\mathcal{L}(A), D(u,u) + D(v,v)\big) \\ &= \big\{\phi\big(D(u,u) + D(v,v)\big) \colon \phi \in S_{\mathcal{L}(A)}\big\}.\end{aligned}$$

But, by hypothesis, for an element ϕ of $S_{\mathcal{L}(A)}$,

$$0 = \phi(0) \leqslant \phi\big(D(u,u) + D(v,v)\big) \leqslant \phi(\mathrm{id}_A) = 1,$$

and

$$\sigma_{\mathcal{B}(A)}\big(D(u,u) + D(v,v)\big) \subseteq [0,1],$$

as required.

(iii) ⇒ (iv). Since $D(u,u) + D(v,v)$ is a positive element of $\mathcal{B}(A)_h$, by (2.3),

$$\big\|D(u,u) + D(v,v)\big\| = r_{\mathcal{B}(A)}\big(D(u,u) + D(v,v)\big) \leqslant 1,$$

by hypothesis.

(iv) ⇒ (i). Suppose that x is an element of the norm-closed face $\{v\}_\prime$ of the unit ball A_{*1} in the predual A_* of A, and let ϕ_x be the corresponding element of $S_{\mathcal{L}(A)}$. Since $\{v\}_\prime$ is the state space of the JBW*-algebra $A_2(v)$, it is a subset of $A_2(v)_*$ and, by Lemma 4.6,

$$\phi_x\big(D(v,v)\big) = 1.$$

Therefore, using Corollary 4.5,

$$\begin{aligned}\|u\|^2_{\Omega_{\phi_x}} + 1 &= \phi_x\big(D(u,u) + D(v,v)\big) \\ &\leqslant \|\phi_x\|\big\|D(u,u) + D(v,v)\big\| \leqslant 1.\end{aligned}$$

It follows that

$$\|u\|_{\Omega_{\phi_x}} = 0,$$

and, by Corollary 4.5(i), that u is contained in $A_0(e_x)$. Using Lemma 4.3,

$$u \in \bigcap_{x \in \{v\}'} A_0(e_x) = \left(\overline{\mathrm{lin}_{\mathbb{C}}\{e_x \colon x \in \{v\}'\}}^{w^*}\right)^{\perp}$$
$$= A_2(v)^{\perp} = A_0(v).$$

Hence, u and v are orthogonal, as required. □

Let u and v be tripotents in the JBW*-triple A. Then, u is said to be a *pre-associate* of v if the Peirce two-space $A_2(u)$ corresponding to u is contained in the Peirce two-space $A_2(v)$ corresponding to v. In the case in which u is a pre-associate of v and v is a pre-associate of u then u and v are said to be *associated*. This relation was discussed in [32] in the more general situation in which A is a Jordan*-triple. In the present situation considerably more can be proved by taking advantage of the properties of the order structure of the Banach–Lie algebra $\mathcal{L}(A)$ of multiplication operators on the JBW*-triple A. The results obtained in [32] are contained in the following lemma.

Lemma 4.8. *Let A be a JBW*-triple, and let u and v be elements of the partially ordered set $\mathcal{U}(A)$ of tripotents in A. Then, the following results hold.*

(i) *The following conditions are equivalent*:
 (a) *u and v are associated*;
 (b) *for j equal to* 0, 1 *and* 2, *the Peirce j-spaces $A_j(u)$ and $A_j(v)$ coincide*;
 (c) *$D(u,u)$ and $D(v,v)$ coincide.*

(ii) *For an element w of $\mathcal{U}(A)$, for u and v associated, and for j and k equal to* 0, 1, *and* 2, *if u lies in the Peirce j-space $A_j(w)$ and v lies in the Peirce k-space $A_k(w)$, then j and k coincide.*

Before attacking the second main theorem some further preliminary results are required.

Lemma 4.9. *Let A be a JBW*-triple and let u and v be elements of the partially ordered set $\mathcal{U}(A)$ of tripotents in A such that u is a pre-associate of v. Then, the following results hold.*

(i) *The tripotents u and v form a compatible pair.*
(ii) *If w is an element of $\mathcal{U}(A)$ which is orthogonal to v then w is orthogonal to u.*

Proof. (i) Since u is contained in $A_2(v)$, this follows from [30, Corollary 1.8].

(ii) Since $A_2(u)$ is contained in $A_2(v)$ it follows that

$$w \in A_0(v) = A_2(v)^{\perp} \subseteq A_2(u)^{\perp} = A_0(u),$$

and w is orthogonal to u, as required. □

For each non-empty subset B of the JBW*-triple A, the *kernel* $\mathrm{Ker}(B)$ of B is the weak*-closed subspace of elements a in A for which $\{B\,a\,B\}$ is equal to $\{0\}$. It follows that the annihilator $B^{\perp}$ of B is contained in $\mathrm{Ker}(B)$ and that $B \cap \mathrm{Ker}(B)$ is contained in $\{0\}$. A subtriple B of A is said to be *complemented* [13] if A coincides with $B \oplus \mathrm{Ker}(B)$. It can easily be seen that every complemented subtriple is a weak*-closed inner ideal. A linear projection R on the JBW*-triple A is said to be a *structural projection* [29] if, for each element a in A,

$$RQ(a)R = Q(Ra). \tag{4.15}$$

The main results of [12,16] and [13] show that the range RA of a structural projection R is a complemented subtriple, that the kernel $\ker(R)$ of the map R coincides with $\mathrm{Ker}(RA)$, that every structural projection is contractive and weak*-continuous, and, most significantly, that every weak*-closed inner ideal is complemented. Let $\mathcal{I}(A)$ denote the complete lattice of weak*-closed inner ideals in the JBW*-triple A and let $\mathcal{S}(A)$ denote the set of structural projections on A. The results of [16] can be used to show that the set $\mathcal{S}(A)$ of structural projections on A is a complete lattice with respect to the ordering defined, for elements R_1 and R_2, by $R_1 \leqslant R_2$ if R_2R_1 is equal to R_1 and the mapping $R \mapsto RA$ is an order isomorphism from $\mathcal{S}(A)$ onto the complete lattice $\mathcal{I}(A)$ of weak*-closed inner ideals in A.

Lemma 4.10. *Let A be a JBW*-triple, let w and v be elements of the partially ordered set $\mathcal{U}(A)$ of tripotents in A, with w a pre-associate of v, and, for j equal to 0, 1, and 2, let $P_j(w)$ and $A_j(w)$, and $P_j(v)$ and $A_j(v)$ be the Peirce projections and Peirce spaces corresponding to w and v, respectively. Then, the linear mapping R on A defined by*

$$R = P_2(w) + P_1(w)P_1(v) + P_0(v)$$

is a structural projection on A with range the weak-closed inner ideal*

$$RA = A_2(w) \oplus A_1(w) \cap A_1(v) \oplus A_0(v),$$

and kernel the weak-closed subspace*

$$\ker(R) = \big(A_1(w) \cap A_2(v)\big) \oplus \big(A_0(w) \cap A_2(v)\big) \oplus \big(A_0(w) \cap A_1(v)\big).$$

Proof. By Lemma 4.9(i), w and v form a compatible pair and, since $A_2(w)$ is contained in $A_2(v)$ and, hence, $A_0(v)$ is contained in $A_0(w)$, it is clear that their intersection table is given by

$\cap$	$A_2(w)$	$A_1(w)$	$A_0(w)$
$A_2(v)$	$A_2(w)$	$A_1(w) \cap A_2(v)$	$A_0(w) \cap A_2(v)$
$A_1(v)$	$\{0\}$	$A_1(w) \cap A_1(v)$	$A_0(w) \cap A_1(v)$
$A_0(v)$	$\{0\}$	$\{0\}$	$A_0(v)$

It will first be shown that the weak*-closed subspace

$$J = A_2(w) \oplus A_1(w) \cap A_1(v) \oplus A_0(v)$$

is an inner ideal in A. For elements a and c in J an b in A, with joint Peirce decompositions

$$a = a_{22} + a_{11} + a_{00}, \qquad c = c_{22} + c_{11} + c_{00},$$
$$b = b_{22} + b_{12} + b_{02} + b_{11} + b_{01} + b_{00},$$

using the Peirce arithmetical relations (4.3)–(4.5) and the intersection table above,

$$\begin{aligned}\{a\,b\,c\} &= \{a_{22}\,b_{22}\,c_{22}\} + \{a_{22}\,b_{22}\,c_{11}\} + \{a_{22}\,b_{22}\,c_{00}\} + \{a_{22}\,b_{11}\,c_{11}\} \\ &\quad + \{a_{22}\,b_{11}\,c_{00}\} + \{a_{11}\,b_{22}\,c_{22}\} + \{a_{11}\,b_{22}\,c_{11}\} + \{a_{11}\,b_{11}\,c_{22}\} \\ &\quad + \{a_{11}\,b_{11}\,c_{11}\} + \{a_{11}\,b_{11}\,c_{00}\} + \{a_{11}\,b_{00}\,c_{11}\} + \{a_{11}\,b_{00}\,c_{00}\} \\ &\quad + \{a_{00}\,b_{11}\,c_{22}\} + \{a_{00}\,b_{11}\,c_{11}\} + \{a_{00}\,b_{11}\,c_{22}\} \\ &\in A_2(w) \oplus A_1(w) \cap A_1(v) \oplus A_0(v) = J.\end{aligned}$$

Therefore, J is a weak*-closed inner ideal in A. Now, let b lie in the weak*-closed subspace

$$L = A_1(w) \cap A_2(v) \oplus A_0(w) \cap A_2(v) \oplus A_0(w) \cap A_1(v),$$

and let a and c be elements of J. Then, again using the joint Peirce decompositions of a, b and c, (4.3)–(4.5), and the intersection table above,

$$\{a\,b\,c\} = \{a_{22} + a_{11} + a_{00}\,b_{12} + b_{02} + b_{01}\,c_{22} + c_{11} + c_{00}\} = 0.$$

It follows that L is contained in $\mathrm{Ker}(J)$. However, using the compatibility of w and v, (4.6), and the intersection table above,

$$A = J \oplus L \subseteq J \oplus \mathrm{Ker}(J) = A,$$

and it can be seen that L and $\mathrm{Ker}(J)$ coincide. It follows that R is a linear projection on A with range J and kernel $\mathrm{Ker}(J)$ and, therefore, by [16, Theorem 3.4], is the unique structural projection onto J. □

It is now possible to prove the final result in the paper.

Theorem 4.11. *Let A be a JBW*-triple, let $\mathcal{L}(A)$ be the Banach–Lie algebra of multiplication operators on A, let $\mathcal{L}(A)_h$ be the unital GM-space of hermitian elements of $\mathcal{L}(A)$, and let u and v be elements of the partially ordered set $\mathcal{U}(A)$ of tripotents in A. Then, u is a pre-associate of v if and only if, in the unital GM-space $\mathcal{L}(A)_h$,*

$$D(u,u) \leqslant D(v,v).$$

Proof. Suppose that u is a pre-associate of v. Then, u lies in the JBW*-triple $A_2(v)$ and, by Lemma 4.1(iv), there exists a maximal tripotent w in $A_2(v)$ such that u is majorized by w. Then, u and $w-u$ are orthogonal tripotents and

$$\begin{aligned}D(w,w) &= D(w-u+u, w-u+u) \\ &= D(w-u, w-u) + D(w-u, u) + D(u, w-u) + D(u,u) \\ &= D(w-u, w-u) + D(u,u).\end{aligned}$$

Hence, in the ordering of $\mathcal{L}(A)_h$,

$$D(u,u) \leqslant D(w,w). \tag{4.16}$$

Since w lies in $A_2(v)$, w is a pre-associate of v and, by Lemma 4.9(i), w and v form a compatible pair. Moreover, since w is maximal in $A_2(v)$, by Lemma 4.1(iii),

$$A_0(w) \cap A_2(v) = 0, \tag{4.17}$$

which implies that

$$P_0(w)P_2(v) = P_2(v)P_0(w) = 0. \tag{4.18}$$

It follows that Lemma 4.10 can be applied to conclude that the linear mapping

$$R = P_2(w) + P_1(w)P_1(v) + P_0(v)$$

is a structural projection on A. Using (4.2) and (4.6), along with the joint Peirce decomposition used in the proof of Lemma 4.10,

$$\begin{aligned} 2\big(D(v,v) - D(w,w)\big) &= \mathrm{id}_A + P_2(v) - P_0(v) - \mathrm{id}_A - P_2(w) + P_0(w) \\ &= \big(P_2(w) + P_1(w)P_2(v) + P_0(w)P_2(v)\big) - P_0(v) \\ &\quad - P_2(w) + \big(P_0(v) + P_0(w)P_1(v) + P_0(w)P_2(v)\big) \\ &= P_1(w)P_2(v) + P_0(w)P_1(v). \end{aligned}$$

Therefore, using (4.18),

$$\begin{aligned} \mathrm{id}_A - 2\big(D(v,v) - D(w,w)\big) &= \mathrm{id}_A - P_1(w)P_2(v) - P_0(w)P_1(v) \\ &= P_2(w) + P_1(w)P_1(v) + P_0(v) = R. \end{aligned}$$

Since R is a structural projection it is of norm one and it follows that the element $\mathrm{id}_A - 2(D(v,v) - D(w,w))$ lies in the unit ball in the GM-space $\mathcal{L}(A)_h$. Therefore, since id_A is the order unit in $\mathcal{L}(A)_h$,

$$\mathrm{id}_A - 2\big(D(v,v) - D(w,w)\big) \leqslant \mathrm{id}_A,$$

and, consequently,

$$D(w,w) \leqslant D(v,v). \tag{4.19}$$

The result then follows from (4.17) and (4.19).

Conversely, if

$$D(u,u) \leqslant D(v,v),$$

then, for any element ϕ of the state space $S_{\mathcal{L}(A)}$ of $\mathcal{L}(A)$,

$$\phi\big(D(u,u)\big) \leqslant \phi\big(D(v,v)\big).$$

Observe that, by (3.2), the positive element $D(v,v)$ of the unital GM-space $\mathcal{L}(A)_h$ is of norm one and, since id_A is the order unit in $\mathcal{L}(A)_h$,

$$0 \leqslant D(v,v) \leqslant \mathrm{id}_A$$

which implies that, for any element ϕ of $S_{\mathcal{L}(A)}$,

$$0 \leqslant \phi\big(D(v,v)\big) \leqslant 1.$$

In particular, for any element x of norm one in the predual $A_2(u)_*$ of the JBW*-algebra $A_2(u)$, by Lemma 4.6,

$$1 = \phi_x\big(D(u,u)\big) \leqslant \phi_x\big(D(v,v)\big) \leqslant 1.$$

Therefore,

$$\phi_x\big(D(v,v)\big) = 1,$$

and, again using Lemma 4.6, x lies in the predual $A_2(v)_*$ of the JBW*-algebra $A_2(u)$. It follows that $A_2(u)_*$ is contained in $A_2(v)_*$ and, using the results of [17], $A_2(u)$ is contained in $A_2(v)$, as required. □

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