

Time Inconsistent Portfolio Selection and Indifference Pricing



Siyuan Li
St Hugh's College
University of Oxford

A thesis submitted for the degree to
Doctor of Philosophy

July 11, 2017

“The air has changed, the scene has changed. I have to adapt to this world-with-a-question-mark as soon as I can.”

——Murakami Haruki, *1Q84*

Acknowledgements

First of all, I would like to thank my supervisor, Prof. Hanqing Jin, for his long-term help and comments to my DPhil study. Besides Hanqing, I wish to thank Prof. Xunyu Zhou, the formal director of the mathematical finance group in Mathematical Institute, University of Oxford. I learned a lot about stochastic control and mathematical finance from the reading group organised by Prof. Zhou and Jin. Without their comments in the reading group, this thesis is impossible to finish.

My sincere thanks also go to other members of the reading group, Wei Wei, Xinchun Wang, Prof. Yunfei Peng, and Pengyu Wei, who provided me numerous comments and suggestions on the work of my DPhil study. Special thanks go to Wei Wei; I learn lots of mathematics, finance, and history from him and enjoy our countless discussions in our office, the common room of Mathematical Institute, and the restaurants.

I thank all the colleagues in Mathematical Institute from whom I received kind help. In particular, I would like to thank Prof. Michael Monoyios, for his helpful advice in the field of utility indifference pricing, and Dr. Jeff Dewynne, for his rigorous feedback on some proofs in my confirmation thesis. I learned lots of things about PDE, especially Burger's equation, from Dr. Jeff Dewynne and I am very grateful for his generous support.

I would like to thank my parents for supporting me financially and spiritually throughout the whole four years of the DPhil study. I own them everything. I also thank many of my friends in both U.K. and P.R. of China; without the support from them, my DPhil study would have been much more difficult.

Finally, I want to say "Thank you" to mathematics. Before the undergraduate study, I never thought that you would stay with me for more than ten years. Sometimes I loved you, and sometimes I hated you. At this end of my student life, I find myself so grateful to have you in my life.

Abstract

In this thesis, we study three time inconsistent portfolio selection and option pricing problems with regime switching via the equilibrium approach. We first completely solve a type of time inconsistent utility maximisation problems. We obtain the equilibrium policy laws for both the time inconsistent CARA and CRRA utility maximisation problems and point out that, for the CARA utility maximisation problem, the equilibrium policy law coincides with the naive policy law. We next study a time inconsistent option pricing problem in an incomplete market via the indifference pricing approach. We provide both the equilibrium utility and mean-variance indifference pricing rule. For the equilibrium utility indifference pricing rule, we give the PDE system characterising the equilibrium utility indifference price and show the local existence and uniqueness of this PDE system in a simplified model; for the equilibrium mean-variance indifference pricing rule, we give the explicit formula for the equilibrium mean-variance indifference price. We also discuss the marginal equilibrium indifference prices and some properties of the equilibrium pricing rules.

Contents

1	Introduction	1
1.1	The Problem and A Literature Review	1
1.2	Main Results and Contributions	8
1.3	Thesis Structure	10
1.4	Some Basic Notations	12
2	A General Time Inconsistent Stochastic Control Problem with Regime Switching	13
2.1	Statement of Time Inconsistent Control Problem with Regime Switching	13
2.2	Differences from Classical Problem and Related Difficulties	15
2.3	Equilibrium Solution: A Game-Theoretic Formulation	19
3	Time Inconsistent Utility Maximisation with Regime Switching	23
3.1	Market Model and Basic Problem	23
3.1.1	The Model	23
3.1.2	The Basic Problem	25
3.2	Case of CARA (Exponential) Utility	26
3.2.1	Extended Hamilton-Jacobi-Bellman Equation	26
3.2.2	A Verification Theorem	29
3.2.3	Is A “Naive” Strategy Really Naive?	33
3.2.3.1	A Brief Explanation from Discrete-Time Analogy	36
3.3	Case of CRRA (Power) Utility	38
3.3.1	Extended Hamilton-Jacobi-Bellman Equation	39
3.3.2	A Verification Theorem	42
3.4	Economic Analysis	43

4	Equilibrium Utility Indifference Pricing with Regime Switching	47
4.1	Market Model and Basic Problem	47
4.1.1	The Model	47
4.1.2	The Basic Problem	49
4.2	Time Inconsistent CARA Utility Maximisation with Regime Switching	50
4.2.1	Utility Maximisation without Contingent Claim	50
4.2.2	Utility Maximisation with Contingent Claim	51
4.3	General Equilibrium Exponential Utility Indifference Pricing Rule	53
4.4	Equilibrium Utility Indifference Pricing Rule in Basis Risk Model	56
4.4.1	Local Existence and Uniqueness for Two-Market-State Case	58
4.4.2	Marginal Equilibrium Utility Indifference Price	68
4.4.3	Constant Risk Aversion Parameter Case	71
4.4.4	Complete Market Case	72
4.4.5	Numerical Experiments	73
4.5	Equilibrium Utility Indifference Pricing for Logarithmic Payoff	76
5	Equilibrium Mean-Variance Indifference Pricing with Regime Switching	81
5.1	Basic Problem	81
5.1.1	Mean Variance Portfolio Selection with Regime Switching	81
5.1.2	Mean Variance Indifference Pricing with Regime Switching	82
5.2	General Equilibrium Mean-Variance Indifference Pricing Rule	83
5.2.1	Mean-Variance Optimisation without Contingent Claim	84
5.2.2	Mean-Variance Optimisation with Contingent Claim	90
5.2.3	Equilibrium Mean-Variance Indifference Pricing Rule	93
5.3	Some Special Cases	94
5.3.1	Two-Market-State Basis Risk Model	94
5.3.2	Mean-Variance Indifference Pricing without Regime Switching	95
5.3.2.1	Equilibrium Mean Variance Indifference Pricing Rule	96
5.3.2.2	Precommitment Mean-Variance Indifference Pricing Rule	98
5.3.2.3	Comparison between Equilibrium and Precommitment Pricing Rules	101
6	Conclusions	103

CONTENTS

A	Reviews of Continuous-Time Markovian Chain	107
A.1	Definitions and Basic Facts	107
A.2	A Generalised Dynkin's Formula	109
B	Precommitment Solution to Time Inconsistent CARA Utility Maximisation with Regime Switching	111
C	Precommitment Solution to Mean-Variance Optimisation with Claims in Port- folio	113
	Bibliography	116

List of Figures

3.1	The Comparison between Equilibrium Strategy and Merton's Strategy.	33
3.2	The Simulation of Equilibrium Strategy and Merton's Strategy.	34
3.3	The Simulation of Equilibrium Strategy and Precommitment Strategy for CRRA Utility.	44
4.1	Equilibrium Exponential Utility Indifference Price	74
4.2	Equilibrium and Precommitment Exponential Utility Indifference Price	75
4.3	Equilibrium Utility Indifference Price For Different Numbers of Claims	76
4.4	Equilibrium Utility Indifference Price For Different Risk Aversion Parameters . .	79
4.5	Equilibrium Utility Indifference Price For Different Drifts Of The Nontradable Asset	80

Chapter 1

Introduction

1.1 The Problem and A Literature Review

The mathematical finance field is a beautiful mix of mathematical techniques and financial problems. Mathematics provides powerful tools with many applications in finance, for instance, in optimal control, partial differential equations and probability theory. The field of mathematical finance is still fast-growing; there have been numerous elegant and useful results established already, and new problems arise from academia and industry every day.

In this thesis, we focus on continuous-time models in mathematical finance, in particular, for one riskless asset, i.e., the bond and several risky assets, i.e., the stocks. Suppose there is an investor who can trade these assets continuously over a pre-specified finite time interval $[0, T]$, i.e., the investment horizon. By constructing her portfolio, the investor accumulates wealth. One of the fundamental questions of mathematical finance is how the investor should make her investment decisions.

Among the various branches of mathematical finance, there are two main topics which have attracted a lot of interest. One is the so-called portfolio selection problem. A portfolio is a collection of investments held by the investor. The investor trades continuously and allocates her wealth among different assets in the market. The portfolio selection problem is finding the optimal strategy for the investor that achieves the best performance of her portfolio, and also what the best investment performance is. The first step towards solving this problem is clarifying a criterion that the investor bases her investment decisions on. Hence, we can only analyse the optimal strategy for the investor once the investor's preferences are set.

One common characterisation of the preference is via the utility function. The utility function is a well-acknowledged way to describe the preference of a rational economic man, see the

monograph [88] for details. For the purpose in this thesis, a utility function is a continuously differentiable function which is strictly increasing and concave. The criterion for portfolio selection of a utility-type investor is then *utility maximisation*. More precisely, the investor tries to determine the strategy that maximises the expected utility of the terminal wealth of her portfolio. Portfolio selection with the (expected) utility maximisation criterion is very popular in mathematical finance and has attracted numerous studies.

The utility maximisation problem was first introduced by Samuelson in [76] under discrete-time settings and Merton in [62] and [63] under continuous-time settings. The authors obtained the optimal trading strategy and the maximum utility using dynamic programming techniques. Since then, the problem has been studied extensively in [46], [47], [13] and [49] in a financial market driven by a Brownian motion, and in [52], [14], [18], [78], [79] under a general semimartingale market setting via the duality method. In [41], a backward stochastic differential equation (BSDE) approach was utilised to solve the maximisation problem with three representative utility functions: the exponential type, the power type, and the logarithmic type.

Utility theory has achieved great success; however, it is not perfect. One criticism is that in practice it is by no means easy to determine the exact utility function for an investor. The difficulty in finding the precise utility function of a specific investor limits the application of the utility maximisation criterion for portfolio selection in the financial industry. In practice, a popular criterion for portfolio selection is *mean-variance optimisation*. For an investor whose preference is characterised by the mean-variance optimisation criterion, her objective is to select a strategy that minimises the variance of the associated terminal wealth while keeping its expectation at a pre-specified level. The criterion of mean-variance optimisation has a clear representation of the “*risk*” as the variance of the terminal value of the portfolio; moreover, due to the relatively simple structure, it is often possible to find explicitly the optimal strategies for mean-variance portfolio selection problems.

The origin of the portfolio selection problem can be traced back to the Nobel-prize-winning work of Harry Markowitz [58] in 1952, which even preceded Merton’s work [62] on utility maximisation. [58] and the later work [59] have been viewed as the foundation of modern theoretical finance and have inspired hundreds of extensions and applications. Subsequently, in [64] an analytical expression of the mean-variance efficient frontier was reported. It is worth mentioning that both [58] and [64] considered a single-period framework. There were attempts to extend the results for the mean-variance portfolio selection problem to multiperiod models, see for instance [12], but it was not until in [55] that the analytical expressions of the optimal multi-period

mean-variance portfolio policy and the efficient frontier were obtained. Later in [95], the authors extended the underlying financial market model to a continuous-time model. One of the difficulties of the multi-period and continuous-time mean-variance portfolio selection problems is the failure of the dynamic programming principle, which brings about both computational and conceptual challenges. The computational challenge was overcome in [55] and [95]; however, at that time the conceptual difficulty of the problem had not even been recognised.

Another important topic in mathematical finance is the option pricing problem. An option or a contingent claim is a contract whose value depends on the state of some (basic) assets in the market. In this thesis, “option” and “contingent claim” will be used interchangeably. The option pricing problem was pioneered by Fischer Black and Myron Scholes in [9], who introduced the replication methodology for arbitrage-free option pricing. Specifically, to price an option one should try to “replicate” it by means of a hedging portfolio; if possible, the option price is then the replicating cost, i.e., the initial value of the hedging portfolio.

However, this replication is not always possible; for instance, in an incomplete market (as is the case in this thesis) it is impossible to hedge the option perfectly in general, and there is more than one arbitrage-free price for this contingent claim. In these situations, we need to resort to another approach to determine the “fair” option price. This “fair” price is subjective to the investor, and the investor’s preference is inevitably involved in option pricing.

The methodology we adopt in this thesis is the so-called indifference pricing approach. The idea of indifference pricing can be traced back to “*certainty equivalent amount*” in the standard economic theory. In an incomplete financial market, although the replication of the contingent claim is impossible, the investor can still optimise her portfolio selection according to her preference. The investor considers two portfolio selection problems with and without the contingent claim in her portfolio. She should be willing to accept a certain cost for this claim such that she is no worse off than without the claim. This cost can then be treated as the price of the contingent claim.

The concept of certainty equivalence within option pricing first appeared in [39]. After that, the utility indifference pricing approach (i.e., the indifference pricing with the assumption that the investor’s preference is given by the utility maximisation criterion) has attracted numerous studies during the last two decades in various models. In [17], a pricing problem with transaction costs was considered; in [92] and [82], the utility indifference pricing approach was applied to stochastic volatility models; in [37], [66], [67] and [16], the utility indifference pricing rule was developed in the so-called “basis risk” models to price a claim written on a nontradable asset

which is hedged with a correlated tradable asset. In [15], a marginal version of utility indifference pricing was introduced; the marginal price can be viewed as the utility indifference price for an infinitesimal position in contingent claims. The marginal utility indifference pricing was then extended to a general Brownian market in [45]. The general theory of utility indifference pricing was developed in [18], [5], [43] and [53] via the duality method. [75] studied a pricing problem in a Brownian market with constraints on the portfolios and gave a quadratic-BSDE characterisation of the exponential utility indifference price. We refer to the excellent surveys [38] and [65] and the monograph [11] for further results and applications of utility indifference pricing.

The classical continuous-time financial market model utilises only the geometric Brownian motion to model the stock prices, see for instance [9], [62] and [95]; the continuous movements of the stock prices are driven by Brownian motion. However, the cyclical nature of stock prices indicates that the general macroeconomic condition should also be included in the financial model. This consideration leads to the regime switching model in which the general market condition is modeled by a finite-state continuous-time Markovian chain. In this thesis, we consider a regime switching market model where the dynamics of the risky assets depend on the general market condition, which is modelled by an observable continuous-time Markovian chain. Specifically, there is an exogenous finite-state continuous-time Markovian chain in the market model which is observable and models the general market condition; each of its states stands for a stage of the economic cycle of the financial market, for instance, bubble or economic recession. The assumption that the Markovian chain is observable indicates that at any time one can tell which regime the current market is in; in practice, one can determine the current market state from various economic indicators, for instance, the Consumer Price Index (CPI). The drift and diffusion coefficients of the stocks in the market depend on the regime of this Markovian chain.

Regime switching models were originally introduced in [35] to model stock return time series. After that, models with regime switching have been studied in different areas of mathematical finance. As to the portfolio selection problem, [91] considered an infinite horizon investment-consumption utility maximisation problem and a market model in which the stock price was driven by a continuous-time Markovian chain only. In [85] the author developed equivalent linear programming formulations for regime switching utility maximisation problems. Both [40] and [77] considered a hidden Markovian chain in a financial market model for investment-consumption and pure investment problems respectively, and only numerical results were provided. [93] studied the nearly optimal strategies for a utility maximisation problem in a regime switching market. In [68] the authors studied a regime switching CRRA utility maximisation problem via the

PDE approach. In all the aforementioned works, utility functions were fixed and, in particular, independent of the market regime. [84] considered an investment-consumption problem in which both the market parameters and the utility function depend on the regime of the market, and the authors obtained the explicit optimal consumption and trading strategy for specific HARA utility functions via the dynamic programming technique. In [96], the authors studied a portfolio selection problem under the mean-variance optimisation criterion with regime dependent market parameters.

For the option pricing problem, the price of a Russian option was given explicitly via the market completion technique in [34]. Later, the same idea was applied to price European and American options with regime switching in [10]. An explicit solution for the price of a perpetual American put option in a regime switching market was given in [94] and [44]. In [19], the pricing problem for a barrier option was studied in the case when the dividend process of the underlying is regime dependent. In [56], the authors derived the explicit price of a European option when both the short rate and the volatility of the underlying are regime dependent. [25] studied a pricing problem in a regime switching market; an equivalent martingale pricing measure was selected by a regime switching random Esscher transform.

In this thesis, both the portfolio selection and the option pricing problem will be studied in a regime switching model. Unlike the standard assumptions in the literature, in the current model, the investor's preference also depends on the current market state. More precisely, the investor's preferences, which are characterised by either utility functions or mean-variance optimisation, depends on the financial market condition. In this thesis, the criterion of the investor's portfolio selection is with respect to her *terminal* wealth; more precisely, for the utility maximisation criterion the investor aims to maximise the expected utility of her terminal wealth, while for the mean-variance optimisation criterion the investor tries to balance the mean and variance of her terminal wealth. Although at any time $t \in [0, T)$ the criterion of the investor for portfolio selection is with respect to her wealth at the terminal time T , the criterion itself is determined by the investor at the current time t and should be influenced by the market state, in particular, by the regime that the current market is in.

To the best of our knowledge, the concept of state-dependent utility can be traced back to [50] and [51], where the author studied a utility maximisation problem with a utility function depending on "a set of states of nature", and discussed the application to insurance; also see [20] for more discussions. In [61], the authors suggested that the state-dependent utility can help explain the equity premium puzzle stated in [60]. A market-regime-dependent utility function was

first introduced in [32], where a consumption-based capital asset pricing model for CRRA utility function with regime dependent risk aversion parameter was studied. Later, the market-regime-dependent utility functions were adopted also in [33] and [84], where the investment-consumption problems were discussed; however, in the portfolio selection criterion the utility functions were dependent on the market state for the time *when the consumption was made*. In contrast, for the problems in this thesis the utility functions depend on the *current* market regime. Our consideration comes from the fact that in expected utility theory, the utility function gauges the satisfaction of the investor from her consumption and terminal wealth level. At time t , if we study the maximisation of the expected utility from her consumption and terminal wealth at time $T > t$, the investor's utility function should be the current understanding or estimation of the utility on her future consumption and wealth. Hence, we believe that it is more natural for the utility function to depend on the *current* market state. In [72], to study an exponential indifference pricing under discrete-time settings, the authors also adopted an exponential utility function with market-regime-dependent risk aversion parameter. In this thesis, for the utility maximisation criterion, the utility function of the investor (with respect to her terminal wealth) depends on the regime of the continuous-time Markovian chain, while for the mean-variance optimisation criterion, the weighting parameter balancing the mean and variance (of her terminal wealth) depends on the regime of the continuous-time Markovian chain. The precise formulations will be given in Chapters 3 and 5.

It is obvious that regime dependent criteria for portfolio selection grant a more accurate representation of the investor's preference. However, the dependence of portfolio selection criteria on the market regime introduces the so-called *time inconsistency* into the problems in this thesis. Essentially, in the current model, the investor's preference evolves throughout the investment horizon $[0, T]$ such that Bellman's optimality principle fails. Therefore, we can no longer derive the classical Hamilton-Jacobi-Bellman equation used to find the optimal strategy and the value function of the investor.

Besides the computational difficulty, a more severe problem that arises from the time inconsistency is that the meaning of "optimal" strategy is no longer clear. Since the dynamic programming principle fails, the optimal strategy determined at the current time is no longer optimal if we reconsider the portfolio management problem at some future time. Thus, if the reconsideration of the problem is permitted, we shall abandon the previous optimal strategy and switch to some new one which is optimal at the later time. This makes the previously determined optimal strategy meaningless.

In the literature, there are three basic approaches that deal with time inconsistent stochastic control problems. One is the so-called *precommitment* approach, where an initial time is fixed, and the optimality is with respect to this very initial time. In other words, we prohibit reconsideration of the problem or “changing our mind” after the initial time. Since reconsideration of the control problem after the initial time is not allowed, the time inconsistency is avoided. Another approach is to solve for the precommitment-sense optimal strategy at every time instance and implement the strategy only at this time instance. This is known as the *naïve* approach and can be viewed as a natural (and naive) extension of the precommitment approach.

The last approach is to treat time inconsistency in a more serious way and studies the time inconsistent control problem under a game-theoretic framework. Roughly speaking, we view the control problem as a game where the players are the incarnations with our own preferences at each time. We then look for the subgame perfect Nash equilibrium point for this game as our “time consistent optimal strategy”, or the *equilibrium* strategy as known in the literature. It is this *equilibrium* approach that we adopt to tackle the time-inconsistency issue.

The phenomenon of time inconsistency was first noticed in [86]. Further studies following the equilibrium approach in discrete and continuous time were carried out in [74], [71], [70], [31], [54], [69], [87], [36], to cite but a few. Recently, the time inconsistent control problem has attracted many scholars from the field of mathematical finance. In [21] and [22], the authors studied a time inconsistent investment-consumption problem under hyperbolic discounting in both deterministic and stochastic models. In [3], the authors studied a mean-variance portfolio selection problem by obtaining an equilibrium solution. Later in [4] a quadratic hedging problem was studied. [6] provided a fairly general class of time inconsistent Markovian stochastic control problems. Later, [7] studied general discrete-time time inconsistent stochastic control problems and [8] studied a mean-variance problem with state-dependent risk aversion parameter. In [73], the author also adopted a regime switching model to study a time inconsistent utility maximisation problem, but the regime dependence was in the market parameters and the stochastic interest rate but not in the utility function, which is different from the work in the current thesis. In [42], the authors considered a time inconsistent LQ control problem and its application to a mean-variance portfolio selection problem.

In this thesis, we consider the portfolio selection and the option pricing problems with regime switching. Utility maximisation problems of both CARA- and CRRA-type with regime-dependent risk aversion parameters are discussed, and indifference pricing approaches with both utility maximisation and mean-variance optimisation criteria are utilised to price the contingent

claim in an incomplete market with regime switching. Due to the regime dependent preference of the investor, all the problems considered are time inconsistent, and we adopt the equilibrium approach to handling the time inconsistency.

1.2 Main Results and Contributions

In this thesis, three topics are studied. The first one is a type of time inconsistent utility maximisation problem with regime switching. We consider a financial market with regime switching and an investor whose preference is characterised by a regime dependent CARA- or CRRA-type utility function and study the corresponding utility maximisation problem of the investor. Due to the dependence on the current market state, the underlying stochastic control problem is *time inconsistent* and we adopt the *equilibrium approach* to overcome the resulting difficulties. Our first contribution is solving this type of time inconsistent utility maximisation problem completely. By introducing suitable auxiliary functions, the extended Hamilton-Jacobi-Bellman equations are derived, and the equilibrium policy laws are obtained for both time inconsistent CARA and CRRA utility maximisation problems. Moreover, the corresponding verification theorems are provided. Our work also leads to an interesting observation that in the case of regime dependent CARA utility maximisation, the equilibrium policy law coincides with the naive policy law. The naive approach for time inconsistent control problems has been viewed as myopic and simple-minded and with no convincing logic behind it. Nevertheless, we provide here an example where the myopic naive policy law can turn out to be an equilibrium solution. Moreover, we show in a rigorous fashion that the investor will allocate more capital to risky assets in a bullish market than in a bearish market, which complements previous empirical studies.

In the second part, we study a time inconsistent option pricing problem with regime switching. We consider the pricing of a European-type contingent claim written on a nontradable asset in a financial market with regime switching. Due to the incompleteness of the market and the failure of the replication method, we resort to a utility indifference pricing approach. The investor's utility function is of exponential type and regime dependent as in the first part of this thesis. The underlying stochastic control problem is again *time inconsistent*. We obtain the equilibrium policy laws and the value functions for the underlying stochastic control problems—utility maximisation with and without the contingent claim in the portfolio—and establish an equilibrium utility indifference pricing rule for the contingent claim. This pricing rule consists of two parts; one part is the equilibrium utility indifference price, which is given by the solution

to an algebraic equation with the two equilibrium value functions involved. The other part is the equilibrium utility indifference hedging policy law, which is the difference of the equilibrium policy laws for the two underlying time inconsistent utility maximisation problems.

This general equilibrium utility indifference pricing rule is complicated and hard to characterise. A simplified basis risk model is then studied extensively. We first give the PDE system to characterise the equilibrium utility indifference price; compared to the classical utility indifference price which can be depicted by a single PDE, we have to characterise the equilibrium utility indifference price via a PDE system. Under the two-regime case, we are able to show the local existence and uniqueness of this PDE system. We then discuss the marginal equilibrium utility indifference price by formally letting the number of claims in the portfolio approach zero. It turns out that this marginal equilibrium utility indifference price can be characterised by a linear system of partial difference equations, and hence the well-posedness is guaranteed by the standard PDE theory. Moreover, if the price of the risky asset is regime independent, the marginal equilibrium utility indifference price coincides with the expected terminal payoff of the contingent claim under the minimal martingale measure. Some properties of the equilibrium utility indifference pricing rule are discussed subsequently; for instance, the equilibrium utility indifference price reduces to the classical utility indifference price when the risk aversion parameter is the regime independent, and to the risk neutral price derived from the replication method when the market is complete. Some numerical tests are also provided. Finally, we analyse a claim with the logarithmic payoff and give an explicit formula for the equilibrium utility indifference price.

In the third part, we consider the same option pricing problem as before but with an investor whose preference is characterised by the mean-variance optimisation criterion for portfolio selection; also, the weighting parameter is regime dependent. Once more, the underlying stochastic control problem is *time inconsistent* and we adopt the equilibrium indifference pricing approach. We first give the equilibrium policy laws and value functions for the corresponding two time inconsistent mean-variance portfolio selection problems—with and without the contingent claim in the portfolio. For the case without any claim in the portfolio, it turns out that the equilibrium policy law for the regime dependent mean-variance portfolio selection problem coincides with the one for the regime dependent CARA utility maximisation problem. With the equilibrium policy laws and value functions, we then establish the equilibrium mean-variance indifference pricing rule. Now the auxiliary functions used to determine the equilibrium mean-variance indifference pricing rule are characterised by linear partial differential equations, and we are able to give the

explicit formulas for the equilibrium mean-variance indifference price and hedging policy law. A marginal equilibrium mean-variance indifference price is studied subsequently.

We then analyse some special cases. First, we study the equilibrium mean-variance indifference pricing rule in a two-market-state basis risk model. It turns out that the equilibrium mean-variance indifference price depends on the market regime only through the weighting parameter corresponding to the regime, and there is no relationship between different equilibrium mean-variance indifference prices for different market regimes. We focus next on the special case of the equilibrium mean-variance indifference pricing rule without regime switching. We give the explicit formulas for both the equilibrium mean-variance indifference price and the corresponding marginal price. It turns out that the marginal equilibrium mean-variance indifference price coincides with the expected terminal payoff of the claim under the minimal martingale measure. Furthermore, we show that the equilibrium mean-variance indifference price is concave with respect to the claim's payoff function, and is reduced to the risk neutral price derived from the replication method when the market is complete. We then compare the equilibrium mean-variance indifference pricing rule with its precommitment counterpart and point out that only their marginal mean-variance indifference prices are identical.

1.3 Thesis Structure

The structure of this thesis is as follows. In Chapter 2, we discuss a general time inconsistent stochastic control problem with regime switching. We introduce the time inconsistent control problem, discuss the difficulties of the problem and provide the framework to deal with the time inconsistency. We state the game-theoretic formulation for this time inconsistent stochastic control problem and the *equilibrium* approach for solving this problem in great detail. This chapter is the foundation for the analysis in later chapters.

In Chapter 3, we study a type of utility maximisation problem with regime switching. Both the cases of CARA- and CRRA-type regime dependent utility functions are studied. Due to the regime dependence of the utility functions, the underlying stochastic control problems are time inconsistent, and we use the *equilibrium* approach stated in Chapter 2 to solve the problems. We first present the market model and the precise formulation of the time inconsistent utility maximisation problems. We then introduce a suitable auxiliary function for each problem and derive the corresponding extended Hamilton-Jacobi-Bellman(HJB) equation. With the extended HJB equations, we then obtain the equilibrium policy laws for both CARA and CRRA utility

maximisation and obtain the ODE systems characterising the corresponding equilibrium value functions. Rigorous verification theorems are provided subsequently. We also show the coincidence of the equilibrium policy law and the naive policy law for this time inconsistent CARA utility maximisation problem. Some economic analysis for the equilibrium strategy is included at the end of this chapter.

In Chapter 4, we turn to an option pricing problem for an investor with regime dependent CARA utility. In the market, there is an asset which is *nontradable* and a European-type contingent claim is written on this nontradable asset. We discuss the (subjective) fair price of this contingent claim for the investor. It is well known that the regime switching market model is *incomplete*, see, for example, [34], [84] and references therein; together with the fact that the underlying asset is nontradable, we fail to price the option via the replication method and instead adopt the utility indifference pricing approach. Essentially, this approach involves solving two utility maximisation problems with and without the claims in the portfolio. However, due to the regime dependence of the CARA utility function of the investor, the underlying two stochastic control problems are time inconsistent, and we utilise the *equilibrium* approach stated in Chapter 2 to solve them. The results of the equilibrium policy laws and value functions for these two problems are cited from Chapter 3. With the equilibrium policy laws and value functions for the associated two time inconsistent utility maximisation problems, we then obtain the equilibrium utility indifference pricing rule. This equilibrium utility indifference pricing rule consists of two parts; one part is the equilibrium utility indifference price, which is given by the solution to an algebraic equation with the two equilibrium value functions involved, and the other part is the equilibrium utility indifference hedging policy law, which is the difference of the equilibrium policy laws for the two underlying time inconsistent utility maximisation problems. We then discuss a simplified basis risk model (see [65] for more details) and give the characterising PDE for the equilibrium utility indifference price. Under the two-regime case, we are able to show the local existence and uniqueness of this PDE system. We then discuss the marginal equilibrium utility indifference price by formally letting the number of the claims in the portfolio approach to zero. Some properties of the equilibrium utility indifference pricing rule and numerical tests are also provided. Finally, we analyse a claim with a logarithmic payoff and present an explicit formula for the equilibrium utility indifference price.

Chapter 5 is also devoted to an option pricing problem in an incomplete market, similar to Chapter 4. Once more, we adopt the indifference pricing approach in this chapter to find the subjective price of the contingent claim for the investor by comparing the results of two

portfolio selection problems with and without the claim. However, we now assume that the investor's preference is characterised by the mean-variance optimisation criterion for portfolio selection and that the weighting parameter is regime dependent. This weighting parameter reflects the investor's balance between expected return and acceptable risk. It is well-known that the mean-variance portfolio selection problem itself is time inconsistent, see [3] and [4] for more details; moreover, the regime dependence of the weighting parameter introduces additional time inconsistency. In this chapter, we first determine the equilibrium policy laws and value functions for the corresponding two time inconsistent mean-variance portfolio selection problems. With the equilibrium policy laws and value functions, we then establish the equilibrium mean-variance indifference pricing rule, which is parallel to the equilibrium utility indifference pricing rule in Chapter 4. We then analyse the special case of the equilibrium mean-variance indifference pricing rule without regime switching.

Chapter 6 concludes the thesis. Appendix A is a review of some facts about the continuous-time Markovian chain and the corresponding version of Dynkin's formula. Appendix B and C contains the details of finding the precommitment policy laws for the time inconsistent CARA utility maximisation and mean-variance optimization problems appeared in Chapters 3 and 5 respectively.

1.4 Some Basic Notations

In the end of this chapter we state some common notations which will be used throughout the thesis. In the thesis, a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a finite positive $T > 0$ are always given and fixed. For any given matrix A , we use superscript T for the transpose operator; i.e. A^T stands for the transpose of A . We use $|\cdot|$ to denote the usual Euclidean norm (on $\mathbb{R}^n$, possibly $n > 1$), and $\|\cdot\|_B$ the norm on some (functional) Banach space B . For $p \geq 1$, $\mathcal{L}^p_{\mathcal{F}}(0, T; \mathbb{R}^n)$ denotes the set of all $\mathbb{R}^n$ -valued $\mathbb{F}$ -progressively measurable stochastic process f satisfying $\mathbb{E} \left[\int_0^T |f_t|^p dt \right] < \infty$.

Chapter 2

A General Time Inconsistent Stochastic Control Problem with Regime Switching

This chapter is devoted to a general time inconsistent stochastic control problem with regime switching. We state the general problem in concern, discuss its differences from the classical stochastic control problem and the related difficulties, and provide the approach, which is based on [6], to deal with this time inconsistent control problem. This chapter is the foundation for the analysis in later chapters.

2.1 Statement of Time Inconsistent Control Problem with Regime Switching

The underlying probability space is $(\Omega, \mathcal{F}, \mathbb{P})$ and the terminal time is $T > 0$, as introduced in the end of Chapter 1. On this probability space are defined a (m -dimensional) Brownian motion (BM) W . and a continuous-time Markovian chain (CTMC) Ξ . with finite discrete value space $\mathcal{E} = \{1, 2, \dots, K\}$; moreover, W . and Ξ . are assumed to be independent with each other. Some basic facts about CTMC are summarised in Appendix A. Our objective is to “maximise” (in a certain sense which will be precisely clarified later) the following functional

$$J(t, x, \alpha; \boldsymbol{\pi}.) = \mathbb{E}\left[F(\alpha, x_T^{\boldsymbol{\pi}}) \middle| x_t = x, \Xi_t = \alpha\right] + G\left(\alpha, \mathbb{E}\left[x_T^{\boldsymbol{\pi}} \middle| x_t = x, \Xi_t = \alpha\right]\right), \quad (2.1)$$

where $F, G : \mathcal{E} \times \mathbb{R}^m \rightarrow \mathbb{R}$ are given terminal gain functions, $\boldsymbol{\pi}.$ is the (l -dimensional) control process and $x^{\boldsymbol{\pi}}$ is the corresponding (n -dimensional) state process which is a controlled Markovian

process satisfying the following stochastic differential equation (SDE)

$$\begin{cases} dx_s^\pi = p(s, x_s^\pi, \Xi_s, \pi_s)ds + q(s, x_s^\pi, \Xi_s, \pi_s)dW_s; \\ x_t = x, \Xi_t = \alpha \end{cases} \quad (2.2)$$

where the coefficients $p : [0, T] \times \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^l \rightarrow \mathbb{R}^n$ and $q : [0, T] \times \mathbb{R}^n \times \mathcal{E} \times \mathbb{R}^l \rightarrow \mathbb{R}^{n \times m}$ are given deterministic functions. Throughout this thesis, we always use boldface letter, say π , to denote a control *process* or a control policy law (which is a *function*), and use ordinary letter without boldface, say π , to denote a possible *value* (which is an l -dimensional vector) of the control process. We denote $\mathcal{A}(t, x, \alpha; T)$ the collection of all *admissible* control process π . such that the SDE (2.2) of the corresponding state process x^π has a unique strong solution and the corresponding objective functional $J(t, x, \alpha; \pi)$ is well-defined. Furthermore,

$$\mathcal{AD}[t, T] = \bigcup_{x \in \mathbb{R}^n, \alpha \in \mathcal{E}} \mathcal{A}(t, x, \alpha; T)$$

is called the *admissible set*.

We emphasise that throughout this thesis, Ξ . is exogenous and does *not* depend on the control process π .

Here are two examples of the general stochastic control problem with regime switching introduced above.

1. A Regime Switching Utility maximisation Problem.

The objective functional takes the following form

$$J(t, x, \alpha; \pi) = \mathbb{E} \left[U(\alpha, x_T^\pi) \middle| x_t = x, \Xi_t = \alpha \right];$$

equivalently, the terminal gain functions are given by

$$\begin{cases} F(\alpha, x) = U(\alpha, x); \\ G(\alpha, x) = 0, \end{cases}$$

where $\{U(\alpha, \cdot)\}_{\alpha \in \mathcal{E}}$ is a set of utility functions, i.e., for each $\alpha \in \mathcal{E}$, $U(\alpha, x)$ is continuously differentiable, strictly increasing and strictly concave with respect to variable x . The financial background of this problem and the specific choices for $U(\alpha, x)$ will be given in great detail in Chapter 3.

2. A Regime Switching Mean-Variance Optimisation Problem.

The objective functional takes the following form

$$\begin{aligned} J(t, x, \alpha; \boldsymbol{\pi}.) &= \mathbb{E} \left[x_T^\pi \middle| x_t = x, \Xi_t = \alpha \right] - \frac{\gamma(\alpha)}{2} \text{Var} \left(x_T^\pi \middle| x_t = x, \Xi_t = \alpha \right) \\ &= \mathbb{E} \left[x_T^\pi - \frac{\gamma(\alpha)}{2} \left(x_T^\pi \right)^2 \middle| x_t = x, \Xi_t = \alpha \right] + \frac{\gamma(\alpha)}{2} \left(\mathbb{E} \left[x_T^\pi \middle| x_t = x, \Xi_t = \alpha \right] \right)^2; \end{aligned}$$

equivalently, the terminal gain functions are given by

$$\begin{cases} F(\alpha, x) = x - \frac{\gamma(\alpha)}{2} x^2; \\ G(\alpha, x) = \frac{\gamma(\alpha)}{2} x^2, \end{cases}$$

where $\{\gamma(\alpha)\}_{\alpha \in \mathcal{E}}$, the weighting parameters of the mean-variance optimisation problem, is a given set of positive numbers. The detailed financial background of this problem will be included in Chapter 5.

2.2 Differences from Classical Problem and Related Difficulties

Before the analysis of the current problem in concern, let us first take a brief review of the classical regime switching control problem. A classical stochastic control problem with regime switching is to maximise the following functional

$$J^C(t, x, \alpha; \boldsymbol{\pi}.) = \mathbb{E} \left[F^C(x_T^\pi) \middle| x_t = x, \Xi_t = \alpha \right] \quad (2.3)$$

for some given terminal gain function $F^C : \mathbb{R}^m \rightarrow \mathbb{R}$, where the control process $\boldsymbol{\pi}.$ and the state process x^π are the same as in Section 2.1. In this chapter, we always use the superscript “C” to emphasise that the current object—for instance the terminal gain function, the objective functional or the value function—is corresponding to the classical problem in the literature. One can immediately find two differences between the classical control problem and the control problem in Section 2.1.

1. First, for the classical problem, although the dynamics of the state process x^π depends on the CTMC $\Xi.$, the objective functional $J^C(t, x, \alpha; \boldsymbol{\pi}.)$ only implicitly depends on $\Xi.$ through x^π ; more precisely, the terminal gain function $F^C(x)$ does *not* depend on $\Xi.$. In contrast, the objective functional $J(t, x, \alpha; \boldsymbol{\pi}.)$ for the current problem at time $t \in [0, T]$ depends on the current state of $\Xi_t = \alpha$ explicitly, i.e., the corresponding terminal gain functions $F(\alpha, x)$ and $G(\alpha, x)$ depend on the state of the CTMC explicitly. Consequently,

as time evolves, the terminal gain functions and the objective functionals corresponding to the different intermediate times are different in general.

2. Second, compared to the classical control problem, there is an extra part

$$G\left(\alpha, \mathbb{E}\left[x_T^\pi \middle| x_t = x, \Xi_t = \alpha\right]\right)$$

in the objective functional (2.1). In general, this is a *nonlinear* transform of the conditional expectation $\mathbb{E}[x_T^\pi | x_t = x, \Xi_t = \alpha]$. The most severe consequence for this nonlinear transform is that it destroys the tower property for the conditional expectation; as a matter of fact, for any $h \in (0, T - t)$,

$$G\left(\alpha, \mathbb{E}\left[x_T^\pi \middle| x_t = x, \Xi_t = \alpha\right]\right) \neq \mathbb{E}\left[G\left(\alpha, \mathbb{E}\left[x_T^\pi \middle| x_{t+h}^\pi, \Xi_{t+h}\right]\right) \middle| x_t = x, \Xi_t = \alpha\right]$$

in general.

These two differences above both result in the failure of the *dynamic programming principle* for the current control problem in Section 2.1. Recall that if we introduce the following value function

$$\begin{aligned} V^C(t, x, \alpha) &= \sup_{\pi. \in \mathcal{AD}^C[t, T]} \left\{ J^C(t, x, \alpha; \pi.) \right\} \\ &= \sup_{\pi. \in \mathcal{AD}^C[t, T]} \left\{ \mathbb{E}\left[F^C(x_T^\pi) \middle| x_t = x, \Xi_t = \alpha\right] \right\} \end{aligned}$$

for the classical control problem, where $\mathcal{AD}^C[t, T]$ is the corresponding admissible set, then for any $0 < h < T - t$ the *dynamic programming principle* (see, for example, [28] and [90]) gives

$$\begin{aligned} V^C(t, x, \alpha) &= \sup_{\pi. \in \mathcal{AD}^C[t, t+h]} \left\{ \mathbb{E}\left[\sup_{\pi. \in \mathcal{AD}^C[t+h, T]} \left\{ \mathbb{E}\left[F^C(x_T^\pi) \middle| x_{t+h}^\pi, \Xi_{t+h}\right] \right\} \middle| x_t = x, \Xi_t = \alpha \right] \right\} \\ &= \sup_{\pi. \in \mathcal{AD}^C[t, t+h]} \left\{ \mathbb{E}\left[\sup_{\pi. \in \mathcal{AD}^C[t+h, T]} \left\{ J^C(t+h, x_{t+h}^\pi, \Xi_{t+h}; \pi.) \right\} \middle| x_t = x, \Xi_t = \alpha \right] \right\} \\ &= \sup_{\pi. \in \mathcal{AD}^C[t, t+h]} \left\{ \mathbb{E}\left[V^C(t+h, x_{t+h}^\pi, \Xi_{t+h}) \middle| x_t = x, \Xi_t = \alpha \right] \right\}. \end{aligned}$$

However, the success of the dynamic programming principle relies at least on two facts; one is that the terminal gain function should be independent of the current state of CTMC $\Xi.$, and the other is the tower property of the conditional expectation. Unfortunately, the two facts fail both

2.2. DIFFERENCES FROM CLASSICAL PROBLEM AND RELATED DIFFICULTIES

for the current problem; as a matter of fact, in general

$$\begin{aligned}
& \sup_{\pi. \in \mathcal{AD}[t, T]} \left\{ J(t, x, \alpha; \pi.) \right\} \\
&= \sup_{\pi. \in \mathcal{AD}[t, T]} \left\{ \mathbb{E} \left[\mathbb{E} \left[F(\alpha, x_T^\pi) | x_{t+h}^\pi, \Xi_{t+h} \right] \middle| x_t = x, \Xi_t = \alpha \right] + G(\alpha, \mathbb{E} [x_T^\pi | x_t = x, \Xi_t = \alpha]) \right\} \\
&\neq \sup_{\pi. \in \mathcal{AD}[t, T]} \left\{ \mathbb{E} \left[\mathbb{E} \left[F(\Xi_{t+h}, x_T^\pi) | x_{t+h}^\pi, \Xi_{t+h} \right] + G(\Xi_{t+h}, \mathbb{E} [x_T^\pi | x_{t+h}^\pi, \Xi_{t+h}]) \right] \middle| x_t = x, \Xi_t = \alpha \right\} \\
&= \sup_{\pi. \in \mathcal{AD}[t, T]} \left\{ \mathbb{E} \left[J(t+h, x_{t+h}^\pi, \Xi_{t+h}; \pi.) \middle| x_t = x, \Xi_t = \alpha \right] \right\} \\
&= \sup_{\pi. \in \mathcal{AD}[t, t+h]} \left\{ \mathbb{E} \left[\sup_{\pi. \in \mathcal{AD}[t+h, T]} \left\{ J(t+h, x_{t+h}^\pi, \Xi_{t+h}; \pi.) \right\} \middle| x_t = x, \Xi_t = \alpha \right] \right\}.
\end{aligned}$$

Therefore, the dynamic programming principle does *not* hold any more for the current problem in general.

The failure of the dynamic programming principle introduces extra difficulties for the current problem compared with the classical one from both the computational and the conceptual perspective. Computationally, dynamic programming principle is a powerful tool for solving classical control problem and is the foundation of the (classical) Hamilton-Jacobi-Bellman (HJB) equation. For the classical case, by solving the HJB equation, one can simultaneously get the value function and the optimal control; also, we will get a feedback type optimal control, i.e., an optimal control *law*. Now, without the dynamic programming principle, one can no longer obtain the HJB equation, and finding the optimal control for the current control problem becomes challenging.

More severely, the failure of the dynamic programming principle also introduces conceptual difficulty—the meaning of “*optimality*” becomes ambiguous. For the classical control problem, due to the dynamic programming principle and assuming the existence of the optimal control $\bar{\pi}^t$ for the control problem at *any* time t (we use the superscript “ t ” to emphasise that this optimal

control is for the control problem at time t), it holds

$$\begin{aligned}
 V^C(t, x, \alpha) &= J^C(t, x, \alpha; \bar{\pi}^t) = \mathbb{E} \left[F^C \left(x_T^{\bar{\pi}^t} \right) \middle| x_t = x, \Xi_t = \alpha \right] \\
 &= \sup_{\pi_{\cdot} \in \mathcal{AD}^C[t, t+h]} \left\{ \mathbb{E} \left[V^C(t+h, x_{t+h}^{\pi_{\cdot}}, \Xi_{t+h}) \middle| x_t = x, \Xi_t = \alpha \right] \right\} \\
 &= \sup_{\pi_{\cdot} \in \mathcal{AD}^C[t, t+h]} \left\{ \mathbb{E} \left[J^C(t+h, x_{t+h}^{\pi_{\cdot}}, \Xi_{t+h}; \bar{\pi}^{t+h}_{\cdot}) \middle| x_t = x, \Xi_t = \alpha \right] \right\} \\
 &= \sup_{\pi_{\cdot} \in \mathcal{AD}^C[t, t+h]} \left\{ \mathbb{E} \left[\mathbb{E} \left[F^C \left(x_T^{\bar{\pi}^{t+h}} \right) \middle| x_{t+h}^{\pi_{\cdot}}, \Xi_{t+h} \right] \middle| x_t = x, \Xi_t = \alpha \right] \right\} \\
 &= \mathbb{E} \left[F^C \left(x_T^{\left(\bar{\pi}^t|_{[t, t+h]}, \bar{\pi}^{t+h}|_{[t+h, T]} \right)} \right) \middle| x_t = x, \Xi_t = \alpha \right] \\
 &= J^C \left(t, x, \alpha; \left(\bar{\pi}^t|_{[t, t+h]}, \bar{\pi}^{t+h}|_{[t+h, T]} \right) \right),
 \end{aligned}$$

where $\bar{\pi}^t|_{[t, t+h]}$ stands for the restriction of the control process $\bar{\pi}^t$ on the time interval $[t, t+h] \subset [t, T]$. Furthermore, under some mild concavity condition on $F^C(\cdot)$, it can be shown that the optimal control is unique. Therefore, it holds

$$\bar{\pi}^t|_{[t+h, T]} = \bar{\pi}^{t+h} \quad \text{on } [t+h, T].$$

This means that, at any later time $t+h \in (t, T)$, if we reconsider the control problem, the optimal control $\bar{\pi}^{t+h}$ obtained at $t+h$ (conditioning on the state $(x_{t+h}^{\bar{\pi}^t}, \Xi_{t+h})$) coincides with the optimal control $\bar{\pi}^t$ obtained previously at t ; in other words, we will not deviate the optimal control obtained previously in any later time. Consequently, reconsideration of the problem later is unnecessary, and once we obtain the optimal control $\bar{\pi}^t$ at time t , we can safely follow it at any later time between t and T . This nice property is known as *time consistency* in the literature: the optimal control remains optimal in any later time, conditioning on the state followed this optimal control.

However, the property of time consistency no longer holds for the current control problem in Section 2.1. The failure of the dynamic programming principle results in the following unpleasant fact that if we still aim to find the “optimal” control $\bar{\pi}^t$ such that the corresponding objective functional at time t achieves the maximum:

$$J(t, x, \alpha; \bar{\pi}^t) = \sup_{\pi_{\cdot} \in \mathcal{AD}[t, T]} \left\{ J(t, x, \alpha; \pi_{\cdot}) \right\},$$

2.3. EQUILIBRIUM SOLUTION: A GAME-THEORETIC FORMULATION

then in general

$$\begin{aligned} J(t, x, \alpha; \bar{\pi}^t) &\neq \sup_{\pi \in \mathcal{AD}[t, t+h]} \left\{ \mathbb{E} \left[\sup_{\pi \in \mathcal{AD}[t+h, T]} \left\{ J(t+h, x_{t+h}^{\pi}, \Xi_{t+h}; \pi) \right\} \middle| x_t = x, \Xi_t = \alpha \right] \right\} \\ &= \sup_{\pi \in \mathcal{AD}[t, t+h]} \left\{ \mathbb{E} \left[J(t+h, x_{t+h}^{\pi}, \Xi_{t+h}; \bar{\pi}^{t+h}) \middle| x_t = x, \Xi_t = \alpha \right] \right\}, \end{aligned}$$

where $\bar{\pi}^{t+h}$ is the optimal control such that the objective functional at time $t+h$ achieves the maximum among all the admissible controls in $\mathcal{AD}[t+h, T]$. Consequently, for some later time $t+h \in [t, T]$,

$$\bar{\pi}^t|_{[t+h, T]} \neq \bar{\pi}^{t+h} \quad \text{on } [t+h, T]$$

even conditioning on the state $(x_{t+h}^{\bar{\pi}^t}, \Xi_{t+h})$. In other words, the “optimal” control obtained at time t is *no longer* “optimal” at the later time $t+h \in (t, T)$. Thus, if we reconsider the control problem at the later time $t+h$, we shall abandon the previous “optimal” control $\bar{\pi}^t$ and switch to some new “optimal” one $\bar{\pi}^{t+h}$; in other words, the “optimal” control obtained at time t will *not* be fully implemented on the whole time horizon $[t, T]$ and thus meaningless. (In fact, the maximum value $J(t, x, \alpha; \bar{\pi}^t)$ can only be achieved when $\bar{\pi}^t$ is fully implemented on $[t, T]$!) This phenomenon is known as *time inconsistency* in the literature. The stochastic control problems encountered in this thesis are all *time inconsistent*.

2.3 Equilibrium Solution: A Game-Theoretic Formulation

As mentioned above, time inconsistency introduces both the conceptual and the computational difficulties in time inconsistent control problem. There are at least three approaches to deal with the conceptual difficulty of time inconsistency.

1. We fix some time (without loss of generality we may take it to be 0) as the *reference time* and try to find the strategy $\bar{\pi}^0$ such that the corresponding objective functional $J(0, x, \alpha; \bar{\pi}^0)$ at the reference time 0 achieving the maximum among all the admissible controls in $\mathcal{AD}[0, T]$. We then simply follow this strategy and ignore that in later time this strategy may no longer be optimal (with respect to the objective at that time); in other words, all the optimality is with respect to the reference time. Such “optimal” strategy is known as the *precommitment* strategy. For this approach, essentially there is only *one* control problem, and we thus avoid the issue of time inconsistency conceptually. Nevertheless, the success of the precommitment approach relies on that we should have a

strong self-control discipline not to “*change our minds*” (i.e., reconsider the problem) at any later time $s \in (t, T)$; otherwise, the *precommitment* strategy $\{\bar{\pi}_u^t\}_{u \in [t, T]}$ obtained at time t will be abandoned after time t and, therefore, this precommitment approach becomes meaningless.

2. At each time we find and follow the precommitment strategy corresponding to that time; at later time, however, we switch our action to the new precommitment strategy corresponding to that new time. More precisely, at any time $t \in [0, T)$ we follow the *precommitment* strategy $\bar{\pi}_t^t$ with respect to the reference time t and, after a short time $h > 0$, we *reconsider* the control problem and switch our action following some new *precommitment strategy* $\bar{\pi}_{t+h}^{t+h}$ with respect to the new reference time $t + h$; we then repeat this procedure afterwards. At the first glance, it seems that at any time t we are “*walking on the right path*”—following the “optimal” strategy $\bar{\pi}_t^t$ at this time t ; nevertheless, *no* “optimal” strategy $\bar{\pi}_t^t$ at any time t will be fully implemented on the rest of the time horizon $[t, T)$ and thus *no* maximum value of objective functional $J(t, x, \alpha; \bar{\pi}_t^t)$ at any time will be achieved. Therefore, this approach for handling time inconsistency has no solid logic behind it. The resulting strategy is called the *naive* strategy (see, for instance, [69]).
3. We treat the issue of time inconsistency seriously and define the desired “optimal” solution, which will be fully implemented even the reconsideration of the problem is permitted in the future, in a game-theoretic point of view. The resulting “optimal” strategy is called the *equilibrium* strategy, which will be explained in detail below.

We admit that there are situations under which any of the three approaches above is meaningful; however, we will focus mainly on the third *equilibrium* approach in this thesis. To this end, we adopt the game theoretic formulation of the time inconsistent stochastic control problem. The essential idea behind the *equilibrium* approach is that, when we decide what to do at any time $t \in [0, T)$, we should keep in mind that in the future our objective functional (or preference) will change and our future-selves will choose the “*optimal*” actions according to the objective functional at the future time. More precisely, we are “wise enough” to be aware the following two facts; first, in the future the state of Ξ . could change and second, *if* the state of Ξ . changes at some future time, the objective functional will also change accordingly by our future-selves, and we will do the “optimisation” with respect to the objective functional at that future time which is different from the current one. Therefore, we view the entire stochastic control problems at all time points as a non-cooperative game, in which at each time point t there is a player with the

2.3. EQUILIBRIUM SOLUTION: A GAME-THEORETIC FORMULATION

reward which is the objective functional $J(t, x, \alpha; \pi.)$ at this time t . Under this game-theoretic point of view, the strategy $\hat{\pi}.$ we look for should be the “*subgame perfect Nash equilibrium point*” of this game. This Nash equilibrium point is exactly the *equilibrium* strategy we defined below.

Intuitively, the *equilibrium* strategy $\hat{\pi}.$ is the one that for any time $t \in [0, T)$, if on (t, T) we follow $\hat{\pi}.$, then the “optimal” action for us at t is $\hat{\pi}_t$. However, the difficulty for applying this idea directly in defining the equilibrium strategy for the continuous-time time inconsistent stochastic control problem is that one single time $\{t\}$ is a set of Lebesgue measure zero and any specific value of a control process on a measure-zero set does not influence the controlled system. Thus, for the continuous-time control problem, we cannot define the equilibrium strategy via perturbing the control at only one single time point. In this thesis, we adopt the following definition of the *equilibrium* strategy from [6].

Definition 1. A deterministic function $\hat{\pi}(t, x, \alpha)$ is called an *equilibrium policy law*, if:

1. the corresponding (*feedback type*) strategy $\hat{\pi}_t = \hat{\pi}(t, x_t, \Xi_t)$ is admissible;
2. for any (*fixed*) $(t, x, \alpha) \in [0, T) \times \mathbb{R}^n \times \mathcal{E}$, any $h \in (0, T - t)$ and any $\pi \in \mathbb{R}^l$, we define the following perturbed policy law

$$\pi^{t,h,\pi}(s, x, \alpha) = \begin{cases} \pi, & s \in [t, t+h), \\ \hat{\pi}(s, x, \alpha), & s \in [t+h, T) \end{cases}$$

and $\pi^{t,h,\pi}$ is the corresponding strategy. Then it holds

$$\liminf_{h \rightarrow 0+} \frac{J(t, x, \alpha; \hat{\pi}.) - J(t, x, \alpha; \pi^{t,h,\pi})}{h} \geq 0.$$

For the equilibrium policy law $\hat{\pi}(t, x, \alpha)$, the corresponding strategy $\hat{\pi}_t = \hat{\pi}(t, x_t, \Xi_t)$ is called the *equilibrium* strategy and the corresponding objective functional

$$\hat{V}(t, x, \alpha) = J(t, x, \alpha; \hat{\pi}.)$$

is called the *equilibrium value function*.

Remark 1. Definition 1 does not mention the *uniqueness* of equilibrium policy law. As a matter of fact, there are such cases in which *multiple* equilibrium policy laws exist, see, for instance, [23], [24] and [87]. In this thesis, we do *not* discuss the uniqueness of equilibrium policy law, but provide the characterization of an equilibrium policy law only in later works.

We want to emphasise that the above definition of *equilibrium strategy* is, in fact, not on the control process directly but on the *control law*, which is a function of time t and state (x, α) ; in other words, it is *equilibrium policy law* that we actually defined in Definition 1. With an *equilibrium policy law*, the (corresponding) *equilibrium strategy* is naturally obtained by substituting the state processes into this *equilibrium policy law*. Some papers, for instance, [42] defines the *equilibrium strategy* directly on the control process by perturbing equilibrium control process itself. More precisely, in the definition of *equilibrium strategy* in [42], the perturbation is against an equilibrium control, while in Definition 1, the perturbation is against an equilibrium policy law. Due to the popularity of Markovian model and feedback-type strategy in the economic study, we adopt the “*feedback-type*” definition of *equilibrium strategy* from [6] in this thesis.

In later chapters, we will solve several time inconsistent stochastic control problems with regime switching by the equilibrium approach defined in this chapter.

Chapter 3

Time Inconsistent Utility Maximisation with Regime Switching

This chapter is devoted to a type of time inconsistent portfolio selection problem. We analyse both the CARA and CRRA utility maximisation problems with regime switching. Due to the dependence of the utility functions on the market state, the underlying stochastic control problems for utility maximisation are time inconsistent. We provide the equilibrium policy laws and value functions for both time inconsistent CARA and CRRA utility maximisation problems. For the CARA utility case, we show the coincidence of the equilibrium and the naive policy laws. Some economic analysis is included at the end of the chapter. The results in this chapter will be cited in Chapter 4 for the equilibrium utility indifference pricing rule.

3.1 Market Model and Basic Problem

In this section, we state the market model and the portfolio selection problem under consideration.

3.1.1 The Model

The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and time horizon $T > 0$ are as stated in Chapter 1. On this probability space defined a d -dimensional standard Brownian motion (BM) $W. = (W.^1, \dots, W.^d)^T$ and a continuous-time Markovian chain (CTMC) $\Xi.$ taking values in a finite discrete state space $\mathcal{E} = \{1, 2, \dots, K\}$; moreover, $W.$ and $\Xi.$ are independent of each other. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ is the completion of the natural filtration generated by $W.$ and $\Xi.$

We use the exogenous process $\Xi.$ to model the financial market state; for any $t \in [0, T]$, the market state at this time t is $\Xi_t \in \mathcal{E}$. More precisely, we assume the market has K states $1, 2, \dots, K$; the state 1 stands for the case that the current market is “*bullish*” or in a bubble and the state K for the market is “*bearish*” or in a crash. Furthermore, $\Xi.$ is *observable*; in other words, at any time one can tell which regime the current market is in. In practice, one can determine the current market state from various economic indicators, for instance, the Consumer Price Index (CPI).

We denote $\Lambda = (\lambda_{\alpha\beta})_{K \times K}$ the intensity matrix of $\Xi.$ To avoid the trivial case, we assume $\lambda_{\alpha\beta} > 0$ for any distinct states α and β and thus the CTMC $\Xi.$ has no *absorbing* state; in other words, we assume the financial market will not stay at any particular state forever.

The financial market consists of one bond $B.$ and n stocks $S. = (S^1, \dots, S^n)^T$ which are traded continuously. For the sake of simplicity, we assume the interest rate is constantly zero and

$$B_t \equiv 1, \quad t \in [0, T].$$

The price process of k -th stock follows the SDE

$$\begin{aligned} dS_t^k &= S_t^k \{ \mu^k(t, \Xi_t) dt + \sigma^k(t, \Xi_t) dW_t \} \\ &= S_t^k \left\{ \mu^k(t, \Xi_t) dt + \sum_{l=1}^d \sigma_{kl}(t, \Xi_t) dW_t^l \right\}, \quad t \in [0, T], \end{aligned}$$

where for each $k = 1, \dots, n$, $\mu^k : [0, T] \times \mathcal{E} \rightarrow \mathbb{R}$ is a given real (measurable) deterministic function and

$$\sigma^k(t, \alpha) = (\sigma_{k1}(t, \alpha), \dots, \sigma_{kd}(t, \alpha)),$$

where for each $l = 1, \dots, d$, $\sigma_{kl} : [0, T] \times \mathcal{E} \rightarrow \mathbb{R}$ is a given real (measurable) deterministic function. Furthermore, we denote the appreciation rate vector and the volatility matrix by

$$\mu(t, \alpha) = (\mu^1(t, \alpha), \dots, \mu^n(t, \alpha))^T$$

and

$$\sigma(t, \alpha) = \begin{pmatrix} \sigma^1(t, \alpha) \\ \vdots \\ \sigma^n(t, \alpha) \end{pmatrix} = (\sigma_{kl}(t, \alpha))_{n \times d}$$

respectively, for each $\alpha \in \mathcal{E}$. Moreover, we denote

$$\sigma(t, \alpha) \sigma(t, \alpha)^T = a(t, \alpha).$$

The following assumptions are made throughout this chapter.

3.1. MARKET MODEL AND BASIC PROBLEM

Assumption 2. All the (deterministic) functions $\mu^k(\cdot, \alpha)$, $\sigma_{kl}(\cdot, \alpha)$ are bounded, for $k = 1, \dots, n$, $l = 1, \dots, d$ and $\alpha \in \mathcal{E}$.

Assumption 3 (Non-degeneracy Condition). There exist some $\kappa > 0$, such that

$$\xi^T \sigma(t, \alpha) \sigma(t, \alpha)^T \xi \geq \kappa |\xi|^2, \quad \forall t \in [0, T], \alpha \in \mathcal{E} \text{ and } \xi \in \mathbb{R}^n.$$

We construct the self-financing portfolio by trading on the bond and stocks without any consumption. Suppose a time $t \in [0, T]$ is fixed temporarily and the market state at time t is $\Xi_t = \alpha \in \mathcal{E}$; moreover, the capital at time t is $x > 0$ and the total wealth at time $s \in [t, T]$ is x_s . Furthermore, we assume that trading takes place continuously and transaction cost is not considered. We use the so-called *dollar representation* of the trading strategy. Denoting the amount of capital invested in k -th risky asset S^k at time s by π_s^k , the dynamics of the wealth process corresponding to the trading strategy $\pi. = (\pi^1, \dots, \pi^n)^T$ is

$$\left\{ \begin{array}{l} dx_s^{(t,x),\pi} = \pi_s^T \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\} \\ \quad = \sum_{k=1}^n \mu^k(s, \Xi_s) \pi_s^k ds + \sum_{k=1}^n \sum_{l=1}^d \sigma_{kl}(s, \Xi_s) \pi_s^k dW_s^l, \\ x_t = x, \quad \Xi_t = \alpha. \end{array} \right.$$

In particular, the corresponding terminal wealth $x_T^{(t,x),\pi}$ is

$$x_T^{(t,x),\pi} = x + \int_t^T \pi_s^T \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\}. \quad (3.1)$$

As clarified in Chapter 2, we always use boldface letter, say $\pi.$, to denote a control process or a control policy law (a function), and use ordinary letter without boldface, say π , to denote a possible value (an n -dimensional vector, for the current case) of the control process.

A trading strategy is said to be *admissible* if $\pi. \in \mathcal{L}_{\mathcal{F}}^2(0, T; \mathbb{R}^n)$; furthermore, a trading policy law is said to be *admissible* if its corresponding strategy is admissible. It is well-known that this *admissible* trading strategy rules out the arbitrage opportunity (see, for instance, [41]).

3.1.2 The Basic Problem

In this chapter, the problem in concern is to “maximise” (in the equilibrium sense) the investor’s expected utility of her terminal wealth. The investor looks for a trading strategy to achieve the maximum expected utility of her terminal wealth x_T . Different from the assumptions in the standard literature, however, we assume the utility function of the investor for her terminal wealth is *regime dependent*. As the financial market state changes from one time to another, the

utility function of the investor may change. More precisely, the investor's utility function for her *terminal* wealth at any intermediate time $t \in [0, T]$ is $U_t(x) = U(x, \Xi_t)$, where Ξ_t is the market state at time t and $\{U(x, \alpha)\}_{\alpha \in \mathcal{E}}$ is a set of utility functions. Then, given the market state α and the capital x at time t , the objective functional at time t corresponding to the trading strategy $\{\pi_s\}_{s \in [t, T]}$ is

$$J(t, x, \alpha; \pi) = \mathbb{E} \left[U(x_T^\pi, \alpha) \mid x_t = x, \Xi_t = \alpha \right].$$

We emphasise that, as explained in Chapter 2, the underlying stochastic control problems for the utility maximisation are *time inconsistent* due to the regime dependence of the utility function $U_t(x) = U(x, \Xi_t)$. We will solve this time inconsistent utility maximisation problem via the *equilibrium* approach defined in Chapter 2; more precisely, we will look for the *equilibrium policy laws* for both time inconsistent CARA and CRRA utility maximisation problems in this chapter. The precise definition of the *equilibrium policy law* is given in Definition 1 in Chapter 2.

3.2 Case of CARA (Exponential) Utility

In this section, the CARA utility case is studied; more precisely, we assume $U(x, \alpha)$ in Section 3.1 takes the following form

$$U(x, \alpha) = -\exp(-\gamma(\alpha)x),$$

where $\gamma : \mathcal{E} \rightarrow \mathbb{R}_+$ is the risk aversion parameter, which is no longer a constant but a function of the market state. For each state $\alpha \in \mathcal{E}$, $\gamma(\alpha) > 0$ stands for the investor's risk aversion parameter corresponding to the market state α .

Given the market state α and the capital x at time t , the objective functional at time t corresponding to the trading strategy $\{\pi_s\}_{s \in [t, T]}$ is

$$J(t, x, \alpha; \pi) = \mathbb{E} \left[-\exp(-\gamma(\alpha)x_T^\pi) \mid x_t = x, \Xi_t = \alpha \right].$$

We now try to find the *equilibrium policy law* for this time inconsistent CARA utility maximisation problem.

3.2.1 Extended Hamilton-Jacobi-Bellman Equation

Recall that the terminal wealth corresponding to the strategy π is given by (3.1). Together with the separable property of the exponential function that $e^{x_1+x_2} = e^{x_1}e^{x_2}$ for any real numbers

3.2. CASE OF CARA (EXPONENTIAL) UTILITY

x_1 and x_2 , in this thesis we only consider wealth-independent equilibrium policy law; in other words, we assume that the equilibrium policy law takes the following form

$$\hat{\pi}(t, x, \alpha) = \hat{\pi}(t, \alpha).$$

We assume a priori that a wealth-independent equilibrium policy law $\hat{\pi}(t, \alpha)$ exists and let $\hat{x}$. be the corresponding (*equilibrium*) *wealth process*. We now introduce the following *auxiliary function*

$$\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha) = \mathbb{E} \left[\exp \left(-\gamma(\alpha) \int_t^T \hat{\pi}(s, \Xi_s)^T \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\} \right) \middle| \Xi_t = \beta \right].$$

Remark 2. There are *two* market states involved in the auxiliary function $\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha)$. Intuitively, the first state β stands for the *current* state of the market, and the second state α stands for the *initial* state of the market, i.e., the market state for some initial time *before* time t .

By definition, we have

$$\hat{V}^{\text{CARA}}(t, x, \alpha) = -e^{-\gamma(\alpha)x} \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha).$$

Moreover, for any (t, x, α) , any $h > 0$ and any $\pi \in \mathbb{R}^n$, the objective functional for the perturbed strategy $\pi^{\cdot, t, h, \pi}$ (see Definition 1 in Chapter 2 for reference) is

$$\begin{aligned} J(t, x, \alpha; \pi^{\cdot, t, h, \pi}) &= \mathbb{E} \left[-\exp \left(-\gamma(\alpha) x_T^{\pi^{\cdot, t, h, \pi}} \right) \middle| x_t = x, \Xi_t = \alpha \right] \\ &= -e^{-\gamma(\alpha)x} \mathbb{E} \left[e^{-\gamma(\alpha) \int_t^{t+h} \pi^T \{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \}} \mathcal{A}^{\hat{\pi}}(t+h, \Xi_{t+h}, \alpha) \middle| \Xi_t = \alpha \right]. \end{aligned}$$

Therefore, the condition

$$\begin{aligned} &\liminf_{h \rightarrow 0+} \frac{J(t, x, \alpha; \hat{\pi}) - J(t, x, \alpha; \pi^{\cdot, t, h, \pi})}{h} \\ &= \liminf_{h \rightarrow 0+} \frac{\mathbb{E} \left[e^{-\gamma(\alpha) \int_t^{t+h} \pi^T \{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \}} \mathcal{A}^{\hat{\pi}}(t+h, \Xi_{t+h}, \alpha) \middle| \Xi_t = \alpha \right] - \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha)}{h} \geq 0 \end{aligned}$$

leads to the following inequality

$$\begin{aligned} \mathcal{A}_t^{\hat{\pi}}(t, \alpha, \alpha) - \gamma(\alpha) \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \pi^T \mu(t, \alpha) + \frac{1}{2} \gamma(\alpha)^2 \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \pi^T a(t, \alpha) \pi \\ + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\pi}}(t, l, \alpha) - \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \right\} \geq 0, \end{aligned}$$

according to the generalised Dynkin's formula (see in Appendix A). Moreover, for this *equilibrium policy law* $\hat{\pi}(t, \alpha)$, the above inequality becomes an equality. Therefore, we get the following

extended Hamilton-Jacobi-Bellman (HJB) equation

$$\left\{ \begin{aligned} & \mathcal{A}_t^{\hat{\pi}}(t, \alpha, \alpha) + \gamma(\alpha) \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \inf_{\pi} \left\{ \frac{1}{2} \gamma(\alpha) \pi^T a(t, \alpha) \pi - \pi^T \mu(t, \alpha) \right\} \\ & + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\pi}}(t, l, \alpha) - \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \right\} = 0; \\ & \mathcal{A}^{\hat{\pi}}(T, \alpha, \alpha) = 1. \end{aligned} \right. \quad (3.2)$$

The first-order condition then gives the minimizer

$$\hat{\pi}(t, \alpha) = \frac{1}{\gamma(\alpha)} a(t, \alpha)^{-1} \mu(t, \alpha).$$

Notice that Assumption 3 guarantees that $a(t, \alpha)$ is invertible for any (t, α) .

To characterise the auxiliary function $\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha)$ and the equilibrium value function $\hat{V}^{\text{CARA}}(t, x, \alpha)$ corresponding to $\hat{\pi}(t, \alpha)$, we notice that, by definition, $\mathcal{A}^{\hat{\pi}}(T, \beta, \alpha) = 1$ for any $\beta, \alpha \in \mathcal{E}$ and

$$\exp \left(-\gamma(\alpha) \int_t^u \hat{\pi}(s, \Xi_s)^T \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\} \right) \mathcal{A}^{\hat{\pi}}(u, \Xi_u, \alpha)$$

is a martingale for $u \in [t, T]$ given $\Xi_t = \beta$. Then, via the *Feynman-Kac* approach (see, for instance, [48]), we can characterise $\{\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha)\}_{\alpha, \beta \in \mathcal{E}}$ by the following ordinary differential equations (ODE)

$$\begin{aligned} \mathcal{A}_t^{\hat{\pi}}(t, \beta, \alpha) + \frac{\gamma(\alpha)}{\gamma(\beta)} \left\{ \frac{\gamma(\alpha)}{2\gamma(\beta)} - 1 \right\} \mu(t, \beta)^T a(t, \beta)^{-1} \mu(t, \beta) \mathcal{A}^{\hat{\pi}}(t, \beta, \alpha) \\ + \sum_{\substack{l=1 \\ l \neq \beta}}^K \lambda_{\beta l} \left\{ \mathcal{A}^{\hat{\pi}}(t, l, \alpha) - \mathcal{A}^{\hat{\pi}}(t, \beta, \alpha) \right\} = 0 \end{aligned}$$

with terminal condition $\mathcal{A}^{\hat{\pi}}(T, \beta, \alpha) = 1$ for any $\alpha, \beta \in \mathcal{E}$. Notice that by taking $\beta = \alpha$ we get

$$\left\{ \begin{aligned} & \mathcal{A}_t^{\hat{\pi}}(t, \alpha, \alpha) - \frac{1}{2} \mu(s, \alpha)^T a(s, \alpha)^{-1} \mu(s, \alpha) \mathcal{A}^{\hat{\pi}}(s, \alpha, \alpha) \\ & + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\pi}}(t, l, \alpha) - \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \right\} = 0; \\ & \mathcal{A}^{\hat{\pi}}(T, \alpha, \alpha) = 1. \end{aligned} \right.$$

This recovers the previous *extended HJB equation* for $\mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha)$ if we substitute the minimizer

$$\hat{\pi}(t, \alpha) = \frac{1}{\gamma(\alpha)} a(t, \alpha)^{-1} \mu(t, \alpha)$$

back into (3.2).

3.2. CASE OF CARA (EXPONENTIAL) UTILITY

3.2.2 A Verification Theorem

We now provide a verification theorem corresponding to this time inconsistent CARA utility maximisation problem.

Consider the ODE system

$$\left\{ \begin{array}{l} \mathcal{A}_t^{\hat{\pi}}(t, \alpha, \alpha) + \gamma(\alpha) \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \inf_{\pi} \left\{ \frac{1}{2} \gamma(\alpha) \pi^T a(t, \alpha) \pi - \pi^T \mu(t, \alpha) \right\} \\ \quad + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\pi}}(t, l, \alpha) - \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \right\} = 0; \\ \mathcal{A}_t^{\hat{\pi}}(t, \beta, \alpha) + \gamma(\alpha) \mathcal{A}^{\hat{\pi}}(t, \beta, \alpha) \left\{ \frac{1}{2} \gamma(\alpha) \hat{\pi}(t, \beta)^T a(t, \beta) \hat{\pi}(t, \beta) - \hat{\pi}(t, \beta)^T \mu(t, \beta) \right\} \\ \quad + \sum_{\substack{l=1 \\ l \neq \beta}}^K \lambda_{\beta l} \left\{ \mathcal{A}^{\hat{\pi}}(t, l, \alpha) - \mathcal{A}^{\hat{\pi}}(t, \beta, \alpha) \right\} = 0; \end{array} \right. \quad \mathcal{A}^{\hat{\pi}}(T, \beta, \alpha) = 1, \quad \alpha, \beta \in \mathcal{E} \quad (3.3)$$

we just derived, where $\hat{\pi}(t, \beta)$ attains the infimum in the first equation for $\mathcal{A}(t, \beta, \beta)$. We now state the verification theorem.

Theorem 4. *Suppose the trading strategy corresponding to the policy law $\hat{\pi}(t, \alpha)$ attaining the infimum in the first equation for $\mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha)$ in (3.3) is admissible; moreover, for each α and β in $\mathcal{E}$, there exists a $\mathcal{C}^1([0, T])$ -function $\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha)$ such that $\{\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha)\}_{\alpha, \beta \in \mathcal{E}}$ is the solution of the system (3.3). Then,*

$$\hat{\pi}(t, \alpha) = \frac{1}{\gamma(\alpha)} a(t, \alpha)^{-1} \mu(t, \alpha) \quad (3.4)$$

is an equilibrium policy law and

$$\hat{V}^{\text{CARA}}(t, x, \alpha) = -e^{-\gamma(\alpha)x} \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \quad (3.5)$$

is the corresponding equilibrium value function, for any $t \in [0, T)$, $x \in \mathbb{R}_+$ and $\alpha \in \mathcal{E}$.

Proof. First of all, thanks to Assumption 2 and 3 we see that $\hat{\pi}(t, \alpha)$ is bounded. Moreover, according to the generalised Dynkin's formula and system (3.3),

$$\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha) = \mathbb{E} \left[\exp \left(-\gamma(\alpha) \int_t^u \hat{\pi}(s, \Xi_s)^T \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\} \right) \mathcal{A}^{\hat{\pi}}(u, \Xi_u, \alpha) \middle| \Xi_t = \beta \right].$$

Letting $u \rightarrow T$ together with the terminal condition $\mathcal{A}(T, \beta, \alpha) = 1$, we then have

$$\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha) = \mathbb{E} \left[\exp \left(-\gamma(\alpha) \int_t^T \hat{\pi}(s, \Xi_s)^T \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\} \right) \middle| \Xi_t = \beta \right].$$

Notice that, thanks to Lemma 5 below, we can apply Lebesgue's dominated convergence theorem here.

Next, for any (*fixed*) (t, x, α) , any $h > 0$ and any $\pi \in \mathbb{R}^n$, we define the perturbed policy law

$$\pi^{t,h,\pi}(s, \alpha) = \begin{cases} \pi, & s \in [t, t+h), \\ \hat{\pi}(s, \alpha), & s \in [t+h, T) \end{cases}$$

in the same manner as in Definition 1 in Chapter 2. Then by the generalised Dynkin's formula,

$$\begin{aligned} & \liminf_{h \rightarrow 0+} \frac{J(t, x, \alpha; \hat{\pi}) - J(t, x, \alpha; \pi^{t,h,\pi})}{h} \\ &= e^{-\gamma(\alpha)x} \liminf_{h \rightarrow 0+} \frac{\mathbb{E} \left[e^{-\gamma(\alpha) \int_t^{t+h} \pi^T \{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \}} \mathcal{A}^{\hat{\pi}}(t+h, \Xi_{t+h}, \alpha) \middle| \Xi_t = \alpha \right] - \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha)}{h} \\ &= e^{-\gamma(\alpha)x} \left\{ \mathcal{A}_t^{\hat{\pi}}(t, \alpha, \alpha) + \gamma(\alpha) \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \left\{ \frac{1}{2} \gamma(\alpha) \pi^T a(t, \alpha) \pi - \pi^T \mu(t, \alpha) \right\} \right. \\ & \quad \left. + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\pi}}(t, l, \alpha) - \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha) \right\} \right\} \geq 0 \end{aligned}$$

according to (3.3). Thus by definition, $\{\hat{\pi}(t, \alpha)\}_{t \in [0, T], \alpha \in \mathcal{E}}$ is an *equilibrium policy law* and

$$\hat{V}^{\text{CARA}}(t, x, \alpha) = -e^{-\gamma(\alpha)x} \mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha)$$

is the corresponding *equilibrium value function*, for any $(t, x, \alpha) \in [0, T] \times \mathbb{R}_+ \times \mathcal{E}$. $\square$

The following simple technical lemma is used in the proof of Theorem 4.

Lemma 5. *Assume Y satisfies the following SDE*

$$\begin{cases} dY_s = B_s ds + A_s dW_s; \\ Y_0 = y \in \mathbb{R}^n, \end{cases}$$

where $B_s \in \mathbb{R}^n$ and $A_s \in \mathbb{R}^{n \times d}$ are bounded $\{\mathcal{F}_t\}_{t \in [0, T]}$ -adapted processes. Then, for any $p \in \mathbb{R}^n$, it holds

$$\mathbb{E} [\exp(p^T Y_t)] < C, \quad \forall t \in [0, T],$$

where $C = C(y, T, p)$ depends only on y , T and p but is independent of $t \in [0, T]$.

Proof. We denote $Z_t^p = e^{p^T Y_t}$ for any $p \in \mathbb{R}^n$. By Itô's formula,

$$Z_t^p = e^{p^T y} + \int_0^t Z_s^p \left\{ p^T B_s + \frac{1}{2} p^T A_s A_s^T p \right\} ds + \int_0^t Z_s^p p^T A_s dW_s.$$

3.2. CASE OF CARA (EXPONENTIAL) UTILITY

Thus,

$$\begin{aligned}
\mathbb{E} \left[|Z_t^p|^2 \right] &\leq 3e^{p^\top y} + 3\mathbb{E} \left[\left| \int_0^t Z_s^p \left\{ p^\top B_s + \frac{1}{2} p^\top A_s A_s^\top p \right\} ds \right|^2 \right] + 3\mathbb{E} \left[\left| \int_0^t Z_s^p p^\top A_s dW_s \right|^2 \right] \\
&\leq 3e^{p^\top y} + 3T\mathbb{E} \left[\int_0^t |Z_s^p|^2 \left| p^\top B_s + \frac{1}{2} p^\top A_s A_s^\top p \right|^2 ds \right] + 3\mathbb{E} \left[\sup_{0 \leq u \leq t} \left| \int_0^u Z_s^p p^\top A_s dW_s \right|^2 \right] \\
&\leq 3e^{p^\top y} + K(p, T)\mathbb{E} \left[\int_0^t |Z_s^p|^2 ds \right] + 12\mathbb{E} \left[\int_0^t |Z_s^p|^2 |p^\top A_s A_s^\top p|^2 ds \right] \\
&\leq 3e^{p^\top y} + K(p, T)\mathbb{E} \left[\int_0^t |Z_s^p|^2 ds \right],
\end{aligned}$$

where $K(a)$ stands for a constant depends on some parameter vector a , which may be different from line to line. Notice that we use *Doob inequality* (see, for example, [48]) in the third step. Then, following *Gronwall's inequality*, we have

$$\mathbb{E} \left[|Z_t^p|^2 \right] \leq 3e^{p^\top y} + 3e^{p^\top y} K(p, T) \int_0^t e^{K(p, T)(t-s)} ds = K(p, T, y).$$

Notice that (by definition of Z_t^p)

$$\mathbb{E} \left[|Z_t^p|^2 \right] = \mathbb{E} \left[|Z_t^{2p}| \right],$$

we then get the result. $\square$

We see that

$$\hat{\pi}(t, \alpha) = \frac{1}{\gamma(\alpha)} a(t, \alpha)^{-1} \mu(t, \alpha)$$

is a *well-defined* measurable (deterministic) function with respect to t , for each $\alpha \in \mathcal{E}$. Together with Assumption 2, we see that $\hat{\pi}(t, \alpha)$ is *admissible* indeed.

We now turn to solve $\mathcal{A}^{\hat{\pi}}(\cdot, \beta, \alpha)$. For each $\alpha \in \mathcal{E}$, we see that only $\mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha)$ is the one we are really interested in for it relates to the equilibrium value function $\hat{V}^{\text{CARA}}(t, x, \alpha)$. However, $\mathcal{A}^{\hat{\pi}}(t, \alpha, \alpha)$ is coupled with the other auxiliary functions $\mathcal{A}^{\hat{\pi}}(t, \beta, \alpha)$ for $\beta \in \mathcal{E}$ (notice now the second variable in state, α , is *fixed!*), so that we need to solve a K -dimensional system of ODEs to get the whole vector $(\mathcal{A}^{\hat{\pi}}(t, 1, \alpha), \dots, \mathcal{A}^{\hat{\pi}}(t, K, \alpha))^\top$. To this end, recall that the ODE system for $\mathcal{A}^{\hat{\pi}}(\cdot, \beta, \alpha)$ is

$$\begin{cases} \mathcal{A}_t^{\hat{\pi}}(t, \beta, \alpha) = \left\{ -\frac{\gamma(\alpha)}{\gamma(\beta)} \left(\frac{\gamma(\alpha)}{2\gamma(\beta)} - 1 \right) \eta^2(t, \beta) + \sum_{\substack{l=1 \\ l \neq \beta}}^K \lambda_{\beta l} \right\} \mathcal{A}^{\hat{\pi}}(t, \beta, \alpha) - \sum_{\substack{l=1 \\ l \neq \beta}}^K \lambda_{\beta l} \mathcal{A}^{\hat{\pi}}(t, l, \alpha); \\ \mathcal{A}^{\hat{\pi}}(T, \beta, \alpha) = 1, \end{cases}$$

where $\eta(t, \beta) = \sqrt{\mu(t, \beta)^T a(t, \beta)^{-1} \mu(t, \beta)}$ is scalar. Thus, $(\mathcal{A}^{\hat{\pi}}(t, 1, \alpha), \dots, \mathcal{A}^{\hat{\pi}}(t, K, \alpha))^T$ solves the following linear homogeneous ODE

$$\begin{cases} \dot{X}(t) = A_{\alpha}^{\text{CARA}}(t)X(t); \\ X(T) = (1, \dots, 1)^T \in \mathbb{R}^K, \end{cases}$$

where $X(t) = (X^1(t), \dots, X^K(t))^T$ and

$$A_{\alpha}^{\text{CARA}}(t) = \begin{pmatrix} -\rho_{\alpha}(1)\eta^2(t, 1) + \sum_{l \neq 1} \lambda_{1l} & -\lambda_{12} & \cdots & -\lambda_{1K} \\ -\lambda_{21} & -\rho_{\alpha}(2)\eta^2(t, 2) + \sum_{l \neq 2} \lambda_{2l} & \cdots & -\lambda_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_{K1} & \lambda_{K2} & \cdots & -\rho_{\alpha}(K)\eta^2(t, K) + \sum_{l \neq K} \lambda_{Kl} \end{pmatrix}$$

in which we use the notation $\rho_{\alpha}(k) = \frac{\gamma(\alpha)(\gamma(\alpha) - 2\gamma(k))}{2\gamma(k)^2}$, for each $k \in \mathcal{E}$. By the standard ODE theory, we know that there exists a unique solution

$$X(t) = \exp\left(-\int_t^T A_{\alpha}^{\text{CARA}}(s)ds\right) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$$

to the above equation, where $\exp(\cdot)$ is understood as the matrix function.

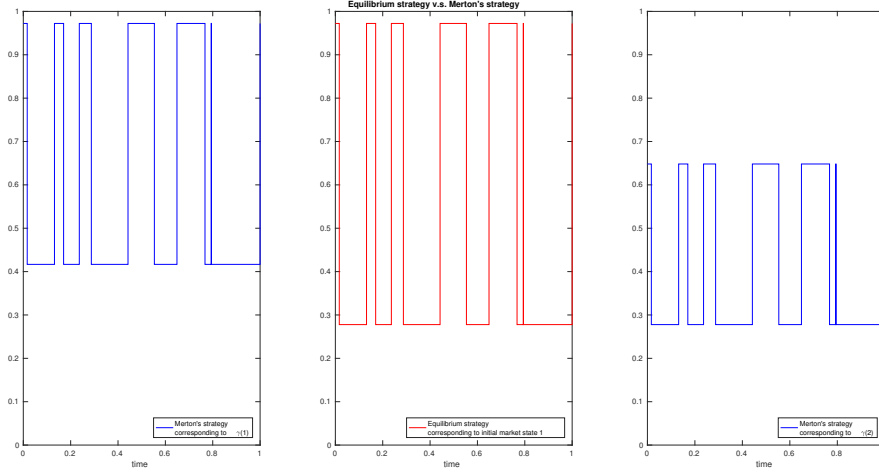
We now report the results of some numerical experiments on the *equilibrium* strategy under the simplification that the number of stocks in the market is one. In Figure 3.1, we provide a simulation of the equilibrium trading strategy and the Merton's trading strategies corresponding to the risk aversion parameters $\gamma(1)$ and $\gamma(2)$ for the initial condition $\Xi_0 = 1$. It will be clear in the next subsection that the Merton's strategy is exactly the *precommitment* strategy with respect to the reference time $t = 0$. We take the time horizon $T = 1$ and the constant market parameters $\mu(t, 1) = 0.07$, $\mu(t, 2) = 0.03$, $\sigma(t, 1) = \sigma(t, 2) = 0.3$, $\lambda_{12} = 0.08$ and $\lambda_{21} = 0.12$. The risk aversion parameters are $\gamma(1) = 0.8$ and $\gamma(2) = 1.2$.

From Figure 3.1, we can clearly find that the strategy paths are all *step* functions. This is intuitive for that the optimal trading strategy for the classical Merton's exponential utility maximisation problem is constant when the market is driven by a geometric Brownian motion. Since the market is driven by a *switched* geometric Brownian motion now and the market parameters change when the general market condition changes, we should adjust our positions in the risk asset accordingly.

Moreover, the jump times for all three trading strategies are the same and coincide with the times at which the market condition changes, i.e., the jump times of the underlying CTMC Ξ ; in other words, trading happens when the market condition changes. However, the magnitudes

3.2. CASE OF CARA (EXPONENTIAL) UTILITY

Figure 3.1: The Comparison between Equilibrium Strategy and Merton's Strategy.



Notes. The figure illustrates the equilibrium trading strategy and the Merton's trading strategies for risk aversion parameter $\gamma(1)$ and $\gamma(2)$, under initial condition $\Xi_0 = 1$. Model parameters: $n = d = 1$, $T = 1$, $\mu(t, 1) = 0.07$, $\mu(t, 2) = 0.03$, $\sigma(t, 1) = \sigma(t, 2) = 0.3$, $\lambda_{12} = 0.08$, $\lambda_{21} = 0.12$, $\gamma(1) = 0.8$, $\gamma(2) = 1.2$.

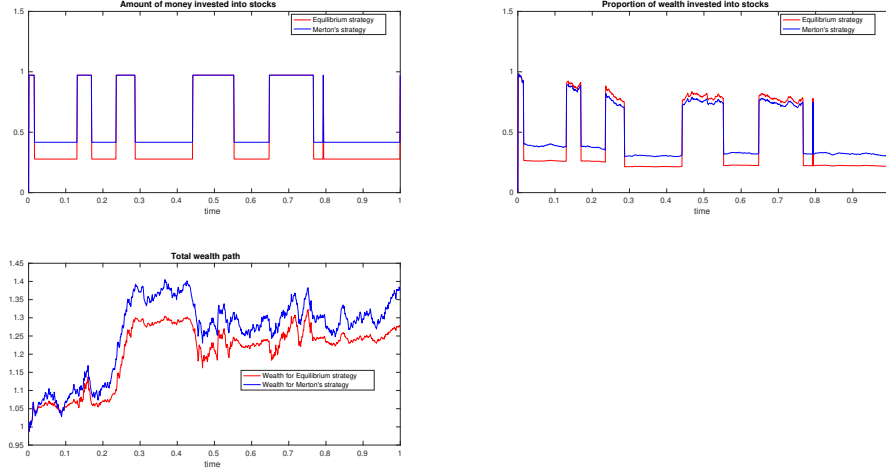
of jumps are different for the equilibrium strategy and the Merton's strategy. The jump sizes of the equilibrium strategy are larger than the ones of the Merton's strategy. In fact, from (3.4) and (3.6) one can see that for the period that the market condition is 1, the equilibrium policy law coincides with the Merton's policy law corresponding to the risk aversion parameter $\gamma(1)$; for the period that the market condition is 2, the equilibrium policy law coincides with the Merton's policy law corresponding to the risk aversion parameter $\gamma(2)$. In other words, the Merton's strategies corresponding to the risk aversion parameter $\gamma(1)$ and $\gamma(2)$ give the upper and lower bounds of the equilibrium strategy. This is clear from Figure 3.1.

In Figure 3.2 we provide some further simulation results for the *equilibrium* strategy and the Merton's strategy with the same initial market state $\Xi_0 = 1$. We show both the money amount and the proportion of the wealth invested in the stock for the two strategies; moreover, we give the simulation of the corresponding wealth paths.

3.2.3 Is A “Naive” Strategy Really Naive?

As mentioned in Chapter 2, there are at least two other approaches to deal with the time inconsistent control problem, known as the *precommitment* and the *naive* approaches. In this

Figure 3.2: The Simulation of Equilibrium Strategy and Merton's Strategy.



Notes. The figure illustrates the equilibrium and the Merton's trading strategies under initial condition $\Xi_0 = 1$. Model parameters: $n = d = 1$, $T = 1$, $\mu(t, 1) = 0.07$, $\mu(t, 2) = 0.03$, $\sigma(t, 1) = \sigma(t, 2) = 0.3$, $\lambda_{12} = 0.08$, $\lambda_{21} = 0.12$, $\gamma(1) = 0.8$, $\gamma(2) = 1.2$.

section we do some comparison between the *equilibrium* strategy and the *naive* strategy. Before stating the main result, we need to clarify the *naive* (and accordingly, the *precommitment*) approach unambiguously. We now give the precise definitions of the *precommitment* policy law and the *naive* policy law below. These definitions are based on [69] (also see, for instance, [21], [22] and [6]).

Definition 6. Suppose an arbitrary time $t \in [0, T)$ is *fixed* as the reference time. A left-continuous (deterministic) function $\bar{\pi}^t(\cdot, \cdot, \cdot) : [t, T) \times \mathbb{R}_+ \times \mathcal{E} \rightarrow \mathbb{R}$ is called the *precommitment* policy law with respect to *this* reference time t , if

$$J(t, x, \alpha; \bar{\pi}^t) = \sup_{\pi \in \mathcal{AD}[t, T)} \left\{ J(t, x, \alpha; \pi) \right\}.$$

where $\bar{\pi}^t$ is the control process corresponding to policy law $\bar{\pi}^t(\cdot, \cdot, \cdot)$. Moreover, $\bar{\pi}^t$, the corresponding control process, is called the *precommitment* strategy (for the reference time t) and $J(t, x, \alpha; \bar{\pi}^t)$ is called the *precommitment value function*.

Notice that in the definition of the *precommitment* policy law $\bar{\pi}^t(\cdot, x, \alpha)$ and strategy $\bar{\pi}^t$, we use the superscript “ t ” to emphasise that this policy law or strategy is “optimal” with respect to this time t as the reference time.

3.2. CASE OF CARA (EXPONENTIAL) UTILITY

For any *fixed* time $t \in [0, T]$ as the reference time, the corresponding *precommitment policy law* $\{\bar{\pi}^t(s, x, \alpha)\}_{s \in [t, T]}$ is the policy law such that its objective functional at the reference time t achieves the maximum among all the admissible strategies. More precisely, given $\Xi_t = \alpha$, if the investor's utility function for her terminal wealth is

$$U_t(x) = -\exp\left(-\gamma(\alpha)x\right),$$

then for the objective functional

$$\begin{aligned} J^t(s, x, \beta; \pi_\cdot) &= \mathbb{E} \left[U_t(x_T^\pi) \middle| x_s = x, \Xi_s = \beta \right] \\ &= \mathbb{E} \left[-\exp\left(-\gamma(\alpha)x_T^\pi\right) \middle| x_s = x, \Xi_s = \beta \right], \quad s \in [t, T] \end{aligned}$$

the *precommitment* policy law $\{\bar{\pi}^t(s, x, \alpha)\}_{s \in [t, T]}$ is *optimal* such that the corresponding objective functional $J^t(t, x, \alpha; \pi_\cdot)$ at the reference time t attains the maximum.

Given the definition of precommitment strategy, we can define the *naive* strategy accordingly (see, for instance, [69], [21] and [22]).

Definition 7. A left-continuous deterministic function $\pi^N(\cdot, \cdot, \cdot) : [0, T] \times \mathbb{R}_+ \times \mathcal{E} \rightarrow \mathbb{R}$ is called the *naive* policy law, if:

$$\pi^N(t, x, \alpha) = \bar{\pi}^t(t, x, \alpha), \quad \forall t \in [0, T],$$

where for any $t \in [0, T]$, $\bar{\pi}^t(\cdot, \cdot, \cdot)$ is the precommitment policy law with respect to this *specific* time t as the reference time. Moreover, π^N , the corresponding control process, is called the *naive* strategy.

Intuitively, the *naive* policy law is a policy law to adjoin the precommitment policy laws at different times. A person who adopts the *naive* policy law follows the precommitment policy law $\bar{\pi}^t(\cdot, x, \alpha)$ at each time $t \in [0, T]$; however, policy law $\bar{\pi}^t(\cdot, x, \alpha)$ is only implemented at this *single* time point t . At later time $t + h$, this person will turn to the precommitment policy law $\bar{\pi}^{t+h}(\cdot, x, \alpha)$ with respect to the new time $t + h$.

As explained in Chapter 2, the logic behind the *naive* approach is not solid for no “optimal” strategy will be fully implemented at any time instance. We now ask ourselves, is this *naive* approach really irrational? The solutions to the time inconsistent CARA utility maximisation problem with regime switching gives a counterexample. To this end, we first make an additional assumption.

Assumption 8. All the coefficients $\mu^k(\cdot, \alpha)$, $\sigma_{kl}(\cdot, \alpha)$ are left-continuous on $[0, T]$, for $k = 1, \dots, n$, $l = 1, \dots, d$ and $\alpha \in \mathcal{E}$.

For a fixed time t as the reference time and given the corresponding market condition $\Xi_t = \alpha$, the objective functional for the *precommitment* control problem corresponding to the strategy π is

$$J^{(t,\alpha)}(s, x, \beta; \pi) = \mathbb{E} \left[-\exp(-\gamma(\alpha)x_T^\pi) \mid x_s = x, \Xi_s = \beta \right], \quad s \in [t, T),$$

where x^π satisfies the following controlled SDE

$$\begin{cases} dx_u^\pi = \pi_u \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\}; \\ x_s = x, \quad \Xi_s = \beta. \end{cases}$$

Similarly, we emphasise the condition at the reference time t for this precommitment problem explicitly by the superscript (t, α) on the objective functional. Via the classical dynamic programming technique, one can easily get the *precommitment* policy law with respect to this reference time t and market condition $\Xi_t = \alpha$ to be

$$\bar{\pi}^{(t,\alpha)}(s, \beta) = \frac{1}{\gamma(\alpha)} a^{-1}(s, \beta) \mu(s, \beta), \quad s \in [t, T), \beta \in \mathcal{E}, \quad (3.6)$$

see Appendix B for the details. Thanks to Assumption 8, we now obtain the corresponding *naive* policy law to be

$$\pi^N(t, \alpha) = \bar{\pi}^{(t,\alpha)}(t, \alpha) = \frac{1}{\gamma(\alpha)} a^{-1}(t, \alpha) \mu(t, \alpha), \quad s \in [t, T), \alpha \in \mathcal{E}.$$

Interestingly, comparing this to (3.4) it turns out that the *naive* policy law *coincides* with the *equilibrium* policy law.

3.2.3.1 A Brief Explanation from Discrete-Time Analogy

We would like to point out that the structure of the current problem is critical for the coincidence of the equilibrium and the naive policy law. To illustrate this, let us make an explanation under a discrete-time two-stage model which will make everything clear. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the underlying probability space and the decision times are now $t = 0, 1$ and the terminal time is $T = 2$. On the probability space defined an exogenous $\mathcal{E}$ -valued process $\{\Xi_t\}_{t=0,1,2}$ and two real random variables R_1 and R_2 ; similar to the continuous-time model, process $\{\Xi_t\}_{t=0,1,2}$ models the market condition at time t , and R_1 and R_2 are the noises in the market at time 1 and 2 respectively. Both R_1 and R_2 are *independent* of $\{\Xi_t\}_{t=0,1,2}$; moreover, R_1 and R_2 are *independent* of each other also. The wealth at time t is denoted by x_t ; $\{x_t\}_{t=1,2}$ evolves according to the following equation

$$x_{t+1} = x_t + g(t, \mathbf{u}_t, \Xi_t, R_{t+1}), \quad \text{for } t = 0, 1,$$

3.2. CASE OF CARA (EXPONENTIAL) UTILITY

where $\mathbf{u}_t$, taking values in some subset $U \subset \mathbb{R}$, is the action we take at time t and $g : \{0, 1\} \times U \times \mathbb{R} \times \mathbb{R}$ is a given function. We use $x^{\mathbf{u}}$ to emphasise that the wealth process is controlled by the control process $\mathbf{u}$. The objective functional at time t given the wealth x and the market state α is

$$J(t, x, \alpha; \mathbf{u}) = \mathbb{E} \left[-e^{-\gamma(\alpha)x_T^{\mathbf{u}}} \middle| x_t = x, \Xi_t = \alpha \right], \quad \text{for } t = 0, 1,$$

where $\gamma : \mathcal{E} \rightarrow \mathbb{R}_+$ is the regime dependent risk aversion parameter. The problem is then to maximise the objective functional by choosing the proper strategy $\{\mathbf{u}_t\}_{t=0,1}$.

We first solve this problem via the *equilibrium* approach. To this end, we do the backward analysis (see, for instance, [6] and [7]) and consider the case $t = 1$ first. Notice that $x_2^{\mathbf{u}} = x_1 + g(1, \mathbf{u}_1, \Xi_1, R_2)$ and the current control problem is

$$\begin{aligned} \hat{V}(1, x, \alpha) &= \sup_{\mathbf{u}_1 \in U} \left\{ \mathbb{E} \left[-\exp \left(-\gamma(\Xi_1) \left\{ x + g(1, \mathbf{u}_1, \Xi_1, R_2) \right\} \right) \middle| x_1 = x, \Xi_1 = \alpha \right] \right\} \\ &= -e^{-\gamma(\alpha)x} \inf_{\mathbf{u}_1 \in U} \left\{ \mathbb{E} \left[\exp \left(-\gamma(\alpha)g(1, \mathbf{u}_1, \alpha, R_2) \right) \right] \right\}. \end{aligned}$$

This is a standard optimisation problem, and under some mild conditions, such as the concavity of function $g(1, u, \alpha, R)$ with respect to u , we can show that there exists a unique solution $\hat{u}_1 = \hat{\mathbf{u}}(1, \alpha)$. Notice that in this step, the *equilibrium* policy law coincides with the *precommitment* and the *naive* policy law automatically, since the underlying problem is a *static single-period* optimisation problem.

For the second step, we consider the case $t = 0$. Notice that we only need to choose the action at time $t = 0$ and, when we come to the time 1, policy law $\hat{\mathbf{u}}(1, \alpha)$ will be implemented by ourselves at that time. Together with $x_1^{\mathbf{u}} = x_0 + g(1, \mathbf{u}_0, \Xi_0, R_1)$, the current control problem becomes

$$\begin{aligned} \hat{V}(0, x, \alpha) &= \sup_{\mathbf{u}_0 \in U} \left\{ \mathbb{E} \left[-\exp \left(-\gamma(\Xi_0) \left\{ x + g(0, \mathbf{u}_0, \Xi_0, R_1) + g(1, \hat{\mathbf{u}}(1, \Xi_1), \Xi_1, R_2) \right\} \right) \middle| x_0 = x, \Xi_0 = \alpha \right] \right\} \\ &= -e^{-\gamma(\alpha)x} \inf_{\mathbf{u}_0 \in U} \left\{ \mathbb{E} \left[\exp \left(-\gamma(\alpha)g(0, \mathbf{u}_0, \alpha, R_1) \right) \right] \right\} \times \\ &\quad \mathbb{E} \left[\exp \left(-\gamma(\alpha)g(1, \hat{\mathbf{u}}(1, \Xi_1), \Xi_1, R_2) \right) \middle| \Xi_0 = \alpha \right], \end{aligned}$$

due to the *independence* of R_1 with Ξ_1 and R_2 . Therefore, essentially the optimisation problem we need to solve at $t = 0$ is

$$\inf_{\mathbf{u}_0 \in U} \left\{ \mathbb{E} \left[\exp \left(-\gamma(\alpha)g(0, \mathbf{u}_0, \alpha, R_1) \right) \right] \right\}.$$

This can be solved via the standard technique, and under some mild conditions, such as the concavity of function $g(0, u, \alpha, R)$ with respect to u , there is a unique solution $\hat{u}_0 = \hat{\mathbf{u}}(0, \alpha)$.

We next turn to solve the problem via the *naive* approach. As mentioned, for the case $t = 1$ the solutions for all three approaches are the same, and we thus go to the case $t = 0$ directly. To find the *naive* strategy $\mathbf{u}_0^N$, we solve the *precommitment* strategy $\bar{\mathbf{u}}^0$ first. Exploiting the dynamic programming technique, the control problem now is

$$\begin{aligned} \bar{V}^0(0, x, \alpha) &= \sup_{\mathbf{u}_0, \mathbf{u}_1 \in U} \left\{ \mathbb{E} \left[-\exp \left(-\gamma(\Xi_0) \left\{ x + g(0, \mathbf{u}_0, \Xi_0, R_1) + g(1, \mathbf{u}_1, \Xi_1, R_2) \right\} \right) \middle| x_0 = x, \Xi_0 = \alpha \right] \right\} \\ &= -e^{-\gamma(\alpha)x} \inf_{\mathbf{u}_0, \mathbf{u}_1 \in U} \left\{ \mathbb{E} \left[\exp \left(-\gamma(\alpha) g(0, \mathbf{u}_0, \alpha, R_1) - \gamma(\alpha) g(1, \mathbf{u}_1, \Xi_1, R_2) \right) \middle| \Xi_0 = \alpha \right] \right\} \\ &= -e^{-\gamma(\alpha)x} \inf_{\mathbf{u}_0 \in U} \left\{ \mathbb{E} \left[\exp \left(-\gamma(\alpha) g(0, \mathbf{u}_0, \alpha, R_1) \right) \right] \right\} \times \\ &\quad \mathbb{E} \left[\inf_{\mathbf{u}_1 \in U} \left\{ \mathbb{E} \left[\exp \left(-\gamma(\alpha) g(1, \mathbf{u}_1, \Xi_1, R_2) \right) \middle| \Xi_1 \right] \right\} \middle| \Xi_0 = \alpha \right], \end{aligned} \tag{3.7}$$

thanks to the independence of Ξ_1 , R_1 and R_2 . Notice that the optimisation problem to find the best action at time $t = 0$ (in *precommitment* sense),

$$\inf_{\mathbf{u}_0 \in U} \left\{ \mathbb{E} \left[\exp \left(-\gamma(\alpha) g(0, \mathbf{u}_0, \alpha, R_1) \right) \right] \right\},$$

is the same as the one for the *equilibrium* policy law at time $t = 0$. Therefore, $\mathbf{u}^N(0, \alpha) = \bar{\mathbf{u}}^0(0, \alpha) = \hat{\mathbf{u}}(0, \alpha)$, i.e., the *naive* policy law coincides with the *equilibrium* policy law.

Let us go back to (3.7). From (3.7), we see that the underlying stochastic control problem, via the precommitment approach, is reduced to a series of one-period optimisation problems. Moreover, the decision at each period has *no* influence on the strategies for later times (this is due to the exponential preference of the investor and the independence of the noise and market condition in the model). This implies the investor's strategy is *myopic*; at each time she needs only to choose the best action for the current period without considering about the future periods.

3.3 Case of CRRA (Power) Utility

In this section, we turn to discuss the CRRA utility case. More precisely, $U(x, \alpha)$ in Section 3.1 now takes the following form

$$U(x, \alpha) = \frac{1}{1 - \gamma(\Xi_t)} x^{1 - \gamma(\Xi_t)},$$

where, similar to Section 3.2, the risk aversion parameter $\gamma : \mathcal{E} \rightarrow (0, 1)$ also depends on the market state. At any time t with market state $\Xi_t = \alpha$, the investor's risk aversion parameter is $\gamma(\alpha) \in (0, 1)$.

3.3. CASE OF CRRA (POWER) UTILITY

In this section, we shall use a different form for the trading strategy; more precisely, we denote the proportion of total wealth invested in k -th risky asset S^k at time s by θ_s^k , and $\boldsymbol{\theta} = (\theta^1, \dots, \theta^n)^\top$ will be the control process in this section. This is known as the *proportion representation* of the trading strategy. The dynamic of the wealth process corresponding to the trading strategy $\boldsymbol{\theta}$ is

$$\begin{cases} dx_s^{(t,x),\boldsymbol{\theta}} = x_s^{(t,x),\boldsymbol{\theta}} \boldsymbol{\theta}_s^\top \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\} \\ \quad = x_s^{(t,x),\boldsymbol{\theta}} \left\{ \sum_{k=1}^n \mu^k(s, \Xi_s) \theta_s^k ds + \sum_{k=1}^n \sum_{l=1}^d \sigma_{kl}(s, \Xi_s) \theta_s^k dW_s^l \right\}, \\ x_t = x, \quad \Xi_t = \alpha, \end{cases}$$

In particular, the corresponding terminal wealth $x_T^{(t,x),\boldsymbol{\theta}}$ is

$$x_T^{(t,x),\boldsymbol{\theta}} = x \exp \left(\int_t^T \left\{ \boldsymbol{\theta}_s^\top \mu(s, \Xi_s) - \frac{1}{2} \boldsymbol{\theta}_s^\top a(s, \Xi_s) \boldsymbol{\theta}_s \right\} ds + \int_t^T \boldsymbol{\theta}_s^\top \sigma(s, \Xi_s) dW_s \right). \quad (3.8)$$

A trading strategy $\boldsymbol{\theta}$ is said to be *admissible* if $\boldsymbol{\theta}$ is $\mathbb{F}$ -progressively measurable and $\int_0^T \boldsymbol{\theta}_s^2 ds < \infty$, $\mathbb{P}$ -a.s.. It is well-known that this definition of admissibility also rules out the arbitrage (see, for instance, [41]).

Given the market state α and the capital x at time t , the objective functional corresponding to the trading strategy $\{\boldsymbol{\theta}_s\}_{s \in [t,T]}$ is

$$J(t, x, \alpha; \boldsymbol{\theta}) = \mathbb{E} \left[\frac{1}{1 - \gamma(\Xi_t)} (x_T^\boldsymbol{\theta})^{1 - \gamma(\Xi_t)} \middle| x_t = x, \Xi_t = \alpha \right].$$

We now deduce the corresponding *extended HJB equation* and gives the *equilibrium policy law* explicitly for this time inconsistent CRRA utility maximisation problem. A verification theorem is also stated.

3.3.1 Extended Hamilton-Jacobi-Bellman Equation

Recall that the terminal wealth corresponding to strategy $\boldsymbol{\theta}$ is given by (3.8). As in Section 3.2, we make the assumption that the equilibrium policy law is independent of the wealth variable x ; in other words, the equilibrium policy law takes the following form

$$\hat{\boldsymbol{\theta}}(t, x, \alpha) = \hat{\boldsymbol{\theta}}(t, \alpha).$$

Similar to the analysis in Section 3.2, we first assume a priori that a wealth-independent equilibrium policy law $\hat{\boldsymbol{\theta}}(t, \alpha)$ exists and denote $\hat{x}$ for the corresponding *equilibrium wealth*

process. We next introduce the following *auxiliary function*

$$\mathcal{A}^{\hat{\theta}}(t, \beta, \alpha) = \mathbb{E} \left[\exp \left((1 - \gamma(\alpha)) \int_t^T \left\{ \hat{\theta}(s, \Xi_s)^T \mu(s, \Xi_s) - \frac{1}{2} \hat{\theta}(s, \Xi_s)^T a(s, \Xi_s) \hat{\theta}(s, \Xi_s) \right\} ds \right. \right. \\ \left. \left. + (1 - \gamma(\alpha)) \int_t^T \hat{\theta}(s, \Xi_s)^T \sigma(s, \Xi_s) dW_s \right) \middle| \Xi_t = \beta \right].$$

Similar to the auxiliary function in Section 3.2, there are *two* market states involved in $\mathcal{A}^{\hat{\theta}}(t, \beta, \alpha)$. For the reader's understanding, the first state variable β stands for the market state at the current time t and the second state variable α represents the market state in some initial time *before* t .

By straightforward computation one has

$$\hat{V}^{\text{CRR}}(t, x, \alpha) = \frac{1}{1 - \gamma(\alpha)} x^{1 - \gamma(\alpha)} \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha)$$

and

$$J(t, x, \alpha; \boldsymbol{\theta}^{t, h, \theta}) = \mathbb{E} \left[\frac{1}{1 - \gamma(\alpha)} \left(x_T^{\boldsymbol{\theta}^{t, h, \theta}} \right)^{1 - \gamma(\alpha)} \middle| x_t = x, \Xi_t = \alpha \right] \\ = \frac{1}{1 - \gamma(\alpha)} x^{1 - \gamma(\alpha)} \mathbb{E} \left[e^{(1 - \gamma(\alpha)) \int_t^{t+h} \left\{ [\boldsymbol{\theta}^T \mu(s, \Xi_s) - \frac{1}{2} \boldsymbol{\theta}^T a(s, \Xi_s) \boldsymbol{\theta}] ds + \boldsymbol{\theta}^T \sigma(s, \Xi_s) dW_s \right\}} \mathcal{A}^{\hat{\theta}}(t + h, \Xi_{t+h}, \alpha) \middle| \Xi_t = \alpha \right].$$

for any (t, x, α) , any $h > 0$ and any $\theta \in \mathbb{R}^n$, where $\boldsymbol{\theta}^{t, h, \theta}$ is the perturbed strategy defined in the same manner as in Definition 1 in Chapter 2. Therefore,

$$\liminf_{h \rightarrow 0+} \frac{J(t, x, \alpha; \hat{\boldsymbol{\theta}}) - J(t, x, \alpha; \boldsymbol{\theta}^{t, h, \theta})}{h} \\ = - \limsup_{h \rightarrow 0+} \frac{\mathbb{E} \left[e^{(1 - \gamma(\alpha)) \int_t^{t+h} \left\{ [\boldsymbol{\theta}^T \mu(s, \Xi_s) - \frac{1}{2} \boldsymbol{\theta}^T a(s, \Xi_s) \boldsymbol{\theta}] ds + \boldsymbol{\theta}^T \sigma(s, \Xi_s) dW_s \right\}} \mathcal{A}^{\hat{\theta}}(t + h, \Xi_{t+h}, \alpha) \middle| \Xi_t = \alpha \right] - \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha)}{h} \\ \geq 0$$

leads to the following inequality

$$\mathcal{A}_t^{\hat{\theta}}(t, \alpha, \alpha) + (1 - \gamma(\alpha)) \left\{ \boldsymbol{\theta}^T \mu(s, \beta) - \frac{1}{2} \gamma(\alpha) \boldsymbol{\theta}^T a(s, \beta) \boldsymbol{\theta} \right\} \mathcal{A}^{\hat{\theta}}(s, \beta, \alpha) \\ + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\theta}}(t, l, \alpha) - \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha) \right\} \leq 0$$

by the generalised Dynkin's formula. Moreover, for the *equilibrium policy law* $\hat{\boldsymbol{\theta}}(t, \alpha)$, the above inequality becomes to an equality. Therefore, we get the following *extended Hamilton-Jacobi-*

3.3. CASE OF CRRA (POWER) UTILITY

Bellman (HJB) equation

$$\left\{ \begin{aligned} & \mathcal{A}_t^{\hat{\theta}}(t, \alpha, \alpha) + (1 - \gamma(\alpha)) \sup_{\theta} \left\{ -\frac{1}{2} \gamma(\alpha) \theta^T a(t, \alpha) \theta + \theta^T \mu(t, \alpha) \right\} \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha) \\ & + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\theta}}(t, l, \alpha) - \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha) \right\} = 0; \end{aligned} \right. \quad (3.9)$$

$$\mathcal{A}(T, \alpha, \alpha) = 1.$$

The first-order condition then gives the maximiser

$$\hat{\theta}(t, \alpha) = \frac{1}{\gamma(\alpha)} a(t, \alpha)^{-1} \mu(t, \alpha).$$

Finally, we turn to characterise the auxiliary function $\mathcal{A}^{\hat{\theta}}(t, \beta, \alpha)$ and the *equilibrium value function* $\hat{V}^{\text{CRRA}}(t, x, \alpha)$. From the definition of $\mathcal{A}^{\hat{\theta}}(t, \beta, \alpha)$ one has $\mathcal{A}^{\hat{\theta}}(T, \beta, \alpha) = 1$ for any $\beta, \alpha \in \mathcal{E}$ and

$$\exp \left((1 - \gamma(\alpha)) \int_t^u \left\{ \left[\hat{\theta}(s, \Xi_s)^T \mu(s, \Xi_s) - \frac{1}{2} \hat{\theta}(s, \Xi_s)^T a(s, \Xi_s) \hat{\theta}(s, \Xi_s) \right] ds + \hat{\theta}(s, \Xi_s)^T \sigma(s, \Xi_s) dW_s \right\} \right) \mathcal{A}(u, \Xi_u, \alpha)$$

is a martingale for $u \in [t, T]$ given $\Xi_t = \beta$. Then, via the *Feynman-Kac* approach, we have the following ODE system

$$\left\{ \begin{aligned} & \mathcal{A}_t^{\hat{\theta}}(t, \beta, \alpha) + \frac{1 - \gamma(\alpha)}{\gamma(\beta)} \left(1 - \frac{\gamma(\alpha)}{2\gamma(\beta)} \right) \mu(t, \beta)^T a(t, \beta)^{-1} \mu(t, \beta) \mathcal{A}^{\hat{\theta}}(t, \beta, \alpha) \\ & + \sum_{\substack{l=1 \\ l \neq \beta}}^K \lambda_{\beta l} \left\{ \mathcal{A}^{\hat{\theta}}(t, l, \alpha) - \mathcal{A}^{\hat{\theta}}(t, \beta, \alpha) \right\} = 0, \end{aligned} \right. \quad \mathcal{A}^{\hat{\theta}}(T, \beta, \alpha) = 1$$

for $\alpha, \beta \in \mathcal{E}$. Notice that by taking $\beta = \alpha$ we get

$$\left\{ \begin{aligned} & \mathcal{A}_t^{\hat{\theta}}(t, \alpha, \alpha) + \frac{1 - \gamma(\alpha)}{2\gamma(\alpha)} \mu(t, \alpha)^T a(t, \alpha)^{-1} \mu(t, \alpha) \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha) \\ & + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\theta}}(t, l, \alpha) - \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha) \right\} = 0; \end{aligned} \right. \quad \mathcal{A}^{\hat{\theta}}(T, \alpha, \alpha) = 1$$

and this recovers the previous *extended HJB equation* for $\mathcal{A}(t, \alpha, \alpha)$ if we substitute the maximiser

$$\hat{\theta}(t, \alpha) = \frac{1}{\gamma(\alpha)} a(t, \alpha)^{-1} \mu(t, \alpha)$$

back into (3.9).

3.3.2 A Verification Theorem

We now provided a verification theorem. The proof is parallel to Theorem 4 and thus omitted.

Consider the ODE system

$$\left\{ \begin{array}{l} \mathcal{A}_t^{\hat{\theta}}(t, \alpha, \alpha) + \gamma(\alpha) \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha) \sup_{\theta} \left\{ -\frac{\gamma(\alpha)}{2} \theta^T a(t, \alpha) \theta + \theta^T \mu(t, \alpha) \right\} \\ \quad + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^{\hat{\theta}}(t, l, \alpha) - \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha) \right\} = 0; \\ \mathcal{A}_t^{\hat{\theta}}(t, \beta, \alpha) + \gamma(\alpha) \mathcal{A}^{\hat{\theta}}(t, \beta, \alpha) \left\{ -\frac{\gamma(\alpha)}{2} \hat{\theta}(t, \beta)^T a(t, \beta) \hat{\theta}(t, \beta) + \hat{\theta}(t, \beta)^T \mu(t, \beta) \right\} \\ \quad + \sum_{\substack{l=1 \\ l \neq \beta}}^K \lambda_{\beta l} \left\{ \mathcal{A}^{\hat{\theta}}(t, l, \alpha) - \mathcal{A}^{\hat{\theta}}(t, \beta, \alpha) \right\} = 0; \\ \mathcal{A}^{\hat{\theta}}(T, \beta, \alpha) = 1, \quad \forall \alpha, \beta \in \mathcal{E}, \end{array} \right. \quad (3.10)$$

we just derived, where $\hat{\theta}(t, \beta)$ attains the supreme in the first equation for $\mathcal{A}^{\hat{\theta}}(t, \beta, \beta)$. We now state the verification theorem.

Theorem 9. *Suppose the trading strategy corresponding to the policy law $\hat{\theta}(t, \alpha)$ attaining the supremum in the first equation for $\mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha)$ in (3.10) is admissible. Moreover, for each α and β in $\mathcal{E}$, there exists a $\mathcal{C}^1([0, T])$ -function $\mathcal{A}^{\hat{\theta}}(t, \beta, \alpha)$ such that $\{\mathcal{A}^{\hat{\theta}}(t, \beta, \alpha)\}_{\alpha, \beta \in \mathcal{E}}$ satisfies the system (3.10). Then,*

$$\hat{\theta}(t, \alpha) = \frac{1}{\gamma(\alpha)} a(t, \alpha)^{-1} \mu(t, \alpha). \quad (3.11)$$

is an equilibrium policy law, and

$$\hat{V}^{\text{CRRRA}}(t, x, \alpha) = \frac{1}{1 - \gamma(\alpha)} x^{1 - \gamma(\alpha)} \mathcal{A}^{\hat{\theta}}(t, \alpha, \alpha) \quad (3.12)$$

is the corresponding equilibrium value function, for any $t \in [0, T]$, $x \in \mathbb{R}_+$ and $\alpha \in \mathcal{E}$.

We see that

$$\hat{\theta}(t, \alpha) = \frac{1}{\gamma(\alpha)} a(t, \alpha)^{-1} \mu(t, \alpha)$$

is a *well-defined* measurable (deterministic) function with respect to t for each $\alpha \in \mathcal{E}$. Together with Assumption 2 we see that $\hat{\theta}(t, \alpha)$ is *admissible* indeed. We point out that this *equilibrium* policy law also coincides with the *naive* policy law, similar to the time inconsistent CARA utility maximisation problem in Section 3.2. The analysis is parallel to the one there and thus omitted.

3.4. ECONOMIC ANALYSIS

We now turn to solve $\mathcal{A}^{\hat{\theta}}(\cdot, \beta, \alpha)$. Parallel to Section 3.2, $\left(\mathcal{A}^{\hat{\theta}}(t, 1, \alpha), \dots, \mathcal{A}^{\hat{\theta}} \mathcal{A}^{\hat{\theta}}(t, K, \alpha)\right)^T$ solves the following linear homogeneous ODE

$$\begin{cases} \dot{X}(t) = A_{\alpha}^{\text{CRRA}}(t)X(t); \\ X(T) = (1, \dots, 1)^T \in \mathbb{R}^K, \end{cases}$$

where $X(t) = (X^1(t), \dots, X^K(t))^T$,

$$A_{\alpha}^{\text{CRRA}}(t) = \begin{pmatrix} -\rho_{\alpha}(1)\eta^2(t, 1) + \sum_{l \neq 1} \lambda_{1l} & -\lambda_{12} & \cdots & -\lambda_{1K} \\ -\lambda_{21} & -\rho_{\alpha}(2)\eta^2(t, 2) + \sum_{l \neq 2} \lambda_{2l} & \cdots & -\lambda_{2K} \\ \vdots & \vdots & \ddots & \vdots \\ -\lambda_{K1} & \lambda_{K2} & \cdots & -\rho_{\alpha}(K)\eta^2(t, K) + \sum_{l \neq K} \lambda_{Kl} \end{pmatrix}$$

in which $\eta(t, \beta) = \sqrt{\mu(t, \beta)^T a(t, \beta)^{-1} \mu(t, \beta)}$ and we use the notation $\rho_{\alpha}(k) = \frac{(1-\gamma(\alpha))(2\gamma(k)-\gamma(\alpha))}{2\gamma^2(k)}$, for each $k \in \mathcal{E}$. By the standard ODE theory, we know that there exists a unique solution to the above equation

$$X(t) = \exp\left(-\int_t^T A_{\alpha}^{\text{CRRA}}(s)ds\right) \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix},$$

where $\exp(\cdot)$ is understood as the matrix function.

In Figure 3.3 we provide the numerical illustration for the *equilibrium* and the *precommitment* strategy under the initial condition $\Xi_0 = 1$ parallel to Section 3.2. The market parameters are the same as the CARA utility case, and the risk aversion parameters are $\gamma(1) = 0.8$ and $\gamma(2) = 0.5$. We show both the money amount and the proportion of the wealth invested in the stock for the *equilibrium* and the *precommitment* strategies; moreover, we give the simulation of the corresponding wealth paths.

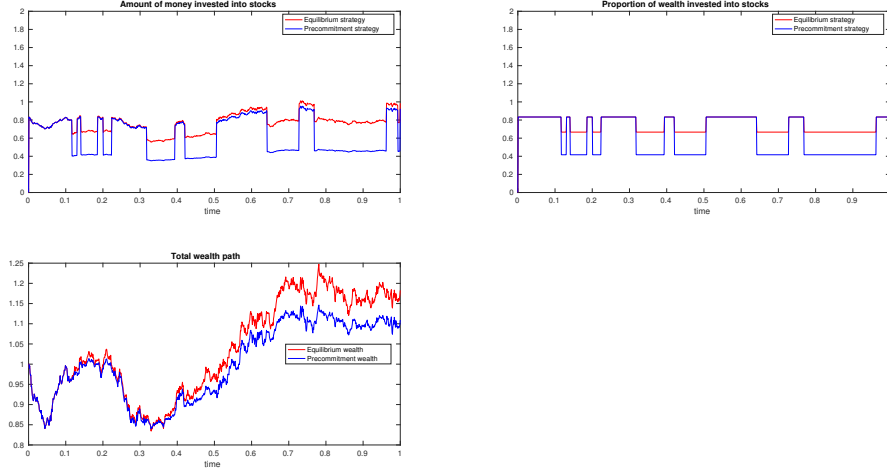
3.4 Economic Analysis

We now do some economic analysis. To facilitate the discussion, we assume the market has only two regimes—bullish and bearish. Let Regime 1 stand for the bullish market and Regime 2 for the bearish market. Moreover, the financial market model, where there is only one stock within, is driven by a switched geometric Brownian motion (i.e., all the market parameters are constant for each market regime).

Recall that given the market condition $\Xi_t = \alpha$, the equilibrium amount of money at time t for the time inconsistent CARA utility maximisation problem is

$$\hat{\pi}(t, \alpha) = \frac{1}{\gamma(\alpha)} \frac{\mu(\alpha)}{\sigma^2(\alpha)} = \frac{1}{\gamma(\alpha)} \frac{\eta(\alpha)}{\sigma(\alpha)}$$

Figure 3.3: The Simulation of Equilibrium Strategy and Precommitment Strategy for CRRA Utility.



Notes. The figure illustrates the equilibrium trading strategy and the precommitment trading strategies under initial condition $\Xi_0 = 1$. Market parameters: $n = d = 1$, $T = 1$, $\mu(t, 1) = 0.06$, $\mu(t, 2) = 0.03$, $\sigma(t, 1) = \sigma(t, 2) = 0.3$, $\lambda_{12} = 0.08$, $\lambda_{21} = 0.12$, $\gamma(1) = 0.8$, $\gamma(2) = 0.5$.

and the equilibrium ratio of the risky investment against wealth for the time inconsistent CRRA utility maximisation problem is

$$\hat{\theta}(t, \alpha) = \frac{1}{\gamma(\alpha)} \frac{\mu(\alpha)}{\sigma^2(\alpha)} = \frac{1}{\gamma(\alpha)} \frac{\eta(\alpha)}{\sigma(\alpha)},$$

where $\eta(\alpha) = \frac{\mu(\alpha)}{\sigma(\alpha)}$ is Sharpe ratio or the market price of risk known in the literature corresponding to the market state α (notice that we assume zero interest rate in this chapter). This indicates that both equilibrium policy laws are constant in each regime of the market. The trading strategies depend on the Sharpe ratio, the stock volatility and the investor's risk aversion parameter only. As long as the market condition remains unchanged, the trading strategies will not change either. These results are consistent with the results in the standard settings; see, for instance, [62], [63], [46] and [84].

A series of empirical studies (see, for instance, [30], [81] and [27]) suggest that when the market condition is good (for instance, the inflation rate is low, the growth of the GDP is fast or the security prices are expected to rise), the ratio of expected excess return over return variance is higher than the one when the market condition is bad (for instance, the market with high

3.4. ECONOMIC ANALYSIS

inflation, with low GDP growth or even decrease or in which the security prices are expected to fall). Therefore, as Regime 1 always stands for the bullish market and Regime 2 for the bearish market, we assume that

$$\frac{\mu(1)}{\sigma^2(1)} > \frac{\mu(2)}{\sigma^2(2)}.$$

Furthermore, in [33], the authors study the state-dependent CRRA utility function and find that the relative Arrow-Pratt risk aversion measure $\gamma(\Xi_t)$ is inclined to be higher at the time “with falls in the business conditions index and/or increases in the recession index”. In other words, the study in [33] indicates that the risk aversion for the market is lower in a bullish market and higher in a bearish market. Thus, it is reasonable to assume that

$$\gamma(1) < \gamma(2).$$

Under the assumptions about the market coefficients and the investor’s risk aversion parameter, one can see that

$$\hat{\pi}(t, 1) > \hat{\pi}(t, 2), \quad \text{for } t \in [0, T]$$

and

$$\hat{\theta}(t, 1) > \hat{\theta}(t, 2), \quad \text{for } t \in [0, T],$$

for CARA and CRRA utility functions respectively. In other words, the investor should allocate a higher amount of money (for the CARA utility case) or proportion of her wealth (for the CRRA utility case) in the risky asset during a bullish period of the market than in a bearish one. This suggests that, when market condition is positive and the expectation for the future is bright, the investor’s trading strategy will become relatively aggressive and she will invest more in the risky assets; when the market condition is negative and the future prospect is dim, the investor will “seek for the safety harbour” and invest less in the risky assets.

Chapter 4

Equilibrium Utility Indifference Pricing with Regime Switching

We consider a time inconsistent option pricing problem in this chapter. Similar to Chapter 3, the investor's preference is characterised by a CARA utility maximisation criterion with regime switching; the portfolio selection problem for the investor is, therefore, time inconsistent. We introduce a general equilibrium utility indifference pricing rule for the investor to price a European-type contingent claim in an incomplete market. In a basis risk model, we provide the PDE system characterising the equilibrium utility indifference pricing rule and show its local existence and uniqueness for a two-state case. Some properties of the equilibrium utility indifference pricing rule are discussed subsequently.

4.1 Market Model and Basic Problem

In this section, we state the financial market model and the option pricing problem under consideration.

4.1.1 The Model

The underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and the terminal time $T > 0$ are as stated in Chapter 1. On this probability space defined two standard Brownian motions (BM) $W, W^\perp$ and one continuous-time stationary Markovian chain (MC) Ξ taking values in a finite discrete state space $\mathcal{E} = \{1, 2, \dots, K\}$; moreover, $W, W^\perp$ and Ξ are independent of each other. The filtration $\mathbb{F} = \{\mathcal{F}_t\}_{t \in [0, T]}$ is the completion of the filtration generated by $W, W^\perp$ and Ξ . As in Chapter 3, we use the exogenous process Ξ to model the financial market state.

In this chapter, the financial market consists of one bond B . and two stocks S . and Y .; moreover, B . and S . are traded continuously while the price of Y . is observable but Y . per se is *not* tradable. As in Chapter 3, we assume the interest rate is constantly zero and

$$B_t \equiv 1, \quad t \in [0, T]. \quad (4.1)$$

The dynamics of S . and Y . follow the SDEs

$$dS_t = S_t \left\{ \mu(t, \Xi_t) dt + \sigma(t, \Xi_t) dW_t \right\} \quad (4.2)$$

and

$$dY_t = b(t, Y_t, \Xi_t) dt + \nu(t, Y_t, \Xi_t) \left\{ \rho(t, \Xi_t) dW_t + \sqrt{1 - \rho^2(t, \Xi_t)} dW_t^\perp \right\} \quad (4.3)$$

respectively, where $\mu, \sigma : [0, T] \times \mathcal{E} \rightarrow \mathbb{R}$, $b, \nu : [0, T] \times \mathbb{R} \times \mathcal{E} \rightarrow \mathbb{R}$ and $\rho : [0, T] \times \mathcal{E} \rightarrow (-1, 1)$ are given deterministic (measurable) functions. Moreover, there is a European-type contingent claim with payoff $F(Y_T)$ for some deterministic function $F(\cdot)$.

The following assumptions are made throughout this chapter and Chapter 5.

Assumption 10. *Coefficients $\mu(t, \alpha)$ and $\sigma(t, \alpha)$ are bounded on $[0, T]$ for any $\alpha \in \mathcal{E}$.*

Assumption 11. *Coefficients $b(t, y, \alpha)$ and $\nu(t, y, \alpha)$ are uniformly bounded with respect to argument $t \in [0, T]$ and uniformly Lipschitz with respect to argument $y \in \mathbb{R}$, i.e., there exists some $M > 0$ such that*

$$|b(t, y, \alpha)| + |\nu(t, y, \alpha)| \leq M, \quad \forall t \in [0, T]$$

for any $(y, \alpha) \in \mathbb{R} \times \mathcal{E}$ and

$$|b(t, y_1, \alpha) - b(t, y_2, \alpha)| + |\nu(t, y_1, \alpha) - \nu(t, y_2, \alpha)| \leq M|y_1 - y_2|, \quad \forall y_1, y_2 \in \mathbb{R}$$

for any $(t, \alpha) \in [0, T] \times \mathcal{E}$.

Assumption 12 (Non-degeneracy Condition). *There exist some $\kappa > 0$, such that*

$$\sigma^2(t, \alpha) \geq \kappa, \quad \forall t \in [0, T] \text{ and } \alpha \in \mathcal{E}.$$

For the sake of simplicity, we introduce the notation $a(t, \alpha) = \sigma^2(t, \alpha)$ for the square of the diffusion and $\eta(t, \alpha) = \frac{\mu(t, \alpha)}{\sigma(t, \alpha)}$ for the instantaneous Sharpe ratio to the tradable asset S .. Moreover, we denote $\bar{\rho}(t, \alpha) = \sqrt{1 - \rho^2(t, \alpha)}$.

We construct the self-financing portfolio by trading B . and S . with no consumption. Suppose we fix a time $t \in [0, T]$ temporarily and the market state at time t is $\Xi_t = \alpha \in \mathcal{E}$; moreover, the

4.1. MARKET MODEL AND BASIC PROBLEM

capital at time t is $x > 0$ and the total wealth at time $s \in [t, T]$ is denoted by x_s . Furthermore, we assume that trading takes place continuously and transaction costs are taken to be zero. Denoting the amount of capital invested in S at time s by π_s , the dynamics of the wealth process corresponding to the trading strategy $\{\pi_s\}_{s \in [t, T]}$ is

$$\begin{cases} dx_s^{(t,x),\pi} = \pi_s \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\} \\ x_t = x, \quad \Xi_t = \alpha. \end{cases} \quad (4.4)$$

In particular, the corresponding terminal wealth $x_T^{(t,x),\pi}$ is

$$x_T^{(t,x),\pi} = x + \int_t^T \pi_s \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\}. \quad (4.5)$$

As stated in Chapter 2, we always use boldface letter, say π , to denote a control process or a control policy law (a function), and use ordinary letter without boldface, say π , to denote a possible value (a scalar, for the current case) of the control process.

A trading strategy π is said to be *admissible* if $\pi \in \mathcal{L}_{\mathcal{F}}^2(0, T; \mathbb{R})$, which is the same as in Section 3.2.

4.1.2 The Basic Problem

The main problem in this chapter is the pricing (and the corresponding hedging) of a contingent claim written on the *nontradable* asset Y_T with payoff function $F(\cdot)$. Since Y is *nontradable*, the replication method fails and we turn to price the contingent claim via the *indifference pricing* approach. The indifference price is the cost of the claim such that the investor feels *indifferent* towards the portfolios with or without the claim. Clearly, the investor's preference is involved in finding her indifference price. In this chapter, we assume the investor's preference is characterised by a CARA utility maximisation criterion for portfolio selection, similar to Chapter 3, and resort to the *utility* indifference pricing approach. More precisely, the investor's utility function for her *terminal* wealth at any intermediate time $t \in [0, T]$ is

$$U_t(x) = -\exp\left(-\gamma(\Xi_t)x\right),$$

where Ξ_t is the market state at time t . There are two choices for the investor: holding zero or k contingent claims (specifically, $k < 0$ stands for shorting instead of holding) in her portfolio. For the first choice, the investor's underlying control problem is to maximise the following objective functional given the market state α and the capital x at time t :

$$J^0(t, x, \alpha; \pi) = \mathbb{E} \left[-\exp\left(-\gamma(\alpha)x_T^\pi\right) \middle| x_t = x, \Xi_t = \alpha \right],$$

where $\pi_\cdot$ is an admissible trading strategy and $x^\pi_\cdot$ is the corresponding wealth process; for the second choice, the underlying control problem is to maximise the following objective functional:

$$J^k(t, x, y, \alpha; \pi_\cdot) = \mathbb{E} \left[-\exp \left(-\gamma(\alpha) \{x_T^\pi + kF(Y_T)\} \right) \middle| x_t = x, Y_t = y, \Xi_t = \alpha \right].$$

Notice that we use the superscript “0” or “k” to emphasise that the corresponding objective functional is with respect to the problem with 0 or k claims in the portfolio. We point out that, similar to Chapter 3, the investor’s preference depends on the market state and the underlying stochastic control problems are *time inconsistent*. As stated in Chapter 2, we exploit the *equilibrium* approach to deal with the time inconsistency; more precisely, we will look for the *equilibrium policy laws* and the *equilibrium value functions* for the underlying two time inconsistent stochastic control problems. After obtaining the equilibrium policy laws and value functions, we then give an equilibrium utility indifference pricing rule.

4.2 Time Inconsistent CARA Utility Maximisation with Regime Switching

In this section, we give the *equilibrium* policy laws and value functions for the two time inconsistent utility maximisation problems—with and without the contingent claim in the portfolio. This section is a preparation for the *equilibrium utility indifference pricing rule* in the next section.

4.2.1 Utility Maximisation without Contingent Claim

We first discuss the time inconsistent utility maximisation problem without any claim in the portfolio. As in Chapter 3, we only consider *wealth-independent* equilibrium policy law; in other words, we assume the equilibrium policy law takes the following form

$$\hat{\pi}^0(t, x, \alpha) = \hat{\pi}^0(t, \alpha).$$

With $\hat{\pi}^0(t, \alpha)$ an equilibrium policy law and $\hat{x}^0_\cdot$ the corresponding *equilibrium* wealth process, we introduce the following *auxiliary function*

$$\mathcal{A}^0(t, \beta, \alpha) = \mathbb{E} \left[\exp \left(-\gamma(\alpha) \int_t^T \hat{\pi}^0(s, \Xi_s) \left\{ \mu(s, \Xi_s) ds + \sigma(s, \Xi_s) dW_s \right\} \right) \middle| \Xi_t = \beta \right]$$

as in Chapter 3; in particular, the first state variable β stands for the market state at the *current* time t and the second state variable α represents the market state at some initial time *before* t . We then have the following result.

Theorem 13. Assume the following extended Hamilton-Jacobi-Bellman (HJB) system

$$\left\{ \begin{array}{l} \mathcal{A}_t^0(t, \alpha, \alpha) + \gamma(\alpha) \mathcal{A}^0(t, \alpha, \alpha) \inf_{\pi} \left\{ \frac{1}{2} \gamma(\alpha) a(t, \alpha) \pi^2 - \mu(t, \alpha) \pi \right\} \\ \quad + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^0(t, l, \alpha) - \mathcal{A}^0(t, \alpha, \alpha) \right\} = 0; \\ \mathcal{A}_t^0(t, \beta, \alpha) + \gamma(\alpha) \mathcal{A}^0(t, \beta, \alpha) \left\{ \frac{1}{2} \gamma(\alpha) a(t, \beta) (\hat{\pi}^0(t, \beta))^2 - \mu(t, \beta) \hat{\pi}^0(t, \beta) \right\} \\ \quad + \sum_{\substack{l=1 \\ l \neq \beta}}^K \lambda_{\beta l} \left\{ \mathcal{A}^0(t, l, \alpha) - \mathcal{A}^0(t, \beta, \alpha) \right\} = 0; \\ \mathcal{A}^0(T, \beta, \alpha) = 1, \end{array} \right. \quad (4.6)$$

for $\alpha, \beta \in \mathcal{E}$ has a (unique) $\mathcal{C}^1([0, T])$ -solution $\{\mathcal{A}^0(t, \beta, \alpha)\}_{\alpha, \beta \in \mathcal{E}}$, and the policy law $\hat{\pi}^0(t, \alpha)$ attaining the infimum in the first equation for $\mathcal{A}^0(t, \alpha, \alpha)$ in (4.6) is admissible. Then, for the time inconsistent utility maximisation problem without any claim in the portfolio,

$$\hat{\pi}^0(t, \alpha) = \frac{\mu(t, \alpha)}{\gamma(\alpha) a(t, \alpha)} \quad (4.7)$$

is an equilibrium policy law and

$$\hat{V}^0(t, x, \alpha) = -e^{-\gamma(\alpha)x} \mathcal{A}^0(t, \alpha, \alpha) \quad (4.8)$$

is the corresponding equilibrium value function, for any $t \in [0, T]$, $x \in \mathbb{R}_+$ and $\alpha \in \mathcal{E}$.

This theorem is a special case of Theorem 4 in Chapter 3.

4.2.2 Utility Maximisation with Contingent Claim

We turn next to the time inconsistent utility maximisation problem with k claims in the portfolio. We briefly deduce the *extended HJB equation* and provide the equilibrium policy law and value function.

Similar to Chapter 3 and Section 4.2.1, we consider only the *wealth-independent* equilibrium policy law taking the following form

$$\hat{\pi}^k(t, x, y, \alpha) = \hat{\pi}^k(t, y, \alpha).$$

With $\hat{\pi}^k(t, y, \alpha)$ an equilibrium policy law and $\hat{x}^k$ the corresponding *equilibrium wealth process*, we introduce the following *auxiliary function*

$$\mathcal{A}^k(t, y, \beta, \alpha) = \mathbb{E} \left[\exp \left(-\gamma(\alpha) \left[\int_t^T \hat{\pi}^k(u, Y_u, \Xi_u) \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \right. \right. \right. \right. \\ \left. \left. \left. + kF(Y_T) \right] \right) \middle| Y_t = y, \Xi_t = \beta \right].$$

Similar to the auxiliary functions in Chapter 3 and in Section 4.2.1, the first state variable β stands for the market state at the *current* time t and the second state variable α represents the market state at some initial time *before* t . Then, by definition one has

$$\hat{V}^k(t, x, y, \alpha) = -e^{-\gamma(\alpha)x} \mathcal{A}^k(t, y, \alpha, \alpha).$$

Moreover, for any (t, y, α) , any $h > 0$ and any $\pi \in \mathbb{R}$, the perturbed policy law $\pi^{k|t,h,\pi}(u, y, \alpha)$

$$\pi^{k|t,h,\pi}(u, y, \alpha) = \begin{cases} \pi, & u \in [t, t+h), \\ \hat{\pi}^k(u, y, \alpha), & u \in [t+h, T) \end{cases}$$

is defined in the same manner as in Definition 1 in Chapter 2, and the corresponding objective functional is

$$\begin{aligned} & J^k(t, x, y, \alpha; \pi^{k|t,h,\pi}) \\ &= -e^{-\gamma(\alpha)x} \mathbb{E} \left[\exp \left(-\gamma(\alpha) \int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \right) \mathcal{A}^k(t+h, Y_{t+h}, \Xi_{t+h}, \alpha) \middle| Y_t = y, \Xi_t = \alpha \right]. \end{aligned}$$

Then

$$\liminf_{h \rightarrow 0+} \frac{J^k(t, x, s, y, \alpha; \hat{\pi}^k) - J^k(t, x, s, y, \alpha; \pi^{k|t,h,\pi})}{h} \geq 0$$

leads to the following *extended Hamilton-Jacobi-Bellman (HJB) equation* for $\mathcal{A}^k(t, y, \alpha, \alpha)$:

$$\begin{cases} \mathcal{A}_t^k(t, y, \alpha, \alpha) + \frac{1}{2} \nu^2(t, y, \alpha) \mathcal{A}_{yy}^k(t, y, \alpha, \alpha) + b(t, y, \alpha) \mathcal{A}_y^k(t, y, \alpha, \alpha) \\ \quad + \sum_{\substack{l=1 \\ l \neq \alpha}}^K \lambda_{\alpha l} \left\{ \mathcal{A}^k(t, y, l, \alpha) - \mathcal{A}^k(t, y, \alpha, \alpha) \right\} \\ \quad + \inf_{\pi} \left\{ \frac{1}{2} \gamma^2(\alpha) a(t, \alpha) \mathcal{A}^k(t, y, \alpha, \alpha) \pi^2 - \gamma(\alpha) \mu(t, \alpha) \mathcal{A}^k(t, y, \alpha, \alpha) \pi \right. \\ \quad \left. - \gamma(\alpha) \rho(t, \alpha) \nu(t, y, s, \alpha) \sigma(t, s, \alpha) \mathcal{A}_y^k(t, s, y, \alpha, \alpha) \pi \right\} = 0; \\ \mathcal{A}^k(T, y, \alpha, \alpha) = e^{-\gamma(\alpha)kF(y)}. \end{cases} \quad (4.9)$$

The first-order condition then gives the minimizer

$$\hat{\pi}^k(t, y, \alpha) = \frac{\eta(t, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)} + \frac{\rho(t, \alpha)\nu(t, y, \alpha)\mathcal{A}_y^k(t, y, \alpha, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)\mathcal{A}^k(t, y, \alpha, \alpha)}.$$

With this $\hat{\pi}^k(t, y, \alpha)$ we can further obtain the partial differential equations (PDE) for $\mathcal{A}^k(t, y, \beta, \alpha)$:

$$\left\{ \begin{aligned} & \mathcal{A}_t^k(t, y, \beta, \alpha) + \frac{1}{2} \nu^2(t, y, \beta) \mathcal{A}_{yy}^k(t, y, \beta, \alpha) + b(t, y, \beta) \mathcal{A}_y^k(t, y, \beta, \alpha) \\ & + \frac{1}{2} \frac{\gamma^2(\alpha)}{\gamma^2(\beta)} \left\{ \eta(t, \beta) + \frac{\rho(t, \beta) \nu(t, y, \beta) \mathcal{A}_y^k(t, y, \beta, \beta)}{\mathcal{A}^k(t, y, \beta, \beta)} \right\}^2 \mathcal{A}^k(t, y, \beta, \alpha) \\ & - \frac{\gamma(\alpha)}{\gamma(\beta)} \left\{ \eta(t, \beta) + \frac{\rho(t, \beta) \nu(t, y, \beta) \mathcal{A}_y^k(t, y, \beta, \beta)}{\mathcal{A}^k(t, y, \beta, \beta)} \right\} \times \left\{ \eta(t, \beta) \mathcal{A}^k(t, y, \beta, \alpha) \right. \\ & \quad \left. + \rho(t, \beta) \nu(t, y, \beta) \mathcal{A}_y^k(t, y, \beta, \alpha) \right\} \\ & + \sum_{\substack{l=1 \\ l \neq \beta}}^K \lambda_{\beta l} \left\{ \mathcal{A}^k(t, y, l, \alpha) - \mathcal{A}^k(t, y, \beta, \alpha) \right\} = 0; \end{aligned} \right. \quad \mathcal{A}^k(T, y, \beta, \alpha) = e^{-\gamma(\alpha)kF(y)} \quad (4.10)$$

for any $\alpha, \beta \in \mathcal{E}$.

We provide the following *verification theorem* to close this section.

Theorem 14. *Suppose the policy law $\hat{\pi}^k(t, y, \alpha)$ attaining the infimum of (4.9) is admissible; moreover, the semilinear parabolic system (4.10) has a (unique) $\mathcal{C}^{1,2}([0, T] \times \mathbb{R})$ -solution $\{\mathcal{A}^k(t, y, \beta, \alpha)\}_{\alpha, \beta \in \mathcal{E}}$. Then, for the time inconsistent utility maximisation problem with k claims in the portfolio,*

$$\hat{\pi}^k(t, y, \alpha) = \frac{\eta(t, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)} + \frac{\rho(t, \alpha)\nu(t, y, \alpha)\mathcal{A}_y^k(t, y, \alpha, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)\mathcal{A}^k(t, y, \alpha, \alpha)}$$

is an equilibrium policy law and

$$\hat{V}^k(t, x, y, \alpha) = -e^{-\gamma(\alpha)x} \mathcal{A}^k(t, y, \alpha, \alpha)$$

is the corresponding equilibrium value function for $t \in [0, T)$, $x \in \mathbb{R}_+$, $y \in \mathbb{R}$ and $\alpha \in \mathcal{E}$.

The proof is completely parallel to the one for Theorem 4 in Chapter 3 and thus omitted.

4.3 General Equilibrium Exponential Utility Indifference Pricing Rule

In this section, we combine the results in Sections 4.2.1 and 4.2.2 to give an equilibrium utility indifference pricing rule.

To price a contingent claim, consider the two choices for the investor—the portfolios with or without the contingent claim. The indifference price is the cost of the claim such that the investor feels indifferent towards the two choices. We give the formal definition below.

Definition 15 (Equilibrium Exponential Utility Indifference Price). An *equilibrium (exponential) utility indifference price* per claim, $p^k(t, x, y, \alpha)$, for a position of k contingent claims with the terminal payoff $kF(Y_T)$ is defined as the solution of the following (algebraic) equation

$$\hat{V}^0(t, x, \alpha) = \hat{V}^k(t, x - kp^k(t, x, y, \alpha), y, \alpha).$$

It is well-known in the literature that the classical exponential utility indifference price is independent of the wealth variable x . This property preserves for the above *equilibrium exponential utility indifference price*, i.e., $p^k(t, y, \alpha)$ is a function of t , y and α only. As a matter of fact, recall that given the current nontradable stock price $Y_t = y$, the market condition $\Xi_t = \alpha$ and the wealth $x_t = x$, the equilibrium value functions for the time inconsistent utility maximisation problems with 0 or k claims in the portfolio are

$$\hat{V}^0(t, x, \alpha) = -e^{-\gamma(\alpha)x} \mathcal{A}^0(t, \alpha, \alpha)$$

and

$$\hat{V}^k(t, x, y, \alpha) = -e^{-\gamma(\alpha)x} \mathcal{A}^k(t, y, \alpha, \alpha)$$

respectively, where $\mathcal{A}^0(t, \alpha, \alpha)$ and $\mathcal{A}^k(t, y, \alpha, \alpha)$ are (part of) the solutions to systems (4.6) and (4.10). Thus, the equilibrium utility indifference price is

$$p^k(t, y, \alpha) = \frac{\log(\mathcal{A}^0(t, \alpha, \alpha)) - \log(\mathcal{A}^k(t, y, \alpha, \alpha))}{k\gamma(\alpha)}.$$

Apart from the price of the contingent claim, another important issue in option pricing problem is the hedging strategy. Suppose that the investor decides to purchase k claims at time t ; she needs not only to pay $kp^k(t, Y_t, \Xi_t)$ for the claims but also to switch her strategy from the previous equilibrium strategy $\hat{\pi}^0(\cdot, \Xi)$ to the new equilibrium one $\hat{\pi}^k(\cdot, Y, \Xi)$. The difference between these two equilibrium strategies can then be viewed as the *hedging strategy* for the contingent claim. We give the definition formally below.

Definition 16 (Equilibrium Exponential Utility Indifference Hedging Strategy). For the equilibrium (exponential) utility indifference hedging strategy $p^k(t, y, \alpha)$ in Definition 15, the corresponding *equilibrium (exponential) utility indifference hedging strategy* for a position of k contingent claims is defined as

$$\hat{\pi}_t^{k,H} = \hat{\pi}^k(t, Y_t, \Xi_t) - \hat{\pi}^0(t, \Xi_t), \quad t \in [0, T).$$

4.3. GENERAL EQUILIBRIUM EXPONENTIAL UTILITY INDIFFERENCE PRICING RULE

Recall that the equilibrium policy laws are

$$\hat{\pi}^0(t, \alpha) = \frac{\eta(t, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)}$$

and

$$\hat{\pi}^k(t, y, \alpha) = \frac{\eta(t, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)} + \frac{\rho(t, \alpha)\nu(t, y, \alpha)\mathcal{A}_y^k(t, y, \alpha, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)\mathcal{A}^k(t, y, \alpha, \alpha)},$$

respectively. Therefore, the *equilibrium (exponential) utility indifference hedging strategy* for a position of k claims is given by

$$\hat{\pi}_t^{k,H} = \frac{\rho(t, \Xi_t)\nu(t, Y_t, \Xi_t)\mathcal{A}_y^k(t, Y_t, \Xi_t, \Xi_t)}{\gamma(\Xi_t)\sigma(t, \Xi_t)\mathcal{A}^k(t, Y_t, \Xi_t, \Xi_t)}, \quad t \in [0, T).$$

Remark 3. The equilibrium (exponential) utility indifference hedging strategy is in the *feedback* form, i.e., $\hat{\pi}_t^{k,H} = \hat{\pi}_H^k(t, Y_t, \Xi_t)$, where

$$\hat{\pi}_H^k(t, y, \alpha) = \hat{\pi}^k(t, y, \alpha) - \hat{\pi}^0(t, \alpha) = \frac{\rho(t, \alpha)\nu(t, y, \alpha)\mathcal{A}_y^k(t, y, \alpha, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)\mathcal{A}^k(t, y, \alpha, \alpha)}.$$

is a deterministic function. We call $\hat{\pi}_H^k(t, y, \alpha)$ the *equilibrium (exponential) utility indifference hedging policy law* for a position of k claims.

Remark 4. The *equilibrium (exponential) utility indifference price* $p^k(t, y, \alpha)$ together with the *equilibrium (exponential) utility hedging policy law* $\hat{\pi}_H^k(t, y, \alpha)$ constitute an *equilibrium (exponential) utility indifference pricing rule*.

Remark 5. The equilibrium utility indifference pricing rule is “*time consistent*” or, more precisely, the equilibrium utility indifference hedging policy law is “*time consistent*”. The “*time consistency*” of the equilibrium utility indifference pricing rule indicates that, at any time when she wants to buy or sell the contingent claim, the investor will follow the *same* equilibrium utility indifference hedging policy law. To illustrate, suppose at some time t the investor decides to purchase k claims in her portfolio; apart from paying the price $p^k(t, Y_t, \Xi_t)$ for each claim, the investor also needs to switch her trading policy law from $\{\hat{\pi}^0(u, \alpha)\}_{u \in [t, T)}$ to $\{\hat{\pi}^k(u, y, \alpha)\}_{u \in [t, T)}$, i.e., she needs to implement the hedging policy law $\hat{\pi}_H^k(t, y, \alpha)$. If, at some future time $s \in (t, T)$, she changes her mind to sell the contingent claims, the investor will also switch her trading policy *back* to $\{\hat{\pi}^0(u, \alpha)\}_{u \in [s, T)}$ from $\{\hat{\pi}^k(u, y, \alpha)\}_{u \in [s, T)}$, i.e., she will implement the (reverse) hedging policy law $-\hat{\pi}_H^k(t, y, \alpha)$ determined previously.

4.4 Equilibrium Utility Indifference Pricing Rule in Basis Risk Model

The above *equilibrium utility indifference pricing rule* is fairly complicated to analyse. We now consider a simpler basis risk model by taking $\phi(t, \alpha) = \phi(t)$ for $\phi = \mu, \sigma, \eta, \rho$ and $\nu(t, y, \alpha) = \nu(t, y)$; in particular, the prices of the tradable stock and the diffusion coefficient of the nontradable asset are independent of the financial market state. However, the drift coefficient of the nontradable asset together with the risk aversion parameters are still regime dependent.

Remark 6. We may also assume the dependence of μ, σ, η and ρ on the market state Ξ . This assumption will, on the one side, makes the PDE systems below (which are considerably complicated already) even more cumbersome and, on the other side, bring about no more difficulty in later analysis.

In the current situation, the *equilibrium utility indifference price* $p^k(t, y, \alpha)$ is

$$p^k(t, y, \alpha) = -\frac{1}{k\gamma(\alpha)} \log \left(\frac{\mathcal{A}^k(t, y, \alpha, \alpha)}{\mathcal{A}^0(t, \alpha, \alpha)} \right)$$

and the *equilibrium utility indifference hedging policy law* $\hat{\pi}_H^k(t, y, \alpha)$ is

$$\hat{\pi}_H^k(t, y, \alpha) = \frac{\rho(t)\nu(t, y)}{\gamma(\alpha)\sigma(t)} \frac{\mathcal{A}_y^k(t, y, \alpha, \alpha)}{\mathcal{A}^k(t, y, \alpha, \alpha)},$$

where $\mathcal{A}^0(t, \beta, \alpha)$ and $\mathcal{A}^k(t, y, \beta, \alpha)$ satisfy the system

$$\left\{ \begin{aligned} &\mathcal{A}_t^0(t, \beta, \alpha) + \left\{ \frac{1}{2} \frac{\gamma(\alpha)}{\gamma(\beta)} \left(\frac{\gamma(\alpha)}{\gamma(\beta)} - 2 \right) \eta^2(t) - \sum_{l \neq \beta} \lambda_{\beta l} \right\} \mathcal{A}^0(t, \beta, \alpha) + \sum_{l \neq \beta} \left\{ \lambda_{\beta l} \mathcal{A}^0(t, l, \alpha) \right\} = 0; \\ &\mathcal{A}^0(T, \beta, \alpha) = 1 \end{aligned} \right. \quad (4.11)$$

and

$$\left\{ \begin{aligned} &\mathcal{A}_t^k(t, y, \beta, \alpha) + \frac{1}{2} \nu^2(t, y) \mathcal{A}_{yy}^k(t, y, \beta, \alpha) \\ &+ \left\{ b(t, y, \beta) - \frac{\gamma(\alpha)}{\gamma(\beta)} \eta(t) \rho(t) \nu(t, y) - \frac{\gamma(\alpha)}{\gamma(\beta)} \rho^2(t) \nu^2(t, y) \frac{\mathcal{A}_y^k(t, y, \beta, \beta)}{\mathcal{A}^k(t, y, \beta, \beta)} \right\} \mathcal{A}_y^k(t, y, \beta, \alpha) \\ &+ \left\{ \frac{1}{2} \frac{\gamma(\alpha)}{\gamma(\beta)} \left(\frac{\gamma(\alpha)}{\gamma(\beta)} - 2 \right) \eta^2(t) + \frac{\gamma(\alpha)}{\gamma(\beta)} \left(\frac{\gamma(\alpha)}{\gamma(\beta)} - 1 \right) \frac{\mathcal{A}_y^k(t, y, \beta, \beta)}{\mathcal{A}^k(t, y, \beta, \beta)} \eta(t) \rho(t) \nu(t, y) \right. \\ &\quad \left. + \frac{1}{2} \left(\frac{\gamma(\alpha)}{\gamma(\beta)} \right)^2 \rho^2(t) \nu^2(t, y) \left(\frac{\mathcal{A}_y^k(t, y, \beta, \beta)}{\mathcal{A}^k(t, y, \beta, \beta)} \right)^2 - \sum_{l \neq \beta} \lambda_{\beta l} \right\} \mathcal{A}^k(t, y, \beta, \alpha) \\ &\quad + \sum_{l \neq \beta} \left\{ \lambda_{\beta l} \mathcal{A}^k(t, y, l, \alpha) \right\} = 0; \\ &\mathcal{A}^k(T, y, \beta, \alpha) = e^{-\gamma(\alpha)kF(y)} \end{aligned} \right. \quad (4.12)$$

4.4. EQUILIBRIUM UTILITY INDIFFERENCE PRICING RULE IN BASIS RISK MODEL

respectively, for $\alpha, \beta \in \mathcal{E}$.

We now introduce the following two auxiliary functions

$$\begin{aligned} f^{0,\beta\alpha}(t) &= \frac{\mathcal{A}^0(t, \beta, \alpha)}{\mathcal{A}^0(t, \alpha, \alpha)}, \\ f^{k,\beta\alpha}(t, y) &= \frac{\mathcal{A}^k(t, y, \beta, \alpha)}{\mathcal{A}^k(t, y, \alpha, \alpha)}, \end{aligned}$$

for $\alpha, \beta \in \mathcal{E}$. From the straightforward calculation we have the following PDE system for the equilibrium utility indifference price $p^k(t, y, \alpha)$:

$$\begin{cases} p_t^k(t, y, \alpha) + \frac{1}{2}\nu^2(t, y)p_{yy}^k(t, y, \alpha) + \left\{ b(t, y, \alpha) - \eta(t)\rho(t)\nu(t, y) \right\} p_y^k(t, y, \alpha) \\ - \frac{1}{2}k\gamma(\alpha)\bar{\rho}^2(t)\nu^2(t, y)(p_y^k(t, y, \alpha))^2 - \frac{1}{k\gamma(\alpha)} \sum_{l \neq \alpha} \left\{ f^{k,l\alpha}(t, y) - f^{0,l\alpha}(t) \right\} = 0; \\ p^k(T, y, \alpha) = F(y) \end{cases} \quad (4.13)$$

for $\alpha \in \mathcal{E}$, where $f^{0,\beta\alpha}(t)$ and $f^{k,\beta\alpha}(t, y)$ satisfy

$$\begin{cases} f_t^{0,\beta\alpha}(t) + \left\{ \frac{1}{2} \left(\frac{\gamma(\alpha)}{\gamma(\beta)} - 1 \right)^2 \eta^2(t) + \sum_{l \neq \alpha} \lambda_{\alpha l} - \sum_{l \neq \beta} \lambda_{\beta l} \right\} f^{0,\beta\alpha}(t) \\ - \left(\sum_{l \neq \alpha} \lambda_{\alpha l} f^{0,l\alpha}(t) \right) f^{0,\beta\alpha}(t) + \sum_{l \neq \beta} \lambda_{\beta l} f^{0,l\alpha}(t) = 0; \\ f^{0,\beta\alpha}(T) = 1 \end{cases} \quad (4.14)$$

and

$$\begin{cases} f_t^{k,\beta\alpha}(t, y) + \frac{1}{2}\nu^2(t, y)f_{yy}^{k,\beta\alpha}(t, y) \\ + \left\{ b(t, y, \beta) - \frac{\gamma(\alpha)}{\gamma(\beta)}\eta(t)\rho(t)\nu(t, y) - k\gamma(\alpha)\nu^2(t, y)[p_y^k(t, y, \alpha) - \rho^2(t)p_y^k(t, y, \beta)] \right\} f_y^{k,\beta\alpha}(t, y) \\ - \left(\sum_{l \neq \alpha} \lambda_{\alpha l} f^{k,l\alpha}(t, y) \right) f^{k,\beta\alpha}(t, y) + \sum_{l \neq \beta} \lambda_{\beta l} f^{k,l\alpha}(t, y) \\ + \frac{1}{2} \left(\left(\frac{\gamma(\alpha)}{\gamma(\beta)} - 1 \right) \eta(t) + k\gamma(\alpha)\rho(t)\nu(t, y)[p_y^k(t, y, \alpha) - p_y^k(t, y, \beta)] \right)^2 f^{k,\beta\alpha}(t, y) \\ + \left\{ k\gamma(\alpha)(b(t, y, \alpha) - b(t, y, \beta))p_y^k(t, y, \alpha) + \sum_{l \neq \alpha} \lambda_{\alpha l} - \sum_{l \neq \beta} \lambda_{\beta l} \right\} f^{k,\beta\alpha}(t, y) = 0; \\ f^{k,\beta\alpha}(T, y) = 1 \end{cases} \quad (4.15)$$

respectively, for $\alpha, \beta \in \mathcal{E}$. We see that ODE system (4.14) for $\{f^{0,\beta\alpha}(t)\}_{\beta, \alpha \in \mathcal{E}}$ is autonomous; however, PDE system (4.13) for $\{p^k(t, y, \alpha)\}_{\beta, \alpha \in \mathcal{E}}$ and (4.15) for $\{f^{k,\beta\alpha}(t, y)\}_{\beta, \alpha \in \mathcal{E}}$ are coupled with each other, and they two together form a system of *semilinear* parabolic PDEs. (4.13),

(4.14) and (4.15) together determine the equilibrium exponential utility indifference pricing rule $p^k(t, y, \alpha)$ and $\hat{\pi}_H^k(t, y, \alpha)$.

Some special cases are discussed below.

4.4.1 Local Existence and Uniqueness for Two-Market-State Case

In this section, we consider the simplest two-market-state model that $\mathcal{E} = \{1, 2\}$; in other words, the market has only *two* states—*bearish* and *bullish*. Moreover, we assume $b(t, y, \alpha) = b_\alpha y$, $\nu(t, y) = \nu y$ and $\phi(t) = \phi$ for $\phi = \mu, \sigma, \eta$ and ρ , where $b_1, b_2, \nu, \mu, \sigma$ and ρ are all constant and $\eta = \frac{\mu}{\sigma}$.

ODE system (4.14) is now reduced to

$$\begin{cases} f_t^{0,21}(t) - \lambda_{12}(f^{0,21}(t))^2 + \left(\frac{1}{2}\left(\frac{\gamma(1)}{\gamma(2)} - 1\right)^2 \eta^2 + \lambda_{12} - \lambda_{21}\right) f^{0,21}(t) + \lambda_{21} = 0, \\ f_t^{0,12}(t) - \lambda_{21}(f^{0,12}(t))^2 + \left(\frac{1}{2}\left(\frac{\gamma(2)}{\gamma(1)} - 1\right)^2 \eta^2 + \lambda_{21} - \lambda_{12}\right) f^{0,12}(t) + \lambda_{12} = 0, \\ f^{0,21}(T) = 1, \quad f^{0,12}(T) = 1. \end{cases} \quad (4.16)$$

Notice that $f^{0,21}(t)$ and $f^{0,12}(t)$ are *decoupled* in this simplest case; moreover, we have $f^{0,11}(\cdot) \equiv 1$ and $f^{0,22}(\cdot) \equiv 1$ by definition.

The following proposition states the well-posedness of (4.16).

Theorem 17. *There exists a unique solution $(f^{0,21}(t), f^{0,12}(t))$ for (4.16). Moreover, we have the following estimates*

$$0 \leq f^{0,21}(t) \leq \kappa_{21}(T - t)$$

$$0 \leq f^{0,12}(t) \leq \kappa_{12}(T - t)$$

for any $t \in [0, T]$, where

$$\begin{aligned} \kappa_{21}(t) &= \left(1 + \frac{\lambda_{21}}{\frac{1}{2}\left(\frac{\gamma(1)}{\gamma(2)} - 1\right)^2 \eta^2 + \lambda_{12} - \lambda_{21}}\right) \exp\left(\left(\frac{1}{2}\left(\frac{\gamma(1)}{\gamma(2)} - 1\right)^2 \eta^2 + \lambda_{12} - \lambda_{21}\right)t\right) \\ &\quad - \frac{\lambda_{21}}{\frac{1}{2}\left(\frac{\gamma(1)}{\gamma(2)} - 1\right)^2 \eta^2 + \lambda_{12} - \lambda_{21}}, \\ \kappa_{12}(t) &= \left(1 + \frac{\lambda_{12}}{\frac{1}{2}\left(\frac{\gamma(2)}{\gamma(1)} - 1\right)^2 \eta^2 + \lambda_{21} - \lambda_{12}}\right) \exp\left(\left(\frac{1}{2}\left(\frac{\gamma(2)}{\gamma(1)} - 1\right)^2 \eta^2 + \lambda_{21} - \lambda_{12}\right)t\right) \\ &\quad - \frac{\lambda_{12}}{\frac{1}{2}\left(\frac{\gamma(2)}{\gamma(1)} - 1\right)^2 \eta^2 + \lambda_{21} - \lambda_{12}}. \end{aligned} \quad (4.17)$$

4.4. EQUILIBRIUM UTILITY INDIFFERENCE PRICING RULE IN BASIS RISK MODEL

Proof. First of all, we do the time-reverse transform $\tau = T - t$ to get the following (decoupled) forward ODE system

$$\begin{cases} x_t^{21}(\tau) = -\lambda_{12}(x^{21}(\tau))^2 + \left(\frac{1}{2}\left(\frac{\gamma(1)}{\gamma(2)} - 1\right)^2 \eta^2 + \lambda_{12} - \lambda_{21}\right) x^{21}(\tau) + \lambda_{21}, \\ x_t^{12}(\tau) = -\lambda_{21}(x^{12}(\tau))^2 + \left(\frac{1}{2}\left(\frac{\gamma(2)}{\gamma(1)} - 1\right)^2 \eta^2 + \lambda_{21} - \lambda_{12}\right) x^{12}(\tau) + \lambda_{12}, \\ x^{21}(0) = 1, \quad x^{12}(0) = 1. \end{cases} \quad (4.18)$$

Then if the well-posedness of the above ODE is established, by taking $f^{0,21}(t) = x^{21}(T - t)$ and $f^{0,12}(t) = x^{12}(T - t)$ we then obtain the solution to (4.16).

We only discuss $x^{21}(\cdot)$; the same argument also applies to $x^{12}(\cdot)$. Since the generator of the ODE system (4.18) is of $\mathcal{C}^1$ and thus *locally Lipschitz*, the uniqueness and local existence of the solution is guaranteed by Picard-Lindelöf theorem. We thus need only to show the global existence of the solution. More precisely, we need to show that $x^{21}(\cdot)$ cannot explode in finite time; in other words, for any time $\tau \geq 0$ we want to give finite upper and lower bounds for $x^{21}(\tau)$ (which can depend on τ).

To this end, we define

$$a_{21} = \frac{1}{2} \left(\frac{\gamma(1)}{\gamma(2)} - 1 \right)^2 \eta^2 + \lambda_{12} - \lambda_{21}$$

and

$$\tilde{x}^{21}(\tau) = e^{-a_{21}\tau} x^{21}(\tau).$$

We then see that $\tilde{x}^{21}(\cdot)$ satisfies the following ODE

$$\begin{cases} \tilde{x}_t^{21}(\tau) = -\tilde{\lambda}_{12}(\tau) (\tilde{x}^{21}(\tau))^2 + \tilde{\lambda}_{21}(\tau); \\ \tilde{x}^{21}(0) = 1, \end{cases}$$

where $\tilde{\lambda}_{12}(\tau) = e^{a_{21}\tau} \lambda_{12} > 0$ and $\tilde{\lambda}_{21}(\tau) = e^{-a_{21}\tau} \lambda_{21} > 0$.

We first give the upper bound of $x^{21}(\tau)$. From

$$\tilde{x}_t^{21}(\tau) = -\tilde{\lambda}_{12}(\tau) (\tilde{x}^{21}(\tau))^2 + \tilde{\lambda}_{21}(\tau) \leq \tilde{\lambda}_{21}(\tau)$$

we have

$$\tilde{x}^{21}(\tau) \leq 1 + \int_0^\tau \tilde{\lambda}_{21}(s) ds.$$

Thus

$$x^{21}(\tau) \leq e^{a_{21}\tau} \left(1 + \int_0^\tau \tilde{\lambda}_{21}(s) ds \right) = \left(1 + \frac{\lambda_{21}}{a_{21}} \right) e^{a_{21}\tau} - \frac{\lambda_{21}}{a_{21}}.$$

We next show the lower bound of $x^{21}(\tau)$ is zero for all $\tau \geq 0$. As a matter of fact,

$$\tilde{x}_t^{21}(\tau) = -\tilde{\lambda}_{12}(\tau) (\tilde{x}^{21}(\tau))^2 + \tilde{\lambda}_{21}(\tau) \geq -\tilde{\lambda}_{12}(\tau) (\tilde{x}^{21}(\tau))^2.$$

Notice that the local solution must be continuous. For any $0 < \tau_1 < \tau_2$, assuming $\tilde{x}^{21}(\cdot) \neq 0$ on $[\tau_1, \tau_2]$, we have

$$\frac{1}{\tilde{x}^{21}(\tau_2)} - \frac{1}{\tilde{x}^{21}(\tau_1)} \leq \int_{\tau_1}^{\tau_2} \tilde{\lambda}_{12}(s) ds.$$

Therefore, if $\tilde{x}^{21}(\tau_2) < 0$, then it must hold $\tilde{x}^{21}(\tau) < 0$ for any $\tau \in (\tau_1, \tau_2)$ by the continuity of $\tilde{x}^{21}(\cdot) < 0$; moreover, $\tilde{x}^{21}(\tau)$ will not approach to 0 as $\tau \rightarrow \tau_1$, otherwise the left handside of the above inequality will approach to $+\infty$ which is a contradiction. Thus, $\tilde{x}^{21}(\tau_1) < 0$ for any $\tau_1 \in (0, \tau_2)$. However, $\tilde{x}^{21}(0) = 1 > 0$ and we have the contradiction. Therefore, $\tilde{x}^{21}(\tau) \geq 0$ and thus $x^{21}(\tau) \geq 0$ for any $\tau \geq 0$. Thus the well-posedness for (4.16) is done. The estimates for $f^{0,21}(t)$ and $f^{0,12}(t)$ are obvious from the above discussion. $\square$

Similarly, (4.13) and (4.15) are now reduced to the following semilinear parabolic system

$$\left\{ \begin{array}{l} p_t^k(t, y, 1) + \frac{1}{2} \nu^2 y^2 p_{yy}^k(t, y, 1) + \{b_1 - \eta \rho \nu\} y p_y^k(t, y, 1) \\ \quad - \frac{1}{2} k \gamma(1) \bar{\rho}^2 \nu^2 y^2 (p_y^k(t, y, 1))^2 = \frac{1}{k \gamma(1)} \{f^{k,21}(t, y) - f^{0,21}(t)\}, \\ p_t^k(t, y, 2) + \frac{1}{2} \nu^2 y^2 p_{yy}^k(t, y, 2) + \{b_2 - \eta \rho \nu\} y p_y^k(t, y, 2) \\ \quad - \frac{1}{2} k \gamma(2) \bar{\rho}^2 \nu^2 y^2 (p_y^k(t, y, 2))^2 = \frac{1}{k \gamma(2)} \{f^{k,12}(t, y) - f^{0,12}(t)\}, \\ f_t^{k,21}(t, y) + \frac{1}{2} \nu^2 y^2 f_{yy}^{k,21}(t, y) \\ \quad + \left\{ b_2 y - \frac{\gamma(1)}{\gamma(2)} \eta \rho \nu y - k \gamma(1) \nu^2 y^2 [p_y^k(t, y, 1) - \rho^2 p_y^k(t, y, 2)] \right\} f_y^{k,21}(t, y) \\ \quad - \lambda_{12} (f^{k,21}(t, y))^2 \\ \quad + \frac{1}{2} \left(\left(\frac{\gamma(1)}{\gamma(2)} - 1 \right) \eta + k \gamma(1) \rho \nu y [p_y^k(t, y, 1) - p_y^k(t, y, 2)] \right)^2 f^{k,21}(t, y) \\ \quad + \{k \gamma(1) (b_1 - b_2) y p_y^k(t, y, 1) + \lambda_{12} - \lambda_{21}\} f^{k,21}(t, y) \\ \quad + \lambda_{21} = 0, \\ f_t^{k,12}(t, y) + \frac{1}{2} \nu^2 y^2 f_{yy}^{k,12}(t, y) \\ \quad + \left\{ b_1 y - \frac{\gamma(2)}{\gamma(1)} \eta \rho \nu y - k \gamma(2) \nu^2 y^2 [p_y^k(t, y, 2) - \rho^2 p_y^k(t, y, 1)] \right\} f_y^{k,12}(t, y) \\ \quad - \lambda_{21} (f^{k,12}(t, y))^2 \\ \quad + \frac{1}{2} \left(\left(\frac{\gamma(2)}{\gamma(1)} - 1 \right) \eta + k \gamma(2) \rho \nu y [p_y^k(t, y, 2) - p_y^k(t, y, 1)] \right)^2 f^{k,12}(t, y) \\ \quad + \{k \gamma(2) (b_2 - b_1) y p_y^k(t, y, 2) + \lambda_{21} - \lambda_{12}\} f^{k,12}(t, y) \\ \quad + \lambda_{12} = 0, \\ p^k(T, y, 1) = F(y), \quad p^k(T, y, 2) = F(y), \quad f^{k,21}(T, y) = 1, \quad f^{k,12}(T, y) = 1. \end{array} \right. \quad (4.19)$$

4.4. EQUILIBRIUM UTILITY INDIFFERENCE PRICING RULE IN BASIS RISK MODEL

Moreover, we have $f^{k,11}(t, y) \equiv 1$ and $f^{k,22}(t, y) \equiv 1$ for $(t, y) \in [0, T] \times \mathbb{R}_+$.

Even for the highly simplified situation, this system is still very complicated; we are only able to show the local existence of the solution to this system.

Theorem 18. *Let the payoff function $F(\cdot)$ be of $\mathcal{C}^1(\mathbb{R}_+)$ and the support of $F(\cdot)$ is contained in $[0, X^*]$ for some $X^* > 0$. Moreover, the derivative of $F(\cdot)$ is Lipschitz continuous, i.e.,*

$$|F_x(x) - F_x(y)| \leq L|x - y|, \quad \forall x, y \in \mathbb{R}_+$$

for some $L > 0$. Then there exists some $T^* > 0$ such that the PDE system (4.19) has a unique smooth solution on $[T - T^*, T]$.

Proof. To facilitate the analysis, we do the time-reverse and logarithmic transform

$$\begin{aligned} p^k(t, y, 1) &= u^1(T - t, \log(y)), & p^k(t, y, 2) &= u^2(T - t, \log(y)), \\ f^{k,21}(t, y) &= u^3(T - t, \log(y)), & f^{k,12}(t, y) &= u^4(T - t, \log(y)) \end{aligned}$$

(for the sake of simplicity we omit the parameter k in the new unknown functions $u^i(t, x)$, $i = 1, 2, 3, 4$) to turn the original PDE system (4.19) to the following forward semilinear parabolic system

$$\left\{ \begin{aligned} -u_t^1 + \frac{1}{2}\nu^2 u_{zz}^1 + \left\{ b_1 - \frac{1}{2}\nu^2 - \eta\rho\nu \right\} u_z^1 - \frac{1}{2}k\gamma(1)\bar{\rho}^2\nu^2(u_z^1)^2 - \frac{1}{k\gamma(1)}(u^3 - x^{21}(t)) &= 0, \\ -u_t^2 + \frac{1}{2}\nu^2 u_{zz}^2 + \left\{ b_2 - \frac{1}{2}\nu^2 - \eta\rho\nu \right\} u_z^2 - \frac{1}{2}k\gamma(2)\bar{\rho}^2\nu^2(u_z^2)^2 - \frac{1}{k\gamma(2)}(u^4 - x^{12}(t)) &= 0, \\ -u_t^3 + \frac{1}{2}\nu^2 u_{zz}^3 + \left\{ b_2 - \frac{1}{2}\nu^2 - \frac{\gamma(1)}{\gamma(2)}\eta\rho\nu - k\gamma(1)\nu^2[u_z^1 - \rho^2 u_z^2] \right\} u_z^3 - \lambda_{12}(u^3)^2 \\ &+ \left\{ \frac{1}{2} \left(\left(\frac{\gamma(1)}{\gamma(2)} - 1 \right) \eta + k\gamma(1)\rho\nu[u_z^1 - u_z^2] \right)^2 + k\gamma(1)(b_1 - b_2)u_z^1 + \lambda_{12} - \lambda_{21} \right\} u^3 + \lambda_{21} = 0, \\ -u_t^4 + \frac{1}{2}\nu^2 u_{zz}^4 + \left\{ b_1 - \frac{1}{2}\nu^2 - \frac{\gamma(2)}{\gamma(1)}\eta\rho\nu - k\gamma(2)\nu^2[u_z^2 - \rho^2 u_z^1] \right\} u_z^4 - \lambda_{21}(u^4)^2 \\ &+ \left\{ \frac{1}{2} \left(\left(\frac{\gamma(2)}{\gamma(1)} - 1 \right) \eta + k\gamma(2)\rho\nu[u_z^2 - u_z^1] \right)^2 + k\gamma(2)(b_2 - b_1)u_z^2 + \lambda_{21} - \lambda_{12} \right\} u^4 + \lambda_{12} = 0, \\ u^1(0, z) = u^2(0, z) = F(e^z), \quad u^3(0, z) = u^4(0, z) = 1, \end{aligned} \right.$$

where $x^{21}(\cdot)$ and $x^{12}(\cdot)$ satisfy the forward ODE system (4.18) in Theorem 17.

We now write the above system as

$$\left\{ \begin{aligned} u_t - \frac{1}{2}\nu^2 u_{zz} &= G(t, u, u_z), \\ u(0, z) &= \varphi(z) = \left(F(e^z), F(e^z), 1, 1 \right)^T, \end{aligned} \right. \quad (4.20)$$

in which the unknown functions are $u(t, z) = (u^1(t, z), u^2(t, z), u^3(t, z), u^4(t, z))^T$ and the non-linear parts $G(t, u, p) = (G^1(t, u, p), G^2(t, u, p), G^3(u, p), G^4(u, p))^T$ are

$$\begin{aligned}
 G^1(t, u, p) &= \tilde{G}^1(u, p) + \frac{1}{k\gamma(1)}x^{21}(t), \\
 \tilde{G}^1(u, p) &= -\frac{1}{2}k\gamma(1)\bar{\rho}^2\nu^2(p^1)^2 + \left\{b_1 - \frac{1}{2}\nu^2 - \eta\rho\nu\right\}(p^1) - \frac{1}{k\gamma(1)}(u^3), \\
 G^2(t, u, p) &= \tilde{G}^2(u, p) + \frac{1}{k\gamma(2)}x^{12}(t), \\
 \tilde{G}^2(u, p) &= -\frac{1}{2}k\gamma(2)\bar{\rho}^2\nu^2(p^2)^2 + \left\{b_2 - \frac{1}{2}\nu^2 - \eta\rho\nu\right\}(p^2) - \frac{1}{k\gamma(2)}(u^4), \\
 G^3(u, p) &= \frac{1}{2}k^2\gamma^2(1)\rho^2\nu^2(p^1)^2(u^3) - k^2\gamma^2(1)\rho^2\nu^2(p^1)(p^2)(u^3) + \frac{1}{2}k^2\gamma^2(1)\rho^2\nu^2(p^2)^2(u^3) \\
 &\quad - k\gamma(1)\nu^2(p^1)(p^3) + k\gamma(1)\rho^2\nu^2(p^2)(p^3) \\
 &\quad + k\gamma(1)\left\{\left(\frac{\gamma(1)}{\gamma(2)} - 1\right)\eta\rho\nu + b_1 - b_2\right\}(p^1)(u^3) - k\gamma(1)\left(\frac{\gamma(1)}{\gamma(2)} - 1\right)\eta\rho\nu(p^2)(u^3) - \lambda_{12}(u^3)^2 \\
 &\quad + \left\{b_2 - \frac{1}{2}\nu^2 - \frac{\gamma(1)}{\gamma(2)}\eta\rho\nu\right\}(p^3) + \left\{\frac{1}{2}\left(\frac{\gamma(1)}{\gamma(2)} - 1\right)^2\eta^2 + \lambda_{12} - \lambda_{21}\right\}u^3 + \lambda_{21}, \\
 G^4(u, p) &= \frac{1}{2}k^2\gamma^2(2)\rho^2\nu^2(p^1)^2(u^4) - k^2\gamma^2(2)\rho^2\nu^2(p^1)(p^2)(u^4) + \frac{1}{2}k^2\gamma^2(2)\rho^2\nu^2(p^2)^2(u^4) \\
 &\quad - k\gamma(2)\nu^2(p^2)(p^4) + k\gamma(1)\rho^2\nu^2(p^1)(p^4) \\
 &\quad + k\gamma(2)\left\{\left(\frac{\gamma(2)}{\gamma(1)} - 1\right)\eta\rho\nu + b_2 - b_1\right\}(p^2)(u^4) - k\gamma(2)\left(\frac{\gamma(2)}{\gamma(1)} - 1\right)\eta\rho\nu(p^1)(u^4) - \lambda_{21}(u^4)^2 \\
 &\quad + \left\{b_1 - \frac{1}{2}\nu^2 - \frac{\gamma(2)}{\gamma(1)}\eta\rho\nu\right\}(p^4) + \left\{\frac{1}{2}\left(\frac{\gamma(2)}{\gamma(1)} - 1\right)^2\eta^2 + \lambda_{21} - \lambda_{12}\right\}u^4 + \lambda_{12},
 \end{aligned}$$

where $u = (u^1, u^2, u^3, u^4)^T$ and $p = (p^1, p^2, p^3, p^4)^T$. One can immediately observe that $\varphi(\cdot) \in \mathcal{C}^1(\mathbb{R}, \mathbb{R}^4)$.

We first define the following working spaces

$$X = \mathcal{C}_0^1(\mathbb{R}; \mathbb{R}^4) = \left\{f : \mathbb{R} \rightarrow \mathbb{R}^4 \mid f \text{ is continuously differentiable and } \sup_{x \in \mathbb{R}} \left\{|f| + |f_x|\right\} < \infty\right\}$$

and

$$Y = \mathcal{C}_0^0(\mathbb{R}; \mathbb{R}^4) = \left\{f : \mathbb{R} \rightarrow \mathbb{R}^4 \mid f \text{ is continuous and } \sup_{x \in \mathbb{R}} \left\{|f|\right\} < \infty\right\}.$$

It is well-known that both X and Y are Banach spaces. Next we introduce the following (non-linear) mapping Φ :

$$\Phi : [0, \infty) \times X \longrightarrow Y$$

$$(s, u) \longmapsto \Phi(s, u) = G(s, u, u_z).$$

Then the above evolution PDE system can be converted to the following integral equation

$$u(t) = \mathcal{S}(t)[\varphi] + \int_0^t \mathcal{S}(t-s)[\Phi(s, u(s))]ds = \Psi[u](t), \quad (4.21)$$

4.4. EQUILIBRIUM UTILITY INDIFFERENCE PRICING RULE IN BASIS RISK MODEL

where Ψ is a mapping from $[0, T] \times X \rightarrow [0, T] \times X$ and $\mathcal{S}(\cdot)$ is the strongly continuous operator semigroup corresponding to $\frac{1}{2}\nu^2\Delta$, i.e., $v(t, x) = \mathcal{S}(t)[f](x)$ solves the following heat equation

$$\begin{cases} v_t = \frac{1}{2}v_{xx}, \\ v(0, x) = f(x). \end{cases}$$

It is well-known that

$$\mathcal{S}(t)[f](x) = \frac{1}{\sqrt{2\pi\nu^2 t}} \int_{\mathbb{R}} f(z) e^{-\frac{(z-x)^2}{2\nu^2 t}} dz$$

for any absolutely integrable function $f(\cdot)$.

Since $\tilde{G}^1(u, p)$, $\tilde{G}^2(u, p)$, $G^3(u, p)$ and $G^4(u, p)$ are all polynomials of u and p with degrees no more than 3, we see that $\Phi(s, \cdot)$ is *locally Lipschitz*: for any $\varepsilon > 0$, any

$$u, v \in B_X(\varphi; \varepsilon) = \{\psi \in X : \|\psi - \varphi\|_X \leq \varepsilon\},$$

it holds

$$\begin{aligned} \|\Phi(s, u) - \Phi(s, v)\|_Y &= \|G^1(s, u, u_x) - G^1(s, v, v_x)\|_Y + \|G^2(s, u, u_x) - G^2(s, v, v_x)\|_Y \\ &\quad + \|G^3(u, u_x) - G^3(v, v_x)\|_Y + \|G^4(u, u_x) - G^4(v, v_x)\|_Y \\ &= \|\tilde{G}^1(u, u_x) - \tilde{G}^1(v, v_x)\|_Y + \|\tilde{G}^2(u, u_x) - \tilde{G}^2(v, v_x)\|_Y \\ &\quad + \|G^3(u, u_x) - G^3(v, v_x)\|_Y + \|G^4(u, u_x) - G^4(v, v_x)\|_Y \\ &\leq K(\varepsilon)\|u - v\|_X \end{aligned}$$

for any $s \geq 0$, where

$$\begin{aligned} K(\varepsilon) &= 2k^2 (\gamma^2(1) + \gamma^2(2)) \rho^2 \nu^2 (\|\varphi\|_X + \varepsilon)^2 \\ &\quad + 2 \left(k\bar{\gamma}\nu^2 + k \max \left\{ \gamma(1) \left| 1 - \frac{\gamma(1)}{\gamma(2)} \right|, \gamma(2) \left| 1 - \frac{\gamma(2)}{\gamma(1)} \right| \right\} \cdot \max \{\eta\rho\nu, 1\} \right) (\|\varphi\|_X + \varepsilon) \\ &\quad + 2 \left(k\bar{\gamma}|b_1 - b_2| + \max \{\lambda_{12}, \lambda_{21}\} \right) (\|\varphi\|_X + \varepsilon) \\ &\quad + \frac{1}{2} \max \left\{ \left(1 - \frac{\gamma(1)}{\gamma(2)} \right)^2, \left(1 - \frac{\gamma(2)}{\gamma(1)} \right)^2 \right\} + \frac{1}{k\underline{\gamma}} + |\lambda_{12} - \lambda_{21}| \\ &\quad + \max_{i=1,2} \left\{ \left| b_i - \frac{1}{2}\nu^2 - \eta\rho\nu \right|, \left| b_i - \frac{1}{2}\nu^2 - \frac{\gamma(1)}{\gamma(2)}\eta\rho\nu \right|, \left| b_i - \frac{1}{2}\nu^2 - \frac{\gamma(2)}{\gamma(1)}\eta\rho\nu \right| \right\}, \end{aligned}$$

in which $\bar{\gamma} = \max\{\gamma(1), \gamma(2)\}$ and $\underline{\gamma} = \min\{\gamma(1), \gamma(2)\}$.

For each $T > 0$ and (fixed) $\varepsilon > 0$, we introduce the following set

$$Z^\varepsilon(T) = \left\{ u \in \mathcal{C}([0, T], X) : u(0) = \varphi, \|u(t) - \varphi\|_X \leq \varepsilon, \quad \forall t \in [0, T] \right\}.$$

Our schedule is to choose a proper $T^* > 0$ such that $\Psi : Z^\varepsilon(T^*) \rightarrow Z^\varepsilon(T^*)$ is a contraction and then Banach fixed point theorem guarantees that (4.21) has a unique solution on $[0, T^*]$ and thus the original (4.19) is uniquely solvable on $[0, T^*]$.

To this end, we first choose T small enough such that Ψ is *onto*. First of all, for any $\varphi \in Z^\varepsilon(T)$ it holds

$$\begin{aligned}\mathcal{S}(t)[\varphi](x) - \varphi(x) &= \frac{1}{\sqrt{2\pi\nu^2t}} \int_{\mathbb{R}} \varphi(z) e^{-\frac{(z-x)^2}{2\nu^2t}} dz - \varphi(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\varphi(x + \nu\sqrt{t}z) - \varphi(x) \right) e^{-\frac{z^2}{2}} dz,\end{aligned}$$

and

$$\begin{aligned}|\mathcal{S}(t)[\varphi](x) - \varphi(x)| &\leq \frac{2}{\sqrt{2\pi}} \int_{\mathbb{R}} \left| F(e^{x+\nu\sqrt{t}z}) - F(e^x) \right| e^{-\frac{z^2}{2}} dz \\ &\leq \frac{4\nu\sqrt{t}}{\sqrt{2\pi}} e^{X^*} \|F_x\|_{\mathcal{C}^0(\mathbb{R}_+)} \int_0^\infty z e^{-\frac{z^2}{2}} dz \\ &= \frac{4\nu\sqrt{t}}{\sqrt{2\pi}} e^{X^*} \|F_x\|_{\mathcal{C}^0(\mathbb{R}_+)}.\end{aligned}$$

Therefore,

$$\|\mathcal{S}(t)[\varphi] - \varphi\|_{\mathcal{C}^0(\mathbb{R})} \leq \frac{4\nu}{\sqrt{2\pi}} e^{X^*} \|F_x\|_{\mathcal{C}^0(\mathbb{R}_+)} \sqrt{t}.$$

Similarly, the partial derivative of $\mathcal{S}(t)[\varphi](x)$ with respect to x is

$$\mathcal{S}(t)[\varphi]_x(x) = \frac{1}{\sqrt{2\pi\nu^2t}} \int_{\mathbb{R}} \varphi_x(z) e^{-\frac{(z-x)^2}{2\nu^2t}} dz.$$

Moreover, since the support of $F(\cdot)$ is contained in $[0, X^*]$, it holds

$$|F_x(e^x)e^x - F_x(e^y)e^y| \leq e^{X^*} \left(e^{X^*} L + \|F_x\|_{\mathcal{C}^0(\mathbb{R}_+)} \right) |x - y|, \quad x, y \in \mathbb{R}.$$

Then the estimate

$$\|\mathcal{S}(t)[\varphi]_x - \varphi_x\|_{\mathcal{C}^0(\mathbb{R})} \leq \frac{4\nu}{\sqrt{2\pi}} e^{X^*} \left(e^{X^*} L + \|F_x\|_{\mathcal{C}^0(\mathbb{R}_+)} \right) \sqrt{t}$$

is parallel to the above calculation, where L is the Lipschitz constant of $F_x(\cdot)$. Hence, we have

$$\|\mathcal{S}(t)[\varphi] - \varphi\|_X \leq \frac{4\nu}{\sqrt{2\pi}} e^{X^*} \left(e^{X^*} L + 2\|F_x\|_{\mathcal{C}^0(\mathbb{R}_+)} \right) \sqrt{t}.$$

By choosing

$$T_1 = \left(\frac{\sqrt{2\pi}\varepsilon}{8\nu e^{X^*} \left(e^{X^*} L + 2\|F_x\|_{\mathcal{C}^0(\mathbb{R}_+)} \right)} \right)^2$$

we then have

$$\|\mathcal{S}(t)[\varphi] - \varphi\|_X \leq \frac{\varepsilon}{2}$$

for any $t \in [0, T_1)$.

Next, for any $\psi \in X$, recall that

$$\mathcal{S}(t)[\psi](x) = \frac{1}{\sqrt{2\pi\nu^2t}} \int_{\mathbb{R}} \psi(z) e^{-\frac{(z-x)^2}{2\nu^2t}} dz.$$

4.4. EQUILIBRIUM UTILITY INDIFFERENCE PRICING RULE IN BASIS RISK MODEL

Thus,

$$\|\mathcal{S}(t)[\psi]\|_{\mathcal{C}^0(\mathbb{R})} \leq \|\psi\|_Y \frac{1}{\sqrt{2\pi\nu^2 t}} \int_{\mathbb{R}} e^{-\frac{(z-x)^2}{2\nu^2 t}} dz = \|\psi\|_Y.$$

Furthermore,

$$\begin{aligned} \mathcal{S}(t)[\psi]_x(x) &= \frac{1}{\sqrt{2\pi\nu^2 t}} \int_{\mathbb{R}} \psi(z) e^{-\frac{(z-x)^2}{2\nu^2 t}} \frac{z-x}{\nu^2 t} dz \\ &= \frac{1}{\sqrt{2\pi\nu^2 t}} \int_{\mathbb{R}} \psi(x + \nu\sqrt{t}z) e^{-\frac{z^2}{2}} z dz \end{aligned}$$

and thus

$$\|\mathcal{S}(t)[\psi]_x\|_{\mathcal{C}^0(\mathbb{R})} \leq \frac{2}{\sqrt{2\pi\nu^2 t}} \|\psi\|_Y.$$

This leads to

$$\|\mathcal{S}(t)[\psi]\|_X = \|\mathcal{S}(t)[\psi]\|_{\mathcal{C}^0(\mathbb{R})} + \|\mathcal{S}(t)[\psi]_x\|_{\mathcal{C}^0(\mathbb{R})} \leq \left(1 + \frac{2}{\sqrt{2\pi\nu^2 t}}\right) \|\psi\|_Y$$

and therefore,

$$\|\mathcal{S}(t)\|_{\mathcal{L}(Y,X)} \leq 1 + \frac{2}{\sqrt{2\pi\nu}} t^{-\frac{1}{2}}.$$

Thirdly, for any $u \in Z^\varepsilon(T_1)$, it holds

$$\begin{aligned} \|\Phi(s, u(s))\|_Y &\leq \|\Phi(s, u(s)) - \Phi(s, \varphi)\|_Y + \|\Phi(s, \varphi)\|_Y \\ &\leq K(\varepsilon)\varepsilon + K_1(T_1), \end{aligned}$$

for any $s \in [0, T_1]$, where

$$\begin{aligned} K_1(T_1) &= \max_{0 \leq s \leq T_1} \left\{ \|\Phi(s, \varphi)\|_Y \right\} \\ &= \|\tilde{G}^1(\varphi, \varphi_x)\|_Y + \|\tilde{G}^2(\varphi, \varphi_x)\|_Y + \|G^3(\varphi, \varphi_x)\|_Y + \|G^4(\varphi, \varphi_x)\|_Y \\ &\quad + \frac{1}{k\gamma(1)} \max_{0 \leq s \leq T_1} \left\{ x^{21}(s) \right\} + \frac{1}{k\gamma(2)} \max_{0 \leq s \leq T_1} \left\{ x^{12}(s) \right\} \\ &= \|\tilde{G}^1(\varphi, \varphi_x)\|_Y + \|\tilde{G}^2(\varphi, \varphi_x)\|_Y + \|G^3(\varphi, \varphi_x)\|_Y + \|G^4(\varphi, \varphi_x)\|_Y + \frac{\kappa_{21}(T_1)}{k\gamma(1)} + \frac{\kappa_{12}(T_1)}{k\gamma(2)}, \end{aligned}$$

where $\kappa_{21}(\cdot)$ and $\kappa_{12}(\cdot)$ are given by (4.17).

We then have

$$\begin{aligned} \left\| \int_0^t \mathcal{S}(t-s)[\Phi(s, u(s))] ds \right\|_X &\leq (K(\varepsilon)\varepsilon + K_1(T_1)) \int_0^t \left(1 + \frac{2}{\sqrt{2\pi\nu}} (t-s)^{-\frac{1}{2}} \right) ds \\ &= (K(\varepsilon)\varepsilon + K_1(T_1)) \left(t + \frac{4}{\sqrt{2\pi\nu}} \sqrt{t} \right). \end{aligned}$$

Therefore, if we choose

$$T_2 = \min \left\{ T_1, \left(\sqrt{\frac{\varepsilon}{2(K(\varepsilon)\varepsilon + K_1(T_1))}} + \frac{2}{\pi\nu^2} - \frac{2}{\sqrt{2\pi\nu}} \right)^2 \right\},$$

then it holds

$$\left\| \int_0^t \mathcal{S}(t-s) [\Phi(s, u(s))] ds \right\|_X \leq \frac{\varepsilon}{2}$$

and thus

$$\|\Psi[u](t) - \varphi\|_X \leq \varepsilon$$

for any $t \in [0, T_2]$. Therefore, $\Psi[\cdot]$ is *onto* from $Z^\varepsilon(T_2)$ to $Z^\varepsilon(T_2)$.

At last we select an T^* such that $\Psi[\cdot]$ is a contraction. To this end, we first notice that

$$\|\Phi(s, u(s)) - \Phi(s, v(s))\|_Y \leq K(\varepsilon) \|u(s) - v(s)\|_X$$

for any $u, v \in Z^\varepsilon(T_2)$. Hence, for any $t \in [0, T_2]$,

$$\begin{aligned} \|\Psi[u](t) - \Psi[v](t)\|_X &= \left\| \int_0^t \mathcal{S}(t-s) [\Phi(s, u(s)) - \Phi(s, v(s))] ds \right\|_X \\ &\leq K(\varepsilon) \left(t + \frac{4}{\sqrt{2\pi\nu}} \sqrt{t} \right) \sup_{s \in [0, t]} \left\{ \|u(s) - v(s)\|_X \right\}. \end{aligned}$$

Then, if we choose

$$T^* = \min \left\{ \left(\sqrt{\frac{\delta}{K(\varepsilon)}} + \frac{2}{\pi\nu^2} - \frac{2}{\sqrt{2\pi\nu}} \right)^2, T_2 \right\},$$

where $\delta \in (0, 1)$ is some number strictly less than 1, then we see

$$\sup_{t \in [0, T^*]} \left\{ \|\Psi[u](t) - \Psi[v](t)\|_X \right\} \leq \delta \sup_{t \in [0, T^*]} \left\{ \|u(t) - v(t)\|_X \right\}.$$

Thus, $\Psi[\cdot]$ is a contraction on $Z^\varepsilon(T^*)$. Taking

$$\delta = \frac{K(\varepsilon)\varepsilon}{2(K(\varepsilon)\varepsilon + K_1(T_1))} < \frac{1}{2}$$

then leads to

$$T^* = \min \left\{ T_1, \left(\sqrt{\frac{\varepsilon}{2(K(\varepsilon)\varepsilon + K_1(T_1))}} + \frac{2}{\pi\nu^2} - \frac{2}{\sqrt{2\pi\nu}} \right)^2 \right\}$$

for any $\varepsilon > 0$. The proof is finished by fix any arbitrary positive ε , for instance taking $\varepsilon = 1$. $\square$

To close this section, we point out that the global existence of (4.19) is fairly difficult. As a matter of fact, consider the following initial value problem

$$\begin{cases} u_t = u_{xx} + uu_x; \\ u(0, x) = u_0(x), \quad x \in \mathbb{R}. \end{cases} \quad (4.22)$$

4.4. EQUILIBRIUM UTILITY INDIFFERENCE PRICING RULE IN BASIS RISK MODEL

This PDE is known as Burger's equation in the literature (see, for instance, [26]), in which the nonlinearity, similar to (4.19), contains the product of the unknown function u and its partial derivative with respect to x .

The solution to Burger's equation can be expressed, via the Cole-Hopf transform (see, for instance, [26]), as

$$u(t, x) = 2 \left[\log (v(t, x)) \right]_x,$$

where $v(t, x)$ is a solution of the linear diffusion problem

$$\begin{cases} v_t = v_{xx}; \\ v(0, x) = v_0(x) = \exp \left(\frac{1}{2} \int_0^x u_0(z) dz \right). \end{cases} \quad (4.23)$$

It is well-known that for (4.23) to have a solution which can be represented as the convolution of the initial condition and the fundamental solution for $t \in [0, \frac{1}{c}]$ for some $c > 0$, the initial condition $v_0(x)$ should be locally integrable and

$$|v_0(x)| \leq M e^{\frac{cx^2}{4}}, \text{ as } x \rightarrow \pm\infty$$

for some finite $M > 0$. This in turn implies that there are restrictions on the initial condition for Burger's equation (4.22), $u_0(x)$, in order that a solution to (4.22) exists on some time interval.

To illustrate, if we set $u_0(x) = \varpi x$, where $\varpi \neq 0$ is a constant, then it holds

$$v_0(x) = e^{\frac{\varpi x^2}{4}}.$$

Notice that if $\varpi < 0$ then $v_0(x)$ vanishes as $x \rightarrow \pm\infty$ and $v(t, x)$ will exist for all $t > 0$; in this case, the global existence for (4.22) holds. In contrast, if $\varpi > 0$ then $v_0(x)$ grows very fast as $x \rightarrow \pm\infty$ and $v(t, x)$ will only exist on time interval $t \in [0, \frac{1}{\varpi})$; in this case, we only have the local existence for (4.22).

Furthermore, if $\left| \frac{u_0(x)}{x} \right| \rightarrow \infty$ as either $x \rightarrow \infty$ or $x \rightarrow -\infty$ then $v_0(x) > e^{cx^2}$ for any $c > 0$ in one of these limits and therefore there is no solution for (4.23) or (4.22).

From the above discussion we see the solvability of Burger's equation highly depends on the initial data, and the global existence for (4.22) may fail even for a simple linear initial condition. Since (4.19) is even much complicated than (4.22), one can image the global existence for (4.19) should be by no means easy to obtain.

4.4.2 Marginal Equilibrium Utility Indifference Price

In this section, we discuss the so-called *marginal equilibrium (exponential) utility indifference price* under the same setting in Section 4.4.1. The definition is given as follows.

Definition 19. For an equilibrium (exponential) utility indifference price $p^k(t, y, \alpha)$, the corresponding *marginal equilibrium (exponential) utility indifference price* $p^0(t, y, \alpha)$ is defined by

$$p^0(t, y, \alpha) = \lim_{k \rightarrow 0} p^k(t, y, \alpha)$$

for $\alpha \in \mathcal{E}$, provided the above limit exists.

Marginal utility indifference price was introduced in [15] first; for more details about this concept, see, for instance, [65].

The dependence of the solution to (4.19) on the parameter k is difficult to analyse, and we are not able to show the existence of the marginal equilibrium utility indifference price. In this section, we *assume* a priori that the limits $p^0(t, y, 1)$ and $p^0(t, y, 2)$ exist and try to give some heuristic characterization of the marginal equilibrium utility indifference price.

To this end, we introduce two more auxiliary functions

$$\begin{aligned} g^{k,21}(t, y) &= \frac{1}{k\gamma(1)} \left\{ f^{k,21}(t, y) - f^{0,21}(t) \right\}; \\ g^{k,12}(t, y) &= \frac{1}{k\gamma(2)} \left\{ f^{k,12}(t, y) - f^{0,12}(t) \right\}. \end{aligned}$$

From (4.16) and (4.19) one can easily obtain the PDE system for $p^k(t, y, 1)$, $p^k(t, y, 2)$, $g^{k,21}(t, y)$

and $g^{k,12}(t, y)$

$$\left\{ \begin{aligned}
 & p_t^k(t, y, 1) + \frac{1}{2}\nu^2 y^2 p_{yy}^k(t, y, 1) + \{b_1 - \eta\rho\nu\} y p_y^k(t, y, 1) \\
 & \quad - \frac{1}{2}k\gamma(1)\bar{\rho}^2 \nu^2 y^2 (p_y^k(t, y, 1))^2 = g^{k,21}(t, y), \\
 & p_t^k(t, y, 2) + \frac{1}{2}\nu^2 y^2 p_{yy}^k(t, y, 2) + \{b_2 - \eta\rho\nu\} y p_y^k(t, y, 2) \\
 & \quad - \frac{1}{2}k\gamma(2)\bar{\rho}^2 \nu^2 y^2 (p_y^k(t, y, 2))^2 = g^{k,12}(t, y), \\
 & g_t^{k,21}(t, y) + \frac{1}{2}\nu^2 y^2 g_{yy}^{k,21}(t, y) \\
 & \quad + \left\{ b_2 - \frac{\gamma(1)}{\gamma(2)}\eta\rho\nu - k\gamma(1)\nu^2 y \left[p_y^k(t, y, 1) - \rho^2 p_y^k(t, y, 2) \right] \right\} y g_y^{k,21}(t, y) \\
 & \quad - k\gamma(1)\lambda_{12} (g^{k,21}(t, y))^2 + \left\{ \lambda_{12} - \lambda_{21} - 2\lambda_{12}f^{0,21}(t) + k\gamma(1)(b_1 - b_2) y p_y^k(t, y, 1) \right\} g^{k,21}(t, y) \\
 & \quad + \frac{1}{2} \left(\left(\frac{\gamma(1)}{\gamma(2)} - 1 \right) \eta + k\gamma(1)\rho\nu y \left[p_y^k(t, y, 1) - p_y^k(t, y, 2) \right] \right)^2 g^{k,21}(t, y) \\
 & \quad + (b_1 - b_2) f^{0,21}(t) y p_y^k(t, y, 1) + \left(\frac{\gamma(1)}{\gamma(2)} - 1 \right) \eta\rho\nu f^{0,21}(t) y \left[p_y^k(t, y, 1) - p_y^k(t, y, 2) \right] \\
 & \quad + \frac{1}{2}k\gamma(1)\rho^2 \nu^2 f^{0,21}(t) y^2 \left[p_y^k(t, y, 1) - p_y^k(t, y, 2) \right]^2 = 0, \\
 & g_t^{k,12}(t, y) + \frac{1}{2}\nu^2 y^2 g_{yy}^{k,12}(t, y) \\
 & \quad + \left\{ b_1 - \frac{\gamma(2)}{\gamma(1)}\eta\rho\nu - k\gamma(2)\nu^2 y \left[p_y^k(t, y, 2) - \rho^2 p_y^k(t, y, 1) \right] \right\} y g_y^{k,12}(t, y) \\
 & \quad - k\gamma(2)\lambda_{21} (g^{k,12}(t, y))^2 + \left\{ \lambda_{21} - \lambda_{12} - 2\lambda_{21}f^{0,12}(t) + k\gamma(2)(b_2 - b_1) y p_y^k(t, y, 2) \right\} g^{k,12}(t, y) \\
 & \quad + \frac{1}{2} \left(\left(\frac{\gamma(2)}{\gamma(1)} - 1 \right) \eta + k\gamma(2)\rho\nu y \left[p_y^k(t, y, 2) - p_y^k(t, y, 1) \right] \right)^2 g^{k,12}(t, y) \\
 & \quad + (b_2 - b_1) f^{0,12}(t) y p_y^k(t, y, 2) + \left(\frac{\gamma(2)}{\gamma(1)} - 1 \right) \eta\rho\nu f^{0,12}(t) y \left[p_y^k(t, y, 2) - p_y^k(t, y, 1) \right] \\
 & \quad + \frac{1}{2}k\gamma(2)\rho^2 \nu^2 f^{0,12}(t) y^2 \left[p_y^k(t, y, 2) - p_y^k(t, y, 1) \right]^2 = 0, \\
 & p^k(T, y, 1) = F(y), \quad p^k(T, y, 2) = F(y), \quad g^{k,21}(T, y) = 0, \quad g^{k,12}(T, y) = 0.
 \end{aligned} \right. \tag{4.24}$$

We now assume a priori that the limits of $g^{k,21}(t, y)$ and $g^{k,12}(t, y)$ also exist as $k \rightarrow 0$ for any (t, y) and denote the corresponding limits as $g^{0,21}(t, y)$ and $g^{0,12}(t, y)$. By formally letting $k \rightarrow 0$ in (4.24), we then get the following PDE system for $p^0(t, y, 1)$, $p^0(t, y, 2)$, $g^{0,21}(t, y)$ and

$g^{0,12}(t, y)$:

$$\left\{ \begin{array}{l} p_t^0(t, y, 1) + \frac{1}{2}\nu^2 y^2 p_{yy}^0(t, y, 1) + \{b_1 - \eta\rho\nu\} y p_y^0(t, y, 1) = g^{0,21}(t, y), \\ p_t^0(t, y, 2) + \frac{1}{2}\nu^2 y^2 p_{yy}^0(t, y, 2) + \{b_2 - \eta\rho\nu\} y p_y^0(t, y, 2) = g^{0,12}(t, y), \\ g_t^{0,21}(t, y) + \frac{1}{2}\nu^2 y^2 g_{yy}^{0,21}(t, y) + \left\{ b_2 - \frac{\gamma(1)}{\gamma(2)} \eta\rho\nu \right\} y g_y^{0,21}(t, y) \\ \quad + \left\{ \lambda_{12} - \lambda_{21} - 2\lambda_{12} f^{0,21}(t) + \frac{1}{2} \left(\frac{\gamma(1)}{\gamma(2)} - 1 \right)^2 \eta^2 \right\} g^{0,21}(t, y) \\ \quad + (b_1 - b_2) f^{0,21}(t) y p_y^0(t, y, 1) + \left(\frac{\gamma(1)}{\gamma(2)} - 1 \right) \eta\rho\nu f^{0,21}(t) y \left[p_y^0(t, y, 1) - p_y^0(t, y, 2) \right] = 0, \\ g_t^{0,12}(t, y) + \frac{1}{2}\nu^2 y^2 g_{yy}^{0,12}(t, y) + \left\{ b_1 - \frac{\gamma(2)}{\gamma(1)} \eta\rho\nu \right\} y g_y^{0,12}(t, y) \\ \quad + \left\{ \lambda_{21} - \lambda_{12} - 2\lambda_{21} f^{0,12}(t) + \frac{1}{2} \left(\frac{\gamma(2)}{\gamma(1)} - 1 \right)^2 \eta^2 \right\} g^{0,12}(t, y) \\ \quad + (b_2 - b_1) f^{0,12}(t) y p_y^0(t, y, 2) + \left(\frac{\gamma(2)}{\gamma(1)} - 1 \right) \eta\rho\nu f^{0,12}(t) y \left[p_y^0(t, y, 2) - p_y^0(t, y, 1) \right] = 0, \\ p^0(T, y, 1) = F(y), \quad p^0(T, y, 2) = F(y), \quad g^{0,21}(T, y) = 0, \quad g^{0,12}(T, y) = 0. \end{array} \right. \quad (4.25)$$

Notice that this is a *linear* parabolic PDE system and the well-posedness of this system is guaranteed by the standard PDE theory. We point out that for the classical case without regime dependence, the marginal utility indifference price is “preference-free”, i.e., it does not depend on the investor’s risk aversion parameter (see, for instance, [65]). In contrast, the marginal equilibrium indifference pricing rule is *not* preference-free, for the risk aversion parameters $\gamma(\cdot)$ is still involved in (4.25).

Furthermore, if we assume

$$b_1 = b_2 = b$$

for some constant b ; in other words, the drift of the nontradable asset is also independent of the market state. One can check that $(\tilde{p}^*(t, y), \tilde{p}^*(t, y), 0, 0)^T$ solves the above system (4.25), where $\tilde{p}^*(t, y)$ solves the following linear second order parabolic PDE

$$\left\{ \begin{array}{l} \tilde{p}_t^* + \frac{1}{2}\nu^2 y^2 \tilde{p}_{yy}^* + \{b - \eta\rho\nu\} y \tilde{p}_y^* = 0, \\ \tilde{p}^*(T, y) = F(y). \end{array} \right.$$

Via the Feynman-Kac approach we have

$$\tilde{p}^*(t, y) = \mathbb{E}^{\tilde{\mathbb{P}}^0} \left[F(Y_T) \middle| Y_t = y \right], \quad (4.26)$$

where $\tilde{\mathbb{P}}^0$ is the *minimal martingale measure* known in the literature (see, for instance, [29] and [65]) and $\mathbb{E}^{\tilde{\mathbb{P}}^0}[\cdot]$ stands for the expectation operator under the minimal martingale measure $\tilde{\mathbb{P}}^0$;

moreover, under $\tilde{\mathbb{P}}^0$ the dynamics of Y is

$$\begin{cases} dY_s = Y_s \{ (b - \eta\rho\nu)ds + \nu[\rho d\tilde{W}_s^0 + \bar{\rho}dW_s^\perp] \}, & s \in [t, T], \\ Y_t = y, \end{cases}$$

where $\tilde{W}^0$ and $W^\perp$ are two standard orthogonal Brownian motions under measure $\tilde{\mathbb{P}}^0$. This gives the same marginal utility indifference price as for the classical case without regime dependence (see, for instance, [65]). In particular, we point out that now the marginal equilibrium utility indifference price is preference-free, i.e., it does not depend on the risk aversion parameter of the investor.

Remark 7. We call $\tilde{p}^*(t, y)$ the *minimal martingale measure price*, or *MMM-price* for short, since it is the expectation of the final payoff of the contingent claim under the minimal martingale measure $\tilde{\mathbb{P}}^0$. The MMM-price will play an important role in the equilibrium mean-variance indifference pricing rule in Chapter 5.

4.4.3 Constant Risk Aversion Parameter Case

In this section, we consider the case that $\mathcal{E} = \{\alpha\}$ is a singleton; in other words, there is only one state for the market and the risk aversion parameter $\gamma(\alpha) = \gamma$ is thus a positive constant. Then, from (4.11) and (4.12), we see the systems for $\mathcal{A}^0(t)$ and $\mathcal{A}^k(t, y)$ are reduced to

$$\begin{cases} \mathcal{A}_t^0(t) - \frac{1}{2}\eta^2(t)\mathcal{A}^0(t) = 0; \\ \mathcal{A}^0(T) = 1 \end{cases}$$

and

$$\begin{cases} \mathcal{A}_t^k(t, y) + \frac{1}{2}\nu^2(t, y)\mathcal{A}_{yy}^k(t, y) + \{b(t, y) - \eta(t)\rho(t)\nu(t, y)\}\mathcal{A}_y^k(t, y) \\ - \frac{1}{2}\eta^2(t)\mathcal{A}^k(t, y) - \frac{1}{2}\rho^2(t)\nu^2(t, y)\frac{(\mathcal{A}_y^k(t, y))^2}{\mathcal{A}^k(t, y)} = 0, \\ \mathcal{A}^k(T, y) = e^{-k\gamma F(y)} \end{cases}$$

respectively. Furthermore, from systems (4.13), (4.14) and (4.15), we can observe that now $f^0(t) = f^k(t, y) = 1$ and $p^k(t, y)$ satisfies the following PDE

$$\begin{cases} p_t^k(t, y) + \frac{1}{2}\nu^2(t, y)p_{yy}^k(t, y) + \{b(t, y) - \eta(t)\rho(t)\nu(t, y)\}p_y^k(t, y) \\ - \frac{1}{2}k\gamma\bar{\rho}^2(t)\nu^2(t, y)(p_y^k(t, y))^2 = 0; \\ p^k(T, y) = F(y). \end{cases} \quad (4.27)$$

(4.27) is the same PDE for the classical exponential utility indifference price (see, for instance, [92] and [65]).

Furthermore, the equilibrium utility indifference hedging policy law now is

$$\hat{\pi}_H^k(t, y) = k\hat{\theta}_H^k(t, y), \quad (4.28)$$

where

$$\hat{\theta}_H^k(t, y) = \frac{\rho(t)\nu(t, y)}{\sigma(t)} \frac{\mathcal{A}_y^k(t, y)}{k\gamma\mathcal{A}^k(t, y)} \quad (4.29)$$

is the equilibrium utility indifference hedging policy law *per claim* of a position of k claims. On the other hand, by definition it holds

$$p^k(t, y) = -\frac{1}{k\gamma} \log \left(\frac{\mathcal{A}^k(t, y)}{\mathcal{A}^{0, \gamma}(t)} \right).$$

Hence, we have

$$p_y^k(t, y) = -\frac{\mathcal{A}_y^k(t, y)}{k\gamma\mathcal{A}^k(t, y)}$$

and thus

$$\hat{\theta}_H^k(t, y) = -\frac{\rho(t)\nu(t, y)}{\sigma(t)} p_y^k(t, y).$$

This is the classical exponential utility indifference hedging policy law (see, for instance, [65]). Therefore, not surprisingly, the equilibrium utility indifference pricing rule is reduced to the classical one for the constant risk aversion parameter case.

4.4.4 Complete Market Case

In this section, we discuss the complete market case; more precisely, we assume $b(t, y, \alpha) = b(t, y)$ for any $\alpha \in \mathcal{E}$ and $\rho(t) \equiv 1$ for $t \in [0, T]$. However, we emphasise that the risk aversion parameter of the investor still depends on the market state; in other words, the investor's preference is still regime dependent. In this situation, replication of the claim is possible and we now examine whether the equilibrium utility indifference price is reduced to the *risk neutral* price derived from the replication method.

For the current case, one can directly check that $(\tilde{p}^*(t, y), f^{0, \beta\alpha}(t))$ solves the coupled PDE system (4.14) and (4.15), where $\tilde{p}^*(t, y)$ is the solution of the linear PDE

$$\begin{cases} \tilde{p}_t^*(t, y) + \frac{1}{2}\nu^2(t, y)\tilde{p}_{yy}^*(t, y) + \left\{ b(t, y) - \eta(t)\nu(t, y) \right\} \tilde{p}_y^*(t, y) = 0; \\ \tilde{p}^*(T, y) = F(y) \end{cases} \quad (4.30)$$

and $f^{0, \beta\alpha}(t)$ is the solution to the autonomous ODE system (4.13). Notice that (4.30) is the exact pricing PDE derived from the replication method under this complete market setting (see, for instance, [65]). Therefore, the equilibrium utility indifference price is reduced to the risk-neutral price in this complete market setting.

4.4.5 Numerical Experiments

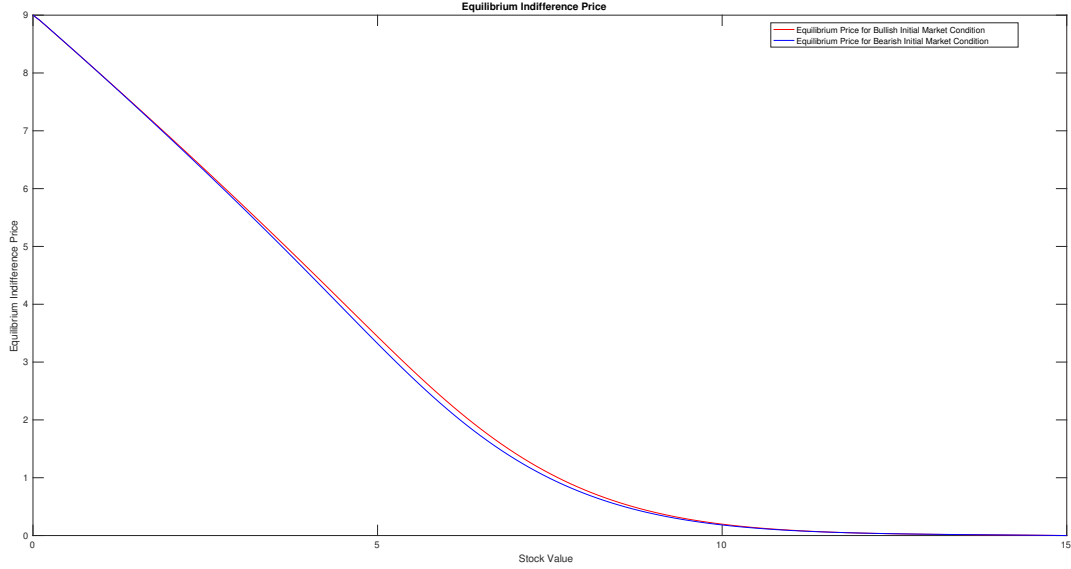
We now report the results of some numerical experiments. Let us first explain the consideration for the choice of the parameters. We always assume that Regime 1 stands for the bullish market and Regime 2 stands for the bearish market. A series of empirical studies (see, for instance, [27], [30] and [81]) suggest that the positive expected excess returns are lower during a bullish market and higher during a bearish market. Hence, according to these studies, we take $b_1 \leq b_2$ in the following numerical tests. Furthermore, as [33] pointed out, the risk aversion parameter is inclined to be higher in the bearish market than in the bullish market (although the results there were for CRRA utility function). Inspired by this, we assume that $\gamma(1) < \gamma(2)$ in the numerical tests.

Figure 4.1 shows the *equilibrium utility indifference prices* of one European put option with strike $K = 9$ and maturity $T = 1$. In the experiment, we take the market parameters $\eta = 0.1$, $b_1 = 0.03$, $b_2 = 0.07$, $\nu = 0.2$ and $\rho = 0.5$; moreover, $\gamma(1) = 0.8$, $\gamma(2) = 1.2$, $\lambda_{12} = 0.08$ and $\lambda_{21} = 0.12$. Both the equilibrium utility indifference prices with respect to the initial market state $\Xi_0 = 1$ and $\Xi_0 = 2$ are contained in Figure 4.1. From Figure 4.1 one can observe that the equilibrium utility indifference price when the initial market is bullish seems to be higher than the one when the initial market is bearish.

We next do the comparison between the *equilibrium* and *precommitment* utility indifference prices. The *precommitment* utility indifference price is defined parallelly as the equilibrium one; however, the precommitment value functions for the utility maximisation problems are used in finding the precommitment utility indifference prices. For the sake of simplicity, we now assume the drift of the nontradable asset is independent of the market regimes; specifically, we take $b_1 = b_2 = b = 0.05$. We show the equilibrium and the precommitment utility indifference prices for both initial market state 1 and 2 in Figure 4.2. Moreover, the marginal equilibrium utility indifference price is also provided. Notice that this price now is the expected terminal payoff under the minimal martingale measure, as we discussed in Section 4.4.2. From Figure 4.2 one can observe that for both cases for bullish and bearish initial market states, the equilibrium utility indifference price seems to be higher than the precommitment utility indifference price but less than the marginal equilibrium utility indifference price.

We also test the dependence of the equilibrium utility indifference prices on the number of claims in the portfolio. We take the numbers of claims to be $k = 1$, $k = 3$, $k = 5$ and $k = 7$. The equilibrium utility indifference prices for both bullish and bearish initial market states are

Figure 4.1: Equilibrium Exponential Utility Indifference Price



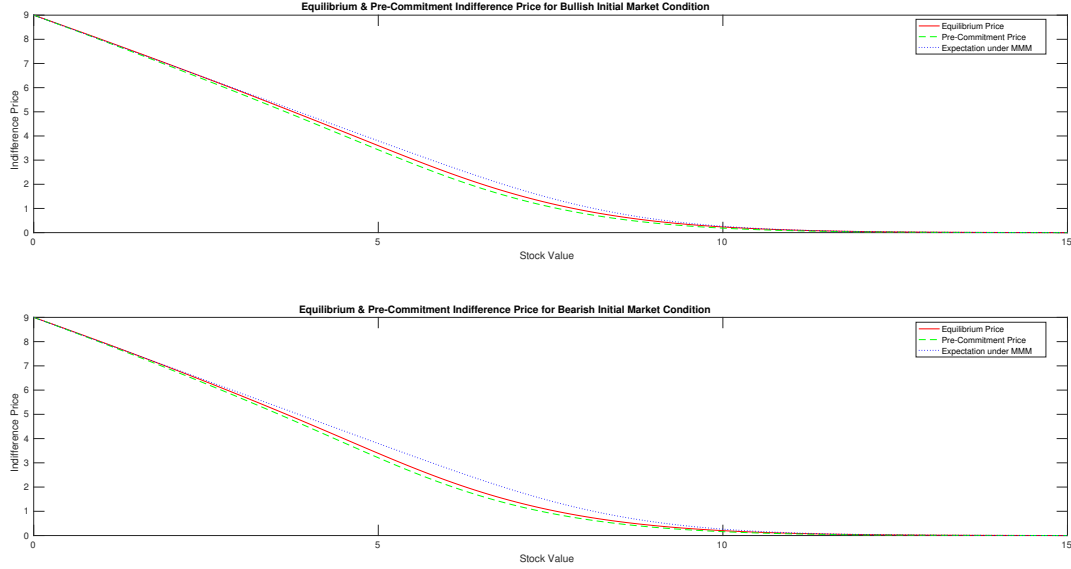
Notes. We give the numerical experiment of *equilibrium* (exponential) utility indifference prices of one European put option with strike $K = 9$ and maturity $T = 1$. In the experiment, we take the market parameters $\eta = 0.1$, $b_1 = 0.03$, $b_2 = 0.07$, $\nu = 0.2$ and $\rho = 0.5$; moreover, $\gamma(1) = 0.8$, $\gamma(2) = 1.2$, $\lambda_{12} = 0.08$ and $\lambda_{21} = 0.12$.

reported in Figure 4.3. From Figure 4.3 one can observe that for both cases for bullish and bearish initial market states, the equilibrium utility indifference prices seem to decline as the number of claims holding grows. This observation is consistent with the intuition that, as the number of claims increasing, the investor should require an extra discount for the claims.

We then test the influence of the risk aversion parameters on the equilibrium utility indifference prices. We fix the risk aversion parameter for the bullish market at $\gamma(1) = 0.8$ while let the risk aversion parameter for the bearish market $\gamma(2)$ increase from 1.0, 1.2, 1.4, to 1.6. The result is shown in Figure 4.4. From the picture, we can observe a slight increase of the deviation between the equilibrium utility indifference prices with different initial market states as $\gamma(2)$ grows. The numerical results of the $\mathcal{L}^2$ -distance of the equilibrium utility indifference prices $p^1(0, y, 1)$ and $p^1(0, y, 2)$ are 0.05556, 0.16876, 0.30656, and 0.44867 accordingly. It supports the intuition that, as the difference of risk aversion parameters $\gamma(1)$ and $\gamma(2)$ increases, the equilibrium utility indifference prices with respect to different initial market states will also deviate more with each other.

4.4. EQUILIBRIUM UTILITY INDIFFERENCE PRICING RULE IN BASIS RISK MODEL

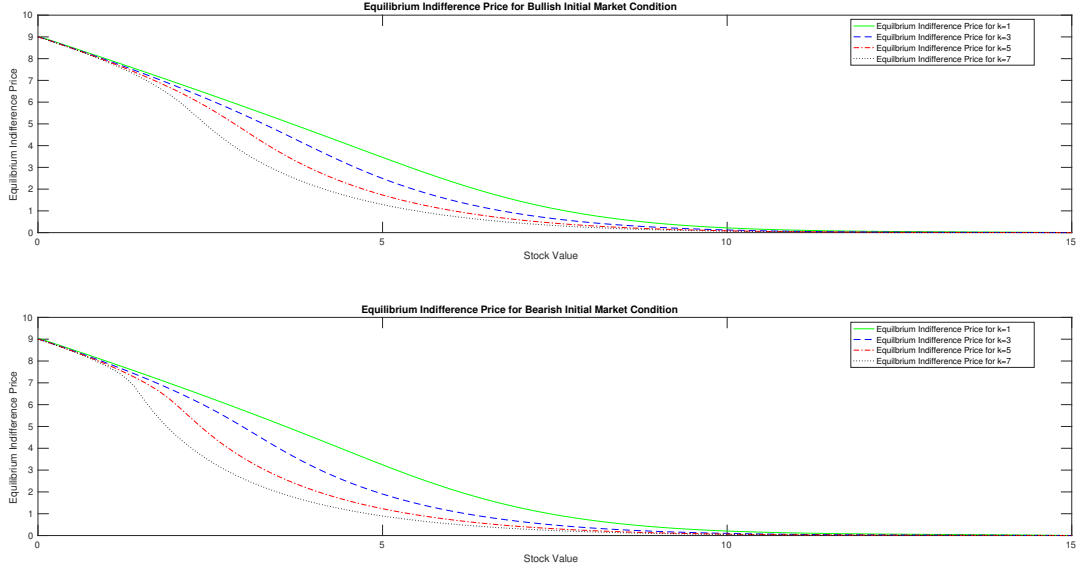
Figure 4.2: Equilibrium and Precommitment Exponential Utility Indifference Price



Notes. We give the numerical experiment of *equilibrium* and *precommitment* utility indifference price of one European put option with strike $K = 9$ and maturity $T = 1$. In the experiment, we take the market parameters $\eta = 0.1$, $b_1 = b_2 = 0.05$, $\nu = 0.2$ and $\rho = 0.5$; moreover, $\gamma(1) = 0.8$, $\gamma(2) = 1.2$, $\lambda_{12} = 0.08$ and $\lambda_{21} = 0.12$.

Finally, we show the tests for the influence of the drift coefficient on the equilibrium utility indifference prices. We fix the drift of the nontradable asset in the bullish market at $b_1 = 0.03$ while let the drift in the bearish market b_2 increase from 0.05, 0.07, 0.09, to 0.11. The result is presented in Figure 4.5. The numerical results of the $\mathcal{L}^2$ -distance of the equilibrium utility indifference prices $p^1(0, y, 1)$ and $p^1(0, y, 2)$, as b_2 grows, are 0.15557, 0.16876, 0.18281, and 0.19670 accordingly. We see there is a very slight growth of the $\mathcal{L}^2$ -distance between the equilibrium utility indifference prices with different initial market conditions as b_2 increases. It implies that as the difference of the drift coefficients b_1 and b_2 increases, which reflects that the market fluctuates more and more heavily among different market regimes, the deviation of the equilibrium utility indifference prices with respect to different initial market states will also increase.

Figure 4.3: Equilibrium Utility Indifference Price For Different Numbers of Claims



Notes. We give the numerical experiment of *equilibrium* utility indifference prices of different numbers of European put options with strike $K = 9$ and maturity $T = 1$. In the experiment, we take the market parameters $\eta = 0.1$, $b_1 = 0.03$, $b_2 = 0.07$, $\nu = 0.2$, $\rho = 0.5$, $\lambda_{12} = 0.08$ and $\lambda_{21} = 0.12$; moreover, the risk aversion parameters are $\gamma(1) = 0.8$ and $\gamma(2) = 1.2$. The numbers of claims in the portfolio are $k = 1$, $k = 3$, $k = 5$ and $k = 7$ accordingly.

4.5 Equilibrium Utility Indifference Pricing for Logarithmic Payoff

In this section, we assume $b(t, y, \alpha) = b(t, \alpha)y$ and $\nu(t, y, \alpha) = \nu(t, \alpha)y$ for some bounded functions $b(t, \alpha)$ and $\nu(t, \alpha)$. We now study a special payoff which is of *logarithmic* type. Specifically, the payoff function is $F(y) = \log(y)$ written on Y_T . In this special case, we are able to obtain an explicit formula for the equilibrium utility indifference price $p^k(t, y, \alpha)$.

Now the dynamics of Y is given by the following SDE

$$\begin{cases} dY_u = Y_u \left\{ b(u, \Xi_u) du + \nu(u, \Xi_u) [\rho(u, \Xi_u) dW_u + \bar{\rho}(u, \Xi_u) dW_u^\perp] \right\}; \\ Y_t = y. \end{cases}$$

We do the logarithmic transform by denoting $\tilde{Y}_t = \log(Y_t)$. Then the SDE for process $\tilde{Y}$ is

$$\begin{cases} d\tilde{Y}_u = \left\{ b(u, \Xi_u) - \frac{1}{2}\nu^2(u, \Xi_u) \right\} du + \nu(u, \Xi_u) \left\{ \rho(u, \Xi_u) dW_u + \bar{\rho}(u, \Xi_u) dW_u^\perp \right\}; \\ \tilde{Y}_t = \log(y). \end{cases}$$

4.5. EQUILIBRIUM UTILITY INDIFFERENCE PRICING FOR LOGARITHMIC PAYOFF

Equivalently, we want to find the *equilibrium utility indifference price* for a new claim with terminal payoff $\tilde{Y}_T$.

To facilitate the analysis, we introduce the auxiliary function

$$\tilde{\mathcal{A}}^k(t, z, \beta, \alpha) = \mathcal{A}^k(t, e^z, \beta, \alpha).$$

In the current setting, the ODE system for $\mathcal{A}^0(t, \beta, \alpha)$ is unchanged as (4.11) and the PDE system for $\tilde{\mathcal{A}}^k(t, z, \beta, \alpha)$ is

$$\left\{ \begin{aligned} & \tilde{\mathcal{A}}_t^k(t, z, \beta, \alpha) + \frac{1}{2} \nu^2(t, \beta) \tilde{\mathcal{A}}_{zz}^k(t, z, \beta, \alpha) + \left\{ b(t, \beta) - \frac{1}{2} \nu^2(t, \beta) \right\} \tilde{\mathcal{A}}_z^k(t, z, \beta, \alpha) \\ & \quad + \sum_{l \neq \beta} \lambda_{\beta l} \left\{ \tilde{\mathcal{A}}^k(t, z, l, \alpha) - \tilde{\mathcal{A}}^k(t, z, \beta, \alpha) \right\} \\ & \quad + \frac{1}{2} \frac{\gamma^2(\alpha)}{\gamma^2(\beta)} \left(\eta(t, \beta) + \rho(t, \beta) \nu(t, \beta) \frac{\tilde{\mathcal{A}}_z^k(t, z, \beta, \beta)}{\tilde{\mathcal{A}}^k(t, z, \beta, \beta)} \right)^2 \\ & \quad - \frac{\gamma(\alpha)}{\gamma(\beta)} \left(\eta^2(t, \beta) + \rho(t, \beta) \nu(t, \beta) \frac{\tilde{\mathcal{A}}_z^k(t, z, \beta, \beta)}{\tilde{\mathcal{A}}^k(t, z, \beta, \beta)} \right) \tilde{\mathcal{A}}^k(t, z, \beta, \alpha) \\ & \quad - \frac{\gamma(\alpha)}{\gamma(\beta)} \left(\rho(t, \beta) \nu(t, \beta) \eta(t, \beta) + \rho^2(t, \beta) \nu^2(t, \beta) \frac{\tilde{\mathcal{A}}_z^k(t, z, \beta, \beta)}{\tilde{\mathcal{A}}^k(t, z, \beta, \beta)} \right) \tilde{\mathcal{A}}_z^k(t, z, \beta, \alpha) = 0, \\ & \tilde{\mathcal{A}}^k(T, z, \beta, \alpha) = e^{-\gamma(\alpha)kz}, \end{aligned} \right.$$

for $\alpha, \beta \in \mathcal{E}$. We now make the following Ansatz:

Ansatz 20. $\tilde{\mathcal{A}}^k(t, z, \beta, \alpha)$ takes the following form

$$\tilde{\mathcal{A}}^k(t, z, \beta, \alpha) = e^{-\gamma(\alpha)kz} \mathcal{B}^k(t, \beta, \alpha).$$

Direct calculation then gives the ODE system for $\mathcal{B}^k(t, \beta, \alpha)$:

$$\left\{ \begin{aligned} & \mathcal{B}_t^k(t, \beta, \alpha) = \left\{ \frac{1}{2} \frac{\gamma(\alpha)}{\gamma(\beta)} \left(2 - \frac{\gamma(\alpha)}{\gamma(\beta)} \right) \eta^2(t, \beta) + \sum_{l \neq \beta} \lambda_{\beta l} - \frac{1}{2} k^2 \gamma^2(\alpha) \bar{\rho}^2(t, \beta) \nu^2(t, \beta) \right. \\ & \quad \left. + k \gamma(\alpha) \left\{ b(t, \beta) - \frac{1}{2} \nu^2(t, \beta) - \eta(t, \beta) \rho(t, \beta) \nu(t, \beta) \right\} \right\} \mathcal{B}^k(t, \beta, \alpha) \\ & \quad - \sum_{l \neq \beta} \lambda_{\beta l} \mathcal{B}^k(t, l, \alpha); \\ & \mathcal{B}^k(T, \beta, \alpha) = 1 \end{aligned} \right.$$

for $\alpha, \beta \in \mathcal{E}$. This is a linear ODE system and the well-posedness is guaranteed by the standard ODE theory.

Now the equilibrium utility indifference price becomes

$$p^k(t, y, \alpha) = \log(y) - \frac{1}{k\gamma(\alpha)} \log \left(\frac{\mathcal{B}^k(t, \alpha, \alpha)}{\mathcal{A}^0(t, \alpha, \alpha)} \right)$$

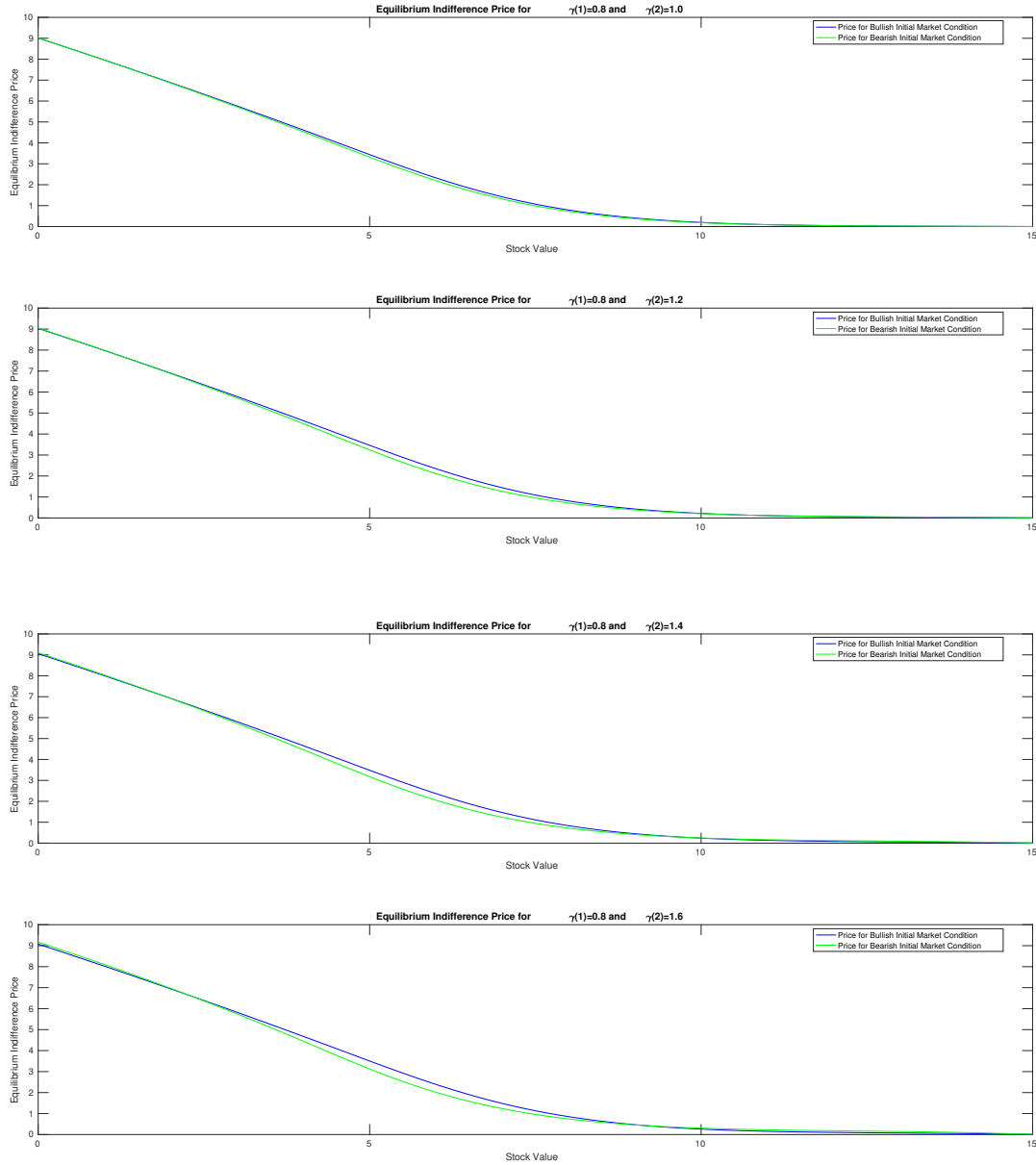
and the equilibrium utility indifference hedging policy law becomes

$$\hat{\pi}_H^k(t, \alpha) = -k \frac{\rho(t, \alpha) \nu(t, \alpha)}{\sigma(t, \alpha)}$$

which is independent of y . Notice that the second part of the equilibrium utility indifference price, $-\frac{1}{k\gamma(\alpha)} \log \left(\frac{\mathcal{B}^k(t, \alpha, \alpha)}{\mathcal{A}^0(t, \alpha, \alpha)} \right)$, can be viewed as the *time value* at time t of the contingent claim with the logarithmic payoff and the maturity T .

4.5. EQUILIBRIUM UTILITY INDIFFERENCE PRICING FOR LOGARITHMIC PAYOFF

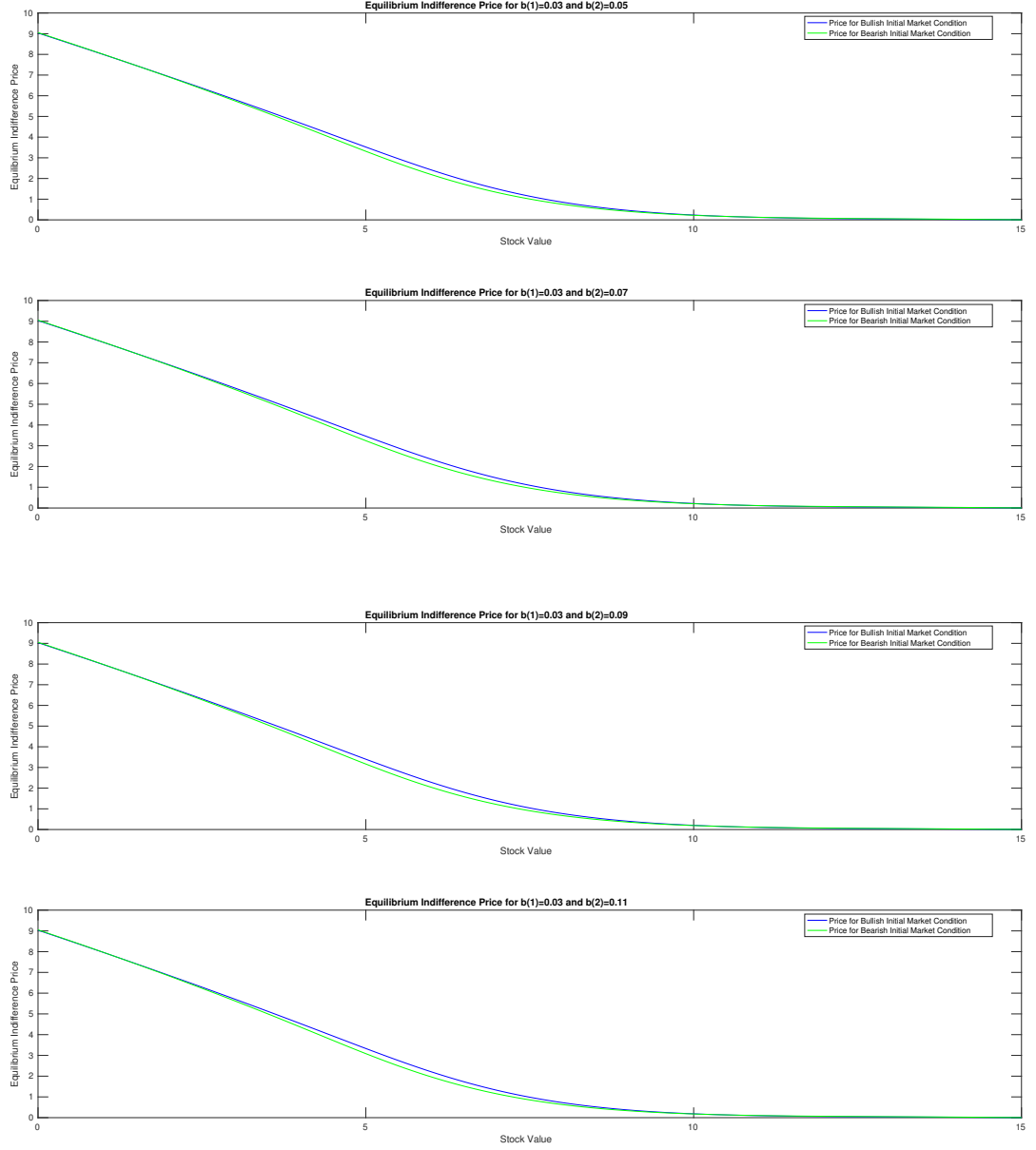
Figure 4.4: Equilibrium Utility Indifference Price For Different Risk Aversion Parameters



Notes. We give the numerical experiment of *equilibrium* utility indifference prices of one European put option with strike $K = 9$ and maturity $T = 1$ for different risk aversion parameters. In the experiment, we take the market parameters $\eta = 0.1$, $b_1 = 0.03$, $b_2 = 0.07$, $\nu = 0.2$, $\rho = 0.5$, $\lambda_{12} = 0.08$ and $\lambda_{21} = 0.12$. The risk aversion parameter for bullish market is fixed at $\gamma(1) = 0.8$, while the risk aversion parameters for bearish market are $\gamma(2) = 1.0$, $\gamma(2) = 1.2$, $\gamma(2) = 1.4$, $\gamma(2) = 1.6$ accordingly.

4. EQUILIBRIUM UTILITY INDIFFERENCE PRICING WITH REGIME SWITCHING

Figure 4.5: Equilibrium Utility Indifference Price For Different Drifts Of The Nontradable Asset



Notes. We give the numerical experiment of *equilibrium* utility indifference prices of one European put option with strike $K = 9$ and maturity $T = 1$ for different risk aversion parameters. In the experiment, we take the model parameters $\eta = 0.1$, $\nu = 0.2$, $\rho = 0.5$, $\lambda_{12} = 0.08$ and $\lambda_{21} = 0.12$; moreover, the risk aversion parameters are $\gamma(1) = 0.8$ and $\gamma(2) = 1.2$. The drift for the nontradable asset in bullish market is fixed at $b_1 = 0.03$, while the drifts in bearish market are $b_2 = 0.05$, $b_2 = 0.07$, $b_2 = 0.09$, $b_2 = 0.11$ accordingly.

Chapter 5

Equilibrium Mean-Variance Indifference Pricing with Regime Switching

We continue the study of time inconsistent option pricing problems in this chapter. The market model is the same as in Chapter 4; however, we now assume the preference of the investor is characterised by the mean-variance optimisation criterion with regime switching. The portfolio selection problem for the investor is thus time inconsistent, similar to Chapter 4. Two mean-variance optimisation problems with regime switching are solved, and an equilibrium mean-variance indifference pricing rule is provided. Some properties of the equilibrium mean-variance indifference pricing rule are discussed subsequently.

5.1 Basic Problem

In this section, we state the option pricing problem under consideration.

5.1.1 Mean Variance Portfolio Selection with Regime Switching

The market model in this chapter is the same as in Chapter 4. There are one riskless bond B , one tradable risky asset S and one nontradable risky asset Y in the market; moreover, the dynamics of B , S and Y are given by (4.1), (4.2) and (4.3). In this chapter, we also employ the indifference pricing approach to value contingent claim written on the *non-tradable* asset Y . However, we assume the investor's preference is characterised by the *mean-variance optimisation* criterion, which is popular in the financial industry. To this end, we first clarify the underlying mean-variance optimisation problems. More precisely, given the trading strategy π , which is

the same as in Chapter 4 and the corresponding wealth x^π whose dynamics follows (4.4), the objective functional for the investor at time t without any claim in her portfolio is given by

$$J^0(t, x, \alpha; \pi_\cdot) = \mathbb{E}\left[x_T^\pi \middle| x_t = x, \Xi_t = \alpha\right] - \frac{\gamma(\alpha)}{2} \text{Var}\left(x_T^\pi \middle| x_t = x, \Xi_t = \alpha\right),$$

where $\gamma : \mathcal{E} \rightarrow \mathbb{R}_+$ is the weighting parameter which is regime dependent; in contrast, if she decided to hold k claims in her portfolio, the objective functional for the investor at time t is given by

$$\begin{aligned} J^k(t, x, y, \alpha; \pi_\cdot) = & \mathbb{E}\left[x_T^\pi + kF(Y_T) \middle| x_t = x, Y_t = y, \Xi_t = \alpha\right] \\ & - \frac{\gamma(\alpha)}{2} \text{Var}\left(x_T^\pi + kF(Y_T) \middle| x_t = x, Y_t = y, \Xi_t = \alpha\right), \end{aligned}$$

where $F(\cdot)$ is the payoff function of the contingent claim. As in Chapter 4, we use superscript “0” or “ k ” to emphasise that the corresponding objective functional is with respect to the problem with 0 or k claims in the portfolio.

It is well-known in the literature that even classical mean-variance optimisation criterion introduces time inconsistency due to the *non*-linearity of variance operator (see, for instance, [3], [4] and [8]). Moreover, in the current setting, the weighting parameter $\gamma(\cdot)$ is no longer a constant but depends on the current market state $\Xi_\cdot$. This weighting parameter balances the *return* and the *risk* of the investor; more precisely, it reflects the investor’s tradeoff between the expectation and the variance of her terminal wealth. We believe this weighting parameter should depend on the current state of the financial market. The dependence of the weighting parameter on the market state, as explained in Chapter 2, also introduces time inconsistency into the portfolio selection problems. Once more, we adopt the *equilibrium* approach to tackling with time inconsistency. We will look for the *equilibrium policy laws* and *value functions* for the underlying two time inconsistent stochastic control problems—with and without the claims in the portfolio.

Due to the linearity of the wealth process x^π with respect to the control process $\pi_\cdot$, we further make the following assumption, similar to Chapters 3 and 4, that the aforementioned equilibrium policy laws are of the form $\hat{\pi}^0(t, \alpha)$ and $\hat{\pi}^k(t, y, \alpha)$ respectively; in other words, the equilibrium policy laws are independent of the wealth variable x .

5.1.2 Mean Variance Indifference Pricing with Regime Switching

After obtained the equilibrium value functions $\hat{V}^0(t, x, \alpha)$ and $\hat{V}^k(t, x, y, \alpha)$ and the equilibrium policy laws $\hat{\pi}^0(t, x, \alpha)$ and $\hat{\pi}^k(t, x, y, \alpha)$ for the underlying time inconsistent stochastic

control problems, we introduce the following definition of *equilibrium mean-variance indifference price rule*.

Definition 21. An *equilibrium mean-variance indifference price* (pre claim) $\hat{p}^k(t, x, y, \alpha)$ for a position of k contingent claims written on Y_T with payoff function $F(\cdot)$ is the solution of the following algebraic equation:

$$\hat{V}^0(t, x, \alpha) = \hat{V}^k(t, x - k\hat{p}^k(t, x, y, \alpha), y, \alpha).$$

Moreover, the difference of the equilibrium policy laws

$$\hat{\pi}_H^k(t, x, y, \alpha) = \hat{\pi}^k(t, x, y, \alpha) - \hat{\pi}^0(t, x, \alpha) \quad (5.1)$$

is the corresponding *equilibrium mean-variance indifference hedging policy law*. The equilibrium mean-variance indifference price $\hat{p}^k(t, x, y, \alpha)$ together with the equilibrium mean-variance indifference hedging policy law $\hat{\pi}_H^k(t, x, y, \alpha)$ constitute an *equilibrium mean-variance indifference pricing rule*.

We give this equilibrium mean-variance indifference pricing rule in the next section.

5.2 General Equilibrium Mean-Variance Indifference Pricing Rule

We first find the equilibrium value functions $\hat{V}^0(t, x, \alpha)$ and $\hat{V}^k(t, x, y, \alpha)$ and the equilibrium policy laws $\hat{\pi}^0(t, \alpha)$ and $\hat{\pi}^k(t, y, \alpha)$, which are critical to the equilibrium mean-variance indifference pricing rule. To this end, we first recall two lemmas. The first one is the celebrated total variance formula.

Lemma 22. For given $\mathcal{F}_T$ -measurable random variable X and $t \in [0, T)$, it holds

$$\text{Var}(X|\mathcal{F}_t) = \mathbb{E} \left[\text{Var}(X|\mathcal{F}_{t+\Delta t}) \middle| \mathcal{F}_t \right] + \text{Var} \left(\mathbb{E}[X|\mathcal{F}_{t+\Delta t}] \middle| \mathcal{F}_t \right),$$

for any $\Delta t \in [0, T - t)$.

It is well-known that adding a constant has no contribution to variance. The second lemma can be viewed as a generalisation to this fact to the *conditional* variance case.

Lemma 23. For given $\mathcal{F}_T$ -measurable random variable X and $\mathcal{F}_t$ -measurable random variable y , we have

$$\text{Var}(y + X|\mathcal{F}_t) = \text{Var}(X|\mathcal{F}_t),$$

for any $t \in [0, T)$ and $\Delta t \in [0, T - t)$.

The proofs for both lemmas above are direct calculations from the definitions, and are thus omitted.

5.2.1 Mean-Variance Optimisation without Contingent Claim

We first assume a priori that an *equilibrium policy law* $\hat{\pi}^0(t, \alpha)$ exists for the mean-variance optimisation problem without any claim in the portfolio; furthermore, $\hat{x}^0$ denotes the corresponding equilibrium wealth process

$$\hat{x}_u^0 = x + \int_t^u \hat{\pi}^0(r, \Xi_r) \left\{ \mu(r, \Xi_r) dr + \sigma(r, \Xi_r) dW_r \right\}$$

given initial capital level $\hat{x}_t^0 = x$.

To facilitate the analysis, we now introduce the following two auxiliary functions:

$$\left\{ \begin{array}{l} \mathcal{B}^0(t, \beta) = \mathbb{E} \left[\int_t^T \hat{\pi}^0(u, \Xi_u) \mu(u, \Xi_u) du \middle| \Xi_t = \beta \right]; \\ \mathcal{C}^0(t, \beta, \alpha) = \mathcal{B}^0(t, \beta) - \frac{\gamma(\alpha)}{2} \text{Var} \left(\int_t^T \hat{\pi}^0(u, \Xi_u) \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \middle| \Xi_t = \beta \right) \\ = \mathbb{E} \left[\int_t^T \hat{\pi}^0(u, \Xi_u) \mu(u, \Xi_u) du \middle| \Xi_t = \beta \right] \\ - \frac{\gamma(\alpha)}{2} \text{Var} \left(\int_t^T \hat{\pi}^0(u, \Xi_u) \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \middle| \Xi_t = \beta \right) \end{array} \right.$$

for any $t \in [0, T)$ and $\alpha, \beta \in \mathcal{E}$. Similar to the auxiliary functions in Chapters 3 and 4, in $\mathcal{C}^0(t, \beta, \alpha)$ the first state variable β stands for the market state at the *current* time t and the second state variable α represents the market state at some initial time *before* t .

The following proposition gives the *extended Hamilton-Jacobi-Bellman (HJB) equation* for the regime switching mean-variance optimisation problem without any claim in the portfolio.

Proposition 24. *Assume the above two auxiliary functions $\mathcal{C}^0(t, \beta, \alpha)$ and $\mathcal{B}^0(t, \beta)$ are continuously differentiable in t . Then $\mathcal{C}^0(t, \beta, \alpha)$ and $\mathcal{B}^0(t, \beta)$ satisfy the following extended Hamilton-Jacobi-Bellman (HJB) equation:*

$$\left\{ \begin{array}{l} \mathcal{C}_t^0(t, \alpha, \alpha) + \sup_{\pi} \left\{ -\frac{\gamma(\alpha)}{2} \sigma^2(t, \alpha) \pi^2 + \mu(t, \alpha) \pi \right\} \\ + \sum_{l \neq \alpha} \lambda_{\alpha l} \left\{ \mathcal{C}^0(t, l, \alpha) - \mathcal{C}^0(t, \alpha, \alpha) \right\} - \frac{\gamma(\alpha)}{2} \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right)^2 = 0; \\ \mathcal{C}^0(T, \alpha, \alpha) = 0, \quad \mathcal{B}^0(T, \alpha) = 0 \end{array} \right. \quad (5.2)$$

for any $t \in [0, T]$ and $\alpha, \beta \in \mathcal{E}$.

5.2. GENERAL EQUILIBRIUM MEAN-VARIANCE INDIFFERENCE PRICING RULE

Proof. We first notice that, according to Lemma 23, the objective functional corresponding to $\hat{\pi}^0(t, \alpha)$ is

$$J^0(t, x, \alpha; \hat{\pi}^0) = x + \mathcal{C}^0(t, \alpha, \alpha).$$

Moreover, for any $\pi \in \mathbb{R}$ the objective functional corresponding to perturbed policy law

$$\pi^{0|t, h, \pi}(u, \alpha) = \begin{cases} \pi, & u \in [t, t+h); \\ \hat{\pi}^0(u, \alpha), & u \in [t+h, T) \end{cases}$$

is

$$\begin{aligned} J^0(t, x, \alpha; \pi^{0|t, h, \pi}) &= x + \mathbb{E} \left[\int_t^{t+h} \pi \mu(u, \Xi_u) du + \mathcal{C}^0(t+h, \Xi_{t+h}, \alpha) \middle| \Xi_t = \alpha \right] \\ &\quad - \frac{\gamma(\alpha)}{2} \text{Var} \left(\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} + \mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right), \end{aligned}$$

where we used Lemma 22 and 23 together with the fact that $\int_t^\cdot \hat{\pi}^0(r, \Xi_r) \sigma(r, \Xi_r) dW_r$ is a martingale, due to Assumption 10 and the admissibility of $\hat{\pi}^0(t, \alpha)$.

To show that (5.2) is the *extended HJB equation*, first we have

$$\begin{aligned} &\lim_{h \rightarrow 0+} \frac{\mathbb{E} \left[\int_t^{t+h} \pi \mu(u, \Xi_u) du + \mathcal{C}^0(t+h, \Xi_{t+h}, \alpha) \middle| \Xi_t = \alpha \right] - \mathcal{C}^0(t, \alpha, \alpha)}{h} \\ &= \mathcal{C}_t^0(t, \alpha, \alpha) + \sum_{l \neq \alpha} \lambda_{al} \left\{ \mathcal{C}^0(t, l, \alpha) - \mathcal{C}^0(t, \alpha, \alpha) \right\} + \mu(t, \alpha) \pi \end{aligned}$$

by the generalised Dynkin's formula. Next, we decompose

$$\frac{\text{Var} \left(\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} + \mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right)}{h} = I_1^0(h) + I_2^0(h) + 2I_3^0(h),$$

where

$$\begin{aligned} I_1^0(h) &= \frac{1}{h} \left\{ \mathbb{E} \left[\left(\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \right)^2 \middle| \Xi_t = \alpha \right] \right. \\ &\quad \left. - \left(\mathbb{E} \left[\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \middle| \Xi_t = \alpha \right] \right)^2 \right\}, \\ I_2^0(h) &= \frac{\mathbb{E} \left[\left(\mathcal{B}^0(t+h, \Xi_{t+h}) \right)^2 \middle| \Xi_t = \alpha \right] - \left(\mathbb{E} \left[\mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right] \right)^2}{h}, \\ I_3^0(h) &= \frac{1}{h} \left\{ \mathbb{E} \left[\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right] \right. \\ &\quad \left. - \mathbb{E} \left[\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \middle| \Xi_t = \alpha \right] \mathbb{E} \left[\mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right] \right\}. \end{aligned}$$

We then discuss $I_1^0(h)$, $I_2^0(h)$ and $I_3^0(h)$ separately. First, from Itô isometry we have

$$\begin{aligned} I_1^0(h) = & h \left\{ \mathbb{E} \left[\left(\frac{1}{h} \int_t^{t+h} \pi \mu(u, \Xi_u) du \right)^2 \middle| \Xi_t = \alpha \right] - \left(\mathbb{E} \left[\frac{1}{h} \int_t^{t+h} \pi \mu(u, \Xi_u) du \middle| \Xi_t = \alpha \right] \right)^2 \right\} \\ & + \mathbb{E} \left[\frac{1}{h} \int_t^{t+h} \pi^2 \sigma^2(u, \Xi_u) du \middle| \Xi_t = \alpha \right] \\ & + 2 \mathbb{E} \left[\left(\frac{1}{h} \int_t^{t+h} \pi \mu(u, \Xi_u) du \right) \left(\int_t^{t+h} \pi \sigma(u, \Xi_u) dW_u \right) \middle| \Xi_t = \alpha \right]. \end{aligned}$$

Furthermore, from Assumption 10 we see as $h \rightarrow 0+$ the first line vanishes and the limit of the second line is

$$\lim_{h \rightarrow 0+} \mathbb{E} \left[\frac{1}{h} \int_t^{t+h} \pi^2 \sigma^2(u, \Xi_u) du \middle| \Xi_t = \alpha \right] = \sigma^2(t, \alpha) \pi^2.$$

As to the last line, by Cauchy-Schwartz inequality we get

$$\begin{aligned} & \left| \mathbb{E} \left[\left(\frac{1}{h} \int_t^{t+h} \pi \mu(u, \Xi_u) du \right) \left(\int_t^{t+h} \pi \sigma(u, \Xi_u) dW_u \right) \middle| \Xi_t = \alpha \right] \right| \\ & \leq \mathbb{E} \left[\left| \frac{1}{h} \int_t^{t+h} \pi \mu(u, \Xi_u) du \right| \left| \int_t^{t+h} \pi \sigma(u, \Xi_u) dW_u \right| \middle| \Xi_t = \alpha \right] \\ & \leq \sqrt{\mathbb{E} \left[\left(\frac{1}{h} \int_t^{t+h} \pi \mu(u, \Xi_u) du \right)^2 \middle| \Xi_t = \alpha \right] \mathbb{E} \left[\left(\int_t^{t+h} \pi^2 \sigma^2(u, \Xi_u) du \right) \middle| \Xi_t = \alpha \right]} \\ & \rightarrow \sqrt{\mu^2(t, \alpha) \pi^2 \times 0} = 0 \end{aligned}$$

as $h \rightarrow 0+$. We then obtain

$$\lim_{h \rightarrow 0+} I_1^0(h) = \sigma^2(t, \alpha) \pi^2. \quad (5.3)$$

Next, by simple algebra we have

$$\begin{aligned} I_2^0(h) = & \frac{\mathbb{E} \left[\left(\mathcal{B}^0(t+h, \Xi_{t+h}) \right)^2 \middle| \Xi_t = \alpha \right] - \left(\mathcal{B}^0(t, \alpha) \right)^2}{h} \\ & - \left(\frac{\mathbb{E} \left[\mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right] - \mathcal{B}^0(t, \alpha)}{h} \right) \left(\mathbb{E} \left[\mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right] + \mathcal{B}^0(t, \alpha) \right). \end{aligned}$$

and, according to the generalised Dynkin's formula,

$$\frac{\mathbb{E} \left[\mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right] - \mathcal{B}^0(t, \alpha)}{h} = \mathcal{B}_t^0(t, \alpha) + \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right)$$

and

$$\begin{aligned} & \frac{\mathbb{E} \left[\left(\mathcal{B}^0(t+h, \Xi_{t+h}) \right)^2 \middle| \Xi_t = \alpha \right] - \left(\mathcal{B}^0(t, \alpha) \right)^2}{h} \\ &= 2\mathcal{B}^0(t, \alpha) \left\{ \mathcal{B}_t^0(t, \alpha) + \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right) \right\} + \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right)^2. \end{aligned}$$

Moreover, by the regularity assumption on $\mathcal{B}^0(\cdot, \alpha)$ we see

$$\lim_{h \rightarrow 0} \mathbb{E} \left[\mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right] = \mathcal{B}^0(t, \alpha).$$

Therefore,

$$\lim_{h \rightarrow 0+} I_2^0(h) = \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right)^2. \quad (5.4)$$

Finally, we have

$$\lim_{h \rightarrow 0+} \mathbb{E} \left[\int_t^{t+h} \pi \mu(u, \Xi_u) du \middle| \Xi_t = \alpha \right] = \mu(t, \alpha) \pi.$$

and, according to the generalised Dynkin's formula,

$$\lim_{h \rightarrow 0+} \frac{\mathbb{E} \left[\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} \mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right]}{h} = \mu(t, \alpha) \mathcal{B}^0(t, \alpha) \pi$$

Hence,

$$\lim_{h \rightarrow 0+} I_3^0(h) = 0. \quad (5.5)$$

Combining (5.3), (5.4) and (5.5), we see

$$\begin{aligned} & \frac{\text{Var} \left(\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} + \mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right)}{h} \\ &= \sigma^2(t, \alpha) \pi^2 + \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right)^2. \end{aligned}$$

We know for any π we have

$$\begin{aligned} & - \liminf_{h \rightarrow 0+} \frac{J^0(t, x, \alpha; \hat{\pi}^0) - J^0(t, x, \alpha; \pi^0|t, h, \pi)}{h} \\ &= \limsup_{h \rightarrow 0+} \left\{ \frac{\mathbb{E} \left[\int_t^{t+h} \pi \mu(u, \Xi_u) du + \mathcal{C}^0(t+h, \Xi_{t+h}, \alpha) \middle| \Xi_t = \alpha \right] - \mathcal{C}^0(t, \alpha, \alpha)}{h} \right. \\ & \quad \left. - \frac{\gamma(\alpha)}{2} \frac{\text{Var} \left(\int_t^{t+h} \pi \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} + \mathcal{B}^0(t+h, \Xi_{t+h}) \middle| \Xi_t = \alpha \right)}{h} \right\} \\ &= \mathcal{C}_t^0(t, \alpha, \alpha) + \sum_{l \neq \alpha} \lambda_{\alpha l} \left\{ \mathcal{C}^0(t, l, \alpha) - \mathcal{C}^0(t, \alpha, \alpha) \right\} - \frac{\gamma(\alpha)}{2} \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right)^2 \\ & \quad + \left\{ -\frac{\gamma(\alpha)}{2} \sigma^2(t, \alpha) \pi^2 + \mu(t, \alpha) \pi \right\} \\ &\leq 0 \end{aligned}$$

from the definition of equilibrium policy law (see Definition 1 in Chapter 2 for reference); moreover, $\hat{\pi}^0(t, \alpha)$ makes the above inequality an equality. Hence, (5.2) is the *extended HJB equation*. $\square$

The next theorem gives the characterisation of *equilibrium policy law* and *value function*.

Theorem 25. *Assume the coefficients $\mu(t, \alpha)$ and $\sigma(t, \alpha)$ are continuous in t for any $\alpha \in \mathcal{E}$. Then*

$$\hat{\pi}^0(t, \alpha) = \frac{\eta(t, \alpha)}{\gamma(\alpha)\sigma(t, \alpha)}, \quad (5.6)$$

is an equilibrium policy law and

$$\hat{V}^0(t, x, \alpha) = x + \mathcal{C}^0(t, \alpha, \alpha), \quad (5.7)$$

is the corresponding equilibrium value function for the mean-variance optimisation problem without any claim in the portfolio, where $\mathcal{B}^0(t, \alpha)$ and $\mathcal{C}^0(t, \beta, \alpha)$ satisfy the following systems of ODEs

$$\begin{cases} \mathcal{B}_t^0(t, \alpha) + \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right) + \frac{1}{\gamma(\alpha)} \eta^2(t, \alpha) = 0; \\ \mathcal{B}^0(T, \alpha) = 0 \end{cases} \quad (5.8)$$

and

$$\begin{cases} \mathcal{C}_t^0(t, \beta, \alpha) + \sum_{l \neq \beta} \lambda_{\beta l} \left\{ \mathcal{C}^0(t, l, \alpha) - \mathcal{C}^0(t, \beta, \alpha) \right\} \\ - \frac{\gamma(\alpha)}{2} \sum_{l \neq \beta} \lambda_{\beta l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \beta) \right)^2 + \frac{1}{\gamma(\beta)} \left(1 - \frac{\gamma(\alpha)}{2\gamma(\beta)} \right) \eta^2(t, \beta) = 0; \\ \mathcal{C}^0(T, \beta, \alpha) = 0 \end{cases} \quad (5.9)$$

respectively, for each $\alpha, \beta \in \mathcal{E}$.

Proof. The concavity of the supremum part in the *extended HJB equation* (5.2) with respect to π is straightforward and, therefore, (5.6) is the maximiser of the supremum part in (5.2) by the first-order condition. Moreover, by Assumptions 10 and 12 we see that $\hat{\pi}^0(t, \alpha)$ is bounded and thus admissible. Furthermore, from the standard ODE theory we can see that $\mathcal{B}^0(t, \alpha)$ and $\mathcal{C}^0(t, \beta, \alpha)$ are continuously differentiable with respect to t . Therefore, all the conditions in Proposition 24 are satisfied, and (5.6) and (5.7) are indeed the equilibrium policy law and value function for the mean-variance optimisation problem without any claim in the portfolio.

Recall that the definition of $\mathcal{B}^0(t, \alpha)$ is

$$\mathcal{B}^0(t, \alpha) = \mathbb{E} \left[\int_t^T \hat{\pi}^0(u, \Xi_u) \mu(u, \Xi_u) du \middle| \Xi_t = \alpha \right].$$

5.2. GENERAL EQUILIBRIUM MEAN-VARIANCE INDIFFERENCE PRICING RULE

Following the Feynman-Kac approach, we have

$$\mu(t, \alpha) \hat{\pi}^0(t, \alpha) + \mathcal{B}_t^0(t, \alpha) + \sum_{l \neq \alpha} \lambda_{\alpha l} \left\{ \mathcal{B}^0(t, l) - \mathcal{B}^0(t, \alpha) \right\} = 0.$$

Substituting (5.6) into above, we then get the ODE system (5.8) to characterise $\mathcal{B}^0(t, \alpha)$.

Similarly, via the Feynman-Kac approach, we have

$$\begin{cases} \mathcal{C}_t^0(t, \beta, \alpha) + \sum_{l \neq \beta} \lambda_{\beta l} \left\{ \mathcal{C}^0(t, l, \alpha) - \mathcal{C}^0(t, \beta, \alpha) \right\} \\ - \frac{\gamma(\alpha)}{2} \sum_{l \neq \beta} \lambda_{\beta l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \beta) \right)^2 - \frac{\gamma(\alpha)}{2} \sigma^2(t, \beta) \left(\hat{\pi}^0(t, \beta) \right)^2 + \mu(t, \beta) \hat{\pi}^0(t, \beta) = 0; \\ \mathcal{C}^0(T, \beta, \alpha) = 0 \end{cases}$$

for any $t \in [0, T]$, and $\alpha, \beta \in \mathcal{E}$. Now substituting (5.6) into above, we then get the ODE system (5.9) to characterise $\mathcal{C}^0(t, \beta, \alpha)$. $\square$

Remark 8. Notice that the equilibrium policy law

$$\hat{\pi}^0(t, \alpha) = \frac{1}{\gamma(\alpha)} \frac{\eta(t, \alpha)}{\sigma(t, \alpha)}$$

is exactly the *same* as the equilibrium policy law for the time inconsistent CARA utility maximization problem we discussed in previous two chapters, see Theorem 4 in Chapter 3 and Theorem 13 in Chapter 4. In other words, the exponential utility maximisation and mean-variance optimisation criteria give the *same* equilibrium policy law for the time inconsistent portfolio selection problem with regime switching.

We now give a probabilistic characterisation of $\mathcal{B}^0(t, \alpha)$ by exploiting Girsanov theorem. More precisely, we make the following change of measure

$$\frac{d\tilde{\mathbb{P}}^0}{d\mathbb{P}} = \exp \left(-\frac{1}{2} \int_0^T \eta^2(u, \Xi_u) du - \int_0^T \eta(u, \Xi_u) dW_u \right). \quad (5.10)$$

Under this measure $\tilde{\mathbb{P}}^0$,

$$\tilde{W}_t^0 = W_t + \int_0^t \eta(u, \Xi_u) du \quad (5.11)$$

is a standard Brownian motion, and the dynamics of S_t becomes

$$dS_t = S_t \sigma(t, \Xi_t) d\tilde{W}_t^0.$$

Remark 9. This probability measure $\tilde{\mathbb{P}}^0$ is known as the *minimal martingale measure (MMM)* in the literature as mentioned in Section 4.4.2.

The probabilistic representation of $\mathcal{B}^0(t, \alpha)$ is then given, via the Feynman-Kac approach, in the next theorem.

Theorem 26. *We have the following representation for $\mathcal{B}^0(t, \alpha)$:*

$$\mathcal{B}^0(t, \alpha) = \mathbb{E}^{\tilde{\mathbb{P}}^0} \left[\int_t^T \frac{1}{\gamma(\Xi_u)} \eta^2(u, \Xi_u) du \middle| \Xi_t = \alpha \right], \quad (5.12)$$

where $\mathbb{E}^{\tilde{\mathbb{P}}^0}[\cdot]$ stands for the expectation operator under measure $\tilde{\mathbb{P}}^0$ whose Radon-Nikodým derivative with respect to $\mathbb{P}$ is given by (5.10).

5.2.2 Mean-Variance Optimisation with Contingent Claim

We now give an equilibrium policy law $\hat{\pi}^k(t, y, \alpha)$ and the corresponding equilibrium value function $\hat{V}^k(t, x, y, \alpha)$ for the mean-variance optimisation problem with k claims in the portfolio. The discussion below is parallel to the previous section.

We assume the existence of the equilibrium policy law $\hat{\pi}^k(t, y, \alpha)$ and denote $\hat{x}_t^k$ the corresponding equilibrium wealth process for the mean-variance optimisation problem with k claims in the portfolio; for any $t < u \leq T$,

$$\hat{x}_u^k = x + \int_t^u \hat{\pi}^k(r, Y_r, \Xi_r) \left\{ \mu(r, \Xi_r) dr + \sigma(r, \Xi_r) dW_r \right\}$$

given initial capital level $\hat{x}_t^k = x$.

Mimicing the analysis in previous section, we introduce the following two auxiliary functions

$$\left\{ \begin{array}{l} \mathcal{B}^k(t, y, \beta) = \mathbb{E} \left[\int_t^T \hat{\pi}^k(u, Y_u, \Xi_u) \mu(u, \Xi_u) du + kF(Y_T) \middle| \Xi_t = \beta, Y_t = y \right]; \\ \mathcal{C}^k(t, y, \beta, \alpha) = \mathcal{B}^k(t, y, \beta) \\ \quad - \frac{\gamma(\alpha)}{2} \text{Var} \left(\int_t^T \hat{\pi}^k(u, Y_u, \Xi_u) \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} + kF(Y_T) \middle| \Xi_t = \beta, Y_t = y \right) \\ \quad = \mathbb{E} \left[\int_t^T \hat{\pi}^k(u, Y_u, \Xi_u) \mu(u, \Xi_u) du + kF(Y_T) \middle| \Xi_t = \beta, Y_t = y \right] \\ \quad - \frac{\gamma(\alpha)}{2} \text{Var} \left(\int_t^T \hat{\pi}^k(u, Y_u, \Xi_u) \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\} + kF(Y_T) \middle| \Xi_t = \beta, Y_t = y \right) \end{array} \right.$$

for any $(t, y) \in [0, T) \times \mathbb{R}$ and $\alpha, \beta \in \mathcal{E}$. Once more, in $\mathcal{C}^k(t, y, \beta, \alpha)$ the first state variable β stands for the market state at the *current* time t and the second state variable α represents the market state at some initial time *before* t .

5.2. GENERAL EQUILIBRIUM MEAN-VARIANCE INDIFFERENCE PRICING RULE

We now report the corresponding *extended HJB equation* and *equilibrium policy law* and *value function* for the regime switching mean-variance optimisation problem with k claims in the portfolio. The proofs are parallel to the ones in Section 5.2.1 and thus omitted.

Proposition 27. *Assume the above two auxiliary functions $\mathcal{C}^k(t, y, \beta, \alpha)$ and $\mathcal{B}^k(t, y, \beta)$ are continuously differentiable in variable t and twice continuously differentiable in variable y for each $\alpha, \beta \in \mathcal{E}$, and each partial derivative of $\mathcal{C}^k(t, y, \beta, \alpha)$ and $\mathcal{B}^k(t, y, \beta)$ has polynomial growth. Then $\mathcal{C}^k(t, y, \beta, \alpha)$ and $\mathcal{B}^k(t, y, \beta)$ satisfy the following extended Hamilton-Jacobi-Bellman (HJB) equation:*

$$\left\{ \begin{aligned} & \mathcal{C}_t^k(t, y, \alpha, \alpha) + \frac{1}{2} \nu^2(t, y, \alpha) \mathcal{C}_{yy}^k(t, y, \alpha, \alpha) + b(t, y, \alpha) \mathcal{C}_y^k(t, y, \alpha, \alpha) \\ & \quad + \sum_{l \neq \alpha} \lambda_{\alpha l} \left\{ \mathcal{C}^k(t, y, l, \alpha) - \mathcal{C}^k(t, y, \alpha, \alpha) \right\} \\ & - \frac{\gamma(\alpha)}{2} \left\{ \nu^2(t, y, \alpha) \left(\mathcal{B}_y^k(t, y, \alpha) \right)^2 + \sum_{l \neq \alpha} \lambda_{\alpha l} \left(\mathcal{B}^k(t, y, l) - \mathcal{B}^k(t, y, \alpha) \right)^2 \right\} \\ & + \sup_{\pi} \left\{ -\frac{\gamma(\alpha)}{2} \sigma^2(t, \alpha) \pi^2 + \left[\mu(t, \alpha) - \gamma(\alpha) \sigma(t, \alpha) \rho(t, \alpha) \nu(t, y, \alpha) \mathcal{B}_y^k(t, y, \alpha) \right] \pi \right\} = 0; \\ & \mathcal{C}^k(T, y, \alpha, \alpha) = 0, \quad \mathcal{B}^k(T, y, \alpha) = 0 \end{aligned} \right. \quad (5.13)$$

for any $t \in [0, T]$, $y \in \mathbb{R}$ and $\alpha, \beta \in \mathcal{E}$.

Theorem 28. *Assume all the conditions in Proposition 27 are satisfied. Then*

$$\hat{\pi}^k(t, y, \alpha) = \frac{\eta(t, \alpha)}{\gamma(\alpha) \sigma(t, \alpha)} - \rho(t, \alpha) \frac{\nu(t, y, \alpha)}{\sigma(t, \alpha)} \mathcal{B}_y^k(t, y, \alpha). \quad (5.14)$$

is an equilibrium policy law and

$$\hat{V}^k(t, x, y, \alpha) = x + \mathcal{C}^k(t, y, \alpha, \alpha), \quad (5.15)$$

is the corresponding equilibrium value function for the mean-variance optimisation problem with k claims in the portfolio, where $\mathcal{B}^k(t, y, \alpha)$ and $\mathcal{C}^k(t, y, \beta, \alpha)$ satisfy the following systems of PDEs

$$\left\{ \begin{aligned} & \mathcal{B}_t^k(t, y, \alpha) + \frac{1}{2} \nu^2(t, y, \alpha) \mathcal{B}_{yy}^k(t, y, \alpha) + \left(b(t, y, \alpha) - \rho(t, \alpha) \eta(t, \alpha) \nu(t, y, \alpha) \right) \mathcal{B}_y^k(t, y, \alpha) \\ & \quad + \sum_{l \neq \alpha} \lambda_{\alpha l} \left\{ \mathcal{B}^k(t, y, l) - \mathcal{B}^k(t, y, \alpha) \right\} + \frac{1}{\gamma(\alpha)} \eta^2(t, \alpha) = 0; \\ & \mathcal{B}^k(T, y, \alpha) = kF(y) \end{aligned} \right. \quad (5.16)$$

and

$$\left\{ \begin{aligned} & \mathcal{C}_t^k(t, y, \beta, \alpha) + \frac{1}{2} \nu^2(t, y, \beta) \mathcal{C}_{yy}^k(t, y, \beta, \alpha) + b(t, y, \beta) \mathcal{C}_y^k(t, y, \beta, \alpha) \\ & \quad + \sum_{l \neq \beta} \lambda_{\beta l} \left\{ \mathcal{C}^k(t, y, l, \alpha) - \mathcal{C}^k(t, y, \beta, \alpha) \right\} \\ & - \frac{\gamma(\alpha)}{2} \left\{ \nu^2(t, y, \beta) \left(\mathcal{B}_y^k(t, y, \beta) \right)^2 + \sum_{l \neq \beta} \lambda_{\beta l} \left(\mathcal{B}^k(t, y, l) - \mathcal{B}^k(t, y, \beta) \right)^2 \right\} \\ & - \frac{\gamma(\alpha)}{2\gamma^2(\beta)} \left\{ \eta(t, \beta) - \gamma(\beta) \rho(t, \beta) \nu(t, y, \beta) \mathcal{B}_y^k(t, y, \beta) \right\}^2 \\ & + \frac{1}{\gamma(\beta)} \left\{ \eta(t, \beta) - \gamma(\alpha) \rho(t, \beta) \nu(t, y, \beta) \mathcal{B}_y^k(t, y, \beta) \right\} \left\{ \eta(t, \beta) - \gamma(\beta) \rho(t, \beta) \nu(t, y, \beta) \mathcal{B}_y^k(t, y, \beta) \right\} = 0; \end{aligned} \right.$$

$$\mathcal{C}^k(T, y, \beta, \alpha) = kF(y) \quad (5.17)$$

respectively, for each $\alpha, \beta \in \mathcal{E}$.

We now perform the same change of measure as in Section 5.2.1 and give a probabilistic characterisation of $\mathcal{B}^k(t, y, \alpha)$. Recall the *minimal martingale measure* $\tilde{\mathbb{P}}^0$ is defined in (5.10) and, under this $\tilde{\mathbb{P}}^0$, the dynamics of Y is

$$dY_t = \left(b(t, Y_t, \Xi_t) - \rho(t, \Xi_t) \eta(t, \Xi_t) \nu(t, Y_t, \Xi_t) \right) dt + \nu(t, Y_t, \Xi_t) \left\{ \rho(t, \Xi_t) d\tilde{W}_t^0 + \tilde{\rho}(t, \Xi_t) dW_t^\perp \right\},$$

where $\tilde{W}^0$ is defined in (5.11) and $W^\perp$ is a standard Brownian motion under measure $\tilde{\mathbb{P}}^0$ and is *orthogonal* to $\tilde{W}^0$. As in Section 5.2.1, via the Feynman-Kac approach, we have the following probabilistic representation for $\mathcal{B}^k(t, y, \alpha)$.

Theorem 29. *We have the following representation for $\mathcal{B}^k(t, y, \alpha)$:*

$$\begin{aligned} \mathcal{B}^k(t, y, \alpha) &= \mathbb{E}^{\tilde{\mathbb{P}}^0} \left[\int_t^T \frac{1}{\gamma(\Xi_u)} \eta^2(u, \Xi_u) du + kF(Y_T) \middle| \Xi_t = \alpha, Y_t = y \right] \\ &= \mathcal{B}^0(t, \alpha) + k\tilde{p}^*(t, y, \alpha), \end{aligned} \quad (5.18)$$

where $\mathcal{B}^0(t, \alpha)$ and $\tilde{p}^*(t, y, \alpha)$ are given by (5.12) and

$$\tilde{p}^*(t, y, \alpha) = \mathbb{E}^{\tilde{\mathbb{P}}^0} \left[F(Y_T) \middle| \Xi_t = \alpha, Y_t = y \right] \quad (5.19)$$

respectively.

Remark 10. $\tilde{p}^*(t, y, \alpha)$ is the expected terminal payoff of the contingent claim written on Y_T with payoff function $F(\cdot)$ under the minimal martingale measure $\tilde{\mathbb{P}}^0$. We call $\tilde{p}^*(t, y, \alpha)$ the *MMM-price* of the contingent claim, which has already been mentioned in Section 4.4.2, Chapter 4 for a special case without regime switching.

5.2.3 Equilibrium Mean-Variance Indifference Pricing Rule

With the equilibrium value functions $\hat{V}^0(t, x, \alpha)$ and $\hat{V}^k(t, x, y, \alpha)$, we can now calculate the equilibrium mean-variance indifference price for the contingent claim written on Y_T with payoff function $F(\cdot)$. Recall an *equilibrium mean-variance indifference price* $\hat{p}^k(t, x, y, \alpha)$ for k claims is defined as the solution of the following algebraic equation:

$$\hat{V}^0(t, x, \alpha) = \hat{V}^k(t, x - k\hat{p}^k(t, x, y, \alpha), y, \alpha).$$

Therefore,

$$\hat{p}^k(t, y, \alpha) = \frac{\mathcal{C}^k(t, y, \alpha, \alpha) - \mathcal{C}^0(t, \alpha, \alpha)}{k}.$$

Hence, by solving systems (5.8), (5.16), (5.9) and (5.17), we can get $\mathcal{B}^0(t, \alpha)$, $\mathcal{B}^k(t, y, \alpha)$, $\mathcal{C}^0(t, \alpha, \alpha)$ and $\mathcal{C}^k(t, y, \alpha, \alpha)$ and this *equilibrium mean-variance indifference price* $\hat{p}^k(t, x, y, \alpha)$ is then obtained. Moreover, according to Theorem 25 and Theorem 28, the corresponding *equilibrium mean-variance indifference hedging policy law* $\hat{\pi}_H^k(t, y, \alpha)$ is

$$\begin{aligned} \hat{\pi}_H^k(t, y, \alpha) &= \hat{\pi}^k(t, y, \alpha) - \hat{\pi}^0(t, \alpha) \\ &= -\rho(t, \alpha) \frac{\nu(t, y, \alpha)}{\sigma(t, \alpha)} \mathcal{B}_y^k(t, y, \alpha) = -k\rho(t, \alpha) \frac{\nu(t, y, \alpha)}{\sigma(t, \alpha)} \tilde{p}_y^*(t, y, \alpha), \end{aligned}$$

where $\tilde{p}^*(t, y, \alpha)$ is the *MMM-price* defined in (5.19). We see, in particular, the equilibrium mean-variance indifference hedging policy law $\hat{\pi}_H^k(t, y, \alpha)$ is *linear* with respect to k , the number of claims in the portfolio.

We now report the probabilistic representation of this equilibrium mean-variance indifference price $\hat{p}^k(t, y, \alpha)$.

Theorem 30. *The equilibrium mean-variance indifference price $\hat{p}^k(t, y, \alpha)$ has the following probabilistic representation:*

$$\begin{aligned} \hat{p}^k(t, y, \alpha) &= \mathbb{E}\left[F(Y_T) \mid \Xi_t = \alpha, Y_t = y\right] - \mathbb{E}\left[\int_t^T g_\alpha(u, Y_u, \Xi_u) du \mid \Xi_t = \alpha, Y_t = y\right] \\ &\quad - k \frac{\gamma(\alpha)}{2} \mathbb{E}\left[\int_t^T G(u, Y_u, \Xi_u) du \mid \Xi_t = \alpha, Y_t = y\right], \end{aligned} \tag{5.20}$$

where

$$\begin{aligned} g_\alpha(t, y, \beta) &= \eta(t, \beta) \rho(t, \beta) \nu(t, y, \beta) \tilde{p}_y^*(t, s, y, \beta) \\ &\quad + \gamma(\alpha) \sum_{l \neq \beta} \lambda_{\beta l} \left(\mathcal{B}^0(t, l) - \mathcal{B}^0(t, \beta) \right) \left(\tilde{p}^*(t, y, l) - \tilde{p}^*(t, y, \beta) \right), \\ G(t, y, \beta) &= \bar{\rho}^2(t, \beta) \nu^2(t, y, \beta) \left(\tilde{p}_y^*(t, y, \beta) \right)^2 + \sum_{l \neq \beta} \lambda_{\beta l} \left(\tilde{p}^*(t, y, l) - \tilde{p}^*(t, y, \beta) \right)^2, \end{aligned} \tag{5.21}$$

and $\tilde{p}^*(t, y, \alpha)$ is the *MMM-price* in (5.19).

Remark 11. We emphasise that the expectation in (5.20) is under the physical measure $\mathbb{P}$; moreover, both $g_\alpha(t, y, \beta)$ and $G(t, y, \beta)$ are *independent* of k , the number of claims in the portfolio. Notice that we use the subscript “ α ” in $g_\alpha(t, y, \beta)$ to emphasise the dependence on state α via $\gamma(\alpha)$ in the second line of (5.21); in contrast, $G(t, y, \beta)$ is independent of α .

Proof. The result is from a straightforward application of the Feynman-Kac approach to (5.9) and (5.17), together with some algebra. $\square$

In the end of this section, we introduce the concept of *marginal equilibrium mean-variance indifference price*.

Definition 31. Given an equilibrium mean-variance indifference price $\hat{p}^k(t, y, \alpha)$, the corresponding *marginal equilibrium mean-variance indifference price* $\hat{p}^0(t, y, \alpha)$ is defined as

$$\hat{p}^0(t, y, \alpha) = \lim_{k \rightarrow 0} \hat{p}^k(t, y, \alpha),$$

provided the above limit exists.

From (5.20), we can easily obtain this *marginal equilibrium mean-variance indifference price* $\hat{p}^0(t, y, \alpha)$ as

$$\hat{p}^0(t, y, \alpha) = \mathbb{E} \left[F(Y_T) \middle| \Xi_t = \alpha, Y_t = y \right] - \mathbb{E} \left[\int_t^T g_\alpha(u, Y_u, \Xi_u) du \middle| \Xi_t = \alpha, Y_t = y \right], \quad (5.22)$$

where $g_\alpha(t, y, \beta)$ is defined in (5.21). We emphasise again that the expectation in (5.22) is under the physical measure $\mathbb{P}$.

5.3 Some Special Cases

In this section, we consider several special cases and hope to deepen our understanding about the *equilibrium mean-variance indifference pricing rule* in these simplified cases.

5.3.1 Two-Market-State Basis Risk Model

We first investigate the equilibrium mean-variance indifference pricing rule in a two-market-state basis risk model with constant coefficients, similar to Chapter 4. Specifically, we assume $\phi(t, \alpha) = \phi$ for $\phi = \mu, \sigma, \eta, \rho$ and $\psi(t, y, \alpha) = \psi y$ for $\psi = b, \nu$; in particular, the prices of both the tradable and nontradable assets do not depend on the general financial market state. Notice that the weighting parameter $\gamma(\cdot)$ is still a function of market state $\alpha \in \mathcal{E}$, thus only the preference of

5.3. SOME SPECIAL CASES

the investor is still regime dependent. Furthermore, we assume that $\mathcal{E} = \{1, 2\}$, i.e., the market has only *two* states—*bearish* and *bullish*.

Under the current setting, the Radon-Nikodým derivative of the minimal martingale measure $\tilde{\mathbb{P}}^0$ with respect to $\mathbb{P}$ is reduced to

$$\frac{d\tilde{\mathbb{P}}^0}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\eta^2 T - \eta W_T\right)$$

and $g_\alpha(t, y, \beta)$ and $G(t, y, \beta)$ are reduced to

$$\begin{aligned} g(t, y) &= \eta\rho\nu y\tilde{p}_y^*(t, y), \\ G(t, y) &= \bar{\rho}^2\nu^2 y^2 (\tilde{p}_y^*(t, y))^2. \end{aligned}$$

In particular, $g_\alpha(t, y, \beta)$, $G(t, y, \beta)$ and the *MMM-price* $\tilde{p}^*(t, y)$ are functions of t and y only and independent of the market state α .

Now the *equilibrium mean-variance indifference price* $\hat{p}^k(t, y, \alpha)$ is reduced to

$$\begin{aligned} \hat{p}^k(t, y, \alpha) &= \mathbb{E}\left[F(Y_T) \middle| Y_t = y\right] - \eta\rho\nu\mathbb{E}\left[\int_t^T Y_u \tilde{p}_y^*(u, Y_u) du \middle| Y_t = y\right] \\ &\quad - k \frac{\gamma(\alpha)}{2} \bar{\rho}^2 \nu^2 \mathbb{E}\left[\int_t^T Y_u^2 (\tilde{p}_y^*(u, Y_u))^2 du \middle| Y_t = y\right]. \end{aligned}$$

Furthermore, the *equilibrium mean-variance indifference hedging policy law* $\hat{\pi}_H^k(t, y)$ is

$$\hat{\pi}_H^k(t, y) = -k \frac{\rho\nu}{\sigma} y \tilde{p}_y^*(t, y).$$

Notice that the hedging policy $\hat{\pi}_H^k(t, y)$ is independent of the market state α , and the indifference price $\hat{p}^k(t, y, \alpha)$ depends on the market state α only via $\gamma(\alpha)$; in other words, the dependence of the equilibrium mean-variance indifference price on the market state is only through the corresponding weighting parameter, and there is no relationship between different equilibrium mean-variance indifference prices for different market regimes, which is much simpler than the case for the equilibrium utility indifference price discussed in Chapter 4. This implies that, for the basis risk model, the feature of regime switching is *not* essential for the equilibrium mean-variance indifference pricing rule. Inspired by this, we turn to study the equilibrium mean-variance indifference pricing rule without regime switching in the next section.

5.3.2 Mean-Variance Indifference Pricing without Regime Switching

In this section, we study the equilibrium mean-variance indifference pricing rule without regime switching; i.e., we assume $\phi(t, \alpha) = \phi(t)$ for $\phi = \mu, \sigma, \eta, \rho$, $\psi(t, y, \alpha) = \psi(t, y)$ for $\psi = b, \nu$ and $\gamma(\alpha) = \gamma$. We study both the *equilibrium* and the *precommitment* mean-variance indifference pricing rules and make the comparison between them.

5.3.2.1 Equilibrium Mean Variance Indifference Pricing Rule

From the discussion in Section 5.3.1 we see that, for the current case, the *equilibrium mean-variance indifference price* $\hat{p}^k(t, y)$ is given by

$$\begin{aligned} \hat{p}^k(t, y) = & \mathbb{E} \left[F(Y_T) - \int_t^T \rho(u) \eta(u) \nu(u, Y_u) \tilde{p}_y^*(u, Y_u) du \middle| Y_t = y \right] \\ & - k \frac{\gamma}{2} \mathbb{E} \left[\int_t^T \bar{\rho}^2(u) \nu^2(u, Y_u) \left(\tilde{p}_y^*(u, Y_u) \right)^2 du \middle| Y_t = y \right], \end{aligned}$$

where $\tilde{p}^*(t, y)$ is the *MMM-price* in (5.19) (which is now independent of α). Recall the PDE for $\tilde{p}^*(t, y)$ is

$$\begin{cases} \tilde{p}_t^*(t, y) + \frac{1}{2} \nu^2(t, y) \tilde{p}_{yy}^*(t, y) + b(t, y) \tilde{p}_y^*(t, y) = \eta(t) \rho(t) \nu(t, y) \tilde{p}_y^*(t, y), \\ \tilde{p}^*(T, y) = F(y). \end{cases}$$

Then, via the Feynman-Kac approach, we see

$$\tilde{p}^*(t, y) = \mathbb{E} \left[F(Y_T) - \int_t^T \rho(u) \eta(u) \nu(u, Y_u) \tilde{p}_y^*(u, Y_u) du \middle| Y_t = y \right].$$

Therefore, we have

$$\hat{p}^k(t, y) = \tilde{p}^*(t, y) - k \frac{\gamma}{2} \mathbb{E} \left[\int_t^T \bar{\rho}^2(u) \nu^2(u, Y_u) \left(\tilde{p}_y^*(u, Y_u) \right)^2 du \middle| Y_t = y \right]. \quad (5.23)$$

Furthermore, the *marginal equilibrium mean-variance indifference price* is

$$\hat{p}^0(t, y) = \tilde{p}^*(t, y) = \mathbb{E}^{\tilde{\mathbb{P}}^0} \left[F(Y_T) \middle| Y_t = y \right]; \quad (5.24)$$

in other words, the *marginal equilibrium mean-variance indifference price* coincides with the *MMM-price*.

On the other hand, the *equilibrium mean-variance indifference hedging policy law* $\hat{\pi}_H^k(t, y)$ is given by

$$\hat{\pi}_H^k(t, y) = k \hat{\pi}_H^*(t, y),$$

where

$$\hat{\pi}_H^*(t, y) = - \frac{\rho(t) \nu(t, y) \tilde{p}_y^*(t, y)}{\sigma(t)} \quad (5.25)$$

is the equilibrium mean-variance indifference hedging policy law per claim. In fact, this $\hat{\pi}_H^*(t, y)$ is related to the equilibrium policy law of minimal variance hedging for this claim (see, for instance, [4]).

5.3. SOME SPECIAL CASES

We now analyse the properties of this *equilibrium mean-variance indifference price* $\hat{p}^k(t, y)$. First, we see that $\hat{p}^k(t, y)$ is linear with respect to k , the number of claims in the portfolio, which is similar to the equilibrium mean-variance indifference hedging policy law $\hat{\pi}_H^k(t, y)$.

Next, we discuss the case of $\rho(t) \equiv 1$ for $t \in [0, T]$, where the market model is *complete* and the replication of the claim is possible. The *equilibrium mean-variance indifference price* and *hedging policy law* now are

$$\hat{p}^k(t, y) = \tilde{p}^*(t, y)$$

and

$$\hat{\pi}_H^k(t, y) = -k \frac{\nu(t, y) \tilde{p}_y^*(t, y)}{\sigma(t)}$$

respectively. Notice that they are the *risk-neutral price* and the *delta-hedging policy law* derived from the replication method respectively (see, for instance, [65]). Therefore, for the complete market case, the equilibrium mean-variance indifference pricing rule is reduced to the risk-neutral pricing rule.

We now show that the equilibrium mean-variance indifference pricing rule is *concave* with respect to the payoff function $F(\cdot)$ of the claim; more precisely, suppose there are two types of contingent claim with payoff functions $F_1(\cdot)$ and $F_2(\cdot)$. Moreover, let $\hat{p}_1^k(t, y)$ and $\hat{p}_2^k(t, y)$ be the corresponding equilibrium mean-variance indifference prices and $\hat{p}_\lambda^k(t, y)$ be the equilibrium mean-variance indifference price of the claim with payoff $\lambda F_1(Y_T) + (1 - \lambda) F_2(Y_T)$, for $\lambda \in (0, 1)$. Then it holds

$$\hat{p}_\lambda^k(t, y) \geq \lambda \hat{p}_1^k(t, y) + (1 - \lambda) \hat{p}_2^k(t, y).$$

We show this via the direct calculation:

$$\begin{aligned} \hat{p}^k(t, y) &= \lambda \tilde{p}_y^{*,1}(t, y) + \tilde{p}_y^{*,1}(t, y) \\ &\quad - k \frac{\gamma}{2} \mathbb{E} \left[\int_t^T (1 - \rho^2(u)) \nu^2(u, Y_u) \left(\lambda \tilde{p}_y^{*,1}(u, Y_u) + (1 - \lambda) \tilde{p}_y^{*,2}(u, Y_u) \right)^2 du \middle| Y_t = y \right] \\ &\geq \lambda \left\{ \tilde{p}_y^{*,1}(t, y) - k \frac{\gamma}{2} \mathbb{E} \left[\int_t^T \bar{\rho}^2(u) \nu^2(u, Y_u) \left(\tilde{p}_y^{*,1}(u, Y_u) \right)^2 du \middle| Y_t = y \right] \right\} \\ &\quad + (1 - \lambda) \left\{ \tilde{p}_y^{*,2}(t, y) - k \frac{\gamma}{2} \mathbb{E} \left[\int_t^T \bar{\rho}^2(u) \nu^2(u, Y_u) \left(\tilde{p}_y^{*,2}(u, Y_u) \right)^2 du \middle| Y_t = y \right] \right\} \\ &= \lambda \hat{p}_1^k(t, y) + (1 - \lambda) \hat{p}_2^k(t, y), \end{aligned}$$

where

$$\tilde{p}_y^{*,i}(t, y) = \mathbb{E}^{\mathbb{P}^0} [F_i(Y_T) | Y_t = y], \quad \text{for } i = 1, 2$$

are the corresponding *MMM-prices* for the contingent claims with payoff $F_1(\cdot)$ and $F_2(\cdot)$ respectively.

We point out that the equilibrium mean-variance indifference pricing rule has no *monotonic* property; more precisely, for two claims with payoff functions $F_1(\cdot)$ and $F_2(\cdot)$ and $F_1(\cdot) \leq F_2(\cdot)$, we cannot assert that $\hat{p}_1^k(t, y) \leq \hat{p}_2^k(t, y)$ in general. This is due to the fact that from (5.23), $\hat{p}^k(t, y)$ also depends on $\tilde{p}_y^*(t, y)$, whose monotonicity depends on the concavity of the MMM-price $\tilde{p}^*(t, y)$.

Notice that for the case $\rho(u) \neq 1$ on $u \in [0, T]$, we have

$$\mathbb{E} \left[\int_t^T \bar{\rho}^2(u) \nu^2(u, Y_u) \left(\tilde{p}_y^*(u, Y_u) \right)^2 du \middle| Y_t = y \right] > 0,$$

thus for $0 < k_1 < k_2$ we see

$$\hat{p}^{k_1}(t, y) > \hat{p}^{k_2}(t, y).$$

This indicates that the investor requires more discount to buy more contingent claims. In particular, for some sufficient large $k > 0$, it must hold $\hat{p}^k(t, y) < 0$. This means that to hold (a sufficiently large amount of) the contingent claim, the investor needs *subsidies* instead of paying for the contingent claims. The reason behind is that the variance part in the mean-variance optimisation criterion penalises both potential gains and losses of the (uncertain) terminal payoff of the claim; in other words, for an investor whose preference is characterised by mean-variance optimisation criterion, her aversion is the *uncertainty* itself. Therefore, the investor whose preference is characterised by mean-variance optimisation criterion will be reluctant to take part in the secondary market heavily.

5.3.2.2 Precommitment Mean-Variance Indifference Pricing Rule

We now study the mean-variance indifference pricing in the *precommitment* sense (the precise definition will be provided later). Recall the definitions of the *precommitment policy law* and *value function* are included in Definition 6 in Chapter 3.

We first provide the *precommitment* value function for the mean-variance optimisation problem without any claim in the portfolio. This problem is to maximise the following objective functional

$$J_{pre}^0(x; \pi) = \mathbb{E} \left[x_T^\pi \middle| x_0 = x \right] - \frac{\gamma}{2} \text{Var} \left(x_T^\pi \middle| x_0 = x \right).$$

We denote $\bar{V}^0(x)$ and $\bar{\pi}^0(t, x)$ the corresponding *precommitment* value function and policy law. We emphasise that the solution is optimal with respect to the initial time zero as the reference time.

5.3. SOME SPECIAL CASES

We report the solution to this problem which is included in [95]. The *precommitment* value function for the mean-variance optimisation problem without any claim in the portfolio is

$$\bar{V}^0(x) = x + \frac{1}{2\gamma} \left(e^{\int_0^T \eta^2(u) du} - 1 \right),$$

and the *precommitment* policy law is

$$\bar{\pi}^0(t, x) = \frac{1}{\gamma} e^{\int_0^T \eta^2(u) du} \frac{\eta(t)}{\sigma(t)} = \bar{\pi}^0(t).$$

We point out, in particular, that the *precommitment* policy law $\bar{\pi}^0(t)$ is *independent* of the wealth variable x , which is similar to the *equilibrium* policy law for the corresponding problem.

We turn next to the mean-variance optimisation problem with k claims in the portfolio. More precisely, we try to maximise the following objective functional

$$J_{pre}^k(x, y; \pi) = \mathbb{E} \left[x_T^\pi + kF(Y_T) \middle| x_0 = x, Y_0 = y \right] - \frac{\gamma}{2} \text{Var} \left(x_T^\pi + kF(Y_T) \middle| x_0 = x, Y_0 = y \right).$$

We denote $\bar{V}^k(x, y)$ and $\bar{\pi}^k(t, x, y)$ the corresponding *precommitment* value function and policy law. Similar to the previous problem, the solution is optimal with respect to the initial time zero as the reference time.

$\bar{V}^k(x, y)$ and $\bar{\pi}^k(t, x, y)$ can be solved via the same approach in [95]; the details in finding them are included in Appendix C and we only report the results. The *precommitment* value function for the mean-variance optimisation problem with k claims in the portfolio is

$$\bar{V}^k(x, y) = x + \frac{1}{2\gamma} \left(e^{\int_0^T \eta^2(u) du} (1 + \varphi^k(0, y))^2 - 1 \right) - \omega^k(0, y)$$

and the *precommitment* policy law is

$$\bar{\pi}^k(t, y) = \frac{1}{\gamma} e^{\int_0^T \eta^2(u) du} \left(1 + \varphi^k(0, y) \right) \frac{\eta(t)}{\sigma(t)} - \frac{1}{\gamma} e^{\int_t^T \eta^2(u) du} \frac{\eta(t) \varphi^k(t, y) + \rho(t) \nu(t, y) \varphi_y^k(t, y)}{\sigma(t)},$$

where $\varphi^k(t, y)$ and $\omega^k(t, y)$ satisfy the following PDEs

$$\begin{cases} \varphi_t^k(t, y) + \frac{1}{2} \nu^2(t, y) \varphi_{yy}^k(t, y) + \left(b(t, y) - \rho(t) \eta(t) \nu(t, y) \right) \varphi_y^k(t, y) = \eta^2(t) \varphi^k(t, y); \\ \varphi^k(T, y) = k\gamma F(y) \end{cases} \quad (5.26)$$

and

$$\begin{cases} \omega_t^k(t, y) + \frac{1}{2} \nu^2(t, y) \omega_{yy}^k(t, y) + b(t, y) \omega_y^k(t, y) = \frac{1}{2\gamma} e^{\int_t^T \eta^2(u) du} \left(\eta(t) \varphi^k(t, y) + \rho(t) \nu(t, y) \varphi_y^k(t, y) \right)^2; \\ \omega^k(T, y) = \frac{\gamma}{2} k^2 F^2(y) \end{cases} \quad (5.27)$$

respectively. Notice that the *precommitment* policy law $\bar{\pi}^k(t, y)$ is also *independent* of the wealth variable x , which is similar to the *equilibrium* policy law for the corresponding problem.

With the *precommitment* value functions $\bar{V}^0(x)$ and $\bar{V}^k(x, y)$, we can define the *precommitment* mean-variance indifference price *at the initial time zero* in a manner similar to Definition 21. We give the formal definition below.

Definition 32. The initial time zero is *fixed* as the reference time. The *precommitment mean-variance indifference price* (pre claim) *at the reference time zero*, $\bar{p}^k(y)$, for a position of k contingent claims written on Y_T with payoff function $F(\cdot)$ is the solution of the following algebraic equation:

$$\bar{V}^0(x) = \bar{V}^k(x - k\bar{p}^k(y), y).$$

Moreover, the difference of the *precommitment* policy laws

$$\bar{\pi}_H^k(t, y) = \bar{\pi}^k(t, y) - \bar{\pi}^0(t) \quad (5.28)$$

is the *precommitment mean-variance indifference hedging policy law*.

From the previous results, we can easily obtain the *precommitment mean-variance indifference price* $\bar{p}^k(y)$ to be

$$\bar{p}^k(y) = \frac{1}{2k\gamma} e^{\int_0^T \eta^2(u) du} \varphi^k(0, y) (2 + \varphi^k(0, y)) - \frac{1}{k} \omega^k(0, y),$$

where $\varphi^k(t, y)$ and $\omega^k(t, y)$ satisfy PDE (5.26) and (5.27) respectively. In addition, the *precommitment mean-variance indifference hedging policy law* is given by

$$\bar{\pi}_H^k(t, y) = \frac{1}{\gamma} e^{\int_0^T \eta^2(u) du} \varphi^k(0, y) \frac{\eta(t)}{\sigma(t)} - \frac{1}{\gamma} e^{\int_t^T \eta^2(u) du} \frac{\eta(t) \varphi^k(t, y) + \rho(t) \nu(t, y) \varphi_y^k(t, y)}{\sigma(t)}.$$

We now perform the same change of measure to the *minimal martingale measure* $\tilde{\mathbb{P}}^0$ to give the probabilistic representations for $\varphi^k(t, y)$, $\omega^k(t, y)$ and hence, $\bar{p}^k(y)$. Via the Feynman-Kac approach, we see

$$\begin{aligned} \varphi^k(t, y) &= k\gamma \mathbb{E}^{\tilde{\mathbb{P}}^0} \left[e^{-\int_t^T \eta^2(u) du} F(Y_T) \middle| Y_t = y \right] \\ &= k\gamma e^{-\int_t^T \eta^2(u) du} \tilde{p}^*(t, y), \end{aligned} \quad (5.29)$$

where $\tilde{p}^*(t, y)$ is the *MMM-price* in (5.19), and

$$\begin{aligned} \omega^k(t, y) &= \mathbb{E} \left[-\frac{1}{2\gamma} \int_t^T e^{\int_u^T \eta^2(r) dr} \left\{ \eta(u) \varphi^k(u, Y_u) + \rho(u) \nu(u, Y_u) \varphi_y^k(u, Y_u) \right\}^2 du + \frac{\gamma}{2} k^2 F^2(Y_T) \middle| Y_t = y \right] \\ &= \frac{\gamma}{2} k^2 \mathbb{E} \left[-\int_t^T e^{-\int_u^T \eta^2(r) dr} \left\{ \eta(u) \tilde{p}^*(u, Y_u) + \rho(u) \nu(u, Y_u) \tilde{p}_y^*(u, Y_u) \right\}^2 du + F^2(Y_T) \middle| Y_t = y \right]. \end{aligned} \quad (5.30)$$

5.3. SOME SPECIAL CASES

Therefore, we have the following probabilistic representation for the *precommitment mean-variance indifference price* $\bar{p}^k(y)$:

$$\begin{aligned} \bar{p}^k(y) = & \tilde{p}^*(0, y) + \frac{k\gamma}{2} e^{-\int_0^T \eta^2(u) du} (\tilde{p}^*(0, y))^2 - \frac{k\gamma}{2} \mathbb{E} \left[F^2(Y_T) \middle| Y_0 = y \right] \\ & + \frac{k\gamma}{2} \mathbb{E} \left[\int_0^T e^{-\int_u^T \eta^2(r) dr} \left\{ \eta(u) \tilde{p}^*(u, Y_u) + \rho(u) \nu(u, Y_u) \tilde{p}_y^*(u, Y_u) \right\}^2 du \middle| Y_0 = y \right]. \end{aligned} \quad (5.31)$$

Moreover, the *precommitment mean-variance indifference hedging policy law* now is

$$\bar{\pi}_H^k(t, y) = k \bar{\pi}_H^*(t, y),$$

where

$$\bar{\pi}_H^*(t, y) = \frac{\eta(t)}{\sigma(t)} \left(\tilde{p}^*(0, y) - \tilde{p}^*(t, y) \right) - \frac{\rho(t) \nu(t, y) \tilde{p}_y^*(t, y)}{\sigma(t)}$$

is the *precommitment mean-variance indifference hedging policy law per claim*, which is *independent* of k .

We close this section by a brief discussion about the *marginal precommitment mean-variance indifference price*.

Definition 33. The *marginal precommitment mean-variance indifference price* $\bar{p}^0(y)$ is defined as

$$\bar{p}^0(y) = \lim_{k \rightarrow 0} \bar{p}^k(y),$$

provided the above limit exists.

From the representation (5.31), we can easily get

$$\bar{p}^0(y) = \tilde{p}^*(0, y) = \mathbb{E}^{\bar{\pi}^0} \left[F(Y_T) \middle| Y_0 = y \right]. \quad (5.32)$$

Thus, the *marginal precommitment mean-variance indifference price* coincides with the *MMM-price* at the initial time zero.

5.3.2.3 Comparison between Equilibrium and Precommitment Pricing Rules

Since the *precommitment mean-variance indifference price* is only defined at the reference time zero, we now compare the *equilibrium* and *precommitment* mean-variance indifference price and their corresponding indifference hedging policy laws at $t = 0$. Not surprisingly, in general the two pricing rules (at time zero) are *different*. The *equilibrium mean-variance indifference price* $\hat{p}^k(0, y)$ is given by

$$\hat{p}^k(0, y) = \tilde{p}^*(0, y) - \frac{k\gamma}{2} \mathbb{E} \left[\int_0^T \bar{\rho}^2(u) \nu^2(u, Y_u) \left(\tilde{p}_y^*(u, Y_u) \right)^2 du \middle| Y_0 = y \right],$$

while the *precommitment mean-variance indifference price* $\bar{p}^k(y)$ is given by

$$\begin{aligned} \bar{p}^k(y) = & \tilde{p}^*(0, y) + \frac{k\gamma}{2} e^{-\int_0^T \eta^2(u) du} (\tilde{p}^*(0, y))^2 - \frac{k\gamma}{2} \mathbb{E} \left[F^2(Y_T) \middle| Y_0 = y \right] \\ & + \frac{k\gamma}{2} \mathbb{E} \left[\int_t^T e^{-\int_u^T \eta^2(r) dr} \left\{ \eta(u) \tilde{p}^*(u, Y_u) + \rho(u) \nu(u, Y_u) \tilde{p}^*(u, Y_u) \right\}^2 du \middle| Y_0 = y \right]. \end{aligned}$$

Moreover, the *equilibrium mean-variance hedging policy* $\hat{\pi}_H^k(t, y)$ is

$$\hat{\pi}_H^k(t, y) = -k \frac{\rho(t) \nu(t, y) \tilde{p}_y^*(t, y)}{\sigma(t)}$$

while the *precommitment mean-variance hedging policy* $\bar{\pi}_H^k(t, y)$ is

$$\begin{aligned} \bar{\pi}_H^k(t, y) = & -k \left\{ \frac{\rho(t) \nu(t, y) \tilde{p}_y^*(t, y)}{\sigma(t)} + \frac{\eta(t)}{\sigma(t)} (\tilde{p}^*(t, y) - \tilde{p}^*(0, y)) \right\} \\ = & \hat{\pi}_H^k(t, y) - k \frac{\eta(t)}{\sigma(t)} (\tilde{p}^*(t, y) - \tilde{p}^*(0, y)). \end{aligned}$$

We notice, however, that although the two hedging policy laws are different, they are both linear in the number of claims in the portfolio.

Moreover, it is interesting to see that the *marginal equilibrium mean-variance indifference price* coincides with the *precommitment* one; in particular, both marginal prices equal to the *MMM-price* (at time zero)—the expectation of the terminal payoff under the minimal martingale measure:

$$\hat{p}^0(0, y) = \bar{p}^0(y) = \tilde{p}^*(0, y) = \mathbb{E}^{\mathbb{P}^0} [F(Y_T) \middle| Y_0 = y].$$

The fact that *minimal martingale measure* is closely related to the quadratic hedging is well-known in the literature (see, for instance, [80]). Our results here indicate that this measure is also deeply involved in both *equilibrium* and *precommitment mean-variance indifference pricing rules*.

Another important fact is that, for the *complete* market case ($\rho(\cdot) \equiv 1$) the *equilibrium mean-variance indifference hedging policy law* reduces to the classical *delta-hedging policy law*:

$$\hat{\pi}_H^k(t, y) = \pi_H^{\text{delta}, k}(t, y) = -k \frac{\nu(t, y) \tilde{p}_y^*(t, y)}{\sigma(t)},$$

see the detailed analysis in Section 5.3.2.1. Nevertheless, this is not the case for the *precommitment mean-variance indifference hedging policy law*. As a matter of fact, the *precommitment mean-variance indifference hedging policy law* for the complete market case is

$$\bar{\pi}_H^k(t, y) = \hat{\pi}_H^k(t, y) - k \frac{\eta(t)}{\sigma(t)} (\tilde{p}^*(t, y) - \tilde{p}^*(0, y)),$$

which coincides with the *equilibrium mean-variance indifference hedging policy law*, and thus the *delta-hedging policy law*, only at the initial time $t = 0$.

Chapter 6

Conclusions

In this thesis, we have studied some time inconsistent portfolio selection and option pricing problems with regime switching. We consider the portfolio selection and indifference option pricing for an investor whose preference is market-regime-dependent, characterised by either utility maximisation or mean-variance optimisation criterion. Due to the time inconsistency, we solve the underlying stochastic control problem via the equilibrium approach.

In Chapter 3 we consider a type of time inconsistent utility maximisation problem where the coefficients of the risky assets in the financial market and the risk aversion parameters of the investor all depend on the market regime. By introducing suitable auxiliary functions, the extended Hamilton-Jacobi-Bellman equations are derived, and the equilibrium policy laws are obtained for both time inconsistent CARA and CRRA utility maximisation problems. Moreover, the corresponding verification theorems are provided.

We also show in Chapter 3 that for this time inconsistent CARA utility maximisation problem with regime switching, the equilibrium policy law coincides with the naive policy law. Naive approach for time inconsistent control problems has been viewed as myopic and simple-minded and with no convincing logic behind it. Nevertheless, in Chapter 3 we provide an interesting example that the myopic naive policy law can turn out to be a deliberate equilibrium solution.

Finally, our results for both time inconsistent CARA and CRRA utility maximisation problem with regime switching, based on the previous empirical studies, show that the investor will allocate more capital to the risky assets in a bullish market than in a bearish market.

In Chapter 4, we study a time inconsistent option pricing problem in an incomplete market with regime switching. The investor wants to determine her subjective price of a European-type contingent claim written on a nontradable asset. Due to the market incompleteness, we resort to the utility indifference pricing approach. The investor's utility function is of exponential type,

and her risk aversion parameter is regime dependent as in Chapter 3. With the equilibrium policy laws and the value functions derived in Chapter 3 we establish an equilibrium utility indifference pricing rule. This pricing rule consists of two parts; one part is the equilibrium utility indifference price, which is given by the solution to an algebraic equation with the two equilibrium value functions involved, and the other part is the equilibrium utility indifference hedging policy law, which is the difference of the equilibrium policy laws for the two underlying time inconsistent utility maximisation problems.

We then discuss a simplified basis risk model extensively and give the PDE system characterising the equilibrium utility indifference price. Compared to the classical indifference price which can be depicted by a single PDE, we have to characterise the equilibrium utility indifference price via a PDE system. Under the two-regime case, we are able to show the local existence and uniqueness of this PDE system. We then discuss the marginal equilibrium utility indifference price by formally letting the number of the claims in the portfolio approach to zero. It turns out that this marginal equilibrium utility indifference price can be characterised by a linear system of partial difference equations and thus the well-posedness is guaranteed by the standard PDE theory. Moreover, if the price of the risky asset is regime independent, the marginal equilibrium utility indifference price coincides with the expected terminal payoff of the contingent claim under the minimal martingale measure. Some properties of the equilibrium utility indifference pricing rule are discussed subsequently; for instance, the equilibrium utility indifference price reduces to the classical utility indifference price when the risk aversion parameter is the regime independent, and to the risk neutral price derived from the replication method when the market is complete. Some numerical tests are also provided. Finally, we analyse a claim with the logarithmic payoff and give an explicit formula for the equilibrium utility indifference price.

In Chapter 5, we consider the same option pricing problem in an incomplete market, similar to Chapter 4. Once more, we adopt the indifference pricing approach to find the subjective price of the contingent claim for the investor by comparing the results of two portfolio selection problems with and without the claim. However, we now assume that the investor's preference is characterised by the mean-variance optimisation criterion for portfolio selection and that the weighting parameter is regime dependent. Both facts introduce time inconsistency into the underlying stochastic control problems. In this chapter, we first give the equilibrium policy laws and value functions for the corresponding two time inconsistent mean-variance portfolio selection problems with and without the contingent claim. For the case without any claim in the portfolio, it turns out that the equilibrium policy law for the regime dependent mean-variance portfolio

selection problem coincides with the one for the regime dependent CARA utility maximisation problem.

With the equilibrium policy laws and value functions, we then establish the equilibrium mean-variance indifference pricing rule, which is parallel to the equilibrium utility indifference pricing rule in Chapter 4. Now the auxiliary functions used to determine the equilibrium mean-variance indifference price are characterised by linear partial differential equations, and we are able to give the explicit formulas for the equilibrium mean-variance indifference price and hedging policy law. A marginal equilibrium mean-variance indifference price is studied subsequently.

We then analyse some special cases. First, we study the equilibrium mean-variance indifference pricing rule in a two-market-state basis risk model. It turns out that the equilibrium mean-variance indifference price depends on the market regime only through the weighting parameter corresponding to the regime, and there is no relationship between different equilibrium mean-variance indifference prices for different market regimes. We focus next on the special case of the equilibrium mean-variance indifference pricing rule without regime switching. We give the explicit formulas for both the equilibrium mean-variance indifference price and the corresponding marginal price. It turns out that the marginal equilibrium mean-variance indifference price coincides with the expected terminal payoff of the claim under the minimal martingale measure. Furthermore, we show that the equilibrium mean-variance indifference price is concave with respect to the claim's payoff function, and is reduced to the risk neutral price derived from the replication method when the market is complete. We then compare the equilibrium mean-variance indifference pricing rule with its precommitment counterpart and point out that only their marginal mean-variance indifference prices are identical.

Appendix A

Reviews of Continuous-Time Markovian Chain

In this appendix, we summarise some basic results about the continuous-time Markovian chain. These results, especially the generalised Dynkin's formula, are the foundations for the analysis in this thesis. All the results are well-known in the literature and can be found in the standard textbooks about continuous-time Markovian chain (see, for instance, [1], [2], [57], [83] and [89]).

A.1 Definitions and Basic Facts

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete filtered probability space and $\mathcal{E}$ be a *countable* set.

Definition 34. A stochastic process Ξ . defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ with values in $\mathcal{E}$ is called a *continuous-time Markovian chain*, if for any finite time set $0 \leq t_1 < t_2 < \dots < t_n < t_{n+1}$ and any points (*states*) $i_1, i_2, \dots, i_{n-1}, i_n, j \in \mathcal{E}$ such that

$$\mathbb{P}(\Xi_{t_n} = i_n, \Xi_{t_{n-1}} = i_{n-1}, \dots, \Xi_{t_1} = i_1) > 0,$$

it holds

$$\mathbb{P}(\Xi_{t_{n+1}} = j | \Xi_{t_n} = i_n, \Xi_{t_{n-1}} = i_{n-1}, \dots, \Xi_{t_1} = i_1) = \mathbb{P}(\Xi_{t_{n+1}} = j | \Xi_{t_n} = i_n).$$

$\mathcal{E}$ is called the *state space* of the process Ξ .

If for all s, t such that $0 \leq s \leq t < \infty$ and all $i, j \in \mathcal{E}$, the conditional probability $\mathbb{P}(\Xi_t = j | \Xi_s = i)$ depends *only* on $t - s$, we then say that the continuous-time Markovian chain Ξ . is

stationary. In this case, we have

$$\mathbb{P}(\Xi_t = j | \Xi_s = i) = \mathbb{P}(\Xi_{t-s} = j | \Xi_0 = i).$$

The function

$$p_{ij}(t) = \mathbb{P}(\Xi_t = j | \Xi_0 = i), \quad i, j \in \mathcal{E}, \quad t \geq 0$$

is called the *transition probability function* of the continuous-time stationary Markovian chain Ξ .

Definition 35. A continuous-time stationary Markovian chain is called *standard*, if

$$\lim_{t \rightarrow 0^+} p_{ii}(t) = 1,$$

for each state $i \in \mathcal{E}$.

Theorem 36. Let Ξ be a continuous-time standard stationary Markovian chain and $p_{ij}(\cdot)$ be its corresponding transition probability function. Then,

$$\lambda_i = \lim_{t \rightarrow 0^+} \frac{1 - p_{ii}(t)}{t}$$

exists (but is possibly ∞) for each state $i \in \mathcal{E}$.

Definition 37. A state $i \in \mathcal{E}$ is called *stable*, if $\lambda_i < \infty$.

Theorem 38. Let Ξ be a continuous-time standard stationary Markovian chain and $p_{ij}(\cdot)$ be its corresponding transition probability function. Suppose state $j \in \mathcal{E}$ is stable. Then

$$\lambda_{ij} = \mathcal{D}_+[p_{ij}](0) = \lim_{t \rightarrow 0^+} \frac{p_{ij}(t)}{t}$$

exists and is finite for all $i \in \mathcal{E}$.

Let $\lambda_{ii} = -\lambda_i$ and $\Lambda = (\lambda_{ij})_{i,j \in \mathcal{E}}$. Λ is called the *generator* or the *intensity matrix* of the corresponding continuous-time standard stationary Markovian chain.

If the state space $\mathcal{E}$ is finite, the corresponding Markovian chain is called *finite*. Without loss of generality, we may write $\mathcal{E} = \{1, 2, \dots, K\}$; in particular, Λ is now an $K \times K$ matrix.

We assume that all the continuous-time Markovian chains in this thesis are *stationary*, *standard* and *finite*, and all their states are *stable*. For such a process, almost every sample path is a *right-continuous step function*.

A.2. A GENERALISED DYNKIN'S FORMULA

Theorem 39. *Let Ξ be a continuous-time finite standard stationary Markovian chain with generator $\Lambda = (\lambda_{ij})_{K \times K}$ and $p(t) = (p_{ij}(t))_{K \times K}$ be its transition probability matrix. Then*

$$p(t) = \exp(t\Lambda),$$

where $\exp(\cdot)$ is understood as a matrix function.

The proofs of all the theorems above can be found in the standard textbooks about continuous-time Markovian chain (see, for instance, [1]).

A.2 A Generalised Dynkin's Formula

We now state the following generalised version of Dynkin's formula (see, for instance, [89], [57] and [2]). Let B be a d -dimensional Brownian motion and Y_t be an n -dimensional Itô process satisfying the following stochastic differential equation

$$\begin{cases} dY_t = b(t, Y_t, \Xi_t)dt + a(t, Y_t, \Xi_t)dB_t; \\ Y_0 = y_0 \in \mathbb{R}^n, \end{cases}$$

where $b : [0, T] \times \mathbb{R}^n \times \mathcal{E} \rightarrow \mathbb{R}^n$ and $a : [0, T] \times \mathbb{R}^n \times \mathcal{E} \rightarrow \mathbb{R}^{n \times d}$ are given deterministic functions satisfying the usual *Lipschitz* and *linear growth* condition, i.e., for $\phi(t, y, k) = b(t, y, k), a(t, y, k)$, it holds

$$\begin{cases} |\phi(t, y_1, k) - \phi(t, y_2, k)| \leq L|y_1 - y_2|, & \forall y_1, y_2 \in \mathbb{R}^n, \forall t \in [0, T], \forall k \in \mathcal{E}, \\ |\phi(t, 0, k)| \leq L, & \forall t \in [0, T], \forall k \in \mathcal{E}, \end{cases}$$

for some constant $L > 0$. Furthermore, for any function $F : [0, T] \times \mathbb{R}^n \times \mathcal{E} \rightarrow \mathbb{R}$ which is continuously differentiable in the first variable t and twice continuously differentiable in the second variable y for any $k \in \mathcal{E}$, we introduce the following operator $\mathcal{L}[\cdot]$ whose action on $F(t, y, k)$ is:

$$\begin{aligned} \mathcal{L}[F](t, y, k) &= F_t(t, y, k) + F_y(t, y, k)b(t, y, k) + \frac{1}{2}Tr[a(t, y, k)^T F_{yy}(t, y, k)a(t, y, k)] \\ &\quad + \sum_{j=1}^K \lambda_{kj} F(t, y, j) \\ &= F_t(t, y, k) + \sum_{i=1}^n F_{y_i}(t, y, k)b^i(t, y, k) \\ &\quad + \frac{1}{2} \sum_{i=1}^n \sum_{j=1}^n F_{y_i y_j}(t, y, k) \left(\sum_{r=1}^d a^{ki}(t, y, k)a^{kj}(t, y, k) \right) \\ &\quad + \sum_{\substack{l=1 \\ l \neq k}}^K \lambda_{kl} \{F(t, y, l) - F(t, y, k)\}, \end{aligned} \tag{A.1}$$

where $F_t(t, y, k)$ stands for the partial derivative with respect to t , $F_y(t, y, k)$ and $F_{yy}(t, y, k)$ are the gradient vector and Hessian matrix with respect to y .

The following theorem gives a generalised version of the *Dynkin's formula*, which is a critical tool in this thesis.

Theorem 40 (*Generalised Dynkin's Formula*). *Let $F : [0, T] \times \mathbb{R}^n \times \mathcal{E} \rightarrow \mathbb{R}$ satisfying $F(t, y, k)$ is continuously differentiable in t and continuously twice differentiable in y for any $k \in \mathcal{E}$, whose partial derivatives with respect to y is of polynomial growth; in other words,*

$$|F_y(t, y, k)| \leq K(1 + |y|^m), \quad \forall y \in \mathbb{R}^n,$$

for some $m \in \mathbb{N}$ and $K > 0$, $\forall t \in [0, T]$, $k \in \mathcal{E}$. Moreover, (Y, Ξ) is defined as before. Then, for any $0 \leq t < s \leq T$, $(y, k) \in \mathbb{R}^n \times \mathcal{E}$, it holds

$$\mathbb{E} \left[F(s, Y_s, \Xi_s) \middle| Y_t = y, \Xi_t = k \right] - F(t, y, k) = \mathbb{E} \left[\int_t^s \mathcal{L}[F](u, Y_u, \Xi_u) du \middle| Y_t = y, \Xi_t = k \right].$$

Moreover,

$$\lim_{h \rightarrow 0+} \frac{\mathbb{E} [F(t+h, Y_{t+h}, \Xi_{t+h}) | Y_t = y, \Xi_t = k] - F(t, y, k)}{h} = \mathcal{L}[F](t, y, k),$$

where $\mathcal{L}[\cdot]$ is defined as (A.1).

This proof can be found in [57].

Appendix B

Precommitment Solution to Time Inconsistent CARA Utility Maximisation with Regime Switching

In this appendix, we provide the precommitment solution to the time inconsistent CARA utility maximisation problem with regime switching in Section 3.2.3, Chapter 3; more precisely, we give the precommitment value function and policy law for that problem. The results are then used to find the corresponding naive policy law and show the coincidence of the equilibrium and the naive policy law in Section 3.2.3, Chapter 3.

To this end, we first fix a $t \in [0, T)$ as the reference time. Recall in Section 3.2.3, Chapter 3 that given the market state $\Xi_t = \alpha$, the corresponding objective functional is

$$J^{(t,\alpha)}(s, x, \beta; \pi_\cdot) = \mathbb{E} \left[-\exp(-\gamma(\alpha)x_T^\pi) \mid x_s = x, \Xi_s = \beta \right], \quad s \in [t, T),$$

where x^π satisfies the following controlled SDE

$$\begin{cases} dx_u^\pi = \pi_u \left\{ \mu(u, \Xi_u) du + \sigma(u, \Xi_u) dW_u \right\}; \\ x_s = x, \quad \Xi_s = \beta. \end{cases}$$

Via the classical dynamic programming technique, we introduce the following value function

$$V^{(t,\alpha)}(s, x, \beta) = \max_{\pi_\cdot} \left\{ \mathbb{E} \left[-\exp(-\gamma(\alpha)x_T^\pi) \mid x_s = x, \alpha_s = \beta \right] \right\}.$$

Dynamic programming principle (see, for example, [28] and [90]) then gives

$$V^{(t,\alpha)}(s, x, \beta) = \max_{\pi_\cdot} \left\{ \mathbb{E} \left[V^{(t,\alpha)}(s+h, x_{s+h}^\pi, \alpha_{s+h}) \mid x_s = x, \alpha_s = \beta \right] \right\}$$

and the corresponding classical *Hamilton-Jacobi-Bellman* equation is

$$\begin{cases} V_\tau^{(t,\alpha)}(s, x, \beta) + \max_\pi \left\{ \frac{1}{2} a(s, \beta) V_{xx}^{(t,\alpha)}(s, x, \beta) \pi^2 + \mu(s, \beta) V_x^{(t,\alpha)}(s, x, \beta) \pi \right\} \\ \quad + \sum_{l \neq \beta} \lambda_{\beta l} \left\{ V^{(t,\alpha)}(s, x, l) - V^{(t,\alpha)}(s, x, \beta) \right\} = 0; \\ V^{(t,\alpha)}(T, x, \beta) = -e^{-\gamma(\alpha)x}, \end{cases}$$

for any $\beta \in \mathcal{E}$. The first-order condition then gives the optimal policy law (with respect to the reference time t) to be

$$\bar{\pi}^{(t,\alpha)}(s, x, \beta) = -\frac{\mu(s, \beta)}{a(s, \beta)} \frac{V_x^{(t,\alpha)}(s, x, \beta)}{V_{xx}^{(t,\alpha)}(s, x, \beta)}, \quad s \in [t, T].$$

To solve this HJB equation, we now make the following Ansatz:

Ansatz 41. *The solution of the HJB equation has the following form:*

$$V^{(t,\alpha)}(s, x, \beta) = -e^{-\gamma(\alpha)x} g^{(t,\alpha)}(s, \beta).$$

Direct calculation then gives the optimal policy law (with respect to the reference time t) to be

$$\bar{\pi}^{(t,\alpha)}(s, \beta) = \frac{1}{\gamma(\alpha)} \frac{\mu(s, \beta)}{a(s, \beta)}, \quad s \in [t, T].$$

Moreover, the HJB equation reduces to the following ODE system:

$$\begin{cases} g_\tau^{(t,\alpha)}(s, \beta) - \frac{1}{2} \eta^2(s, \beta) g^{(t,\alpha)}(s, \beta) - \sum_{l \neq \beta} \lambda_{\beta l} \left\{ g^{(t,\alpha)}(s, l) - g^{(t,\alpha)}(s, \beta) \right\} = 0; \\ g^{(t,\alpha)}(T, \beta) = 1, \end{cases}$$

for any $\beta \in \mathcal{E}$. This is a linear ODE system, and the well-posedness is guaranteed by the standard ODE theory.

Therefore, we see the *precommitment* policy law with respect to this reference time t and the market condition $\Xi_t = \alpha$ at t is

$$\bar{\pi}^{(t,\alpha)}(s, \beta) = \frac{1}{\gamma(\alpha)} a^{-1}(s, \beta) \mu(s, \beta), \quad s \in [t, T], \beta \in \mathcal{E}.$$

Appendix C

Precommitment Solution to Mean-Variance Optimisation with Claims in Portfolio

In this appendix, we give the details for finding the precommitment policy law and value function for the mean-variance optimisation problem with k claims in the portfolio in Section 5.3.2.2. More precisely, we fix the reference time at zero and try to maximise the following objective functional

$$J_{pre}^k(x, y; \pi) = \mathbb{E} \left[x_T^\pi + kF(Y_T) \middle| x_0 = x, Y_0 = y \right] - \frac{\gamma}{2} \text{Var} \left(x_T^\pi + kF(Y_T) \middle| x_0 = x, Y_0 = y \right),$$

where π , x^π and Y are the same as in Chapter 5. Furthermore, recall that $\bar{V}^k(x, y)$ and $\bar{\pi}^k(t, x, y)$ are the corresponding *precommitment* value function and policy law for this problem, as denoted in Section 5.3.2.2.

We solve $\bar{V}^k(x, y)$ and $\bar{\pi}^k(t, x, y)$ via the same approach in [95]. To this end, we first introduce the following (time consistent) auxiliary control problem to maximise a new objective functional

$$A^k(x, y; \pi; \lambda) = -\frac{\gamma}{2} \mathbb{E} \left[(x_T^\pi + kF(Y_T))^2 \middle| x_0 = x, Y_0 = y \right] + \lambda \mathbb{E} \left[x_T^\pi + kF(Y_T) \middle| x_0 = x, Y_0 = y \right]$$

and denote $\mathcal{A}^k(x, y; \lambda)$ and $\bar{\pi}^k(t, x, y; \lambda)$ the corresponding value function and optimal policy law. Via the *dynamic programming technique* one can get the value function

$$\begin{aligned} \mathcal{A}^k(x, y; \lambda) = & -\frac{\gamma}{2} e^{-\int_0^T \eta^2(u) du} x^2 + \left\{ \lambda e^{-\int_0^T \eta^2(u) du} - \varphi^k(0, y) \right\} x \\ & + \frac{\lambda^2}{2\gamma} \left(1 - e^{-\int_0^T \eta^2(u) du} \right) + \frac{\lambda}{\gamma} \varphi^k(0, y) - \omega^k(0, y) \end{aligned} \tag{C.1}$$

and the optimal trading policy

$$\bar{\pi}^k(t, x, y; \lambda) = \frac{\eta(t)}{\sigma(t)} \left(\frac{\lambda}{\gamma} - x \right) - e^{\int_t^T \eta^2(u) du} \frac{\eta(t) \varphi^k(t, y) + \rho(t) \nu(t, y) \varphi_y^k(t, y)}{\gamma \sigma(t)}, \quad (\text{C.2})$$

where $\varphi^k(t, y)$ and $\omega^k(t, y)$ satisfies the PDE (5.26) and (5.27) in Section 5.3.2.2, Chapter 5. We emphasise that both $\varphi^k(t, y)$ and $\omega^k(t, y)$ do *not* depend on λ . Furthermore, we have the following result similar to Theorem 3.1 in [95].

Theorem 42. *If the trading strategy $\bar{\pi}^k$ is optimal for the mean-variance optimisation problem with k claims in the portfolio, i.e.,*

$$\bar{V}^k(x, y) = \mathbb{E} \left[\bar{x}_T^k + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right] - \frac{\gamma}{2} \text{Var} \left(\bar{x}_T^k + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right),$$

in which $\bar{x}^k$ is the corresponding wealth process, then

$$\mathcal{A}^k(x, y; \bar{\lambda}^k) = -\frac{\gamma}{2} \mathbb{E} \left[(\bar{x}_T^k + kF(Y_T))^2 \middle| \bar{x}_0 = x, Y_0 = y \right] + \bar{\lambda}^k \mathbb{E} \left[\bar{x}_T^k + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right],$$

where

$$\bar{\lambda}^k = 1 + \gamma \mathbb{E} \left[\bar{x}_T^k + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right].$$

The proof is entirely parallel to the one for Theorem 3.1 in [95] and we provide it here for the reader's convenience.

Proof. If the result is not true, then there exists a trading strategy and the corresponding wealth process (π, x^π) such that

$$\begin{aligned} & -\frac{\gamma}{2} \mathbb{E} \left[(\bar{x}_T^k + kF(Y_T))^2 \middle| \bar{x}_0 = x, Y_0 = y \right] + \bar{\lambda}^k \mathbb{E} \left[\bar{x}_T^k + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right] \\ & < -\frac{\gamma}{2} \mathbb{E} \left[(x_T^\pi + kF(Y_T))^2 \middle| x_0 = x, Y_0 = y \right] + \bar{\lambda}^k \mathbb{E} \left[x_T^\pi + kF(Y_T) \middle| x_0 = x, Y_0 = y \right]. \end{aligned} \quad (\text{C.3})$$

Define

$$\Theta(x, y) = -\frac{\gamma}{2}x + \frac{\gamma}{2}y^2 + y,$$

then by definition

$$\bar{V}^k(x, y) = \Theta \left(\mathbb{E} \left[(\bar{x}_T^k + kF(Y_T))^2 \middle| \bar{x}_0 = x, Y_0 = y \right], \mathbb{E} \left[\bar{x}_T^k + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right] \right).$$

Notice that $\Theta(x, y)$ is convex and

$$\Theta_x(x, y) = -\frac{\gamma}{2}, \quad \Theta_y(x, y) = \gamma y + 1,$$

therefore

$$\begin{aligned}
& J_{pre}^k(x, y; \pi) - \bar{V}^k(x, y) \\
& \geq -\frac{\gamma}{2} \left\{ \mathbb{E} \left[(x_T^\pi + kF(Y_T))^2 \middle| x_0 = x, Y_0 = y \right] - \mathbb{E} \left[(\bar{x}_T^k + kF(Y_T))^2 \middle| \bar{x}_0 = x, Y_0 = y \right] \right\} \\
& + \left(\gamma \mathbb{E} \left[\bar{x}_T^k + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right] + 1 \right) \times \\
& \quad \left\{ \mathbb{E} \left[x_T^\pi + kF(Y_T) \middle| x_0 = x, Y_0 = y \right] - \mathbb{E} \left[\bar{x}_T^k + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right] \right\} > 0
\end{aligned}$$

according to (C.3). This is then a contradiction with that $\bar{V}^k(x, y)$ is the precommitment value function. $\square$

According to (C.2), the expectation of $\bar{x}_T^{k, \lambda}$ is

$$\begin{aligned}
\mathbb{E} \left[\bar{x}_T^{k, \lambda} \middle| \bar{x}_0 = x, Y_0 = y \right] &= e^{-\int_0^T \eta^2(u) du} x + \frac{\lambda}{\gamma} \left(1 - e^{-\int_0^T \eta^2(u) du} \right) \\
&\quad - \frac{1}{\gamma} \mathbb{E} \left[\int_0^T \left\{ \eta^2(t) \varphi^k(t, Y_t) + \rho(t) \eta(t) \nu(t, Y_t) \varphi_y^k(t, Y_t) \right\} dt \middle| Y_0 = y \right].
\end{aligned}$$

Moreover, via the Feynman-Kac approach, we have

$$\begin{aligned}
\mathbb{E} \left[k\gamma F(Y_T) \middle| Y_0 = y \right] &= \mathbb{E} \left[\varphi^k(T, Y_T) \middle| Y_0 = y \right] \\
&= \varphi^k(0, y) + \mathbb{E} \left[\int_0^T \left\{ \eta^2(t) \varphi^k(t, Y_t) + \rho(t) \eta(t) \nu(t, Y_t) \varphi_y^k(t, Y_t) \right\} dt \middle| Y_0 = y \right].
\end{aligned}$$

Then from Theorem 42 we see $\bar{\lambda}^k$ solves the following algebraic equation

$$\begin{aligned}
\bar{\lambda}^k &= 1 + \gamma \mathbb{E} \left[\bar{x}_T^{k, \bar{\lambda}^k} + kF(Y_T) \middle| \bar{x}_0 = x, Y_0 = y \right] \\
&= 1 + \gamma e^{-\int_0^T \eta^2(u) du} x + \bar{\lambda}^k \left(1 - e^{-\int_0^T \eta^2(u) du} \right) + \varphi^k(0, y)
\end{aligned}$$

and thus

$$\bar{\lambda}^k = \gamma x + e^{\int_0^T \eta^2(u) du} (1 + \varphi^k(0, y)).$$

Therefore, the *precommitment* value function for the mean-variance optimisation problem with k claims in the portfolio is

$$\bar{V}^k(x, y) = x + \frac{1}{2\gamma} \left(e^{\int_0^T \eta^2(u) du} (1 + \varphi^k(0, y))^2 - 1 \right) - \omega^k(0, y)$$

and the *precommitment* policy law is

$$\bar{\pi}^k(t, y) = \frac{1}{\gamma} e^{\int_0^T \eta^2(u) du} (1 + \varphi^k(0, y)) \frac{\eta(t)}{\sigma(t)} - \frac{1}{\gamma} e^{\int_t^T \eta^2(u) du} \frac{\eta(t) \varphi^k(t, y) + \rho(t) \nu(t, y) \varphi_y^k(t, y)}{\sigma(t)},$$

where $\varphi^k(t, y)$ and $\omega^k(t, y)$ satisfy (5.26) and (5.27).

Bibliography

- [1] William J Anderson. *Continuous-time Markov chains: An applications-oriented approach*. Springer Science & Business Media, 2012.
- [2] David Applebaum. *Lévy processes and stochastic calculus*. Cambridge university press, 2009.
- [3] Suleyman Basak and Georgy Chabakauri. Dynamic mean-variance asset allocation. *Review of Financial Studies*, 23(8):2970–3016, 2010.
- [4] Suleyman Basak and Georgy Chabakauri. Dynamic hedging in incomplete markets: a simple solution. *Review of financial studies*, 25(6):1845–1896, 2012.
- [5] Dirk Becherer. Utility–indifference hedging and valuation via reaction–diffusion systems. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 460, pages 27–51. The Royal Society, 2004.
- [6] Tomas Björk and Agatha Murgoci. A general theory of markovian time inconsistent stochastic control problems. *Available at SSRN 1694759*, 2010.
- [7] Tomas Björk and Agatha Murgoci. A theory of markovian time-inconsistent stochastic control in discrete time. *Finance and Stochastics*, 18(3):545–592, 2014.
- [8] Tomas Björk, Agatha Murgoci, and Xun Yu Zhou. Mean–variance portfolio optimization with state-dependent risk aversion. *Mathematical Finance*, 24(1):1–24, 2014.
- [9] Fischer Black and Myron Scholes. The pricing of options and corporate liabilities. *Journal of political economy*, 81(3):637–654, 1973.
- [10] John Buffington and Robert J Elliott. American options with regime switching. *International Journal of Theoretical and Applied Finance*, 5(05):497–514, 2002.

-
- [11] René Carmona. *Indifference pricing: theory and applications*. Princeton University Press, 2009.
 - [12] Andrew HY Chen, Frank C Jen, and Stanley Zions. The optimal portfolio revision policy. *Journal of business*, pages 51–61, 1971.
 - [13] Jakša Cvitanić and Ioannis Karatzas. Convex duality in constrained portfolio optimization. *The Annals of Applied Probability*, pages 767–818, 1992.
 - [14] Jakša Cvitanić, Walter Schachermayer, and Hui Wang. Utility maximization in incomplete markets with random endowment. *Finance and Stochastics*, 5(2):259–272, 2001.
 - [15] Mark HA Davis. Option pricing in incomplete markets. *Mathematics of derivative securities*, 15:216–226, 1997.
 - [16] Mark HA Davis. Optimal hedging with basis risk. In *From stochastic calculus to mathematical finance*, pages 169–187. Springer, 2006.
 - [17] Mark HA Davis, Vassilios G Panas, and Thaleia Zariphopoulou. European option pricing with transaction costs. *SIAM Journal on Control and Optimization*, 31(2):470–493, 1993.
 - [18] Freddy Delbaen, Peter Grandits, Thorsten Rheinländer, Dominick Samperi, Martin Schweizer, and Christophe Stricker. Exponential hedging and entropic penalties. *Mathematical finance*, 12(2):99–123, 2002.
 - [19] Giuseppe Di Graziano and LCG Rogers. Barrier option pricing for assets with markov-modulated dividends. *Journal of Computational Finance*, 9(4):75, 2006.
 - [20] Jacques H Dreze and Aldo Rustichini. State-dependent utility and decision theory. In *Handbook of utility theory*, pages 839–892. Springer, 2004.
 - [21] Ivar Ekeland and Ali Lazrak. Being serious about non-commitment: subgame perfect equilibrium in continuous time. *arXiv preprint math/0604264*, 2006.
 - [22] Ivar Ekeland and Ali Lazrak. Equilibrium policies when preferences are time inconsistent. *arXiv preprint arXiv:0808.3790*, 2008.
 - [23] Ivar Ekeland and Ali Lazrak. The golden rule when preferences are time inconsistent. *Mathematics and Financial Economics*, 4(1):29–55, 2010.

BIBLIOGRAPHY

- [24] Ivar Ekeland, Oumar Mbodji, and Traian A Pirvu. Time-consistent portfolio management. *SIAM Journal on Financial Mathematics*, 3(1):1–32, 2012.
- [25] Robert J Elliott, Leunglung Chan, and Tak Kuen Siu. Option pricing and *Esscher transform* under regime switching. *Annals of Finance*, 1(4):423–432, 2005.
- [26] Lawrence C Evans. Partial differential equations. 2010.
- [27] Eugene F Fama and Kenneth R French. Business conditions and expected returns on stocks and bonds. *Journal of financial economics*, 25(1):23–49, 1989.
- [28] Wendell H Fleming and Halil Mete Soner. *Controlled Markov processes and viscosity solutions*, volume 25. Springer Science & Business Media, 2006.
- [29] Hans Follmer and Martin Schweizer. Hedging of contingent claims. *Applied stochastic analysis*, 5:389, 1991.
- [30] Kenneth R French, G William Schwert, and Robert F Stambaugh. Expected stock returns and volatility. *Journal of financial Economics*, 19(1):3–29, 1987.
- [31] Steven M Goldman. Consistent plans. *The Review of Economic Studies*, 47(3):533–537, 1980.
- [32] Stephen Gordon and Pascal St-Amour. A preference regime model of bull and bear markets. *The American Economic Review*, 90(4):1019–1033, 2000.
- [33] Stephen Gordon and Pascal St-Amour. Asset returns and state-dependent risk preferences. *Journal of Business & Economic Statistics*, 22(3):241–252, 2004.
- [34] Xin Guo. An explicit solution to an optimal stopping problem with regime switching. *Journal of Applied Probability*, 38(02):464–481, 2001.
- [35] James D Hamilton. A new approach to the economic analysis of nonstationary time series and the business cycle. *Econometrica: Journal of the Econometric Society*, pages 357–384, 1989.
- [36] Christopher Harris and David Laibson. Instantaneous gratification. *The Quarterly Journal of Economics*, page qjs051, 2012.
- [37] Vicky Henderson. Valuation of claims on nontraded assets using utility maximization. *Mathematical Finance*, 12(4):351–373, 2002.

-
- [38] Vicky Henderson and David Hobson. Utility indifference pricing-an overview. *Volume on Indifference Pricing*, 2004.
- [39] Stewart D Hodges and Anthony Neuberger. Optimal replication of contingent claims under transaction costs. *Review of futures markets*, 8(2):222–239, 1989.
- [40] Toshiaki Honda. Optimal portfolio choice for unobservable and regime-switching mean returns. *Journal of Economic Dynamics and Control*, 28(1):45–78, 2003.
- [41] Ying Hu, Peter Imkeller, Matthias Müller, et al. Utility maximization in incomplete markets. *The Annals of Applied Probability*, 15(3):1691–1712, 2005.
- [42] Ying Hu, Hanqing Jin, and Xun Yu Zhou. Time-inconsistent stochastic linear-quadratic control. *SIAM journal on Control and Optimization*, 50(3):1548–1572, 2012.
- [43] Julien Hugonnier, Dmitry Kramkov, and Walter Schachermayer. On utility-based pricing of contingent claims in incomplete markets. *Mathematical Finance*, 15(2):203–212, 2005.
- [44] Arnaud Jobert and Leonard CG Rogers. Option pricing with markov-modulated dynamics. *SIAM Journal on Control and Optimization*, 44(6):2063–2078, 2006.
- [45] Ioannis Karatzas and Steve G Kou. On the pricing of contingent claims under constraints. *The annals of applied probability*, pages 321–369, 1996.
- [46] Ioannis Karatzas, John P Lehoczy, Suresh P Sethi, and Steven E Shreve. Explicit solution of a general consumption/investment problem. *Mathematics of Operations Research*, 11(2):261–294, 1986.
- [47] Ioannis Karatzas, John P Lehoczy, and Steven E Shreve. Optimal portfolio and consumption decisions for a “small investor” on a finite horizon. *SIAM journal on control and optimization*, 25(6):1557–1586, 1987.
- [48] Ioannis Karatzas and Steven Shreve. *Brownian motion and stochastic calculus*, volume 113. Springer Science & Business Media, 2012.
- [49] Ioannis Karatzas and Steven E Shreve. *Methods of mathematical finance*, volume 39. Springer Science & Business Media, 1998.
- [50] Edi Karni. Risk aversion for state-dependent utility functions: Measurement and applications. *International Economic Review*, pages 637–647, 1983.

BIBLIOGRAPHY

- [51] Edi Karni. Generalized expected utility analysis of risk aversion with state-dependent preferences. *International Economic Review*, pages 229–240, 1987.
- [52] Dimitri Kramkov and Walter Schachermayer. The asymptotic elasticity of utility functions and optimal investment in incomplete markets. *Annals of Applied Probability*, pages 904–950, 1999.
- [53] Dmitry Kramkov and Mihai Sirbu. Sensitivity analysis of utility-based prices and risk-tolerance wealth processes. *The Annals of Applied Probability*, pages 2140–2194, 2006.
- [54] David Laibson. Golden eggs and hyperbolic discounting. *The Quarterly Journal of Economics*, 112(2):443–478, 1997.
- [55] Duan Li and Wan-Lung Ng. Optimal dynamic portfolio selection: Multiperiod mean-variance formulation. *Mathematical Finance*, 10(3):387–406, 2000.
- [56] Rogemar S Mamon and Marianito R Rodrigo. Explicit solutions to european options in a regime-switching economy. *Operations Research Letters*, 33(6):581–586, 2005.
- [57] Xuerong Mao and Chenggui Yuan. *Stochastic differential equations with Markovian switching*. World Scientific, 2006.
- [58] Harry Markowitz. Portfolio selection. *The journal of finance*, 7(1):77–91, 1952.
- [59] Harry M Markowitz. *Portfolio selection: efficient diversification of investments*, volume 16. Yale university press, 1968.
- [60] Rajnish Mehra and Edward C Prescott. The equity premium: A puzzle. *Journal of monetary Economics*, 15(2):145–161, 1985.
- [61] Angelo Melino and Alan X Yang. State-dependent preferences can explain the equity premium puzzle. *Review of Economic Dynamics*, 6(4):806–830, 2003.
- [62] Robert C Merton. Lifetime portfolio selection under uncertainty: The continuous-time case. *The review of Economics and Statistics*, pages 247–257, 1969.
- [63] Robert C Merton. Optimum consumption and portfolio rules in a continuous-time model. *Journal of economic theory*, 3(4):373–413, 1971.
- [64] Robert C Merton. An analytic derivation of the efficient portfolio frontier. *Journal of financial and quantitative analysis*, 7(04):1851–1872, 1972.

-
- [65] Michael Monoyios. Utility indifference pricing with market incompleteness. *Nonlinear Models in Mathematical Finance: New Research Trends in Option Pricing*, pages 67–100, 2008.
- [66] Michael Monoyios et al. Performance of utility-based strategies for hedging basis risk. *Quantitative Finance*, 4(3):245–255, 2004.
- [67] Marek Musiela and Thaleia Zariphopoulou. An example of indifference prices under exponential preferences. *Finance and Stochastics*, 8(2):229–239, 2004.
- [68] Hideo Nagai and Wolfgang J Runggaldier. Pde approach to utility maximization for market models with hidden markov factors. In *Seminar on Stochastic Analysis, Random Fields and Applications V*, pages 493–506. Springer, 2007.
- [69] Ted O’Donoghue and Matthew Rabin. Doing it now or later. *American Economic Review*, pages 103–124, 1999.
- [70] Bezalel Peleg and Menahem E Yaari. On the existence of a consistent course of action when tastes are changing. *The Review of Economic Studies*, 40(3):391–401, 1973.
- [71] Edmund S Phelps and Robert A Pollak. On second-best national saving and game-equilibrium growth. *The Review of Economic Studies*, 35(2):185–199, 1968.
- [72] Traian A Pirvu and Huayue Zhang. Utility indifference pricing: a time consistent approach. *Applied Mathematical Finance*, 20(4):304–326, 2013.
- [73] Traian A Pirvu and Huayue Zhang. Investment–consumption with regime-switching discount rates. *Mathematical Social Sciences*, 71:142–150, 2014.
- [74] Robert A Pollak. Consistent planning. *The Review of Economic Studies*, 35(2):201–208, 1968.
- [75] Richard Rouge and Nicole El Karoui. Pricing via utility maximization and entropy. *Mathematical Finance*, 10(2):259–276, 2000.
- [76] Paul A Samuelson. Lifetime portfolio selection by dynamic stochastic programming. *The review of economics and statistics*, pages 239–246, 1969.
- [77] Jörn Sass and Ulrich G Haussmann. Optimizing the terminal wealth under partial information: The drift process as a continuous time markov chain. *Finance and Stochastics*, 8(4):553–577, 2004.

- [78] Walter Schachermayer. Optimal investment in incomplete financial markets. In *Mathematical Finance—Bachelier Congress 2000*, pages 427–462. Springer, 2002.
- [79] Walter Schachermayer. *Portfolio optimization in incomplete financial markets*. Cattedra Galileiana, 2004.
- [80] Martin Schweizer. A guided tour through quadratic hedging approaches. 1999.
- [81] G William Schwert. Why does stock market volatility change over time? *The journal of finance*, 44(5):1115–1153, 1989.
- [82] Ronnie Sircar and Thaleia Zariphopoulou. Bounds and asymptotic approximations for utility prices when volatility is random. *SIAM Journal on Control and Optimization*, 43(4):1328–1353, 2004.
- [83] Anatoliĭ Skorokhod. *Asymptotic methods in the theory of stochastic differential equations*.
- [84] Luz Rocío Sotomayor and Abel Cadenillas. Explicit solutions of consumption-investment problems in financial markets with regime switching. *Mathematical Finance*, 19(2):251–279, 2009.
- [85] Richard H Stockbridge. Portfolio optimization in markets having stochastic rates. In *Stochastic Theory and Control*, pages 447–458. Springer, 2002.
- [86] Robert Henry Strotz. Myopia and inconsistency in dynamic utility maximization. *The Review of Economic Studies*, 23(3):165–180, 1955.
- [87] Nicolas Vieille and Jörgen W Weibull. Multiple solutions under quasi-exponential discounting. *Economic Theory*, 39(3):513, 2009.
- [88] John Von Neumann and Oskar Morgenstern. *Theory of games and economic behavior*. Princeton university press, 2007.
- [89] George Yin and Qing Zhang. *Continuous-time Markov chains and applications: a two-time-scale approach*, volume 37. Springer Science & Business Media, 2012.
- [90] Jiongmin Yong and Xun Yu Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*, volume 43. Springer Science & Business Media, 1999.
- [91] Thaleia Zariphopoulou. Investment-consumption models with transaction fees and markov-chain parameters. *SIAM Journal on Control and Optimization*, 30(3):613–636, 1992.

- [92] Thaleia Zariphopoulou. A solution approach to valuation with unhedgeable risks. *Finance and stochastics*, 5(1):61–82, 2001.
- [93] Q Zhang and G Yin. Nearly-optimal asset allocation in hybrid stock investment models. *Journal of Optimization theory and Applications*, 121(2):419–444, 2004.
- [94] Qing Zhang and Xin Guo. Closed-form solutions for perpetual american put options with regime switching. *SIAM Journal on Applied Mathematics*, 64(6):2034–2049, 2004.
- [95] Xun Yu Zhou and Duan Li. Continuous-time mean-variance portfolio selection: A stochastic lq framework. *Applied Mathematics & Optimization*, 42(1):19–33, 2000.
- [96] Xun Yu Zhou and George Yin. Markowitz’s mean-variance portfolio selection with regime switching: A continuous-time model. *SIAM Journal on Control and Optimization*, 42(4):1466–1482, 2003.