

STRONG CONVERGENCE RATES FOR EULER APPROXIMATIONS TO A CLASS OF STOCHASTIC PATH-DEPENDENT VOLATILITY MODELS

ANDREI COZMA* AND CHRISTOPH REISINGER*

Abstract. We consider a class of stochastic path-dependent volatility models where the stochastic volatility, whose square follows the Cox–Ingersoll–Ross model, is multiplied by a (leverage) function of the spot process, its running maximum, and time. We propose a Monte Carlo simulation scheme which combines a log-Euler scheme for the spot process with the full truncation Euler scheme or the backward Euler–Maruyama scheme for the squared stochastic volatility component. Under some mild regularity assumptions and a condition on the Feller ratio, we establish the strong convergence with order $1/2$ (up to a logarithmic factor) of the approximation process up to a critical time. The model studied in this paper contains as special cases Heston-type stochastic-local volatility models, the state-of-the-art in derivative pricing, and a relatively new class of path-dependent volatility models. The present paper is the first to prove the convergence of the popular Euler schemes with a positive rate, which is moreover consistent with that for Lipschitz coefficients and hence optimal.

Key words. Path-dependent volatility, running maximum, Cox–Ingersoll–Ross process, Euler scheme, Monte Carlo simulation, strong convergence order

AMS subject classifications. 60H35, 65C05, 65C30

1. Introduction. The two major families of option pricing models are local volatility (LV) (e.g., [18]) and stochastic volatility (SV) (e.g., [30]). LV models are flexible enough to perfectly replicate the market prices of vanilla options, whereas SV models generate much richer and more realistic spot-vol dynamics. The class of stochastic-local volatility (SLV) models introduced in [38, 44, 46] contain a stochastic volatility component and a local volatility component (the leverage function), and combine advantages of the two. According to [53, 55, 56], they allow for a better calibration to vanilla options and improve the pricing and risk-management performance. SLV models were recently referred to in [45] as the de facto standard for pricing foreign exchange (FX) options.

European options are actively traded on many asset classes, including FX, equities and commodities. Barrier options are also actively traded in these markets, and especially in FX markets. Their popularity can be explained by two key factors. First, they are useful in limiting the risk exposure of an investor. Second, they offer additional flexibility and can match an investor’s view on the market for a lower price than a European option. Barrier options, and in particular no-touch options, are so heavily traded that they are no longer considered exotic options. Hence, a pricing model that allows a perfect calibration to both European and no-touch options is desirable. While the prices of European options depend only on the final distribution of the underlying spot process (e.g., a stock price or a spot FX rate), the prices of no-touch options depend on the entire distribution of the underlying throughout the duration of the contract.

Path-dependent volatility (PDV) models (see, e.g., [25] and the references therein) assume that the volatility depends on the path of the underlying through the current value of the spot and a finite number of path-dependent variables, like the running or the moving average, the running maximum or minimum etc. PDV models are complete, can be perfectly calibrated to both vanilla and no-touch options (e.g., [48]),

*Mathematical Institute, University of Oxford, Woodstock Road, Oxford OX2 6GG, United Kingdom (andrei.s.cozma@gmail.com, christoph.reisinger@maths.ox.ac.uk).

and can produce rich implied volatility dynamics. Furthermore, according to [8], the joint distribution of the spot process and any path-dependent quantity of any SV or SLV model agrees with that of a suitably chosen PDV model. As a consequence, there is always a PDV model that produces the same prices of both vanilla and exotic options. We introduce path-dependency only through the running maximum of the underlying spot process, which allows for an exact calibration to both vanilla and no-touch options [48]. Guided by practically realistic PDV model implementations, we make more specific assumptions on the coefficients than are established for the mimicking process in the general framework in [8].

Stochastic path-dependent volatility (SPDV) models were briefly discussed in [25] as a generalization of PDV models. Although incomplete, they generate richer spot-vol dynamics, and include as special cases both SLV and PDV models. In this paper, we consider a Heston-type SPDV model because the Cox–Ingersoll–Ross (CIR) process [10] for the squared stochastic volatility is widely used in the industry due to its desirable properties, such as mean-reversion, non-negativity and analytical tractability, which allows for a fast calibration of the stochastic volatility parameters.

The SPDV model is non-affine and hence a closed-form solution to the European option valuation problem is not available. Therefore, we use Monte Carlo simulation methods [24] – which can handle path-dependent features easily – and approximate the solution to the stochastic differential equation (SDE) using an explicit or implicit Euler or Milstein discretization. Weak convergence is important when estimating expectations of payoffs. Strong convergence plays a crucial role in multilevel Monte Carlo methods [22, 23, 39] and may be required for some complex path-dependent derivatives. Furthermore, pathwise convergence follows automatically [40].

The usual theorems in [42] on the convergence of numerical simulations assume that the drift and diffusion coefficients are globally Lipschitz continuous and satisfy a linear growth condition, whereas [32] extended the analysis to locally Lipschitz SDEs. The standard convergence theory does not apply to the present work because of the explicit dependence of the drift and diffusion coefficients on the running maximum and also since the square-root diffusion coefficient of the CIR process is not Lipschitz. Strong and weak divergence of Euler approximations to SDEs with super-linearly growing coefficients was proved in [35]. Furthermore, a considerable amount of research has recently been devoted to proving that approximation schemes for some multi-dimensional SDEs with infinitely often differentiable and globally bounded coefficients converge arbitrarily slowly [21, 27, 37, 50, 57]. In particular, it was shown in [21] that for any arbitrarily slow speed of convergence, there exists a 2-dimensional SDE with smooth and bounded coefficients such that no approximation method based on finitely many observations of the driving Brownian motion can converge in L^1 to the solution faster than the given speed of convergence. These slow convergence phenomena raise the natural question of whether Euler approximations to the class of SPDV models considered in this paper also converge in the strong sense arbitrarily slowly, if at all.

The literature on the convergence of Monte Carlo methods under stochastic volatility is scarce. The Heston stochastic volatility model was considered in [31] and the strong convergence without a rate as well as the weak convergence for bounded payoffs were derived for a stopped Euler scheme with a reflection fix in the diffusion coefficient. For the log-Heston model, the convergence in L^p with order $1/2$ up to a logarithmic factor was established in [41] when the Euler scheme and the backward (drift-implicit) Euler–Maruyama (BEM) scheme are employed in the discretization of the (log-)spot process and its squared volatility, respectively. Moreover, [58] proved

the weak convergence with order 2 of the stochastic trapezoidal rule for the (log-)spot process when its squared volatility is simulated exactly. For the Heston model, [3] proved the weak convergence with order 1 of a log-Euler (LE) scheme for the spot process and a drift-implicit Milstein scheme for its squared volatility. Moreover, using the full truncation Euler (FTE) or the BEM scheme instead to discretize the squared volatility, the convergence in L^p with order 1/2 up to a logarithmic factor can easily be deduced by using some strong convergence results of [16, 17] together with a recent moment bound result of [12]. A hybrid Heston-type stochastic-local volatility model with stochastic short interest rates was considered in [12] and the strong convergence without a rate as well as the weak convergence for vanilla and exotic options were derived when the spot process is discretized via the LE scheme and its squared volatility and the short rates are discretized via the FTE scheme. The convergence rate, however, remained an open question and this paper is the first to address it.

In this work, we establish the strong convergence in L^p with order 1/2 up to a logarithmic factor of the Monte Carlo method with the LE scheme for the spot process and the (explicit) FTE scheme proposed in [47] or the (implicit) BEM scheme proposed in [1] for the squared volatility. The FTE scheme is arguably the most widely used scheme in practice because it preserves the positivity of the original process, is easy to implement and, perhaps most importantly, is found empirically to produce the smallest bias of all explicit Euler schemes with different fixes at the boundary [47]. The BEM scheme is often encountered in the finance literature and its convergence properties are well-understood [2, 17, 36, 51]. Hence, we obtain the optimal strong convergence rate for the numerical approximation of SDEs with globally Lipschitz coefficients [33, 49]. As a consequence, the Euler discretization of the spot process also converges with weak order 1/2 (up to a logarithmic factor), which is optimal because the Euler scheme for the running maximum converges with weak order of at most 1/2 (see, e.g., [5, 24]) rather than the weak order 1 typical for SDEs with smooth coefficients.

Finally, we note that [43] already uses Euler's method to show existence of solutions to a class of SDEs with random coefficients (Lipschitz for the diffusion coefficient and monotone for the drift), while [9, Section 16] show convergence of the stochastic Euler method in the semi-martingale topology under a similar setting.

In summary, to the best of our knowledge, this paper is the first to establish a positive strong convergence rate for Euler approximations to models with: (1) path-dependent volatility dynamics; (2) local and stochastic volatility dynamics, even without the path-dependency.

The remainder of this paper is structured as follows. In Section 2, we discuss the model and the postulated assumptions. Then, we define the simulation scheme and discuss the main theorem. In Section 3, we prove the strong convergence with a rate of the approximated spot process. In Section 4, we conduct numerical tests for the strong and weak convergence rates that validate and complement our theoretical findings. Section 5 contains a short discussion. Finally, detailed proofs of some technical results are given in the Appendix.

2. Set-up and main result.

2.1. Model assumptions. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions and $(W_t^s, W_t^v)_{t \geq 0}$ be a two-dimensional standard \mathcal{F}_t -

adapted Brownian motion with $d\langle W^s, W^v \rangle_t = \rho dt$, $\rho \in (-1, 1)$. We study the model

$$(2.1) \quad \begin{cases} dS_t = \mu(t, S_t, M_t)S_t dt + \sqrt{v_t}\sigma(t, S_t, M_t)S_t dW_t^s, & S_0 > 0, \\ dv_t = k(\theta - v_t)dt + \xi\sqrt{v_t}dW_t^v, & v_0 > 0, \\ M_t = \sup_{u \in [0, t]} S_u, \end{cases}$$

For a fixed time horizon $T > 0$, let

$$(2.2) \quad \mu : [0, T] \times \{(x, y) \in \mathbb{R}_+^2 \mid S_0 \vee x \leq y\} \rightarrow \mathbb{R}$$

and

$$(2.3) \quad \sigma : [0, T] \times \{(x, y) \in \mathbb{R}_+^2 \mid S_0 \vee x \leq y\} \rightarrow \mathbb{R}_+.$$

When the running maximum component vanishes, i.e., when $\mu(t, S_t, M_t) = \mu(t, S_t)$ and $\sigma(t, S_t, M_t) = \sigma(t, S_t)$, the SPDV model (2.1) collapses to a SLV model. The SPDV model further reduces to a LV model if the stochastic volatility component also vanishes, i.e., if we take ξ zero, or to a SV model if we take μ and σ constant.

When the stochastic volatility component vanishes, i.e., when ξ is zero, the SPDV model collapses to a PDV model, which can also be regarded as the Markovian projection of an Itô process onto the spot and its running maximum, see Theorem 3.6 and Corollary 3.10 in [8].

In this paper, we work under the following model assumptions:

ASSUMPTION 2.1. *The drift and diffusion functions μ and σ are bounded, i.e., there exist non-negative constants μ_{max} and σ_{max} such that, for all $t \in [0, T]$ and $0 \leq x \leq S_0 \vee x \leq y$, we have*

$$(2.4) \quad |\mu(t, x, y)| \leq \mu_{max}$$

and

$$(2.5) \quad 0 \leq \sigma(t, x, y) \leq \sigma_{max}.$$

ASSUMPTION 2.2. *The drift and diffusion functions μ and σ are bounded and piecewise 1/2-Hölder continuous in time, respectively, and Lipschitz continuous in log-spot and log-running maximum, i.e., there exist $N_T \in \mathbb{N}$ and non-negative constants $C_{\mu, t}, C_{\mu, x}, C_{\mu, m}, C_{\sigma, t}, C_{\sigma, x}, C_{\sigma, m}$ and $(C_{\sigma, t, j})_{1 \leq j \leq N_T}$ such that, for all $t_1, t_2 \in [0, T]$, $0 < x_1 \leq S_0 \vee x_1 \leq y_1$ and $0 < x_2 \leq S_0 \vee x_2 \leq y_2$, we have*

$$(2.6) \quad \begin{aligned} |\mu(t_1, x_1, y_1) - \mu(t_2, x_2, y_2)| &\leq C_{\mu, t} \mathbb{1}_{t_1 \neq t_2} + C_{\mu, x} |\log(x_1) - \log(x_2)| \\ &\quad + C_{\mu, m} |\log(y_1) - \log(y_2)| \end{aligned}$$

and

$$(2.7) \quad \begin{aligned} |\sigma(t_1, x_1, y_1) - \sigma(t_2, x_2, y_2)| &\leq C_{\sigma, t} \sqrt{|t_1 - t_2|} + \sum_{j=1}^{N_T} C_{\sigma, t, j} \mathbb{1}_{t_1 \wedge t_2 < \frac{jT}{N_T} \leq t_1 \vee t_2} \\ &\quad + C_{\sigma, x} |\log(x_1) - \log(x_2)| + C_{\sigma, m} |\log(y_1) - \log(y_2)|. \end{aligned}$$

For the results in this paper to hold, the jumps in the function σ do not have to be equally spaced as long as they occur at the time discretization nodes.

Under Assumptions 2.1 and 2.2, by techniques similar to the proof of Theorem 16.3.11 in [9], we obtain global existence and uniqueness of a strong solution.

REMARK 2.3. If Assumptions 2.1 and 2.2 hold, then for the purpose of this paper we choose the smallest non-negative constants possible. In particular,

$$(2.8) \quad \sigma_{max} = \sup \left\{ \sigma(t, x, y) \mid t \in [0, T], 0 \leq x \leq S_0 \vee x \leq y \right\},$$

$$(2.9) \quad C_{\sigma, x} = \sup \left\{ \frac{|\sigma(t, x_1, y) - \sigma(t, x_2, y)|}{|\log(x_1) - \log(x_2)|} \mid t \in [0, T], 0 < x_1 < x_2 \leq S_0 \vee x_2 \leq y \right\},$$

$$(2.10) \quad C_{\sigma, m} = \sup \left\{ \frac{|\sigma(t, x, y_1) - \sigma(t, x, y_2)|}{|\log(y_1) - \log(y_2)|} \mid t \in [0, T], 0 < x \leq S_0 \vee x \leq y_1 < y_2 \right\}.$$

REMARK 2.4. When modelling asset prices or spot FX rates, the drift function μ is usually a combination of deterministic interest rates and dividend yields, such that $\mu(t, S_t, M_t) = \mu(t)$ satisfies the above assumptions. Moreover, for the diffusion function σ to be consistent with European call and put prices, it has to be given by the ratio between a calibrated Brunick–Shreve volatility and the square-root of the conditional expectation of the squared stochastic volatility [8, 48]. In case of no running maximum component, the leverage function σ that is consistent with vanilla prices is given by the ratio between a calibrated Dupire local volatility and the square-root of the conditional expectation of the squared stochastic volatility [25, 26, 53]. In practice, the leverage function is defined on a grid of points (20 points per year in time and 30 points in space usually suffice for an acceptable calibration error [11, 26]), interpolated flat-forward in time and by cubic splines in spot, and extrapolated flat outside an interval.

Hence it is reasonable from a practical perspective to assume that the drift and diffusion functions μ and σ are constant outside a bounded interval, i.e., we can find $0 \leq S_{min} < S_0 < S_{max}$ such that, for all $t \in [0, T]$ and $0 \leq x \leq S_0 \vee x \leq y$, we have

$$(2.11) \quad \mu(t, x, y) = \mu(t, S_{min} \vee x \wedge S_{max}, S_{min} \vee y \wedge S_{max})$$

and

$$(2.12) \quad \sigma(t, x, y) = \sigma(t, S_{min} \vee x \wedge S_{max}, S_{min} \vee y \wedge S_{max}).$$

Under this assumption, Assumption 2.2 can be replaced by the requirement that the drift and diffusion functions μ and σ are bounded and piecewise 1/2-Hölder continuous in time, respectively, and Lipschitz continuous in spot and running maximum, i.e., there exist non-negative constants $C_{\mu, S}$, $C_{\mu, M}$, $C_{\sigma, S}$ and $C_{\sigma, M}$ such that, for all $t_1, t_2 \in [0, T]$, $0 \leq x_1 \leq S_0 \vee x_1 \leq y_1$ and $0 \leq x_2 \leq S_0 \vee x_2 \leq y_2$, we have

$$(2.13) \quad |\mu(t_1, x_1, y_1) - \mu(t_2, x_2, y_2)| \leq C_{\mu, t} \mathbf{1}_{t_1 \neq t_2} + C_{\mu, S} |x_1 - x_2| + C_{\mu, M} |y_1 - y_2|$$

and

$$(2.14) \quad |\sigma(t_1, x_1, y_1) - \sigma(t_2, x_2, y_2)| \leq C_{\sigma, t} \sqrt{|t_1 - t_2|} + \sum_{j=1}^{N_T} C_{\sigma, t, j} \mathbf{1}_{t_1 \wedge t_2 < \frac{jT}{N_T} \leq t_1 \vee t_2} + C_{\sigma, S} |x_1 - x_2| + C_{\sigma, M} |y_1 - y_2|.$$

2.2. Simulation scheme. Let $N \in \mathbb{N}$ be a multiple of N_T and consider a uniform grid

$$(2.15) \quad T = N\delta t, \quad t_n = n\delta t, \quad \forall n \in \{0, 1, \dots, N\}.$$

We use either the full truncation Euler scheme from [47] or the drift-implicit (square-root) Euler scheme, also known as the backward Euler–Maruyama scheme, proposed in [1] in order to discretize the squared volatility v . For the FTE discretization, we introduce the discrete-time auxiliary process

$$(2.16) \quad \tilde{v}_{t_{n+1}} = \tilde{v}_{t_n} + k(\theta - \tilde{v}_{t_n}^+) \delta t + \xi \sqrt{\tilde{v}_{t_n}^+} \delta W_{t_n}^v, \quad \tilde{v}_0 = v_0,$$

where $v^+ = \max(0, v)$ and $\delta W_{t_n}^v = W_{t_{n+1}}^v - W_{t_n}^v$, its continuous-time interpolation

$$(2.17) \quad \tilde{v}_t = \tilde{v}_{t_n} + k(\theta - \tilde{v}_{t_n}^+)(t - t_n) + \xi \sqrt{\tilde{v}_{t_n}^+} (W_t^v - W_{t_n}^v)$$

as well as the non-negative piecewise constant process

$$(2.18) \quad \bar{v}_t = \tilde{v}_{t_n}^+,$$

whenever $t \in [t_n, t_{n+1})$. For the BEM discretization, assuming the boundary point 0 is inaccessible (i.e., $2k\theta \geq \xi^2$) and using a Lamperti transformation $y = \sqrt{v}$, we get

$$(2.19) \quad dy_t = (\alpha y_t^{-1} + \beta y_t) dt + \gamma dW_t^v,$$

where

$$(2.20) \quad \alpha = \frac{4k\theta - \xi^2}{8}, \quad \beta = -\frac{k}{2} \quad \text{and} \quad \gamma = \frac{\xi}{2}.$$

We introduce the discrete-time auxiliary process

$$(2.21) \quad \tilde{y}_{t_{n+1}} = \tilde{y}_{t_n} + (\alpha \tilde{y}_{t_{n+1}}^{-1} + \beta \tilde{y}_{t_{n+1}}) \delta t + \gamma \delta W_{t_n}^v, \quad \tilde{y}_0 = y_0,$$

as well as the piecewise constant processes

$$(2.22) \quad \bar{y}_t = \tilde{y}_{t_n} \quad \text{and} \quad \bar{v}_t = \tilde{y}_{t_n}^2,$$

whenever $t \in [t_n, t_{n+1})$. If $4k\theta > \xi^2$, then $\alpha > 0$ and, as $\beta < 0$, (2.21) has the unique positive solution

$$(2.23) \quad \tilde{y}_{t_{n+1}} = \frac{\tilde{y}_{t_n} + \gamma \delta W_{t_n}^v}{2(1 - \beta \delta t)} + \sqrt{\frac{(\tilde{y}_{t_n} + \gamma \delta W_{t_n}^v)^2}{4(1 - \beta \delta t)^2} + \frac{\alpha \delta t}{1 - \beta \delta t}}.$$

Note that unlike in [2, 17, 51], it is critical that we use a piecewise constant continuous-time interpolation \bar{v} for the squared volatility v because we only simulate increments of the Brownian driver W^s and hence the diffusion coefficient of the spot process needs to be constant in between time nodes. We use an Euler–Maruyama scheme to discretize the log-spot process $x = (x_t)_{t \geq 0}$, where $x_t = \log(S_t)$, and we define for convenience the log-running maximum $m = (m_t)_{t \geq 0}$, where $m_t = \log(M_t) = \sup_{u \in [0, t]} x_u$. Let \bar{x} be the approximated log-spot process, then the discrete method reads:

$$(2.24) \quad \begin{aligned} \bar{x}_{t_{n+1}} &= \bar{x}_{t_n} + \int_{t_n}^{t_{n+1}} \mu(u, e^{\bar{x}_{t_n}}, e^{\bar{m}_{t_n}}) du - \frac{1}{2} \sigma^2(t_n, e^{\bar{x}_{t_n}}, e^{\bar{m}_{t_n}}) \bar{v}_{t_n} \delta t \\ &\quad + \sigma(t_n, e^{\bar{x}_{t_n}}, e^{\bar{m}_{t_n}}) \sqrt{\bar{v}_{t_n}} \delta W_{t_n}^s, \quad \bar{x}_0 = x_0, \end{aligned}$$

$$(2.25) \quad \bar{m}_{t_{n+1}} = \max_{0 \leq i \leq n+1} \bar{x}_{t_i}, \quad \bar{m}_0 = x_0.$$

We do not freeze the drift coefficient at the previous time step in order to avoid larger discretization errors due to possible discontinuities in the drift function μ in the time variable, which is allowed in our setting by Assumption 2.2. This way, for purely time-dependent drift and no volatility the result is exact.

The continuous-time approximation for $t \in [t_n, t_{n+1})$ is

$$\begin{aligned} \bar{x}_t &= x_0 + \int_0^t \bar{\mu}(u, e^{\bar{x}_u}, e^{\bar{m}_u}) du - \frac{1}{2} \int_0^t \bar{\sigma}^2(u, e^{\bar{x}_u}, e^{\bar{m}_u}) \bar{v}_u du \\ &\quad + \int_0^t \bar{\sigma}(u, e^{\bar{x}_u}, e^{\bar{m}_u}) \sqrt{\bar{v}_u} dW_u^s, \end{aligned} \quad (2.26)$$

where $\bar{\mu}(t, e^{\bar{x}_t}, e^{\bar{m}_t}) = \mu(t, e^{\bar{x}_{t_n}}, e^{\bar{m}_{t_n}})$ and $\bar{\sigma}(t, e^{\bar{x}_t}, e^{\bar{m}_t}) = \sigma(t_n, e^{\bar{x}_{t_n}}, e^{\bar{m}_{t_n}})$. Let $\bar{S} = (\bar{S}_t)_{t \geq 0}$, where $\bar{S}_t = e^{\bar{x}_t}$, be the continuous-time approximation of S , and let $\bar{M}_{t_n} = e^{\bar{m}_{t_n}} = \max_{0 \leq i \leq n} \bar{S}_{t_i}$, for all $0 \leq n \leq N$. Using Itô's formula, we obtain

$$\bar{S}_t = S_0 + \int_0^t \bar{\mu}(u, \bar{S}_u, \bar{M}_u) \bar{S}_u du + \int_0^t \bar{\sigma}(u, \bar{S}_u, \bar{M}_u) \sqrt{\bar{v}_u} \bar{S}_u dW_u^s. \quad (2.27)$$

We prefer the log-Euler scheme to the standard Euler scheme to discretize the spot process because the former preserves positivity and produces no discretization bias in the spot direction when μ is deterministic and σ is constant, which is desirable because the drift function may be discontinuous.

2.3. The main theorem. Before we state the main result, we introduce some necessary notations. Let the Feller ratio be

$$\nu = \frac{2k\theta}{\xi^2}.$$

Theorem 2.5 below establishes the L^p convergence rate of the FTE and BEM schemes for ν above a certain value, p below a certain value, and up to a certain time T as defined in turn below. Throughout this paper, we use a superscript $*$ $\in \{\text{FTE}, \text{BEM}\}$ to differentiate between the two discretization schemes for the squared volatility process.

For brevity, define now

$$\nu^{\text{FTE}} = 2 + \sqrt{3}, \quad \nu^{\text{BEM}} = 2,$$

$$p^{\text{FTE}}(\nu) = \nu^{-1}(\nu - 1)^2, \quad p^{\text{BEM}}(\nu) = \nu.$$

Then, define

$$T_x^*(p) = \frac{2}{\sqrt{(\varphi^*(p) - k^2)^+}} \left[\frac{\pi}{2} + \arctan \left(\frac{k}{\sqrt{(\varphi^*(p) - k^2)^+}} \right) \right], \quad (2.31)$$

$$\varphi^*(p) = \inf_{q \in (p, p^*)} \tilde{\varphi}(p, q), \quad (2.32)$$

where

$$\begin{aligned} \tilde{\varphi}(p, q) &= \left(\sqrt{(2 + \beta_0^2)(C_{\sigma, x} + C_{\sigma, m})^2 q + 2(C_{\sigma, x} + C_{\sigma, m})(2\sigma_{\max} - C_{\sigma, x} - C_{\sigma, m})} \right. \\ &\quad \left. + \beta_0(C_{\sigma, x} + C_{\sigma, m})\sqrt{q} \right)^2 \frac{pq\xi^2}{2(q - p)} \end{aligned} \quad (2.33)$$

and $\beta_0 \approx 1.307$ is the unique positive root of

$$(2.34) \quad \phi_0(s) = -e^{\frac{s^2}{2}} + s \int_0^s e^{\frac{u^2}{2}} du.$$

Moreover, define

$$(2.35) \quad T_S^{\text{FTE}}(p) = \begin{cases} \frac{4k}{\phi(p)} & \text{if } \phi(p) < 4k^2, \\ \frac{1}{\sqrt{\phi(p)} - k} & \text{if } \phi(p) \geq 4k^2, \end{cases}$$

$$(2.36) \quad T_S^{\text{BEM}}(p) = \frac{1}{\sqrt{\phi(p)}},$$

where

$$(2.37) \quad \phi(p) = \xi^2 \sigma_{max}^2 (p + \sqrt{(p-1)p})^2.$$

Finally, define

$$(2.38) \quad T^*(p) = \sup_{q \in (2 \vee p, p^*)} \left[T_x^*(q) \wedge T_S^*(pq(q-p)^{-1}) \right],$$

with T_x^* given in (2.31) and T_S^{FTE} and T_S^{BEM} in (2.35) and (2.36), respectively.

To the best of our knowledge, Theorem 2.5 below is the first result to establish a positive strong convergence rate for Euler approximations to models with local and stochastic volatility dynamics, even without the path-dependency. The proof is postponed to Section 3. In Section 4, we briefly examine the critical time T^* defined in (2.38) with respect to the model parameters in a realistic scenario.

THEOREM 2.5. *Suppose that Assumptions 2.1 and 2.2 hold and that $\nu > \nu^*$, with ν^* defined in (2.29). Then for all $1 \leq p < p^*(\nu)$ and $T < T^*(p)$, with p^* defined in (2.30) and T^* given in (2.38), there exists a constant C such that, for all $N \geq 1$,*

$$(2.39) \quad \sup_{t \in [0, T]} \mathbb{E} \left[|S_t - \bar{S}_t|^p \right]^{\frac{1}{p}} \leq C \sqrt{\frac{\log(2N)}{N}}.$$

If the stochastic volatility component vanishes (e.g., take $v_0 = \theta = 1$ and $\xi = 0$), then the SPDV model (2.1) collapses to a path-dependent volatility model

$$(2.40) \quad \begin{cases} dS_t^{\text{PDV}} = \mu(t, S_t^{\text{PDV}}, M_t^{\text{PDV}}) S_t^{\text{PDV}} dt + \sigma(t, S_t^{\text{PDV}}, M_t^{\text{PDV}}) S_t^{\text{PDV}} dW_t^s, & S_0^{\text{PDV}} > 0, \\ M_t^{\text{PDV}} = \sup_{u \in [0, t]} S_u^{\text{PDV}}. \end{cases}$$

Upon noticing from (2.28), (2.30) and (2.31) – (2.38) that $\nu = p^*(\nu) = \infty$ and $T^*(p) = \infty$, for all $p \geq 1$, the same argument ensures the strong convergence in L^p with order $1/2$ (up to a logarithmic factor), for all $p \geq 1$, of the corresponding approximation process S^{PDV} defined in (2.27). Corollary 2.6 below is the first result to establish a positive strong convergence rate for Euler approximations to models with path-dependent volatility dynamics, to the best of our knowledge.

COROLLARY 2.6. *Suppose that Assumptions 2.1 and 2.2 hold. Then for all $p \geq 1$, there exists a constant C such that, for all $N \geq 1$,*

$$(2.41) \quad \sup_{t \in [0, T]} \mathbb{E} \left[|S_t^{\text{PDV}} - \bar{S}_t^{\text{PDV}}|^p \right]^{\frac{1}{p}} \leq C \sqrt{\frac{\log(2N)}{N}}.$$

We know from Theorem 10.2.2 in [42] the strong convergence in L^1 with order 1/2 of Euler approximations to the LV model

$$(2.42) \quad dS_t^{\text{LV}} = \mu(t, S_t^{\text{LV}})S_t^{\text{LV}}dt + \sigma(t, S_t^{\text{LV}})S_t^{\text{LV}}dW_t^s, \quad S_0^{\text{LV}} > 0,$$

when the drift and diffusion coefficients (i.e., $\mu(t, x)x$ and $\sigma(t, x)x$) satisfy a linear growth condition, are 1/2-Hölder continuous in time and Lipschitz continuous in spot. Hence, Corollary 2.6 extends the strong order 1/2 convergence of numerical simulations for LV models to allow dependence on the running maximum under somewhat different model assumptions.

We have thus shown that the Euler discretization of the spot process in (2.1) attains the optimal strong convergence order of 1/2 up to a logarithmic factor that is characteristic of approximations of SDEs with globally Lipschitz coefficients [33, 49]. As a consequence, the Euler discretization of the spot process also converges with weak order 1/2 (up to a logarithmic factor), which is optimal because the Euler scheme for the running maximum converges with weak order of at most 1/2 (see [5, 24]) instead of the weak order 1 typical for SDEs with smooth coefficients.

3. Convergence analysis.

3.1. The squared volatility process. First, we need to control the moments of the CIR process and its FTE and BEM approximations.

LEMMA 3.1. *The CIR process v from (2.1):*

(1) *has bounded moments, i.e.,*

$$(3.1) \quad \sup_{t \in [0, T]} \mathbb{E} [v_t^p] < \infty, \quad \forall p > -\nu;$$

(2) *has uniformly bounded moments, i.e.,*

$$(3.2) \quad \mathbb{E} \left[\sup_{t \in [0, T]} v_t^p \right] < \infty, \quad \forall p \geq 1.$$

Proof. The first part follows from [17] or Theorem 3.1 in [34] whereas the second part follows from Proposition 3.7 in [12] or Lemma 3.2 in [17]. \square

LEMMA 3.2. *The FTE scheme \tilde{v} from (2.17) has uniformly bounded moments, i.e.,*

$$(3.3) \quad \sup_{N \geq 1} \mathbb{E} \left[\sup_{t \in [0, T]} |\tilde{v}_t|^p \right] < \infty, \quad \forall p \geq 1.$$

Proof. Follows from a simple application of the Burkholder–Davis–Gundy (BDG) inequality and Proposition 3.7 in [12]. \square

LEMMA 3.3. *If $\nu \geq 1$, then the BEM scheme \bar{v} from (2.22) has uniformly bounded moments, i.e.,*

$$(3.4) \quad \sup_{N \geq 1} \mathbb{E} \left[\sup_{t \in [0, T]} \bar{v}_t^p \right] < \infty, \quad \forall p \geq 1.$$

Proof. Follows from an application of Lemma 2.5 in [51] to the CIR process. \square

For convenience of notation, for all $t \in [0, T]$, define

$$(3.5) \quad \bar{t} = \delta t \left\lfloor \frac{t}{\delta t} \right\rfloor.$$

PROPOSITION 3.4. *Suppose that $\nu > 1$ and let $1 \leq p < \nu$. Then the BEM scheme defined in (2.22) converges strongly in L^p with order $1/2$, i.e., there exists a constant C such that, for all $N \geq 1$,*

$$(3.6) \quad \sup_{t \in [0, T]} \mathbb{E} [|y_t - \bar{y}_t|^p]^{\frac{1}{p}} \leq CN^{-\frac{1}{2}}$$

and

$$(3.7) \quad \sup_{t \in [0, T]} \mathbb{E} [|v_t - \bar{v}_t|^p]^{\frac{1}{p}} \leq CN^{-\frac{1}{2}}.$$

Proof. The triangle inequality yields

$$(3.8) \quad \sup_{t \in [0, T]} \mathbb{E} [|y_t - \bar{y}_t|^p] \leq 2^{p-1} \sup_{t \in [0, T]} \mathbb{E} [|y_t - y_{\bar{t}}|^p] + 2^{p-1} \sup_{t \in [0, T]} \mathbb{E} [|y_{\bar{t}} - \bar{y}_{\bar{t}}|^p],$$

and the bound in (3.6) is a consequence of Lemma 3.2 and Proposition 3.3 in [17]. As

$$(3.9) \quad |v_t - \bar{v}_t| = (y_t + \bar{y}_t)|y_t - \bar{y}_t|,$$

choosing any $p < q < \nu$ and applying Hölder's inequality leads to

$$(3.10) \quad \begin{aligned} \sup_{t \in [0, T]} \mathbb{E} [|v_t - \bar{v}_t|^p] &\leq 2^{p-1} \left\{ \sup_{t \in [0, T]} \mathbb{E} \left[v_t^{\frac{pq}{2(q-p)}} \right]^{1-\frac{p}{q}} + \sup_{N > N_0} \sup_{t \in [0, T]} \mathbb{E} \left[\bar{v}_t^{\frac{pq}{2(q-p)}} \right]^{1-\frac{p}{q}} \right\} \\ &\quad \times \sup_{t \in [0, T]} \mathbb{E} [|y_t - \bar{y}_t|^q]^{\frac{p}{q}}. \end{aligned}$$

The bound in (3.7) follows from Lemmas 3.1 and 3.3 and (3.6). \square

3.2. The log-spot process. The next auxiliary result provides upper bounds on the discretization errors in the drift and diffusion functions μ and σ .

LEMMA 3.5. *Under Assumption 2.2 we have that, for all $u \in [0, T]$,*

$$(3.11) \quad \begin{aligned} |\mu(u, S_u, M_u) - \mu(u, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})| &\leq (C_{\mu, x} + 2C_{\mu, m}) \sup_{t \in [0, u]} |\log(S_t) - \log(\bar{S}_{\bar{t}})| \\ &\quad + (C_{\mu, x} + C_{\mu, m}) \sup_{t \in [0, u]} |\log(S_t) - \log(\bar{S}_{\bar{t}})| \end{aligned}$$

and

$$(3.12) \quad \begin{aligned} |\sigma(u, S_u, M_u) - \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})| &\leq C_{\sigma, t} \sqrt{\delta t} + (C_{\sigma, x} + 2C_{\sigma, m}) \sup_{t \in [0, u]} |\log(S_t) - \log(\bar{S}_{\bar{t}})| \\ &\quad + (C_{\sigma, x} + C_{\sigma, m}) \sup_{t \in [0, u]} |\log(S_t) - \log(\bar{S}_{\bar{t}})|. \end{aligned}$$

Proof. See Appendix A. \square

Since the choice of discretization scheme for the squared volatility process makes little difference in the subsequent proofs, we henceforth denote by \bar{v} both the FTE and the BEM discretizations. For the convergence analysis, we need to control the polynomial moments of the log-spot process and its approximation.

LEMMA 3.6. *The following statements hold under Assumption 2.1.*

(1) The log-spot process has uniformly bounded moments, i.e.,

$$(3.13) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |x_t|^p \right] < \infty, \quad \forall p \geq 1.$$

(2) The approximated log-spot process has uniformly bounded moments, i.e.,

$$(3.14) \quad \sup_{N \geq 1} \mathbb{E} \left[\sup_{t \in [0, T]} |\bar{x}_t|^p \right] < \infty, \quad \forall p \geq 1.$$

Proof. (1) Note from Jensen's inequality that it suffices to consider $p \geq 2$. Recall from (2.1) that

$$(3.15) \quad x_t = x_0 + \int_0^t \mu(u, S_u, M_u) du - \frac{1}{2} \int_0^t \sigma^2(u, S_u, M_u) v_u du + \int_0^t \sigma(u, S_u, M_u) \sqrt{v_u} dW_u^s.$$

Using the Hölder and BDG inequalities and Fubini's theorem, we deduce that

$$(3.16) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |x_t|^p \right] \leq 4^{p-1} (|x_0|^p + \mu_{max}^p T^p) + 2^{p-2} \sigma_{max}^{2p} T^p \sup_{t \in [0, T]} \mathbb{E} [v_t^p] \\ + 4^{p-1} \sigma_{max}^p T^{\frac{p}{2}} C \sup_{t \in [0, T]} \mathbb{E} [v_t^{p/2}],$$

for some non-negative constant C , and the right-hand side is finite by Lemma 3.1.

(2) Recall from (2.26) that

$$(3.17) \quad \bar{x}_t = x_0 + \int_0^t \bar{\mu}(u, \bar{S}_u, \bar{M}_u) du - \frac{1}{2} \int_0^t \bar{\sigma}^2(u, \bar{S}_u, \bar{M}_u) \bar{v}_u du + \int_0^t \bar{\sigma}(u, \bar{S}_u, \bar{M}_u) \sqrt{\bar{v}_u} dW_u^s.$$

Proceeding as before, we deduce that

$$(3.18) \quad \sup_{N \geq 1} \mathbb{E} \left[\sup_{t \in [0, T]} |\bar{x}_t|^p \right] \leq 4^{p-1} (|x_0|^p + \mu_{max}^p T^p) + 2^{p-2} \sigma_{max}^{2p} T^p \sup_{N \geq 1} \sup_{t \in [0, T]} \mathbb{E} [\bar{v}_t^p] \\ + 4^{p-1} \sigma_{max}^p T^{\frac{p}{2}} C \sup_{N \geq 1} \sup_{t \in [0, T]} \mathbb{E} [\bar{v}_t^{p/2}],$$

and the conclusion follows from Lemmas 3.2 and 3.3. \square

The following result is concerned with the uniform convergence in L^p with order $1/2$ (up to a logarithmic factor) of the approximated log-spot process. In the special case of constant drift and diffusion functions μ and σ , the SPDV model (2.1) collapses to the Heston stochastic volatility model and we notice from (2.31) – (2.33) that $T_x^*(p) = \infty$, for all $1 \leq p < p^*(\nu)$. For the LE-BEM scheme, i.e., when the LE and the BEM schemes are employed in the discretization of the spot process and its squared volatility, respectively, this result was proved in Corollary 5.5 in [41]. Furthermore, the extension to the LE-FTE scheme is straightforward. However, the simple argument employed to prove Proposition 3.7 under a purely stochastic volatility model does not apply to the general case of non-trivial drift and diffusion functions μ and σ , even without path-dependency. In this case, we require more advanced techniques in order to overcome the technical challenges. We also mention that in the case of no stochastic volatility (e.g., take $v_0 = \theta = 1$ and $\xi = 0$), the SPDV model

(2.1) collapses to a path-dependent volatility model and we notice from (2.28), (2.30) and (2.31) – (2.33) that $\nu = p^*(\nu) = \infty$ and $T_x^*(p) = \infty$, for all $p \geq 1$. In this case, the analysis involved in Proposition 3.7 becomes somewhat simpler, as will be clear from the proof.

PROPOSITION 3.7. *Suppose that Assumptions 2.1 and 2.2 hold and that $\nu > \nu^*$, with ν^* defined in (2.29). Then for all $2 \leq p < p^*(\nu)$ and $T < T_x^*(p)$, with p^* defined in (2.30) and T_x^* given in (2.31), there exists a constant C such that, for all $N \geq 1$,*

$$(3.19) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |x_t - \bar{x}_t|^p \right]^{\frac{1}{p}} \leq C \sqrt{\frac{\log(2N)}{N}}.$$

Proof. By a continuity argument, we can find $p < q < p^*(\nu)$ such that

$$(3.20) \quad T < \frac{2}{\sqrt{(\tilde{\varphi}(p, q) - k^2)^+}} \left[\frac{\pi}{2} + \arctan \left(\frac{k}{\sqrt{(\tilde{\varphi}(p, q) - k^2)^+}} \right) \right].$$

We break the proof into two steps. In a first step, for a random stopping time, we derive an upper bound on the difference between the exact and the approximated stopped log-spot processes in the L^q -sup norm. In a second step, we define a set of stopping times and use this upper bound to infer an upper bound on the difference between the exact and the approximated log-spot processes in the L^p -sup norm.

Step 1. For convenience of notation, define

$$(3.21) \quad \epsilon_t^x = x_t - \bar{x}_t, \quad \epsilon_0^x = 0,$$

and

$$(3.22) \quad \Delta x_t = x_t - x_{\bar{t}}.$$

Let τ be a stopping time and define the function, $\text{sgn}(y) = 1$ if $y > 0$ and $\text{sgn}(y) = -1$ otherwise. Applying Itô's formula to the \mathcal{C}^2 function $f(\epsilon_{t \wedge \tau}^x) = |\epsilon_{t \wedge \tau}^x|^q$ yields

$$(3.23) \quad |\epsilon_{t \wedge \tau}^x|^q = q \int_0^{t \wedge \tau} |\epsilon_u^x|^{q-1} \text{sgn}(\epsilon_u^x) d\epsilon_u^x + \frac{1}{2} q(q-1) \int_0^{t \wedge \tau} |\epsilon_u^x|^{q-2} d\langle \epsilon^x \rangle_u,$$

and hence

$$(3.24) \quad \begin{aligned} |\epsilon_{t \wedge \tau}^x|^q &= q \int_0^{t \wedge \tau} |\epsilon_u^x|^{q-1} \text{sgn}(\epsilon_u^x) (\mu(u, S_u, M_u) - \mu(u, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})) du \\ &\quad - \frac{1}{2} q \int_0^{t \wedge \tau} |\epsilon_u^x|^{q-1} \text{sgn}(\epsilon_u^x) (v_u \sigma^2(u, S_u, M_u) - \bar{v}_u \sigma^2(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})) du \\ &\quad + q \int_0^{t \wedge \tau} |\epsilon_u^x|^{q-1} \text{sgn}(\epsilon_u^x) (\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})) dW_u^s \\ &\quad + \frac{1}{2} q(q-1) \int_0^{t \wedge \tau} |\epsilon_u^x|^{q-2} (\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}}))^2 du. \end{aligned}$$

Taking the supremum over $[0, T]$ and then expectations on both sides, we derive an initial upper bound on the difference between the exact and the approximated stopped log-spot processes in the L^q -sup norm,

$$(3.25) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau}^x|^q \right] \leq q \mathbb{E} \left[\int_0^{T \wedge \tau} |\epsilon_u^x|^{q-1} |\mu(u, S_u, M_u) - \mu(u, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})| du \right]$$

$$\begin{aligned}
& + \frac{1}{2} q \mathbb{E} \left[\int_0^{T \wedge \tau} |\epsilon_u^x|^{q-1} |v_u \sigma^2(u, S_u, M_u) - \bar{v}_u \sigma^2(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})| du \right] \\
& + q \mathbb{E} \left[\sup_{t \in [0, T]} \int_0^{t \wedge \tau} |\epsilon_u^x|^{q-1} \operatorname{sgn}(\epsilon_u^x) (\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})) dW_u^s \right] \\
(3.25) \quad & + \frac{1}{2} q(q-1) \mathbb{E} \left[\int_0^{T \wedge \tau} |\epsilon_u^x|^{q-2} |\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})|^2 du \right].
\end{aligned}$$

We can show that the stochastic integral in (3.24) is a true martingale by a simple application of Hölder's inequality and Lemmas 3.1, 3.2, 3.3 and 3.6.

Let $\lambda \in (0, 1)$. Using a sharp maximal inequality for continuous-path martingales X starting at zero (Corollary 4.4 in [52]),

$$(3.26) \quad \|\sup_t X_t\|_1 \leq \beta_0 \| [X, X]^{1/2} \|_1,$$

where $[X, X]$ denotes its square bracket and β_0 is defined in (2.34), and the arithmetic mean-geometric mean (AM-GM) inequality, $2\sqrt{ab} \leq a + b$, $\forall a, b \geq 0$, yields

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} \int_0^{t \wedge \tau} |\epsilon_u^x|^{q-1} \operatorname{sgn}(\epsilon_u^x) (\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})) dW_u^s \right] \\
& \leq \beta_0 \mathbb{E} \left[\left(\int_0^{T \wedge \tau} |\epsilon_u^x|^{2(q-1)} |\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})|^2 du \right)^{\frac{1}{2}} \right] \\
& \leq \beta_0 \mathbb{E} \left[\left(\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau}^x|^q \int_0^{T \wedge \tau} |\epsilon_u^x|^{q-2} |\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})|^2 du \right)^{\frac{1}{2}} \right] \\
& \leq \frac{q\beta_0^2}{4\lambda} \mathbb{E} \left[\int_0^{T \wedge \tau} |\epsilon_u^x|^{q-2} |\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})|^2 du \right] \\
(3.27) \quad & + \frac{\lambda}{q} \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau}^x|^q \right].
\end{aligned}$$

Substituting back into (3.25) with (3.27) and after some rearrangements, we deduce that

$$\begin{aligned}
& \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau}^x|^q \right] \leq \frac{q}{1-\lambda} \mathbb{E} \left[\int_0^{T \wedge \tau} |\epsilon_u^x|^{q-1} |\mu(u, S_u, M_u) - \mu(u, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})| du \right] \\
& + \frac{q}{2(1-\lambda)} \mathbb{E} \left[\int_0^{T \wedge \tau} |\epsilon_u^x|^{q-1} |v_u \sigma^2(u, S_u, M_u) - \bar{v}_u \sigma^2(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})| du \right] \\
(3.28) \quad & + qc_q(\lambda) \mathbb{E} \left[\int_0^{T \wedge \tau} |\epsilon_u^x|^{q-2} |\sqrt{v_u} \sigma(u, S_u, M_u) - \sqrt{\bar{v}_u} \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})|^2 du \right],
\end{aligned}$$

where we defined, for brevity,

$$(3.29) \quad c_q(\lambda) = \frac{q-1}{2(1-\lambda)} + \frac{q\beta_0^2}{4\lambda(1-\lambda)}.$$

For any $n \in \mathbb{N}$ and $z_k \geq 0$, $d_k > 0$, for all $1 \leq k \leq n$, the Cauchy-Schwarz inequality

yields

$$(3.30) \quad \left(\sum_{k=1}^n z_k \right)^2 \leq \left(\sum_{k=1}^n z_k^2 d_k \right) \left(\sum_{k=1}^n d_k^{-1} \right).$$

Let $\eta > 0$. Employing Lemma 3.5 and (3.30) with $n = 3$, $d_1 = d_2 = 2\eta^{-1}(1 + \eta)$ and $d_3 = 1 + \eta$, we get

$$(3.31) \quad \begin{aligned} |\sigma(u, S_u, M_u) - \sigma(\bar{u}, \bar{S}_u, \bar{M}_u)|^2 &\leq 2\eta^{-1}(1 + \eta) C_{\sigma,t}^2 \delta t + (1 + \eta) (C_{\sigma,x} + C_{\sigma,m})^2 \sup_{t \in [0, u]} |\epsilon_t^x|^2 \\ &\quad + 2\eta^{-1}(1 + \eta) (C_{\sigma,x} + 2C_{\sigma,m})^2 \sup_{t \in [0, u]} |\Delta x_t|^2. \end{aligned}$$

Next, using Lemma 3.5, (3.30) with $n = 2$, $d_1 = \eta^{-1}(1 + \eta)$ and $d_2 = 1 + \eta$, as well as (3.31), after some rearrangements, we deduce from (3.28) that

$$(3.32) \quad \begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau}^x|^q \right] &\leq \frac{q}{1 - \lambda} (C_{\mu,x} + C_{\mu,m}) \mathbb{E} \left[\int_0^{T \wedge \tau} \sup_{t \in [0, u]} |\epsilon_t^x|^q du \right] \\ &\quad + \frac{q}{1 - \lambda} \sigma_{max} (C_{\sigma,x} + C_{\sigma,m}) \mathbb{E} \left[\int_0^{T \wedge \tau} v_u \sup_{t \in [0, u]} |\epsilon_t^x|^q du \right] \\ &\quad + qc_q(\lambda)(1 + \eta)^2 (C_{\sigma,x} + C_{\sigma,m})^2 \mathbb{E} \left[\int_0^{T \wedge \tau} v_u \sup_{t \in [0, u]} |\epsilon_t^x|^q du \right] \\ &\quad + \frac{q}{1 - \lambda} \sigma_{max} C_{\sigma,t} T^{\frac{1}{2}} \mathbb{E} \left[\int_0^{T \wedge \tau} v_u \sup_{t \in [0, u]} |\epsilon_t^x|^{q-1} N^{-\frac{1}{2}} du \right] \\ &\quad + \frac{q}{1 - \lambda} (C_{\mu,x} + 2C_{\mu,m}) \mathbb{E} \left[\int_0^{T \wedge \tau} \sup_{t \in [0, u]} |\epsilon_t^x|^{q-1} \sup_{t \in [0, u]} |\Delta x_t| du \right] \\ &\quad + \frac{q}{1 - \lambda} \sigma_{max} (C_{\sigma,x} + 2C_{\sigma,m}) \mathbb{E} \left[\int_0^{T \wedge \tau} v_u \sup_{t \in [0, u]} |\epsilon_t^x|^{q-1} \sup_{t \in [0, u]} |\Delta x_t| du \right] \\ &\quad + \frac{q}{2(1 - \lambda)} \sigma_{max}^2 \mathbb{E} \left[\int_0^{T \wedge \tau} \sup_{t \in [0, u]} |\epsilon_t^x|^{q-1} |v_u - \bar{v}_u| du \right] \\ &\quad + 2qc_q(\lambda)\eta^{-1}(1 + \eta)^2 C_{\sigma,t}^2 T \mathbb{E} \left[\int_0^{T \wedge \tau} v_u \sup_{t \in [0, u]} |\epsilon_t^x|^{q-2} N^{-1} du \right] \\ &\quad + 2qc_q(\lambda)\eta^{-1}(1 + \eta)^2 (C_{\sigma,x} + 2C_{\sigma,m})^2 \mathbb{E} \left[\int_0^{T \wedge \tau} v_u \sup_{t \in [0, u]} |\epsilon_t^x|^{q-2} \sup_{t \in [0, u]} |\Delta x_t|^2 du \right] \\ &\quad + (1 - \gamma^*) qc_q(\lambda)\eta^{-1}(1 + \eta) \sigma_{max}^2 \mathbb{E} \left[\int_0^{T \wedge \tau} \sup_{t \in [0, u]} |\epsilon_t^x|^{q-2} |\sqrt{v_u} - \sqrt{\bar{v}_u}|^2 du \right] \\ &\quad + \gamma^* qc_q(\lambda)\eta^{-1}(1 + \eta) \sigma_{max}^2 \mathbb{E} \left[\int_0^{T \wedge \tau} \sup_{t \in [0, u]} |\epsilon_t^x|^{q-2} |\sqrt{v_u} - \sqrt{\bar{v}_u}|^2 du \right], \end{aligned}$$

where $\gamma^{\text{FTE}} = 0$ and $\gamma^{\text{BEM}} = 1$. Since $\nu \geq 1$, the process v has almost surely strictly positive paths and we can bound the term before the last on the right-hand side of (3.32) from above by using

$$(3.33) \quad |\sqrt{v_u} - \sqrt{\bar{v}_u}|^2 \leq v_u^{-1} |v_u - \bar{v}_u|^2.$$

498 For any $a, b \geq 0$ and $j \in \{1, 2\}$, Young's inequality yields

499 (3.34)
$$a^{q-j}b^j = \left(\eta^{\frac{j(q-j)}{q}}a^{q-j}\right)\left(\eta^{-\frac{j(q-j)}{q}}b^j\right) \leq \frac{q-j}{q}\eta^j a^q + \frac{j}{q}\eta^{j-q}b^q.$$

500 Going back to (3.32), using (3.33), (3.34) (with $\eta^{\frac{3}{2}}$ instead of η for the term before
501 the last) and Fubini's theorem leads to an upper bound

502
$$\mathbb{E}\left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau}^x|^q\right] \leq \left\{ \frac{\eta^{1-q}}{1-\lambda} [4(1-\lambda)c_q(\lambda)(1+\eta)^2 C_{\sigma, t}^2 T^2 + \sigma_{max} C_{\sigma, t} T^{\frac{3}{2}}] \sup_{t \in [0, T]} \mathbb{E}[v_t] N^{-\frac{q}{2}} \right.$$

503
$$+ 2(1-\gamma^*)\eta^{2-\frac{3}{2}q}(1+\eta)c_q(\lambda)\sigma_{max}^2 T \sup_{t \in [0, T]} \mathbb{E}[v_t^{-1}|v_t - \bar{v}_t|^q]$$

504
$$+ 2\gamma^*\eta^{1-q}(1+\eta)c_q(\lambda)\sigma_{max}^2 T \sup_{t \in [0, T]} \mathbb{E}[|\sqrt{v_t} - \sqrt{\bar{v}_t}|^q]$$

505
$$+ \frac{\eta^{1-q}}{2(1-\lambda)}\sigma_{max}^2 T \sup_{t \in [0, T]} \mathbb{E}[|v_t - \bar{v}_t|^q] + \frac{\eta^{1-q}}{1-\lambda}(C_{\mu, x} + 2C_{\mu, m})T \mathbb{E}\left[\sup_{t \in [0, T]} |\Delta x_t|^q\right]$$

506
$$+ 4\eta^{1-q}(1+\eta)^2 c_q(\lambda)(C_{\sigma, x} + 2C_{\sigma, m})^2 T \mathbb{E}\left[\sup_{t \in [0, T]} v_t \sup_{t \in [0, T]} |\Delta x_t|^q\right]$$

507
$$+ \frac{\eta^{1-q}}{1-\lambda}\sigma_{max}(C_{\sigma, x} + 2C_{\sigma, m})T \mathbb{E}\left[\sup_{t \in [0, T]} v_t \sup_{t \in [0, T]} |\Delta x_t|^q\right]\Bigg\}$$

508
$$+ \mathbb{E}\left[\int_0^{T \wedge \tau} \sup_{t \in [0, u]} |\epsilon_t^x|^q \left\{ v_u \left(\frac{q}{1-\lambda}\sigma_{max}(C_{\sigma, x} + C_{\sigma, m}) + qc_q(\lambda)(C_{\sigma, x} + C_{\sigma, m})^2 \right. \right. \right.$$

509
$$+ \eta(2+\eta)qc_q(\lambda)(C_{\sigma, x} + C_{\sigma, m})^2 + \frac{\eta(q-1)}{1-\lambda}\sigma_{max}[C_{\sigma, t}T^{\frac{1}{2}} + C_{\sigma, x} + 2C_{\sigma, m}]$$

510
$$+ 2\eta(1+\eta)^2(q-2)c_q(\lambda)[C_{\sigma, t}^2 T + (C_{\sigma, x} + 2C_{\sigma, m})^2] \Bigg) + \left(\frac{q}{1-\lambda}(C_{\mu, x} + C_{\mu, m}) \right.$$

511
$$+ \frac{\eta(q-1)}{1-\lambda}(C_{\mu, x} + 2C_{\mu, m}) + \frac{\eta(q-1)}{2(1-\lambda)}\sigma_{max}^2 + \gamma^*\eta(1+\eta)(q-2)c_q(\lambda)\sigma_{max}^2 \Bigg)$$

512
$$(3.35) + v_u^{-1}(1-\gamma^*)\eta^2(1+\eta)(q-2)c_q(\lambda)\sigma_{max}^2 \Bigg\} du \Bigg].$$

513

514 We choose $\lambda = \lambda_q$ that minimizes the function $f_q(\lambda) : (0, 1) \rightarrow \mathbb{R}$ given by
(3.36)

515
$$f_q(\lambda) = \frac{q}{1-\lambda}\sigma_{max}(C_{\sigma, x} + C_{\sigma, m}) + q\left(\frac{q-1}{2(1-\lambda)} + \frac{q\beta_0^2}{4\lambda(1-\lambda)}\right)(C_{\sigma, x} + C_{\sigma, m})^2.$$

516 For brevity, define

517 (3.37)
$$\Delta_\sigma = \frac{C_{\sigma, x} + C_{\sigma, m}}{\sigma_{max}}.$$

518 Looking at the first derivative of f_q , we find its unique positive root

519 (3.38)
$$\lambda_q = \frac{-q\beta_0^2\Delta_\sigma + \beta_0\sqrt{q^2\beta_0^2\Delta_\sigma^2 + 2q\Delta_\sigma(2 + (q-1)\Delta_\sigma)}}{4 + 2(q-1)\Delta_\sigma},$$

which clearly satisfies $\lambda_q \in (0, 1)$. Some straightforward computations lead to

$$\begin{aligned} f_q(\lambda_q) &= \frac{q^2 \beta_0^2}{4\lambda_q^2} (C_{\sigma,x} + C_{\sigma,m})^2 \\ &= \frac{1}{4} \sigma_{max}^2 \left(\sqrt{q^2(2 + \beta_0^2)\Delta_\sigma^2 + 2q\Delta_\sigma(2 - \Delta_\sigma)} + q\beta_0\Delta_\sigma \right)^2. \end{aligned}$$

Next, we bound the first seven terms on the right-hand side of (3.35) from above. On the one hand, for the FTE discretization of the squared volatility process, note that we can find $r > 1$ such that

$$\frac{\nu}{\nu - 1} < \frac{r}{r - 1} < \frac{\nu - 1}{q}.$$

Using Hölder's inequality, Theorem 1.1 in [16] and Lemma 3.1, we deduce that there exists a constant C such that, for all $N \geq 1$,

$$\sup_{t \in [0, T]} \mathbb{E} [v_t^{-1} |v_t - \bar{v}_t|^q] \leq \sup_{t \in [0, T]} \mathbb{E} [v_t^{-r}]^{\frac{1}{r}} \sup_{t \in [0, T]} \mathbb{E} \left[|v_t - \bar{v}_t|^{\frac{rq}{r-1}} \right]^{\frac{r-1}{r}} \leq CN^{-\frac{q}{2}}$$

and

$$\sup_{t \in [0, T]} \mathbb{E} [|v_t - \bar{v}_t|^q] \leq CN^{-\frac{q}{2}}.$$

On the other hand, for the BEM discretization of the squared volatility process, we know from Proposition 3.4 that there exists a constant C such that, for all $N \geq 1$,

$$\sup_{t \in [0, T]} \mathbb{E} [|\sqrt{v_t} - \sqrt{\bar{v}_t}|^q] \leq CN^{-\frac{q}{2}}$$

and

$$\sup_{t \in [0, T]} \mathbb{E} [|v_t - \bar{v}_t|^q] \leq CN^{-\frac{q}{2}}.$$

Furthermore, using the Cauchy–Schwarz inequality and Lemma 3.1 and applying Theorem 1 in [19] to the log-spot process from (3.15), we conclude that there exists a constant C such that, for all $N \geq 1$,

$$\mathbb{E} \left[\sup_{t \in [0, T]} v_t \sup_{t \in [0, T]} |\Delta x_t|^q \right] \leq \mathbb{E} \left[\sup_{t \in [0, T]} v_t^2 \right]^{\frac{1}{2}} \mathbb{E} \left[\sup_{t \in [0, T]} |\Delta x_t|^{2q} \right]^{\frac{1}{2}} \leq C \left(\frac{N}{\log(2N)} \right)^{-\frac{q}{2}}$$

and

$$\mathbb{E} \left[\sup_{t \in [0, T]} |\Delta x_t|^q \right] \leq C \left(\frac{N}{\log(2N)} \right)^{-\frac{q}{2}}.$$

For convenience, define the strictly increasing stochastic process

$$g_{q,\eta}(t) = \int_0^t \left(a_{q,\eta} + (f_q(\lambda_q) + \eta b_{q,\eta}) v_u + \eta^2 c_{q,\eta} v_u^{-1} \right) du,$$

where

$$a_{q,\eta} = \frac{q}{1 - \lambda_q} (C_{\mu,x} + C_{\mu,m}) + \frac{\eta(q-1)}{1 - \lambda_q} (C_{\mu,x} + 2C_{\mu,m}) + \frac{\eta(q-1)}{2(1 - \lambda_q)} \sigma_{max}^2$$

$$(3.48) \quad + \gamma^* \eta (1 + \eta) (q - 2) c_q(\lambda_q) \sigma_{max}^2,$$

$$(3.49) \quad b_{q,\eta} = (2 + \eta) q c_q(\lambda_q) (C_{\sigma,x} + C_{\sigma,m})^2 + \frac{q-1}{1-\lambda_q} \sigma_{max} [C_{\sigma,t} T^{\frac{1}{2}} + C_{\sigma,x} + 2C_{\sigma,m}]$$

$$(3.49) \quad + 2(1 + \eta)^2 (q - 2) c_q(\lambda_q) [C_{\sigma,t}^2 T + (C_{\sigma,x} + 2C_{\sigma,m})^2],$$

$$(3.50) \quad c_{q,\eta} = (1 - \gamma^*) (1 + \eta) (q - 2) c_q(\lambda_q) \sigma_{max}^2.$$

Substituting back into (3.35) with (3.41) – (3.47), we conclude that there exists a constant $C_{q,\eta} > 0$ such that, for all $N \geq 1$,

$$(3.51) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau}^x|^q \right] \leq C_{q,\eta} \left(\frac{N}{\log(2N)} \right)^{-\frac{q}{2}} + \mathbb{E} \left[\int_0^{T \wedge \tau} \sup_{t \in [0, T]} |\epsilon_{t \wedge u}^x|^q dg_{q,\eta}(u) \right].$$

Step 2. Next, consider the set of stopping times $\{\tau_{q,\eta}^\kappa, \kappa \geq 0\}$ defined by

$$(3.52) \quad \tau_{q,\eta}^\kappa = \inf\{t \geq 0 \mid g_{q,\eta}(t) \geq \kappa\}, \quad \tau_{q,\eta}^0 = 0,$$

and note that they are finite, since

$$(3.53) \quad \tau_{q,\eta}^\kappa \leq \frac{2\kappa(1 - \lambda_q)}{\eta(q - 1)\sigma_{max}^2},$$

and strictly increasing, and that $g_{q,\eta}(\tau_{q,\eta}^\kappa) = \kappa$ by continuity. Fix $\kappa > 0$ and set $\tau = \tau_{q,\eta}^\kappa$ in (3.51). Using an idea from [6], define a stochastic time-change $s = g_{q,\eta}(u)$ such that $u = \tau_{q,\eta}^s$ and note that

$$(3.54) \quad g_{q,\eta}(T \wedge \tau_{q,\eta}^\kappa) = g_{q,\eta}(\tau_{q,\eta}^\kappa) \wedge g_{q,\eta}(T) = \kappa \wedge g_{q,\eta}(T).$$

By Lebesgue's change-of-time formula (see, e.g., Theorem A4.3 in [54]), we get

$$(3.55) \quad \begin{aligned} \mathbb{E} \left[\int_0^{T \wedge \tau_{q,\eta}^\kappa} \sup_{t \in [0, T]} |\epsilon_{t \wedge u}^x|^q dg_{q,\eta}(u) \right] &= \mathbb{E} \left[\int_0^{\kappa \wedge g_{q,\eta}(T)} \sup_{t \in [0, T]} |\epsilon_{t \wedge \tau_{q,\eta}^s}^x|^q ds \right] \\ &\leq \int_0^\kappa \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau_{q,\eta}^s}^x|^q \right] ds. \end{aligned}$$

Substituting back into (3.51) with this upper bound yields

$$(3.56) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau_{q,\eta}^\kappa}^x|^q \right] \leq C_{q,\eta} \left(\frac{N}{\log(2N)} \right)^{-\frac{q}{2}} + \int_0^\kappa \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau_{q,\eta}^s}^x|^q \right] ds,$$

and applying Gronwall's inequality leads to

$$(3.57) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau_{q,\eta}^\kappa}^x|^q \right] \leq C_{q,\eta} e^\kappa \left(\frac{N}{\log(2N)} \right)^{-\frac{q}{2}},$$

for all $\kappa > 0$. Proceeding in a similar way as in the argument of (3.51) and setting $\tau = T$, we get

$$(3.58) \quad \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_t^x|^p \right] \leq C_{p,\eta} \left(\frac{N}{\log(2N)} \right)^{-\frac{p}{2}} + \mathbb{E} \left[\int_0^T \sup_{t \in [0, T]} |\epsilon_{t \wedge u}^x|^p dg_{p,\eta}(u) \right].$$

However, note from (3.38) that both

$$(3.59) \quad \frac{\sqrt{q}}{\lambda_q} = \sqrt{q} + \sqrt{q(1 + 2\beta_0^{-2}) - 2\beta_0^{-2}(1 - 2\Delta_\sigma^{-1})}$$

and

$$(3.60) \quad \frac{q}{1 - \lambda_q} = q + \frac{q\sqrt{q}}{\sqrt{q(1 + 2\beta_0^{-2}) - 2\beta_0^{-2}(1 - 2\Delta_\sigma^{-1})}}$$

are increasing in q , and hence that

$$(3.61) \quad \frac{d}{dt} g_{p,\eta}(t) \leq \frac{d}{dt} g_{q,\eta}(t), \quad \forall t \in [0, T].$$

Using the same time-change from before, Fubini's theorem and Hölder's inequality, we deduce that

$$(3.62) \quad \begin{aligned} \mathbb{E} \left[\int_0^T \sup_{t \in [0, T]} |\epsilon_{t \wedge u}^x|^p dg_{p,\eta}(u) \right] &\leq \mathbb{E} \left[\int_0^T \sup_{t \in [0, T]} |\epsilon_{t \wedge u}^x|^p dg_{q,\eta}(u) \right] \\ &\leq \int_0^\infty \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau_{q,\eta}^s}^x|^p \mathbf{1}_{s \leq g_{q,\eta}(T)} \right] ds \\ &\leq \int_0^\infty \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_{t \wedge \tau_{q,\eta}^s}^x|^q \right]^{\frac{p}{q}} \mathbb{P} \left(s \leq g_{q,\eta}(T) \right)^{1 - \frac{p}{q}} ds. \end{aligned}$$

Combining (3.57), (3.58) and (3.62) yields

$$(3.63) \quad \begin{aligned} \mathbb{E} \left[\sup_{t \in [0, T]} |\epsilon_t^x|^p \right] &\leq \left\{ C_{p,\eta} + C_{q,\eta}^{\frac{p}{q}} \int_0^\infty \exp \left\{ \frac{sp}{q} \right\} \mathbb{P} \left(s \leq g_{q,\eta}(T) \right)^{1 - \frac{p}{q}} ds \right\} \\ &\quad \times \left(\frac{N}{\log(2N)} \right)^{-\frac{p}{2}}. \end{aligned}$$

All that is left to do is bound the probability on the right-hand side from above. For brevity, define

$$(3.64) \quad w_{p,q} = \frac{p}{q - p}$$

and let $w > w_{p,q}$. Markov's inequality yields

$$(3.65) \quad \mathbb{P} \left(s \leq g_{q,\eta}(T) \right) \leq \exp \{-ws\} \mathbb{E} \left[\exp \{wg_{q,\eta}(T)\} \right].$$

From (3.47) and Hölder's inequality, we get

$$(3.66) \quad \begin{aligned} \mathbb{E} \left[\exp \{wg_{q,\eta}(T)\} \right] &\leq \mathbb{E} \left[\exp \left\{ w(1 + \eta)(f_q(\lambda_q) + \eta b_{q,\eta}) \int_0^T v_t dt \right\} \right]^{\frac{1}{1+\eta}} \\ &\quad \times \mathbb{E} \left[\exp \left\{ w\eta(1 + \eta)c_{q,\eta} \int_0^T v_t^{-1} dt \right\} \right]^{\frac{\eta}{1+\eta}} \exp \{wa_{q,\eta}T\}. \end{aligned}$$

However, note from (2.33) and (3.39) that

$$\begin{aligned} w(1+\eta)(f_q(\lambda_q) + \eta b_{q,\eta}) &= w_{p,q}f_q(\lambda_q) + (w - w_{p,q})f_q(\lambda_q) + \eta w(f_q(\lambda_q) + (1+\eta)b_{q,\eta}) \\ &= \frac{\tilde{\varphi}(p,q)}{2\xi^2} + (w - w_{p,q})f_q(\lambda_q) + \eta w(f_q(\lambda_q) + (1+\eta)b_{q,\eta}). \end{aligned}$$

Using exponential integrability properties of the CIR process from Proposition 3.5 in [12] (for the integrated CIR process) and also from Lemma A.2 in [7] or Theorem 3.1 in [34] (for the integrated inverse CIR process), together with the upper bound on T in (3.20) and a continuity argument, since $\nu > 1$, we conclude that there exist $\eta > 0$ sufficiently small and w sufficiently close to $w_{p,q}$ such that the two expectations on the right-hand side of (3.66), and hence the one on the left-hand side, are finite. Finally, substituting back into (3.63) with (3.65) and upon noticing that

$$\int_0^\infty \exp\left\{\frac{sp}{q} - ws\left(1 - \frac{p}{q}\right)\right\} ds = \frac{q}{(w - w_{p,q})(q - p)},$$

we deduce that

$$\begin{aligned} \mathbb{E}\left[\sup_{t \in [0,T]} |\epsilon_t^x|^p\right] &\leq \left\{C_{p,\eta} + \frac{q}{(w - w_{p,q})(q - p)} C_{q,\eta}^{\frac{p}{q}} \mathbb{E}\left[\exp\{wg_{q,\eta}(T)\}\right]^{1-\frac{p}{q}}\right\} \\ &\quad \times \left(\frac{N}{\log(2N)}\right)^{-\frac{p}{2}}, \end{aligned}$$

and the conclusion follows. \square

3.3. Moment bounds. Many models with stochastic volatility dynamics have the undesirable feature that moments of order higher than 1 can explode in finite time [4]. The finiteness of moments of order higher than 1 of the exact and numerical solutions of an SDE is an important ingredient in the convergence analysis [32]. The finiteness of higher moments was established in [12] for explicit Euler approximations to stochastic-local volatility models. We now extend this result to stochastic path-dependent volatility models and include the proof here for completeness.

LEMMA 3.8. Suppose that Assumption 2.1 holds and let $p > 1$.

(1) If $T < T_S^{\text{CIR}}(p)$, then the spot process has a bounded p th moment, i.e.,

$$\sup_{t \in [0,T]} \mathbb{E}[S_t^p] < \infty,$$

where

$$T_S^{\text{CIR}}(p) = \frac{2}{\sqrt{(\phi(p) - k^2)^+}} \left[\frac{\pi}{2} + \arctan\left(\frac{k}{\sqrt{(\phi(p) - k^2)^+}}\right) \right],$$

with ϕ given in (2.37).

(2) If $\nu \geq 1$ and $T < T_S^*(p)$, with T_S^{FTE} and T_S^{BEM} given in (2.35) and (2.36), respectively, then the approximated spot process has a bounded p th moment, i.e., there exists $N_0 \in \mathbb{N}$ such that

$$\sup_{N > N_0} \sup_{t \in [0,T]} \mathbb{E}[\bar{S}_t^p] < \infty.$$

Proof. See Appendix B. \square

3.4. Proof of Theorem 2.5. With these results at our disposal, we are now ready to prove the main theorem.

Proof. By a continuity argument, we can find $2 \vee p < q < p^*(\nu)$ such that

$$(3.73) \quad T < T_x^*(q) \wedge T_S^*(pq(q-p)^{-1}).$$

Fix $r > 1$ and recall the definition of ϕ from (2.37). If $\phi(r) \geq 4k^2$, then

$$(3.74) \quad \frac{2}{\sqrt{\phi(r) - k^2}} \left[\frac{\pi}{2} + \arctan \left(\frac{k}{\sqrt{\phi(r) - k^2}} \right) \right] \geq \frac{1}{\sqrt{\phi(r) - k^2}} \sqrt{\frac{\sqrt{\phi(r)} + k}{\sqrt{\phi(r)} - k}} \\ = \frac{1}{\sqrt{\phi(r) - k}} \geq \frac{1}{\sqrt{\phi(r)}}.$$

If $k^2 < \phi(r) < 4k^2$, then

$$(3.75) \quad \frac{2}{\sqrt{\phi(r) - k^2}} \left[\frac{\pi}{2} + \arctan \left(\frac{k}{\sqrt{\phi(r) - k^2}} \right) \right] \geq \frac{4}{\sqrt{\phi(r) - k^2}} \sqrt{\frac{k^2}{\phi(r)} \left(1 - \frac{k^2}{\phi(r)} \right)} \\ = \frac{4k}{\phi(r)} \geq \frac{1}{\sqrt{\phi(r)}}.$$

Hence, we have that $T_S^{\text{CIR}}(r) \geq T_S^{\text{FTE}}(r) \geq T_S^{\text{BEM}}(r)$ for all $r > 1$. From the Mean-Value Theorem and Hölder's inequality, for some $N_0 \in \mathbb{N}$ suitably chosen, we deduce that

$$(3.76) \quad \sup_{t \in [0, T]} \mathbb{E} \left[|S_t - \bar{S}_t|^p \right]^{\frac{1}{p}} \leq \sup_{t \in [0, T]} \mathbb{E} \left[\max\{S_t^p, \bar{S}_t^p\} |x_t - \bar{x}_t|^p \right]^{\frac{1}{p}} \\ \leq \left\{ \sup_{t \in [0, T]} \mathbb{E} \left[S_t^{\frac{pq}{q-p}} \right]^{\frac{q-p}{pq}} + \sup_{N > N_0} \sup_{t \in [0, T]} \mathbb{E} \left[\bar{S}_t^{\frac{pq}{q-p}} \right]^{\frac{q-p}{pq}} \right\} \\ \times \sup_{t \in [0, T]} \mathbb{E} \left[|x_t - \bar{x}_t|^q \right]^{\frac{1}{q}}.$$

The conclusion follows from Proposition 3.7 and Lemma 3.8. \square

4. Numerical tests. In this section, we assume the spot process dynamics from (2.1) with $\mu = 0$ and perform a numerical analysis of the strong and weak convergence of the approximation process with the LE-FTE scheme, i.e., when the LE and the FTE schemes are employed in the discretization of the spot process and its squared volatility, respectively. Throughout this section, we fix the time horizon $T = 1$ and assign the following values to the underlying model parameters as a base case, and vary a selection individually:

$$(4.1) \quad S_0 = 1, \quad v_0 = 0.025, \quad k = 8, \quad \theta = 0.02, \quad \xi = 0.2, \quad \rho = -0.1.$$

These values are consistent with empirical observations in equity and FX markets and are close to the calibrated values in Table 2 in [11] and Table 1 in [14]. Furthermore, they lead to a Feller ratio of $\nu = 8$, which is greater than ν^{FTE} defined in (2.29).

4.1. Strong convergence. We define on $\mathcal{D} = \{(t, x, y) \in [0, T] \times \mathbb{R}_+^2 \mid S_{\min} \leq x \leq S_0 \vee x \leq y \leq S_{\max}\}$ a parametric leverage function σ and extrapolate it flat outside these bounds, where

$$(4.2) \quad S_{\min} = S_0 e^{-\frac{6}{2} \sqrt{v_0 T}} \quad \text{and} \quad S_{\max} = S_0 e^{\frac{6}{2} \sqrt{v_0 T}}.$$

In particular, we use a stochastic volatility inspired (SVI) parameterization (see [20, 28]) in both spot and running maximum, i.e.,

$$\sigma(t, x, y) = \frac{1}{2} \left[\sigma_1(t+1, \log(S_{\min} \vee x \wedge S_{\max}) - \log(S_0)) + \sigma_2(t+1, \log(S_{\min} \vee y \wedge S_{\max}) - \log(S_0)) \right],$$

where, for all $i \in \{1, 2\}$,

$$\sigma_i(u, z) = \frac{1}{\sqrt{u}} \sqrt{a_i + b_i \left(c_i(z - d_i) + \sqrt{(z - d_i)^2 + e_i^2} \right)}.$$

We assign the following values to the parameters:

$$a_{1,2} = 1, \quad b_{1,2} = 2, \quad c_{1,2} = 0, \quad d_{1,2} = 0, \quad e_{1,2} = 0.25.$$

In Figure 1, we plot the leverage function σ at three different time slices: $t = 0$, $t = 0.5$ and $t = 1$. Note that this leverage function is constant outside a bounded interval by definition. Furthermore, one can easily show that σ is $1/2$ -Hölder continuous in time and Lipschitz continuous in spot and running maximum. Then, by Remark 2.4, Assumptions 2.1 and 2.2 are satisfied.

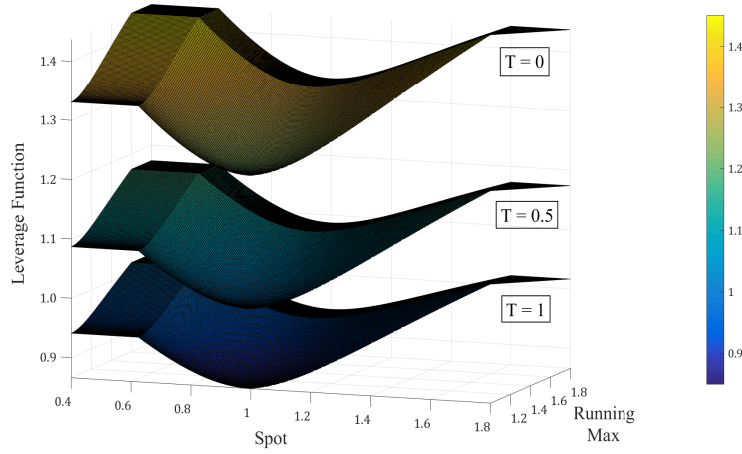


Fig. 1: The leverage function σ with the SVI parameterization plotted against the spot and the running maximum at three different time slices.

To establish the strong convergence in L^p with order $1/2$ (up to a logarithmic factor) of the approximation process, we need to compute the critical time $T^{\text{FTE}}(p)$ from (2.38). Recall the definitions of σ_{\max} , $C_{\sigma,x}$ and $C_{\sigma,m}$ from (2.8) – (2.10). A straightforward technical analysis of the leverage function yields $\sigma_{\max} = 1.437$, $C_{\sigma,x} = 0.307$ and $C_{\sigma,m} = 0.307$. Therefore, we obtain $T^{\text{FTE}}(1) = 132.58$ and $T^{\text{FTE}}(2) = 12.57$, both greater than $T = 1$, and hence all conditions in the statement of Theorem 2.5 are satisfied. For illustration, we plot in Figure 2 the critical time against the power p (in the L^p norm) and the mean reversion rate k of the squared volatility process. First, we infer from Figures 2a and 2b that $\lim_{p \rightarrow p^{\text{FTE}}} T^{\text{FTE}}(p) = 0$, a fact which can easily

be verified from the definition of the critical time. Second, we infer from Figures 2c and 2d that $\lim_{k \rightarrow \infty} T^{\text{FTE}}(p) = \infty$. The limiting case corresponds to a purely path-dependent volatility, where the strong convergence result holds for all $T > 0$.

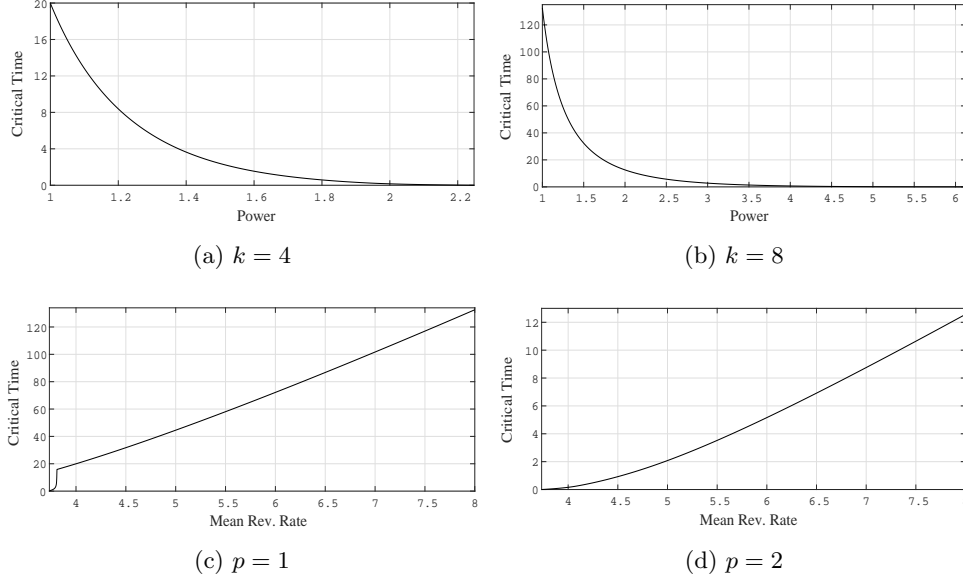


Fig. 2: The critical time defined in (2.38) plotted against the power and the mean reversion rate when $k \in \{4, 8\}$ and $p \in \{1, 2\}$, respectively.

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Next, we denote by $\bar{S}_{T,N}$ the value at time T of the approximation process corresponding to an equidistant discretization with N time steps, and study the L^p error

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$$(4.6) \quad \varepsilon_S(N) = \mathbb{E} \left[|S_T - \bar{S}_{T,N}|^p \right]^{\frac{1}{p}}$$

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when $p = 1$ (convergence in L^1 implies weak convergence for a large class of options, see, e.g., [12]) and $p = 2$ (convergence in L^2 is useful for multilevel Monte Carlo methods, see, e.g., [22]). Due to the difficulty in computing the quantity in (4.6), we employ Proposition 4.1 and estimate as proxy the difference between the values of the approximation process corresponding to N time steps ($\bar{S}_{T,N}$) and $2N$ time steps ($\bar{S}_{T,2N}$) for the same Brownian path. A proof of Proposition 4.1 for $p = 1$ can be found, for instance, in [1]. However, the extension to $p \geq 1$ is non-trivial and we include the proof for the general case in the extended arXiv version of this paper [15].

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PROPOSITION 4.1. *Let $T > 0$ and $p \geq 1$, and suppose that there exists $\eta > 1 - \frac{1}{p}$ such that*

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$$(4.7) \quad \mathbb{E} \left[|S_T - \bar{S}_{T,N}|^p \right]^{\frac{1}{p}} = \mathcal{O} \left((\log(2N))^{-\eta} \right).$$

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Then, for any $\alpha > 0$ and $\beta \geq 0$,

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$$\mathbb{E} \left[|S_T - \bar{S}_{T,N}|^p \right]^{\frac{1}{p}} = \mathcal{O} \left(\frac{(\log(2N))^\beta}{N^\alpha} \right)$$

$$(4.8) \quad \Leftrightarrow \quad \mathbb{E} \left[|\bar{S}_{T,N} - \bar{S}_{T,2N}|^p \right]^{\frac{1}{p}} = \mathcal{O} \left(\frac{(\log(2N))^\beta}{N^\alpha} \right).$$

In Theorem 1 in [29], a lower error bound was established for all discretization schemes for the CIR process based on equidistant evaluations of the Brownian motion in the accessible boundary regime. As a consequence of this result, the FTE scheme achieves at most a strong convergence order of ν when $\nu < 1/2$. In fact, we demonstrated the L^1 order of $\min\{\nu, 1/2\}$ of the FTE scheme for the CIR process numerically in [16]. Therefore, due to the CIR dynamics driving the squared volatility of the spot process in (2.1), we expect a strong convergence order of the LE-FTE scheme strictly less than $1/2$ when $\nu < 1/2$. The data in Figure 3 suggest an empirical L^1 order between 0 and $1/2$ when $\nu < 1/2$ and an order of $1/2$ when $\nu \geq 1/2$, which is in line with the previous observation and also with our theoretical results when $\nu > 2 + \sqrt{3}$.

4.2. Weak convergence. We conclude this section with a numerical analysis of the rate of weak convergence. In particular, we consider a European call option with strike $K = 0.9$ and time to maturity $T = 1$, and assign the same values to the underlying model parameters as in (4.1). In order to observe the asymptotic rate of convergence in a reasonable computational time, we define a new parametric leverage function σ with a stronger dependence on the running maximum, namely

$$(4.9) \quad \sigma(t, x, y) = 1 + \arctan(\log(y) - \log(S_0)).$$

Note that this leverage function is bounded, constant in time and spot, and Lipschitz continuous in log-running maximum. Hence, Assumptions 2.1 and 2.2 are satisfied.

In order to establish the strong convergence in L^1 with order $1/2$ (up to a logarithmic factor) – and hence the weak convergence of the same order – of the approximation process, we compute the critical time $T^{\text{FTE}}(1)$ from (2.38). A straightforward technical analysis of the leverage function yields $\sigma_{\max} = 2.571$, $C_{\sigma,x} = 0$ and $C_{\sigma,m} = 1$. Therefore, we obtain $T^{\text{FTE}}(1) = 38.92$, which is greater than $T = 1$, and hence all conditions in the statement of Theorem 2.5 are satisfied.

Next, we study the weak error

$$(4.10) \quad \varepsilon_w(N) = |\mathbb{E}[f(S_T)] - \mathbb{E}[f(\bar{S}_{T,N})]|,$$

where $f(S) = (S - K)^+$ is the European call option payoff. We employ Proposition 4.2 and estimate as proxy the difference between the values of the approximated call price corresponding to N time steps ($\bar{S}_{T,N}$) and $2N$ time steps ($\bar{S}_{T,2N}$). The proof is similar to that of Proposition 4.1 and is included in the extended arXiv version of this paper [15].

PROPOSITION 4.2. *Let $T > 0$ and $f : \mathcal{C}(\mathbb{R}_+) \rightarrow \mathbb{R}_+$, and suppose that*

$$(4.11) \quad \lim_{N \rightarrow \infty} \mathbb{E}[f(\bar{S}_{T,N})] = \mathbb{E}[f(S_T)].$$

Then, for any $\alpha > 0$ and $\beta \geq 0$,

$$(4.12) \quad \begin{aligned} & |\mathbb{E}[f(S_T)] - \mathbb{E}[f(\bar{S}_{T,N})]| = \mathcal{O} \left(\frac{(\log(2N))^\beta}{N^\alpha} \right) \\ \Leftrightarrow & |\mathbb{E}[f(\bar{S}_{T,N})] - \mathbb{E}[f(\bar{S}_{T,2N})]| = \mathcal{O} \left(\frac{(\log(2N))^\beta}{N^\alpha} \right). \end{aligned}$$

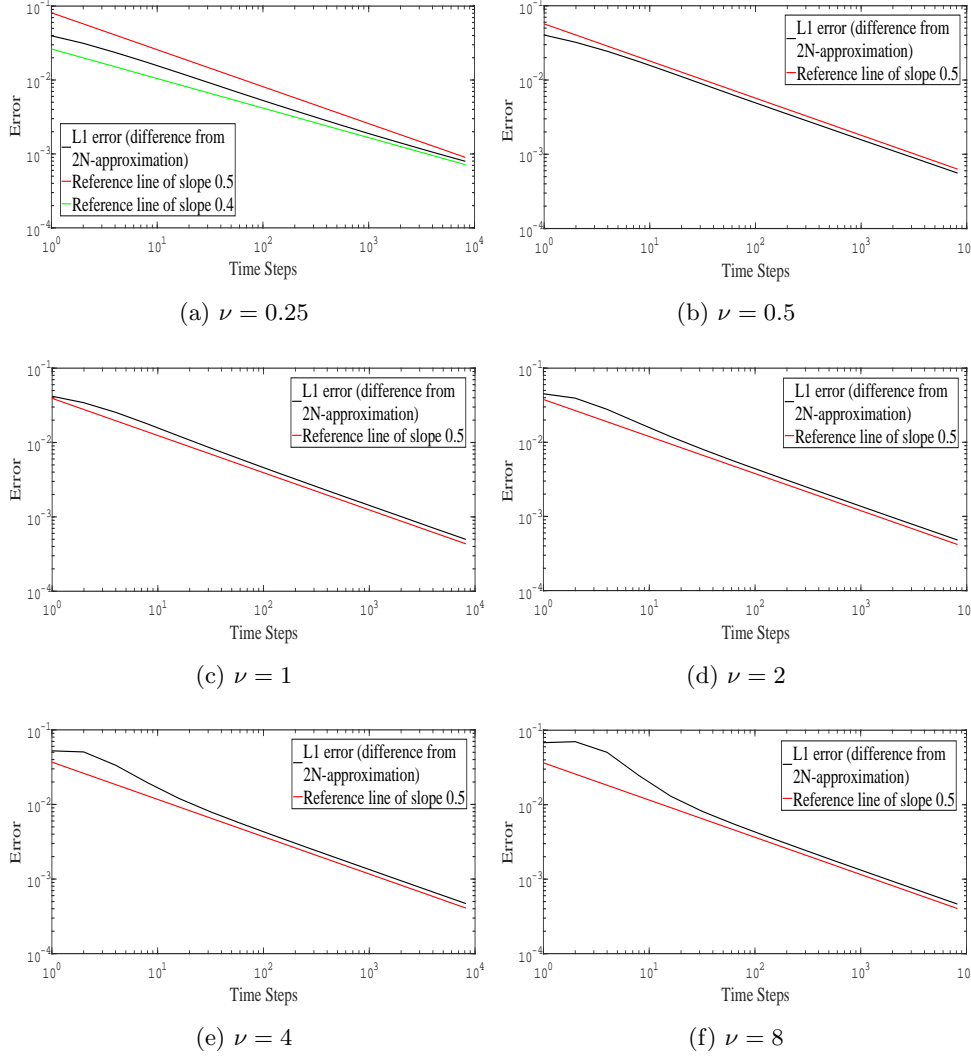


Fig. 3: The L^1 errors against the number of time steps when $k \in \{0.25, 0.5, 1, 2, 4, 8\}$ and the other parameters are as defined in (4.1), computed using up to 2.6×10^6 Monte Carlo paths (for a relative error less than 10bp).

In order to improve the weak convergence rate, we employ Brownian bridge interpolation. Given the approximated log-spot process at two subsequent time nodes, \bar{x}_{t_n} and $\bar{x}_{t_{n+1}}$, instead of taking the maximum over a piecewise linear interpolation as in (2.25), we simulate the maximum of the interpolating Brownian bridge, i.e.,

$$\hat{m}_{[t_n, t_{n+1}]} = \frac{1}{2} \left[\bar{x}_{t_{n+1}} + \bar{x}_{t_n} + \sqrt{(\bar{x}_{t_{n+1}} - \bar{x}_{t_n})^2 - 2\sigma^2(t_n, e^{\bar{x}_{t_n}}, e^{\bar{m}_{t_n}}) \bar{v}_{t_n} \delta t \log(U_n)} \right],$$

where $(U_n)_{0 \leq n \leq N-1}$ are independent $\mathcal{U}[0, 1]$ random variables, and update the running

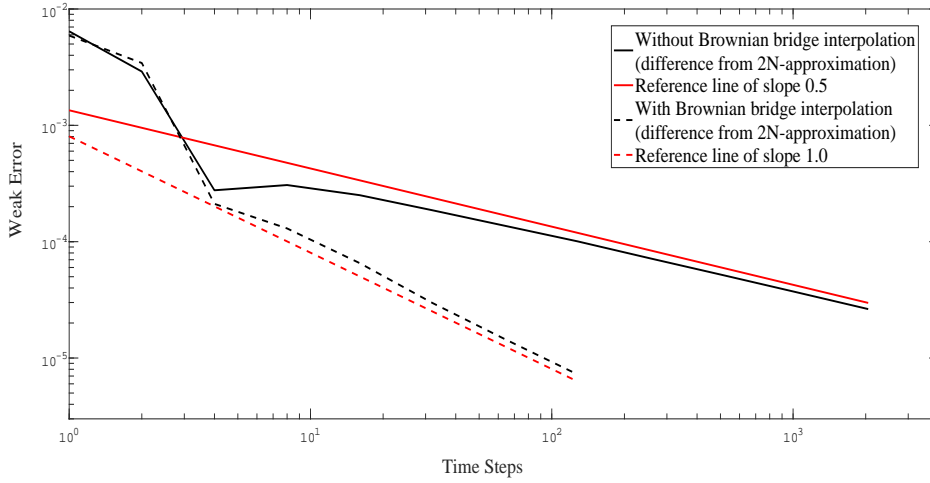


Fig. 4: The weak errors for a European call payoff (with and without Brownian bridge interpolation) against the number of time steps when the parameters are as defined in (4.1) and the strike is $K = 0.9$, computed using up to 3.2×10^9 Monte Carlo paths (for a relative error less than 1%).

759 maximum via

$$760 \quad (4.14) \quad \bar{m}_{t_{n+1}} = \max \{ \bar{m}_{t_n}, \hat{m}_{[t_n, t_{n+1}]} \}, \quad \bar{m}_0 = x_0.$$

761 Finally, the data in Figure 4 suggest an empirical weak convergence order of 1/2
 762 with piecewise linear interpolation and an order of 1 with Brownian bridge interpola-
 763 tion, as expected.

764 **5. Conclusions.** The efficient pricing and hedging of vanilla and exotic options
 765 requires an adequate model that takes into account both the local and the stochastic
 766 features of the volatility dynamics. In this paper, we have studied a stochastic path-
 767 dependent volatility model together with a simple and efficient Monte Carlo simulation
 768 scheme. We have made some realistic model assumptions and established, up to a
 769 critical time, the strong convergence in L^p with order 1/2 up to a logarithmic factor of
 770 the Euler approximation. In particular, this enables the use of multilevel simulation,
 771 as in [23], with substantial efficiency improvements for the estimation of expected
 772 financial payoffs. Inevitably, this work also raises some questions, such as whether we
 773 can relax the condition on the stochastic volatility parameters and still deduce similar
 774 convergence properties of the scheme, as suggested by our numerical results.

775 Appendix A. Proof of Lemma 3.5.

776 *Proof.* Since N is a multiple of N_T , (2.6), (2.7) and the triangle inequality yield

$$777 \quad |\mu(u, S_u, M_u) - \mu(u, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})| \leq C_{\mu,x} |x_u - x_{\bar{u}}| + C_{\mu,x} |x_{\bar{u}} - \bar{x}_{\bar{u}}| \\ 778 \quad (A.1) \quad + C_{\mu,m} |m_u - m_{\bar{u}}| + C_{\mu,m} |m_{\bar{u}} - \bar{m}_{\bar{u}}|$$

780 and

$$781 \quad |\sigma(u, S_u, M_u) - \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}})| \leq C_{\sigma,t} \sqrt{\delta t} + C_{\sigma,x} |x_u - x_{\bar{u}}| + C_{\sigma,x} |x_{\bar{u}} - \bar{x}_{\bar{u}}|$$

$$(A.2) \quad + C_{\sigma, m} |m_u - m_{\bar{u}}| + C_{\sigma, m} |m_{\bar{u}} - \bar{m}_{\bar{u}}|.$$

First, we clearly have that

$$(A.3) \quad |x_u - x_{\bar{u}}| \leq \sup_{t \in [0, u]} |x_t - x_{\bar{t}}| \quad \text{and} \quad |x_{\bar{u}} - \bar{x}_{\bar{u}}| \leq \sup_{t \in [0, u]} |x_t - \bar{x}_t|.$$

Second, note that

$$(A.4) \quad \begin{aligned} |m_u - m_{\bar{u}}| &= \sup_{t \in [0, u]} x_t - \sup_{t \in [0, \bar{u}]} x_t \leq \sup_{t \in [0, u]} (x_{\bar{t}} + |x_t - x_{\bar{t}}|) - \sup_{t \in [0, u]} x_{\bar{t}} \\ &\leq \sup_{t \in [0, u]} |x_t - x_{\bar{t}}|. \end{aligned}$$

Third, note that

$$(A.5) \quad \begin{aligned} |m_{\bar{u}} - \bar{m}_{\bar{u}}| &= \left| \sup_{t \in [0, \bar{u}]} x_t - \sup_{t \in [0, \bar{u}]} \bar{x}_t \right| \leq \sup_{t \in [0, \bar{u}]} |x_t - \bar{x}_t| \leq \sup_{t \in [0, \bar{u}]} |x_t - x_{\bar{t}}| + \sup_{t \in [0, \bar{u}]} |x_{\bar{t}} - \bar{x}_t| \\ &\leq \sup_{t \in [0, u]} |x_t - x_{\bar{t}}| + \sup_{t \in [0, u]} |x_t - \bar{x}_t|. \end{aligned}$$

Substituting back into (A.1) and (A.2) with the upper bounds derived in (A.3) – (A.5) leads to the conclusion. \square

Appendix B. Proof of Lemma 3.8.

Proof. (1) The argument follows that of Proposition 3.12 in [12]. Fix $p > 1$ and note that

$$(B.1) \quad S_t^p \leq S_0^p \exp \left\{ p\mu_{max}t - \frac{p}{2} \int_0^t \sigma^2(u, S_u, M_u) v_u du + p \int_0^t \sigma(u, S_u, M_u) \sqrt{v_u} dW_u^s \right\}.$$

Consider the Hölder pair (q_1, q_2) given by

$$(B.2) \quad q_1 = 1 + \sqrt{\frac{p-1}{p}} \quad \text{and} \quad q_2 = 1 + \sqrt{\frac{p}{p-1}}.$$

Next, define the stochastic process

$$(B.3) \quad Y_t = pq_1 \int_0^t \sigma(u, S_u, M_u) \sqrt{v_u} dW_u^s$$

with quadratic variation

$$(B.4) \quad \langle Y \rangle_t = p^2 q_1^2 \int_0^t \sigma^2(u, S_u, M_u) v_u du.$$

Taking expectations in (B.1), we deduce that

$$(B.5) \quad \mathbb{E}[S_t^p] \leq S_0^p e^{p\mu_{max}t} \mathbb{E} \left[\exp \left\{ \frac{1}{q_1} \left[Y_t - \frac{1}{2} \langle Y \rangle_t \right] + \frac{1}{2} p(pq_1 - 1) \int_0^t \sigma^2(u, S_u, M_u) v_u du \right\} \right].$$

Applying Hölder's inequality with the pair from (B.2) yields

$$(B.6) \quad \sup_{t \in [0, T]} \mathbb{E}[S_t^p] \leq S_0^p e^{p\mu_{max}T} \sup_{t \in [0, T]} \mathbb{E} \left[\exp \left\{ Y_t - \frac{1}{2} \langle Y \rangle_t \right\} \right]^{\frac{1}{q_1}}$$

$$(B.6) \quad \times \mathbb{E} \left[\exp \left\{ \frac{1}{2} p q_2 (p q_1 - 1) \sigma_{max}^2 \int_0^T v_u du \right\} \right]^{\frac{1}{q_2}}.$$

The stochastic exponential is a martingale if Novikov's condition holds, and hence if

$$(B.7) \quad \mathbb{E} \left[\exp \left\{ \frac{1}{2} \langle Y \rangle_T \right\} \right] \leq \mathbb{E} \left[\exp \left\{ \frac{1}{2} p^2 q_1^2 \sigma_{max}^2 \int_0^T v_u du \right\} \right] < \infty.$$

The finiteness of the two expectations on the right-hand side of (B.6) follows from Proposition 3.1 in [4] or Proposition 3.5 in [12].

(2) The argument follows that of Proposition 3.13 in [12]. Fix $p > 1$ and note that

$$(B.8) \quad \bar{S}_t^p \leq S_0^p \exp \left\{ p \mu_{max} t - \frac{p}{2} \int_0^t \sigma^2(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}}) \bar{v}_u du + p \int_0^t \sigma(\bar{u}, \bar{S}_{\bar{u}}, \bar{M}_{\bar{u}}) \sqrt{\bar{v}_u} dW_u^s \right\}.$$

We argue as before and use Theorem 3.6 in [12] and Proposition 3.4 in [13]. \square

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