The Atkin operator on spaces of overconvergent modular forms and arithmetic applications

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Abstract

We investigate the action of the Atkin operator on spaces of overconvergent p-adic modular forms. Our contributions are both computational and geometric. We present several algorithms to compute the spectrum of the Atkin operator, as well as its p-adic variation as a function of the weight. As an application, we explicitly construct Heegner-type points on elliptic curves. We then make a geometric study of the Atkin operator, and prove a potential semi-stability theorem for correspondences. We explicitly determine the stable models of various Hecke operators on quaternionic Shimura curves, and make a purely geometric study of canonical subgroups.

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Contents

Notation and conventions

The completion of $\mathbf{Q}$ with respect to the p-adic valuation v_p for some prime p, is denoted $\mathbf{Q}_p$, and its ring of integers $\mathbf{Z}_p$. The completion of an algebraic closure of $\mathbf{Q}_p$ with respect to the unique extension of v_p will be denoted $\mathbf{C}_p$, with valuation ring $\mathscr{O}_{\mathbf{C}_p}$. Given a number field F, we denote $\mathbf{A}_F^\infty$ for the ring of finite adèles, or simply $\mathbf{A}^\infty$ when $F = \mathbf{Q}$. For a scheme $\mathscr{X}$ over a valuation ring R, we denote $\mathscr{X}_s$ its special fibre, and $\mathscr{X}_{\bar{s}}$ its geometric special fibre. The singular locus of the special fibre $\mathscr{X}_s$ is denoted $\mathscr{X}_s^{\text{sing}}$. If Z is a component of $\mathscr{X}_s$, then we denote Z^{sm} for its smooth part $Z \backslash \mathscr{X}_s^{\text{sing}}$.

Adic spaces. Let K be a non-Archimedean field whose topology is induced from a rank 1 valuation, and R its ring of integers. We will frequently need notions of p-adic geometry to investigate geometric objects defined over K, and adopt the viewpoint of *adic spaces* as developed in [Hub93] and [Hub94]. Curves over K have 5 types of points, the first 4 of which correspond to valuations of rank 1 and are already considered in Berkovich's theory. The points corresponding to rank 2 valuations are called Type-V points. There is a fully faithful functor from the category of rigid analytic varieties to the category of adic spaces, which we denote $X \mapsto X^{\text{ad}}$. This functor identifies admissible opens and admissible open covers with opens and open covers respectively. This defines an adification functor from the category of admissible formal R-schemes via the Raynaud generic fibre functor. For any scheme $\mathscr{X}$ over R, we set $\mathscr{X}^{\text{ad}}$ to be the adification of the formal completion of $\mathscr{X}$ along its special fibre. A morphism $f : \mathscr{X} \to \mathscr{Y}$ of schemes over R determines a morphism $f^{\text{ad}} : \mathscr{X}^{\text{ad}} \to \mathscr{Y}^{\text{ad}}$, which we often denote f for simplicity.

Chapter 1

Introduction

The aim of this dissertation is to study the p-adic variation of modular forms. We approach this topic in two different ways. Firstly, we develop several algorithms that allow for an explicit computational study of overconvergent forms and the p-adic variation of their eigenvalues. Secondly, we make a geometric study of Hecke correspondences and their semi-stable reduction at bad primes.

Outline. We start by recalling the definitions of overconvergent p-adic modular forms and the Atkin operator in 1.1, and briefly discuss two celebrated applications of Hida's theory of ordinary p-adic families of modular forms. These applications lead us to formulate four motivating questions, which will serve as motivation for the investigations in this thesis. We state these motivating questions in 1.2, and summarise our main contributions in 1.3.

1.1 Modular forms in p-adic families

In this section, we recall the definitions of overconvergent modular forms and the Atkin operator. We include a brief overview of the parts of the literature on p-adic variation of modular eigenforms most relevant for our purposes.

Modular forms are ubiquitous in modern number theory, and form an essential tool for the investigation of geometric and arithmetic objects that arise from the study of Diophantine equations. The arithmetic encoded in modular eigenforms transpires through their Fourier coefficients, whose properties are of tremendous importance. Let us start by noting the following elementary congruence of power series in $\mathbf{Z}[\![q]\!]$:

$$q\prod_{n\geq 1}(1-q^n)^2(1-q^{11n})^2 \equiv q\prod_{n\geq 1}(1-q^n)^{24} \pmod{11}. \tag{1.1}$$

The left hand side is the q-expansion of the unique normalised cuspidal newform on $\Gamma_0(11)$ of weight 2, whereas the right hand side is the q-expansion of the modular discriminant, which is the unique normalised cuspidal form of level 1 and weight 12. For $p \geq 3$ prime, the Kummer congruences for Bernoulli numbers show us that

$$1 - \frac{2k}{B_{2k}} \cdot \sum_{n\geq 1} \sum_{d|n} d^{k-1} q^n \equiv 1 \pmod{p^{n+1}} \qquad \text{when } k \equiv 0 \bmod p^n(p-1), \tag{1.2}$$

where the left hand side is the q-expansion of an Eisenstein series of level 1 and weight k, and the right hand side may be interpreted as a modular form of level 1 and weight 0.

Many other congruences between classical modular forms were known to Ramanujan. The congruences (1.1) and (1.2) follow from the explicit and elementary nature of the Fourier coefficients of the modular forms under consideration, but are in fact special instances of a much deeper phenomenon which is explained by the theory of *overconvergent modular forms*, which we now briefly recall.

Main definitions

We now come to the definition of our two central objects of study. We first define spaces of *overconvergent modular forms* $M_k^{\dagger,r}$, and then the *Atkin operator* U_p.

Definition 1.1. Let $N \geq 5$ and $p \nmid N$ be a prime. We let $\mathscr{X}_{/\mathbf{Z}_p}$ be the moduli space of generalised elliptic curves with $\Gamma_1(N)$-level structure, universal curve $\pi : \mathscr{E} \to \mathscr{X}$, and closed subscheme of cusps $\mathscr{I}_C$. Set

$$\omega := \pi_* \Omega^1_{\mathscr{E}/\mathscr{X}}(\log \pi^{-1}\mathscr{I}_C),$$

which is a line bundle on $\mathscr{X}$. As discussed in [Kat73], there exists a unique section $A \in \mathrm{H}^0(\mathscr{X}_s, \omega^{\otimes p-1})$ with q-expansion 1, called the *Hasse invariant*. Now let $r \in \mathbf{C}_p$ with $0 < v_p(r) \leq 1$, then we can define an open subspace X_r of $\mathscr{X}^{\mathrm{ad}}$ for every $r \in \mathbf{C}_p$, such that

$$X_r(\mathbf{C}_p) := \{x : v_p(E_x) \leq r\},$$

where E_x is a local lift of the Hasse invariant A at x, see [Col96, Section 1]. Note we do not require a global lift of the Hasse invariant to exist, which is known to fail in general when $p \leq 5$. We define the space of *r-overconvergent modular forms* of integer weight k on $\Gamma_1(N)$ to be $M_k^{\dagger,r} := \mathrm{H}^0(X_r, \omega^{\otimes k})$.

We now define a norm $\|\cdot\|_r$ that makes $M_k^{\dagger,r}$ into a p-adic Banach space. Pick a point $x \in X_r$, let K be a finite extension of the residue field of x, and let $\mathrm{Spec}(K) \to \mathscr{X}_{\mathbf{Q}_p}$ be a point whose image corresponds to x. The properness of $\mathscr{X}$ implies that this extends uniquely to a point $\varphi : \mathrm{Spec}(\mathscr{O}_K) \to \mathscr{X}$. Now let $f \in M_k^{\dagger,r}$, then $\varphi^* f = a_f s$ for some section s generating the trivial line bundle $\varphi^*\omega^{\otimes k}$ and some $a \in \mathscr{O}_K$. We set $|f(x)| := |a_f|$, which is independent of the choice of s. The norm $\|f\|_r := \sup\{|f(x)| : x \in X_r\}$ makes $M_k^{\dagger,r}$ into a p-adic Banach space. This induces the structure of a p-adic Fréchet space on

$$M_k^\dagger := \varinjlim_{r>0} M_k^{\dagger,r},$$

which we call the space of *overconvergent modular forms*.

These spaces come with a collection of Hecke operators T_l for $l \nmid Np$, U_l for $l \mid N$, which can be defined by restricting correspondences on $\mathscr{X}$ and have the usual effect on q-expansions. Our main object of study is the Atkin operator U_p, which we define now. Recall that if we set $X_1(N;p)$ and $X_{\mathrm{sc}}(N)$ to be the modular curves parametrising tuples (E_N, G_1) and (E_N, G_1, G_2), where E_N is an elliptic curve with $\Gamma_1(N)$-level structure and G_1, G_2 are distinct subgroups of order p, then the operator U_p on classical modular forms is defined by the correspondence

$$\begin{array}{ccc} & X_{\mathrm{sc}}(N) & \\ \swarrow^{\pi_1} & & \searrow^{\pi_2} \\ X_1(N;p) & & X_1(N;p) \end{array} \qquad \begin{array}{l} \pi_1 : (E_N, G_1, G_2) \mapsto (E_N, G_1) \\ \pi_2 : (E_N, G_1, G_2) \mapsto (E_N/G_2, E[p]/G_2), \end{array}$$

where the maps are induced by the maps on the moduli problem via Yoneda's lemma.

Definition 1.2. The Atkin operator U_p on $M_k^{\dagger,r}$ is defined whenever $r < p/(p+1)$. For such r, any elliptic curve over $\mathbf{C}_p$ corresponding to a point in $X_r(\mathbf{C}_p)$ has a canonical subgroup of order p, see [Kat73] and [Lub79]. This gives rise to a section $X_r \hookrightarrow X_1(N;p)^{\mathrm{ad}}$, and as $\pi_{2,*} \circ \pi_1^*(X_r) \subseteq X_r$, we obtain a well-defined action of U_p on $M_k^{\dagger,r}$ by restricting the correspondence. Evaluating an element of $M_k^{\dagger,r}$ or $M_k^{\dagger}$ on the formal neighbourhood around the cusp ∞ on X_r yields a q-expansion principle as in the classical case, and on q-expansions we obtain the familiar rule

$$U_p : \sum_{n \geq 0} a_n q^n \mapsto \sum_{n \geq 0} a_{np} q^n.$$

Remark 1.3. As can be seen in the above definition, we may have equally, and perhaps more naturally, defined X_r as an open subspace of $\mathscr{X}_1(N;p)^{\mathrm{ad}}$. This is indeed the viewpoint we adopt in Chapter 5. However, we have chosen to work on the modular curve $X_1(N)$ for several reasons, the most important being that the good reduction at p makes for a more concise presentation. Moreover, for reasons of computational efficiency we will be forced to work on $X_1(N)$ in the next chapter.

Some history

A notion of p-adic modular forms of arbitrary p-adic weight was introduced by Serre [Ser73], working solely with q-expansions. Serre defines the space of p-adic modular forms formally as the space of p-adic limits of q-expansions of classical modular forms of different weights. This yields a space of forms with weights parametrised by the group $\mathbf{Z}_p \times \mathbf{Z}/(p-1)\mathbf{Z}$. His definition is simple and allows for powerful applications such as a proof of the Kubota–Leopoldt theorem, but it has the disadvantage that the spaces considered are *too large* and lack a meaningful spectral theory for their Hecke operators. At the same time, Katz [Kat73] gave the geometric definition of p-adic overconvergent modular forms of weights $k \in \mathbf{Z}$ we recalled above. This yields a much smaller, though still infinite-dimensional, p-adic Banach or Fréchet space with a good Hecke theory. The Hecke operator U_p is compact, and hence has a discrete spectrum.

Let $\Lambda_N := \mathbf{Z}_p[[(\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Z}_p^\times]]$ be the Iwasawa algebra of level N, and set $\mathscr{W}_N := (\mathrm{Spf}\,\Lambda_N)^{\mathrm{ad}}$, which we call *weight space*. Whereas the geometric definition of Katz yields an interesting spectral theory for U_p, it is not obvious how to define overconvergent modular forms for a more general weight $\kappa \in \mathscr{W}_N(\mathbf{C}_p) = \mathrm{Hom}_{\mathrm{cont}}((\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Z}_p^\times, \mathbf{C}_p^\times)$. This was first done for ordinary modular forms by Hida [Hid86b], [Hid86a] and for arbitrary slopes by Coleman [Col97b]. Coleman uses the explicit nature of the q-expansion of the Eisenstein series to define overconvergent p-adic Eisenstein families, and defines the space $M_\kappa^\dagger$ of overconvergent modular forms of sufficiently small weight κ as the set of q-expansions obtained from the space of weight 0 by multiplication by the Eisenstein series of weight κ. Recently, Andreatta–Iovita–Stevens [AIS12] and Pilloni [Pil13] have given geometric constructions of line bundles whose sections yield spaces of overconvergent modular forms for more general p-adic weights, without relying on the Eisenstein family. Their method is closer in spirit to the original approach by Hida using Igusa towers.

The eigencurve. Perhaps the most streamlined way of thinking about the p-adic variation of modular forms is in terms of the *eigencurve*. It was first introduced by Coleman–Mazur [CM98], and its construction was later axiomatised by Buzzard [Buz07] to allow generalisations to automorphic forms on more general types of reductive groups. Informally, the eigencurve is a rigid analytic curve $\mathscr{C}_N$ equipped with a map $\pi : \mathscr{C}_N \to \mathscr{W}_N$, such that any point in the fibre of $\kappa \in \mathscr{W}_N(\mathbf{C}_p)$ corresponds to a finite slope overconvergent form in $M_\kappa^\dagger$ that is an eigenform for all Hecke operators. By the work of Coleman [Col97b], we may define spaces $M_V^\dagger$ where $V = \mathrm{Spa}(A, A^+)$ is a sufficiently small open subspace of $\mathscr{W}_N$, which come equipped with a set of Hecke operators. Spectral theory assures discreteness of the spectrum of the compact operator U_p, and several convenient notions such as trace and characteristic series are at hand. We may look at the vanishing locus in $V \times \mathbf{G}_m$ of $\det\left(1 - tU_p \mid M_V^\dagger\right)$, and these regions for various V patch together nicely into an object over $\mathscr{W}_N$, which is called the *spectral curve of U_p*. The eigencurve now arises as a modification of this spectral curve, roughly by separating the multiplicities of U_p using the other Hecke operators.

Two applications of Hida theory

As noted above, the theory of ordinary p-adic families of modular forms developed by Hida, is particularly satisfactory. For instance, the ordinary part of the eigencurve is finite over $\mathscr{W}_N$ and étale at integer weights $k \geq 2$. This allows for a nice description of Hida families, their associated deformations of Galois representations, and a 2-variable "Mazur–Kitagawa" p-adic L-function, attached to classical modular forms. We now discuss two celebrated results whose proofs feature aspects of Hida theory whose analogous statements for non-ordinary forms are at present much more poorly understood. In the next section, we discuss the guiding questions for this thesis, which can be motivated by the successes of Hida theory we discuss here.

The Birch and Swinnerton-Dyer conjecture. This celebrated conjecture states that for an elliptic curve $E/\mathbf{Q}$ the order of vanishing of its Hasse-Weil L-function $L(E,s)$ at $s = 1$ is equal to the rank r of the Mordell-Weil group $E(\mathbf{Q})$, and predicts the leading coefficient in terms of various invariants of E. By the work of Amice-Vélu [AV75] and Vishik [Vis76], we may attach a p-adic L-function $L_p(E,s)$ to a modular elliptic curve, as reviewed in [MTT86, Section 1]. While investigating p-adic analogues of the Birch and Swinnerton-Dyer conjecture, Mazur–Tate–

Teitelbaum [MTT86] formulated their *exceptional zero conjecture*. For an elliptic curve $E/\mathbf{Q}$ with split multiplicative reduction at $p \geq 5$, this conjecture states that $L_p(E,1) = 0$, and

$$L_p'(E,1) = \mathscr{L} \cdot L(E,1) \cdot \Omega_E^{-1} \qquad \text{where} \qquad \mathscr{L} = \log_p(q_E)/v_p(q_E),$$

with $q_E \in p\,\mathbf{Z}_p$ the Tate period of E and $\log_p$ the branch of the p-adic logarithm with $\log_p(p) = 0$.

The exceptional zero conjecture was proved for elliptic curves $E/\mathbf{Q}$ as above by Greenberg-Stevens [GS93]. The ingredient that is of most importance to us, is the Mazur–Kitagawa p-adic L-function $L_p(k,s)$. This is an analytic function of $k,s \in \mathbf{Z}_p$ with $L_p(2,s) = L_p(E,s)$, whose existence and analytic properties we can establish using Hida theory. The exceptional zero conjecture for E as above is now proved by calculating in two different ways the linear term of the Taylor expansion of $L_p(k,s)$ around $(2,1)$. This is a striking example of how a good description of the p-adic variation of a modular form, in this case the one attached to E, leads to a deep theorem. One of the guiding questions in this dissertation is to describe explicitly the p-adic variation of modular forms that are not necessarily ordinary at p.

The Artin conjecture. Consider an irreducible continuous representation $\rho : G_{\mathbf{Q}} \to \mathrm{GL}_n(\mathbf{C})$, where $G_{\mathbf{Q}}$ is the absolute Galois group of $\mathbf{Q}$. Now fix an embedding $\iota : \overline{\mathbf{Q}} \hookrightarrow \overline{\mathbf{Q}_p}$ for every p, and define the Artin L-function as the Euler product

$$L_\rho(s) = \prod_p \det\left(1 - \rho^{I_p}(\mathrm{Frob}_p)p^{-s}\right)^{-1},$$

where ρ^{I_p} is the induced representation on vectors invariant under the inertia group I_p at p. This Euler product is independent of our choice of embeddings, and converges for $\mathrm{Re}(s) > 1$. It was conjectured by Artin [Art24] that $L_\rho(s)$ has an analytic continuation to the entire complex plane, with the exception of a possible pole at $s = 1$ when ρ is trivial. The Brauer induction theorem shows that we can meromorphically continue $L_\rho(s)$ to the entire complex plane, but only very few instances of holomorphic continuation are known. When $n = 1$, the Artin conjecture follows from class field theory and the work of Hecke. When $n = 2$, modularity of all odd representations ρ was established by Khare–Wintenberger [KW09a, KW09b]. Such ρ may be distinguished by their projective image in $\mathrm{PGL}_2(\mathbf{C})$, which is necessarily cyclic, dihedral, or isomorphic to A_4, S_4 or A_5. With the exception of the last case, which we call the class of *icosahedral representations*, these images are solvable. The deep work of Langlands-Tunnell in [Lan80, Tun81] proves the Artin conjecture in those cases, leaving us to wonder what happens for icosahedral representations. A series of papers by Taylor and collaborators proved the Artin conjecture for many interesting icosahedral representations. A crucial argument in their line of attack relies on the theory of overconvergent modular forms. It is this ingredient we wish to discuss here, following Buzzard–Taylor [BT99]. See also [BDSBT01] for many more cases.

We will give an informal sketch of the argument, ignoring the precise assumptions we need to make on ρ for this strategy to work. Using the isomorphism $A_5 \simeq \mathrm{SL}_2(\mathbf{F}_4)$, Shepherd-Barron and Taylor [SBT97] find an appropriate weight 2 modular form over $\mathbf{F}_4$ by realising ρ in the 2-torsion of an abelian surface A with real multiplication by the ring of integers of $\mathbf{Q}(\sqrt{5})$, whose modularity is obtained with the techniques in [Wil95, TW95, BCDT01]. This implies the existence of two *companion forms*, which live in ordinary Hida families that can be specialised to weight one forms f_α, f_β with the desired Fourier coefficients outside p, and whose U_p-eigenvalues α, β are the two eigenvalues of $\rho(\mathrm{Frob}_p)$. To obtain the desired classical modular form of weight 1, we set f to be the overconvergent weight 1 form $(\alpha f_\alpha - \beta f_\beta)/(\alpha - \beta)$, which is shown to be classical by constructing a section h of ω on an open subset of $X_1(N)^{\mathrm{ad}}$ that glues together with f to give a classical form of weight 1. The analyticity of the Artin L-function then follows from the modularity of ρ.

1.2 Main questions

Motivated by the success of the well-understood theory of families of ordinary forms due to Hida [Hid86b, Hid86a], we now identify the key questions we wish to investigate in this dissertation. If f is an overconvergent U_p-eigenform with eigenvalue λ, then $v_p(\lambda)$ is called the *slope* of f. The eigenforms in $M_k^{\dagger,r}$ of slope 0 are exactly the ordinary forms. There is a finite number of them, which leaves us to wonder:

> **Question I:** For a given weight k, can we describe or compute the slopes that occur in the spaces $M_k^{\dagger,r}$? What are their properties?

A first step towards an understanding of the spectrum of the Atkin operator U_p acting on spaces of overconvergent modular forms could be its explicit computation. Lauder [Lau11] developed such an algorithm for $p \geq 5$, and we remove this restriction on p in Chapter 2. As an example, we compute that the first few slopes of U_3 acting on the space of 3-adic overconvergent weight 1 forms $M_1^{\dagger,1/4}(11,\chi)$, where χ is the unique quadratic Dirichlet character modulo 11, are

$$\mathbf{0}_3, \mathbf{1/2}_4, \mathbf{5/2}_{12}, \mathbf{13/2}_{12}, \mathbf{17/2}_{12}, \mathbf{21/2}_{12}, \mathbf{33/2}_{12}, \mathbf{37/2}_{12}, \mathbf{41/2}_{12}, \mathbf{49/2}_{12}, \mathbf{53/2}_{12}, \ldots$$

where the subscripts denote multiplicities. The forms of slope 0 span the ordinary subspace, whose properties we understand well thanks to Hida theory. In contrast, the spaces of larger slope consist entirely of non-classical forms, and remain largely mysterious. In certain well-behaved cases, not including the above example, Buzzard [Buz05] gives a conjectural recipe for the set of slopes occurring in this sequence. We investigate these conjectures in Chapter 2, as well as properties of these slopes in general situations where Buzzard's conjectures do not apply.

> **Question II:** How do slope sequences vary as a function of the weight? Can we describe or compute this variation precisely?

The variation of ordinary forms was described by Hida, and the work of Coleman [Col97b] and Coleman–Mazur [CM98] generalises this to arbitrary finite slope eigenforms, where the p-adic variation of finite slope eigenforms is described by the eigencurve. Unfortunately, precise information on the geometry of eigencurves remains at present mostly unknown away from the ordinary part. As a consequence of the work of Coleman [Col97b], we know that the dimension of the slope α subspace, for some positive rational number α, is locally constant in the weight.

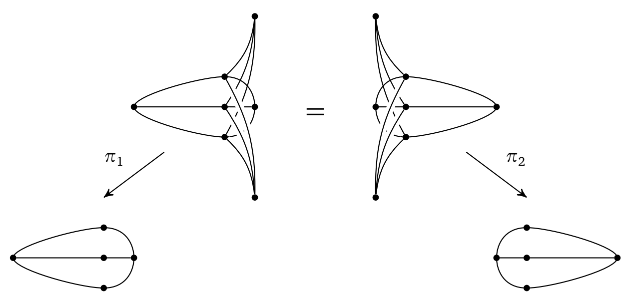
Gouvea–Mazur [GM92] conjectured that this dimension is in fact constant on disks in $\mathscr{W}_N$ of radius linear in α. When $p \geq 5$, a lower bound of order $O_{N,p}(1/\sqrt{\alpha})$ for this radius was found by Wan [Wan98], and we remove this restriction on p in Chapter 2. In the same chapter, we describe an algorithm that computes an equation for the spectral curve $\mathscr{Z} \subseteq \mathscr{W}_N \times \mathbf{G}_m$. For instance, let $P(\kappa, t)$ be the characteristic power series for U_3 over the component of $\mathscr{W}_{11}$ corresponding to the character χ as above. Setting κ to be the parameter which gives us the series for weight $k \in \mathbf{Z}$ and character χ at $\kappa = 4^k - 4$, we are able to compute expansions of $P(\kappa, t)$ of the form

$$\begin{aligned} P(\kappa,t) \;=\; & 1 + 36t + 72t^2 + 53t^3 + 42t^4 + 57t^5 + 18t^6 + 45t^7 \\ & + \kappa\ (5t + 47t^2 + 4t^3 + 14t^4 + 23t^5 + 36t^6 + 33t^7) \\ & + \kappa^2(44t + 38t^2 + 9t^3 + 72t^4 + 56t^5 + 55t^6 + 14t^7 + 9t^8 + 27t^9) \\ & + \kappa^3(51t + 62t^2 + 15t^3 + 4t^4 + 43t^5 + 66t^6 + 17t^7 + 75t^8 + 45t^9 + 27t^{10}) \\ & + O(\kappa^4, 3^4), \end{aligned}$$

which recover the series whose slopes we computed above for $k = 1$, and describes the variation to third order in κ. Note that we have kept both our p-adic and κ-adic precisions extremely low to produce manageable output, but our algorithms can take this *much* further.

> **Question III:** Can we study U_p as a *geometric object*? Are there meaningful integral models for Hecke operators? Can we use these models to compute information about their spectrum?

The determination of good integral models of modular curves at primes of bad reduction is at the heart of our understanding of modular forms, so it is natural to ask whether a similar notion is available for U_p, or indeed general correspondences between curves. When investigating Galois representations occuring in the étale cohomology of curves, the knowledge of a semi-stable integral model is an invaluable tool. Such a model always exists for smooth proper geometrically connected curves over a non-Archimedean valued field, at least after a finite separable base change, by the theorem of Deligne–Mumford [DM69]. In Chapter 4, we propose a notion of semi-stable reduction for *correspondences* between curves, and prove an analogous potential semi-stability theorem. Our theorem generalises and strengthens the work of Coleman [Col03] and Liu [Liu06]. As a consequence, we obtain a notion of *skeleta* for correspondences, which allows for a combinatorial analysis of the action on étale cohomology induced by any correspondence. We explicitly determine the stable model various Hecke correspondences on quaternionic Shimura curves. For instance, we obtain the skeleton of the stable model of U_p on $X_0(Np)$ which may be depicted as:



More details can be found in Chapter 5. As a consequence of the determination of this skeleton, we generalise the moduli interpretation of the irreducible components in the special fibre of the semi-stable model of $\mathscr{X}_0(Np^2)$ found by Coleman [Col05] in level 1. Our potential semi-stability theorem also allows for a geometric study of many key theorems in the theory of overconvergent modular forms, avoiding a systematic use of the specific moduli interpretation available. This has been used to great success by Goren–Kassaei, and we generalise some of their work in Chapter 5.

Question IV: How do our results apply to the study of other arithmetic questions?

We present a number of applications that follow a line of investigation initiated by the work of Darmon, Lauder and Rotger. This work centres around the Birch and Swinnerton-Dyer conjecture in both its global and p-adic incarnations. One of the principal difficulties in attacking this problem is finding a natural supply of points on E. This is most fruitfully studied via the construction of cycles on various covers of E, such as Heegner points on Shimura curves that cover E. Such cycles are hard to come by, even though large supplies are often predicted by the Tate conjecture, and when we can show their existence, their explicit determination is often out of reach. Following the approach of [DR14] and [DLR14], we investigate Gross–Kudla–Schoen cycles attached to a triple of modular forms. The resulting global point can be related to this triple via an explicit p-adic formula in the spirit of Gross–Zagier [GZ86].

1.3 Summary

This dissertation attempts to address the four questions above. **Our main contributions are:**

- Extensions to general primes p of the work of Wan [Wan98] on the Gouvêa–Mazur conjecture, as well as an extension of the algorithm of Lauder [Lau11] to compute the characteristic series of the Atkin operator U_p on $M_k^{\dagger,r}$. See sections 2.1 and 2.2.
- An algorithm to compute the two-variable characteristic power series $P(\kappa, t)$ of U_p, or alternatively the defining equation for the spectral curve in $\mathscr{W}_N \times \mathbf{G}_m$. See section 2.3.
- A potential semi-stable reduction theorem for *correspondences* analogous to the theorem of Deligne–Mumford [DM69] for curves, generalising and strengthening work of Coleman [Col03] and Liu [Liu06] on simultaneous semi-stable reduction for finite maps between curves. See chapter 4.
- An explicit determination of the stable models of various Hecke operators on quaternionic Shimura curves. As a consequence, we generalise work of Coleman [Col05] on supersingular components of $X_0(Np^2)$ and Goren–Kassaei [GK06] on canonical subgroups. See chapter 5.

We apply these results throughout to a variety of explicit examples, most notably the computation of slope sequences and exceptional denominators of slopes, and the construction of Chow–Heegner and Stark points on elliptic curves via special values of Rankin triple product p-adic L-functions.

The outline is as follows. In **Chapter 2**, we investigate Question I from a computational perspective. The explicit bases for $M_k^{\dagger,r}$ and stability of the corresponding integral structure under U_p in [Kat73] rely on the existence of a lift to $\mathbf{Z}_p$ of the Hasse invariant. We investigate the situations where such a lift does not necessarily exist, thereby removing the restriction $p \geq 5$ from the work of Lauder [Lau11] and Wan [Wan98]. We computationally verify Buzzard's conjectures [Buz05] in the regular case, and report some large denominators in the irregular case. Finally, we present an algorithm to compute the equation of the spectral curve in $\mathscr{W}_N \times \mathbf{G}_m$, in the form of a 2-variable characteristic series $P(\kappa, t) \in \mathbf{Z}_p[\![\kappa, t]\!]$ up to some precision $O(p^m, \kappa^n)$, which forms a computational contribution to the study of Question II.

In **Chapter 3**, we use our algorithms to explicitly construct Chow–Heegner points in Section 3.2, and Stark points in Section 3.3, on elliptic curves. These are global points, whose p-adic logarithm may be computed from the special value of a Rankin triple product p-adic L-function through a p-adic analogue of the Gross-Zagier formula developed by Darmon–Rotger [DR14] for Chow–Heegner points, and by Darmon–Lauder–Rotger [DLR14] for Stark points. All this material concerns Question IV above.

We then turn to a detailed study of the Atkin operator U_p as a *geometric object*. For correspondences between curves, we prove in **Chapter 4** that we can find semi-stable models for all the constituents simultaneously over some finite separable extension of the base, such that the transition maps remain finite. In fact, we prove a stronger *skeletal* version of this theorem in many cases, which gives rise to the notion of the *skeleton* of a correspondence, analogous to the familiar notion for curves in p-adic geometry. We show how this skeleton allows for a combinatorial understanding of the induced maps between cohomology groups, by restricting to the graded pieces for the weight-monodromy filtration. This provides a contribution to the study of Question III for a general class of correspondences.

In **Chapter 5**, we explicitly determine the stable models of various Hecke operators on quaternionic Shimura curves. As a corollary, we generalise the work of Coleman [Col05] to arbitrary levels, to find a moduli interpretation of the components that appear in the special fibre of the stable model $\mathfrak{U}_p$ of U_p on $X_0(Np)$ in terms of *too supersingular* and *nearly too supersingular* elliptic curves. We then investigate to what extent the theories of canonical subgroups and analytic continuation of overconvergent eigenforms are implied by the underlying *geometry* of U_p, avoiding the use of the specific moduli interpretation which might not be available in more general settings. This adheres to the philosophy put forth by Goren–Kassaei [GK06] and Kassaei [Kas09], and we revisit and generalise some of their results.

Chapter 2

Computing overconvergent modular forms

In this chapter, we describe how to explicitly compute with overconvergent modular forms. Our contribution is twofold. Firstly, we remove the restriction $p \geq 5$ that exists in work of Katz [Kat73], and as an application extend the results of [Wan98] on the Gouvêa–Mazur conjecture, and the algorithms in [Lau11]. Secondly, we present an algorithm to explicitly compute an equation for the spectral curve of U_p, which is a 2-variable power series $P(\kappa, t) \in \mathbf{Z}_p[\![\kappa, t]\!]$ cutting out the spectral curve in $\mathscr{W}_N \times \mathbf{G}_m$. This power series describes the action of U_p on $M_\kappa^{\dagger,r}$ for all p-adic weights $\kappa \in \mathscr{W}_N$ simultaneously, up to some chosen precision.

Outline. In section 2.1, we construct an explicit basis for spaces of overconvergent forms. This provides us with a non-canonical $\mathbf{Z}_p$-lattice which we compare to the ones considered in [Kat73] to understand their behaviour under the action of U_p. In section 2.2, we present a number of applications of these results by removing the restriction $p \geq 5$ from both Wan's quadratic bound for the Gouvêa–Mazur conjecture [Wan98], and Lauder's algorithms for computing characteristic series for U_p. This yields experimental tools to investigate conjectures of Buzzard about slopes, and of Gouvêa–Mazur on the p-adic variation of them. Finally, in section 2.3, we describe how to compute an equation $P(\kappa, t)$ for the spectral curves $\mathscr{X} \subseteq \mathscr{W}_N \times \mathbf{G}_m$ for arbitrary tame levels N and primes p, up to any given p-adic and κ-adic precision.

Remark. With the exception of the material on spectral curves, this chapter is a reworked and expanded version of the paper [Von15], published in LMS Journal of Computational Mathematics (2015).

2.1 Explicit bases for small primes

We now construct explicit bases for r-overconvergent forms for any prime p, removing the restriction $p \geq 5$ from [Kat73, Proposition 2.6.2]. We then study how U_p interacts with the corresponding lattice.

Notation. Let $\mathscr{X}$ be the modular curve over $\mathbf{Z}_p$ with $\Gamma_1(N)$-level structure for $N \geq 5$ and line bundle ω as in Definition 1.1. Let n be the smallest power of p such that A^n lifts to a level 1 Eisenstein series E of weight $k_E = n(p-1)$. Throughout this section, we assume $v_p(r^n) \leq 1$.

	$p=2$	$p=3$	$p \geq 5$
E	E_4	E_6	E_{p-1}
n	4	3	1
k_E	4	6	$p-1$

Remark 2.1. We can make similar definitions for arbitrary congruence subgroups Γ, and the computations in Section 2.2 are for $\Gamma = \Gamma_0(N)$. The justification lies in the fact that the arguments below can be done on the coarse moduli scheme $\mathscr{X}_0(N)$ over $\mathbf{Z}_p$ instead, working with the line bundles ω_k as in [Maz77, Lemma II.4.5]. A very careful analysis is given in [BC05, Appendix].

2.1.1 Explicit bases

We now describe an explicit basis for the p-adic Banach space $M_k^{\dagger,r}$. Let $\mathscr{I}_r$ be the sheaf of ideals in $\mathrm{Sym}(\omega^{\otimes k_E})$ generated by $E - r^n$, and define the line bundle $\mathscr{L} = Spec_{\mathscr{X}}\left(\mathrm{Sym}(\omega^{\otimes k_E})/\mathscr{I}_r\right) \xrightarrow{\pi_{\mathscr{L}}} \mathscr{X}$. Assuming that $k \neq 1$, we can apply the base change theorems from [Kat73, Theorem 1.7.1] to show that

$$M_k^{\dagger,r} = \mathrm{H}^0\left(\mathscr{L}^{\mathrm{ad}}, \pi_{\mathscr{L}}^* \omega^{\otimes k}\right) = \mathrm{H}^0\left(\mathscr{X}, \omega^{\otimes k} \otimes \mathrm{Sym}(\omega^{\otimes k_E})\right) / \mathrm{H}^0(\mathscr{X}, \mathscr{I}_r). \tag{2.1}$$

Having this concrete description in hand, we now attempt to eliminate the relation $E = r^n$ by investigating the map given by multiplication by E on modular forms as in [Kat73, Lemma 2.6.1]. The proof is nearly identical.

Lemma 2.1. *Let $k \neq 1$, then the injection given by the multiplication by E-map*

$$- \times E : \mathrm{H}^0\left(\mathscr{X}, \omega^{\otimes k}\right) \longrightarrow \mathrm{H}^0\left(\mathscr{X}, \omega^{\otimes k + k_E}\right)$$

splits as a map of $\mathbf{Z}_p$-modules.

Proof. The result is clear for $k \leq 0$. For $k \geq 2$, we have $\mathrm{H}^1(\mathscr{X}, \omega^{\otimes k}) = 0$ by computing the degree of ω as in [Kat73, Theorem 1.7.1]. We obtain the short exact sequence

$$0 \to \mathrm{H}^0\left(\mathscr{X}, \omega^{\otimes k}\right) \xrightarrow{\times E} \mathrm{H}^0\left(\mathscr{X}, \omega^{\otimes k + k_E}\right) \longrightarrow \mathrm{H}^0\left(\mathscr{X}, \mathscr{F}\right) \to 0,$$

where $\mathscr{F}$ is the quotient sheaf. This sheaf $\mathscr{F}$ is flat over $\mathbf{Z}_p$, and since $\mathscr{F}$ is a skyscraper sheaf over $\mathbf{F}_p$ it follows that $\mathrm{H}^1(\mathscr{X}_s, \mathscr{F}) = 0$ and hence $\mathrm{Supp}\, \mathrm{R}^1 f_* \mathscr{F} = \emptyset$, where $f : \mathscr{X} \to \mathrm{Spec}(\mathbf{Z}_p)$ is the defining morphism for $\mathscr{X}$. We conclude that $\mathrm{H}^0(\mathscr{X}, \mathscr{F})$ is a free $\mathbf{Z}_p$-module, from which the conclusion follows. □

For every $i \geq 0$, choose generators $\{a_{i,j}\}_j$ for a complement of the submodule $\mathrm{Im}(- \times E) \subseteq \mathrm{H}^0(\mathscr{X}, \omega^{\otimes k + ik_E})$. This choice is not canonical, but we will fix it once and for all in what follows. By running through the proof of [Kat73, Proposition 2.6.2], one can check that the following theorem is a direct consequence of equation (2.1) and Lemma 2.1.

Theorem 2.1. *The set* $\left\{r^{ni}a_{i,j}E^{-i}\right\}_{i,j}$ *is a basis for the p-adic Banach space* $M_k^{\dagger,r}$.

Remark 2.2. Note that we have avoided the case $k=1$, as the standard base change results are known to fail for many levels. However, we can deal with weight 1 forms in all applications by Frobenius semi-linearity of U_p in the sense of [Col97b, Eqn. (3.3)], hence reducing the question to one in higher weight for which the results above hold. See also [Lau11, Section 2.2].

2.1.2 Comparing lattices

We now have two integral structures on $M_k^{\dagger,r}$. The first, which we will call $\mathscr{M}_H(r)$, is the one defined by Katz [Kat73, Section 2.2] using the interpretation of modular forms as certain rules on test objects. It has the advantage of being well suited for analysing its interaction with various operators. It has the disadvantage of only being explicit, in the sense of Theorem 2.1, when a lift of the Hasse invariant to $\mathbf{Z}_p$ exists. The second is the $\mathbf{Z}_p$-Banach space $\mathscr{M}_E(r)$ spanned by the basis we just defined in Theorem 2.1. It has the advantage of being explicit and computational, but the disadvantage of being non-canonical and hence having a rather mysterious interaction with U_p. We will now attempt to compare $\mathscr{M}_H$ and $\mathscr{M}_E$, in order to get the best of both worlds.

Lemma 2.2. *Assume that there exists a lift H of the Hasse invariant to* $\mathbf{Z}_p$*, then we have*

$$r^{n-1}\mathscr{M}_H(r)\subseteq\mathscr{M}_E(r).$$

Proof. Fix a choice of complementary subspaces for both E and H as above. Let $f\in\mathscr{M}_H(r)$, then by Theorem A we can write $f=\sum_{i\geq 0} r^i b_i H^{-i}$, where b_i is in the i-th complementary subspace for H. We rewrite this as

$$f=\sum_{i\geq 1}\sum_{j=0}^{n-1} r^{ni-j}\ b_{ni-j}\ H^{-(ni-j)}=\sum_{i\geq 1} r^{ni}H^{-ni}\left(\sum_{j=0}^{n-1} r^{-j}\ b_{ni-j}\ H^j\right). \tag{2.2}$$

The inner sum in the above expression is guaranteed to be in $\mathrm{H}^0(\mathscr{X},\omega^{\otimes k+ni})$ when multiplied by r^{n-1}. We can decompose this multiple as $\sum_{m=0}^{i} c_m E^m$, where c_m is in the m-th complementary subspace for E. Recall that n is a power of p, from which we get $E\equiv H^n \pmod{pn}$. If we substitute all this into (2.2), we obtain that $r^{n-1}f\in\mathscr{M}_E(r)$ as desired. □

With the aid of this lemma, we now investigate the interaction of our explicit lattice $\mathscr{M}_E(r)$ with the operators U_p and multiplication by $G:=E/V_pE$, where V_p is the Frobenius operator defined in [Col96, Section 2]. Both operators will play a crucial role in the applications.

Theorem 2.2. *Let* $v_p(r)<\frac{1}{p+1}$*, then we have*

$$r^{n-1}pU_p\left(\mathscr{M}_E(r)\right)\subseteq\mathscr{M}_E(r^p)\quad \textit{and}\quad G\cdot\mathscr{M}_E(r)\subseteq\mathscr{M}_E(r).$$

Proof. Assume first that there exists a lift of the Hasse invariant to $\mathbf{Z}_p$. The first statement follows immediately from Lemma 2.2 and the inclusion $pU_p\left(\mathscr{M}_H(r)\right) \subseteq \mathscr{M}_H(r^p)$, which is [Gou88, Proposition II.3.6]. The second statement follows from [Wan98, Lemma 2.1] for $p \geq 5$ and from the formulae in [Cal08, Section 3] for $p \leq 3$, where it is proved that $G - 1 \in r^{n-1}\mathscr{M}_H(r)$ for $r < 1/(p+1)$. It follows that $G \cdot \mathscr{M}_E \subseteq \mathscr{M}_E + r^{n-1}\mathscr{M}_H \subseteq \mathscr{M}_E$.

In the general case, add two additional level structures Γ_1, Γ_2 that both assure the existence of a $\mathbf{Z}_p$-lift of the Hasse invariant and intersect trivially, see [Col96, Section 6] and [Col97b, Section B2]. Upon picking a set of complementary subspaces for both, we obtain two $\mathbf{Z}_p$-Banach spaces $\mathscr{M}_{E,1}(r)$ and $\mathscr{M}_{E,2}(r)$. We claim that we may choose the complementary subspaces so that

$$\mathscr{M}_{E,1}(r) \cap \mathscr{M}_{E,2}(r) = \mathscr{M}_E(r),$$

for any value of r, which clearly implies the theorem by reducing to the above case. By remark 2.2, we may assume $k \geq 2$. Denoting $\mathscr{X}_{\Gamma_1}$ for the modular curve over $\mathbf{Z}_p$ with additional Γ_1-level structure, it suffices to prove that for any choice of splitting s in the diagram

$$\begin{array}{ccc} \mathrm{H}^0(\mathscr{X}, \omega^{\otimes k}) & \xrightarrow[\underset{s}{\dashleftarrow}]{\times E} & \mathrm{H}^0(\mathscr{X}, \omega^{\otimes k+k_E}) \\ \downarrow & & \downarrow \\ \mathrm{H}^0(\mathscr{X}_{\Gamma_1}, \omega^{\otimes k}) & \xrightarrow{\times E} & \mathrm{H}^0(\mathscr{X}_{\Gamma_1}, \omega^{\otimes k+k_E}) \end{array}$$

where the vertical arrows are inclusion, we may find a splitting of the bottom horizontal arrow that restricts to s. Let M be the free $\mathbf{Z}_p$-module spanned by the images of the bottom horizontal and right vertical maps. As M arises as the sections of a sheaf which has zero-dimensional support in the special fibre, we use the same argument as in the proof of Lemma 2.1 to obtain a splitting

$$M \underset{\overset{s_M}{\dashleftarrow}}{\longrightarrow} \mathrm{H}^0(\mathscr{X}_{\Gamma_1}, \omega^{\otimes k+k_E})$$

which, together with a splitting of the left vertical map, yields the result. □

2.2 Applications

We now explain how Theorems 2.1 and 2.2 enable us to generalise previous work in the literature due to Wan [Wan98] and Lauder [Lau11]. We work with $\Gamma = \Gamma_0(N)$ for computational simplicity when appropriate, see Remark 2.1.

2.2.1 The Gouvêa–Mazur conjecture

An enormous amount of arithmetic information is encoded in the *slopes* of overconvergent modular forms, which are the valuations of their U_p-eigenvalues. One of the consequences of the theory of Coleman [Col97b] is that for any $\alpha > 0$, there exists a smallest integer N_α with the following property: If $k_1, k_2 \in \mathbf{Z}$ such that $k_1 \equiv k_2 \mod p^{N_\alpha}(p-1)$, then the collection of slopes $\leq \alpha$ in weights k_1 and k_2 agree, with multiplicities. Gouvêa and Mazur conjectured in [GM92] that $N_\alpha \leq \lfloor \alpha \rfloor$, to which a counterexample was given in [BC04]. However, Wan [Wan98] exhibits an explicit quadratic upper bound for N_α, depending on p and the level, provided that $p \geq 5$. We will now remove this restriction on p.

Wan's analysis relies on a good knowledge of an explicit basis, along with an understanding of how U_p and G act on the integral lattice. This is exactly the content of Theorems 2.1 and 2.2, making the proof a straightforward adaptation of the methods in [Wan98]. We estimate the size of the coefficients of the characteristic series $P_k(t)$ of U_p on the space $M_k^{\dagger,r}$. This is done by analysing the entries of the matrix of U_p on the Katz basis. After twisting U_p by E, Theorem B enables us to do this uniformly with respect to variations of the weight.

Notation. Choose generators $a_{u,v}$ for the u-th complementary subspace, giving rise to a Katz basis $e_{u,v} = r^{nu} a_{u,v} E^{-u}$. Multiplication by E^j defines an isomorphism $M_k^{\dagger,r} \to M_{k+jk_E}^{\dagger,r}$, so we conclude by an application of Frobenius linearity of U_p [Col96, Eqn. (3.3)] that $P_{k+jk_E}(t)$ equals the characteristic series of $U_p \circ G^j$ on $M_k^{\dagger,r}$, where we recall that $G = E/V_pE$. We write

$$U \circ G^j(e_{u,v}) = \sum_{w,z} A_{u,v}^{w,z}(j)\, e_{w,z},$$

for some $A_{u,v}^{w,z} \in K$. The following lemma estimates the size of these numbers, independently of j. This is the analogue of Wan [Wan98, Lemma 3.1], but note the extra term $v_p(r)(n-1)$ appearing.

Lemma 2.3. *We have*

$$v_p\left(A_{u,v}^{w,z}(j)\right) \geq wk_E v_p(r) - 1 - v_p(r)(n-1).$$

Proof. It follows from Theorem B that

$$U \circ G^j(e_{u,v}) = \frac{1}{r^{n-1}p} \sum_w \frac{r^{npw}}{E^w} b_w(u,v,j) = \frac{1}{r^{n-1}p} \sum_w r^{k_E w} \frac{r^{nw}}{E^w} b_w(u,v,j),$$

where $b_w(u,v,j)$ is in the j-th complementary subspace, and hence an integral combination of the $a_{w,z}$. This gives us the desired bound on $A_{u,v}^{w,z}(j)$. □

The key observation is that the above lower bound is independent of j. After taking determinants, we obtain a lower bound on the coefficients of $P_{k+jk_E}(t)$, again independent of j. Wan now proceeds by proving a very general reciprocity lemma on Newton polygons, which allows him to transform the lower bound for $P_k(t)$ into an upper bound for N_α. The analysis goes through without modifications, and using Wan's results we deduce from Lemma 2.3 that

Theorem 2.3. *There is an explicitly computable quadratic polynomial $P \in \mathbf{Q}[x]$, depending only on p and the level, such that $N_\alpha \leq P(\alpha)$.*

2.2.2 Lauder's algorithm

Lauder [Lau11] describes an algorithm to compute the characteristic power series of the compact operator U_p on $M_k^{\dagger,r}$ for $p \geq 5$. By an application of Frobenius semi-linearity [Col96, Eqn. (3.3)], often referred to as *Coleman's trick*, it is particularly efficient when k becomes large. We now remove the assumption $p \geq 5$ from his algorithm.

The crucial ingredients are the determination of an explicit basis for $M_k^{\dagger,r}$, as well as a good understanding of the interaction of U_p with this basis. Theorems 2.1 and 2.2 give unconditional analogues for these two ingredients, and it now becomes straightforward to remove the restriction on p from the algorithms in [Lau11]. The code for our extension to small primes can be found on the author's webpage.

Algorithm 2.1. *Take as input (p, N, k, m) where p is a prime number, N and m are positive integers, and $k \in \mathbf{Z}$. Then compute:*

1. ***Complementary subspaces.*** *Compute the unique $k_0, j \in \mathbf{Z}$ with $0 \leq k_0 < k_E$ and $k_0 + jk_E = k$. Compute $c := \left\lfloor \frac{p+1}{k_E}(m + 1 + \frac{n-1}{p+1}) \right\rfloor$. Set working precision $m' = m + \lceil \frac{nc}{p+1} \rceil$. Now compute q-expansions with coefficients in $\mathbf{Z}/p^{m'}\mathbf{Z}$ for the complementary subspaces $a_{i,j}$ from Theorem 2.1 by first computing a $\mathbf{Z}$-basis for spaces of small weight, and multiplying them together at random.*
2. ***Katz Expansions.*** *Compute the q-expansions of $p^{\lfloor \frac{in}{p+1} \rfloor} E^{-i} a_{i,j}$, which are an approximation of the first few basis elements of $M_k^{\dagger,r}$ as in Theorem 2.1.*
3. ***Twisted Atkin operator.*** *Compute $G(q)^j$ and multiply every basis element $e_{i,j}$ by it. Compute U_p on this multiplied basis, and extract its characteristic series.*

Theorem 2.4. *Algorithm 2.1 gives as output the power series $P_k(t) \pmod{p^m}$, and has polynomial running time in p, N, m and linear running time in $\log(k)$.*

Proof. The increase of working precision to m' in Step I enables us to work over $\mathbf{Z}_p$, by replacing r^{in} with $p^{\lfloor \frac{in}{p+1} \rfloor}$. This leads to a loss of precision when solving the system of equations necessary to obtain the matrix of U_p in Step III, see [Lau11, Section 3.2.1] for more details. Furthermore, Lemma 2.3 guarantees that the matrix we compute this way for U_p is indeed correct up to the given p-adic precision. It follows that Algorithm 2.1 outputs the power series $P_k(t)$ up to p-adic precision m. The complexity statements follow from the analysis in Lauder [Lau11, Section 3.2.2]. □

Remark 2.3. Lauder [Lau11], constructs complementary spaces by multiplying together random forms of weights bounded by a small constant. For $p = 2, 3$ this seemed very inefficient, and we implemented instead an enhancement that uses all forms of lower weights that we constructed up to any given point during the algorithm.

Example 2.4. The case $p = 2$ is prolific soil for finding counterexamples to the Gouvêa–Mazur conjecture. As noted above, the first counterexample was given in [BC04] for $p = 59$ and level 1, and a further one for $p = 79$ in [Lau11]. For $p = 2$, we obtain the following slope sequences in level $\Gamma_0(19)$:

$$\begin{aligned}
k = -2: &\quad \mathbf{0}_4, \mathbf{1/2}_2, \mathbf{1}_3, \mathbf{2}_5, \mathbf{9/4}_4, \mathbf{4}_3, \mathbf{5}_2, \mathbf{6}_{21}, \mathbf{15/2}_2, \dots \\
k = 0: &\quad \mathbf{0}_4, \mathbf{1/2}_2, \mathbf{1}_5, \mathbf{3}_{11}, \mathbf{13/4}_4, \mathbf{7}_{25}, \mathbf{25/2}_4, \mathbf{13}_{11}, \dots \\
k = 2: &\quad \mathbf{0}_4, \mathbf{1/2}_2, \mathbf{1}_3, \mathbf{3/2}_2, \mathbf{2}_5, \mathbf{4}_{11}, \mathbf{17/4}_4, \mathbf{8}_{25}, \mathbf{27/2}_4, \dots \\
k = 4: &\quad \mathbf{0}_4, \mathbf{1/2}_2, \mathbf{1}_5, \mathbf{5/2}_2, \mathbf{3}_6, \mathbf{7/2}_2, \mathbf{4}_3, \mathbf{5}_5, \mathbf{21/4}_4, \dots \\
k = 6: &\quad \mathbf{0}_4, \mathbf{1/2}_2, \mathbf{1}_3, \mathbf{2}_7, \mathbf{5/2}_2, \mathbf{4}_3, \mathbf{9/2}_2, \mathbf{5}_6, \mathbf{11/2}_2, \dots \\
k = 8: &\quad \mathbf{0}_4, \mathbf{1/2}_2, \mathbf{1}_5, \mathbf{3}_{13}, \mathbf{7/2}_2, \mathbf{6}_5, \mathbf{13/2}_2, \mathbf{7}_6, \mathbf{15/2}_2, \dots
\end{aligned}$$

Notice the aberration in the dimensions of the slope 1 subspaces, as well as the slope 3 subspaces in weights 0 and 8. Whereas these are all *near misses*, in that the smallest slopes for which discrepancies arise are exactly equal to the valuation of the weight difference, we note a 2-dimensional slope 3/2 subspace in weight 2, which is completely absent in weight 6, whereas $3/2 < v_2(6-2) = 2$. Similarly, the slope 9/4 subspace in weight -2 does not exist in weight $6 = -2 + 2^3$.

2.2.3 Buzzard's slope conjectures

In [Buz05], Buzzard made conjectures on the sequences of slopes for $M_k^{\dagger,r}$ on $\Gamma_0(N)$, and gives a precise conjectural recipe for these sequences when p is $\Gamma_0(N)$-regular. This is a condition which essentially ensures that the slopes at small weights are as small as Hida theory allows them to be. For a precise definition and a reformulation in terms of Galois representations, see [Buz05, Section 1]. A summary of the current state of the theorems and conjectures on slopes may be found in a recent preprint by Buzzard–Gee [BG15].

Example 2.5. We compute that the first few slopes of U_3 acting on $M_{278}^{\dagger}\left(\Gamma_0(41)\right)$ are

$$\mathbf{0}_{12}, \mathbf{1}_{14}, \mathbf{3}_{48}, \mathbf{6}_{14}, \mathbf{7}_{22}, \mathbf{8}_6, \mathbf{9}_{22}, \mathbf{10}_{14}, \mathbf{12}_{48}, \mathbf{14}_{14}, \mathbf{16}_{22}, \mathbf{17}_6, \mathbf{18}_{22}, \dots$$

where the subscripts denote multiplicities. We check that 3 is $\Gamma_0(41)$-regular, and that the slopes agree with Buzzard's prediction. Note that this slope sequence equals the one in weight 8 for all the terms we display here, suggesting a very strong form of the Gouvêa–Mazur conjecture.

Example 2.6. To illustrate a case where regularity fails in a striking way, we compute the first few slopes of U_2 acting on $M_{10}^{\dagger}\left(\Gamma_0(89)\right)$, where as before the subscripts denote multiplicities:

$$\mathbf{0}_{16}, \mathbf{1}_{22}, \mathbf{2}_{22}, \mathbf{14/5}_5, \mathbf{3}_1, \mathbf{4}_{68}, \mathbf{9/2}_4, \mathbf{6}_1, \mathbf{31/5}_5, \mathbf{7}_{22}, \mathbf{8}_{22}, \mathbf{9}_{30}, \mathbf{10}_{22}, \mathbf{21/2}_{16}, \mathbf{12}_{52}, \dots$$

The appearance of denominators as large as 5 does not seem to have been recorded before. Note that by Coleman's classicality criterion [Col96, Theorem 6.1], the overconvergent forms giving rise to these denominators are in fact even classical.

Example 2.7. A more systematic computation of 2-adic overconvergent forms of levels $\Gamma_0(53)$ and $\Gamma_0(61)$ suggests a remarkable relationship between the corresponding eigencurves, for which we have no explanation. The table below lists the first few entries of the 2-adic slope sequences in weights 14 and 16.

	$k = 14$
$\Gamma_0(53)$	$\mathbf{0}_{10}, \mathbf{1}_{13}, \mathbf{2}_{23}, \mathbf{4}_{13}, \mathbf{6}_{59}, \mathbf{9}_{13}, \mathbf{11}_{23}, \mathbf{12}_{13}, \mathbf{13}_{18}, \mathbf{14}_{13}, \mathbf{29/2}_{10}, \mathbf{16}_{18}, \mathbf{17}_{13}, \mathbf{18}_{23}, \mathbf{21}_{13}, \ldots$
$\Gamma_0(61)$	$\mathbf{0}_{12}, \mathbf{1}_{15}, \mathbf{2}_{25}, \mathbf{4}_{15}, \mathbf{6}_{69}, \mathbf{9}_{15}, \mathbf{11}_{25}, \mathbf{12}_{15}, \mathbf{13}_{22}, \mathbf{14}_{15}, \mathbf{29/2}_{10}, \mathbf{16}_{22}, \mathbf{17}_{15}, \mathbf{18}_{25}, \mathbf{21}_{15}, \ldots$

	$k = 16$
$\Gamma_0(53)$	$\mathbf{0}_{10}, \mathbf{1}_{13}, \mathbf{3/2}_{10}, \mathbf{3}_{31}, \mathbf{17/3}_{3}, \mathbf{6}_{1}, \mathbf{7}_{67}, \mathbf{15/2}_{2}, \mathbf{9}_{1}, \mathbf{28/3}_{3}, \mathbf{12}_{31}, \mathbf{27/2}_{10}, \mathbf{14}_{13}, \mathbf{15}_{18}, \mathbf{16}_{13}, \ldots$
$\Gamma_0(61)$	$\mathbf{0}_{12}, \mathbf{1}_{15}, \mathbf{3/2}_{10}, \mathbf{3}_{37}, \mathbf{17/3}_{3}, \mathbf{6}_{1}, \mathbf{7}_{78}, \quad \mathbf{8}_{1}, \quad \mathbf{9}_{1}, \mathbf{28/3}_{3}, \mathbf{12}_{37}, \mathbf{27/2}_{10}, \mathbf{14}_{15}, \mathbf{15}_{22}, \mathbf{16}_{15}, \ldots$

This computation was carried out to a large precision, and for a much larger range of weights. We chose to include the start of the sequence for $k = 14, 16$ as it illustrates the general behaviour rather well. The set of slopes, without multiplicities, seems to agree for both levels in all weights, with the exception of a small deviation. This deviation, if it occurs, seems to be accounted for by the 2-stabilisations of the largest classical cuspidal slope of level N.

Remark 2.8. Example 2.7 is akin to the examples given in [Buz05, Consequence 4.6], where the 5-adic slopes in levels $\Gamma_0(6), \Gamma_0(8)$ and $\Gamma_0(30), \Gamma_0(40)$ are observed to coincide for all weights, even with multiplicities.

2.3 Computing the spectral curve of U_p

We now study the variation of the characteristic series of U_p as a function of the weight, and develop an algorithm to compute the 2-variable series $P(\kappa, t)$. The curve in $\mathscr{W}_N \times \mathbf{G}_m$ cut out by this equation is the *spectral curve* of U_p, which yields the eigencurve after an additional modification, see [CM98].

Notation. In what follows, we set $q = p$, unless $p = 2$ in which case $q = 4$. Let Λ_N to be the level N Iwasawa algebra $\mathbf{Z}_p[\![(\mathbf{Z}/N\mathbf{Z})^\times \times \mathbf{Z}_p^\times]\!]$. The weight space $\mathscr{W}_N = (\mathrm{Spf}\,\Lambda_N)^{\mathrm{ad}}$ is isomorphic to a disjoint union of open unit disks, indexed by $(\mathbf{Z}/qN\mathbf{Z})^\times$.

2.3.1 The algorithm

Recall that the equation for the spectral curve of U_p is a power series in $\Lambda_N[\![t]\!]$, by a result of Coleman [Col97a]. Let κ be the parameter of $\mathscr{W}_N$ with respect to which the integers $k \in \mathbf{Z}$ are embedded via $\kappa = (1+q)^k - 1$. Over any connected component of $\mathscr{W}_N$, we may now describe the spectral curve $\mathscr{Z} \subseteq D \times \mathbf{G}_m$ by a 2-variable power series $P(\kappa, t) \in \mathbf{Z}_p[\![\kappa, t]\!]$. Our extension of the algorithms in Lauder [Lau11] as presented above a priori computes the characteristic series of U_p in a fixed integer weight k. We now describe a way to efficiently interpolate these computations to obtain an algorithm for computing $P(\kappa, t)$.

Let $f : \mathscr{W}_N \to \mathbf{C}_p$ be a function in the Iwasawa algebra, and $\{\kappa_0, \kappa_1, \ldots, \kappa_n\}$ a finite set of Type-I points. Then we denote $f[\kappa_0] = f(\kappa_0)$ and we inductively define the *divided difference* of order n to be

$$f[\kappa_0, \kappa_1, \ldots, \kappa_n] := \frac{f[\kappa_1, \ldots, \kappa_n] - f[\kappa_0, \ldots, \kappa_{n-1}]}{\kappa_n - \kappa_0}.$$

We now define the n-th Newton series to be

$$P_n(\kappa, t) = \sum_{i=0}^{n} P[\kappa_0, \kappa_1, \ldots, \kappa_i](t) \times (\kappa - \kappa_0)(\kappa - \kappa_1) \cdots (\kappa - \kappa_i), \tag{2.3}$$

where $P[\kappa_0, \ldots, \kappa_n](t)$ is the power series in t obtained by taking the corresponding finite differences on the coefficients of $P(\kappa, t)$ of t, which are elements of the Iwasawa algebra by Coleman [Col97a]. The theory of finite differences then shows that upon increasing the number of interpolation points, the n-th Newton series p-adically approaches the series $P(\kappa, t)$. This means that all we need to do to compute an approximation for $P(\kappa, t)$, is to choose our interpolation points carefully and estimate the error term.

We will perform all computations for every component of $\mathscr{W}_N$ separately. When choosing our interpolation points, it is helpful to choose weights which are congruent mod k_E, which is the weight of the Eisenstein series lifting our chosen power of the Hasse invariant as above. Indeed, this enables us to compute U_p in one weight, and twist the obtained matrix to obtain the characteristic series in the other weights. This leads to the algorithm below. The author has implemented a version in `Magma` [BCP97], and we compute some examples below.

Algorithm 2.2. *Take the input (p, N, k_0, m, n), where p is a prime number, N, m and n are positive integers, and $k_0 \in \mathbf{Z}$. Then compute:*

1. ***Interpolation points.*** *Compute $n' = \lceil \frac{m}{v_p(qk_E)} + n \rceil - 1$, and choose n' weights $k_i = k_0 + a_i k_E$, corresponding to $\kappa_i = (1+q)^{a_i k_E} - 1$, where we recall $k_E = (p-1)$ unless $p = 2, 3$, see Section 2.1. Let m_{loss} be the maximal valuation of an n'-fold product of terms $(\kappa_i - \kappa_j)$ for $i \neq j$, and set working precision $m' = m + m_{loss}$.*
2. ***Twisted Atkin operator.*** *Use Algorithm 2.1 above to compute the characteristic series $P(\kappa_i, t)$ up to precision m'. As all the weights are chosen to be congruent mod k_E, this can be done in parallel for all weights, using Frobenius semi-linearity of U_p.*
3. ***Finite differences.*** *Now recursively compute the finite differences $P[\kappa_0, \kappa_1, \ldots, \kappa_i](t)$ for all $i \leq n'$. Use the Newton interpolation 2.3 to obtain a polynomial in the two variables κ, t and discard any terms of degree at least n in κ.*

Theorem 2.5. *Let κ be the parameter on weight space interpolating $(1+q)^{k-k_0} - 1$ for integer weights k. The above algorithm computes the two-variable power series $P(\kappa, t)$ in the ring $\mathbf{Z}_p[\![\kappa, t]\!]/(p^m, \kappa^n)$, with running time polynomial in p, N, m, n, and linear in $\log k_0$.*

Proof. As the coefficients of t in $P(\kappa, t)$ are elements of $\mathbf{Z}_p[\![\kappa]\!]$, we may compute $P(\kappa, t)$ using the finite difference formula. We now determine how many terms of 2.3 we need to compute to obtain the desired result. First off, we note that the j-th term

$$P[\kappa_0, \kappa_1, \ldots, \kappa_j](t) \times \kappa \left(\kappa - (\kappa_1 - \kappa_0)\right) \cdots \left(\kappa - (\kappa_{j-1} - \kappa_0)\right)$$

contributes to the coefficient of κ^d for $d < j$, but this contribution will vanish up to our desired precision if the valuation of every $(j-d)$-fold product of terms $(\kappa_0 - \kappa_i)$ is divisible by p^m. Therefore, by increasing the number of interpolation points to n', we are guaranteed to obtain the correct answer for $P(\kappa, t)$ up to the given precisions.

When computing the finite differences of the characteristic series for U_p we obtain from our extension of Lauder's algorithm 2.1, we lose precision whenever we divide by $(\kappa_i - \kappa_j)$. Therefore the increased working precision m' is sufficient to guarantee that the end result is correct up to p-adic precision m. This proves that our output is correct up to the given precisions. The complexity statements now follow from the analysis by Lauder [Lau11, Section 3.2.2]. □

Remark 2.9. In the above algorithm, we have a choice of interpolation points to make. The complexity of Lauder's algorithm in m is larger than the complexity in the number of weights. This is because as we increase the number of interpolation points, the only extra cost is to compute the characteristic series of an increased number of twist of the matrix of U_p in weight k_0. Therefore it is vastly more efficient to choose the set of interpolation points κ_i such that the loss of precision, or equivalently the sum of the $v_p(\kappa_i - \kappa_j)$ for $i \neq j$, is as small as possible.

2.3.2 Examples

We explicitly compute some examples, and relate our computations to previous work on slopes by Buzzard–Calegari [BC05], Buzzard–Kilford [BK05], Roe [Roe14] and the ongoing work on boundary slopes and the spectral halo by Andreatta–Iovita–Pilloni and Bergdall–Pollack.

Example 2.10. We revisit the celebrated example due to Buzzard–Calegari [BC05] and Buzzard–Kilford [BK05], and set $(p, N) = (2, 1)$. With notation κ as before, we compute that

$$\begin{aligned} P(\kappa, t) \quad = \quad & 1 + 519736167t + 413685912t^2 + 148708352t^3 + 1065353216t^4 \\ & + \kappa\ (36306799t + 374998993t^2 + 380696768t^3 + 281739264t^4) \\ & + \kappa^2(43984100t + 481404364t^2 + 496002384t^3 + 387895296t^4 + 1811939328t^5) \\ & + \kappa^3(874017364t + 890496879t^2 + 487943741t^3 + 4077568t^4 + 964689920t^5) \\ & + \kappa^4(392124398t + 264203079t^2 + 839291211t^3 + 908503936t^4 + 817102848t^5) \\ & + O(\kappa^5, 2^{30}), \end{aligned}$$

We actually computed $P(\kappa, t)$ to precision $O(\kappa^{25}, 2^{70})$, which took about 5 minutes, but the enormous output would make for poor exposition. It was proved in [BC05] by an explicit parametrisation of the relevant region in $X_0(2)$, which has genus 0, that in weight 0 the n-th slope is equal to

$$1 + 2v_2\left(\frac{(3n)!}{n!}\right),$$

with multiplicity 1. We indeed recover the first few terms of this sequence by setting $\kappa = 0$. As for the other extreme, the main result of Buzzard–Kilford [BK05] states that the slopes for $1/8 < |\kappa| < 1$ form an arithmetic progression with n-th term $n v_2(\kappa)$, all with multiplicity 1. Indeed, by substituting $\kappa = 2$ we obtain the slope sequence $0, 1, 2, 3, 4, \ldots$, while for $\kappa = 4$ we recover $0, 2, 4, 6, 8, \ldots$. Our computed power series $P(\kappa, t)$ hence combines the best of both worlds, by describing the spectral curve over the inner regions of $\mathscr{W}$ as well as the outskirts.

We now investigate properties of the coefficients of t in the series $P(\kappa, t)$. This is the subject of recent work of Bergdall–Pollack, which has yet to appear. We let $a_i(\kappa)$ be the coefficient of t^i in $P(\kappa, t)$, and compute its 2-adic Newton polygons. Let $\lambda(i)$ be the number of roots of $a_i(\kappa)$ in the open unit disk. The following table displays the 2-adic valuations of these roots, along with their multiplicities:

Coefficient	Valuations	λ
$a_0(\kappa) = 1$	$\emptyset$	0
$a_1(\kappa)$	$\emptyset$	0
$a_2(\kappa)$	$\mathbf{3}_1$	1
$a_3(\kappa)$	$\mathbf{3}_2, \mathbf{4}_1$	3
$a_4(\kappa)$	$\mathbf{3}_4, \mathbf{4}_1, \mathbf{7}_1$	6
$a_5(\kappa)$	$\mathbf{3}_6, \mathbf{4}_2, \mathbf{5}_1, \mathbf{7}_1$	10
$a_6(\kappa)$	$\mathbf{3}_9, \mathbf{4}_3, \mathbf{5}_2, \mathbf{6}_1$	15
$a_7(\kappa)$	$\mathbf{3}_{12}, \mathbf{4}_5, \mathbf{5}_2, \mathbf{6}_1, \mathbf{8}_1$	21

By inspecting the 2-adic valuations of the coefficients we computed, we see that this output is provably correct and complete. Note that

$$\lambda(i) = \binom{i}{2},$$

which follows from the main result of Buzzard–Kilford [BK05]. It is our understanding that the forthcoming work of Bergdall–Pollack has more precise conjectures about the location of the zeroes of a_i.

Example 2.11. We now turn to $(p, N) = (2, 3)$ and compute $P(\kappa, t)$ up to precision $O(2^{60}, \kappa^{20})$. This computation took about 90 minutes on a standard laptop. Let $a_i(\kappa)$ be the coefficient of t^i in $P(\kappa, t)$ as above, and $\lambda(i)$ the number of roots of $a_i(\kappa)$ in the closed unit disk. Furthermore, set $\mu(i)$ to be the largest power of p that divides $a_i(\kappa)$. The recent work of Bergdall–Pollack uses Koike's trace formula to prove that $\mu(i) = 0$ whenever $N = 1$. However, in our situation μ appears to be larger for several i. Our computations are summarised in the following table:

Coefficient	Valuations	λ	μ
$a_0(\kappa) = 1$	$\emptyset$	0	0
$a_1(\kappa)$	–	–	–
$a_2(\kappa)$	$\emptyset$	0	0
$a_3(\kappa)$	$\emptyset$	0	≤ 1
$a_4(\kappa)$	$\mathbf{4}_1$	1	0
$a_5(\kappa)$	$\mathbf{3}_2$	2	≤ 1
$a_6(\kappa)$	$\mathbf{3}_2, \mathbf{4}_1$	3	0
$a_7(\kappa)$	$\mathbf{3}_2, \mathbf{4}_1, \mathbf{8}_1$	4	≤ 1
$a_8(\kappa)$	$\mathbf{3}_3, \mathbf{4}_1, \mathbf{5}_1, \mathbf{6}_1$	6	0
$a_9(\kappa)$	$\mathbf{3}_4, \mathbf{4}_3, \mathbf{6}_1$	8	≤ 1
$a_{10}(\kappa)$	$\mathbf{3}_5, \mathbf{4}_3, \mathbf{5}_1, \mathbf{8}_1$	10	0
$a_{11}(\kappa)$	$\mathbf{3}_6, \mathbf{4}_4, \mathbf{5}_2$	12	≤ 1
$a_{12}(\kappa)$	$\mathbf{3}_7, \mathbf{4}_5, \mathbf{5}_2, \mathbf{7}_1$	15	0

We remark that the missing entry means that $a_1(\kappa)$ was 0 up to our precision $O(\kappa^{20}, 2^{30})$, and it can indeed be proved using Koike's trace formula that $a_1(\kappa) = 0$. We understand that the above table is consistent with the forthcoming work of Bergdall–Pollack for those cases.

Example 2.12. Let us set $(p, N) = (3, 1)$ and compute $P(\kappa, t)$ up to precision $O(3^{90}, \kappa^{60})$. This computation took about 3 hours. With the same notation as above, we find the following table for the slopes of the zeroes of the coefficients $a_i(\kappa)$:

Coefficient	Valuations	λ
$a_0(\kappa) = 1$	$\emptyset$	0
$a_1(\kappa)$	$\emptyset$	0
$a_2(\kappa)$	$\mathbf{1}_2$	2
$a_3(\kappa)$	$\mathbf{1}_5, \mathbf{3}_1$	6
$a_4(\kappa)$	$\mathbf{1}_9, \mathbf{2}_2, \mathbf{3}_1$	12
$a_5(\kappa)$	$\mathbf{1}_{15}, \mathbf{2}_4, \mathbf{3}_1$	20
$a_6(\kappa)$	$\mathbf{1}_{22}, \mathbf{2}_5, \mathbf{3}_2, \mathbf{4}_1$	30
$a_7(\kappa)$	$\mathbf{1}_{30}, \mathbf{2}_8, \mathbf{3}_2, \mathbf{4}_2$	42
$a_8(\kappa)$	$\mathbf{1}_{40}, \mathbf{2}_{11}, \mathbf{3}_2, \mathbf{4}_3$	56

Again, this output is complete and provably correct. Notice that

$$\lambda(i) = 2\binom{i}{2},$$

which follows from the main result of Roe [Roe14]. Roe tackled this more complicated situation using the same techniques as Buzzard–Kilford [BK05].

Remark 2.13. In connection with the above examples, we mention ongoing work of Andreatta–Iovita–Pilloni on *boundary slopes* of overconvergent modular forms. They embed weight space $\mathscr{W}_N$ into a *wide open ball* as in section 4.2, and extend the definition of the spaces $M_\kappa^\dagger$ to the situation where κ is the apex point of this wide open ball. This yields a space of overconvergent forms over $\mathbf{F}_p((t))$ with a compact operator U_p, which shows that the sequence of slopes of weight κ only depends on the distance of κ to the center of its component of $\mathscr{W}_N$ whenever that distance is sufficiently close to 1. The sequence of *boundary slopes* multiplied by $v_p(\kappa)$ gives us the slope sequence for U_p acting on $M_\kappa^\dagger$, and can be computed efficiently using our algorithms. We give some examples below.

Example 2.14. As above, set $(p,N)=(2,3)$ and compute $P(\kappa,t)$ up to precision $O(2,\kappa^{30})$, which takes about one minute. We compute the degrees of the t-coefficients, which suggest the boundary slope sequence

$$\mathbf{0}_2, \mathbf{1/2}_2, \mathbf{1}_2, \mathbf{3/2}_2, \mathbf{2}_2, \mathbf{5/2}_2, \mathbf{3}_2, \mathbf{7/2}_2, \dots$$

which is indeed in accordance with the Newton polygon of $\lambda+\mu$ computed above, up to the chosen precisions. Notice the similarity with the slope sequence for $(p,N)=(2,1)$.

Example 2.15. As above, set $(p,N)=(11,1)$ and compute $P(\kappa,t)$ up to precision $O(11,\kappa^{60})$, which takes about two minutes. We compute the degrees of the t-coefficients, which suggest the boundary slope sequence:

$$\mathbf{0}_1, \mathbf{1}_1, \mathbf{2}_1, \mathbf{3}_1, \mathbf{4}_2, \mathbf{5}_1, \mathbf{6}_1, \mathbf{7}_1, \mathbf{9}_2, \dots$$

2.3.3 Minimal slopes and Emerton's thesis

We compute the two-variable characteristic series $P(\kappa,t)$ for U_2 in tame level 1, up to precision $(2^{21},\kappa^7)$. This takes 0.370 seconds on a standard laptop.

$$\begin{aligned} P(\kappa,t) \quad \equiv \quad & 1+t(1739623+655215\kappa+2041060\kappa^2+1602132\kappa^3+2054126\kappa^4+779022\kappa^5+1634724\kappa^6) \\ & +t^2(546968+1705937\kappa+1156556\kappa^2+1304431\kappa^3+2059079\kappa^4+1677821\kappa^5+644339\kappa^6) \\ & +t^3(1907712+1112256\kappa+1074512\kappa^2+1404477\kappa^3+430411\kappa^4+51909\kappa^5+1261732\kappa^6) \\ & +t^4(720896\kappa+2019328\kappa^2+1980416\kappa^3+437120\kappa^4+1161264\kappa^5+1648837\kappa^6) \\ & +t^5(1310720\kappa^4+524288\kappa^5+1101824\kappa^6) \\ & +O(2^{21},\kappa^7) \end{aligned}$$

Investigating the coefficients $a_i(\kappa)$ of $P(\kappa,t)$ for small values, we see that their valuation on $\kappa\in\mathbf{Z}_2$ only seems to depend on $\kappa \pmod{2^6}$. This can be made into a rigorous proof of this fact, by using the uniform estimates in Wan [Wan98] for the Newton polygon in t of $P(\kappa,t)$ recalled above. After possibly redoing the computation to a higher precision, to assure that all the slopes are indeed correct, we recover the following theorem, which may be found in Emerton [Eme98, Theorem 1.1].

Theorem 2.6 (Emerton)**.** *The minimal non-zero slope of U_2 on $M_k^\dagger$ in tame level 1, along with its multiplicity, depends only on $k \pmod{16}$. In particular, the minimal U_2 slope with multiplicity is $\mathbf{3}_1$ when $k\equiv 0 \pmod 4$, $\mathbf{4}_1$ when $k\equiv 2 \pmod 8$, $\mathbf{5}_1$ when $k\equiv 6 \pmod{16}$ and $\mathbf{6}_2$ when $k\equiv 14 \pmod{16}$.*

We note that the calculations of Emerton [Eme98] rely crucially on the explicit uniformisations of 2-adic regions on the genus 0 modular curves $X_0(2^n)$ for small values of n, which are hard to come by in higher levels and primes. Our algorithms do not rely on any specifics of the situation $(p, N) = (2, 1)$, and therefore similar arguments work in more general settings.

Chapter 3

Global points on elliptic curves and p-adic iterated integrals

In this chapter we will discuss two recent strategies for constructing global points on elliptic curves defined over number fields, due to Darmon–Rotger [DR14] and Darmon–Lauder–Rotger [DLR14]. Our small contribution is purely computational, and consists of a set of algorithms to explicitly compute these points in practice, extending the work of [Lau14].

Outline. We start with a quick overview of the motives attached to modular forms by Scholl, and the standard conjectures relating special values of motivic L-functions to algebraic cycles. We give a brief summary in 3.2 of the theory of *Chow–Heegner* points [DR14], whose definition is algebraic in flavour and quite well-understood. In 3.3, we discuss the recently introduced *Stark points* [DLR14], which are more analytic in nature and largely conjectural. Both come with a proposed p-adic Gross–Zagier formula relating their formal logarithms to certain p-adic iterated integrals attached to triples of modular forms. Using the algorithms developed in chapter 2, we extend the work of Lauder [Lau14] on the explicit computation of these points to general primes p, removing the restriction $p \geq 5$. We present computed data in sections 3.2.3 and 3.3.2. Our ability to deal with $p = 2, 3$ allows for experimentation with cases where the conductor is divisible by a large power of p. We also present evidence for a natural generalisation of the conjectures in [DR14] to include modular forms of infinite slope. The `Magma` code for our extension of the algorithms developed by Lauder in [Lau14], [DLR14] may be found on the author's webpage.

Remark. As our contributions in this chapter are solely computational, we content ourselves with a very brief overview of the general theory on Chow–Heegner and Stark point constructions. These topics represent at best a small part of a largely conjectural family of point constructions. The reader interested in precise details on the construction of these various types of cycles is referred to [BDP12], [DRS12], [BDP13], [DDLR15], [DR14], [DLR14] and the references contained therein.

3.1 Motives and periods

Our study of Chow–Heegner points and their properties is perhaps best motivated by placing it in the broader abstract framework of motives and special values of their L-functions. In this section, we briefly recall some of the basic definitions concerning motives attached to modular forms, and conjectural links between motivic L-functions and algebraic cycles.

3.1.1 Scholl motives

We start with a brief review of Scholl motives attached to modular forms, following [Sch90]. Let $N \geq 3$ and $Y(N)$ the open modular curve of full level N with compactification $j : Y(N) \hookrightarrow X(N)$. The k-th *Kuga–Sato variety*, for $k \geq 0$, is the canonical desingularisation of the k-fold product of the universal generalised elliptic curve $\pi : \mathscr{E} \to X(N)$ constructed in [Del71a], which we denote by $\mathscr{E}^k$. The level structure and inversion on the fibres of π, together with the natural permutation action of the symmetric group S_k on the factors, yield an action of the wreath product

$$\Gamma_k := (\mathrm{C}_N^2 \rtimes \mathrm{C}_2) \wr \mathrm{S}_k,$$

on the Kuga–Sato variety $\mathscr{E}^k$. The sign characters on the symmetric group S_k and C_2^k give rise to a projection operator e in the group algebra $\mathbf{Q}[\Gamma_k]$. First, Scholl shows that the parabolic l-adic cohomology group attached to the space of cusp forms of weight $k+2$ and level $\Gamma(N)$ is realised by the Chow motive $(\mathscr{E}^k, e, 0)$. That is, we have an isomorphism

$$\mathrm{H}^1_{\text{ét}}\left(X(N)_{\overline{\mathbf{Q}}}, j_* \mathrm{Sym}^k R^1 \pi_* \mathbf{Q}_l\right) \simeq \mathrm{H}^*_{\text{ét}}(\mathscr{E}^k, \mathbf{Q}_l)(e),$$

and a similar isomorphism holds for the Betti realisation of $(\mathscr{E}^k, e, 0)$.

To further decompose this motive, Scholl passes to the category of Grothendieck motives with coefficients in the number field K_f generated by the Fourier coefficients of an eigenform $f \in S_{k+1}(\Gamma(N))$. Recall that this category has the same objects as the category of Chow motives, but the algebraic cycles giving rise to morphisms are tensored with K_f and considered up to homological equivalence. This suffices for our purposes, and we will not concern ourselves with decomposing $(\mathscr{E}^k, e, 0)$ in the category of Chow motives. The construction now proceeds by defining suitable Hecke operators on the Kuga–Sato variety $\mathscr{E}^k$ by pulling back the usual Hecke operators on $Y(N)$ and taking the closure of the resulting graph in $\mathscr{E}^k \times \mathscr{E}^k$. Using these operators, Scholl refines the projection operator e to obtain a motive $\mathfrak{M}_f$ attached to f whose étale realisation is the 2-dimensional Galois representation attached to f in [Del71a]. This motive $\mathfrak{M}_f$ will be called the *Scholl motive* attached to f. To deal with modular forms for general congruence subgroups, one applies an additional projector as in [Sch90, Section 4.2.0].

3.1.2 The Beilinson–Bloch conjecture

We now recall some of the standard conjectures relating special values of L-functions of motives to the theory of algebraic cycles. Our goal is the modest one of predicting the existence of interesting algebraic cycles on the product of three Kuga–Sato varieties attached to appropriate triples of classical modular forms. As a consequence, we merely scratch the surface of this vast web of deep conjectures, and refer the reader to more thorough sources such as [Nek94] and [Jan88].

The far reaching conjectures of Beilinson [Bei85] give precise relations between special values of motivic L-functions and algebraic cycles. A very weak form of one of these conjectures may be viewed as a generalisation of the Birch and Swinnerton-Dyer conjecture.

Conjecture 3.1 (Weak Beilinson–Bloch). *Let $\mathfrak{M} = (X, e, n)$ be a self-dual pure $\mathbf{Q}$-motive of weight -1 with coefficients in K, then*

$$\mathrm{ord}_{s=0} L(\mathfrak{M}, s) = \dim_K e\mathrm{CH}^n(X)_0(\mathbf{Q}).$$

We may occasionally be able to find an analytic continuation and functional equation for the L-function attached to a motive $\mathfrak{M}$ as above. In such cases, it is often possible to show the vanishing of this L-function at the central value due to sign considerations. In such cases, the above conjecture predicts the existence of large supplies of interesting cycles defined over number fields which are of infinite order in the Chow group. As cycles are often highly sought-after, for instance in the form of rational points on varieties, this provides us with valuable motivation to attempt an explicit construction. In fact, the conjectures of Beilinson [Bei85] allow one to obtain precise information about such cycles, by relating the special values of motivic L-functions to Deligne periods. An excellent exposition on these conjectures may be found in [Nek94, Section 2 and Conjecture 6.5]. As precise definitions of such periods are not needed in what follows, we merely remark that the main theorem of [DR14] is concerned with proving a p-adic version of such a formula for the special value of a Rankin triple product p-adic L-function. We briefly recall the work of Darmon–Rotger below, and show how it allows us to construct *"Chow–Heegner"* rational points on elliptic curves.

3.2 Chow–Heegner points

We now review the construction of Gross–Kudla–Schoen cycles on Kuga–Sato varieties, and the relation between their images under the p-adic Abel–Jacobi map to special values of Rankin triple product p-adic L-functions. In 3.2.3, we use the algorithms in chapter 2 to explicitly compute these special values and construct the corresponding Chow–Heegner points on elliptic curves in particular examples. We also carry out these computations for modular forms of *infinite slope*, where the work of Darmon–Rotger [DR14] does not apply, and present evidence for a more general p-adic Gross–Zagier formula in 3.2.4.

The reader is referred to Darmon–Rotger [DR14] for more details on the proof of the p-adic Gross–Zagier formula. A precise description of the algorithms we use can be found in Lauder

[Lau14]. An extension to general primes p can be found in [Von15], along with some of the examples we included below.

3.2.1 Gross–Kudla–Schoen cycles

We now construct certain cycles on the product of three Kuga–Sato varieties, following [DR14]. They are attached to suitable triples of modular forms, and to motivate their existence, we investigate triple products of Scholl motives as discussed above. Consider the motive

$$\mathfrak{M}_{f,g,h} := \mathfrak{M}_f \otimes \mathfrak{M}_g \otimes \mathfrak{M}_h \otimes \mathbf{Z}(c), \qquad \text{with } c := \frac{k+l+m-2}{2},$$

where (f,g,h) is a triple of modular forms of weights at least 2 whose sum is even, and $\mathbf{Z}(n)$ is the pure Tate motive of weight $-2n$ given by $(\mathbf{P}^1_{\mathbf{Q}}, e, 0)$, where e is the projector that annihilates $\mathrm{H}^0(\mathbf{P}^1_{\mathbf{Q}})$ and acts trivially on $\mathrm{H}^2(\mathbf{P}^1_{\mathbf{Q}})$ for any Weil cohomology theory H^i. The motive $\mathfrak{M}_{f,g,h}$ is pure of weight -1.

Now pick a triple of modular forms (f,g,h) which is *balanced* in the sense that any one weight is strictly smaller than the sum of the other two, and such that the product of nebentypes $\chi_f \cdot \chi_g \cdot \chi_h = \mathbf{1}$. These conditions assure that the motive $\mathfrak{M}_{f,g,h}$ is self-dual. The work of Piatetski-Shapiro and Rallis [PSR87] and Prasad [Pra90] shows that under mild hypotheses on the triple (f,g,h), the L-function $L(\mathfrak{M}_{f,g,h},s)$ has meromorphic continuation to the complex plane, and satisfies a functional equation relating its values at s and $k+l+m-2-s$. Prasad [Pra90] makes the sign of the functional equation explicit, and shows that in our case the order of vanishing of $L(\mathfrak{M}_{f,g,h},s)$ at $s=c$ is odd. As this motive arises in the middle cohomology of the product $\mathcal{V} := \mathcal{E}^{k-2} \times \mathcal{E}^{l-2} \times \mathcal{E}^{m-2}$, the weak Beilinson–Bloch conjecture recalled above predicts the existence of a large supply of cycles in the Chow group $\mathrm{CH}^c(\mathcal{V})_0(\mathbf{Q})$, which is the group of null-homologous cycles of codimension c.

So how do we construct such cycles? The approach of Darmon–Rotger [DR14] is to investigate diagonal cycles, arising from embeddings of Kuga–Sato varieties into $\mathcal{V}$. The *Gross–Kudla–Schoen* cycle

$$\Delta_{k,l,m} \in \mathrm{CH}^c(\mathcal{V})_0$$

is constructed in [DR14, Section 3.1], essentially just by modifying the natural embedding of $\mathcal{E}^{c-2}$ into $\mathcal{V}$. There is an action of the Hecke algebra, as recalled in 3.1.1, and we define $\Delta_{f,g,h}$ as the (f,g,h) isotypic component of $\Delta_{k,l,m}$ under this action. These cycles are referred to as Gross–Kudla–Schoen cycles. The standard conjectures on motives predict a precise relationship between numerical invariants attached to these cycles and special values of L-functions. In [DR14, Section 5], such a relation is proved between the images of Gross–Kudla–Schoen cycles under the p-adic Abel–Jacobi map recalled below, and the special value of a p-adic L-function attached to (f,g,h). It takes the form of a formula, reminiscent of the Gross–Zagier formula in the setting of Heegner points.

3.2.2 A p-adic Gross–Zagier formula

Let (f,g,h) be a triple of modular eigenforms as above, which are henceforth assumed to be p-ordinary. Pick a prime $p \nmid N$, and recall that we have a p-adic Abel–Jacobi map

$$\mathrm{AJ}_p : \mathrm{CH}^c(\mathscr{V})_0 \to \mathrm{Fil}^c\, \mathrm{H}^{2c-1}_{dR}(\mathscr{V})^\vee,$$

as defined in [Bes00], [Nek00]. Hence from the cycle $\Delta_{f,g,h}$ we obtain a map

$$\mathrm{AJ}_{p,\Delta_{f,g,h}} : \mathrm{Fil}^c\, \mathrm{H}^{2c-1}_{dR}(\mathscr{V}) \to \overline{\mathbf{Q}}_p.$$

To the triple (f,g,h) we may also attach a differential form $\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h$ in the space $\mathrm{Fil}^c\, \mathrm{H}^{2c-1}_{dR}(\mathscr{V})$ as described in [DR14, Section 3.4], which we can then use to evaluate $\mathrm{AJ}_{\Delta,p}$.

We can also attach a Garrett–Rankin triple product p-adic L-function $\mathscr{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})$ to the Hida families $\mathbf{f},\mathbf{g},\mathbf{h}$ passing to our chosen triple of modular forms. For a precise definition, see [DR14, Definition 4.4]. We will be interested in the special value

$$\mathscr{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})(k,l,m),$$

which is outside of the classical interpolation range. The work of Darmon–Rotger identifies this value as an explicit factor times a period which may be computed as follows. If we let $e_{ord} = \lim_n U_p^{n!}$ be Hida's ordinary projector, then it is the coefficient of the ordinary p-stabilisation $f^{(p)}$ of f when we write $e_{ord}(d^{-1}g^{[p]} \times h)$ in the Hecke eigenbasis of the ordinary subspace of the overconvergent forms of weight k. Here, $d^{-1} = \lim_n d^{p^n-1}$ is a p-adic limit of powers of Serre's differential operator d which acts on q-expansions as $\frac{d}{dq}$, and $g^{[p]}$ denotes the p-depletion $(1-V_pU_p)g$ of g.

The main theorem of [DR14] can be seen as a p-adic analogue of the Gross–Zagier formula for diagonal cycles. It states that

$$\mathscr{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})(k,l,m) = C_{f,g,h} \times \mathrm{AJ}_{p,\Delta}(\eta_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h),$$

where C is an elementarily computable quadratic number depending on (f,g,h). When $k=l=m=2$, and f is defined over $\mathbf{Q}$ and corresponds to an elliptic curve $E/\mathbf{Q}$, it follows from [DRS12, Lemma 2.4] that $\mathrm{AJ}_{\Delta,p}(\omega_f^{\text{u-r}} \otimes \omega_g \otimes \omega_h)$ equals $\log_p(P)$ for some explicit $P \in E(\mathbf{Q})$. We usually refer to points obtained in this way as *Chow–Heegner points*.

3.2.3 Computing Chow–Heegner points

The above description of $\mathscr{L}_p(\mathbf{f},\mathbf{g},\mathbf{h})(k,l,m)$ shows that our algorithms on overconvergent forms from chapter 2 allow us to explicitly compute these special values of Rankin triple product p-adic L-functions. By the p-adic Gross–Zagier formula described above, appropriate choices of modular forms allow us to obtain the formal logarithm of global points on elliptic curves, up to some small algebraic factor.

An algorithm to compute this special value was described by Lauder [Lau14], and extended by us to general primes p in [Von15]. The crux in computing the special value of the Rankin triple product p-adic L-function is the efficient computation of the U_p-operator on the space $M_k^{\dagger,r}$. The previous chapter removed the restriction $p \geq 5$ from the algorithm in [Lau11] to compute this action, and it is now straightforward to compute the desired special value of the Rankin triple product p-adic L-function, as described in detail in [Lau14]. We have implemented a version in `Magma` that works for all p, which is available on the author's webpage. Let us turn to some numerical examples.

Example 3.1. Let $E_{\mathbf{Q}} : y^2 + xy = x^3 - x^2 - x + 1$ be the unique rank 1 elliptic curve of conductor 58, with associated newform f, and let g be the unique newform on $\Gamma_0(58)$ different from f. Both f and g are 2-ordinary. Letting $P = (0, 1)$ be a generator for $E(\mathbf{Q})$, we compute that

$$\mathscr{L}_2(\mathbf{g}, \mathbf{f}, \mathbf{g})(2, 2, 2) \equiv 3 \log_E(P) \pmod{2^{200}},$$

as predicted by the theory in [DR14].

Example 3.2. Let $E_{\mathbf{Q}} : y^2 + y = x^3 - x^2 - 2x + 2$ be the unique rank 1 elliptic curve of conductor 57, and let f be its associated newform. Let g be the newform of level 57 with q-expansion starting with

$$g = q + q^2 + q^3 - q^4 - 2q^5 + q^6 + \ldots,$$

where we point out that f and g are both 3-ordinary forms. Now pick the generator $P = (2, -2)$ for $E(\mathbf{Q})$, then we find experimentally that

$$\mathscr{L}_3(\mathbf{g}, \mathbf{f}, \mathbf{g})(2, 2, 2) \equiv \frac{32}{9} \log_E(P) \pmod{3^{200}}.$$

Remark 3.3. The above examples and [Lau14], [DLR14] only consider the setting where p divides the conductor of E exactly once. The algorithms we implemented do not depend on the existence of Hida families, so we might wonder what happens when larger powers of p divide the conductor. The next subsection addresses this question.

3.2.4 Extending the conjectures

When the Tate module of $E_{\mathbf{Q}}$ is wildly ramified at 2 or 3, we might wonder whether the Chow–Heegner point construction just described continues to work. The associated newform f will be of infinite slope, so we lack a notion of Hida or Coleman family passing through f. It is therefore not obvious whether the theoretical framework of [DR14] will generalise to such a setting. Nonetheless, we are often able to run our extension of Lauder's algorithms [Lau11] and [Lau14], and recover a rational point on E, as the following examples show.

Example 3.4. Let $E : y^2 + y = x^3 + 9x - 10$, which is an elliptic curve over $\mathbf{Q}$ of conductor $4617 = 3^5 \cdot 19$ and rank 1. Let f be the associated newform. Let $g = q - 2q^3 - 2q^4 + 3q^5 - q^7 + \ldots$

be the unique cuspidal newform of weight 2 on $\Gamma_0(19)$. Despite f being of infinite 3-adic slope, we can run the computation and find a numerical value for $\mathscr{L}_2(\mathbf{g}, \text{“}f\text{”}, \mathbf{g})(2,2,2)$. We find that

$$\mathscr{L}_3(\mathbf{g}, \text{“}f\text{”}, \mathbf{g})(2,2,2) \equiv t \cdot \log_E(P) \pmod{3^{200}} \quad \text{where} \quad 2t^2 + 48t + 729 = 0,$$

where $P = (4,9)$ is a generator of $E(\mathbf{Q})$. The fact that both quantities are related by a quadratic number t of small height suggests that a more general analogue of the theory for ordinary forms in [DR14] might exist.

Example 3.5. Let $E : y^2 = x^3 + x^2 - 62893x - 6091893$, which is an elliptic curve over $\mathbf{Q}$ of rank 1 and conductor $15104 = 2^8 \cdot 59$. Let f be its associated newform, and let $g = q - q^2 - q^3 + q^4 - 3q^5 + \ldots$ be the newform of level 118 associated to the elliptic curve with Cremona label `118.a1`. Note that g is 2-ordinary. We compute that

$$\mathscr{L}_2(\mathbf{g}, \text{“}f\text{”}, \mathbf{g})(2,2,2) \equiv 6 \log_E(P) \pmod{2^{100}},$$

where $P = (20821, 3004216)$ is a generator of $E(\mathbf{Q})$. As in the previous example, this suggests that an analogue of the theory in [DR14] holds for f of infinite slope. Note that this would work the other way: once we compute the value of $\mathscr{L}_2(\mathbf{g}, \text{“}f\text{”}, \mathbf{g})(2,2,2)$, we can use a formal exponentiation routine as in [Lau14] to recover a point of infinite order, which is of considerable height in this example.

3.3 Stark points

Chow–Heegner points have a well-understood geometric origin and can also be constructed by complex analytic methods, see [DRS12] and [DDLR15]. In this section we turn to applications of triple product p-adic L-functions to the p-adic construction of points in more mysterious settings. To suitable triples of modular forms (f, g, h) of weights $(1, 2, 1)$, we attach a conjectural collection of *Stark points*. These points were introduced in [DLR14], and are certain distinguished points on the modular abelian variety attached to f, defined over the number field cut out by the Artin representations attached to g and h.

We refer the reader to [DLR14] for precise definitions and conjectures, as well as a proof of certain special cases of the conjecture stated below. We use the algorithms developed in chapter 2 and [Von15] to compute some examples in 3.3.2.

3.3.1 The main conjecture

We now discuss the conjecture stated in [DLR14, Conjecture ES], making several simplifications for the purpose of clarity. Let $E/\mathbf{Q}$ be an elliptic curve, and let $\varrho_\flat, \varrho_\sharp$ be 2-dimensional odd Artin representations with coefficients in a number field L, both cutting out the same number field H. The modularity results obtained in the previous two decades allow us to attach modular forms f, g and h of weights 2, 1 and 1 to E, $\varrho_\flat$ and $\varrho_\sharp$. Pick a prime p that does not divide any of the levels of these forms, an embedding $H \hookrightarrow \overline{\mathbf{Q}}_p$, and pick a p-stabilisation g_α of g. Under a number of technical assumptions, Darmon–Lauder–Rotger attach to g_α two objects. The first [DLR14, Section 1.2] is a unit $u \in \mathcal{O}_H[1/p]^\times$, the second [DLR14, Page 6] is a 2×2 matrix $R_{g_\alpha}(E, \varrho_\flat \otimes \varrho_\sharp)$ with entries in $H \otimes L$ considered as p-adic numbers via a chosen embedding of H. We make a number of further technical assumptions, most importantly that $\mathrm{ord}_{s=1} L(E, \varrho_\flat \otimes \varrho_\sharp, s) = 2$, where $L(E, \rho, s)$ is the L-series attached to E, twisted by an Artin representation ρ. Then the main conjecture of [DLR14] reads that

$$\int_{\breve{g}_\alpha} \breve{f} \cdot \breve{h} = \frac{\det\left(R_{g_\alpha}(E, \varrho_\flat \otimes \varrho_\sharp)\right)}{\log_p(u_{g_\alpha})},$$

where the integral symbol denotes a certain p-adic iterated integral defined in [DLR14, Page 5]. The interest of this formula lies in the quantity $\det\left(R_{g_\alpha}(E, \varrho_\flat \otimes \varrho_\sharp)\right)$, which we did not define here. We simply note that in nice situations, where this regulator is called *factorisable* [DLR14, Top of page 8], it is the product of the formal p-adic elliptic logarithms of two global points on E, defined over a suitable number fields. These are the points that we refer to as *Stark points*.

Remark 3.6. The p-adic iterated integral is by definition essentially the same as the output of the projection algorithms used for the calculation of Chow–Heegner points above. It equals the special value of a suitably defined Garrett–Hida p-adic L-function, see [DLR14, Proposition 2.5]. The reason we opt for the integral notation as opposed to the notation $\mathscr{L}_p(\mathbf{g}_\alpha, \mathbf{f}, \mathbf{h})(1, 2, 1)$ is to emphasise the different nature of the obtained points. The algorithms used in both cases are very similar.

3.3.2 Computing Stark points

We now turn to the numerical calculation of Stark points, as considered in [DLR14]. The conjectural formula presented above is proved in certain cases, but remains open in most. It is therefore important to be able to experiment with the setting where the theory is not understood. These computations are very difficult, even given the algorithms for overconvergent modular forms that we used for the construction of Chow–Heegner points. The main issues arise from the need to perform computations in large extensions of $\mathbf{Q}_p$. A second difficulty is that in some cases considered, the Stark points we are trying to construct lie at present beyond the capabilities of known algorithms for elliptic curves.

Example 3.7. Let $E/\mathbf{Q}$ be the unique elliptic curve over $\mathbf{Q}$ of conductor 83, and let f be its associated newform. Set g to be the Eisenstein series in $M_1(\Gamma_0(83), \chi)$, where χ is the quadratic Dirichlet character mod 83. Now pick $p = 3$, and notice that both of these forms are 3-ordinary. We compute

$$\int_{\breve{E}_{1,\chi}} \breve{f} \cdot \breve{E}_{1,\chi} = \frac{25}{18} \cdot \frac{\log_3(P)^2}{\log_3(u)} \pmod{3^{100}},$$

where $P = (1, -3)$ is the Heegner point attached to $\mathbf{Q}(\sqrt{-83})$ and $u \in \mathbf{Z}[1/3]^\times$ is the Stark unit, which is a root of $x^2 - 5x + 27$.

Example 3.8. Let E be the form with Cremona label `849.a1`, which has conductor $849 = 3 \cdot 283$

and rank 1 over $\mathbf{Q}$ with generator $P = (2, 3)$. Let f be its associated newform. The space $S_1(\Gamma_1(283), 1)$ is 3-dimensional with one dihedral newform, and a pair of conjugate S_4 newforms. Let

$$g = q - \alpha q^2 + \alpha q^3 - q^4 + \alpha q^5 + 2q^6 - q^7 - q^9 + 2q^{10} + \dots,$$

where $\alpha = \sqrt{-2}$. This is one of the newforms with projective Galois image S_4. Now pick $p = 3$, and set g_α, g_β to be the 3-stabilisations of g. We computed

$$\int_{\breve{g}_\alpha} \breve{f} \cdot \breve{g}$$

up to precision 3^{40}. The point construction is work in progress. This should allow us to find a point in the octic extension arising as the compositum of the number fields defined by $x^4 - x - 1$ and $x^2 + 283$. We note that current algorithms to find points in this extension fail to find generators for this Mordell-Weil group.

Chapter 4

Stable models of correspondences

In this chapter, we investigate the geometry of *correspondences* between curves, and prove that correspondences over a non-Archimedean valued field have potential semi-stable reduction. Such models allow for an explicit description of the action of the correspondence on the cohomology of the generic fibre.

Motivation. This work is motivated by the desire to study the Atkin operator *as a geometric object*. By interpreting U_p as a correspondence between algebraic curves, we now turn to the question of whether there is a sensible concept of integral models and reduction of correspondences. The celebrated conjecture of Serre [Ser87] on level lowering of modular Galois representations was proved by Ribet [Rib90]. The main geometric ingredient in his proof is the analysis of the action of Hecke operators on various parts of the Jacobians of modular curves via Picard-Lefschetz theory. This relies crucially on the availability of satisfactory answers to the following two questions:

1. Can we find semi-stable models for these curves at places of bad reduction?
2. Can we describe how Hecke operators interact with these models?

Ribet worked in the setting where the prime of bad reduction p divides the level at most once, in which case answers to both questions were implicit in the work of Deligne-Rapoport [DR73] and Katz-Mazur [KM85]. Much work has gone into answering question 1 in situations where higher powers of the prime p divide the level, culminating in the recent work of Weinstein [Wei12].

We now investigate the geometry of correspondences, and formalise the precise notion needed to answer both questions by introducing the concept of *skeletal semi-stable models* for correspondences. We prove a potential semi-stable reduction theorem for correspondences between curves, generalising the work of Coleman [Col03] and Liu [Liu06]. This suggests a systematic geometric study of spectral properties of Hecke operators, avoiding the use of moduli interpretations which are absent in more general settings of automorphic forms not coming from PEL Shimura varieties. Instead, we propose to focus on the underlying *geometry* of the correspondences. Let l be a prime different from the residue characteristic of K, and C a correspondence between smooth, proper, geometrically connected curves over K as above. We obtain the linear map

$$C^* : \mathrm{H}^{\mathrm{i}}_{\text{ét}}(Y_{1,\overline{K}}, \mathbf{Q}_l) \to \mathrm{H}^{\mathrm{i}}_{\text{ét}}(Y_{2,\overline{K}}, \mathbf{Q}_l),$$

defined by $\pi_{2,*} \circ \pi_1^*$. Now suppose we have a skeletal semi-stable model $\mathfrak{C}$ of C. This makes the computation of parts of the action of C on $\mathrm{H}^1_{\text{ét}}$ combinatorial, as C^* is *strict* in the sense that it respects the weight-monodromy filtrations. We can therefore compute information about the map on cohomology induced by C, by restricting to the graded pieces with respect to this filtration. We may equally describe this as the étale realisation of the induced map

$$C^* : \mathscr{J}_s^0(Y_2) \to \mathscr{J}_s^0(Y_1)$$

between the connected components of the Néron models of the Jacobian. The Néron component group Φ can be characterised in terms of the monodromy pairing on the toric parts, and we may consequently compute the induced map $C^* : \Phi_1 \to \Phi_2$. All of this is discussed in 4.3.1.

Guide to the literature. Simultaneous stable reduction for finite maps is investigated in [LL99]. Existence of simultaneous semi-stable models for finite maps was obtained over $\mathbf{C}_p$ by Coleman [Col03], and for curves over function fields of Dedekind schemes by Liu [Liu06]. Stronger *skeletal* versions were proved by Cornelissen-Kato-Kool [CKK15], and using techniques of Berkovich geometry in [ABBR15]. There has been recent interest in the geometry of correspondences motivated by a study of their dynamical systems, see [DS06]. A theory of canonical heights for correspondences can be found in [Ing14].

Semi-stable models of curves were first used by Deligne and Mumford [DM69] to prove the irreducibility of the moduli space $\mathscr{M}_g$ of curves of genus g. The work of Raynaud [Ray70] and Grothendieck [Gro72] establishes a strong connection between semi-stable models of curves and Galois representations arising from their cohomology, which put the notion of semi-stability at the centre of arithmetic geometry. We require techniques from p-adic geometry that allow us to keep track of different semi-stable models, using the combinatorial notion of *semi-stable vertex sets* recalled below. These techniques have their origins in the work of Bosch and Lütkebohmert [BL85] and Berkovich [Ber90]. We will make extensive use of the powerful theorems in [BPR11], [BR13], and especially the recent work of Amini-Baker-Brugallé-Rabinoff [ABBR15].

When constructing the stable models of Hecke operators, we make essential use of the literature on semi-stable models of quaternionic Shimura curves. A semi-stable model for $X_0(Np)$ was constructed in [DR73]. The seminal work of Katz-Mazur [KM85] constructs regular models for modular curves with a clear moduli interpretation, which are the starting point of the construction of a semi-stable model for $X_0(Np^2)$ by Edixhoven [Edi90]. The semi-stable models for quaternionic Shimura curves we consider in 5.1.1 were constructed by Buzzard [Buz03], Carayol [Car86] and Jarvis [Jar99]. Recently, Weinstein [Wei12] has constructed a semi-stable covering for the tower of modular curves, which yields semi-stable models for modular curves in great generality.

Outline. We recall results on the specialisation map of adic spaces and functoriality of semi-stable models of curves in section 4.2, and define the important notion of the skeleton attached to a semi-stable model of a punctured curve. In section 4.3, we use these techniques to prove that correspondences of curves over non-Archimedean valued fields are potentially semi-stable, which aids us in computing the action of correspondences on étale cohomology and Néron component groups.

4.1 Specialisation maps and formal fibres

We start by recalling the notion of semi-stable vertex sets of smooth quasi-projective curves. They provide us with a combinatorial tool to understand semi-stable models and finite maps between them, via the specialisation map. Most of the material in this section is explained in greater detail in Baker–Payne–Rabinoff [BPR11] and Amini–Baker–Brugallé–Rabinoff [ABBR15], but for future applications we instead adopt the language of adic spaces as developed by Huber [Hub93, Hub94].

Definition 4.1. Let K be an algebraically closed, complete, non-Archimedean field with topology induced by a non-trivial valuation $|\cdot|$ of rank 1. Let R be its valuation ring with maximal ideal $\mathfrak{m}$ and residue field k. For a curve $\mathscr{X}$ over R, we write $\mathscr{X}_s$ for its special fibre, and X for its generic fibre. We let $f : X \to Y$ be a finite morphism between smooth proper connected curves over K, and $X^{\mathrm{ad}}, Y^{\mathrm{ad}}$ the adic spaces associated to X, Y. We will abuse notation and simply write f for the induced map f^{ad} between these adic spaces. Given a point $x \in X^{\mathrm{ad}}(K)$, we write $k(x)$ for the residue field of $\mathscr{O}_{X^{\mathrm{ad}},x}$ and $k(x)^+$ for the image of $\mathscr{O}^+_{X^{\mathrm{ad}},x}$.

4.1.1 Hyperbolic curves and stable reduction.

To add flexibility to our treatment of stable reduction, so as to include curves of small genus into our discussion, we will allow *punctures* $D_X \subset X(K)$ and $D_Y \subset Y(K)$, which are finite sets of Type-I points such that $f^{-1}(D_Y) = D_X$. Recall that a smooth, proper, punctured curve (X, D_X) is said to be *hyperbolic* if $\chi(X, D_X) < 0$, where $\chi(X, D_X) = 2 - 2g(X) - |D_X|$ is the Euler characteristic. A *semi-stable formal model* of the punctured curve (X, D_X) is an integral proper admissible formal R-scheme $\mathfrak{X}$ such that its generic fibre as an adic space is isomorphic to X^{ad}, and moreover

- its special fibre $\mathfrak{X}_s$ is a reduced connected algebraic curve over k with at most ordinary double points for singularities,
- all points in D_X reduce to distinct smooth points on $\mathfrak{X}_s$.

When (X, D_X) is hyperbolic, there exists a unique minimal such model, which we call the *stable formal model*. It is characterised by the properties of semi-stability, together with the additional constraint that the Euler characteristic of every rational component of the special fibre $\mathfrak{X}_s$, minus its singular locus and the set of punctures, is negative. The category $\mathbf{FMod}^{\mathrm{ss}}_{\mathrm{X}}$ consists of semi-stable formal models of (X, D_X), together with an isomorphism between the adic generic fibre and X^{ad}. A morphism between two such models is a morphism of formal R-schemes that induces the identity on X^{ad}.

Coleman's wide open spaces. We recall the notion of wide open balls and annuli due to Coleman [Col89], in the language of adic spaces. A wide open ball is an adic space which is isomorphic to the complement of the open set $|t|=1$ in $\mathrm{Spa}(K\langle t\rangle, R\langle t\rangle)$. A wide open annulus is an adic space isomorphic to the complement in a wide open ball of the open set $|t| \leq p^w$ for some $w \in \mathbf{R}_{>0}$ which we call the *width* of the annulus. We note that a wide open ball possesses exactly one Type-V point which is not the specialisation of any Type-II point. This Type-V point is called the *apex point* of the wide open ball. Similarly, wide open annuli have exactly 2 apex points.

4.1.2 The specialisation map.

Let $\mathfrak{X}$ be an admissible formal R-scheme with generic fibre X^{ad}. As in [Sch12, Theorem 2.22] we can define a specialisation map, which is a morphism of locally ringed topological spaces

$$\mathrm{sp}_{\mathfrak{X}} : \left(X^{\mathrm{ad}}, \mathcal{O}^{+}_{X^{\mathrm{ad}}}\right) \to (\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}),$$

whose fibres are called *formal fibres*. When $\mathfrak{X}$ is semi-stable, it is possible to determine the nature of the formal fibres by combining [BL85, Propositions 2.2 and 2.3] and [Ber90, Proposition 2.4.4]. We obtain the following theorem, see also [BPR11, Theorem 4.6].

Theorem 4.1 (Bosch–Lütkebohmert, Berkovich)**.** *Let ξ be a point of $\mathfrak{X}_s$. Then*

- *ξ is a generic point if and only if $\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi)$ consists of a single Type-II point of X^{ad},*
- *ξ is a smooth closed point if and only if $\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi)$ is a wide open ball,*
- *ξ is an ordinary double point if and only if $\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi)$ is a wide open annulus.*

Semi-stable vertex sets. A *semi-stable vertex set* of (X, D_X) is a finite set V of Type-II points of X^{ad} such that

- the space $X^{\mathrm{ad}}\backslash V$ is a disjoint union of wide open balls and finitely many wide open annuli,
- the points in D_X belong to distinct wide open balls in $X^{\mathrm{ad}}\backslash V$.

For a Type-II point x in a semi-stable vertex set V, call its *valency* the number of apex points of wide open annuli in $X^{\mathrm{ad}}\backslash V$ in the topological closure of x. The category $\mathbf{Vert}^{\mathrm{ss}}_{\mathrm{X}}$ consists of semi-stable vertex sets of (X, D_X), where a morphism between two such sets is given by inclusion. Theorem 4.1 allows us to attach to a semi-stable formal model $\mathfrak{X}$ the finite set $V_{\mathfrak{X}} := \{\mathrm{sp}_{\mathfrak{X}}^{-1}(\xi)\}_{\xi}$, where ξ ranges over the generic points of the irreducible components of $\mathfrak{X}_s$. It follows immediately from Theorem 4.1 that $V_{\mathfrak{X}}$ is a semi-stable vertex set for (X, D_X). This defines a functor between $\mathbf{FMod}^{\mathrm{ss}}_{\mathrm{X}}$ and $\mathbf{Vert}^{\mathrm{ss}}_{\mathrm{X}}$. It turns out that it is in fact an anti-equivalence of categories, which is [BPR11, Theorem 4.11].

Theorem 4.2. *The functor $\mathbf{FMod}^{ss}_{\mathrm{X}} \to \mathbf{Vert}^{ss}_{\mathrm{X}} \ : \ \mathfrak{X} \mapsto V_{\mathfrak{X}}$ induces an anti-equivalence of categories.*

Choose semi-stable formal models $\mathfrak{X},\mathfrak{Y}$ of (X,D_X) and (Y,D_Y) respectively. This yields a rational map $\mathfrak{X} \dashrightarrow \mathfrak{Y}$ induced by f. We will investigate when this rational map extends to a morphism $\mathfrak{X} \to \mathfrak{Y}$, and whether we can arrange for such an extension to be a finite map. The equivalence we just established gives us a direct way to investigate this problem, as semi-stable vertex sets naturally live in X^{ad}, where we understand the behaviour of f. The following theorem is [ABBR15, Theorem 5.13].

Theorem 4.3. *Let $\mathfrak{X},\mathfrak{Y}$ be semi-stable formal models of (X,D_X) and (Y,D_Y), then f extends to a morphism $\mathfrak{X} \to \mathfrak{Y}$ if and only if $f^{-1}(V_{\mathfrak{Y}}) \subseteq V_{\mathfrak{X}}$. This extension is finite if and only if $f^{-1}(V_{\mathfrak{Y}}) = V_{\mathfrak{X}}$.*

4.1.3 Faithfully flat descent.

We will be interested in base fields K_0 that are not necessarily algebraically closed or complete. We now recall how to descend some results to base fields K_0 equipped with a non-trivial non-Archimedean valuation v of rank 1, with valuation ring R. A nice exposition of the theory can be found in [ABBR15, Section 5], to which we refer the reader for more details.

Let (X,D) be a smooth, projective, geometrically connected curve over a field K_0 with a non-trivial non-Archimedean valuation of rank 1 and valuation ring R_0. Let $K = \widehat{\overline{K}_0}$ be the completion of an algebraic closure of K_0, which is itself algebraically closed, and let R be the valuation ring of K. The tools outlined above can be used to analyse semi-stable models of (X_K,D_K), after which we pass back to (X,D) by the theory of faithfully flat descent. More precisely, any semi-stable model of X_K is isomorphic to $\mathscr{X}_R$, where $\mathscr{X}$ is a semi-stable model of X_{K_1} over some finite separable extension K_1 of K_0. Moreover, the following Lemma is proved in [ABBR15, Lemma 5.5].

Lemma 4.1. *Let $f : X \to Y$ be a finite morphism between smooth, projective, geometrically connected curves over K_0, with semi-stable models $\mathscr{X}$ and $\mathscr{Y}$ respectively. Suppose that f extends to a finite morphism $\mathscr{X}_R \to \mathscr{Y}_R$, then f also extends uniquely to a finite morphism $\mathscr{X} \to \mathscr{Y}$ defined over R_0.*

Proof. The rational map $f : \mathscr{X} \dashrightarrow \mathscr{Y}$ extends to a morphism if and only if the projection $\pi : \Gamma_f \to \mathscr{X}$ of its graph Γ_f in $\mathscr{X} \times_{R_0} \mathscr{Y}$ is an isomorphism. The isomorphism $\pi_R : \Gamma_{f_R} \simeq \Gamma_f \times \mathrm{Spec}(R) \to \mathscr{X}_R$ descends to an isomorphism $\pi : \Gamma_f \to \mathrm{Spec}(R_0)$, and hence f extends to a morphism $\mathscr{X} \to \mathscr{Y}$. This extension is necessarily finite, as it becomes finite after faithfully flat base change. □

4.2 Stable models of correspondences

In this section, we prove an analogue for correspondences of the stable reduction theorem of Deligne–Mumford [DM69]. This generalises the stable reduction theorems for finite maps proved by Coleman [Col03] and Liu [Liu06]. The proof is via analytic geometry, using the powerful techniques outlined in the previous section.

Main definitions. Let K be a field equipped with a non-trivial non-Archimedean valuation $|\cdot|$ of rank 1, whose valuation ring R has maximal ideal $\mathfrak{m}$ and residue field k. A *punctured* *correspondence* is a diagram

$$\mathbf{C}: \quad (Y_1, D_1) \xleftarrow{\pi_1} (X, D_X) \xrightarrow{\pi_2} (Y_2, D_2)$$

where

- X, Y_1, Y_2 are smooth, projective, geometrically connected K-curves,
- π_1, π_2 are finite K-morphisms,
- D_X, D_1, D_2 are finite sets of punctures with $\pi_1^{-1}(D_1) = D_X = \pi_2^{-1}(D_2)$.

A punctured correspondence **C** is said to be *hyperbolic* if its objects are hyperbolic punctured curves in the sense of section 4.1.1. A *semi-stable R-model* of a punctured correspondence **C** with objects as above is a diagram

$$\mathfrak{C}: \quad \mathscr{Y}_1 \longleftarrow \mathscr{X} \longrightarrow \mathscr{Y}_2$$

where

- $\mathscr{X}, \mathscr{Y}_1$ and $\mathscr{Y}_2$ are integral flat proper semi-stable R-curves, together with isomorphisms $\mathscr{X}_K \simeq X$ as well as $\mathscr{Y}_{1,K} \simeq Y_1$ and $\mathscr{Y}_{2,K} \simeq Y_2$, so that their formal completions along the special fibre are semi-stable formal R-models for $(X, D_X), (Y_1, D_1)$ and (Y_2, D_2) in the sense of 4.1.1,
- the morphisms are **finite**, and restrict to π_1, π_2 on the generic fibres, via the given isomorphisms.

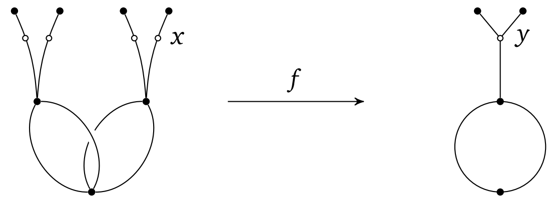
We say that a semi-stable R-model $\mathfrak{C}_1$ *dominates* another semi-stable R-model $\mathfrak{C}_2$, if the objects of $\mathfrak{C}_1$ dominate the objects of $\mathfrak{C}_2$ pairwise. A semi-stable R-model is called *stable* if it is minimal with respect to the relation of domination. The stable model of a hyperbolic correspondence is unique up to isomorphism if it exists, but in general the objects of the stable model are **not** the stable models of its objects.

4.2.1 Stable models of Galois morphisms.

Before coming to a proof of potential stable reduction of correspondences, which is the content of Theorem A, we prove a lemma about Galois maps $f : X \to Y$. The general case will be reduced to this one in the proof of Theorem A. The weaker statement, where we do not insist that the extension of f should be finite, was proved by Liu-Lorenzini [LL99, Proposition 4.4]. The argument for $K = \mathbf{C}_p$ is given in Coleman [Col03], whereas the proof we include is close in spirit to the methods of [ABBR15].

Lemma 4.2. *Let $f : (X, D_X) \to (Y, D_Y)$ be a finite Galois morphism between smooth, proper, geometrically connected, hyperbolic punctured curves over K. Assume (X, D_X) has stable reduction over K, with stable model $\mathscr{X}$. Then there exists a unique semi-stable model $\mathscr{Y}$ of (Y, D_Y) such that the map f extends to a finite morphism $f : \mathscr{X} \to \mathscr{Y}$.*

Proof. Extend scalars to $\overline{K}^{\wedge}$. Let $V \subset X^{\mathrm{ad}}$ be the stable vertex set of (X, D_X), and set $W = f(V)$. There is a minimal semi-stable vertex set $W' \subset Y^{\mathrm{ad}}$ for (Y, D_Y) containing W, which may be obtained by adding a finite number of Type-II points if necessary, see [ABBR15, Lemma 3.15]. We will prove that $W' = W$. Pick any element $y \in W' \backslash W$, and any element $x \in f^{-1}(y)$. We see that this means that y must be of valency at least 3, whereas x must be of valency 2.



Because the Galois action on fibres over Type-I points is transitive and Y^{ad} is connected, the map $f : X^{\mathrm{ad}} \to Y^{\mathrm{ad}}$ is a Galois covering of topological spaces. It follows that the valency of x must be at least the valency of y, which is a contradiction. Therefore $W = W'$ and hence W is semi-stable. Theorem 4.3 assures the existence of a finite morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of semi-stable formal models over the ring of integral elements in $\overline{K}^{\wedge}$, where $\mathfrak{X}$ is in fact stable. These can be made into algebraic models $\mathscr{X}$ and $\mathscr{Y}$ by the algebraisation theorem [Abb10, Corollaire 2.3.19], and both $\mathscr{X}$ and $\mathscr{Y}$ descend to the integral closure of R in some finite separable extension of K, see [ABBR15, Lemma 5.4]. Finally, the morphism f descends to a finite morphism over the same field by faithfully flat descent as in Lemma 4.1. □

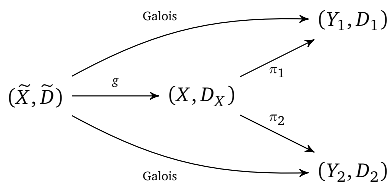
Remark 4.2. We note that this theorem is false without the assumption that f is Galois. However, it is true that a general f extends to a morphism between the stable models of its source and target, although such an extension will in general fail to be finite. See [LL99] for more details.

4.2.2 Potential stable reduction for correspondences.

We now come to the main result of this paper, which should be viewed as an analogue of the theorem of Deligne–Mumford [DM69, Corollary 2.7] on potential semi-stable reduction of smooth projective curves. We note that we already made essential use of the result of Deligne–Mumford in the proof of Lemma 4.2, so we do not obtain a new proof by specialising to the degenerate case where our correspondence is a single curve with identity morphism.

Theorem A. *Let $\mathbf{C}$ be a hyperbolic punctured correspondence over K. There is a finite separable extension of K over which $\mathbf{C}$ has a stable R-model, which is unique up to isomorphism.*

Proof. Change scalars to $\overline{K}^{\wedge}$. Consider the Galois closure $g : (\widetilde{X}, \widetilde{D}) \to (X, D_X)$ of both π_1 and π_2, with respect to some embedding of their function fields into $\overline{K(t)}$. This yields a diagram



where we set $\widetilde{D} = g^{-1}(D_X)$. This makes $(\widetilde{X}, \widetilde{D})$ hyperbolic, and as such there is a unique stable vertex set $\widetilde{V} \subset \widetilde{X}^{\mathrm{ad}}$ by Deligne–Mumford. Set $V := g(\widetilde{V})$ and $W_i = \pi_i(V)$, which are semi-stable vertex sets by Lemma 4.2. We must have $V = \pi_i^{-1}(W_i)$ by transitivity of the Galois action on fibres, and hence π_i extends to a finite morphism between the corresponding formal models. By the algebraisation theorem [Abb10, Corollaire 2.3.19], this yields three semi-stable curves over the ring of integral elements in $\overline{K}^{\wedge}$, which descend to a finite separable extension of K by [ABBR15, Lemma 5.4]. We use faithfully flat descent as in Lemma 4.1 to show that the morphisms descend to the same extension, giving us a semi-stable model $\mathfrak{C}$.

To show the existence of a stable model over the same field extension, we note that any semi-stable model must have the property that the semi-stable vertex sets of its objects contain the stable vertex sets. Hence, we may consider the stable vertex set $S \subset X^{\mathrm{ad}}$ of (X, D_X) and the semi-stable vertex sets $\pi_i(S) = T_i$. Now repeat the following procedure: Enlarge S to contain $\pi_i^{-1}(T_i)$, and enlarge the sets T_i to contain $\pi_i(S)$. This procedure is guaranteed to end, as all the new Type-II points introduced at every step are necessarily contained in the finite semi-stable vertex sets corresponding to the semi-stable model $\mathfrak{C}$ constructed above via the Galois closure. After this procedure terminates, we find a corresponding semi-stable model $\mathfrak{C}'$ for $\mathbf{C}$, which is minimal with respect to the relation of domination among all the semi-stable models. Hence $\mathfrak{C}'$ is the stable model of $\mathbf{C}$, defined over the same finite separable extension of K as $\mathfrak{C}$. □

Remark 4.3. We note that in general, the stable model of $\mathbf{C}$ does **not** consist of the stable models of its objects. As can be seen below for the case of Hecke operators, typically some extra components appear. We also note that it is still possible to find semi-stable models for correspondences that fail to be hyperbolic, but in general there will not be a minimal such model. To overcome this problem, we can fix semi-stable models for all the objects of $\mathbf{C}$, and run through the above argument to construct their *stable hull* $\mathfrak{C}$, which is the minimal semi-stable model such that all the objects dominate their corresponding fixed semi-stable models.

4.3 Weight-monodromy and skeleta of correspondences

We now investigate how the knowledge of a semi-stable model for a correspondence **C** allows us to deduce information about the spectrum of **C**. More precisely, linearisation of **C** yields a map

$$\mathbf{C}^* : \mathrm{H}^{\mathrm{i}}_{\text{ét}}(Y_{1,\overline{K}}, \mathbf{Q}_l) \to \mathrm{H}^{\mathrm{i}}_{\text{ét}}(Y_{2,\overline{K}}, \mathbf{Q}_l),$$

whose properties are often of tremendous importance. In the case of Hecke operators, which we treat explicitly in section 5.1, we have $Y_1 = Y_2$ and the eigenvalues of $\mathbf{C}^*$ are Fourier coefficients of weight two modular forms. Rather than attempting to compute this action on the full cohomology groups, we investigate how the geometry of a semi-stable model of **C** allows us to read off the action on the graded pieces with respect to the weight-monodromy filtration. To this end, we introduce the skeleton of a semi-stable model, discuss the important notion of harmonic morphisms between them, and show how the knowledge of a skeletal semi-stable model implies much sought after spectral information.

Even when we restrict ourselves to the algebraic situation of smooth projective curves, we encounter an infinitude of components in the special fibre as in the example of the Hecke operator T_{p} in 5.1.3. It should be possible to rephrase the results in 4.3.5 for completely general correspondences, and we hope to generalise this material in the future to more general types of adic spaces, so as to also include such perfectoid spaces as Lubin–Tate towers [Wei12].

4.3.1 The weight-monodromy filtration.

We now recall the definition of the monodromy filtration on the l-adic étale cohomology of a proper, smooth, separated scheme X of finite type over a non-Archimedean local field K, and the identification of the graded pieces with quantities that can be computed from a semi-stable model in the case of curves. By the monodromy theorem of Grothendieck [Gro72], there exists a nilpotent operator

$$N \in \mathrm{End}(\mathrm{H}^{\mathrm{i}}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_l))$$

such that every σ in a sufficiently small open subgroup of the inertia group $I \subset \mathrm{Gal}(K^s/K)$ acts as $\exp(t_l(\sigma)N)$, where $t_l : I \to \mathbf{Z}_l(1)$ is the maximal pro-l quotient map. We obtain an ascending *monodromy filtration* $M_\bullet$ on $\mathrm{H}^{\mathrm{i}}_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_l)$, characterised by $NM_i \subseteq M_{i-2}(-1)$ and

$$N^i : \mathrm{Gr}^M_i \xrightarrow{\sim} \mathrm{Gr}^M_{-i}(-i)$$

From the description of the action of inertia in terms of the operator N, we obtain a well-defined action of the geometric Frobenius Fr_q determined by unipotent inertia, and in particular a notion of *weights*.

Conjecture 4.1 (Weight-monodromy conjecture). *The eigenvalues of* Fr_q *on* $\mathrm{Gr}^M_j \mathrm{H}^{\mathrm{i}}_{\acute{e}t}(X_{\overline{K}}, \mathbf{Q}_l)$ *are algebraic numbers whose conjugates all have complex absolute values equal to* $q^{(i+j)/2}$.

This conjecture is known in many cases. We will only need the case of curves, where everything was proved by Grothendieck [Gro72]. The case of semi-stable surfaces was settled by Rapoport–Zink [RZ82], using properties of the weight spectral sequence which we discuss next. The analogous conjecture over function fields was proved by Deligne [Del71b], and many subsequent developments reduce the case of mixed characteristic to Deligne's setting, see for instance [Sch12]. Let $\mathscr{X}$ be a semi-stable model for X with components Y_i in the special fibre, then the analysis of the vanishing and nearby cycles functor in [RZ82] yields the construction of the *weight spectral sequence*, with first page

$$E_1^{p,q} = \bigoplus_{i \geq \max(0,-p)} \mathrm{H}^{q-2i}_{\text{ét}}\left(Y_{\bar{s}}^{[p+2i+1]}, \mathbf{Q}_l(-i)\right) \quad \Rightarrow \quad \mathrm{H}^{p+q}_{\text{ét}}\left(X_{\overline{K}}, \mathbf{Q}_l\right),$$

where $Y^{[m]}$ is the disjoint union of the m-fold intersections of the Y_i. This induces a filtration on the cohomology which is known to coincide with the weight-monodromy filtration in many cases, including that of curves.

4.3.2 Spectral properties of correspondences.

Let X be a smooth proper curve over K, then the knowledge of an explicit semi-stable model $\mathscr{X}$ for X helps us compute the graded pieces of the weight filtration. Indeed, the weight spectral sequence [RZ82] gives us the short exact sequence

$$0 \to \mathrm{H}^1_{\text{ét}}(\widetilde{\mathscr{X}_{\bar{s}}}, \mathbf{Q}_l) \to \mathrm{H}^1_{\text{ét}}(X_{\overline{K}}, \mathbf{Q}_l) \to \mathrm{H}^1(\Gamma_{\mathscr{X}}, \mathbf{Q}_l)(-1) \to 0,$$

where $\Gamma_{\mathscr{X}}$ is the dual graph attached to $\mathscr{X}$, and $\widetilde{\mathscr{X}_{\bar{s}}}$ denotes the normalisation of its geometric special fibre. Let $\mathscr{J}$ be the Néron model of the Jacobian of X. The quotient of $\mathscr{J}_{\bar{s}}$ by its identity connected component $\mathscr{J}_{\bar{s}}^0$ defines the Néron component group Φ, so we get the exact sequence

$$0 \to \mathscr{J}_{\bar{s}}^0 \to \mathscr{J}_{\bar{s}} \to \Phi \to 0.$$

Recall that the results of [Ray70] imply that $\mathscr{J}_{\bar{s}}^0 \simeq \mathrm{Pic}^0(\mathscr{X}_{\bar{s}})$, see [Gro72, Exposé IX, 12.1.11]. This means we can further decompose $\mathscr{J}_{\bar{s}}^0$ as follows

$$0 \to \mathrm{H}^1(\Gamma_{\mathscr{X}}, \mathbf{Z}) \otimes \mathbf{G}_m \to \mathscr{J}_{\bar{s}}^0 \to \mathrm{Pic}^0(\widetilde{\mathscr{X}_{\bar{s}}}) \to 0.$$

Note that it is not necessary to assume that $\mathscr{X}$ is regular, see [Rib90, Section 2]. The algebraic group $\mathrm{H}^1(\Gamma_{\mathscr{X}}, \mathbf{Z}) \otimes \mathbf{G}_m$ is called the *toric part* of $\mathscr{J}$. The Néron component group Φ may be computed as the cokernel of the map from the toric part to its dual, given by the monodromy pairing [Gro72, Théorème 11.5].

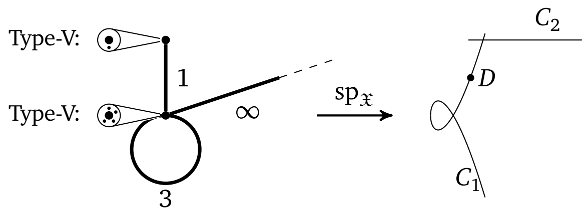
We identify a correspondence $\mathbf{C} : Y_1 \leftarrow X \rightarrow Y_2$ between K-curves with an element $\mathbf{C} \in \mathrm{CH}^1(Y_1 \times Y_2)$, and define the morphism $\mathbf{C}^* : \mathrm{H}^i_{\text{ét}}(Y_{1,\overline{K}}, \mathbf{Q}_l) \to \mathrm{H}^i_{\text{ét}}(Y_{2,\overline{K}}, \mathbf{Q}_l)$ by $\pi_{2,*} \circ (\mathbf{C} \cup -) \circ \pi_1^*$. It is proved in [Sai03] that the weight spectral sequence is equivariant under the actions of $\mathrm{Gal}(K^s/K)$ and correspondences $\mathbf{C}$, so $\mathbf{C}^*$ respects the weight-monodromy filtration. We see that an explicit knowledge of a semi-stable model $\mathfrak{C}$ for $\mathbf{C}$ is exactly what is needed to describe the induced map on the first step of the weight-monodromy filtration. The reduction of $\mathfrak{C}$ decomposes as a finite set of correspondences between irreducible curves over the residue field of R, providing a geometric description of the induced morphism $\mathrm{Pic}^0(\widetilde{\mathscr{Y}}_{1,\bar{s}}) \to \mathrm{Pic}^0(\widetilde{\mathscr{Y}}_{2,\bar{s}})$. For the case of the Hecke operator $T_\mathfrak{p}$ on modular curves X_0^B treated in 5.1.3, this yields the celebrated Eichler-Shimura relation $T_\mathfrak{p} = \mathrm{Frob} + \mathrm{Ver}$. As this curve has good reduction at $\mathfrak{p}$, the first step of the filtration exhaust the entire cohomology group.

To describe the induced morphism between both the graded pieces of weight 2 and the Néron component groups $\Phi_1 \to \Phi_2$, we use Raynaud's theorem and the monodromy pairing to reduce the problem to finding a geometric interpretation of the maps between the toric parts of the Jacobians. As recalled above, we may describe the toric part in terms of the dual graph of a semi-stable model, which has a natural interpretation as a *skeleton* in the setting of adic spaces. We now recall the relevant theory, and use it to motivate the important definition of a *skeletal semi-stable model* in 4.3.5. Such a model allows us to combinatorially read off the map on the graded piece of weight 2 by pulling back and pushing forward cycles. We mention the work of Yoshida [Yos11] in this context, which describes a different algorithm to compute this action via intersection theory on regular models of the objects of $\mathbf{C}$, in arbitrary dimensions.

4.3.3 The skeleton of a semi-stable vertex set.

To a semi-stable vertex set V for (X, D_X), we can associate a combinatorial structure which we call the *skeleton* Σ_V of X with respect to V. First, we define Γ_V as the set of points of X^{ad} that are not contained in a wide open ball which is disjoint from $V \cup D_X$, endowed with the subspace topology. Recall that the maximal Hausdorff quotient $\overline{\Gamma}_V$, often referred to as the *Berkovich skeleton* of X with respect to V, can naturally be viewed as a metric realisation of the dual graph. Its vertex set is $V \cup D_X$, and edges come in two flavours: There is an edge e_Q between two vertices in V for every intersection point Q of the corresponding components in the special fibre of $\mathfrak{X}_V$, and we set the length $l(e_Q)$ of e_Q to be the width of the wide open annulus $\mathrm{sp}_V^{-1}(Q)$. Every vertex in D_X is adjacent to exactly one edge of length ∞. Define the skeleton of X^{ad} with respect to V, or $\mathfrak{X}_V$, to be the pair $\Sigma_V = (\Gamma_V, \mathscr{L}_V)$, where $\mathscr{L}_V = \{l(e) : e \text{ edge of } \overline{\Gamma}_V\}$ consists of the set of edge lengths appearing in the maximal Hausdorff quotient $\overline{\Gamma}_V$, viewed as a metric graph. We define the *genus* of a Type-II point x in Γ_V to be the genus of the residue field $k(x)$ as defined in 4.1.

Example 4.4. Consider the projective closure of $y^2 = x^3 + x^2 + p^3$, over $\mathscr{O}_{\mathbb{C}_p}$, and let $\mathscr{X}$ be the blow-up at a smooth point of the special fibre. Then $\mathscr{X}$ is a normal semi-stable model whose special fibre consists of a nodal curve C_1 and a projective line C_2, crossing transversally. Now set X to be the generic fibre of $\mathscr{X}$, and $D = \{(0,1,0)\}$ the point at infinity. The formal completion $\mathfrak{X}$ of $\mathscr{X}$ is a semi-stable formal model for (X, D), and its corresponding skeleton $\Sigma_{\mathfrak{X}}$ can be visualised as



The Type-II points in $V_{\mathfrak{X}}$ have valencies 1 and 4, and are both of genus 0.

4.3.4 Harmonic morphisms.

Given a finite morphism $f : \mathfrak{X} \to \mathfrak{Y}$ of semi-stable formal models for (X, D_X) and (Y, D_Y), we know that $f^{-1}(V_{\mathfrak{Y}}) = V_{\mathfrak{X}}$. In general, it is not true that $f^{-1}(\Gamma_{\mathfrak{Y}}) = \Gamma_{\mathfrak{X}}$, see [ABBR15, Remark 5.23]. When this holds, we say that f is a *skeletal* finite morphism. For maps f, we can always modify f so as to make it skeletal, see [ABBR15, Corollary 4.18]. Let x be an apex point of a wide open annulus in $X^{\mathrm{ad}} \backslash V_{\mathfrak{X}}$, and set l_x to be the width of the annulus. It can be shown that the ratio $d_x := l_{f(x)}/l_x$ equals the ramification index of the map $f : \mathfrak{X}_s \to \mathfrak{Y}_s$ at the point $\mathrm{sp}_{\mathfrak{X}}(x)$, see [Col05, Lemma 2.1]. We use this to define d_x for all apex points attached to $\mathfrak{X}$, by simply setting d_x to be the corresponding ramification index for the map f. The integer d_x is often referred to as the *expansion factor* of the edge of $\Gamma_{\mathfrak{X}}$ adjacent to x. It can be shown that the morphism $f : \Gamma_{\mathfrak{X}} \to \Gamma_{\mathfrak{Y}}$ is *harmonic* in the sense that the sum

$$\sum_{x \mapsto y} d_x,$$

is independent of the Type-V point $y \in \overline{V}_{\mathfrak{Y}} \cap \Gamma_Y$, and equals the degree of f. A skeletal finite morphism $f : \mathfrak{X} \to \mathfrak{Y}$ gives rise to a harmonic morphism on skeleta in this sense. This notion was introduced in [ABBR15, Section 4.27] for Berkovich spaces. **Remark 4.5.** Note that these conditions give rise to a notion similar to that of harmonic morphisms of metrised complexes of curves in [ABBR15, Section 2.16]. The list of properties included in *loc. cit.* is considerably longer. Whereas the study of this more abstract category is a fascinating one, we shall not be concerned with liftings of tropicalisation maps in the present work, as our metrised complexes will always arise from algebraic curves.

4.3.5 Skeletal stable models of correspondences.

Pick a semi-stable model $\mathfrak{C}$ for **C**, with objects $\mathscr{X}, \mathscr{Y}_1, \mathscr{Y}_2$ and morphisms π_1, π_2. The semi-stable model $\mathfrak{C}$ is *skeletal* if both π_1 and π_2 are skeletal in the sense of 4.3.4. It is not true that any semi-stable model $\mathfrak{C}$ of **C** is skeletal, already when $\mathfrak{C}$ degenerates to a map, see [ABBR15, Remark 5.23]. The question arises whether it is always possible to modify a semi-stable model of a punctured correspondence, so as to make it skeletal, and whether there is always a minimal such skeletal semi-stable model if **C** is hyperbolic.

We continue with the notation of the proof of Theorem A, where we found semi-stable vertex sets V, W_1, W_2 for the objects of **C**, such that $\pi_1^{-1}(W_1) = \pi_2^{-1}(W_2) = V$. From [ABBR15, Remark 4.19], we know that $\pi_i(\overline{\Gamma}_V)$ is the union of $\overline{\Gamma}_{W_i}$ and a finite set of edges. Now enlarge W_i to contain the endpoints of those edges, and run through the following procedure: Enlarge V to contain $\pi_i^{-1}(W_i)$, and enlarge the sets W_i to contain $\pi_i(V)$. Suppose this procedure terminates for a given correspondence **C** over K, then we find a skeletal semi-stable model $\mathfrak{C}$, which gives rise to a diagram

$$\Sigma_{\mathfrak{C}} : \quad \Sigma_{\mathfrak{Y}_1} \xleftarrow{\sigma_1} \Sigma_{\mathfrak{X}} \xrightarrow{\sigma_2} \Sigma_{\mathfrak{Y}_2}$$

where σ_1, σ_2 are finite harmonic morphisms as defined in 4.3.4. We call this diagram the *skeleton* of the skeletal semi-stable model $\mathfrak{C}$ of $\mathbf{C}$.

The question now becomes whether this procedure always terminates. The answer in general is *no*, as the example in section 5.1.3 of T_{p} on quaternionic Shimura varieties X^B shows. However, in some cases the answer is *yes*. Assume that either (i) π_1, π_2 are Galois, or (ii) either π_1 or π_2 is the identity. Every time a new vertex $v \in V$ is introduced, it appears on the stable skeleton of (X, D_X), and the set of edge lengths $\mathscr{L}_V$ of the skeleton $\Sigma_V = (\Gamma_V, \mathscr{L}_V)$ remains constant by Lemma 4.2. This shows that the procedure terminates. Starting with semi-stable models $\mathscr{X}, \mathscr{Y}_1, \mathscr{Y}_2$ for the objects of $\mathbf{C}$, this argument is insensitive to enlarging the stable vertex sets of X, Y_1, Y_2 to contain the vertex sets of the chosen models at the start of the procedure. We summarise this discussion in the following statement.

Theorem B. *Let* $\mathbf{C}$ *be a hyperbolic punctured correspondence over* K*, and assume that either (i) both morphisms are Galois, or (ii) one of the morphisms is the identity. Given semi-stable models* $\mathscr{X}, \mathscr{Y}_1, \mathscr{Y}_2$ *for the objects of* $\mathbf{C}$*, there is a finite separable extension of* K *over which* $\mathbf{C}$ *has a* ***skeletal*** *semi-stable model whose objects dominate the given semi-stable models pairwise, and which is minimal with respect to the relation of domination.*

The semi-stable model in the above theorem is called the *skeletal stable hull* of $\mathbf{C}$ with respect to the triple $(\mathscr{X}, \mathscr{Y}_1, \mathscr{Y}_2)$. This strengthens the relative theorem for finite maps by Liu [Liu06]. When we take the triple of semi-stable models to consists of the unique stable models of the objects in $\mathbf{C}$, we call the resulting minimal skeletal semi-stable model the *skeletal stable model*. As in the proof of Theorem A, we may construct skeletal semi-stable models for general hyperbolic correspondences by reducing to the Galois case. However, we make no claims about the existence of skeletal stable models in this case.

Corollary 4.1. *Let* $\mathbf{C}$ *be a hyperbolic punctured correspondence over* K*. There is a finite separable extension of* K *over which* $\mathbf{C}$ *has a* ***skeletal*** *semi-stable* R*-model.*

Proof. Resume the notation of the proof of Theorem A, and let $(\widetilde{X}, \widetilde{D}) \xrightarrow{g} (X, D)$ be the Galois closure of the morphisms of $\mathbf{C}$. Let $\widetilde{\mathfrak{C}}$ be the skeletal stable model of the resulting Galois correspondence, which is guaranteed by Theorem B, and let Γ be the image of the skeleton of $(\widetilde{X}, \widetilde{D})$. It is now easy to see that this defines a skeleton of (X, D) which, together with the skeleta for (Y_1, D_1) and (Y_2, D_2) implicit in $\widetilde{C}$, define a skeletal semi-stable model of $\mathbf{C}$. $\square$

The moduli space of curves and tropicalisation. Theorem A should be considered as an analogue of the Deligne–Mumford theorem. It would be interesting to investigate what it tells us about compactification and irreducibility of a suitably defined moduli space of correspondences. Skeleta of correspondences should be considered as tropical objects in a natural way, and the association of a skeleton to a correspondence might be viewed as an analogue of the map

$$\mathrm{Trop}: \overline{\mathscr{M}}^{\mathrm{ad}}_{g,n} \longrightarrow \overline{M}^{\mathrm{trop}}_{g,n},$$

from the Deligne–Mumford–Knudsen moduli stack of n-pointed curves of genus g, to the corresponding moduli space of tropical curves. This map is studied by Abramovich, Caporaso and Payne [ACP15], and may be interpreted as the retraction map of the moduli stack of curves onto its canonical skeleton in the sense of Thuillier [Thu07].

Chapter 5

Hecke operators on Shimura curves

We present a number of applications of the main results in the previous chapter. We explicitly describe stable models of Hecke operators on various quaternionic Shimura curves over totally real fields, which leads to a geometric interpretation of the *graph algorithm* of Mestre-Oesterlé [Mes86] and Dembélé-Voight [DV13] that allows us to compute Fourier coefficients of Hilbert modular forms, and gives a concrete geometric description of a special case of Jacquet-Langlands functoriality. We also give a proof of the existence of canonical subgroups [Kat73, Lub79], which applies to situations which are vastly more general than the case of modular curves. The proof is consistent with the philosophy of Goren-Kassaei on a systematic focus on geometry, avoiding the use of arguments that involve a specific moduli interpretation which might not be available in more general settings.

Outline. After recalling some results on semi-stable models of quaternionic Shimura varieties by Deligne-Rapoport [DR73], Katz-Mazur [KM85], Carayol [Car86], Edixhoven [Edi90], Buzzard [Buz97] and Jarvis [Jar99], we turn to the explicit determination of Hecke operators away from $\mathfrak{p}$ in 5.1.2, and at $\mathfrak{p}$ in 5.1.3. As a consequence, we generalise a theorem of Coleman in 5.1.5. We then generalise results on canonical subgroups by Goren-Kassaei [GK06] in F, and discuss some future directions.

5.1 Hecke operators on Shimura curves

We now compute some explicit examples of stable models of Hecke correspondences, and revisit many classical results such as the Eichler–Shimura relation, the Mestre–Oesterlé *méthode des graphes* [Mes86], the algorithms due to Dembélé–Voight on Fourier coefficients of Hilbert modular forms [DV13], results on the action on Néron component groups by Ribet [Rib90], and the moduli interpretation of the semi-stable model of $X_0(Np^2)$ constructed by Edixhoven [Edi90] due to Coleman [Col05].

The explicit determination of stable models of correspondences $\mathbf{C}$ between curves is a difficult problem, and already the degenerate case of a single curve has been subject of much research in arithmetic geometry. For correspondences, we not only need to describe explicit semi-stable models for the objects, but we also need a good understanding of the geometry of the morphisms. This task is simplified when the objects and morphisms of **C** have a clear moduli interpretation. For all the examples discussed below, descriptions of integral models may be found by identifying a closely related PEL Shimura variety, for which one may describe integral models by computing the deformation theory of the associated p-divisible groups with appropriate Drinfeld level structure. This is done via the Serre–Tate theorem, which was proved by Messing [Mes72] using the theory of crystals attached to p-divisible groups. A simpler proof was given by Drinfeld [Dri76], of which an excellent exposition may be found in [Kat81, Section 1].

We start by reviewing parts of the literature on semi-stable models of Shimura curves in section 5.1.1. We then determine the stable models at $\mathfrak{p}$ for Hecke operators $T_{\mathfrak{l}}$ for $\mathfrak{l} \neq \mathfrak{p}$ in section 5.1.2, and $\mathfrak{l} = \mathfrak{p}$ in section 5.1.3. As an application, we generalise a theorem of Coleman in section 5.1.5.

5.1.1 Models of quaternionic Shimura curves.

We recall some of the literature on integral models of quaternionic Shimura curves, due to Katz–Mazur [KM85], Carayol [Car86], Edixhoven [Edi90], Buzzard [Buz97] and Jarvis [Jar04]. Let us first fix some notation. Let F be a totally real number field of degree d over $\mathbf{Q}$, and $B_{/F}$ a quaternion algebra which is split at exactly one infinite prime. Let R be a maximal order of B, and set $G := \mathrm{Res}_{F/\mathbf{Q}}(B^{\times})$ to be the Weil restriction of the unit group of B. For K a compact open subgroup of $G(\mathbf{A}^{\infty})$, the quaternionic Shimura curve attached to B and K is a curve X_K^B whose $\mathbf{C}$-points are

$$X_K^B(\mathbf{C}) = G(\mathbf{Q})\backslash G(\mathbf{A}^{\infty}) \times \mathfrak{H}^{\pm}/K.$$

By the theory of canonical models due to Shimura [Shi70], the curve X_K^B is defined over F. Henceforth, we will fix a prime $\mathfrak{p}$ above the rational prime p at which B is split, and a sufficiently small level structure away from $\mathfrak{p}$. We denote X^B for the corresponding Shimura curve, and $X_0^B(\mathfrak{p})$ for the Shimura curve with additional Iwahori level structure at $\mathfrak{p}$. For convenience, we will consider integral models over the strict Henselisation $\mathcal{O}_{F,\mathfrak{p}}^{\mathrm{sh}}$, and set W to be the ring of Witt vectors of $\overline{\mathbf{F}}_p$.

When $d = 1$, the curves $X_0^B(p)$ are PEL Shimura varieties defined over $\mathbf{Q}$. The curves $X_0^B(p)$ are fine moduli schemes for elliptic curves when $B = \mathrm{M}_2(\mathbf{Q})$ and for so called *false elliptic curves* when B is non-split. False elliptic curves are abelian schemes of relative dimension 2 with quaternionic multiplication by R, together with a level structure. Once level structures are understood in the sense of Drinfeld [Dri74], we obtain integral models $\mathscr{X}_0^B(p)$ over W. Buzzard [Buz97] notes that the p-divisible group of a false elliptic curve A over an algebraically closed field of characteristic p is of the form $E[p^{\infty}] \times E[p^{\infty}]$, where E is an elliptic curve. This allows him to apply the theorem of Serre–Tate to prove the following result, which is [Buz97, Theorem 4.7]. The case where B is split is due to Deligne–Rapoport [DR73].

Theorem 5.1 (Deligne–Rapoport, Buzzard). *The scheme $\mathscr{X}_0^B(p)$ over W is regular, with special fibre consisting of two irreducible components isomorphic to $\mathscr{X}_s^B$, crossing transversally at the supersingular points. The degeneration map $\pi_1 : \mathscr{X}_0^B(p) \to \mathscr{X}^B$ is finite flat.*

We recall that Edixhoven determined a semi-stable model when $B = \mathrm{M}_2(\mathbf{Q})$, in the case of $\Gamma_0(p^2)$ structure at p. Edixhoven's semi-stable model for $X_0(Np^2)$ is defined over $W[\varpi]$ where $\varpi^{(p^2-1)/2} = p$. The special fibre contains four ordinary components. Two of these, which we call the *inner ordinary components*, are isomorphic to $\mathscr{X}_0(N)_s$. The other two, X^+ and X^-, are called the *outer ordinary components*, and are isomorphic to the Igusa curves $\overline{X}_0(N, \mathrm{Ig}(p)/\pm)$. There is one *supersingular component* Z^σ for every supersingular point σ on $\mathscr{X}_0(Np^2)_s$, for which equations may be found in Edixhoven [Edi90, Theorem 2.1.1]. Every supersingular component intersects every ordinary component, and there are no other intersection points. The formal neighbourhoods at these intersection points are all regular, except where the supersingular components meet the two inner ordinary components. If E_σ is the elliptic curve with auxiliary $\Gamma_0(N)$-structure corresponding to a supersingular point σ on $X_0(Np)$, we set $\alpha_\sigma := |\mathrm{Aut}(E_\sigma)|/2$. The formal completion of the local ring at such a singularity is isomorphic to

$$\widehat{\mathscr{O}}_{\mathscr{X}_0(Np^2),\sigma} \simeq W[\varpi][\![x,y]\!] / \left(xy - \varpi^{(p-1)\alpha_\sigma}\right).$$

The Katz–Mazur model for the curves $X_0(Np^n)$ for $n \geq 2$ is not semi-stable, and a description of the local ring at the supersingular points was only recently given by Weinstein [Wei12]. This yields a procedure to determine a semi-stable model for $X_0(Np^n)$ for any n.

When $d > 1$, the situation is more complex, as the Shimura curves $X_0^B(\mathfrak{p})$ are no longer PEL. Carayol [Car86] constructs integral models for these curves by comparing them to unitary Shimura curves which arise from picking an auxiliary CM field E containing F in which $\mathfrak{p}$ splits. These unitary Shimura curves are PEL, and integral models may be found by imposing level structure in the sense of Drinfeld [Dri74]. The geometric properties of these integral models can then be investigated by identifying the completed local rings at closed points as universal objects for a suitable deformation problem of p-divisible groups, in the spirit of the Serre–Tate theorem. This allows Carayol to construct integral models for quaternionic Shimura curves, by identifying their connected components with connected components of appropriate unitary Shimura curves. Jarvis [Jar99, Jar04] makes this analysis explicit in the case of Iwahori level structure at $\mathfrak{p}$. The following theorem may be found in [Jar99, Section 10].

Theorem 5.2 (Carayol, Jarvis)**.** *There is a regular model $\mathscr{X}_0^B(\mathfrak{p})$ over $\mathscr{O}_{F,\mathfrak{p}}^{sh}$, with a finite flat degeneration map $\pi_1 : \mathscr{X}_0^B(\mathfrak{p}) \to \mathscr{X}^B$ to the model constructed by Carayol [Car86]. The special fibre of $\mathscr{X}_0^B(\mathfrak{p})$ consists of two components isomorphic to $\mathscr{X}_s^B$, crossing transversally at the supersingular points.*

5.1.2 Hecke operators away from $\mathfrak{p}$.

Let $\mathfrak{l}$ be coprime to both $\mathfrak{p}$ and the implicit tame level. The analysis of the special fibre of $\mathscr{X}_0^B(\mathfrak{p})$ by Deligne–Rapoport, Buzzard and Jarvis immediately yields a skeletal semi-stable model $\mathfrak{T}_\mathfrak{l}$ for the Hecke operators $T_\mathfrak{l}$ on $X_0^B(\mathfrak{p})$.

Theorem C. *The correspondence $T_\mathfrak{l}$ on $X_0^B(\mathfrak{p})$ has skeletal semi-stable reduction over $\mathscr{O}_{F,\mathfrak{p}}^{sh}$. This model is stable for large $\mathfrak{p}$, and the stable skeleton is as depicted in Figure 5.1.*

Proof. We note that the regular models for $X_0^B(\mathfrak{p})$ described above are semi-stable over $\mathscr{O}_{F,\mathfrak{p}}$, regardless of the tame level structure imposed. Let $X_0^B(\mathfrak{p},\mathfrak{l})$ be the Shimura curve obtained by imposing additional Iwahori level structure at $\mathfrak{l}$, then the map $\pi_1 : X_0^B(\mathfrak{p},\mathfrak{l}) \longrightarrow X_0^B(\mathfrak{p})$ is finite, and as π_2 is the Atkin-Lehner involution at $\mathfrak{p}$ composed with π_1, we obtain a semi-stable model $\mathfrak{T}_\mathfrak{l}$, which is stable whenever $X_0^B(\mathfrak{p})$ is. The description of the special fibre now follows from the above theorems. □

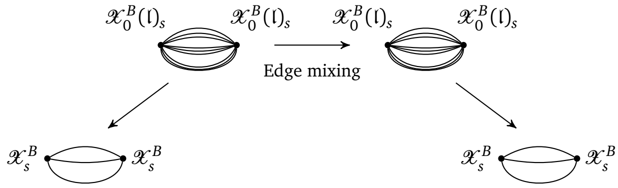


Figure 5.1: The skeleton of $\mathfrak{T}_\mathfrak{l}$

We now describe how this yields a geometric description of a special case of the Jacquet–Langlands correspondence, and the consequent graph algorithms due to Mestre–Oesterlé [Mes86] and Dembélé–Voight [DV13]. Let $\overline{B}$ be the quaternion algebra over F obtained from B by making the split infinite place and $\mathfrak{p}$ ramified, and set $\overline{G} = \mathrm{Res}_{F/\mathbf{Q}}(\overline{B}^\times)$. The Jacquet–Langlands correspondence now provides us with a functor from automorphic representations for $\overline{G}$ to automorphic representations for G which are square integrable modulo center at $\mathfrak{p}$ and the split infinite place. We may identify the action on the toric part $\mathrm{H}^1(\Gamma_{\mathcal{X}_0^B(\mathfrak{p})}, \mathbf{Z}) \otimes \mathbf{G}_m$ of $\mathscr{J}_s^0$ with the action on the set of supersingular points

$$\overline{G}(\mathbf{Q})\backslash\overline{G}(\mathbf{A}^\infty)/K^\mathfrak{p} \times \mathcal{O}_{\overline{B}_\mathfrak{p}}^\times,$$

which is a finite set amenable to computation. In fact, elements in this set may be represented as right ideal classes in an Eichler order of $\overline{B}$ depending on the level, as is explained in [DV13]. By means of an example, set $\alpha = \zeta_7 + \zeta_7^{-1}$ and $F = \mathbf{Q}(\zeta_7)^+ = \mathbf{Q}(\alpha)$ to be the totally real cubic number field unramified outside 7, and let f be the unique normalised Hilbert modular form of prime level $3\mathcal{O}_F$ and parallel weight $(2,2,2)$. Set B to be the quaternion algebra ramified at two of the three infinite places. By computing the relevant Brandt matrices for $\overline{G}$, we find the following Fourier coefficients:

$\mathfrak{l}$	(2)	(5)	(11)	$(13, 3+\alpha)$	(17)	(19)
$a_\mathfrak{l}$	−4	22	58	1	0	−56

5.1.3 Hecke operators at $\mathfrak{p}$.

We start with the operator $T_\mathfrak{p}$ on X^B, which is an example of a correspondence whose semi-stable model cannot be made skeletal in the above sense. The morphisms $\pi_1, \pi_2 : \mathcal{X}_0^B(\mathfrak{p}) \rightrightarrows \mathcal{X}^B$ are finite flat, and hence define a semi-stable model for $T_\mathfrak{p}$. However, the procedure outlined in

4.3.5 does not terminate, and keeps introducing new components, which accumulate around the ordinary components. Figure 5.2 depicts the first few iterations of this process. Note that we could have omitted the finiteness condition in our definition of a semi-stable vertex set, and modify the concept of skeletal semi-stable models of correspondences accordingly. This analysis would then determine a skeletal semi-stable model for $T_{\mathfrak{p}}$. For applications, this added generality seems harmless and may be adopted should the need arise.

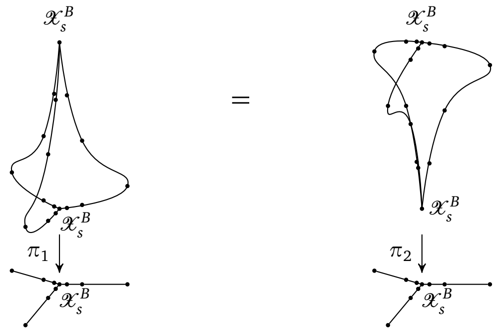


Figure 5.2: The stable model $\mathfrak{T}_{\mathfrak{p}}$

As a next step, we wish to increase the level structure at $\mathfrak{p}$. As it is difficult to find precise statements on integral models for the Shimura curves we wish to discuss, we restrict ourselves to the setting $B = \mathrm{M}_2(\mathbf{Q})$. We show how to deduce a semi-stable model for U_p on $X_0(Np^2)$ from the work of Edixhoven [Edi90]. The top object of U_p is the split Cartan curve $X_{\mathrm{sc}}(N)$.

Theorem D. *Let $p \geq 5$ coprime to N. The correspondence U_p on $X_0(Np)$ has a semi-stable model $\mathfrak{U}_p$ over $W[\varpi]$, where $\varpi^{(p^2-1)/2} = p$. The skeleton of $\mathfrak{U}_p$ is depicted in Figure 5.3.*

Proof. We have an isomorphism $X_{\mathrm{sc}}(N) \simeq X_0(p^2N)$. The work of Edixhoven [Edi90] shows that we may invoke [Liu02, Proposition 10.4.6] to establish the semi-stability of the normalisation $\mathscr{X}$ of the base change to $W[\varpi]$ of the blow-up the Katz–Mazur model $\mathscr{X}_{\mathrm{sc}}(N)$ at the supersingular points. Now extend scalars to $\mathscr{O}_{\mathbf{C}_p}$. The constructive proof of Theorem A given above shows that we may obtain a semi-stable model $\mathfrak{U}_p$ for U_p by enlarging $V_{\mathscr{X}_0(Np)}$ to contain $\pi_1(V_{\mathscr{X}})$ and $\pi_2(V_{\mathscr{X}})$ respectively.

The expansion factors of the edges we obtain equal the ramification indices of the corresponding points on the ordinary components of π_1 and π_2. As π_1 and π_2 restrict to the identity and Frobenius maps on the inner ordinary components, and are totally ramified above supersingular points on $X^{\pm}$, we obtain the expansion factors $1, p$ and $(p-1)/2$ as in Figure 5.3. This determines a skeletal semi-stable model $\mathfrak{U}_p$, which is defined over $W[\varpi]$ by faithfully flat descent as in 4.1.3. □

To increase the number of cases in which U_p is hyperbolic, and hence has a unique stable skeleton, we may puncture the modular curves $X_0(Np)$ and $X_{\mathrm{sc}}(N)$ at the cusps. Figure 5.3 uses the notation of Edixhoven for the ordinary components, which correspond to the ordinary components of $\mathscr{X}_0(Np^2)$ under the isomorphism $X_{\mathrm{sc}}(N) \simeq X_0(Np^2)$. The numbers marking the edges are the

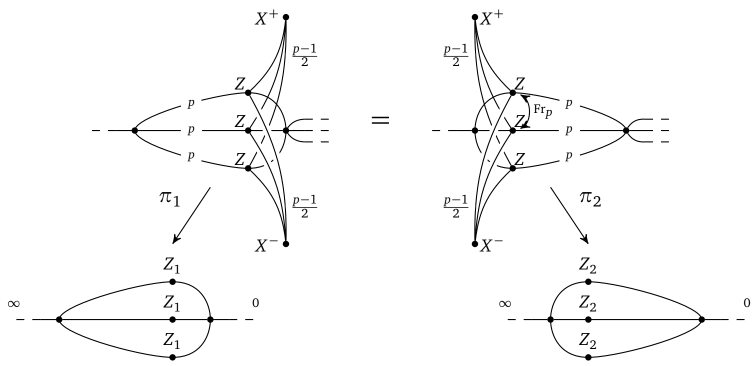


Figure 5.3: The skeleton of $\mathfrak{U}_p$

expansion factors under the maps involved, which we omitted whenever it is 1. As an immediate consequence of Theorem D, we obtain that the action of U_p on the toric part of $\mathrm{H}^1_{\text{ét}}(X_0(Np)_{\overline{\mathbf{Q}}_p}, \mathbf{Q}_l)$ equals the action of Frobenius. This is [Rib90, Proposition 3.8]. We also immediately recover the action of U_p on the Néron component group Φ.

5.1.4 Hecke operators at the infinite level.

As we increment the level at $\mathfrak{p}$ of our Hecke operators, it becomes increasingly difficult to find semi-stable models the modular curves. In view of the recent work of Weinstein [Wei12], it seems prudent to work at the infinite level instead. The theory we have set up in this paper was phrased in the language of adic spaces, partly because of the more streamlined treatment of the main definitions due to the presence of Type-V points, but also to indicate that the correspondence between semi-stable models and semi-stable vertex sets, along with their behaviour under finite maps, should carry over to more general settings. In particular, we expect the statement and proof of our main theorem A to remain valid in the setting of perfectoid spaces. If one can give a satisfactory description of the harmonic maps constituting U_p at the infinite level, this would likely reveal interesting information. We hope to return to this problem in the near future.

5.1.5 Too supersingular elliptic curves.

Coleman [Col05] gives a moduli interpretation of the components in the special fibre of Edixhoven's semi-stable model $\mathscr{X}_0(Np^2)$ in tame level $N = 1$. As an immediate consequence of the determination of the stable model $\mathfrak{U}_p$ given above, we now generalise this theorem to general N. We note that the necessary machinery on finite maps between integral models of curves developed by Coleman [Col05] is encoded in the notion of expansion factors and harmonicity of maps, as recalled in Section 4.3.4. The approach taken by Coleman is roughly equivalent to ours, and should extend to any tame level N.

Terminology. For any supersingular elliptic curve E_σ with $\Gamma_0(N)$ structure, we obtain components $Z, Z_1^\sigma, Z_2^\sigma$ on the objects of $\mathfrak{U}_p$. An elliptic curve over $\mathbf{C}_p$ is said to be *too supersingular* if it does not have a canonical subgroup as in [Kat73] and [Lub79]. It is *nearly too supersingular* if it is p-isogenous to a too supersingular curve.

The following corollary of Theorem D provides a moduli interpretation for the components of $\mathfrak{U}_p$ lying over the supersingular points of the Katz–Mazur models. We remark that the ordinary components of the special fibre of $\mathfrak{U}_p$ arise as strict transforms of components in the Katz–Mazur models, or normalisations thereof, and hence carry a natural moduli interpretation. What remains is to find a good moduli interpretation of the supersingular components, which is the subject of the following result, which was proved in tame level 1 by Coleman [Col05].

Theorem E. *Let $p \geq 5$ coprime to N. The points on $X_0(Np)$ reducing to a smooth point of some Z_1, respectively Z_2, parametrise too supersingular curves, respectively nearly too supersingular curves, with $\Gamma_0(N)$-level structure.*

Proof. For an open annulus A with outer radius 1 and inner radius r, the open subannulus $A_s := \{|.|_x : 1 \geq |T|_x \geq s\}$ for $s \geq r$ is independent of the chosen parameter T for A. Letting E_{p-1} be a global parameter for the supersingular annuli in $X_0(Np)$, lifting the Hasse invariant, we see that the smooth points of Z_1 and Z_2 classify those elliptic curves over $\mathscr{O}_{\mathbf{C}_p}$ with Hasse invariants $p/(p+1)$ and $1/(p+1)$ respectively. By the work of Katz [Kat73] and Lubin [Lub79], these are exactly the too supersingular, resp. nearly too supersingular, elliptic curves. $\square$

5.2 Overconvergent p-adic modular forms

We now discuss some consequences for the theory of overconvergent p-adic modular forms. We generalise the work of Goren–Kassaei [GK06] on canonical subgroups for a general class of curves, including Shimura curves over totally real fields, and make a number of remarks about the work of Kassaei [Kas09] on analytic continuation and generalisations of our results to higher dimensions.

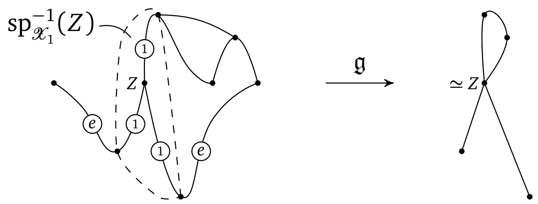
We will revisit two of the main results of the theory of overconvergent forms: canonical subgroups and analytic continuation. Over $\mathrm{GL}_{2,\mathbf{Q}}$ these results were proved using the moduli interpretation of modular curves. The work of Goren–Kassaei [GK06] and Kassaei [Kas09] shifted away from moduli interpretations, towards an approach focussed instead on the geometry of the curves involved. This approach has proved extremely useful for subsequent generalisations to other reductive groups, most notably the theory of overconvergent Hilbert modular forms [GK09, GK12].

5.2.1 Canonical subgroups for general curves.

We now generalise a result of Goren–Kassaei [GK06]. Set K to be a field complete with respect to a non-trivial non-Archimedean valuation $|\cdot|$ of rank 1, with valuation ring R. Let $g : \mathscr{X} \to \mathscr{Y}$ be a finite flat morphism of degree $e+1$ between semi-stable curves over R, and let $\mathfrak{g} : \mathscr{X}_1 \to \mathscr{Y}_1$ be the minimal skeletal semi-stable model of g dominating $\mathscr{X}$ and $\mathscr{Y}$. We may now prove the following generalisation of [GK06, Theorem 3.9].

Theorem F. *If g is an isomorphism on a component Z of $\mathscr{X}_s$, then g is an isomorphism on $\mathrm{sp}_{\mathscr{X}_1}^{-1}(Z)$.*

Proof. Every edge adjacent to Z in the skeleton of $\mathfrak{g}$ must have expansion factor 1, as this is the ramification index of the corresponding point on Z. We obtain the following picture: Following Goren–Kassaei [GK06, Proposition 3.1], the section $\mathfrak{g}(Z) \to Z$ gives rise to a section



$s : \mathrm{sp}^{-1}(\mathfrak{g}(Z)^{\mathrm{sm}}) \to X^{\mathrm{ad}}$ on the tube of the smooth locus $\mathfrak{g}(Z)^{\mathrm{sm}}$, as R is complete and therefore Henselian. By Coleman–Gouvêa–Jochnowitz [CGJ95, Lemma 6], we may extend s to a section $s^\dagger : U \to X^{\mathrm{ad}}$ for $U \subset Y^{\mathrm{ad}}$ an open subspace strictly containing the domain of s. As the expansion factors of the edges in the skeleton of $\mathscr{X}_1$ adjacent to Z are all 1, we see that g defines an isomorphism between the wide open $V = \mathrm{sp}_{\mathscr{X}_1}^{-1}(Z^{\mathrm{sing}})$ and its image. Its inverse agrees with $s^\dagger$ on $U \cap \mathfrak{g}(V)$, and hence we may glue it together with $s^\dagger$ to produce the desired section

$$s : U \cup \mathfrak{g}(V) = \mathrm{sp}_{\mathscr{Y}_1}^{-1}(\mathfrak{g}(Z)) \to \mathrm{sp}_{\mathscr{X}_1}^{-1}(Z).$$

□

Remark 5.1. The value of Theorem F lies in the fact that it allows one to identify the maximal region over which a section to g^{ad} exists, in terms of the skeletal stable model of the map g. A special case was treated in Goren–Kassaei [GK06, Theorem 3.9], where $\mathscr{X}_s$ was assumed to be composed of two components, intersecting transversally, $\mathscr{Y}$ was assumed to have good reduction, and K was a finite extension of $\mathbf{Q}_p$. These assumptions are satisfied in the case of Shimura curves $X_0^B(\mathfrak{p}) \to X^B$ and related unitary Shimura varieties. Under their assumptions all the expansion factors adjacent to Z are 1, whereas the others are e by harmonicity of the skeleton of $\mathfrak{g}$. This shows that g^{ad} maps $\mathrm{sp}_{\mathscr{X}_1}^{-1}(Z)$ isomorphically to the adic space attached to the rigid subspace $\mathfrak{Y}_{\mathrm{rig}}[0, e/(e+1))$ defined by Goren–Kassaei [GK06, Theorem A], and we recover their result.

Analytic continuation. One crucial ingredient in the theory of overconvergent modular forms which seems hard to generalise to other settings is Coleman's classicality criterion. It guarantees that an overconvergent modular eigenform of weight k is in fact a classical modular form if the p-adic valuation of its U_p-eigenvalue is less than $k-1$. Coleman [Col96] initially proved this criterion cohomologically, an approach which is taken by Johansson [Joh13] but which seems difficult to generalise further. Buzzard [Buz03] and Kassaei [Kas06] develop an approach which involves p-adically continuing an overconvergent modular form to the entire supersingular locus, and gluing it together with an appropriately constructed form on a complementary region arising from the Atkin operator. Kassaei [Kas09] proves analytic continuation results following this strategy, but only referring to the underlying geometry of the U_p operator. His results may naturally be viewed in the framework of semi-stable models for correspondences developed in this paper.

Higher dimensions. It is natural to wonder whether Theorem A may be generalised to correspondences between varieties of arbitrary dimension. If X is a smooth and proper variety over a non-Archimedean valued field, we may attach to any semi-stable formal model $\mathfrak{X}$ a skeleton of X^{ad}, and one might hope to be able to set up a similar framework. There are many difficulties to overcome, but Goren-Kassaei [GK09, GK12] prove the existence of canonical subgroups over Hilbert modular varieties using methods that seem to be close in spirit to our skeletal approach. This suggests that some form of generalisation for correspondences in higher dimensions might exist.

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