Categorical bialgebras arising from 2-Segal spaces



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Abstract

Topological field theory (TFT) is the study of representations of the cobordism category of manifolds. Such representations are often constructed from purely algebraic inputs. For example, 2d TFTs are constructed from commutative Frobenius algebras. Categories of representations of finite dimensional involutory Hopf algebras and quantum groups at roots of unity define 3d TFTs. In general, the algebraic inputs for 4d TFTs are much less well understood. Crane–Frenkel proposed a construction of a 4d TFT from a *bimonoidal category*, that is, a category equipped with both a monoidal and comonoidal structure such that the data defining the comonoidal structure is itself monoidal. Said another way, a bimonoidal category is a *categorification* of a bialgebra. This proposal has been hugely influential, essentially kick-starting the whole field of categorification. Despite this, it was not until the work of Gal–Gal almost 20 years later that the first non-trivial examples of bimonoidal categories appeared.

A rich source of bialgebraic structures, including the (positive part of) quantum groups, are the *Hall algebras* of finitary abelian categories. A new perspective, introduced independently by Dyckerhoff–Kapranov and Gálvez-Carrillo–Kock–Tonks, separates the construction of the Hall algebra of an abelian category A into three steps. In the first step one associates to A its Waldhausen S-construction: a simplicial groupoid $S_\bullet(A)$ with $S_n(A)$ the groupoid of length n flags in A. The second stop associates to the simplicial groupoid $S_\bullet(A)$ its *universal Hall algebra*, an algebra in the category of spans of groupoids whose underlying groupoid is the groupoid of objects of A. In the final step this algebra is *linearised* by passing to the finitely supported functions on the isomorphism classes of objects of A and implementing the product via a pull-push construction.

This new perspective on Hall algebras opens up the possibility of vastly generalising the construction, as was realised by both sets of authors. The associativity of the product on the universal Hall algebra of A was a direct consequence of $S_\bullet(A)$ being a particular example of a 2-*Segal groupoid*, a simplicial groupoid satisfying a 2-dimensional analogue of the Segal conditions. Both sets of authors define the universal Hall algebra for a general 2-Segal space. A second, independent direction of generalisation is the freedom of choosing alternate linearisation procedures.

We exploit these possibilities to construct new examples of bimonoidal categories. The main contribution of this thesis is the definition of the *Hall bimonoidal category*, $\mathfrak{H}(A)$, of a *bifinitary* abelian category A. On a purely formal level, the universal Hall algebra can be dualised to yield the *universal Hall coalgebra*, a coassociative coproduct on the groupoid of objects of A. Our first step in the definition of the Hall bimonoidal category of A is to extend the universal Hall algebra and coalgebra of A to a *universal Hall bialgebra*. We show that the product and coproduct are *laxly compatible*, endowing the groupoid of objects of A with the structure of a *lax bialgebra* as an object in the 2-category of iterated spans of groupoids. In fact, we construct the universal Hall bialgebra on the level of generality of so-called *double* 2*-Segal spaces*, which are bisimplicial spaces $\mathcal{X}_{\bullet,\bullet}$ such that the simplicial spaces $\mathcal{X}_{\bullet,1}$ and $\mathcal{X}_{1,\bullet}$ are 2-Segal spaces.

The Hall bimonoidal category of A is obtained by applying a *categorical linearisation*, due to Morton, to the universal Hall bialgebra. As a result, the monoidal and comonoidal structures on the universal Hall bimonoidal category $\mathfrak{H}(A)$ are, in general, only laxly compatible. We prove that $\mathfrak{H}(A)$ is a bimonoidal category if and only if A is semisimple. In particular, taking $A = \mathrm{Vect}_{\mathbb{F}_q}$ we obtain a bimonoidal structure on the category of *representations of the general linear groupoid* extending the monoidal structure introduced by Joyal–Street. Furthermore, the bimonoidal category $\mathfrak{H}(\mathrm{Vect}_{\mathbb{F}_q})$ is a categorification of one of the primary examples of Zelevinsky's *positive, self-adjoint Hopf algebras*.

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Chapter 1

Introduction

1.1 Background and motivation

1.1.1 4d topological field theories and bimonoidal categories

Originating in physics as the study of quantum systems having no local degrees of freedom, in mathematics *topological field theory (TFT)* is the study of representations of the cobordism category of manifolds. Since manifolds can be decomposed along submanifolds of arbitrary codimension, such a representation is inherently higher categorical in nature. Instead of merely considering representations on vector spaces (as is the case in the representation theory of groups, algebras etc.), for n-dimensional topological field theories one considers representations on k-categorical analogues of vector spaces for $k = 1, \ldots, n$. More precisely, an *n-dimensional TFT of codimension k* is a symmetric monoidal functor

$$Z : \mathrm{Bord}_{n,k} \to \mathcal{C}, \tag{1.1.1}$$

where $\mathrm{Bord}_{n,k}$ is the symmetric monoidal (∞, k)-category having as objects $(n-k)$-dimensional closed oriented manifolds, as 1-morphisms $(n-(k-1))$-dimensional oriented bordisms and so on, and $\mathcal{C}$ is some symmetric monoidal (∞, k)-category of k-categorical vector spaces [2, 48, 55, 65]. For example, for $k = 2$ one could take $\mathcal{C}$ to be the 2-category of linear categories, or for $k = 3$ the 3-category of tensor categories.

The representation theory of 3-dimensional cobordisms, that is, 3d TFT, is intimately connected with *bialgebras*. The exact data necessary to define a codimension 2, 3d TFT is a so-called modular tensor category [4], and the categories of representations of quantum groups at roots of unity provide a rich source of modular tensor categories [58]. Finite dimensional, involutory Hopf algebras define 3d TFTs of codimension 3 [3].

Based on the fundamental role played by bialgebras in 3d TFT, Crane–Frenkel proposed that the algebraic data needed to define 4d TFTs would be a categorification of a bialgebra, that is, a *bimonoidal category* [11]. More precisely, a bimonoidal category is a linear category carrying both a monoidal and comonoidal structure such that the functors and natural transformations defining the

comonoidal structure are themselves monoidal. Their proposal is best understood through the lens of the *Cobordism Hypothesis* which, in one form, asserts that an n-dimensional TFT of codimension n is determined by the object it assigns to a point [2, 48]. From this perspective, one can reinterpret the Crane–Frenkel proposal as the suggestion that bimonoidal categories are naturally the objects of a symmetric monoidal $(\infty, 4)$-category.

The rigorous construction of such an $(\infty, 4)$-category is within reach given current technology but has not yet been carried out. The main tool is due to Johnson-Freyd–Scheimbauer [36]. They construct from a symmetric monoidal (∞, n)-category a symmetric monoidal $(\infty, n + k)$-category of E_k-algebra objects. Starting with the symmetric monoidal 2-category of linear categories and applying their construction twice with $k = 1$ one obtains a symmetric monoidal $(\infty, 4)$-category. An object of this $(\infty, 4)$-category is a tensor category H equipped with a second 'product' implemented by an $(H^{\boxtimes 2}, H)$-bimodule category. Every bimonoidal category H defines such an object by making $H^{\boxtimes 2}$ into an $(H^{\boxtimes 2}, H)$-bimodule category, where the left action is induced from the monoidal structure on H and the right action from the comonoidal structure.

Even taking for granted the rigorous construction of this symmetric monoidal $(\infty, 4)$-category of bimonoidal categories there is considerably more work that must be done to define 4d TFTs from them[1]. Nonetheless, this provides strong motivation to study these algebraic objects, and before undertaking a systematic study of bimonoidal categories one would like to have a number of examples. Despite having been introduced by Crane–Frenkel over 20 years ago, to our knowledge the first non-trivial examples of bimonoidal categories were the categories of *equivariant polynomial functors* recently constructed by Gal–Gal [22].

A rich source of bialgebraic structures, including the positive part of quantum groups, are the *Hall algebras* of finitary abelian categories. Recent works by Dyckerhoff–Kapranov [17] and Gálvez-Carrillo–Kock–Tonks [23, 27] put forward a new perspective on the Hall construction which can be applied fruitfully to the construction of new examples of bimonoidal categories.

1.1.2 2-Segal spaces and Hall algebras

A classical construction associates to a suitably finite abelian category A its *Hall algebra*, $H(A)$, which serves to encode the structure of extensions in A [32, 57, 59]. More precisely, $H(A)$ is the algebra having as basis the isomorphism classes of objects in A, denoted $\pi_0(A)$, and product given by

$$[a] \cdot [c] = \sum_{[b] \in \pi_0(A)} \frac{\# \left\{ 0 \to a \hookrightarrow b \twoheadrightarrow c \to 0 \right\}}{|\mathrm{Aut}(a)||\mathrm{Aut}(c)|} [b].$$

[1] As per Lurie's formulation of the Cobordism Hypothesis, one must compute the homotopy fixed point space of the $SO(4)$-action on the ∞-groupoid of *fully dualisable* bimonoidal categories.

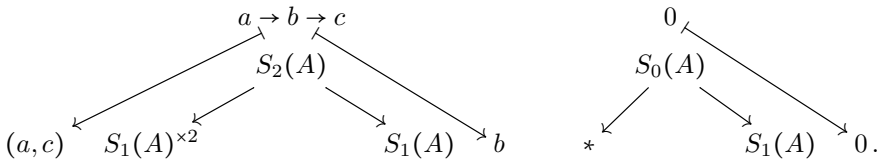
The vector space $H(A)$ can also be endowed with a coproduct given by

$$\Delta([b]) = \sum_{[a],[c]\in\pi_0(A)} \frac{\#\left\{ 0 \longrightarrow a \hookrightarrow b \twoheadrightarrow c \longrightarrow 0 \right\}}{|\mathrm{Aut}(b)|} [a] \otimes [c].$$

Green's theorem asserts that $H(A)$ with the above defined product and coproduct is, upon twisting the product on $H(A)^{\otimes 2}$, a bialgebra [31].

Dyckerhoff–Kapranov [17] and Gálvez-Carrillo–Kock–Tonks [23, 27] have recently introduced a new perspective on the Hall construction of an abelian category A. They emphasis that it is a *three step process*: One first applies the *Waldhausen S-construction* to A, yielding a simplicial groupoid $S_\bullet(A)$ with $S_n(A)$ the groupoid of length n flags in A [67]. In particular, $S_0(A)$ is trivial, $S_1(A)$ is obtained from A by discarding all non-invertible morphisms, and $S_2(A)$ is the groupoid of short exact sequences in A.

In the second step, one constructs from the simplicial groupoid $S_\bullet(A)$ the *universal Hall algebra* of A: An algebra in the *category of spans* of groupoids whose underlying groupoid is $S_1(A)$ with product and unit given, respectively, by the spans



(1.1.2)

Dually, these diagrams can be read from right to left, endowing $S_1(A)$ with a coproduct and counit. We call this coalgebra in the category of spans of groupoids the *universal Hall coalgebra* of A.

Finally, the third step is to obtain the Hall algebra of A as a *linearisation* of the universal Hall algebra of A in the following sense. The vector space $H(A)$ is the vector space of finitely supported, complex valued functions on $\pi_0(S_1(A))$. The diagrams in Eq. 1.1.2 define the product and unit of $H(A)$ by pulling back along the left morphisms and pushing forward along the right. Dually, pulling back along the right morphisms and pushing forward along the left defines the coproduct and counit. Another way to say this is that the pull-push construction defines a symmetric monoidal functor from the category of spans of groupoids to the category of vector spaces. The Hall algebra and Hall coalgebra of A are the images under this functor of, respectively, the universal Hall algebra and universal Hall coalgebra of A.

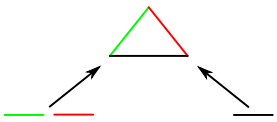
Remark 1.1.1. As explained by Gálvez-Carrillo–Kock–Tonks ([27] 2), there is a precise sense in which the category of spans of groupoids should be considered to be linear. Namely, for a groupoid $\mathbb{G}$ the slice category $\mathrm{Grpd}_{/\mathbb{G}}$ is the homotopy-sum-completion of $\mathbb{G}$. The category whose objects are slice categories $\mathrm{Grpd}_{/\mathbb{G}}$ and whose morphisms are homotopy-sum-preserving functors is equivalent to the category of spans of groupoids. From this perspective *algebraisation* is a more appropriate term than linearisation. Nonetheless, we will follow Morton [53] and use the latter terminology.

The key observation made by both Dyckerhoff–Kapranov and Gálvez-Carrillo–Kock–Tonks is that the only property of the simplicial groupoid $S_\bullet(A)$ used in the construction of the universal Hall algebra of A was that it was a 2-*Segal space*: a simplicial space satisfying a 2-dimensional analogue of the Segal conditions. Rather than state the condition directly, let us instead consider how the space of 1-simplices $\mathcal{X}_1$ of a simplicial space $\mathcal{X}_\bullet$ could carry an associative product, or coassociative coproduct, in the category of spans of spaces.

Recall that a simplicial space $\mathcal{X}_\bullet$ can be evaluated on arbitrary simplicial sets X,

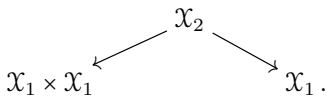
$$\mathcal{X}_X \coloneqq \operatorname*{holim}_{\Delta[n]\to X} \mathcal{X}_n.$$

From the following diagram of simplicial sets



(1.1.3)

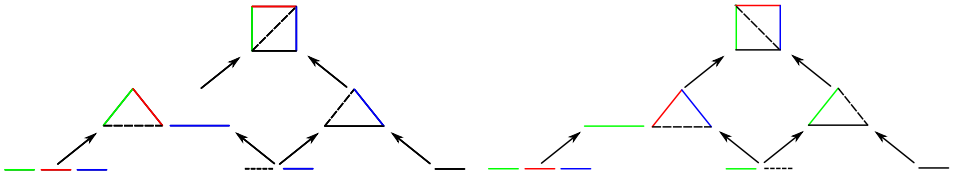
one obtains a diagram



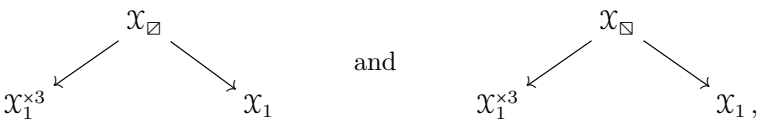
(1.1.4)

This defines morphisms μ from $\mathcal{X}_1 \times \mathcal{X}_1$ to $\mathcal{X}_1$ and Δ from $\mathcal{X}_1$ to $\mathcal{X}_1 \times \mathcal{X}_1$ in the category of spans of spaces which are candidates for a product and coproduct on $\mathcal{X}_1$.

The key question now is whether or not μ is associative, or equivalently, whether Δ is coassociative. Denote by $\boxslash$ and $\boxbslash$ the simplicial sets making the middle squares in the following two diagrams pushouts:

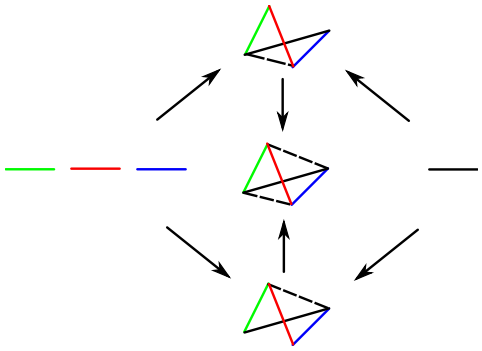


The spans



are, respectively, the morphisms $\mu(\mu \times \mathrm{Id})$ and $\mu(\mathrm{Id} \times \mu)$. As $\boxslash$ and $\boxbslash$ are non-isomorphic simplicial sets it is not necessarily the case that $\mathcal{X}_{\boxslash}$ and $\mathcal{X}_{\boxbslash}$ are equivalent spaces. Therefore, for an arbitrary simplicial space $\mathcal{X}_\bullet$ it is not necessarily the case that the product μ defined above is associative.

It is here that the 2-Segal condition comes in. Observe that both $\boxslash$ and $\boxbslash$ are canonically sub-simplicial sets of $\Delta[3]$,



This yields the following diagram in spaces

$$\begin{array}{ccccc} & & \mathcal{X}_{\boxslash} & & \\ & \swarrow & \uparrow & \searrow & \\ \mathcal{X}_1^{\times 3} & \longleftarrow & \mathcal{X}_3 & \longrightarrow & \mathcal{X}_1\,. \\ & \nwarrow & \downarrow & \nearrow & \\ & & \mathcal{X}_{\boxbslash} & & \end{array} \tag{1.1.5}$$

The 2-Segal condition requires, in particular, that the two vertical maps in this diagram are equivalences, and hence, that the product μ is associative up to equivalence. Even more, the 2-Segal condition ensures that μ is associative up to coherent homotopy ([17] 11.1.19, [23] 7.1).

This new perspective offers two independent directions to generalise the Hall construction. In the first direction, there are numerous examples of 2-Segal spaces which are not the Waldhausen S-construction of an abelian category. Every Segal space is a 2-Segal space ([17] 2.5.3, [23] 3.5), a fact which has been exploited by Gálvez-Carrillo–Kock–Tonks [26] to realise many (co)algebras in combinatorics [37] as instances of the Hall construction. Dyckerhoff–Kapranov have found examples of 2-Segal spaces of geometric origin [17, 18]. In the second direction one can vary the linearisation procedure, which is explored in some depth by Dyckerhoff–Kapranov ([17], 8) and Walde [66].

We exploit both of these directions in what follows. Going further, we expand upon the second step by defining not just a universal Hall (co)algebra of an abelian category, but a *universal Hall bialgebra*.

1.2 Outline

The main contribution of this thesis is the construction of the *Hall bimonoidal category*, $\mathfrak{H}(A)$, of a *bifinitary* abelian category A. This is accomplished in two steps. We first construct the *universal Hall bialgebra* of A, a *lax bialgebra* in the $(\infty, 2)$-category of *bispans of spaces*. We have already seen that $S_1(A)$, the groupoid of objects in A, carries both a unital product μ and a counital coproduct

Δ given by the spans in Eq. 1.1.2. A natural question to ask is whether or not $(S_1(A), \mu, \Delta)$ is a bialgebra, or in particular, whether or not it is the case that

$$\Delta\mu \simeq \overline{\mu}^2\Delta^2.$$

The operations $\Delta\mu$ and $\overline{\mu}^2\Delta^2$ are implemented, respectively, by the spans

$$\begin{array}{ccc} & S_+(A) & \\ \swarrow & & \searrow \\ S_1(A)^2 & & S_1(A)^2 \end{array} \qquad \text{and} \qquad \begin{array}{ccc} & S_\square(A) & \\ \swarrow & & \searrow \\ S_1(A)^2 & & S_1(A)^2 \end{array},$$

where $S_+(A)$ and $S_\square(A)$ are, respectively, the groupoids of diagrams in A of the form

$$\begin{array}{ccccc} & & b & & \\ & & \downarrow & & \\ a' & \longrightarrow & b' & \longrightarrow & c' \\ & & \downarrow & & \\ & & b'' & & \end{array} \qquad \text{and} \qquad \begin{array}{ccccc} a & \longrightarrow & b & \longrightarrow & c \\ \downarrow & & & & \downarrow \\ a' & & & & c' \\ \downarrow & & & & \downarrow \\ a'' & \longrightarrow & b'' & \longrightarrow & c'' \end{array}$$

having each row and column a short exact sequence. To investigate the compatibility of μ and Δ, consider the diagram

$$\begin{array}{ccccc} & & S_+(A) & & \\ & \swarrow & \uparrow & \searrow & \\ S_1(A)^2 & \longleftarrow & S_{2,2}(A) & \longrightarrow & S_1(A)^2, \\ & \nwarrow & \downarrow & \nearrow & \\ & & S_\square(A) & & \end{array} \tag{1.2.1}$$

where $S_{2,2}(A)$ is the groupoid of diagrams in A of the form

$$\begin{array}{ccccc} a & \longrightarrow & b & \longrightarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ a' & \longrightarrow & b' & \longrightarrow & c' \\ \downarrow & & \downarrow & & \downarrow \\ a'' & \longrightarrow & b'' & \longrightarrow & c'' \end{array}$$

having each row and column a short exact sequence. The morphism $S_{2,2}(A) \to S_+(A)$ is an equivalence since every cross of short exact sequences can be completed to a 3x3 grid of short exact sequences in an essentially unique way using pullbacks, pushouts, kernels and cokernels. Unfortunately, the morphism $S_{2,2}(A) \to S_\square(A)$ is an equivalence if and only if A is the trivial abelian category ([16] 2.38).

We are therefore forced to conclude that $(S_1(A), \mu, \Delta)$ *is not*, in general, a bialgebra. Nonetheless, the diagram in Eq. 1.2.1 *does* define a non-invertible 2-morphism

$$\Delta\mu \Rightarrow \overline{\mu}^2\Delta^2$$

in the $(\infty, 2)$-category of *bispans of spaces* [33]. In other words, the 2-morphism $\Delta\mu \Rightarrow \overline{\mu}^2\Delta^2$ is a witness for the *lax compatibility* of the universal Hall algebra and coalgebra of A. We show that the universal Hall algebra and coalgebra, along with this 2-morphism, form a *lax bialgebra object* in the $(\infty, 2)$-category of bispans of spaces. We call this the universal Hall bialgebra of A.

In fact, we obtain the universal Hall bialgebra of an abelian category as a special case of a more general construction. By iterating the Waldausen S-construction, every abelian category A defines a bisimplicial space $S_{\bullet,\bullet}(A)$, where $S_{n,k}(A)$ is the groupoid of length n flags of length k flags in A. This is a particular example of a *double* 2*-Segal space*: A bisimplicial space $\mathcal{X}_{\bullet,\bullet}$ such that $\mathcal{X}_{\bullet,1}$ and $\mathcal{X}_{1,\bullet}$ are 2-Segal spaces. We show that the universal Hall algebra of $\mathcal{X}_{\bullet,1}$, the universal Hall coalgebra of $\mathcal{X}_{1,\bullet}$, and a 2-morphism analogous to the one in Eq. 1.2.1 define a lax bialgebra, the universal Hall bialgebra of the double 2-Segal space $\mathcal{X}_{\bullet,\bullet}$.

The second step in the construction of the Hall bimonoidal category of an abelian category A is to apply a *categorical linearisation* to the universal Hall bialgebra of A. Recall that the linearisation used above to define the ordinary Hall algebra of A was given in terms of a symmetric monoidal functor from the category of spans in spaces to the category of vector spaces. Since the universal Hall bialgebra is an inherently 2-categorical object the category of vector spaces is not, a priori, rich enough to detect this structure. From this perspective, the twisting of the product on $H(A)^{\otimes 2}$ in Green's theorem should be understood as an attempt to correct this defect of the category of vector spaces.

A linearisation which more faithfully preserves the universal Hall bialgebra of A would also be 2-categorical. A natural categorification of the vector space of functions on the isomorphism classes of objects of a groupoid is its linear category of representations. Therefore one might search for a symmetric monoidal functor from the 2-category of bispans of groupoids to the 2-category of linear categories, which, on objects, sends a groupoid to its category of representations. A functor of this sort has already been defined by Morton [53], and we define the Hall bimonoidal category of an abelian category to be the image of its universal Hall bialgebra under this functor.

As a result, the monoidal and comonoidal structures on $\mathfrak{H}(A)$ are, in general, only laxly compatible. We prove that $\mathfrak{H}(A)$ is a bimonoidal category if and only if A is semisimple. In particular, taking $A = \mathrm{Vect}_{\mathbb{F}_q}$ we obtain a bimonoidal structure on the category of *representations of the general linear groupoid* extending the monoidal structure introduced by Joyal–Street [41]. Further, the bimonoidal category $\mathfrak{H}(\mathrm{Vect}_{\mathbb{F}_q})$ is a categorification of one of the primary examples of Zelevinsky's *positive, self-adjoint Hopf algebras* [69].

Having given a general overview of the contents of this thesis, we now give a detailed breakdown of the content found in each chapter.

1.2.1 Background on $(\infty, 2)$-categories

In the preceding discussion we introduced two particular cases of a *lax bialgebra object* in a symmetric monoidal $(\infty, 2)$-category, one in the $(\infty, 2)$-category of bispans and the other in the 2-category of linear categories. The purpose of Chapter 2 is to develop the technical background necessary to give precise definitions of these notions.

Let us consider first the notion of a bialgebra object in a symmetric monoidal ∞-category. Naively, this should be an object equipped with a product and coproduct along with an elaborate system of higher morphisms coherently witnessing their compatibility. As it is completely infeasible to specify this data by hand it is necessary to find a formalism which efficiently encodes the same information. For bialgebras in ordinary symmetric monoidal categories this has been accomplished by Pirashvili [54]. He introduced a symmetric monoidal category Bialg which corepresents bialgebras in the sense that the category of symmetric monoidal functors from Bialg into a given symmetric monoidal category C is equivalent to the category of bialgebra objects in C. Said another way, Bialg is the PROP for bialgebras. Contained within Bialg are two wide subcategories Alg and Coalg, which, respectively, corepresent algebras and coalgebras. A straightforward extension of this formalism provides a definition of (co)algebra and bialgebra objects in symmetric monoidal ∞-categories, and indeed symmetric monoidal $(\infty, 2)$-categories.

As the informal description of a lax bialgebra given above explicitly makes use of non-invertible 2-morphisms it should come as no surprise that to define a lax bialgebra object in a symmetric monoidal $(\infty, 2)$-category one must make use of ideas which are inherently $(\infty, 2)$-categorical. Specifically, one needs *lax functors* between $(\infty, 2)$-categories. Recall that a lax functor between ordinary 2-categories differs from a usual functor in that the 2-morphisms which witness the preservation of units and composition are not required to be invertible. Fortunately, in the particular model of $(\infty, 2)$-categories that we use in this thesis there already exists in the literature a definition of lax functors via the *unstraightening construction*, the ∞-categorical generalisation of the ordinary Grothendieck construction. A lax bialgebra object in a symmetric monoidal $(\infty, 2)$-category is then defined to be a symmetric monoidal lax functor out of Bialg such that the induced (co)algebra structures are (co)associative.

The universal Hall bialgebra of a double 2-Segal space will be a lax bialgebra object in the symmetric monoidal $(\infty, 2)$-category of bispans of spaces. One of the technical hurdles that one must surmount when working with lax functors of $(\infty, 2)$-categories is the difficulty of providing an explicit description of the unstraightening construction. As such, we devote a considerable part of this chapter to the unstraightening of the $(\infty, 2)$-category of bispans, rendering it into a form which is adequate for our purposes in the remaining chapters.

Chapter 2 is structured as follows. Section 2.1 simply reviews our chosen model of symmetric monoidal $(\infty, 2)$-categories and their associated homotopy bicategories. Section 2.2 is similarly a revision, this time of the definition of symmetric monoidal lax functors. Section 2.3 is a brief discussion on the twisted arrow construction, a construction which appears throughout this work. All of the necessary material for the $(\infty, 2)$-category of bispans is covered Section 2.4, beginning with a review of Haugseng's original definition in Section 2.4.1. Sections 2.4.2 and 2.4.3 contain technical material culminating in a description of the unstraightening of the $(\infty, 2)$-category of

bispans. Finally, Section 2.5 provides the details of the formalism sketched above for defining lax (co)algebra and bialgebra objects.

1.2.2 The universal Hall algebra

This chapter concerns the universal Hall (co)algebra of a 2-Segal space, which has already been constructed independently by Dyckerhoff–Kapranov [17] and Gálvez-Carrillo–Kock–Tonks [23]. We give a third construction of the universal Hall (co)algebra, and do so for two reasons. Firstly, our true aim is the universal Hall bialgebra of a double 2-Segal space, the construction of which will be a natural extension of the construction of the universal Hall (co)algebra given in this chapter. Secondly, our new construction will serve to elucidate the exact role played by the 2-Segal condition.

There are two key observations we should take from the informal discussion of the 2-Segal condition in Section 1.1.2. First, we saw in Eq. 1.1.4 that for an arbitrary simplicial space $\mathcal{X}_\bullet$, the space $\mathcal{X}_1$ carries a product μ and coproduct Δ. While they failed to be (co)associative in general, the diagram in Eq. 1.1.5 gains new meaning in an $(\infty,2)$-categorical context: it defines a 2-morphism between $\mu(\mu\times\mathrm{Id})$ and $\mu(\mathrm{Id}\times\mu)$ (or $\Delta(\Delta\times\mathrm{Id})$ and $\Delta(\mathrm{Id}\times\Delta)$) in the $(\infty,2)$-category of bispans of spaces. Summarising Theorems 3.2.9 and 3.3.1 we show the following:

Theorem 1. *Every simplicial space $\mathcal{X}_\bullet$ endows the space $\mathcal{X}_1$ with the structure of a lax (co)algebra in the $(\infty,2)$-category of bispans of spaces. Furthermore, $\mathcal{X}_\bullet$ is a 2-Segal space if and only if it endows $\mathcal{X}_1$ with the structure of a (co)algebra. We call this the* universal Hall (co)algebra *of $\mathcal{X}_\bullet$.*

Said another way, the 2-Segal condition is *precisely* the condition one needs to impose on a simplicial space to ensure (co)associativity of its universal Hall (co)algebra.

The second observation, which hints at the approach we take towards the construction of the universal Hall (co)algebra of a 2-Segal space, is that all of the diagrams used to define the lax (co)algebra arose from diagrams of simplicial sets. This is no coincidence, as the Yoneda embedding defines a simplicial object $\Delta[\bullet]$ in the opposite of the category of simplicial sets. There is a sense in which $\Delta[\bullet]$ is the initial simplicial object, and we leverage this to provide a purely combinatorial construction of the lax (co)algebra structure arising from a simplicial space. We endow $\Delta[1]$ with the structure of a lax (co)algebra as an object of the $(\infty,2)$-category bispans in the opposite of the category of simplicial sets. Pushing this structure into spaces using $\mathcal{X}_\bullet$ yields the first statement of Theorem 1. In fact, the category of spaces plays no special role here, and our proof applies to a simplicial object in an arbitrary ∞-category having finite limits.

Remark 1.2.1. The sense in which $\Delta[\bullet]$ is the initial simplicial object is the following. Consider the category of pairs $(C, X_\bullet)$ with C a category having finite limits and $X_\bullet$ a simplicial object in C. A morphism $(C, X_\bullet)\to(C', X'_\bullet)$ is a finite limit preserving functor $C\to C'$ sending $X_\bullet$ to $X'_\bullet$. The initial object in this category is $((\mathrm{Fin}_\Delta)^{\mathrm{op}}, \Delta[\bullet])$, where Fin_Δ is the category of levelwise finite simplicial sets. The unique morphism to any $(C, X_\bullet)$ is given by right Kan extension along $X_\bullet$.

Chapter 3 begins, in Section 3.1, with a review of the definition of 2-Segal spaces as well as a discussion of the 2-Segal spaces $S_\bullet(A)$ for A a *proto-exact category*. A proto-exact category is a non-additive generalisation of an abelian category ([17] 2.4.2). Section 3.2 proves the first part of Theorem 1, that is, the construction of the lax (co)algebra of a simplicial space. The second part of Theorem 1, the equivalence between the 2-Segal condition and (co)associativity of the universal Hall (co)algebra, is proven in Section 3.3. Finally, we carefully describe the truncation of the universal Hall (co)algebra to the homotopy bicategory of bispans in Section 3.4.

1.2.3 The universal Hall bialgebra

We saw in Theorem 1 that every simplicial space endows its space of 1-simplices with the structure of a lax (co)algebra in the $(\infty, 2)$-category of bispans of spaces.

In the special case of the bisimplicial space $S_{\bullet,\bullet}(A)$, the diagram in Eq. 1.2.1 provides a witness for the lax compatibility between the algebra and coalgebra structures. This suggests that for a general bisimplicial space $\mathcal{X}_{\bullet,\bullet}$, the space of $(1,1)$-simplices $\mathcal{X}_{1,1}$ is a *totally lax bialgebra* in the $(\infty, 2)$-category of bispans: A space carrying both the structures of a lax algebra and a lax coalgebra in the $(\infty, 2)$-category of bispans of which are laxly compatible. Formally, this is simply a symmetric monoidal lax functor from Bialg to the $(\infty, 2)$-category of bispans. The aim of this chapter is to prove the following.

Theorem 2. *Every bisimplicial space $\mathcal{X}_{\bullet,\bullet}$ endows the space $\mathcal{X}_{1,1}$ with the structure of a totally lax bialgebra in the $(\infty, 2)$-category of bispans of spaces. Furthermore, $\mathcal{X}_{\bullet,\bullet}$ is a double 2-Segal space if and only if it endows $\mathcal{X}_{1,1}$ with the structure of a lax bialgebra. We call this the* universal Hall bialgebra *of $\mathcal{X}_{\bullet,\bullet}$.*

The proof of this Theorem follows a similar structure to that of Theorem 1, but we instead exploit the fact that the initial bisimplicial object is the collection of standard bisimplices $\Delta[\bullet, \bullet]$ in the opposite of the category of bisimplicial sets. That is, we first endow $\Delta[1,1]$ with a totally lax bialgebra structure and push this structure into spaces using $\mathcal{X}_{\bullet,\bullet}$. The lax algebra and coalgebra structures on $\Delta[1,1]$ arise from the simplicial objects $\Delta[\bullet, 1]$ and $\Delta[1, \bullet]$. The second statement of Theorem 2 then follows from the second statement of Theorem 1.

Chapter 4 begins with Section 4.1, a short section introducing double 2-Segal spaces. The first part of Theorem 2 is proven in Section 4.2 while the second is proven in 4.3. Finally, we conclude the chapter with a thorough account of the truncation of the universal Hall bialgebra to the homotopy bicategory of bispans in Section 4.4.

1.2.4 The Hall bimonoidal category

As outlined above, one would like to obtain the Hall bimonoidal category of a double 2-Segal groupoid as a categorical linearisation of its universal Hall bialgebra. Such a categorical linearisation procedure

has already been defined by Morton [53] in the form of a symmetric monoidal functor from the bicategory of bispans in the category of *locally finite groupoids* and *finite functors* to the 2-category of linear categories. It is necessary to restrict to the class of *locally bifinitary* double 2-Segal groiupoids so that the universal Hall bialgebra is a lax bialgebra object in this bicategory. We therefore define the Hall bimonoidal category of a locally bifinitary double 2-Segal groupoid to be the image of its universal Hall bialgebra under Morton's linearisation functor.

As the universal Hall bialgebra of a locally bifinitary double 2-Segal groupoid is a lax bialgebra, its Hall bimonoidal category is, in general, only a lax bimonoidal category. We show that the Hall bimonoidal category of a bifinitary abelian category is bimonoidal if and only if the abelian category is semisimple. Furthermore, we show that bifinitary semisimple abelian categories are direct sums of $\mathrm{Vect}_{\mathbb{F}_q}$, the category of finite dimensional vector spaces over a finite field. The Hall bimonoidal category of $\mathrm{Vect}_{\mathbb{F}_q}$ is

$$\mathfrak{H}(\mathrm{Vect}_{\mathbb{F}_q}) = \bigoplus_{n\geq 0} \mathrm{Rep}(\mathrm{GL}_n(\mathbb{F}_q)),$$

with monoidal structure given by parabolic induction [30] and comonoidal structure given by the adjoint to parabolic induction. This is the category of representations of the general linear groupoid over a finite field and the monoidal structure we obtain agrees with the one introduced by Joyal–Street [41].

Outside of the abelian context one loses the tools of homological algebra making such a general statement harder to make. An interesting class of non-abelian proto-exact categories are those which are linear over $\mathbb{F}_1$, the 'field with one element'. One can think of a finite group Γ as a finite dimensional division algebra over $\mathbb{F}_1$, and so the categories of $\mathbb{F}_1$-modules over a finite group Γ, denoted Γ_0-$\mathrm{Mod}_{\mathbb{F}_1}$, are the $\mathbb{F}_1$ analogues of a semisimple abelian category with one simple object. We show that for each finite group Γ, the universal Hall bialgebra of Γ_0-$\mathrm{Mod}_{\mathbb{F}_1}$ is a bialgebra. The Hall bimonoidal category of Γ-$\mathrm{Mod}_{\mathbb{F}_1}$ is

$$\mathfrak{H}(\Gamma_0\text{-}\mathrm{Mod}_{\mathbb{F}_1}) = \bigoplus_{n\geq 0} \mathrm{Rep}(\Gamma \wr \Sigma_n)$$

with monoidal and comonoidal structures given by ordinary induction and restriction. It turns out these are equivalent to the bimonoidal categories of Γ-*equivariant polynomial functors* originally constructed by Gal–Gal [22].

The Grothendieck groups of the Hall bimonoidal categories of $\mathrm{Vect}_{\mathbb{F}_q}$ and Γ-$\mathrm{Mod}_{\mathbb{F}_1}$ are the primary examples of *positive self-adjoint Hopf (PSH) algebras* as introduced by Zelevinsky [69]. We have therefore shown the following.

Theorem 3. *The Hall bimonoidal categories of the categories* $\mathrm{Vect}_{\mathbb{F}_q}$ *and* Γ-$\mathrm{Mod}_{\mathbb{F}_1}$ *are categorifications of Zelevinsky's positive self-adjoint Hopf algebras.*

Chapter 5 consists of three sections. The first, Section 5.1, reviews Morton's categorical linearisation. The second, Section 5.2, introduces locally bifinite double 2-Segal spaces and defines their Hall bimonoidal categories. The thesis concludes with the proof of Theorem 3 in Section 5.3.

1.3 Notation conventions

Throughout this thesis we make extensive use of the theory of ∞-categories as developed by Joyal [38, 39] and Lurie [46]. In particular, by an ∞-category we shall always mean a *quasi-category*, a simplicial set having fillers for all inner horns.

Ordinary, simplicially-enriched or model categories will always be denoted by either Greek letters (e.g. Δ) or in ordinary font (e.g. C). ∞-categories will either be blackboard Greek letters (e.g. $\mathbb{\Delta}$) or have the first character in calligraphic font (e.g. $\mathcal{C}$).

For ordinary categories C and D, $\mathrm{Fun}(C,D)$ is the category of functors and natural transformations and $\mathrm{Map}(C,D)$ is the groupoid of functors and natural isomorphisms. Analogously, for ∞-categories $\mathcal{C}$ and $\mathcal{D}$, $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})$ is the ∞-category of functors and $\mathcal{M}\mathrm{ap}(\mathcal{C},\mathcal{D})$ is the largest Kan complex inside $\mathcal{F}\mathrm{un}(\mathcal{C},\mathcal{D})$. In particular, the ($\infty$-)category of *$k$-fold simplicial objects* in an ordinary category C or ∞-category $\mathcal{C}$ are

$$C_{\Delta^k} = \mathrm{Fun}\left((\Delta^{\mathrm{op}})^{\times k}, C\right) \text{ and } \mathcal{C}_{\mathbb{\Delta}^k} = \mathcal{F}\mathrm{un}\left((\mathbb{\Delta}^{\mathrm{op}})^{\times k}, \mathcal{C}\right),$$

where Δ is the category of non-empty linearly ordered finite sets and the ∞-category $\mathbb{\Delta}$ is the nerve of Δ.

Let qCat denote the simplicially-enriched category of quasi-categories with mapping spaces given by $\mathcal{M}\mathrm{ap}$. The *∞-category of ∞-categories*, $\mathcal{C}\mathrm{at}_\infty$, is the coherent nerve of qCat ([46] 3.0.0.1). The full subcategory of qCat of Kan complexes is denoted Kan, and its coherent nerve, $\mathcal{S}$, is the ∞-category of spaces. The inclusion $\mathcal{S} \hookrightarrow \mathcal{C}\mathrm{at}_\infty$ admits a right adjoint ([46] 1.2.5.3, 5.2.4.5) denoted

$$(-)^{\simeq} : \mathcal{C}\mathrm{at}_\infty \to \mathcal{S}.$$

We denote by CSS the category of bisimplicial sets carrying the Rezk model structure [56]. This is a simplicial model category whose fibrant-cofibrant objects are the *complete Segal spaces*. The coherent nerve of the full subcategory of complete Segal spaces, denoted $\mathcal{C}\mathrm{SS}$, is canonically equivalent to $\mathcal{C}\mathrm{at}_\infty$ ([42] 4.11).

The category of finite sets is denoted by Fin and every object is isomorphic to one of the form

$$\underline{n} = \{1, \cdots, n\} \in \mathrm{Fin}.$$

Similarly, Fin_* is the category of pointed finite sets. Objects of Fin_* are denoted X_*, with $*$ the basepoint and X the complement of the basepoint. The ∞-categories $\mathcal{F}\mathrm{in}$ and $\mathcal{F}\mathrm{in}_*$ are, respectively, the nerves of Fin and Fin_*.

We adopt the topologists convention for the category Δ of non-empty, linearly ordered finite sets in that we label its objects by

$$[n] = \{0 < \cdots < n\} \in \Delta.$$

The *active* and *inert* morphisms form a factorization system on Δ. The former are those morphisms which preserve the bottom and top elements while the latter are the inclusions of subintervals. Active morphisms will be denoted by arrows of the form $(\twoheadrightarrow)$ while inert morphisms by arrows of the form $(\rightarrowtail)$. Their corresponding wide subcategories are denoted, respectively, by Δ_{ac} and Δ_{in}. Every morphism in Δ can be uniquely factored as an active followed by an inert morphism. The ∞-categories $\mathbb{\Delta}_{\mathrm{ac}}$ and $\mathbb{\Delta}_{\mathrm{in}}$ are, respectively, the nerves of Δ_{ac} and Δ_{in}.

Remark 1.3.1. The active-inert factorisation of morphisms in Δ is a particular example of the general notion of generic-free factorisations in the theory of monads as developed by Weber [68] and Berger-Mellies-Weber [8]. Following Lurie [49] and Haugseng [34], we adopt the former terminology as we feel that it is more descriptive.

The category Δ is a full subcategory of Δ_+, the category of (possibly empty) linearly ordered finite sets, the objects of which are

$$\langle n \rangle = \{1 < \cdots < n\} \in \Delta_+.$$

The category ∇ has the same objects as Δ_+ and morphisms given by spans of the form $\langle n \rangle \leftarrow \langle k \rangle \rightarrowtail \langle m \rangle$ ([28] 8)[2]. The category of *k-fold nabla objects* in a category C, denoted C_{∇^k}, is the category of functors $(\nabla^{\mathrm{op}})^{\times k} \to C$.

There is a bijective on objects and full functor $\mathfrak{G} : \Delta \to \nabla$ which restricts to isomorphisms $\Delta_{\mathrm{ac}} \simeq \Delta_+^{\mathrm{op}}$ and $\Delta_{\mathrm{in}}^{\geq 1} \simeq (\Delta_+)_{\mathrm{in}}^{\geq 1}$([28] 8.2). Restriction along $\mathfrak{G}$ induces a fully faithful functor

$$\mathfrak{G}^* : C_{\nabla^k} \to C_{\Delta^k}. \tag{1.3.1}$$

The category Δ_{ac} has a canonical monoidal structure

$$[n] \vee [m] = [n + m]$$

having unit $[0]$. The functor $\mathfrak{G}$ restricts to a monoidal equivalence if we endow Δ_+ with the monoidal structure

$$\langle n \rangle + \langle m \rangle = \langle n + m \rangle \tag{1.3.2}$$

having unit $\langle 0 \rangle$.

Finally, all algebra objects are assumed to be unital and all coalgebra objects counital.

[2]Note that our category ∇ is the opposite of the one defined in [28]

Chapter 2

Background on $(\infty, 2)$-categories

The purpose of this chapter is to cover the technical material concerning $(\infty, 2)$-categories that we shall use throughout the remainder of this thesis. Section 2.1 simply reviews our chosen model of symmetric monoidal $(\infty, 2)$-categories and their associated homotopy bicategories. Section 2.2 is similarly a revision, this time of the definition of symmetric monoidal lax functors. Section 2.3 is a brief discussion on the twisted arrow construction, a construction which appears throughout this work. All of the necessary material for the $(\infty, 2)$-category of bispans is covered Section 2.4, beginning with a review of Haugseng's original definition in Section 2.4.1. Sections 2.4.2 and 2.4.3 contain technical material culminating in a description of the unstraightening of the $(\infty, 2)$-category of bispans. Finally, Section 2.5 provides the details of the formalism used to define lax (co)algebra and bialgebra objects in symmetric monoidal $(\infty, 2)$-categories.

2.1 Symmetric monoidal $(\infty, 2)$-categories

The model of $(\infty, 2)$-categories that we use in this work is the one originally introduced by Lurie in [47].

Definition 2.1.1. An $(\infty, 2)$-*category* is a simplicial ∞-category $\mathcal{B} \in (\mathcal{C}\mathrm{at}_\infty)_{\Delta}$ such that

1. $\mathcal{B}$ is a Segal object, that is, for each $n \geq 2$ the functor $\mathcal{B}_n \to \mathcal{B}_1 \times_{\mathcal{B}_0} \cdots \times_{\mathcal{B}_0} \mathcal{B}_1$ is an equivalence;
2. The ∞-category $\mathcal{B}_0$ is a space, that is, $\mathcal{B}_0 \in \mathcal{S} \subset \mathcal{C}\mathrm{at}_\infty$; and,
3. The Segal space $\Delta^{\mathrm{op}} \xrightarrow{\mathcal{B}} \mathcal{C}\mathrm{at}_\infty \xrightarrow{(-)^{\simeq}} \mathcal{S}$ is complete.

The ∞-category of $(\infty, 2)$-categories is a full subcategory $\mathcal{C}\mathrm{at}_{(\infty,2)} \hookrightarrow (\mathcal{C}\mathrm{at}_\infty)_{\Delta}$. In other words, functors between $(\infty, 2)$-categories are simply natural transformations.

Remark 2.1.2. A widely used model of $(\infty, 2)$-categories in the literature are 2-fold complete Segal spaces as introduced by Barwick [5]. Barwick's model is recovered from the one used in this work by presenting $\mathcal{C}\mathrm{at}_\infty$ as $\mathcal{C}$SS.

Let us take a moment to consider the three conditions placed on a simplicial ∞-category $\mathcal{B} \in (\mathcal{C}\mathrm{at}_\infty)_{\mathbb{\Delta}}$ for it to be an $(\infty, 2)$-category. Condition 1 says that $\mathcal{B}$ is a double ∞-category, that is, an ∞-category internal to $\mathcal{C}\mathrm{at}_\infty$. Informally, one can think of $\mathcal{B}_n$ as having objects strings of n composable morphisms and morphisms diagrams of the form

$$\begin{array}{ccccccccc} s_0 & \longrightarrow & s_1 & \longrightarrow & \cdots & \longrightarrow & s_{n-1} & \longrightarrow & s_n \\ \downarrow & \swarrow & \downarrow & & & & \downarrow & \swarrow & \downarrow \\ t_0 & \longrightarrow & t_1 & \longrightarrow & \cdots & \longrightarrow & t_{n-1} & \longrightarrow & t_n \end{array}$$

The second condition implies that the vertical 1-morphisms in this diagram are equivalences so that there is only one direction in $\mathcal{B}$ having non-trivial composition of 1-morphisms, as one would expect from the ∞-categorical analogue of a bicategory. Finally, the completeness condition ensures that object-wise equivalences of $(\infty, 2)$-categories yield the 'correct' notion of equivalence.

One can think of an $(\infty, 2)$-category $\mathcal{B}$ as a category weakly enriched in ∞-categories. For any two objects x and y of $\mathcal{B}$ consider the pullbacks

$$\begin{array}{ccc} \mathcal{B}(x, \bullet, \ldots, \bullet, y) & \longrightarrow & \mathcal{B}_1 \times_{\mathcal{B}_0} \cdots \times_{\mathcal{B}_0} \mathcal{B}_1 \\ \downarrow & & \downarrow \\ * & \xrightarrow[(x,y)]{} & \mathcal{B}_0^2 \end{array} \qquad \begin{array}{ccc} \mathcal{B}^{(n)}(x, y) & \longrightarrow & \mathcal{B}_n \\ \downarrow & & \downarrow \\ * & \xrightarrow[(x,y)]{} & \mathcal{B}_0^2 , \end{array}$$

where the morphisms on the right in each diagram map a string of n composable morphisms to the initial and final object of the string. These categories are candidates for the ∞-category of strings of n composable morphisms from x to y. Since $\mathcal{B}$ is a Segal object the induced map

$$\mathcal{B}^{(n)}(x, y) \xrightarrow{\sim} \mathcal{B}(x, \bullet, \ldots, \bullet, y)$$

is an equivalence. In particular, $\mathcal{B}(x, y) = \mathcal{B}^{(1)}(x, y)$ is the ∞-category of 1-morphisms from x to y. For any triple of objects x, y and z of $\mathcal{B}$, the sequence of functors

$$\mathcal{B}(x, y) \times \mathcal{B}(y, z) \longrightarrow \mathcal{B}(x, \bullet, z) \xleftarrow{\sim} \mathcal{B}^{(2)}(x, z) \longrightarrow \mathcal{B}(x, z) \tag{2.1.1}$$

give a weak composition on the Hom ∞-categories of $\mathcal{B}$.

This perspective leads to a natural definition of the homotopy bicategory of an $(\infty, 2)$-category.

Definition 2.1.3 ([10] 2.7)**.** Let $\mathcal{B}$ be an $(\infty, 2)$-category. The *homotopy bicategory* of $\mathcal{B}$, denoted $\mathrm{Ho}(\mathcal{B})$, has the same objects as $\mathcal{B}$, Hom-categories between any two objects x and y given by the homotopy category of the ∞-category $\mathcal{B}(x, y)$, and composition given by the diagram in Eq. 2.1.1.

Let $\mathrm{op} : \Delta \to \Delta$ denote the automorphism sending $[n]$ to $[n]^{\mathrm{op}}$.

Definition 2.1.4. For an $(\infty, 2)$-category $\mathcal{B}$ its *opposite* $(\infty, 2)$*-category* is the composite

$$\mathcal{B}^{\mathrm{op}} : \mathbb{\Delta}^{\mathrm{op}} \xrightarrow{\mathrm{op}} \mathbb{\Delta}^{\mathrm{op}} \xrightarrow{\mathcal{B}} \mathcal{C}\mathrm{at}_\infty .$$

There is a universal way to extract from a Segal object $\mathcal{B}$ in $\mathcal{C}\mathrm{at}_\infty$ a new Segal object $\mathcal{UB}$ whose ∞-category of 0-simplices is a space: let $\mathcal{S}\mathrm{eg}\left(\mathcal{C}\mathrm{at}_\infty\right)$ denote the full subcategory of $\left(\mathcal{C}\mathrm{at}_\infty\right)_{\Delta}$ spanned by Segal objects and let $\mathcal{S}\mathrm{eg}_0\left(\mathcal{C}\mathrm{at}_\infty\right)$ be the full subcategory of the former spanned by those objects satisfying Condition 2 above. Then the inclusion $\mathcal{S}\mathrm{eg}_0\left(\mathcal{C}\mathrm{at}_\infty\right) \hookrightarrow \mathcal{S}\mathrm{eg}\left(\mathcal{C}\mathrm{at}_\infty\right)$ admits a right adjoint ([33] 2.13)

$$\mathcal{U} : \mathcal{S}\mathrm{eg}\left(\mathcal{C}\mathrm{at}_\infty\right) \to \mathcal{S}\mathrm{eg}_0\left(\mathcal{C}\mathrm{at}_\infty\right).$$

Explicitly, given $\mathcal{B} \in \mathcal{S}\mathrm{eg}\left(\mathcal{C}\mathrm{at}_\infty\right)$, we have $\mathcal{UB}_0 = \mathcal{B}_0^{\simeq}$ and for each $n \geq 1$ a pullback square

$$\begin{array}{ccc} \mathcal{UB}_n & \longrightarrow & \mathcal{B}_n \\ \downarrow & & \downarrow \\ \left(\mathcal{B}_0^{\simeq}\right)^{n+1} & \hookrightarrow & \left(\mathcal{B}_0\right)^{n+1} \end{array} \tag{2.1.2}$$

There are a number of ways in the literature to define symmetric monoidal $(\infty, 2)$-categories. We follow Lurie ([49] 2.0.0.7) in choosing the one generalising Segal's notion of a special Γ-space [60].

Definition 2.1.5. A *symmetric monoidal* $(\infty, 2)$*-category* is a functor

$$\mathcal{B}^{\otimes} : \mathcal{F}\mathrm{in}_* \times \Delta^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$$

such that

1. For each $S_* \in \mathcal{F}\mathrm{in}_*$, the simplicial ∞-category $\mathcal{B}^{\otimes}(S_*, \bullet)$ is an $(\infty, 2)$-category.
2. For each $[n] \in \Delta^{\mathrm{op}}$ and $S_* \in \mathcal{F}\mathrm{in}_*$, the map $\mathcal{B}^{\otimes}(S_*, [n]) \to \prod_{s \in S} \mathcal{B}^{\otimes}(\{s\}_*, [n])$ is an equivalence.

The ∞-category of symmetric monoidal $(\infty, 2)$-categories is a full subcategory

$$\mathcal{C}\mathrm{at}^{\otimes}_{(\infty,2)} \hookrightarrow \mathcal{F}\mathrm{un}\left(\mathcal{F}\mathrm{in}_* \times \Delta^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty\right).$$

In other words, a symmetric monoidal functor between symmetric monoidal $(\infty, 2)$-categories is simply a natural transformation, that is, a morphism in $\mathcal{F}\mathrm{un}\left(\mathcal{F}\mathrm{in}_* \times \Delta^{\mathrm{op}}, \mathcal{C}\mathrm{at}_\infty\right)$.

One should think of such functors as endowing the $(\infty, 2)$-category $\mathcal{B} = \mathcal{B}^{\otimes}(\underline{1}_*, \bullet)$ with a symmetric monoidal structure. Letting $m : \underline{2}_* \to \underline{1}_*$ denote the morphism in $\mathcal{F}\mathrm{in}_*$ coming from the unique map $\underline{2} \to \underline{1}$, the commutative product on $\mathcal{B}$ is defined by

$$\mathcal{B} \times \mathcal{B} = \mathcal{B}^{\otimes}(\underline{1}_*) \times \mathcal{B}^{\otimes}(\underline{1}_*) \xleftarrow{\sim} \mathcal{B}^{\otimes}(\underline{2}_*) \xrightarrow{\mathcal{B}^{\otimes}(m)} \mathcal{B}^{\otimes}(\underline{1}_*). \tag{2.1.3}$$

Remark 2.1.6. The homotopy bicategory of $\mathcal{B}$ inherits a monoidal structure from the diagram in Eq. 2.1.3. One can similarly find all the data one needs to define a symmetric monoidal structure on $\mathrm{Ho}\left(\mathcal{B}\right)$.

2.2 Symmetric monoidal lax functors via the unstraightening construction

Recall that a lax functor between ordinary 2-categories $L : A \rightsquigarrow B$ differs from a functor in that it no longer respects identity arrows and composition of 1-morphisms [44]. Instead, for each object a of A and each pair of composable morphisms f and g, one has (not necessarily invertible) 2-morphisms $\mathrm{Id}_{L(a)} \Rightarrow L(\mathrm{Id}_a)$ and $L(g) \circ L(f) \Rightarrow L(g \circ f)$ in B witnessing the lax preservation of unitality and composition. Furthermore, these 2-morphisms must satisfy associativity and unitality coherence equations.

The notion of a lax functor between $(\infty, 2)$-categories requires the use of the theory of *cocartesian fibrations* of ∞-categories and the *unstraightening construction* as developed by Lurie in ([46] 2.4). Recall that the unstraightening construction defines an equivalence between functors $\mathcal{C} \to \mathcal{C}\mathrm{at}_\infty$ and cocartesian fibrations over $\mathcal{C}$ ([46] 3.2.0.1),

$$\mathrm{Un} : \ \mathcal{F}\mathrm{un}(\mathcal{C}, \mathcal{C}\mathrm{at}_\infty) \xrightarrow{\sim} \mathcal{C}\mathrm{ocart}/\mathcal{C} \ .$$

Under this equivalence a natural transformation $\eta : F \Rightarrow G$ is sent to a morphism

$$\begin{array}{ccc} \mathrm{Un}(F) & \xrightarrow{\mathrm{Un}(\eta)} & \mathrm{Un}(G) \\ & \searrow \quad \swarrow & \\ & \mathcal{C} & \end{array}$$

such that $\mathrm{Un}(\eta)$ sends cocartesian morphisms in $\mathrm{Un}(F)$ to cocartesian morphisms in $\mathrm{Un}(G)$. The unstraightening construction is natural in $\mathcal{C}$ in the sense that from a composite

$$\mathcal{D} \xrightarrow{G} \mathcal{C} \xrightarrow{F} \mathcal{C}\mathrm{at}_\infty$$

one has a pullback diagram ([29] A.31)

$$\begin{array}{ccc} \mathrm{Un}(FG) & \longrightarrow & \mathrm{Un}(F) \\ \downarrow & & \downarrow \\ \mathcal{D} & \xrightarrow[G]{} & \mathcal{C}. \end{array}$$

Taking $\mathcal{C} = \mathbb{\Delta}^{\mathrm{op}}$ one can unstraighten an $(\infty, 2)$-category, and functors become morphisms of cocartesian fibrations over $\mathbb{\Delta}^{\mathrm{op}}$ which preserve cocartesian morphisms. Lax functors will still be morphisms of fibrations but will preserve fewer cocartesian morphisms.

Definition 2.2.1. A *lax functor* $L : \mathcal{A} \rightsquigarrow \mathcal{B}$ between $(\infty, 2)$-categories $\mathcal{A}, \mathcal{B} : \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ is a morphism

$$\begin{array}{ccc} \mathrm{Un}\,(\mathcal{A}) & \xrightarrow{L} & \mathrm{Un}\,(\mathcal{B}) \\ & \searrow \quad \swarrow & \\ & \mathbb{\Delta}^{\mathrm{op}} & \end{array}$$

such that L sends cocartesian lifts of morphisms in $(\mathbb{\Delta}_{\mathrm{in}})^{\mathrm{op}}$, the subcategory of inert morphisms, to cocartesian morphisms.

Remark 2.2.2. We are not certain as to the exact history of this approach to defining lax functors between $(\infty, 2)$-categories. We first learned it from Dyckerhoff–Kapranov ([17] 9.2.8) and Lurie's definition of a morphism of ∞-operads ([46] 2.1.2.7) is qualitatively similar. We believe an analogous statement must be known for lax functors between 2-categories but do not know any references.

Remark 2.2.3. Since $\mathrm{op} : \Delta \to \Delta$ preserves inert morphisms, the naturality of the unstraightening construction implies that from a lax functor $L : \mathcal{A} \rightsquigarrow \mathcal{B}$ one gets a lax functor $L^{\mathrm{op}} : \mathcal{A}^{\mathrm{op}} \rightsquigarrow \mathcal{B}^{\mathrm{op}}$.

It is worth taking a moment to understand how this intuitively captures the correct notion of a lax functor between $(\infty, 2)$-categories. A lax functor $L : \mathcal{A} \rightsquigarrow \mathcal{B}$ has the desired property that

$$L : \left(x_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} x_n \right) \mapsto \left(L(x_0) \xrightarrow{L(f_1)} \cdots \xrightarrow{L(f_n)} L(x_n) \right),$$

as the operations which pick out the constituents of a string of composable morphisms are cocartesian lifts of inert morphisms, or equivalently, the images of inert morphisms under the original functors $\mathcal{A}$ and $\mathcal{B}$. For example, the cocartesian lifts of the inert morphisms $[1] \twoheadleftarrow [0]$ are the source and targets maps of a 1-morphism, while the cocartesian lifts of the inert morphisms $[2] \twoheadleftarrow [1]$ map a pair of composable morphisms to either the first or second morphism in the pair.

The cocartesian lifts of active maps encode composition. Letting (f, g) be a pair of composable 1-morphisms in $\mathcal{A}$, the cocartesian lift of the unique active map $[2] \mapsfrom [1]$ having source (f, g) is the composition map $\circ : (f, g) \to gf$. Under the lax functor L this is sent to a morphism $L(\circ) : (L(f), L(g)) \to L(gf)$ in $\mathrm{Un}(\mathcal{B})$ covering $[2] \mapsfrom [1]$. As L is not required to preserve cocartesian morphisms, it is not necessarily the case that $L(\circ)$ is cocartesian, that is, it *is not the composition operation* in $\mathcal{B}$. Instead, the defining property of cocartesian morphisms says that there is an essentially unique $\mathcal{L}_{g,f}$ covering $\mathrm{Id}_{[1]}$ making the following diagram commute

$$\begin{array}{ccc} (L(f), L(g)) & \xrightarrow{\circ} & L(g) \circ L(f) \\ & \searrow^{L(\circ)} & \downarrow{\scriptstyle \mathcal{L}_{g,f}} \\ & & L(gf)\,. \end{array}$$

In the language of ordinary 2-categories, $\mathcal{L}_{g,f} : L(g) \circ L(f) \Rightarrow L(gf)$ is a component of the lax functor L witnessing the lax preservation of composition. A similar analysis for the cocartesian lifts of the active map $[0] \mapsfrom [1]$ yields the witness for the lax preservation of the unit.

It is straightforward to extend this notion to define symmetric monoidal lax functors, that is, functors between symmetric monoidal $(\infty, 2)$-categories which preserve the symmetric monoidal structure but only laxly preserve composition, as follows.

Definition 2.2.4. A *symmetric monoidal lax functor* $L : \mathcal{A}^{\otimes} \rightsquigarrow \mathcal{B}^{\otimes}$ between symmetric monoidal $(\infty, 2)$-categories $\mathcal{A}^{\otimes}, \mathcal{B}^{\otimes} : \mathcal{F}\mathrm{in}_* \times \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$ is a morphism

$$\begin{array}{ccc} \mathrm{Un}\,(\mathcal{A}^{\otimes}) & \xrightarrow{\quad L \quad} & \mathrm{Un}\,(\mathcal{B}^{\otimes}) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}\mathrm{in}_* \times \mathbb{\Delta}^{\mathrm{op}} & \end{array}$$

such that L sends cocartesian lifts of morphisms in $\mathcal{F}\mathrm{in}_* \times (\mathbb{\Delta}_{\mathrm{in}})^{\mathrm{op}}$ to cocartesian morphisms.

Remark 2.2.5. The above discussion shows that a symmetric monoidal lax functor $L : \mathcal{A}^{\otimes} \rightsquigarrow \mathcal{B}^{\otimes}$ induces a symmetric monoidal lax functor

$$\mathrm{Ho}\,(\mathcal{A})^{\otimes} \overset{\mathrm{Ho}(L)}{\rightsquigarrow} \mathrm{Ho}\,(\mathcal{B})^{\otimes}\ .$$

2.3 The twisted arrow category and normal oplax functors

The *twisted arrow category* ([50] IX.6.3), Σ^D, of an ordinary category D will be a recurring character in this work. It will first appear in the definition of the $(\infty, 2)$-category of bispans in Section 2.4, and its subsequent reappearances will be tied to various constructions involving bispans.

For an ordinary category D, let Σ^D be the category having as objects arrows $f : d \to d'$ in D, and morphisms from f_1 to f_2

$$\begin{array}{ccc} d_1 & \xrightarrow{f_1} & d_1' \\ \downarrow & & \uparrow \\ d_2 & \xrightarrow[f_2]{} & d_2' \end{array}$$

Remembering only the source and target of an object of Σ^D defines a forgetful functor $\Sigma^D \to D \times D^{\mathrm{op}}$. Furthermore, the twisted arrow categories assemble into a functor $\Sigma^{-} : \mathrm{Cat} \to \mathrm{Cat}$.

Remark 2.3.1. There are two equally canonical conventions for the definition of the twisted arrow category, the second of which has morphisms given by diagrams

$$\begin{array}{ccc} d_1 & \xrightarrow{f_1} & d_1' \\ \uparrow & & \downarrow \\ d_2 & \xrightarrow[f_2]{} & d_2' \end{array}$$

We follow the convention used by Haugseng [33], while the second convention is used by Barwick [6] and Lurie ([49] 5.2.1).

A number of constructions in later sections involve writing explicit functors of the form $\Sigma^D \to C$ for C a category having finite limits. It turns out that such functors can be equivalently described as *normal oplax functors* $D \nrightarrow \mathrm{sp}\,(C)$, where $\mathrm{sp}\,(C)$ is a bicategory which we shall describe shortly. For our purposes this latter description will often be more convenient.

Given a category C having finite limits one can define a bicategory $\mathrm{sp}\,(C)$ [7] having the same objects as C, 1-morphisms given by spans, and 2-morphisms given by diagrams

$$\begin{array}{ccccc} & & d & & \\ & \swarrow & \downarrow & \searrow & \\ c & \nwarrow & \downarrow & \nearrow & c'\,. \\ & & d' & & \end{array}$$

Horizontal composition is given by pullbacks, chosen once and for all. The identity 1-morphism for an object c is the span $c = c = c$.

Remark 2.3.2. Note that this bicategory is similar to, but distinct from, the $(\infty, 2)$-category of bispans in C that we shall introduce in Section 2.4. They have different 2-morphisms and, unlike the $(\infty, 2)$-category of bispans in C, $\mathrm{sp}(C)$ has no k-morphisms for $k > 2$.

A *normal oplax functor* $F : D \nrightarrow \mathrm{sp}(C)$ consists of, for each object $d \in D$ an object $F(d) \in C$, for each morphism $f : c \to d$ in D a span

$$\begin{array}{ccc} & F(f) & \\ \swarrow & & \searrow \\ F(c) & & F(d) \end{array}$$

and for each pair of composable morphisms $c \xrightarrow{f} d \xrightarrow{g} e$ a diagram

$$\begin{array}{ccccc} & & F(g \circ f) & & \\ & \swarrow & \downarrow{\scriptstyle \Phi_{g,f}} & \searrow & \\ F(c) & \nwarrow & & \nearrow & F(e)\,. \\ & & F(g) \times_{F(d)} F(f) & & \end{array}$$

For each object $d \in D$ it must be that $F(\mathrm{Id}_d) = \mathrm{Id}_{F(d)}$, for each morphism $c \xrightarrow{f} d$ it must be that $\Phi_{\mathrm{Id}_d, f} = \mathrm{Id}_{F(f)} = \Phi_{f, \mathrm{Id}_c}$, and, suppressing associator isomorphisms, for each string of composable morphisms $b \xrightarrow{f} c \xrightarrow{g} d \xrightarrow{h} e$,

$$\left(\mathrm{Id}_{F(h)} \times_{F(d)} \Phi_{g,f}\right) \circ \Phi_{h,gf} = \left(\Phi_{h,g} \times_{F(c)} \mathrm{Id}_{F(f)}\right) \circ \Phi_{hg,f}, \tag{2.3.1}$$

as morphisms $F(hgf) \to F(h) \times_{F(d)} F(g) \times_{F(c)} F(f)$.

Theorem 2.3.3 ([19] 3.4.1)**.** *For any category D and any category C having finite limits there is a natural isomorphism*

$$\mathrm{Hom}\left(\Sigma^D, C\right) \simeq \mathrm{Hom}^{\mathrm{n.oplax}}\left(D, \mathrm{sp}(C)\right),$$

where $\mathrm{Hom}^{\mathrm{n.oplax}}$ *denotes the set of normal oplax functors.*

The isomorphism is given as follows. A normal oplax functor $F : D \nrightarrow \mathrm{sp}(C)$ associates to each diagram

$$\begin{array}{ccc} d_1 & \xrightarrow{f_1} & d_1' \\ {\scriptstyle g}\downarrow & & \uparrow{\scriptstyle h} \\ d_2 & \xrightarrow[f_2]{} & d_2'\,. \end{array}$$

a diagram

$$\begin{array}{ccccccc} & & & F(f_1) & & & \\ & & \swarrow & {\scriptstyle F_{g,h}}\downarrow & \searrow & & \\ & F(g) & & F(f_2) & & F(h) & \\ \swarrow & & \searrow & \swarrow & \searrow & \swarrow & \searrow \\ F(d_1) & & F(d_2) & & F(d_2') & & F(d_1')\,. \end{array}$$

The corresponding functor $\tilde{F} : \Sigma^D \to C$ is then

$$\tilde{F} : f \mapsto F(f), \quad ((g,h) : f_1 \to f_2) \mapsto F_{g,h}.$$

By mimicking the standard proof that left adjoints preserve colimits, one has as a corollary of Theorem 2.3.3 the following.

Corollary 2.3.4. *The twisted arrow construction* $\Sigma^- : \mathrm{Cat} \to \mathrm{Cat}$ *preserves colimits.*

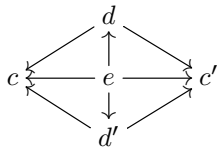
Proof. Since bicategories and normal oplax functors between them assemble into a category, the set-valued functor $\mathrm{Hom}^{\mathrm{n.oplax}}(-,-)$ sends colimits in the first variable to limits in sets. Let $L : J \to \mathrm{Cat}$ be a diagram in Cat. By Theorem 2.3.3 one has the following string of bijections

$$\begin{aligned}\mathrm{Hom}\left(\Sigma^{\mathrm{colim}\, L}, C\right) &\simeq \mathrm{Hom}^{\mathrm{n.oplax}}\left(\mathrm{colim}\, L, \mathrm{sp}\,(C)\right) \simeq \lim_{j\in J} \mathrm{Hom}^{\mathrm{n.oplax}}\left(L(j), \mathrm{sp}\,(C)\right) \\ &\simeq \lim_{j\in J} \mathrm{Hom}\left(\Sigma^{L(j)}, C\right) \simeq \mathrm{Hom}\left(\operatorname*{colim}_{j\in J} \Sigma^{L(j)}, C\right).\end{aligned}$$

The Yoneda lemma implies that Σ^- preserves colimits. □

2.4 The $(\infty, 2)$-category of bispans

The construction which associates the ∞-category $\mathcal{S}\mathrm{pan}\,(\mathcal{C})$ to an ∞-category $\mathcal{C}$ having finite limits can be iterated to form (∞, n)-categories for each n. Informally, a 2-morphism between spans is a 'span of spans', that is a diagram of the form



while a 3-morphism is a 'span of spans of spans', and so on. The cartesian product in $\mathcal{C}$ endows these (∞, n)-categories with a symmetric monoidal structure. The rigorous construction of these symmetric monoidal (∞, n)-categories has been carried out by Haugseng [33].

In Section 2.4.1 we review Haugseng's construction in the case that concerns us, namely the construction of the symmetric monoidal $(\infty, 2)$-category $\mathcal{S}\mathrm{pan}_2\,(\mathcal{C})^\times$ of *bispans* in $\mathcal{C}$. In Section 2.4.2 we prove that $\mathcal{S}\mathrm{pan}_2\,(\mathcal{C})^\times$ is *semistrict*, a technical condition which simplifies the description of lax functors. Finally, in Section 2.4.3 we determine explicitly the unstraightening of $\mathcal{S}\mathrm{pan}_2\,(N(C))^\times$ for C an ordinary category having finite limits.

2.4.1 Definition of $\mathcal{S}\mathrm{pan}_2\,(\mathcal{C})^\times$

Haugseng's construction of the symmetric monoidal $(\infty, 2)$-category $\mathcal{S}\mathrm{pan}_2\,(\mathcal{C})^\times$ is an iteration of Barwick's construction of the ∞-category $\mathcal{S}\mathrm{pan}\,(\mathcal{C})$ [6]. One first defines a symmetric monoidal

double ∞-category $\overline{\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}} \in \mathcal{S}\mathrm{eg}(\mathcal{C}\mathrm{at}_\infty^{\otimes})$ which fails to be in $\mathcal{S}\mathrm{eg}_0(\mathcal{C}\mathrm{at}_\infty^{\otimes})$. One then remedies this problem by defining $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times} := \mathcal{U}\left(\overline{\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}}\right)$.

The morphisms in the $(\infty,2)$-category of bispans are given by diagrams in $\mathcal{C}$ of the following form. Let $\Sigma^n := \Sigma^{[n]}$ be the twisted arrow construction of $[n]$. Explicitly, this is the opposite of the poset of non-empty intervals in $[n]$. Similarly, the category $\Sigma^{n,k} := \Sigma^{[n]\times[k]} = \Sigma^n \times \Sigma^k$ is opposite of the poset of non-empty rectangles in $[n]\times[k]$. These assemble into a functor

$$\Sigma^{\bullet,\bullet} : (\Delta)^2 \to \mathrm{Cat}.$$

There is a full subcategory $\Lambda^n \hookrightarrow \Sigma^n$ consisting of intervals $[i;j]$ with $|j-i| \leq 1$ and hence a full subcategory $\Lambda^{n,k} \hookrightarrow \Sigma^{n,k}$. The ∞-categories $\mathbb{\Lambda}^n$ and $\mathbb{\Sigma}^n$ are, respectively, the nerves of Λ^n and Σ^n.

Observe that for $\mathcal{C}$ an ∞-category and $F : \mathcal{C}_1 \to \mathcal{C}_2$ a fully faithful functor, both the right and left Kan extensions,

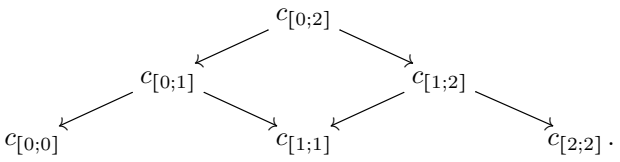
$$F_*, F_! : \mathcal{F}\mathrm{un}(\mathcal{C}_1, \mathcal{C}) \to \mathcal{F}\mathrm{un}(\mathcal{C}_2, \mathcal{C}),$$

are fully faithful should they exist.

Definition 2.4.1. Let $\mathcal{C}$ be an ∞-category and $F : \mathcal{C}_1 \to \mathcal{C}_2$ be a fully faithful functor. We call functors in the image of F_* *cartesian* and those in the image of $F_!$ *cocartesian*. We denote these full subcategories, respectively, by $\mathcal{F}\mathrm{un}^{\mathrm{cart}}(\mathcal{C}_2, \mathcal{C})$ and $\mathcal{F}\mathrm{un}^{\mathrm{cocart}}(\mathcal{C}_2, \mathcal{C})$, with the functor F to be understood implicitly from the context.

Remark 2.4.2. We follow Haugseng's terminology [33] as our definition of a (co)cartesian functor is a generalisation of the one given there. In Section 2.2 we discussed cocartesian morphisms and cocartesian fibrations. These are distinct notions from the one being introduced now, but as these terms are used in different contexts there is little fear of confusion.

Example 2.4.3. A functor $\mathbb{\Sigma}^2 \to \mathcal{C}$ is a diagram,



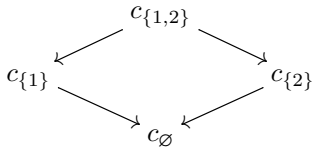
Such a diagram is cartesian if it is the right Kan extension of its restriction to $\mathbb{\Lambda}^2$, i.e., if the middle square is a pullback in $\mathcal{C}$. In general, a functor $\mathbb{\Sigma}^n \to \mathcal{C}$ is pyramid of spans on $n+1$ objects. It being cartesian says higher tiers of this pyramid consist of a coherent choice of pullbacks of the n spans along the bottom two tiers.

For each set S, denote by $\Pi(S)$ the poset of subsets of S ordered by inclusion. These assemble into a functor $\Pi(-) : \mathrm{Fin}_*^{\mathrm{op}} \to \mathrm{Cat}$ by declaring the image of a pointed map $f : S_* \to T_*$ to be

$$\Pi(f) : \Pi(T) \to \Pi(S), \quad U \mapsto f^{-1}(U).$$

For each $S_* \in \mathrm{Fin}_*$ there is a full subcategory $P(S) \hookrightarrow \Pi(S)$ consisting of the singleton subsets. The ∞-categories $\mathcal{P}(S)$ and $\mathbb{\Pi}(S)$ are, respectively, the nerves of $P(S)$ and $\Pi(S)$.

Example 2.4.4. A functor $\mathbb{\Pi}(\{1,2\})^{\mathrm{op}} \to \mathcal{C}$ is a diagram,



Such a diagram is cartesian if it presents $c_{\{1,2\}}$ as the product of $c_{\{1\}}$ and $c_{\{2\}}$ and $c_{\varnothing}$ is terminal. Similarly, a cartesian functor $\mathbb{\Pi}(S)^{\mathrm{op}} \to \mathcal{C}$ encodes a coherent choice of products for a collection of objects of $\mathcal{C}$ labelled by the elements of S.

Presenting $\mathcal{C}\mathrm{at}_\infty$ as $\mathcal{C}\mathrm{SS}$ we can now define the symmetric monoidal $(\infty, 2)$-category of bispans. For an ∞-category $\mathcal{C}$ having finite limits, consider the functor

$$\mathrm{Fin}_* \times (\Delta^{\mathrm{op}})^2 \to \mathrm{Kan}, \quad (S_*, [n], [k]) \mapsto \mathcal{M}\mathrm{ap}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k}, \mathcal{C}\right),$$

which is well-defined by Proposition 3.8 of [33]. Taking the coherent nerve defines a functor $\overline{\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}} : \mathcal{F}\mathrm{in}_* \times (\mathbb{\Delta}^{\mathrm{op}})^2 \to \mathcal{S}$.

Proposition 2.4.5. *Let $\mathcal{C}$ be an ∞-category with finite limits. Then the functor $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times} := \mathcal{U}\left(\overline{\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}}\right)$ is a symmetric monoidal $(\infty, 2)$-category. We call this the* symmetric monoidal $(\infty, 2)$-category of bispans in $\mathcal{C}$.

Remark 2.4.6. Our description of the symmetric monoidal structure arising from the cartesian product on $\mathcal{C}$ differs from Haugseng. He instead presents the symmetric monoidal structure by giving a sequence of $(\infty, 2+k)$-categories delooping the $(\infty, 2)$-category of bispans ([33] 9.1).

Proof. By Corollary 6.5 of [33], for each fixed $S_* \in \mathcal{F}\mathrm{in}_*$, the simplicial ∞-category $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}(S_*, \bullet, \bullet)$ is a $(\infty, 2)$-category. Consider the following commutative diagram in $\mathcal{S}$,

$$\begin{array}{ccc} \overline{\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}}(S_*, [n], [k]) & \longrightarrow & \prod_{s \in S} \overline{\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}}(\{s\}_*, [n], [k]) \\ \wr\downarrow & & \downarrow\wr \\ \mathcal{M}\mathrm{ap}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k}, \mathcal{C}\right) & \xrightarrow[j^*]{} & \mathcal{M}\mathrm{ap}\left(\mathcal{P}(S)^{\mathrm{op}} \times \mathbb{\Lambda}^{n,k}, \mathcal{C}\right), \end{array}$$

where j denotes the fully faithful functor including $\mathcal{P}(S)^{\mathrm{op}} \times \mathbb{\Lambda}^{n,k}$ into $\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k}$. Since $\mathcal{U}$ preserves limits it suffices to show that the top map is an equivalence.

By Definition 2.4.1, the category $\mathcal{F}\mathrm{un}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k}, \mathcal{C}\right)$ is the image of $\mathcal{F}\mathrm{un}\left(\mathcal{P}(S)^{\mathrm{op}} \times \mathbb{\Delta}^{n,k}, \mathcal{C}\right)$ under j_*, the right adjoint of j^*. Since j_* is fully faithful it is an equivalence onto its image with inverse j^*. It follows that the top map in the above diagram is an equivalence, as desired. □

Remark 2.4.7. The homotopy bicategory of $\mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}$ is the ordinary bicategory of bispans in $\mathcal{C}$, denoted $\mathrm{Span}_2^{\times}\left(\mathcal{C}\right)$.

2.4.2 $\mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}$ is semistrict

Describing symmetric monoidal lax functors for general symmetric monoidal $(\infty, 2)$-categories can be quite complicated due to the use of the unstraightening construction. For the following class of symmetric monoidal $(\infty, 2)$-categories it simplifies greatly.

Recall that a functor $N(C) \to \mathcal{C}\mathrm{at}_\infty$ is equivalent to a functor $\mathfrak{C}N(C) \to \mathrm{qCat}$ of simplicially enriched categories, where $\mathfrak{C}$ is the left adjoint of the coherent nerve ([46] 1.1.5).

Definition 2.4.8. For an ordinary category C, a functor $N(C) \to \mathcal{C}\mathrm{at}_\infty$ is called *semistrict* if there is a functor of ordinary categories $C \to \mathrm{qCat}^0$, for qCat^0 the ordinary category of quasi-categories and functors between them, such that the following commutes

$$\begin{array}{ccc} \mathfrak{C}N(C) & \longrightarrow & \mathrm{qCat} \\ {\scriptstyle\epsilon}\downarrow & & \uparrow \\ C & \longrightarrow & \mathrm{qCat}^0, \end{array}$$

where ϵ is the counit of the adjunction $\mathfrak{C} \dashv N$. In particular, a symmetric monoidal $(\infty, 2)$-category is semistrict if it is semistrict as a functor $\mathcal{F}\mathrm{in}_* \times \mathbb{\Delta}^{\mathrm{op}} \to \mathcal{C}\mathrm{at}_\infty$.

The aim of this section is to show that $\mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}$ is semistrict. To that end, we must first compute the pullback of ∞-categories in diagram 2.1.2, or equivalently, the homotopy pullback of

$$\begin{array}{ccc} & & \overline{\mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}}(S_*, [n], \bullet) \\ & & \downarrow{\scriptstyle v} \\ \left(\left(\overline{\mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}}(S_*, [0], \bullet)\right)^{\simeq}\right)^{n+1} & \hookrightarrow & \left(\overline{\mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}}(S_*, [0], \bullet)\right)^{n+1} \end{array} \tag{2.4.1}$$

in CSS, the category of bisimplicial sets endowed with the Rezk model structure [56].

The computation is rendered trivial by the following lemma.

Lemma 2.4.9. *For each* $(S_*, [n]) \in \mathrm{Fin}_* \times \Delta^{\mathrm{op}}$*, the morphism*

$$v : \overline{\mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}}(S_*, [n], \bullet) \to \left(\overline{\mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}}(S_*, [0], \bullet)\right)^{n+1}$$

is a fibration in CSS*.*

Proof. It suffices to show that v is a Reedy fibration since CSS is a left Bousfield localisation of the Reedy model structure on the category of bisimplicial sets, and both the source and the target of v are fibrant in CSS ([35] 3.3.16).

For the purposes of this proof we will use the shorthand $\mathcal{M}(-,-) \coloneqq \mathcal{M}\mathrm{ap}(-,-)$. Consider the following commutative diagram in Set_Δ,

$$\begin{array}{ccc} \mathcal{M}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k}, \mathcal{C}\right) & \longrightarrow & \left(\mathcal{M}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,\bullet}, \mathcal{C}\right)\right)^{\partial\Delta[k]} \\ \downarrow & & \downarrow \\ \left(\mathcal{M}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{0,k}, \mathcal{C}\right)\right)^{n+1} & \longrightarrow & \left(\left(\mathcal{M}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{0,\bullet}, \mathcal{C}\right)\right)^{n+1}\right)^{\partial\Delta[k]} \end{array} \tag{2.4.2}$$

The morphism v is a Reedy fibration if and only if the induced map $\overline{\nu}$ from $\mathcal{M}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k}, \mathcal{C}\right)$ to the pullback is a Kan fibration of simplicial sets ([56] 2.4).

Let $B(n,k)$ be the pushout of simplicial sets

$$\begin{array}{ccc} (\Delta[0])^{\amalg n+1} \times \partial\Delta[k] & \longrightarrow & \Delta[n] \times \partial\Delta[k] \\ \downarrow & & \downarrow \\ (\Delta[0])^{\amalg n+1} \times \Delta[k] & \longrightarrow & B(n,k)\,. \end{array}$$

The pullback of diagram 2.4.2 is $\lim_{\Delta[m]\to B(n,k)} \mathcal{M}^{\mathrm{cart}}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{m}, \mathcal{C}\right)$ since $\mathcal{M}(-,\mathcal{C}) : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$ sends colimits to limits. To prove that $\overline{\nu}$ is a Kan fibration it suffices to show that

$$\nu : \mathcal{M}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k}, \mathcal{C}\right) \to \lim_{\Delta[m]\to B(n,k)} \mathcal{M}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{m}, \mathcal{C}\right)$$

is a Kan fibration as the target of $\overline{\nu}$ is a union of connected components of the target of ν.

By left Kan extension along the Yoneda embedding one can extend $\mathbb{\Sigma}^\bullet$ to a functor $\mathbb{\Sigma}^\bullet : \mathrm{Set}_\Delta \to \mathrm{Set}_\Delta$. Let $\epsilon : \Delta \to \Delta$ to be the edgewise subdivison functor sending $[n]$ to $[n] \star [n]^{\mathrm{op}}$ ([6] 2.5). Since $\mathbb{\Sigma}^n = \epsilon^* \Delta[n]$ and both functors preserve colimits one has that $\mathbb{\Sigma}^\bullet = \epsilon^*$. From this we conclude that $\mathbb{\Sigma}^\bullet$ preserves products and monomorphisms.

It follows, therefore, that the target of ν is $\mathcal{M}\left(\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{B(n,k)}, \mathcal{C}\right)$. The morphism ν is induced by the functor $\mathbb{\Sigma}^{B(n,k)} \to \mathbb{\Sigma}^{n,k}$, which is itself induced by the inclusion $B(n,k) \hookrightarrow \Delta[n] \times \Delta[k]$ and so is a monomorphism. Hence, by Lemma 3.1.3.6 of [46], ν is a Kan fibration. □

Next, denote by $V_{n,k}$ the sub-poset of $[n] \times [k]$ on those morphisms of the form (Id, g). We say a functor $\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k} \to \mathcal{C}$ is *vertically constant* if morphisms of the form (Id, v) for $v \in \mathbb{\Sigma}^{V_{n,k}}$ are sent to equivalences.

Definition 2.4.10. Let $\mathcal{C}$ be an ∞-category with finite limits and $(S_*, [n]) \in \mathrm{Fin}_* \times \Delta^{\mathrm{op}}$. Define $\mathcal{S}\mathrm{p}_{S,n}(\mathcal{C})$ to be the simplicial set having k-simplices the cartesian, vertically constant functors $\mathbb{\Pi}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k} \to \mathcal{C}$.

Proposition 2.4.11. *Let $\mathcal{C}$ be an ∞-category with finite limits. Then for each $(S_*, [n])$ in $\mathrm{Fin}_* \times \Delta^{\mathrm{op}}$, the simplicial set $\mathcal{S}\mathrm{p}_{S,n}(\mathcal{C})$ is a quasi-category. Furthermore, the symmetric monoidal $(\infty, 2)$-category of bispans $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$ is given by the functor*

$$\mathrm{Fin}_* \times \Delta^{\mathrm{op}} \to \mathrm{qCat}^0, \quad (S_*, [n]) \mapsto \mathcal{S}\mathrm{p}_{S,n}(\mathcal{C}),$$

and so is semistrict.

Proof. Recall that for a complete Segal space $\mathcal{X}$ each vertex of $\mathcal{X}_k$ determines a sequence of composable morphisms $[f_i]$ in the homotopy category of $\mathcal{X}$. Then $\mathcal{X}_k^{\simeq}$ is the full sub simplicial set of $\mathcal{X}_k$ on those vertices having each $[f_i]$ invertible in the homotopy category ([47] 1.1.11).

In particular, $\left(\overline{\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}}(S_*, [0], \bullet)\right)_k^{\simeq}$ is the full sub simplicial set of $\mathcal{M}\mathrm{ap}^{\mathrm{cart}}(\mathbb{\Gamma}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^k, \mathcal{C})$ generated by those functors sending morphisms of the form (Id, f) to equivalences. Since all of the objects in diagram 2.4.1 are fibrant, Lemma 2.4.9 implies that the ordinary pullback in CSS is a complete Segal space presenting the homotopy pullback ([46] A.2.4.4). We can therefore conclude that $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}(S_*, [n], [k])$ is the full sub simplicial set of $\mathcal{M}\mathrm{ap}^{\mathrm{cart}}(\mathbb{\Gamma}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{n,k}, \mathcal{C})$ generated by those functors sending morphisms of the form (Id, v) for $v \in \mathbb{\Sigma}^{V_{n,k}}$ to equivalences.

The canonical equivalence $\mathcal{C}\mathrm{SS} \simeq \mathcal{C}\mathrm{at}_\infty$ is presented by a Quillen equivalence ([42] 4.11)

$$\mathrm{CSS} \underset{(p_1)_*}{\overset{(i_1)^*}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} (\mathrm{Set}_\Delta)_{\mathrm{Joyal}}, \quad (p_1)^* : X_{\bullet,\bullet} \mapsto X_{\bullet,0}\ .$$

By definition $\mathcal{S}\mathrm{p}_{S,n}(\mathcal{C}) = (p_1)^* \mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}(S_*, [n], \bullet)$, which is a quasi-category since $(p_1)^*$ preserves fibrations. □

Corollary 2.4.12. *For any ∞-category $\mathcal{C}$ having finite limits its symmetric monoidal $(\infty, 2)$-category of bispans is equivalent to its opposite.*

Proof. By Proposition 2.4.11, the opposite of the symmetric monoidal $(\infty, 2)$-category of bispans is

$$\mathrm{Fin}_* \times \Delta^{\mathrm{op}} \to \mathrm{qCat}^0, \quad (S_*, [n]) \mapsto \mathcal{S}\mathrm{p}_{S,[n]^{\mathrm{op}}}(\mathcal{C}),$$

where the quasi-category $\mathcal{S}\mathrm{p}_{S,[n]^{\mathrm{op}}}(\mathcal{C})$ has k-simplices the cartesian and vertically constant functors $\mathbb{\Gamma}(S)^{\mathrm{op}} \times \mathbb{\Sigma}^{[n]^{\mathrm{op}},k} \to \mathcal{C}$. Since $[n]$ is canonically isomorphic to $[n]^{\mathrm{op}}$, one has an equivalence $\mathbb{\Sigma}^{[n]^{\mathrm{op}},k} \simeq \mathbb{\Sigma}^{n,k}$ and hence $\mathcal{S}\mathrm{p}_{S,[n]^{\mathrm{op}}}(\mathcal{C}) \simeq \mathcal{S}\mathrm{p}_{S,n}(\mathcal{C})$. □

2.4.3 Unstraightening $\mathcal{S}\mathrm{pan}_2(N(C))^{\times}$

The aim of this section is to give a description, sufficient for our purposes, of the unstraightening of the symmetric monoidal $(\infty, 2)$-category of bispans for an ordinary category C having finite limits. In general, determining the quasi-category $\mathrm{Un}(F)$ unstraightening a functor $F : \mathcal{D} \to \mathcal{C}\mathrm{at}_\infty$ can be quite difficult. This can, however, be done in the special case when $\mathcal{D} = N(D)$ for an ordinary

category D and F is semistrict ([46] 3.2.5.2). In this case, the set of k-simplices $\mathrm{Un}(F)_k$ is the set of pairs

$$\Big(\sigma \in N_k(D),\ \big\{\tau(J) : J \to F(\sigma(\overline{J}))\big\}_{\varnothing \neq J \subset [k]}\Big), \tag{2.4.3}$$

where $\overline{J}$ is the maximal element of J. The family of functors τ must be such that for each $\varnothing \neq J \subset L \subset [k]$ the following diagram commutes

$$\begin{array}{ccc} J & \xrightarrow{\tau(J)} & F(\sigma(\overline{J})) \\ \big\downarrow & & \big\downarrow{\scriptstyle F(\sigma(\overline{J} \leq \overline{L}))} \\ L & \xrightarrow[\tau(L)]{} & F(\sigma(\overline{L})) \end{array}$$

By Proposition 2.4.11, $\mathcal{S}\mathrm{pan}_2\,(\mathcal{C})^{\times}$ is semistrict for any ∞-category $\mathcal{C}$ having finite limits, in particular when $\mathcal{C} = N(C)$ as we shall assume for the remainder of this section[1]. We can therefore apply the above to compute its unstraightening.

Remark 2.4.13. As a slight abuse of notation we will not distinguish between an element $(f, \phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$ and its opposite functor

$$(f, \phi) : [k]^{\mathrm{op}} \to \mathrm{Fin}_*^{\mathrm{op}} \times \Delta.$$

Furthermore, for $j \leq j' \in [k]$ we write the composite morphisms as

$$f_{j',j} : f(j) \to f(j') \in \mathrm{Fin}_* \text{ and } \phi_{j',j} : \phi(j') \to \phi(j) \in \Delta.$$

Let Δ_{inj} be the wide subcategory of Δ on the injective maps and let $I(k)$ denote the full subcategory of $[k]^{\mathrm{op}} \times \Delta_{\mathrm{inj}/[k]}$ on those objects (j, J) such that $\varnothing \neq J \subset [j]$. For each $(f, \phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$, denote by $E(f, \phi)$ the composite

$$I(k) \hookrightarrow [k]^{\mathrm{op}} \times \Delta_{\mathrm{inj}/[k]} \xrightarrow{(f,\phi)\times O} \mathrm{Fin}_*^{\mathrm{op}} \times \Delta^2 \xrightarrow{\Pi(-)^{\mathrm{op}} \times \Sigma^{\bullet,\bullet}} \mathrm{Cat},$$

where O is the forgetful map from $\Delta_{\mathrm{inj}/[k]}$ to Δ.

Lemma 2.4.14. *The k-simplices of the unstraightening of $\mathcal{S}\mathrm{pan}_2\,(N(C))^{\times}$ are the pairs*

$$\big((f, \phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}}),\ \tau : E(f, \phi) \Rightarrow \mathrm{const}_C\big),$$

such that each component $\tau_{j,J} : \Pi(f(j))^{\mathrm{op}} \times \Sigma^{\phi(j),J} \to C$ is cartesian and vertically constant.

Proof. By Eq. 2.4.3 and Proposition 2.4.11, a k-simplex in the unstraightening of $\mathcal{S}\mathrm{pan}_2\,(N(C))^{\times}$ is an element $(f, \phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$ along with, for each $J \in \Delta_{\mathrm{inj}/[k]}$, a cartesian and vertically constant functor

$$\tau(J) : \Pi(f(\overline{J}))^{\mathrm{op}} \times \Sigma^{\phi(\overline{J}),J} \to C$$

[1] In fact, we only ever make use of the particular cases of $C = (\mathrm{Fin}_{\Delta})^{\mathrm{op}}$ or $(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}$. Performing a similar analysis as we present in this section for a general ∞-category would involve explicitly determining certain colimits in $\mathcal{C}\mathrm{at}_{\infty}$. This is both considerably more difficult and unnecessary for our purposes.

such that for $J \to L \in \Delta_{\mathrm{inj}/[k]}$ the following diagram commutes

$$\begin{array}{ccc} \Pi(f(\overline{L}))^{\mathrm{op}} \times \Sigma^{\phi(\overline{L}),J} & \longrightarrow & \Pi(f(\overline{J}))^{\mathrm{op}} \times \Sigma^{\phi(\overline{J}),J} \\ \downarrow & & \downarrow{\scriptstyle \tau(J)} \\ \Pi(f(\overline{L}))^{\mathrm{op}} \times \Sigma^{\phi(\overline{L}),L} & \xrightarrow[\tau(L)]{} & C \end{array} \tag{2.4.4}$$

where the functor along the top is induced by $\overline{J} \leq \overline{L} \in [k]$ and the functor along the left is induced by $J \subset L$. Note that for every string of n composable morphisms in $\Delta_{\mathrm{inj}/[k]}$ one can build an analogous commutative $(n+1)$-cube based on the commutativity of diagram 2.4.4 and the functoriality of $\Pi(-)^{\mathrm{op}} \times \Sigma^{\bullet,\bullet}$.

From this family $\{\tau(J)\}_{J \in \Delta_{\mathrm{inj}/[k]}}$ we shall construct a natural transformation $\tau : E(f,\phi) \Rightarrow \mathrm{const}_C$. For each $(j,J) \in I(k)$ one has a morphism $J \to [j] \in \Delta_{\mathrm{inj}/[k]}$, and hence the diagram 2.4.4 with L replaced by $[j]$ commutes. We define $\tau_{j,J}$ to be the composite functor along the diagonal of this diagram, which is cartesian and vertically constant by construction. Given a morphism $(j,J) \to (l,L) \in I(k)$ there is a triple of composable morphisms

$$J \to L \to [l] \to [j] \in \Delta_{\mathrm{inj}/[k]}.$$

The naturality of τ is a consequence of the commutative 4-cube constructed from this triple of composable morphisms.

Conversely, given a natural transformation $\tau : E(f,\phi) \Rightarrow \mathrm{const}_C$ having cartesian and vertically constant components, define $\tau(J) \coloneqq \tau_{\overline{J},J}$. The commutativity of diagram 2.4.4 follows directly from the naturality of τ. □

By the universal property of colimits in Cat, a natural transformation $E(f,\phi) \Rightarrow \mathrm{const}_C$ is equivalently a functor $\operatorname{colim} E(f,\phi) \to C$. Our first task in this section is to compute $\operatorname{colim} E(f,\phi)$. We shall then determine conditions under which a functor $\tau : \operatorname{colim} E(f,\phi) \to C$ restricts to cartesian and vertically constant functors $\tau_{j,J}$.

Computing the colimit of the diagram $E(f,\phi)$. Recall that the *Grothendieck construction* of a functor $G : D \to \mathrm{Cat}$ is the category having as objects

$$\{(g,d) \mid d \in D,\ g \in G(d)\},$$

and morphisms

$$(\alpha,\delta) : (g,d) \to (g',d'),\ \delta : d \to d' \text{ and } \alpha : G(\delta)(d) \to d'.$$

The key category underlying the computation of $\operatorname{colim} E(f,\phi)$ is the category M_ϕ, for $\phi \in N_k(\Delta^{\mathrm{op}})$, which is the Grothendieck construction of the functor

$$[k]^{\mathrm{op}} \xrightarrow{\ \phi\ } \Delta \hookrightarrow \mathrm{Cat}\ .$$

Explicitly, M_ϕ is the poset having object set

$$\{(a,b) \mid b \in [k]^{\mathrm{op}}, a \in \phi(b)\},$$

and ordering defined by declaring $(a,b) \leq (a',b')$ if and only if $b \leq b' \in [k]^{\mathrm{op}}$ and $\phi_{b,b'}(a) \leq a'$.

Example 2.4.15. For $\phi \in N_k(\Delta^{\mathrm{op}})$ the constant map on $[n]$, the poset M_ϕ is $[n] \times [k]^{\mathrm{op}}$.

Example 2.4.16. For the unique active morphism $\phi = ([2] \twoheadleftarrow [1]) \in N_1(\Delta^{\mathrm{op}})$, the poset M_ϕ is

$$\begin{array}{ccccc} (0,1) & & \longrightarrow & & (1,1) \\ \downarrow & & & & \downarrow \\ (0,0) & \longrightarrow & (1,0) & \longrightarrow & (2,0) \end{array}$$

Lemma 2.4.17. *The poset M_ϕ is the colimit of the diagram*

$$I(k) \hookrightarrow [k]^{\mathrm{op}} \times \Delta_{\mathrm{inj}/[k]} \xrightarrow{\phi \times \iota} \Delta^2 \xrightarrow{\bullet \times \bullet} \mathrm{Cat}\ . \tag{2.4.5}$$

Proof. As the category M_ϕ is obtained via the Grothendieck construction it is the colimit of the diagram

$$\Sigma^{([k]^{\mathrm{op}})} \longrightarrow [k]^{\mathrm{op}} \times [k] \xrightarrow{\phi \times [k]^{\mathrm{op}}_{-/}} \mathrm{Cat}\ .$$

Observe that, since $[k]^{\mathrm{op}}_{i/} = [i]^{\mathrm{op}} \simeq [i]$, one can obtain this diagram by precomposing the diagram in Eq. 2.4.5 with the functor

$$F : \Sigma^{([k]^{\mathrm{op}})} \to I(k), \quad [j;i] \mapsto (j, [i]).$$

The functor F is final, as for any $(j,J) \in I(k)$ with $\overline{J} = \max J$, the category $F^{(j,J)/}$ is the full subcategory of $\Sigma^{([k]^{\mathrm{op}})}$ on those objects

$$\{[a;b] \mid \overline{J} \leq b \leq a \leq j\},$$

which is non-empty and connected. □

Lemma 2.4.18. *For any $(f,\phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$, one has the isomorphism of categories*

$$\operatorname{colim} E(f,\phi) \simeq \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi}.$$

Proof. The diagonal inclusion of $I(k)$ into the full subcategory of $([k]^{\mathrm{op}})^2 \times \Delta_{\mathrm{inj}/[k]}$ on objects (i,j,J), where $J \subset [j]$ is final. Therefore $\operatorname{colim} E(f,\phi)$ is isomorphic to the product of the colimits over $\Pi(f)^{\mathrm{op}} : [k]^{\mathrm{op}} \to \mathrm{Cat}$ and the diagram

$$I(k) \hookrightarrow [k]^{\mathrm{op}} \times \Delta_{\mathrm{inj}/[k]} \xrightarrow{\phi \times \iota} \Delta^2 \xrightarrow{\Sigma^{\bullet,\bullet}} \mathrm{Cat}\ .$$

The colimit of the former is simply $\Pi(f(0))^{\mathrm{op}}$. For the latter colimit, Lemma 2.3.4 says that the twisted arrow construction preserves colimits, and so by Lemma 2.4.17 it is Σ^{M_ϕ}. □

The cartesian and vertical constancy conditions. Having determined $\operatorname{colim} E(f,\phi)$ we shall now define conditions under which a functor $\tau : \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \to C$ induces cartesian and vertically constant functors $\tau_{j,J} : \Pi(f(j))^{\mathrm{op}} \times \Sigma^{\phi(j),J} \to C$ for each $(j,J) \in I(k)$.

Definition 2.4.19. Define V_ϕ to be the sub-poset of M_ϕ on those morphisms $(a,b) \to (\phi_{b,b'}(a), b')$ and Λ^{M_ϕ} to be the full sub-poset of Σ^{M_ϕ} on those intervals $[(a,b);(\phi_{b,b'}(a'),b')]$ satisfying $|b'-b| \leq 1$ and $|a'-a| \leq 1$.

Example 2.4.20. For ϕ the constant map on $[n]$, one has that $V_\phi \simeq V_{n,k}$ and $\Lambda^{M_\phi} \simeq \Lambda^{n,k}$.

Example 2.4.21. For the unique active morphism $\phi = ([2] \twoheadleftarrow [1]) \in N_1(\Delta^{\mathrm{op}})$, the poset V_ϕ is

$$\begin{array}{ccccc} (0,1) & & & & (1,1) \\ \downarrow & & & & \downarrow \\ (0,0) & & (1,0) & & (2,0) \end{array}$$

and the poset Λ^{M_ϕ} is

$$\begin{array}{ccccccc} [(0,1);(0,1)] & \longleftarrow & & [(0,1);(1,1)] & & \longrightarrow & [(1,1);(1,1)] \\ \uparrow & & & \uparrow & & & \uparrow \\ [(0,1);(0,0)] & \longleftarrow & & [(0,1);(2,0)] & & \longrightarrow & [(1,1);(2,0)] \\ \downarrow & & \swarrow & & \searrow & & \downarrow \\ [(0,0);(0,0)] & \leftarrow & [(0,0);(1,0)] & \rightarrow & [(1,0);(1,0)] & \leftarrow & [(1,0);(2,0)] \rightarrow [(2,0);(2,0)] \end{array}$$

Definition 2.4.22. Let C be a category with finite limits and $(f,\phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$. Then we say a functor $\tau : \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \to C$ is:

1. *cartesian* if it is the right Kan extension of its restriction to $P(f(0))^{\mathrm{op}} \times \Lambda^{M_\phi}$.
2. *vertically constant* if morphisms of the form (Id, v) for $v \in \Sigma^{V_\phi}$ are sent to isomorphisms.

Remark 2.4.23. When $(f,\phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$ is the constant map on $(S_*, [n])$ this definition reduces to the notions already introduced for functors $\Pi(S)^{\mathrm{op}} \times \Sigma^{n,k} \to C$.

Since the categories M_ϕ for various $\phi \in N_k(\Delta^{\mathrm{op}})$ are obtained by the Grothendieck construction they are compatible in the following sense. For a natural transformation $\eta : \phi' \Rightarrow \phi$ one has a functor

$$M(\eta) : M_{\phi'} \to M_\phi,\ (a,b) \mapsto (\eta_b(a), b)$$

and for a morphism $\gamma : [n] \to [k]$ there is a functor

$$M(\gamma) : M_{\phi\gamma} \to M_\phi,\ (a,b) \mapsto (a, \gamma(b)).$$

Lemma 2.4.24. *Let C be an category having finite limits, $\phi \in N_k(\Delta^{\mathrm{op}})$ and $F : \Sigma^{M_\phi} \to C$ a cartesian functor. Then for a natural transformation $\eta : \phi' \Rightarrow \phi$ and a morphism $\gamma : [n] \to [k]$, the composite functors*

$$\Sigma^{M_{\phi'}} \xrightarrow{M(\eta)} \Sigma^{M_\phi} \xrightarrow{F} C \quad \text{and} \quad \Sigma^{M_{\phi\gamma}} \xrightarrow{M(\gamma)} \Sigma^{M_\phi} \xrightarrow{F} C$$

are cartesian.

The proof of this Lemma is somewhat technical and the details are unnecessary for the remainder of the text. We therefore separate its proof into Section 2.4.3.1.

Proposition 2.4.25. *Let C be a category with finite limits, and let $\langle C \rangle_k$ be the set*

$$\left\{ \left((f, \phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}}),\, \tau : \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \to C \right) \mid \tau \text{ cartesian, vertically constant} \right\}.$$

Then the sets $\langle C \rangle_k$ assemble into a sub simplicial set of the unstraightening of $\mathbb{S}\mathrm{pan}_2\left(N(C)\right)^{\times}$.

Remark 2.4.26. In general, $\langle C \rangle$ is a proper sub simplicial set of the unstraightening. This will nonetheless suffice for our purposes as we will make use of this explicit form to map *into* the unstraightening.

Proof. To begin, we observe that by Lemma 2.4.24, if τ is cartesian then so is $\gamma^*\tau$ for any $\gamma : [n] \to [k]$, where

$$\gamma^*\tau :\ \Pi(f\gamma(0))^{\mathrm{op}} \times \Sigma^{M_{\phi\gamma}} \xrightarrow{\Pi(f_{\gamma(0),0})^{\mathrm{op}} \times \Sigma^{M(\gamma)}} \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \xrightarrow{\tau} C\ .$$

Furthermore, since $M(\gamma)$ maps $V_{\phi\gamma}$ to V_ϕ, the functor $\gamma^*\tau$ is vertically constant whenever τ is. Therefore, the sets $\langle C \rangle_k$ assemble into a simplicial set.

By Lemma 2.4.18 the set of k-simplices of the unstraightening of $\mathbb{S}\mathrm{pan}_2\left(N(C)\right)^{\times}$ consists of pairs

$$\left\{ \left((f, \phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}}),\, \tau : \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \to C \right) \right\}$$

such that the induced functors $\tau_{j,J}$ are cartesian and vertically constant. To show that $\langle C \rangle$ is a sub simplicial set it therefore suffices to show that if τ is cartesian and vertically constant than so are the functors $\tau_{j,J}$. For the remainder of this proof we shall fix a cartesian, vertically constant functor τ.

Observe that $\phi(j) \times J^{\mathrm{op}}$ is $M_{c_{\phi(j)}}$, where $c_{\phi(j)} : J^{\mathrm{op}} \to \mathrm{Cat}$ is the constant functor on $\phi(j)$, and that one has a diagram

$$\begin{array}{ccc} J^{\mathrm{op}} & & \\ {\scriptstyle\rho}\downarrow & \overset{\eta}{\swarrow} & \searrow^{c_{\phi(j)}} \\ [k]^{\mathrm{op}} & \xrightarrow[\phi]{} & \mathrm{Cat}\,, \end{array}$$

where η is the natural transformation having components $\eta_i = \phi_{j,i} : \phi(j) \to \phi(i)$. Letting $\psi_{j,J}$ denote the composite functor

$$\psi_{j,J} :\ \phi(j) \times J \xrightarrow{\sim} M_{c_{\phi(j)}} \xrightarrow{M(\eta)} M_{\phi\rho} \xrightarrow{M(\rho)} M_\phi\ ,$$

one has that $\tau_{j,J}$ is

$$\Pi(f(j))^{\mathrm{op}} \times \Sigma^{\phi(j),J} \xrightarrow{\Pi(f_{j,0})^{\mathrm{op}} \times \Sigma^{\psi_{j,J}}} \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \xrightarrow{\tau} C\ .$$

It follows from Lemma 2.4.24 that $\tau_{j,J}$ is cartesian. The vertically constancy of $\tau_{j,J}$ follows from noting that $M(\eta)$ sends $V_{c_{\phi(j)}}$ to $V_{\phi\rho}$ and $M(\rho)$ sends $V_{\phi\rho}$ to V_ϕ.

□

2.4.3.1 Proof of Lemma 2.4.24

Throughout this subsection we shall fix $\phi, \phi' \in N_k(\Delta^{\mathrm{op}})$, a natural transformation $\eta : \phi' \Rightarrow \phi$, a morphism $\gamma : [n] \to [k]$ and a cartesian functor $F : \Sigma^{M_\phi} \to C$. We shall also make use of some notational shorthands. Set $\alpha = \Lambda^{M_{\phi'}}$, $A = \Sigma^{M_{\phi'}}$, $\theta = \Lambda^{M_{\phi\gamma}}$, $\Theta = \Sigma^{M_{\phi\gamma}}$, $\omega = \Lambda^{M_\phi}$, $\Omega = \Sigma^{M_\phi}$, $\xi = \Sigma^{M(\eta)}$ and $\zeta = \Sigma^{M(\gamma)}$.

We first show that $F \circ \xi$ is cartesian. Since F is cartesian, for each $x \in A$, the object $F\xi(x)$ is the limit of the diagram

$$\omega(\eta)^{\xi(x)/} \hookrightarrow \Omega \xrightarrow{F} C \ ,$$

where $\omega(\eta)$ is the full subcategory of Ω containing ω as well as those objects p such that $p \geq \xi(q)$ for some $q \in \alpha$. On the other hand, the right Kan extension of the restriction of $F\xi$ to α evaluated at $x \in A$ is the limit of the diagram

$$\alpha^{x/} \xrightarrow{\xi^{x/}} \omega(\eta)^{\xi(x)/} \hookrightarrow \Omega \xrightarrow{F} C \ .$$

To show that $F\xi$ is cartesian it therefore suffices to show that for each $x \in A$, the functor $\xi^{x/}$ is initial ([50] IX.3). That is, we must show that for each $y \in \omega(\eta)^{\xi(x)/}$, the poset

$$\xi^{x/}_{/y} = \{z \in \alpha \mid z \geq x,\ \xi(z) \leq y\}$$

is non-empty and connected.

Letting $y = [(c,d);(c',d')] \in \omega(\eta)^{\xi(x)/}$ we define the following

$$\hat{c} = \max\{i \in \phi'(d) \mid \eta_d(i) \leq c\} \text{ and } \hat{c}' = \min\{i \in \phi'(d) \mid \eta_{d'}(\phi'_{d,d'}(i)) \geq c'\}.$$

There are a now two cases to consider. First, if $\hat{c} \leq \hat{c}'$ then $\hat{y} = [(\hat{c},d);(\phi_{d,d'}(\hat{c}'),d')]$ is the maximal element of α satisfying $\xi(\hat{y}) \leq y$. Therefore $\xi^{x/}_{/y} = \alpha^{x/}_{/\hat{y}}$ and so $\xi^{x/}_{/y}$ is non-empty and connected as it has terminal object $\hat{y}$. Second, if $\hat{c}' \leq \hat{c}$ then from the inequalities

$$\phi_{d,d'}(c) \leq c' \leq \phi_{d,d'}\left(\eta_d(\hat{c}')\right) \leq \phi_{d,d'}\left(\eta_d(\hat{c})\right) \leq \phi_{d,d'}(c)$$

one concludes that $c' = \phi_{d,d'}(c)$, $\eta_d(\hat{c}') = \eta_d(\hat{c}) = c$, and $[\hat{c}';\hat{c}]$ is the largest subinterval of $\phi'(d)$ sent to $[c;c]$ under η_d. The poset $\xi^{x/}_{/y}$ is non-empty as it contains $[(\hat{c},d);(\phi'_{d,d'}(\hat{c}),d']$. Letting $[p;q]$ denote the maximal subinterval of $[\hat{c}';\hat{c}]$ such that $x \leq [(p,d);(q,d)]$, one has that for any $z \in \xi^{x/}_{/y}$ there is at least one $r \in [p;q]$ such that $z \leq [(r,d);(r,d)]$. As the poset $\xi^{x/}_{/y}$ contains the connected sub-poset

$$\{[(u,d);(v,d)] \mid [u;v] \subset [p;q],\ |u-v| \leq 1\}$$

and each element maps into this sub-poset it follows that $\xi^{x/}_{/y}$ is connected.

Next, we will show that $F \circ \zeta$ is cartesian. The argument is quite similar to the above, and we shall recycle certain notation. It suffices to show that for each $x \in \Theta$ and $y \in \omega(\gamma)^{\zeta(x)/}$, the poset

$$\zeta^{x/}_{/y} = \{z \in \theta \mid z \geq x,\ \zeta(z) \leq y\} \ ,$$

is non-empty and connected, where $\omega(\gamma)$ is full subcategory of Ω containing ω and everything above the image of θ under ζ. Set $y = [(c,d);(c',d')] \in \omega(\gamma)^{\xi(x)/}$ and

$$\hat{d} = \max\{i \in [n]^{\mathrm{op}} \mid \gamma(i) \leq d \in [k]^{\mathrm{op}}\} \text{ and } \hat{d}' = \min\{i \in [n]^{\mathrm{op}} \mid \gamma(i) \geq d' \in [k]^{\mathrm{op}}\}.$$

There are again two cases. The first is when $\hat{d} \leq \hat{d}'$, then $\hat{y} = [(c,\hat{d});(c',\hat{d}')]$ is the maximal element of θ satisfying $\zeta(\hat{y}) \leq y$, proving as above that $\zeta^{x/}_{/y}$ is non-empty and connected. Next, if $\hat{d}' \leq \hat{d}$ then $d' = d$ and $[\hat{d}';\hat{d}]$ is the largest subinterval of $[n]^{\mathrm{op}}$ sent to $[d;d]$. The same analysis as above, mutatis mutundi, shows that $\zeta^{x/}_{/y}$ is non-empty and connected.

2.5 Corepresenting algebraic structures

2.5.1 Algebra and coalgebra objects

To specify an algebra object in a symmetric monoidal $(\infty,2)$-category one must provide not just an associative and unital binary operation, but a coherent choice of higher associativity and unitality data. To package together all of this data we will make use of a category originally introduced by Pirashvili [54].

Denote by Alg the category having as objects finite sets. A morphism in Alg is a function $p : X \to Y$ along with a choice of linear ordering of the (possibly empty) preimages $p^{-1}(y)$ for each $y \in Y$. The composition of a pair of composable morphisms

$$X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_2} X_2$$

is the composition of the underlying functions, with linear ordering on $(p_2p_1)^{-1}(x_2)$ given by

$$(p_2p_1)^{-1}(x_2) = \bigvee_{x_1 \in p_2^{-1}(x_2)} p_1^{-1}(x_1),$$

where the wedge denotes the monoidal structure on Δ_+, the category of finite linear orders, introduced in Eq. 1.3.2. The disjoint union endows Alg with a symmetric monoidal structure.

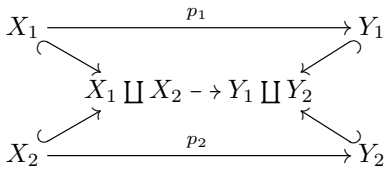
The category Alg corepresents algebra objects in the sense that, for a symmetric monoidal category C, symmetric monoidal functors $\mathrm{Alg} \to C$ are the same as algebra objects in C. This is because Alg is the *category of operators* [51] for the Σ-operad of associative algebras, that is, $\mathrm{Hom}_{\mathrm{Alg}}(\underline{n},\underline{1}) \simeq \Sigma_n$ and all morphisms in Alg are, up to precomposition with an isomorphism, disjoint unions of these. Dually, the symmetric monoidal category $\mathrm{Coalg} := \mathrm{Alg}^{\mathrm{op}}$ corepresents coalgebra objects.

Remark 2.5.1. One only needs a monoidal structure on a category to define algebra objects in it. The simpler category Δ_+ corepresents algebra objects in *monoidal* categories and so one can equivalently define an algebra object in a symmetric monoidal category C to be a *monoidal* functor from Δ_+ to C. However, one needs at least a braiding on a monoidal category to define bialgebra objects in it. For our purposes it is therefore necessary to work the category Alg instead of the simpler Δ_+.

To corepresent algebra and coalgebra objects in a symmetric monoidal $(\infty, 2)$-category it suffices to present Alg and Coalg in the model of symmetric monoidal $(\infty, 2)$-categories that we use in this paper. Recall that the symmetric monoidal structure on an ∞-category $\mathcal{D}$ having finite coproducts is given by the functor

$$\mathcal{D}^{\amalg} : \mathcal{F}\mathrm{in}_* \to \mathcal{C}\mathrm{at}_\infty, \quad S_* \mapsto \mathcal{F}\mathrm{un}^{\mathrm{cocart}}\left(\mathbb{\Pi}(S), \mathcal{D}\right).$$

While the disjoint union is *not* the coproduct in Alg, it is the case that given morphisms $p_i : X_i \to Y_i$ there is a unique morphism making the following diagram commute



Define a functor $F : \Pi(S) \times [n] \to \mathrm{Alg}$ to be *cocartesian* if for each $i \in [n]$, the composite

$$\Pi(S) \xrightarrow{F(-,i)} \mathrm{Alg} \xrightarrow{\mathrm{forget}} \mathrm{Fin}$$

is cocartesian. One has by the above that the functor

$$\mathcal{A}\mathrm{lg}^{\amalg} : \mathrm{Fin}_* \times \Delta^{\mathrm{op}} \to \mathrm{qCat}^0$$

sending $(S_*, [n])$ to the set of cocartesian functors $\Pi(S) \times [n] \to \mathrm{Alg}$ is a symmetric monoidal $(\infty, 2)$-category.

Similarly, define a functor $F : \Pi(S)^{\mathrm{op}} \times [n] \to \mathrm{Coalg}$ to be *cartesian* if for each $i \in [n]$, the composite

$$\Pi(S)^{\mathrm{op}} \xrightarrow{F(-,i)} \mathrm{Coalg} \xrightarrow{\mathrm{forget}} \mathrm{Fin}^{\mathrm{op}}$$

is cartesian. Then the functor

$$\mathcal{C}\mathrm{oalg}^{\amalg} : \mathrm{Fin}_* \times \Delta^{\mathrm{op}} \to \mathrm{qCat}^0$$

sending $(S_*, [n])$ to the set of cartesian functors $\Pi(S)^{\mathrm{op}} \times [n] \to \mathrm{Coalg}$ is a symmetric monoidal $(\infty, 2)$-category. By construction, $\mathcal{C}\mathrm{oalg}^{\amalg}$ is the opposite of $\mathcal{A}\mathrm{lg}^{\amalg}$.

We can now define the algebraic structures which are the focus of Chapter 3.

Definition 2.5.2. Let $\mathcal{B}^{\otimes}$ be a symmetric monoidal $(\infty, 2)$-category.

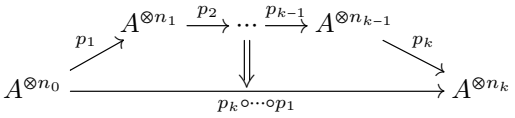
- An *algebra object* in $\mathcal{B}^{\otimes}$ is a symmetric monoidal functor $\mathcal{A}\mathrm{lg}^{\amalg} \to \mathcal{B}^{\otimes}$.
- A *coalgebra object* in $\mathcal{B}^{\otimes}$ is a symmetric monoidal functor $\mathcal{C}\mathrm{oalg}^{\amalg} \to \mathcal{B}^{\otimes}$.
- A *lax algebra object* in $\mathcal{B}^{\otimes}$ is a symmetric monoidal lax functor $\mathcal{A}\mathrm{lg}^{\amalg} \rightsquigarrow \mathcal{B}^{\otimes}$.
- A *lax coalgebra object* in $\mathcal{B}^{\otimes}$ is a symmetric monoidal lax functor $\mathcal{C}\mathrm{oalg}^{\amalg} \rightsquigarrow \mathcal{B}^{\otimes}$.

Remark 2.5.3. In each of the definitions above, one should think of such a functor as endowing the image of $\underline{1}$ in $\mathcal{B}^{\otimes}$ with the corresponding algebraic structure.

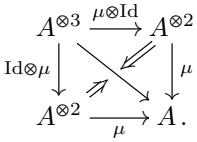
It is worth taking a moment to informally discuss the exact nature of a lax algebra object A, as it is slightly more subtle than one might initially expect. For each string of composable morphisms in Alg,

$$\underline{n}_0 \xrightarrow{p_1} \underline{n}_1 \xrightarrow{p_2} \cdots \xrightarrow{p_k} \underline{n}_k$$

one has 2-morphisms



which are compatible with disjoint union and the composition of morphisms in Alg. In particular, the witness to the lax associativity of the product on A is a diagram



2.5.2 Bialgebra objects

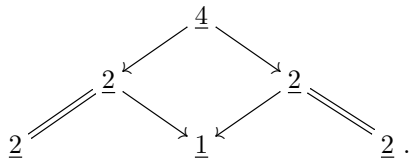
A bialgebra object in a symmetric monoidal $(\infty, 2)$-category is, just like in an ordinary symmetric monoidal category, an object equipped with compatible algebra and coalgebra structures. The compatibility condition requires that the data defining the coalgebra structure be algebra homomorphisms, or equivalently, that the data defining the algebra structure be coalgebra homomorphisms. The data required for defining a bialgebra object can be corepresented by a combinatorially defined symmetric monoidal ∞-category $\mathcal{B}\mathrm{ialg}^{\amalg}$ originally introduced by Pirashvili [54] and Lack [43].

Naively, one would like to define $\mathcal{B}\mathrm{ialg}$ to be $\mathcal{S}\mathrm{pan}\,(\mathrm{Alg})$, but unfortunately Alg does not admit all pullbacks. Temporarily putting that issue aside, let us consider how $\mathcal{S}\mathrm{pan}\,(\mathrm{Alg})$ could corepresent bialgebra objects. By considering morphisms of the form



one observes that $\mathcal{S}\mathrm{pan}\,(\mathrm{Alg})$ contains Alg and Coalg as wide subcategories. Therefore a symmetric monoidal functor out of $\mathcal{S}\mathrm{pan}\,(\mathrm{Alg})$ specifies an object carrying both an algebra and a coalgebra structure. The compatibility of these structures arises from the requirement that functors preserve composition of spans. For example, the condition that the coproduct respects products arises from

the composite



To give a proper definition of $\mathcal{B}$ialg we must find a way to compose span diagrams in Alg despite its lack of pullbacks.

Definition 2.5.4 ([54])**.** A (not necessarily commutative) square in Alg

$$\begin{array}{ccc} X' & \xrightarrow{p'} & Y' \\ {\scriptstyle q'}\downarrow & & \downarrow{\scriptstyle q} \\ X & \xrightarrow[p]{} & Y \end{array}$$

is called a *pseudo-pullback square* if

1. the image of the square under the forgetful functor Alg $\to$ Fin is a pullback square;
2. for each $x \in X$, the induced map $p_* : (q')^{-1}(x) \to q^{-1}(p(x))$ is an isomorphism of ordered sets; and,
3. for each $y \in Y'$ the induced map $q_* : (p')^{-1}(y) \to p^{-1}(q(y))$ is an isomorphism of ordered sets.

Example 2.5.5. Consider the following pullback square of finite sets

$$\begin{array}{ccc} \underline{4} & \xrightarrow{\overline{m}^2} & \underline{2} \\ {\scriptstyle m^2}\downarrow & & \downarrow{\scriptstyle m} \\ \underline{2} & \xrightarrow[m]{} & \underline{1} \end{array}$$

Denote by $\underline{2} = \{y_1, y_2\}$, $\underline{4} = \{y_{1,1}, y_{1,2}, y_{2,1}, y_{2,2}\}$, $m^2(y_{i,j}) = y_i$ and $\overline{m}^2(y_{i,j}) = y_j$. If one lifts this to a diagram in Alg by setting the ordering on the fibres of m to be $y_1 \leq y_2$, such a square is a pseudo-pullback when one equips m^2 and $\overline{m}^2$ with the orderings

$$(m^2)^{-1}(y_i) = (y_{i,1} \leq y_{i,2}) \text{ and } (\overline{m}^2)^{-1}(y_i) = (y_{1,i} \leq y_{2,i}).$$

The resulting diagram in Alg *does not commute*: the composites $m \circ m^2$ and $m \circ \overline{m}^2$ define, respectively, the following inequivalent orderings on $\underline{4}$:

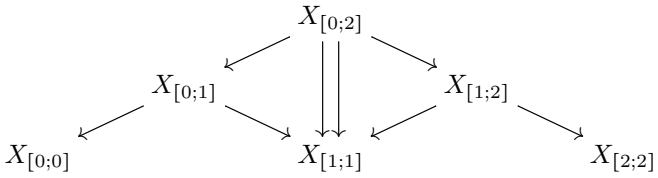
$$(y_{1,1} \leq y_{1,2} \leq y_{2,1} \leq y_{2,2}) \text{ and} (y_{1,1} \leq y_{2,1} \leq y_{1,2} \leq y_{2,2}).$$

The failure of commutativity in the above example is a general property of pseudo-pullback squares in Alg. Nonetheless, since pseudo-pullback squares Alg are sent to pullback squares in Fin under the forgetful functor and isomorphisms in Fin have unique lifts in Alg, such squares do satisfy all of the properties required of pullback squares to define the composition in a category of spans.

The only care that must be taken is that one can no longer use diagrams of the form Σ^n to define strings of composable morphisms.

For each $n \geq 0$, let G^n be the reflexive directed graph having vertices subintervals $[a;b] \subset [n]$. There is an edge $[a;b] \to [a';b']$ if and only if $[a';b'] \subset [a;b]$ and $[a;b]$ contains at most one more element than $[a';b']$. Define the categories Λ^n to be the free categories on G^n.

Example 2.5.6. A functor $\Lambda^2 \to \mathrm{Alg}$ is a diagram,



A Λ^n-diagram is a not-necessarily-commutative diagram in the same shape as a Σ^n-diagram.

The categories Λ^n assemble into a functor $\Lambda^\bullet : \Delta \to \mathrm{Cat}$. A morphism $\phi : [n] \to [m]$ is sent to the functor $\phi_* : \Lambda^n \to \Lambda^m$ which is defined on objects by sending $[a;b]$ to $[\phi(a);\phi(b)]$ and on the generating morphisms by

$$\begin{aligned} ([a;b] \to [a+1;b]) &\mapsto \Big([\phi(a);\phi(b)] \to [\phi(a)+1;\phi(b)] \to \cdots \to [\phi(a+1);\phi(b)] \Big) \\ ([a;b] \to [a;b-1]) &\mapsto \Big([\phi(a);\phi(b)] \to [\phi(a);\phi(b)-1] \to \cdots \to [\phi(a+1);\phi(b-1)] \Big) . \end{aligned} \tag{2.5.1}$$

We define a functor $F : \Lambda^n \to \mathrm{Alg}$ to be *pseudo-cartesian* if for each $\phi : [2] \to [n]$ the middle square of $\phi^* F : \Lambda^2 \to \mathrm{Alg}$ is a pseudo-pullback.

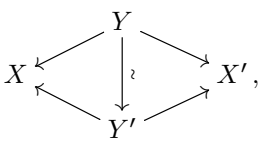
Definition 2.5.7. For each $(S_*, [n]) \in \mathrm{Fin}_* \times \Delta^{\mathrm{op}}$, let $\mathcal{B}\mathrm{ialg}_{S,n}$ denote the nerve of the groupoid $\mathrm{Bialg}_{S,n}$ of functors $F : \Pi(S) \times \Lambda^n \to \mathrm{Alg}$ such that

1. For each $U \in \Pi(S)$, the functor $F(U, -) : \Lambda^n \to \mathrm{Alg}$ is pseudo-cartesian.
2. For each $x \in \Lambda^n$, the functor $F(-, x) : \Pi(S) \to \mathrm{Alg}$ is cocartesian.

These assemble into a symmetric monoidal $(\infty, 2)$-category

$$\mathcal{B}\mathrm{ialg}^{\sqcup} : \mathrm{Fin}_* \times \Delta^{\mathrm{op}} \to \mathrm{qCat}^0, \quad (S_*, [n]) \mapsto \mathcal{B}\mathrm{ialg}_{S,n}.$$

Remark 2.5.8. The homotopy bicategory of $\mathcal{B}\mathrm{ialg}^{\sqcup}$, denoted $\mathrm{Bialg}^{\sqcup}$, has the same objects as Alg, 1-morphisms spans, 2-morphisms diagrams



and a symmetric monoidal structure given by disjoint union.

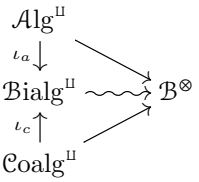
Note that projecting on to the right and left endpoints define, respectively, natural transformations $\curlywedge^{\bullet} \Rightarrow [\bullet]$ and $\curlywedge^{\bullet} \Rightarrow [\bullet]^{\mathrm{op}}$. These induce, respectively, symmetric monoidal functors

$$\mathcal{A}\mathrm{lg}^{\sqcup} \xrightarrow{\iota_a} \mathcal{B}\mathrm{ialg}^{\sqcup} \qquad \text{and} \qquad \mathcal{C}\mathrm{oalg}^{\sqcup} \xrightarrow{\iota_c} \mathcal{B}\mathrm{ialg}^{\sqcup}.$$

The following bialgebraic structures will be the focus of Chapters 4 and 5.

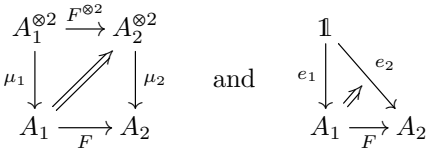
Definition 2.5.9. Let $\mathcal{B}^{\otimes}$ be a symmetric monoidal $(\infty, 2)$-category.

- A *bialgebra object* in $\mathcal{B}^{\otimes}$ is a symmetric monoidal functor $\mathcal{B}\mathrm{ialg}^{\sqcup} \to \mathcal{B}^{\otimes}$.
- A *totally lax bialgebra object* in $\mathcal{B}^{\otimes}$ is a symmetric monoidal lax functor $\mathcal{B}\mathrm{ialg}^{\sqcup} \rightsquigarrow \mathcal{B}^{\otimes}$.
- A *lax bialgebra object* in $\mathcal{B}^{\otimes}$ is a totally lax bialgebra object in $\mathcal{B}^{\otimes}$ which restricts to symmetric monoidal functors $\mathcal{A}\mathrm{lg}^{\sqcup} \to \mathcal{B}^{\otimes}$ and $\mathcal{C}\mathrm{oalg}^{\sqcup} \to \mathcal{B}^{\otimes}$, i.e.,

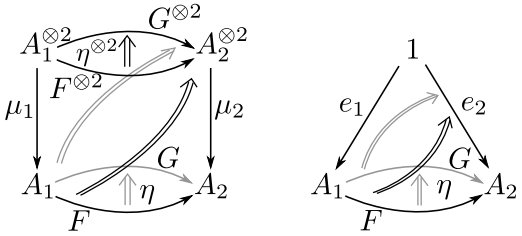


It is both instructive and necessary for our applications in Chapter 5 to unpackage the data of a lax bialgebra in a symmetric monoidal bicategory. To do so we must first recall a few notions.

Let $B^{\otimes}$ be a symmetric monoidal bicategory and let (A_1, μ_1, e_1) and (A_2, μ_2, e_2) be algebra objects in $B^{\otimes}$. An *oplax* 1*-homomorphism* $A_1 \nrightarrow A_2$ is a morphism $F : A_1 \to A_2$ along with 2-morphisms



which must satisfy the usual associativity and unitality constraints. A 2*-homomorphism* between oplax 1-homomorphisms is a natural transformation $\eta : F \Rightarrow G$ such that the following diagrams commute



Putting these together one has a symmetric monoidal bicategory $\mathrm{Alg}_{\mathrm{oplax}}(B)^{\otimes}$ having objects algebras in $B^{\otimes}$, 1-morphisms oplax 1-homomorphisms and 2-morphisms 2-homomorphisms.

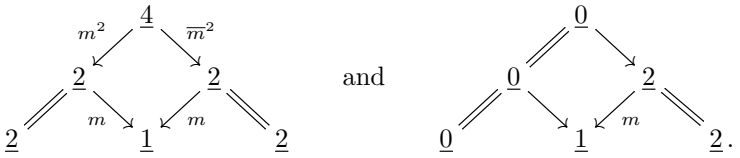
Remark 2.5.10. When $B^{\otimes}$ is the 2-category of categories this is the 2-category of monoidal categories, oplax monoidal functors and monoidal natural transformations.

Proposition 2.5.11. *Let $B^{\otimes}$ be a symmetric monoidal bicategory. Then a lax bialgebra object in $B^{\otimes}$ is the same thing as a coalgebra object in* $\mathrm{Alg}_{\mathrm{oplax}}(B)^{\otimes}$.

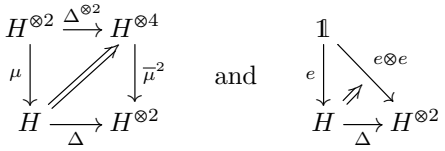
Remark 2.5.12. This should be compared with the familiar statement that a bialgebra in an ordinary symmetric monoidal category is a coalgebra object in the category of algebras.

Proof. Let $\beta : \mathrm{Bialg}^{\amalg} \rightsquigarrow B^{\otimes}$ be a symmetric monoidal lax functor such that $\beta\iota_a$ and $\beta\iota_c$ are symmetric monoidal. Then $H \coloneqq \beta(\underline{1})$ is both an algebra and coalgebra object in $B^{\otimes}$. Letting $(\mu, e, \alpha, \lambda, \rho)$ and $(\Delta, \epsilon, \alpha^c, \lambda^c, \rho^c)$ denote the data defining, respectively, the algebra and coalgebra structures, it remains to show Δ and ϵ come endowed with 2-morphisms making them oplax 1-homomorphisms and that α^c, λ^c and ρ^c are 2-homomorphisms.

This amounts to specifying four 2-morphisms in B, two each for the oplax structures on Δ and ϵ, and checking fourteen coherence equations. These coherence equations all trivially follow from compositions in Bialg, so we shall only describe the oplax structures. The lax structure on β corresponding to the composites

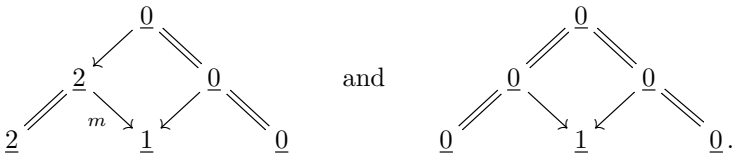


are 2-morphisms

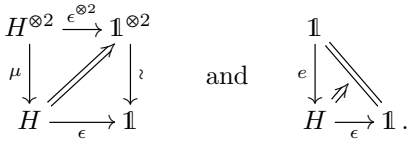


(2.5.2)

and the composites



yield 2-morphisms



(2.5.3)

□

Chapter 3

The universal Hall algebra

In this chapter we provide a new construction of the universal Hall algebra and universal Hall coalgebra of a 2-Segal space. Summarising this chapter, we show the following.

Theorem. *Every simplicial space $\mathcal{X}_\bullet$ endows the space $\mathcal{X}_1$ with the structure of a lax (co)algebra in the $(\infty, 2)$-category of bispans of spaces. Furthermore, $\mathcal{X}_\bullet$ is a 2-Segal space if and only if it endows $\mathcal{X}_1$ with the structure of a (co)algebra. We call this the* universal Hall (co)algebra *of $\mathcal{X}_\bullet$.*

In Section 3.1 we begin with a review of the definition of 2-Segal spaces as well as a discussion of the 2-Segal spaces $S_\bullet(A)$ for A a *proto-exact category*. Section 3.2 proves the first part of the above Theorem, namely, the construction from a simplicial space a lax (co)algebra. The second part of the above Theorem is proven in Section 3.3. Finally, we conclude the chapter with a thorough account of the truncation of the universal Hall (co)algebra to the homotopy bicategory of bispans in Section 3.4.

3.1 2-Segal spaces

2-Segal spaces, as independently introduced by Dyckerhoff–Kapranov [17] and Gálvez-Carrillo–Kock–Tonks [23], are simplicial spaces satisfying a 2-dimensional generalisation of the Segal conditions. Let $\mathcal{C}$ be an ∞-category having finite limits and $\mathcal{X} \in \mathcal{C}_{\Delta}$ a simplicial object in $\mathcal{C}$. Then by right Kan extension of $\mathcal{X}$ along $\Delta^{\mathrm{op}} \to (\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}}$ there is a functor $\mathcal{X} : (\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}} \to \mathcal{C}$. Recall that $\mathcal{X}$ satisfies the Segal condition if for each $n \geq 2$ and $0 \leq i \leq n$ the image of the square

$$\begin{array}{ccc} \Delta[\{i\}] & \longrightarrow & \Delta[\{i, \dots, n\}] \\ \downarrow & & \downarrow \\ \Delta[\{0, \dots, i\}] & \longrightarrow & \Delta[n] \end{array}$$

under $\mathcal{X}$ is a pullback in $\mathcal{C}$.

Definition 3.1.1. [17] 2.3.2 A simplicial object $X \in \mathcal{C}_{\Delta}$ is a 2-Segal object if and only if for every $n \geq 3$ and every $0 \leq i < j \leq n$, the image of the squares

$$\begin{array}{ccc} \Delta[\{0,i\}] & \longrightarrow & \Delta[\{0,i,\ldots,n\}] \\ \downarrow & & \downarrow \\ \Delta[\{0,\ldots,i\}] & \longrightarrow & \Delta[n] \end{array} \qquad \text{and} \qquad \begin{array}{ccc} \Delta[\{j,n\}] & \longrightarrow & \Delta[\{0,\ldots,j\}] \\ \downarrow & & \downarrow \\ \Delta[\{j,\ldots,n\}] & \longrightarrow & \Delta[n] \end{array} \tag{3.1.1}$$

under X are pullbacks in $\mathcal{C}$ and for each $n \geq 2$ and $0 \leq i < n$, the image of the square

$$\begin{array}{ccc} \Delta[\{i,i+1\}] & \longrightarrow & \Delta[n] \\ \downarrow & & \downarrow \\ \Delta[\{i\}] & \longrightarrow & \Delta[n-1] \end{array} \tag{3.1.2}$$

under X is a pullback in $\mathcal{C}$.

In particular, a 2-*Segal space* is a 2-Segal object in $\mathcal{S}$, the ∞-category of spaces.

Remark 3.1.2. What we call a 2-Segal object is called a *unital* 2-*Segal object* in [17] and a *decomposition space* in [23].

Remark 3.1.3. For a simplicial object X to be 2-Segal it suffices for the images under X of the squares in Eq. 3.1.1 to be equivalences for $i = 0$ or $j = n$ ([17] 2.3.2).

A large number of examples of 2-Segal objects are constructed in [17, 23, 25, 26, 28]. In this thesis, particularly in Chapter 5, we will primarily be concerned with 2-Segal objects in the ∞-category $\mathcal{G}\mathrm{rpd}$, the ∞-category associated to the canonical model structure on category of groupoids [1].

A important class of examples of 2-Segal objects in $\mathcal{G}\mathrm{rpd}$ come from applying the Waldhausen S-construction to so-called *proto-exact categories*.

Definition 3.1.4 ([17] 2.4.2)**.** A *proto-exact category* is a category A having zero object 0 equipped with two isomorphism containing subcategories: the admissible monomorphisms, denoted $\mathfrak{M}$, and the admissible epimorphisms, denoted $\mathfrak{E}$. These subcategories must satisfy the following:

1. For each object $a \in A$, the unique morphism $0 \to a$ must be in $\mathfrak{M}$ and the unique morphism $a \to 0$ must be in $\mathfrak{E}$;

2. Every square
$$\begin{array}{ccc} a & \xrightarrow{e_0} & b \\ {\scriptstyle m_0}\downarrow & & \downarrow{\scriptstyle m_1} \\ c & \xrightarrow[e_1]{} & d \end{array} \tag{3.1.3}$$
with $m_i \in \mathfrak{M}$ and $e_i \in \mathfrak{E}$ is a pullback square if and only if it is a pushout square; and

3. Every diagram $c \xleftarrow{m_0} a \xrightarrow{e_0} b$ with $m_0 \in \mathfrak{M}$ and $e_0 \in \mathfrak{E}$ can be completed to a pushout square of the form 3.1.3 with $m_1 \in \mathfrak{M}$ and $e_1 \in \mathfrak{E}$; and,

4. Every diagram $c \xrightarrow{e_1} d \xleftarrow{m_1} b$ with $m_1 \in \mathfrak{M}$ and $e_1 \in \mathfrak{E}$ can be completed to a pullback square of the form 3.1.3 with $m_0 \in \mathfrak{M}$ and $e_0 \in \mathfrak{E}$.

Abelian categories are examples of proto-exact categories, where one takes the admissible mono- and epi- morphisms to be the categorical mono- and epi- morphisms. Here are further non-abelian examples, the first two of which will be of relevance in Chapter 5.

Example 3.1.5. Let $\mathrm{Vect}_{\mathbb{F}_1}$ denote the category whose objects are pointed finite sets $S_0 = S \coprod \{0\}$ and whose morphisms are pointed maps $f : S_0 \to T_0$ such that $f|_{S_0 \smallsetminus f^{-1}(0)}$ is injective. Declaring the admissible monomorphisms to be the injections and the admissible epimorphisms to be the surjections endows $\mathrm{Vect}_{\mathbb{F}_1}$ with the structure of a proto-exact category.

A skeleton of $\mathrm{Vect}_{\mathbb{F}_1}$ has objects indexed by $\mathbb{N}$, the non-negative integers. The set of morphisms from the object $\underline{n}$ to the object $\underline{m}$ is the set of $m \times n$ matrices having entries in $\{0, 1\}$ such that every row and column has at most one non-zero entry. Composition is given by matrix multiplication. We will not distinguish between this skeleton and $\mathrm{Vect}_{\mathbb{F}_1}$ in the remainder of this work.

Remark 3.1.6. As suggested by the notation, $\mathrm{Vect}_{\mathbb{F}_1}$ should be thought of as the category of finite dimensional vector spaces over $\mathbb{F}_1$, the mythical 'field with one element' [12]. The linear nature of $\mathrm{Vect}_{\mathbb{F}_1}$ is revealed by combining the observation that it is equivalent to the category of spans in the category of finite sets and injections with the discussion in Remark 1.1.1.

The category $\mathrm{Vect}_{\mathbb{F}_1}$ has a symmetric monoidal structure given by smash product with unit $\{0\}$. A *finite dimensional algebra over* $\mathbb{F}_1$, that is, an algebra object in $\mathrm{Vect}_{\mathbb{F}_1}$, is a pointed finite monoid M_0: A monoid M_0 with zero element $0 \in M_0$ satisfying $m \cdot 0 = 0 = 0 \cdot m$ for all $m \in M_0$. The subset M of M_0 is a monoid, and any finite monoid M' can be extended to a pointed finite monoid M_0' by adjoining the zero element. These operations are inverses to one another, defining an isomorphism between the category of finite dimensional algebras over $\mathbb{F}_1$ and the category of finite monoids.

Similarly, a *finite dimensional division algebra over* $\mathbb{F}_1$, a group object in $\mathrm{Vect}_{\mathbb{F}_1}$, is a pointed finite monoid Γ_0 such that the submonoid Γ is a finite group. The category of finite dimensional division algebras over $\mathbb{F}_1$ is isomorphic to the category of finite groups.

Example 3.1.7. For a finite dimensional division algebra Γ_0 over $\mathbb{F}_1$, define $\Gamma_0\text{-Mod}_{\mathbb{F}_1}$ to be the category of finite dimensional modules over Γ_0. Every group homomorphism $\Gamma \to \Gamma'$ induces a functor $\Gamma_0\text{-Mod}_{\mathbb{F}_1} \to \Gamma_0'\text{-Mod}_{\mathbb{F}_1}$. In particular, the homomorphism $\Gamma \to \{e\}$ induces a functor $\Gamma_0\text{-Mod}_{\mathbb{F}_1} \to \{e\}_0\text{-Mod}_{\mathbb{F}_1} = \mathrm{Vect}_{\mathbb{F}_1}$. The categories $\Gamma_0\text{-Mod}_{\mathbb{F}_1}$ inherit a proto-exact structure by declaring a morphism to be an admissible mono- or epi- morphism if and only if its image in $\mathrm{Vect}_{\mathbb{F}_1}$ is such.

A skeleton of $\Gamma_0\text{-Mod}_{\mathbb{F}_1}$ has objects indexed by $\mathbb{N}$. The set of morphisms from the object $\underline{n}$ to the object $\underline{m}$ is the set of $m \times n$ matrices having entries in Γ_0 such that every row and column has

at most one non-zero entry. Composition is again given by matrix multiplication using the product on Γ_0. Note that the automorphism group of the object $\underline{n}$ is $\Gamma \wr \Sigma_n$, the n-th wreath product of Γ.

Example 3.1.8. Many more examples arise in the study of geometry over $\mathbb{F}_1$, such as in the work of Szczesny [64, 61, 62, 63] and Deitmar [13, 14].

Given a proto-exact category A, the Waldhausen S-construction [67] produces a simplicial groupoid

$$S_\bullet(A) : \Delta^{\mathrm{op}} \to \mathrm{Grpd}$$

as follows. One begins by defining the cosimplicial category $\mathrm{Arr}_\bullet = \mathrm{Fun}([1], \bullet) : \Delta \to \mathrm{Cat}$. Then $S_\bullet(A)$ is the sub simplicial groupoid of $\mathrm{Map}(\mathrm{Arr}_\bullet, A)$ where for each n, the groupoid $S_n(A)$ is the full subgroupoid of functors $F \in \mathrm{Fun}(\mathrm{Arr}_n, A)$ of the following form

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F_{01} & \longrightarrow & F_{02} & \longrightarrow & \cdots & \longrightarrow & F_{0n} \\
 & & \downarrow & & \downarrow & & & & \downarrow \\
 & & 0 & \longrightarrow & F_{12} & \longrightarrow & \cdots & \longrightarrow & F_{1n} \\
 & & & & \downarrow & & & & \downarrow \\
 & & & & 0 & \longrightarrow & \cdots & \longrightarrow & F_{2n} \\
 & & & & & & & & \downarrow \\
 & & & & & & \ddots & & \vdots \\
 & & & & & & & & \downarrow \\
 & & & & & & & & 0
\end{array} \tag{3.1.4}$$

where all horizontal morphisms are in $\mathfrak{M}$, vertical morphisms in $\mathfrak{E}$, and every square is a pushout, or equivalently a pullback..

In particular, $S_0(A)$ is the trivial groupoid, $S_1(A)$ is the groupoid of objects in A and $S_2(A)$ is the groupoid of *admissible short exact sequences* in A: the groupoid of pullback, or equivalently a pushout squares, of the form

$$\begin{array}{ccc}
a & \xrightarrow{f} & b \\
\downarrow & & \downarrow g \\
0 & \longrightarrow & c
\end{array}$$

with $f \in \mathfrak{M}$ and $g \in \mathfrak{E}$. In general, $S_n(A)$ is equivalent to the groupoid of length n *admissible flags* in A. Both Dyckerhoff–Kapranov ([17] 2.4.8) and Gálvez-Carrillo–Kock–Tonks ([23] 3.5) have shown that the simplicial groupoid $S(A) \in \mathcal{G}\mathrm{rpd}_\Delta$ is a 2-Segal object.

Remark 3.1.9. Bergner–Osorno–Ozornova–Rovelli–Scheimbauer have introduced an extension of the Waldhausen S-construction which induces an equivalence between the category of *augmented stable double categories* and 2-Segal sets [9]. A work in progress from the same authors is to demonstrate that a further extension of the Waldhausen S-construction induces an equivalence between *augmented stable double Segal spaces* and 2-Segal spaces.

We now turn our attention to the construction of the universal Hall algebra of a general 2-Segal space, returning to the special case of the 2-Segal spaces $S_\bullet(A)$ in Section 3.4.

3.2 Simplicial spaces define lax algebras in $\mathcal{S}\mathrm{pan}_2(\mathcal{S})^{\times}$

As the first step in our construction of the universal Hall (co)algebra of a 2-Segal space, in this section we shall construct, from a simplicial object $X \in \mathcal{C}_{\Delta}$ in an ∞-category $\mathcal{C}$ having finite limits, a lax algebra structure on X_1 as an object of $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$. That is, we shall construct a symmetric monoidal lax functor $\alpha_X : \mathcal{A}\mathrm{lg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$ with $\alpha_X(\underline{1}) = X_1$, as per Definition 2.5.2. As a formal consequence we also obtain a symmetric monoidal lax functor $\chi_X : \mathcal{C}\mathrm{oalg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$ which endows X_1 with a lax coalgebra structure. We show in Section 3.3 that when X is a 2-Segal space α_X and χ_X are symmetric monoidal functors. The resulting (co)algebra in $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$ is the universal Hall (co)algebra of X.

To carry out this construction we exploit the fact that the opposite of the Yoneda embedding,

$$\Delta[\bullet] : \Delta^{\mathrm{op}} \to (\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}},$$

is the *initial simplicial object* in an ∞-category having finite limits. From any simplicial object $X_\bullet \in \mathcal{C}_{\Delta}$ in an ∞-category having finite limits one has a finite limit preserving functor $X : (\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}} \to \mathcal{C}$ given by right Kan extension,

$$\begin{array}{ccc} (\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}} & \xrightarrow{X} & \mathcal{C} \\ \uparrow & \nearrow_{X} & \\ \Delta^{\mathrm{op}} & & \end{array}$$

The simplicial object $X_\bullet \in \mathcal{C}_{\Delta}$ is then the image of $\Delta[\bullet]$ under the right Kan extension.

We first, in Section 3.2.1, explicitly construct a symmetric monoidal lax functor

$$\mathcal{A}\mathrm{lg}^{\amalg} \overset{\alpha}{\rightsquigarrow} \mathcal{S}\mathrm{pan}_2((\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}})^{\times},$$

endowing the standard 1-simplex $\Delta[1]$ with the structure of a lax algebra. Formally dualising this construction yields a lax coalgebra structure on $\Delta[1]$. Then, in Section 3.2.2, we show how the object X_1 in a simplicial object $X_\bullet$ inherits these same structures from the universal property of $\Delta[\bullet]$.

3.2.1 The lax algebra structure on $\Delta[1]$

Our first step in the construction of the lax (co)algebra coming from a general simplicial object $X_\bullet \in \mathcal{C}_{\Delta}$ will be to carry out the construction for the initial simplicial object $\Delta[\bullet] \in (\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}}$. According to Definitions 2.2.4 and 2.5.2, this amounts to constructing a morphism of fibrations

$$\begin{array}{ccc} \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg}) & \xrightarrow{\alpha} & \mathrm{Un}\left(\mathcal{S}\mathrm{pan}_2((\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}})^{\times}\right) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}\mathrm{in}_* \times \Delta^{\mathrm{op}} & \end{array}$$

preserving cocartesian lifts of morphisms of the form $(f, \phi) \in \mathcal{F}\mathrm{in}_* \times (\Delta_{\mathrm{in}})^{\mathrm{op}}$. Furthermore, the image of the object $\underline{1}$ in the fibre over $(\underline{1}_*, [0])$ must be $\Delta[1]$.

Before diving into the detailed construction of α, let us first give an informal description. On objects, the lax functor α is simply

$$\alpha : X \mapsto X \cdot \Delta[1] = \coprod_{x \in X} \Delta[1].$$

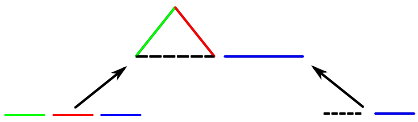
Recall that a morphism in $\mathcal{A}\mathrm{lg}^{\sqcup}$ is a function $p : X \to Y$ along with a linear ordering of $p^{-1}(y)$ for each $y \in Y$. The lax functor α sends the morphism p to the morphism $\alpha(p) \in \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}$ given by the diagram

$$\begin{array}{ccccc} & & \coprod_{y \in Y} \Delta\left[|p^{-1}(y)|\right] & & \\ & \overset{\sigma}{\nearrow} & & \overset{\lambda}{\nwarrow} & \\ X \cdot \Delta[1] & & & & Y \cdot \Delta[1], \end{array}$$

The morphism λ sends the 1-simplex associated to the element $y \in Y$ to the long edge of the standard simplex $\Delta\left[|p^{-1}(y)|\right]$. Note that one can label the edges along the spine of $\Delta\left[|p^{-1}(y)|\right]$ by the elements of $p^{-1}(y)$ using the linear ordering. The morphism σ sends the 1-simplex associated to $x \in X$ to the appropriate edge along the spine of $\Delta\left[|p^{-1}(p(x))|\right]$.

Example 3.2.1. For the morphism $m : \underline{2} \to \underline{1}$, the morphism $\alpha(m)$ in $\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}$ is given by the diagram in Eq. 1.1.3.

Example 3.2.2. Consider the morphism $(m \sqcup \mathrm{Id}) : \underline{3} = \underline{2} \sqcup \underline{1} \to \underline{1} \sqcup \underline{1} = \underline{2}$. Then $\alpha(m \sqcup \mathrm{Id})$ is given by the diagram

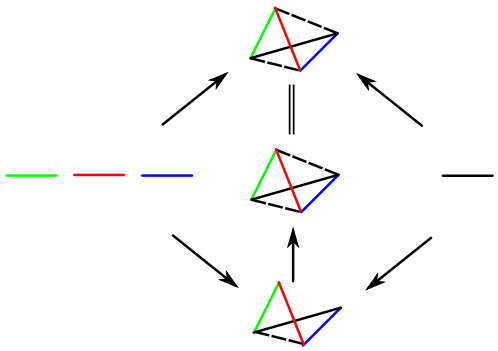


The lax structure on the functor α is given by associating to each pair of composable morphisms $X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_2} X_2$ in $\mathcal{A}\mathrm{lg}^{\sqcup}$ a 2-morphism in $\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}$ of the form

$$\begin{array}{ccccc} & & \alpha(p_2 p_1) & & \\ & \nearrow & \| & \nwarrow & \\ \alpha(X_0) & \longrightarrow & \alpha(p_2 p_1) & \longleftarrow & \alpha(X_2). \\ & \searrow & \uparrow & \swarrow & \\ & & \alpha(p_2) \coprod_{\alpha(X_1)} \alpha(p_1) & & \end{array}$$

Example 3.2.3. Consider the pair of composable morphisms $\underline{3} \xrightarrow{m \sqcup \mathrm{Id}} \underline{2} \xrightarrow{m} \underline{1}$. The lax structure on

α is given by the diagram



Outline of the construction. Recall from Section 2.5.1 that $\mathcal{A}\mathrm{lg}^{\amalg}$ is semistrict by construction, and so it is straightforward to determine from Eq. 2.4.3 that its unstraightening is given by

$$\mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_k = \{((f,\phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}}),\ \theta : \Pi(f(0)) \times \phi(0) \to \mathrm{Alg}) \mid \theta \text{ cocartesian}\}.$$

From such data we will build a cartesian and vertically constant functor

$$\alpha(\theta) : \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \to (\mathrm{Fin}_\Delta)^{\mathrm{op}}.$$

By Proposition 2.4.25 this defines a k-simplex in the unstraightening of $\mathcal{S}\mathrm{pan}_2((\mathcal{F}\mathrm{in}_\Delta)^{\mathrm{op}})^{\times}$. To better understand the approach we take to the construction of $\alpha(\theta)$ it will be instructive to describe a special case.

Consider $((f,\phi),\theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_1$ where f is constant on $\underline{1}_*$ and $\phi : [2] \twoheadleftarrow [1]$ is the unique active morphism. Then the functor θ is a pair of composable morphisms

$$X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_2} X_2\ .$$

Since $\alpha(\theta)$ is cartesian it is determined by its restriction to Λ^{M_ϕ}, the diagram described in Example 2.4.21. The restriction of $\alpha(\theta)$ to Λ^{M_ϕ} is

$$\begin{array}{ccccccc}
\alpha(X_0) & \longrightarrow & \alpha(p_2p_1) & \longleftarrow & \alpha(X_2) \\
\| & & \| & & \| \\
\alpha(X_0) & \longrightarrow & \alpha(p_2p_1) & \longleftarrow & \alpha(X_2) \\
\| & \nearrow & & \nwarrow & \| \\
\alpha(X_0) & \longrightarrow \alpha(p_1) \longleftarrow & \alpha(X_1) & \longrightarrow \alpha(p_2) \longleftarrow & \alpha(X_2)
\end{array}$$

Observe that all of the data in this diagram can be obtained from the following diagram

$$\begin{array}{ccccc}
 & & \alpha(p_2p_1) & & \\
 & \nearrow & & \nwarrow & \\
 & \alpha(p_1) & & \alpha(p_2) & \\
\nearrow & & \nwarrow\ \nearrow & & \nwarrow \\
\alpha(X_0) & & \alpha(X_1) & & \alpha(X_2)
\end{array} \tag{3.2.1}$$

which is a functor $\overline{\alpha}(\theta) : \Sigma^{\phi(0)} \to (\mathrm{Fin}_\Delta)^{\mathrm{op}}$. Specifically, the restriction of $\alpha(\theta)$ to Λ^{M_ϕ} is obtained from $\overline{\alpha}(\theta)$ by restricting along a functor $\Lambda^{M_\phi} \to \Sigma^{\phi(0)}$.

In general, observe that for each $\phi \in N_k(\Delta^{\mathrm{op}})$ one has a natural transformation $\phi \Rightarrow c_{\phi(0)}$ having components $\phi_{i,0}$. The naturality of the Grothendieck construction implies that this natural transformation induces a functor

$$p_\phi : M_\phi \to \phi(0).$$

Our construction of $\alpha(\theta)$ for general $((f,\phi),\theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_k$ proceeds by first defining a functor

$$\overline{\alpha}(\theta) : P(f(0))^{\mathrm{op}} \times \Sigma^{\phi(0)} \to (\mathrm{Fin}_\Delta)^{\mathrm{op}}$$

generalising the one in Eq. 3.2.1. The functor $\alpha(\theta)$ is then defined so that the following diagram is a right Kan extension

$$\begin{array}{ccccccc} \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} & & & \xrightarrow{\qquad\qquad\alpha(\theta)\qquad\qquad} & & & (\mathrm{Fin}_\Delta)^{\mathrm{op}} \\ \uparrow & & & & & & \| \\ P(f(0))^{\mathrm{op}} \times \Lambda^{M_\phi} & \hookrightarrow & P(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} & \xrightarrow[\mathrm{Id}\times\Sigma^{p_\phi}]{} & P(f(0))^{\mathrm{op}} \times \Sigma^{\phi(0)} & \xrightarrow[\overline{\alpha}(\theta)]{} & (\mathrm{Fin}_\Delta)^{\mathrm{op}} \end{array} \tag{3.2.2}$$

The functor $\alpha(\theta)$ is manifestly cartesian, and is vertically constant as every morphism in V_ϕ is sent to the identity in $\phi(0)$ by p_ϕ.

Construction of the functor $\overline{\alpha}(\theta)$. By Theorem 2.3.3 it suffices to define a normal oplax functor

$$\overline{\alpha}(\theta) : P(f(0))^{\mathrm{op}} \times \phi(0) \nrightarrow \mathrm{sp}\left((\mathrm{Fin}_\Delta)^{\mathrm{op}}\right).$$

Recall from Section 1.3 that ∇ is the category of spans of the form $\langle n\rangle \leftarrow \langle k\rangle \rightarrowtail \langle m\rangle$ in Δ_+. The category of *levelwise finite nabla sets*, denoted Fin_∇ is the category of functors $\nabla^{\mathrm{op}} \to \mathrm{Fin}$. From a morphism $p : X \to Y$ in Alg one can define the following diagram of levelwise finite nabla sets

$$X \leftarrow \coprod_{y\in Y} \nabla\left[p^{-1}(y)\right] \rightarrow Y,$$

where X and Y are constant nabla sets and $\nabla[n]$ is the nabla set represented by $\langle n\rangle$. The first and second map arise, respectively, from the following morphisms in ∇:

$$\begin{array}{ccc} & \{x\} & \\ \swarrow & & \searrow \\ \{x\} & & p^{-1}(p(x)) \end{array} \qquad \begin{array}{ccc} & p^{-1}(y) & \\ \swarrow & & \searrow \\ \{y\} & & p^{-1}(y). \end{array}$$

We can then apply the functor $\mathfrak{G}^*$ of Eq. 1.3.1 to obtain a span in $(\mathrm{Fin}_\Delta)^{\mathrm{op}}$.

Now, define $\overline{\alpha}(\theta)$ on objects as

$$\overline{\alpha}(\theta)(s,i) = \mathfrak{G}^*\theta(s,i) = \theta(s,i) \cdot \Delta[1].$$

To define $\overline{\alpha}(\theta)$ on a morphism $f : (s,i) \to (s,j)$ in $P(f(0))^{\mathrm{op}} \times \phi(0)$, one applies the above construction to the morphism $\theta(f)$ in Alg, yielding a span in $(\mathrm{Fin}_\Delta)^{\mathrm{op}}$

$$\begin{array}{ccc} & \overline{\alpha}(\theta)(f) & \\ \nearrow & & \nwarrow \\ \overline{\alpha}(\theta)(s,i) & & \overline{\alpha}(\theta)(s,j)\,. \end{array}$$

From a pair of composable morphisms $X \xrightarrow{p} Y \xrightarrow{q} Z$ in Alg, one has a commutative diagram of nabla sets

$$\begin{array}{ccccccccc} & & & & \coprod_{z\in Z} \nabla\left[(qp)^{-1}(z)\right] & & & & \\ & \nearrow & & \nearrow{\scriptstyle a} & & \nwarrow{\scriptstyle b} & & \nwarrow & \\ X & \longrightarrow & \coprod_{y\in Y} \nabla\left[p^{-1}(y)\right] & \longleftarrow & Y & \longrightarrow & \coprod_{z\in Z} \nabla\left[q^{-1}(z)\right] & \longleftarrow & Z\,, \end{array}$$

where a and b arise, respectively, from the following morphisms in ∇:

$$\begin{array}{ccc} & p^{-1}(y) & \\ = & & \searrow \\ p^{-1}(y) & & (qp)^{-1}(q(y)) \end{array} \qquad \begin{array}{ccc} & (qp)^{-1}(a) & \\ {\scriptstyle p}\swarrow & & = \\ q^{-1}(z) & & (qp)^{-1}(z)\,. \end{array}$$

Applying the functor $\mathfrak{G}^*$ of Eq. 1.3.1 yields a diagram in Fin_Δ.

For a pair of composable morphisms $(s,i) \xrightarrow{f} (s,j) \xrightarrow{g} (s,k)$ in $P(f(0))^{\mathrm{op}} \times \phi(0)$ we define the corresponding component $\overline{A}(\theta)_{g,f}$ of the oplax structure on $\overline{\alpha}(\theta)$ as follows. Applying the construction of the preceding paragraph to the image under θ of the pair of composable morphisms one has, in particular, a commutative square

$$\begin{array}{ccc} \overline{\alpha}(\theta)(s,j) & \longrightarrow & \overline{\alpha}(\theta)(g) \\ \downarrow & & \downarrow \\ \overline{\alpha}(\theta)(f) & \longrightarrow & \overline{\alpha}(\theta)(g\circ f)\,. \end{array} \tag{3.2.3}$$

Then the component $\overline{A}(\theta)_{g,f}$ is the universal morphism

$$\overline{A}(\theta)_{g,f} : \overline{\alpha}(\theta)(g) \coprod_{\overline{\alpha}(\theta)(s,j)} \overline{\alpha}(\theta)(f) \to \overline{\alpha}(\theta)(g\circ f),$$

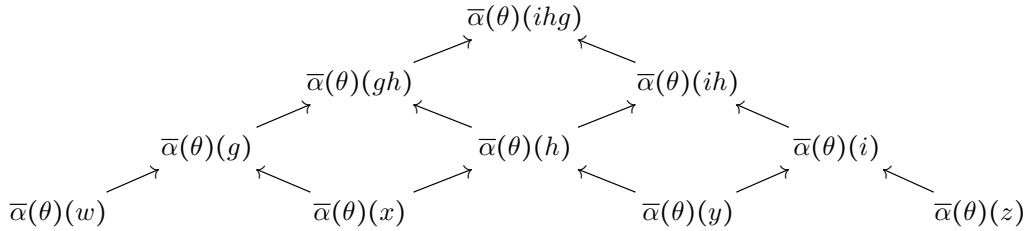
induced by the diagram in Eq. 3.2.3.

Lemma 3.2.4. *For each $((f,\phi),\theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_k$, the above data defines a normal oplax functor and hence a functor*

$$\overline{\alpha}(\theta) : P(f(0))^{\mathrm{op}} \times \Sigma^{\phi(0)} \to (\mathrm{Fin}_\Delta)^{\mathrm{op}}.$$

Proof. The unitality conditions $\overline{A}(\theta)_{\mathrm{Id},f} = \mathrm{Id}_{\overline{\alpha}(\theta)(f)} = \overline{A}(\theta)_{f,\mathrm{Id}}$ are straightforward to verify. Given a triple $w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{i} z$ of composable morphisms in $P(f(0))^{\mathrm{op}} \times \phi(0)$, along similar lines as

above one has a diagram in Fin_Δ



Then the associativity condition,

$$\overline{A}(\theta)_{i,hg} \circ \left(\mathrm{Id}_{\overline{\alpha}(\theta)(i)} \coprod_{\overline{\alpha}(\theta)(y)} \overline{A}(\theta)_{h,g} \right) = \overline{A}(\theta)_{ih,g} \circ \left(\overline{A}(\theta)_{i,h} \coprod_{\overline{\alpha}(\theta)(x)} \mathrm{Id}_{\overline{\alpha}(\theta)(g)} \right),$$

holds since both sides of the equation are the universal morphism for the colimit of the bottom two rows of the above diagram. □

The morphism α is a symmetric monoidal lax functor. The diagram in Eq. 3.2.2 defines, for each $(f, \phi) \in N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$ a functor

$$R_{(f,\phi)} : \mathrm{Fun}^{\mathrm{cart}}\left(P(f(0))^{\mathrm{op}} \times \Sigma^{\phi(0)}, (\mathrm{Fin}_\Delta)^{\mathrm{op}}\right) \to \mathrm{Fun}^{\mathrm{cart}}\left(\Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi}, (\mathrm{Fin}_\Delta)^{\mathrm{op}}\right).$$

We define $\alpha(\theta)$, for $((f, \phi), \theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_k$, to be

$$\alpha(\theta) \coloneqq R_{(f,\phi)}\overline{\alpha}(\theta) \in \langle (\mathrm{Fin}_\Delta)^{\mathrm{op}} \rangle_k ,$$

where $\langle (\mathrm{Fin}_\Delta)^{\mathrm{op}} \rangle$ is the sub simplicial set of the unstraightening of $\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}$ from Proposition 2.4.25.

Lemma 3.2.5. *The assignment $((f, \phi), \theta) \mapsto \alpha(\theta)$ defines a morphism of simplicial sets*

$$\alpha : \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg}) \to \langle (\mathrm{Fin}_\Delta)^{\mathrm{op}} \rangle .$$

Proof. Fix $((f, \phi), \theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_k$ and a morphism $\gamma : [n] \to [k]$. We must show that $\gamma^*\alpha(\theta) = \alpha(\gamma^*\theta)$, that is, that the following diagram commutes,

$$\begin{array}{ccc} \Pi(f\gamma(0))^{\mathrm{op}} \times \Sigma^{M_{\phi\gamma}} & & \\ \downarrow & \searrow^{\alpha(\gamma^*\theta)} & \\ \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} & \xrightarrow[\alpha(\theta)]{} & (\mathrm{Fin}_\Delta)^{\mathrm{op}} , \end{array}$$

where $\gamma^*\theta$ is the composite

$$\Pi(f\gamma(0)) \times \phi\gamma(0) \longrightarrow \Pi(f(0)) \times \phi(0) \xrightarrow{\theta} \mathrm{Alg} .$$

Letting $\overline{P(f(0))^{\mathrm{op}}}$ denote the full subcategory of $\Pi(f(0))^{\mathrm{op}}$ containing $P(f(0))^{\mathrm{op}}$ as well as those $U \subset f(0)$ such that $U \subset f^{-1}_{\gamma(0),0}(s)$ for some $s \in f\gamma(0)$, one has that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Fun}^{\mathrm{cart}}\left(\Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi}, (\mathrm{Fin}_\Delta)^{\mathrm{op}}\right) & \xrightarrow{\gamma^*} & \mathrm{Fun}^{\mathrm{cart}}\left(\Pi(f\gamma(0))^{\mathrm{op}} \times \Sigma^{M_{\phi\gamma}}, (\mathrm{Fin}_\Delta)^{\mathrm{op}}\right) \\ {\scriptstyle R_{(f,\phi)}}\uparrow & & \uparrow{\scriptstyle R_{(f\gamma,\phi\gamma)}} \\ \mathrm{Fun}^{\mathrm{cart}}\left(\overline{P(f(0))^{\mathrm{op}}} \times \Sigma^{\phi(0)}, (\mathrm{Fin}_\Delta)^{\mathrm{op}}\right) & \xrightarrow[\gamma^*]{} & \mathrm{Fun}^{\mathrm{cart}}\left(P(f\gamma(0))^{\mathrm{op}} \times \Sigma^{\phi\gamma(0)}, (\mathrm{Fin}_\Delta)^{\mathrm{op}}\right) \end{array}$$

It therefore suffices to show that for each $s \in f\gamma(0)$,

$$\overline{\alpha}(\gamma^*\theta)(s,-) = \coprod_{t \in f^{-1}_{\gamma(0),0}(s)} \gamma^*\overline{\alpha}(\theta)(t,-)$$

as normal oplax functors $\phi\gamma(0) \twoheadrightarrow \mathrm{sp}\left((\mathrm{Fin}_\Delta)^{\mathrm{op}}\right)$. This follows from the fact that θ is cocartesian. □

We have therefore constructed a morphism of fibrations

$$\begin{array}{ccc} \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg}) & \xrightarrow{\alpha} & \mathrm{Un}\left(\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}\right) \\ & \searrow \quad \swarrow & \\ & \mathcal{F}\mathrm{in}_* \times \mathbb{\Delta}^{\mathrm{op}} & \end{array}$$

Furthermore, the image of the object $\underline{1}$ in the fibre over $(\underline{1}_*, [0])$ is $\Delta[1]$. To show that the α endows $\Delta[1]$ with a lax algebra structure it remains only to show that α defines a symmetric monoidal lax functor.

Proposition 3.2.6. *The morphism of simplicial sets α is a symmetric monoidal lax functor*

$$\alpha : \mathcal{A}\mathrm{lg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}.$$

Hence, α endows the standard 1-simplex $\Delta[1]$ with the structure of a lax algebra..

Proof. We must show that for every $((f,\phi),\theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_1$ with ϕ inert the functor

$$\alpha(\theta)_{1,[1]} : \Pi(f(1))^{\mathrm{op}} \times \Sigma^{\phi(1),[1]} \to (\mathrm{Fin}_\Delta)^{\mathrm{op}}$$

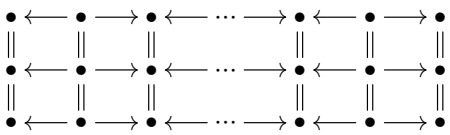
defines an equivalence in the quasicategory $\mathcal{S}\mathrm{p}_{f(1),\phi(1)}((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}})$ ([46] 3.2.5.2).

For ϕ inert, the poset M_ϕ is of the form



with the image of $\phi(1) \times [1]$ indicated in red.

Therefore for each $U \in \Pi(f(1))^{\mathrm{op}}$ the functor $\alpha(\theta)_{1,[1]}(U,-)$ is the right Kan extension of a diagram in $(\mathrm{Fin}_\Delta)^{\mathrm{op}}$ of the form



As the equivalences in $\mathcal{S}\mathrm{p}_{f(1),\phi(1)}$ are exactly those cartesian functors $\Pi(f(1))^{\mathrm{op}} \times \Sigma^{\phi(1),[1]} \to (\mathrm{Fin}_{\Delta})^{\mathrm{op}}$ which send all morphisms of the form $(\mathrm{Id}, \mathrm{Id}, v)$ to equivalences ([33] 6.2) the claimed result follows. □

By Remark 2.2.3, the opposite of α gives a symmetric monoidal lax functor

$$\chi = \alpha^{\mathrm{op}} : \mathcal{C}\mathrm{oalg}^{\amalg} \rightsquigarrow \left(\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}\right)^{\mathrm{op}}.$$

However, Corollary 2.4.12 says that $\left(\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}\right)^{\mathrm{op}} \simeq \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}$. Therefore, as an immediate corollary to Proposition 3.2.6, we obtain a lax coalgebra structure on $\Delta[1]$.

Proposition 3.2.7. *The symmetric monoidal lax functor*

$$\chi = \alpha^{\mathrm{op}} : \mathcal{C}\mathrm{oalg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}$$

endows the standard 1*-simplex* $\Delta[1]$ *with the structure of a lax coalgebra.*

3.2.2 The lax algebra structure inherited from $\Delta[1]$

Let $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}}$ be a simplicial object in an ∞-category having finite limits. We can now show how $\mathcal{X}_1$ inherits both a lax algebra and coalgebra structure in $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$ from the ones endowed upon standard 1-simplex $\Delta[1]$ in Propositions 3.2.6 and 3.2.7. These structures will define the universal Hall algebra and coalgebra of $\mathcal{X}$ in Section 3.3.

We make use of the following result due to Li-Bland.

Theorem 3.2.8 ([45] 4.1)**.** *The construction which assigns to an* ∞*-category having finite limits its symmetric monoidal* $(\infty, 2)$*-category of bispans defines a functor*

$$\mathcal{S}\mathrm{pan}_2(-)^{\times} : \mathcal{C}\mathrm{at}_{\infty}^{\mathrm{lex}} \to \mathcal{C}\mathrm{at}_{(\infty,2)}^{\otimes}.$$

where $\mathcal{C}\mathrm{at}_{\infty}^{\mathrm{lex}}$ *is the* ∞*-category of* ∞*-categories having finite limits and finite limit preserving functors between them.*

Recall that any simplicial object $\mathcal{X}_{\bullet} \in \mathcal{C}_{\mathbb{\Delta}}$ defines a finite limit preserving functor $\mathcal{X} : (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}} \to \mathcal{C}$ by right Kan extension,

$$\begin{array}{ccc} (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}} & \xrightarrow{\mathcal{X}} & \mathcal{C} \\ \uparrow & \nearrow_{\mathcal{X}} & \\ \mathbb{\Delta}^{\mathrm{op}} & & \end{array}$$

By Theorem 3.2.8 this induces a symmetric monoidal functor

$$\mathcal{X} : \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times} \to \mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}.$$

Then as immediate corollaries of Propositions 3.2.6 and 3.2.7 one has the following.

Theorem 3.2.9. *Let $\mathcal{C}$ be an ∞-category with finite limits and let $X_\bullet \in \mathcal{C}_{\Delta}$ be a simplicial object. Then the composite*

$$\alpha_X : \mathcal{A}\mathrm{lg}^{\amalg} \xrightsquigarrow{\alpha} \mathcal{S}\mathrm{pan}_2((\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}})^{\times} \xrightarrow{X} \mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times},$$

is a symmetric monoidal lax functor endowing X_1 with the structure of a lax algebra and the composite

$$\chi_X : \mathcal{C}\mathrm{oalg}^{\amalg} \xrightsquigarrow{\chi} \mathcal{S}\mathrm{pan}_2((\mathcal{F}\mathrm{in}_{\Delta})^{\mathrm{op}})^{\times} \xrightarrow{X} \mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times},$$

is a symmetric monoidal lax functor endowing X_1 with the structure of a lax coalgebra.

3.3 Associativity and the 2-Segal condition

Having equipped the object of 1-simplices X_1 of a simplicial object $X_\bullet \in \mathcal{C}_{\Delta}$ with a lax (co)algebra structure in $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$ in Theorem 3.2.9, we now demonstrate that the 2-Segal condition is exactly the right condition that enforces the (co)associativity of this structure. This serves both to elucidate the precise meaning of the 2-Segal condition and to provide our definition of the universal Hall (co)algebra of a 2-Segal object.

Theorem 3.3.1. *Let $\mathcal{C}$ be an ∞-category with finite limits. Then a simplicial object $X \in \mathcal{C}_{\Delta}$ is a 2-Segal object if and only if the symmetric monoidal lax functor α_X of Theorem 3.2.9 endows X_1 with the structure of a algebra. Dually, X is 2-Segal if and only if the symmetric monoidal lax functor χ_X of Theorem 3.2.9 makes X_1 a coalgebra.*

Definition 3.3.2. The *universal Hall (co)algebra* of a 2-Segal object $X \in \mathcal{C}_{\Delta}$ is X_1 equipped with the (co)algebra structure of Theorem 3.3.1.

Remark 3.3.3. Dyckerhoff–Kapranov ([17] 11.1.19) and Gálvez-Carrillo–Kock–Tonks ([23] 7.3) have independently constructed the universal Hall (co)algebra of a 2-Segal space. Theorem 3.3.1 is a strengthening of their results, showing that the 2-Segal condition is not only sufficient, but *necessary*, to define the universal Hall (co)algebra.

Proof. The second statement is simply the dual of the first, therefore it suffices to prove only the first. We must show that X is a 2-Segal object if and only if $\alpha_X(\theta)_{1,[1]}$ is an equivalence in $\mathcal{S}\mathrm{p}_{f(1),\phi(1)}(\mathcal{C})$ for every $((f,\phi),\theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_1$.

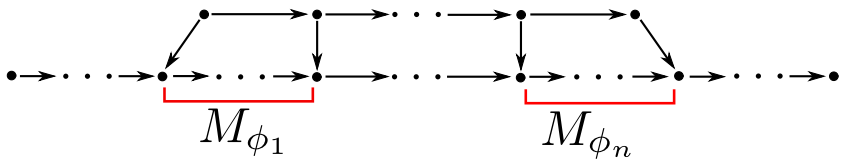
Every active morphism $\psi : [n] \twoheadrightarrow [m]$ can be decomposed as

$$\bigvee_{i=1}^{n} \psi_i : \bigvee_{i=1}^{n} [1] \twoheadrightarrow \bigvee_{i=1}^{n} [m_i],$$

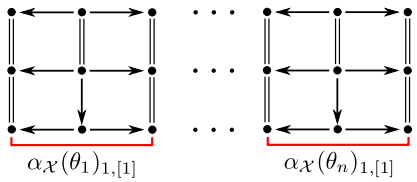
where ψ_i is the unique active morphism $[1] \twoheadrightarrow [m_i]$ and $[m] = \vee_i [m_i]$. Every inert morphism is of the form $[n] \rightarrowtail [a] \vee [n] \vee [b]$. Since morphisms in Δ can be uniquely factored as an active morphism followed by an inert morphism, one can decompose the morphism $\phi : [m] \leftarrow [n] \in N_1(\Delta^{\mathrm{op}})$ as

$$\phi = \bigvee_{i=1}^{n} \phi_i : [a_\phi] \vee \left(\bigvee_{i=1}^{n} [m_i] \right) [b_\phi] \leftarrow \bigvee_{i=1}^{n} [1].$$

One can therefore write the poset M_ϕ schematically as



Denoting by $\theta_i : \Pi(f(0)) \times [1] \to \mathrm{Alg}$ the restriction of θ to the i-th summand of $\phi(1)$, it follows that for each $U \in \Pi(f(1))^{\mathrm{op}}$, the functor $\alpha_{\mathcal{X}}(\theta)_{\phi(1),[1]}(U,-)$ is the right Kan extension of a diagram in $\mathcal{C}$ of the form



Therefore $\alpha_{\mathcal{X}}(\theta)_{1,[1]}$ is an equivalence if and only if $\alpha_{\mathcal{X}}(\theta_i)_{1,[1]}$ is an equivalence for each i.

We conclude that $\alpha_{\mathcal{X}} : \mathcal{A}\mathrm{lg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$ is a symmetric monoidal functor if and only if $\alpha_{\mathcal{X}}(\theta)_{1,[1]}$ is an equivalence in $\mathcal{S}\mathrm{p}_{f(1),\phi(1)}(\mathcal{C})$ for every $((f,\phi),\theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_1$ where $\phi : [n] \twoheadleftarrow [1]$ is the unique active morphism. By Lemmas 3.3.4 and 3.3.5, proven below, this latter condition is equivalent to $\mathcal{X}$ satisfying the 2-Segal condition. □

To conclude our proof of Theorem 3.3.1 we must prove two lemmas.

Lemma 3.3.4. *Let $\mathcal{X} \in \mathcal{C}_{\Delta}$ be a simplicial object in an ∞-category having finite limits. If, for every $((f,\phi),\theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_1$ with $\phi : [n] \twoheadleftarrow [1]$ being the unique active morphism $\alpha_{\mathcal{X}}(\theta)_{1,[1]}$ is an equivalence in $\mathcal{S}\mathrm{p}_{f(1),\phi(1)}(\mathcal{C})$, then $\mathcal{X}$ satisfies the 2-Segal condition.*

Proof. Consider, for each $n \geq 3$ and $i,j = 1,\dots,n-1$, the following pairs of composable morphisms in Alg:

$$\{1,\dots,n\} \xrightarrow{p_1^i} \{i,\dots,n\} \xrightarrow{p_2^i} \underline{1} \tag{3.3.1}$$

$$\{1,\dots,n\} \xrightarrow[p_1^j]{} \{1,\dots,j+1\} \xrightarrow[p_2^j]{} \underline{1}\,.$$

The morphism p_1^i maps a to i when $1 \leq a \leq i$ and a otherwise, while p_1^j maps a to $j+1$ when $j+1 \leq a \leq n$ and a otherwise. The linear orders on the fibres are induced by the ordering on $\{1,\dots,n\}$. Observe that

$$\alpha(p_2^i) \coprod_{\alpha(\{i,\dots,n\})} \alpha(p_1^i) \quad \text{and} \quad \alpha(p_2^j) \coprod_{\alpha(\{1,\dots,j+1\})} \alpha(p_1^j)$$

are, respectively, the pushouts of the left and right square of Eq. 3.1.1. Furthermore, $\alpha(p_2^i p_1^i) = \Delta[n] = \alpha(p_2^j p_1^j)$, and the corresponding components of the lax structure on α agree with the ones arising from the squares in Eq. 3.1.1.

Next, consider for each $n \geq 2$ and $0 \leq i < n$ the pair of composable morphisms in Alg:

$$\{1, \dots, n-1\} \xrightarrow{e_1^i} \{1, \dots, n\} \xrightarrow{e_2^i} \underline{1} \,, \tag{3.3.2}$$

where e_1^i is the evident map which skips the element $i+1$ in $\{1, \dots, n\}$. Observe that

$$\alpha(e_2^i) \coprod_{\alpha(\{1,\dots,n\})} \alpha(e_1^i)$$

is the pushout of Eq. 3.1.2. We also have that $\alpha(e_2^i e_1^i) = \Delta\,[n-1]$, and the corresponding components of the lax structure on α agree with the ones arising from the square in Eq. 3.1.2.

Finally, consider $((f, \phi), \theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_1$, where f is constant on $\underline{1}_*$ and $\phi : [2] \twoheadleftarrow [1]$ is the unique active map. Then the functor θ is a pair of composable morphisms

$$X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_2} X_2$$

and $\alpha_{\mathcal{X}}(\theta)_{1,[1]}$ is the diagram

$$\begin{array}{ccccc}
\alpha_{\mathcal{X}}(X_1) & \longleftarrow & \alpha_{\mathcal{X}}(p_2 p_1) & \longrightarrow & \alpha_{\mathcal{X}}(X_2) \\
\| & & \| & & \| \\
\alpha_{\mathcal{X}}(X_1) & \longleftarrow & \alpha_{\mathcal{X}}(p_2 p_1) & \longrightarrow & \alpha_{\mathcal{X}}(X_2) \\
\| & & \downarrow{\scriptstyle \delta} & & \| \\
\alpha_{\mathcal{X}}(X_1) & \longleftarrow & \alpha_{\mathcal{X}}(p_2) \times_{\alpha_{\mathcal{X}}(X_1)} \alpha_{\mathcal{X}}(p_1) & \longrightarrow & \alpha_{\mathcal{X}}(X_2)
\end{array}$$

The functor $\alpha_{\mathcal{X}}(\theta)_{1,[1]}$ is an equivalence in $\mathcal{S}\mathrm{p}_{\underline{1},[1]}$ precisely when δ is an equivalence.

Taking θ to be the pairs of morphisms defined in Eqs. 3.3.1 and 3.3.2 it follows that $\mathcal{X}$ is a 2-Segal object. □

For the second of the two lemmas which complete Theorem 3.3.1 we must make use of an equivalent formulation of the 2-Segal condition due to Gálvez-Carrillo–Kock–Tonks ([23] 3). They show that $\mathcal{X} \in \mathcal{C}_{\Delta}$ is 2-Segal if and only if the image under $\mathcal{X}$ of every pushout square in Δ of the form

$$\begin{array}{ccc}
[n] & \twoheadrightarrow & [m] \\
\downarrow & & \downarrow \\
[k] & \twoheadrightarrow & [l]
\end{array}$$

is a pullback in $\mathcal{C}$.

Lemma 3.3.5. *Let $\mathcal{X} \in \mathcal{C}_{\Delta}$ be a 2-Segal object. Then $\alpha_{\mathcal{X}}(\theta)_{1,[1]}$ is an equivalence in $\mathcal{S}\mathrm{p}_{f(1),\phi(1)}(\mathcal{C})$ for every $((f, \phi), \theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\amalg})_1$ with $\phi : [n] \twoheadleftarrow [1]$ being the unique active morphism.*

Proof. It suffices to show that for every sequence of composable morphisms in Alg

$$X_0 \xrightarrow{p_1} X_1 \xrightarrow{p_2} \cdots \xrightarrow{p_n} X_n \,,$$

the image of

$$\alpha(p_n) \coprod_{\alpha(X_{n-1})} \cdots \coprod_{\alpha(X_1)} \alpha(p_1) \to \alpha(p_n \cdots p_1)$$

under $\mathcal{X}$ is an equivalence. Since every morphism in Alg is a disjoint union of morphisms having target the singleton set, and the functors α and $\mathcal{X}$ are symmetric monoidal, it suffices to consider the case when $X_n = \underline{1}$. The statement for general n follows by iterating the special case of $n = 2$.

First, in the trivial case of $X_1 = \underline{0}$, then also $X_0 = \underline{0}$ and

$$\alpha(p_2) \coprod_{\alpha(X_1)} \alpha(p_1) = \Delta[0] = \alpha(p_2 p_1).$$

For $X_1 \neq \underline{0}$, write $X_1 = \{1, \dots, k\}$ and denote by

$$\kappa = \mathfrak{G}^{-1}(p_2) \text{ and } \kappa_i = \mathfrak{G}^{-1}\left(p_1|_{p_1^{-1}(i)}\right),\ i \in X_1,$$

where $\mathfrak{G}^{-1}(\langle n \rangle) = [n]$ is the inverse on objects of the functor in Eq. 1.3.1. Recall that $\Delta[\kappa]$ can be thought of as a standard simplex having the edges along its spine labelled by the elements of X_1 according to the linear ordering defined by p_2. The pushout $\alpha(p_2) \coprod_{\alpha(X_1)} \alpha(p_1)$ is the simplicial set obtained by gluing each simplex $\Delta[\kappa_i]$ by its long edge to the corresponding edge on the spine of $\Delta[\kappa]$. Therefore $\alpha(p_2) \coprod_{\alpha(X_1)} \alpha(p_1)$ is the iterated pushout

$$\Delta[\kappa] \coprod_{i=1}^{k} \Delta[\kappa_i] = \left(\cdots \left(\left(\Delta[\kappa] \coprod_{\Delta[(0,1)]} \Delta[\kappa_1] \right) \coprod_{\Delta[(1,2)]} \Delta[\kappa_2] \right) \cdots \right) \coprod_{\Delta[(k-1,k)]} \Delta[\kappa_k] \ .$$

Next, set $\kappa_{\leq i}$ to be the inductively defined pushouts in Δ

$$\begin{array}{ccc} [1] & \twoheadrightarrow & \kappa_1 \\ {\scriptstyle g_0}\downarrow & & \downarrow \\ \kappa & \twoheadrightarrow & \kappa_{\leq 1} \end{array} \qquad \text{and} \qquad \begin{array}{ccc} [1] & \twoheadrightarrow & \kappa_{i+1} \\ {\scriptstyle g_i}\downarrow & & \downarrow \\ \kappa_{\leq i} & \twoheadrightarrow & \kappa_{\leq i+1} \end{array} \tag{3.3.3}$$

where g_0 has image the smallest two elements of κ, and g_i has image the k_i'th and $k_i + 1$'th elements of $\kappa_{\leq i}$ where $k_i = \sum_{j=1}^{i} |\kappa_j| - 1$. Then since $\kappa_{\leq k} = \vee_{i=1}^{k} \kappa_i$ one has that $\Delta[\kappa_{\leq k}] = \alpha(p_2 p_1)$. Furthermore, the morphism $\alpha(p_2) \coprod_{\alpha(X_1)} \alpha(p_1) \to \alpha(p_2 p_1)$ factors as the composite

$$\Delta[\kappa] \coprod_{i=1}^{k} \Delta[\kappa_i] \longrightarrow \Delta[\kappa_{\leq 1}] \coprod_{i=2}^{k} \Delta[\kappa_i] \longrightarrow \cdots \longrightarrow \Delta[\kappa_{\leq k}] \tag{3.3.4}$$

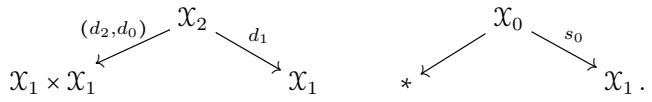
where each morphism arises from the pushout squares in Eq. 3.3.3.

It follows from the Gálvez-Carrillo–Kock–Tonks form of the 2-Segal condition that each morphism the sequence in Eq. 3.3.4 is sent to an equivalence under $\mathcal{X}$, proving the claimed result. □

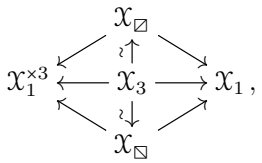
3.4 The universal Hall algebra in the homotopy bicategory

The universal Hall (co)algebra of a 2-Segal object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}}$ of Definition 3.3.2 defines, in particular, a (co)algebra object in the homotopy bicategory of $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times}$. In this final section we will provide an explicit description of this (co)algebra for a general 2-Segal object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}}$ and then specialise to the case $\mathcal{C} = \mathcal{G}\mathrm{rpd}$ and $\mathcal{X} = S(A)$, the Waldhausen S-construction of a proto-exact category A. As the data for the coalgebra is dual to that of the algebra we shall only describe the latter.

Recall that an algebra object in a symmetric monoidal bicategory consists of an object along with a 5-tuple $(\mu, e, \alpha, \lambda, \rho)$, where μ is the product, e is the unit, α the associator, and λ and ρ, respectively, the left and right unitor. The underlying object of the universal Hall algebra of a 2-Segal object $X \in \mathcal{C}_{\Delta}$ is X_1. The product and unit are given by the spans



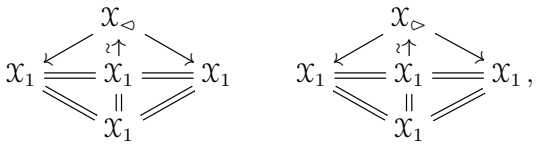
The associator is the 2-morphism given by the diagram



where $X_{\boxslash}$ and $X_{\boxbslash}$ are the pullbacks in $\mathcal{C}$

$$\begin{array}{ccc} X_{\boxslash} & \longrightarrow & X_2 \\ \downarrow & & \downarrow{\scriptstyle (d_2,d_0)} \\ X_2 \times X_1 & \xrightarrow[d_1 \times \mathrm{Id}]{} & X_1^{\times 2} \end{array} \qquad \begin{array}{ccc} X_{\boxbslash} & \longrightarrow & X_2 \\ \downarrow & & \downarrow{\scriptstyle (d_2,d_0)} \\ X_1 \times X_2 & \xrightarrow[\mathrm{Id} \times d_1]{} & X_1^{\times 2} . \end{array}$$

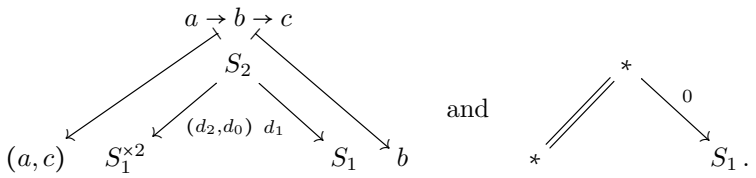
The left and right unitors are the 2-morphisms given, respectively, by the diagrams



where $X_{\triangleleft}$ and $X_{\triangleright}$ are the pullbacks in $\mathcal{C}$

$$\begin{array}{ccc} X_{\triangleleft} & \longrightarrow & X_2 \\ \downarrow & & \downarrow{\scriptstyle (d_2,d_0)} \\ X_0 \times X_1 & \xrightarrow[s_0 \times \mathrm{Id}]{} & X_1^{\times 2} \end{array} \qquad \begin{array}{ccc} X_{\triangleright} & \longrightarrow & X_2 \\ \downarrow & & \downarrow{\scriptstyle (d_2,d_0)} \\ X_1 \times X_0 & \xrightarrow[\mathrm{Id} \times s_0]{} & X_1^{\times 2} . \end{array}$$

For the universal Hall algebra of $S = S(A)$ the underlying groupoid is S_1, the groupoid of objects in A. The product and unit are given by the spans



(3.4.1)

Before giving the associator and unitors we must take a brief detour to discuss pullbacks in $\mathcal{G}$rpd.

Pullbacks in $\mathcal{G}\mathrm{rpd}$ are *homotopy pullbacks* in Grpd under the canonical model structure ([46] A.2.4.4). The diagram

$$\begin{array}{ccc} \mathbb{G} \times^h_{\mathbb{H}} \mathbb{G}' & \longrightarrow & \mathbb{G}' \\ \downarrow & & \downarrow{\scriptstyle F'} \\ \mathbb{G} & \xrightarrow[F]{} & \mathbb{H} \end{array}$$

is a pullback in $\mathcal{G}\mathrm{rpd}$, where $\mathbb{G} \times^h_{\mathbb{H}} \mathbb{G}'$ is the groupoid having objects triples

$$(g \in \mathbb{G}, g' \in \mathbb{G}', \phi : F(g) \simeq F'(g')).$$

A morphism between such triples is a morphism in $\mathbb{G}$ and a morphism in $\mathbb{G}'$ making the evident diagram commute. The homotopy pullback is not guaranteed to be equivalent to the ordinary pullback unless one of F or F' is an isofibration [40].

Definition 3.4.1. A functor $F : \mathbb{G} \to \mathbb{G}'$ between groupoids is an *isofibration* if for $g \in \mathbb{G}$ and morphism $\phi' : F(g) \to g'$ in $\mathbb{G}'$ there is a morphism $\phi : g \to h$ in $\mathbb{G}$ such that $F(\phi) = \phi'$.

Remark 3.4.2. The condition that a functor is an isofibration of groupoids is equivalent to requiring that its nerve is a Kan fibration of simplicial sets.

The functors on the bottom of both of the squares defining $S_{\boxslash}$ and $S_{\boxbslash}$ in Eq. 3.4 are isofibrations as, for every admissible short exact sequence $a \to b \to c$, an isomorphism of b induces an isomorphism of the whole sequence. One can therefore compute $S_{\boxslash}$ and $S_{\boxbslash}$ in terms of ordinary pullbacks, showing that they are, respectively, the groupoid of diagrams in A of the form

$$\begin{array}{ccccc} a & \longrightarrow & b & \longrightarrow & c \\ & & \downarrow & & \\ & & b' & & \\ & & \downarrow & & \\ & & b'' & & \end{array} \qquad\qquad \begin{array}{ccccc} & & & & c \\ & & & & \downarrow \\ a' & \longrightarrow & b' & \longrightarrow & c' \\ & & & & \downarrow \\ & & & & c'' \end{array}$$

where each row and column is a admissible short exact sequence.

One can directly compute that $S_{\triangleleft}$ and $S_{\triangleright}$ are, respectively, the groupoids of admissible short exact sequences of the form $0 \to b \to c$ and $a \to b \to 0$.

Chapter 4

The universal Hall bialgebra

In this chapter we present the first significant contribution contained in this thesis. Namely, the construction of the universal Hall bialgebra of a double 2-Segal space. More specifically, we show the following.

Theorem. *Every bisimplicial space $\mathcal{X}_{\bullet,\bullet}$ endows the space $\mathcal{X}_{1,1}$ with the structure of a totally lax bialgebra in the $(\infty,2)$-category of bispans of spaces. Furthermore, $\mathcal{X}_{\bullet,\bullet}$ is a double 2-Segal space if and only if it endows $\mathcal{X}_{1,1}$ with the structure of a lax bialgebra. We call this the* universal Hall bialgebra *of $\mathcal{X}_{\bullet,\bullet}$.*

We begin, in Section 4.1, with a short section introducing double 2-Segal spaces. The first part of the above Theorem is proven in Section 4.2 while the second is proven in 4.3. Finally, we conclude the chapter with a thorough account of the truncation of the universal Hall bialgebra to the homotopy bicategory of bispans in Section 4.4.

4.1 Double 2-Segal spaces

In Section 3.1 we saw that the Waldhausen S-construction is an important source of examples of 2-Segal spaces. Waldhausen introduced this construction to define the algebraic K-theory spectrum of what is now known as a Waldhausen category, of which proto-exact categories are an example. The key observation made by Waldhausen is that the S-construction can be *iterated* and he leveraged this to define the m-th space of the K-theory spectrum using the m-th iteration of the construction.

More precisely, for a proto-exact category A let $\overline{S}_\bullet(A)$ be the full sub simplicial category of $\mathrm{Fun}(\mathrm{Arr}_\bullet, A)$ on those diagrams of the form given in Eq. 3.1.4. The categories $\overline{S}_k(A)$ inherit a proto-exact structure from the one on A by declaring the admissible mono- and epi- morphisms to be those natural transformations of diagrams all of whose components are admissible mono- and epi-morphisms. One can therefore apply the S-construction level-wise to obtain a bisimplicial space

$$S_{n,k}(A) \coloneqq S_n(\overline{S}_k(A)).$$

In particular one has that $S_{1,\bullet}(A) \simeq S_\bullet(A) \simeq S_{\bullet,1}(A)$ and $S_{2,2}(A)$ is the groupoid of admissible short exact sequences of admissible short exact sequences. In general, $S_{n,k}(A)$ is equivalent to the groupoid of length n admissible flags of length k admissible flags in A.

By construction this bisimplicial groupoid satisfies the condition that for each k the simplicial groupoids $S_{\bullet,k}(A)$ and $S_{k,\bullet}(A)$ are 2-Segal. Just as the natural generalisation of the simplicial groupoid $S_\bullet(A)$ were 2-Segal spaces, the following is a natural generalisation of the bisimplicial groupoid $S_{\bullet,\bullet}(A)$.

Definition 4.1.1. Let $\mathcal{C}$ be an ∞-category with finite limits. Then a *double 2-Segal object* in $\mathcal{C}$ is a bisimplicial object $\mathcal{X}_{\bullet,\bullet} \in \mathcal{C}_{\mathbb{\Delta}^2}$ such that the simplicial objects $\mathcal{X}_{\bullet,1}$ and $\mathcal{X}_{1,\bullet}$ are 2-Segal spaces.

Remark 4.1.2. The bisimplicial groupoids $S_{\bullet,\bullet}(A)$ are double 2-Segal objects satisfying the strictly stronger condition that $S_{\bullet,k}(A)$ and $S_{k,\bullet}(A)$ are 2-Segal for all k. As we show in Theorem 4.3.2 that one only needs the weaker condition above to define the universal Hall bialgebra.

Besides the examples coming from iterating the Waldhausen S-construction, one also has the following examples.

Example 4.1.3. Since every Segal space is a 2-Segal space ([17] 2.5.3, [23] 3.5), double Segal spaces are also double 2-Segal spaces. In particular, by presenting $\mathcal{C}\mathrm{at}_\infty$ as $\mathcal{C}\mathrm{SS}$ every monoidal ∞-category, or more generally every $(\infty, 2)$-category, is a double 2-Segal space.

Example 4.1.4. Let $2\mathcal{S}\mathrm{eg}(\mathcal{C})$ be the full subcategory of $\mathcal{C}_{\mathbb{\Delta}}$ on the 2-Segal objects. Then a Segal object in $2\mathcal{S}\mathrm{eg}(\mathcal{C})$ is a double 2-Segal object in $\mathcal{C}$. In particular, the *monoidal decomposition spaces* introduced by Gálvez-Carrillo–Kock–Tonks ([23] 10) are examples of double 2-Segal spaces.

We now turn our attention to the construction of the universal Hall bialgebra of a general double 2-Segal space, returning to the special case of the double 2-Segal spaces $S_{\bullet,\bullet}(A)$ in Section 4.4.

4.2 Bisimplicial spaces define totally lax bialgebras

As the first step in our construction of the universal Hall bialgebra of a double 2-Segal space, in this section we shall construct, from a bisimplicial object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ in an ∞-category $\mathcal{C}$ having finite limits, a totally lax bialgebra structure on $\mathcal{X}_{1,1}$ as an object of $\mathcal{S}\mathrm{pan}_2(\mathcal{C})^\times$. That is, we shall construct a symmetric monoidal lax functor $\beta_\mathcal{X} : \mathcal{B}\mathrm{ialg}^\amalg \rightsquigarrow \mathcal{S}\mathrm{pan}_2(\mathcal{C})^\times$ with $\beta_\mathcal{X}(\underline{1}) = \mathcal{X}_{1,1}$ as per Definition 2.5.9. As we show in Section 4.3, when $\mathcal{X}$ is a double 2-Segal space $\beta_\mathcal{X}$ induces an algebra and a coalgebra structure on $\mathcal{X}_{1,1}$. The resulting lax bialgebra is the universal Hall bialgebra of $\mathcal{X}$.

To carry out this construction we exploit the fact that the opposite of the Yoneda embedding,

$$\Delta[\bullet,\bullet] : (\mathbb{\Delta}^{\mathrm{op}})^2 \to (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}},$$

is the *initial bisimplicial object* in an ∞-category having finite limits. From any bisimplicial object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ in an ∞-category having finite limits one has a finite limit preserving functor $\mathcal{X} : (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}} \to \mathcal{C}$ given by right Kan extension,

$$\begin{array}{ccc} (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}} & \xrightarrow{\mathcal{X}} & \mathcal{C} \\ \uparrow & \nearrow_{\mathcal{X}} & \\ (\mathbb{\Delta}^{\mathrm{op}})^2 & & \end{array}$$

The bisimplicial object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ is then the image of $\Delta[\bullet,\bullet]$ under the right Kan extension.

We first, in Section 4.2.1, explicitly construct a symmetric monoidal lax functor

$$\mathcal{B}\mathrm{ialg}^{\amalg} \overset{\beta}{\rightsquigarrow} \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times},$$

endowing the standard $(1,1)$-simplex $\Delta[1,1]$ with the structure of a totally lax bialgebra. Then, in Section 4.2.2, we show how the object $\mathcal{X}_{1,1}$ in a bisimplicial object $\mathcal{X}$ inherits a totally lax bialgebra structure from the universal property of $\Delta[\bullet,\bullet]$.

In Section 3.2.1 we constructed a symmetric monoidal lax functor $\alpha : \mathcal{A}\mathrm{lg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}$ endowing $\Delta[1]$ with the structure of a lax algebra and dually, a symmetric monoidal lax functor $\chi : \mathcal{C}\mathrm{oalg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times}$ endowing $\Delta[1]$ with a lax coalgebra structure. By right Kan extension along the functors

$$\Delta[\bullet,1],\ \Delta[1,\bullet] : \mathbb{\Delta}^{\mathrm{op}} \to (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}$$

one obtains, respectively, finite limit preserving functors

$$(-)^h, (-)^v : (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}} \to (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}, \quad X^h := X \boxtimes \Delta[1], X^v = \Delta[1] \boxtimes X.$$

The symmetric monoidal lax functor β will be an extension of both α and χ in the sense that one has a commutative diagram

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}^{\amalg} & \overset{\alpha}{\rightsquigarrow} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times} \\ {\scriptstyle \iota_a}\downarrow & & \downarrow{\scriptstyle (-)^h} \\ \mathcal{B}\mathrm{ialg}^{\amalg} & \overset{\beta}{\rightsquigarrow} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times} \\ {\scriptstyle \iota_c}\uparrow & & \uparrow{\scriptstyle (-)^v} \\ \mathcal{C}\mathrm{oalg}^{\amalg} & \overset{\chi}{\rightsquigarrow} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times} \end{array}$$

Informal description of the symmetric monoidal lax functor β. On objects, β is given by

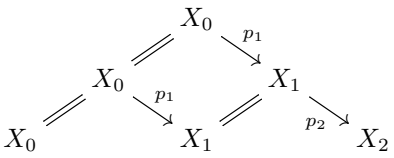
$$\beta : X \mapsto X \cdot \Delta[1,1] = \coprod_{x \in X} \Delta[1,1].$$

The image under β of a span $f :\ X_0 \xleftarrow{p} Y \xrightarrow{q} X_1$ is the span in $(\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}$ given by the outer edges of the diagram

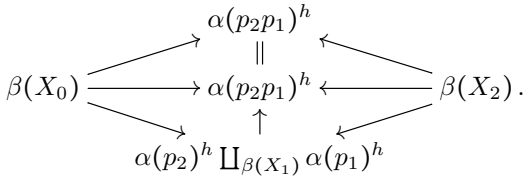
$$\begin{array}{ccccc} & & \beta(f) & & \\ & \nearrow & & \nwarrow & \\ & \chi(p)^v & & & \alpha(q)^h \\ \nearrow & & \nwarrow & \nearrow & & \nwarrow \\ \beta(X_0) & & & \beta(Y) & & \beta(X_1) \end{array} \tag{4.2.1}$$

where the middle square is a pushout.

The lax structure for β is considerably more difficult to describe. Let us look at some special cases. First, consider a pair of composable morphisms in the image of $\mathcal{A}\mathrm{lg}^{\amalg}$, that is, a composite of the form

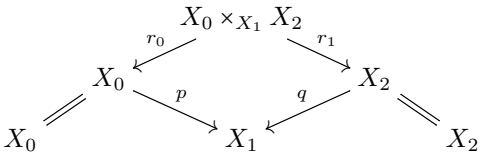


As we wish for β to extend α, the component of β's lax structure for this composite will be the diagram

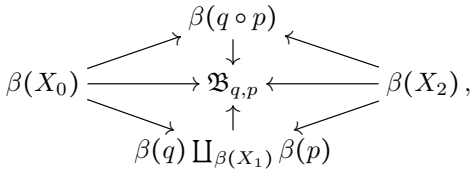


For a composite in the image of $\mathcal{C}\mathrm{oalg}^{\amalg}$ one does the same, replacing α for χ and $(-)^h$ for $(-)^v$.

Next, consider a pair of composable morphisms p and q given by the composite



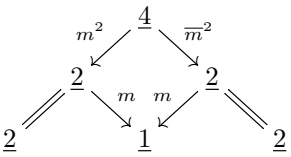
Then the component of β's lax structure for this composite will be a diagram



where $\mathfrak{B}_{q,p}$ is the bisimplicial set

$$\mathfrak{B}_{q,p} = \coprod_{x \in X_1} \Delta\left[|p^{-1}(x)|, |q^{-1}(x)|\right].$$

Example 4.2.1. For the composite



the lax structure of β is the bisimplicial set $\Delta[2,2]$, which is exactly the witness for the lax compatibility of the product and coproduct in the universal Hall algebra of a proto-exact category that we described in Eq. 1.2.1.

For a general composition, the component of the lax structure will be given as a certain colimit of the special cases considered above. Most of the difficulty in giving the rigorous construction of β comes from the need to give a coherent parametrisation of these colimit diagrams for all strings of composable morphisms in $\mathcal{B}\mathrm{ialg}^{\amalg}$.

Outline of the construction. According to Definitions 2.2.4 and 2.5.9, a totally lax bialgebra structure on $\Delta[1,1]$ as an object of $\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times}$ is a functor

$$\begin{array}{ccc} \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg}) & \xrightarrow{\quad\beta\quad} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times} \\ & \searrow \quad \swarrow & \\ & \mathcal{F}\mathrm{in}_* \times \mathbb{\Delta}^{\mathrm{op}} & \end{array}$$

which preserves cocartesian lifts of morphisms in $\mathcal{F}\mathrm{in} \times \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{in}}$. Furthermore, the image of the object $\underline{1}$ in the fibre over $(\underline{1}_*, [0])$ must be $\Delta[1,1]$.

By construction, $\mathcal{B}\mathrm{ialg}^{\amalg}$ is semistrict and so one can readily see from Eq. 2.4.3 that $\mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg})$ is the nerve of the ordinary Grothendieck construction of the functor

$$\mathrm{Fin}_* \times \Delta^{\mathrm{op}} \to \mathrm{Cat}, \quad (S_*, [n]) \mapsto \mathrm{Bialg}_{S,n}.$$

That is, an element of $\mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg})_k$ is a triple $((f,\phi),\theta,\gamma)$, where:

- (f,ϕ) is an element of $N_k(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$;
- θ is a family of functors

$$\theta_i : \Pi(f(i)) \times \mathbb{\Lambda}^{\phi(i)} \to \mathrm{Alg} \in \mathrm{Bialg}_{f(i),\phi(i)}, \quad i \in [k];$$

- γ is a family of natural isomorphisms

$$\gamma_i : \ (f_i,\phi_i)_*\theta_i \xRightarrow{\sim} \theta_{i+1} \ , \quad i = 0,\dots,k-1.$$

From such data we will build a cartesian, vertically constant functor

$$\beta(\theta,\gamma) : \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \to (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}.$$

By Proposition 2.4.25 this defines a k-simplex in the unstraightening of $\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times}$. As the general construction of $\beta(\theta,\gamma)$ will be somewhat involved, let us first consider a very special case.

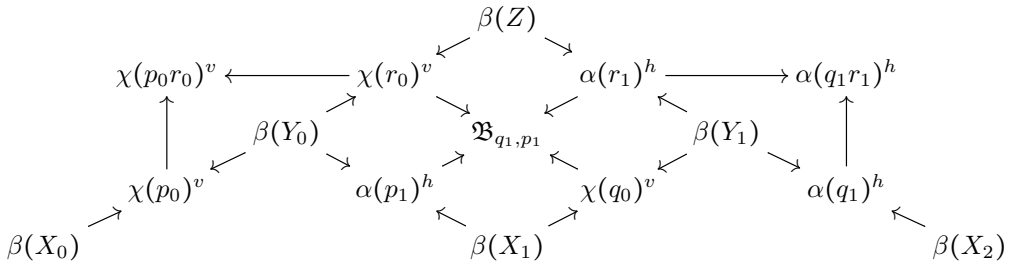
Consider $((f,\phi),\theta,\gamma) \in \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg})_1$, where f is constant on $\underline{1}_*$, $\phi : [2] \twoheadleftarrow [1]$ is the unique active morphism and the natural isomorphism γ_0 is the identity. Then the family θ is determined by a pair of composable morphisms p and q in $\mathcal{B}\mathrm{ialg}^{\amalg}$, that is, a diagram in Alg

$$\begin{array}{ccccccc} & & & Z & & & \\ & & {\scriptstyle r_0}\swarrow & & \searrow{\scriptstyle r_1} & & \\ & & Y_0 & & Y_1 & & \\ & {\scriptstyle p_0}\swarrow & & {\scriptstyle p_1}\searrow \quad {\scriptstyle q_0}\swarrow & & {\scriptstyle q_1}\searrow & \\ X_0 & & & X_1 & & & X_2 \end{array}$$

where the middle square is a pseudo-pullback. Since $\beta(\theta,\gamma)$ is cartesian it is determined by its restriction to Λ^{M_ϕ}, the diagram described in Example 2.4.21. The restriction of $\beta(\theta,\gamma)$ to Λ^{M_ϕ} is

$$\begin{array}{ccccc}
\beta(X_0) & \longrightarrow & \beta(qp) & \longleftarrow & \beta(X_2) \\
\| & & \downarrow & & \| \\
\beta(X_0) & \longrightarrow & \mathfrak{B}_{q,p} & \longleftarrow & \beta(X_2) \\
\| & \nearrow & & \nwarrow & \| \\
\beta(X_0) & \longrightarrow \beta(p) \longleftarrow & \beta(X_1) & \longrightarrow \beta(q) \longleftarrow & \beta(X_2)
\end{array}$$

The bisimplicial set $\mathfrak{B}_{q,p}$ is the colimit of the diagram



This diagram is a functor $\overline{\beta}(\theta,\gamma)$ from the twisted arrow category of Σ^2 to $(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}$.

Looking back at the description of β on 1-morphisms in Eq. 4.2.1, one observes that *every* object in the restriction of $\beta(\theta,\gamma)$ to Λ^{M_ϕ} is a colimit over some subdiagram of $\overline{\beta}(\theta,\gamma)$. We therefore refer to $\overline{\beta}(\theta,\gamma)$ as the *master diagram*, as the functor $\beta(\theta,\gamma)$ is obtained from it by systematically extracting colimits over subdiagrams.

For a general $((f,\phi),\theta,\gamma) \in \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\sqcup})_k$, the master diagram is a functor

$$\overline{\beta}(\theta,\gamma) : P(f(0))^{\mathrm{op}} \times \Sigma^{\Omega_\phi} \to (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}, \tag{4.2.2}$$

where Ω_ϕ is a suitable generalisation of Σ^2 that we shall define in Section 4.2.1.

The process of systematically extracting colimits over subdiagrams of the master diagram $\overline{\beta}(\theta,\gamma)$ is formalised in the construction of the *cone functor*

$$\kappa_\phi : \Sigma^{M_\phi} \to \widehat{\Sigma^{\Omega_\phi}}, \tag{4.2.3}$$

where $\widehat{\Sigma^{\Omega_\phi}}$ is the free completion of Σ^{Ω_ϕ}.

The functor $\beta(\theta,\gamma)$ is defined by a sequence of right Kan extensions and restrictions as described by the following diagram

$$\begin{array}{ccccccc}
\Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} & & & \xrightarrow{\beta(\theta,\gamma)} & & & (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}} \\
\uparrow & & & & & & \| \\
P(f(0))^{\mathrm{op}} \times \Lambda^{M_\phi} & \hookrightarrow & P(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} & \xrightarrow[\mathrm{Id}\times\kappa_\phi]{} & P(f(0))^{\mathrm{op}} \times \widehat{\Sigma^{\Omega_\phi}} & \longrightarrow & (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}} \\
 & & & & \uparrow & & \| \\
 & & & & P(f(0))^{\mathrm{op}} \times \Sigma^{\Omega_\phi} & \xrightarrow[\overline{\beta}(\theta,\gamma)]{} & (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}
\end{array} \tag{4.2.4}$$

where each square is a right Kan extension.

4.2.1 The totally lax bialgebra structure on $\Delta[1,1]$

Our first step in the construction of the totally lax bialgebra arising from a general bisimplicial object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ will be to carry out the construction for the initial bisimplicial object $\Delta[\bullet,\bullet] \in (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}$. As per Definitions 2.2.4 and 2.5.9, a totally lax bialgebra structure on $\Delta[1,1]$ as an object of $\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times}$ is a functor

$$\begin{array}{ccc} \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\sqcup}) & \xrightarrow{\quad\beta\quad} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times} \\ & \searrow \quad \swarrow & \\ & \mathcal{F}\mathrm{in}_* \times \mathbb{\Delta}^{\mathrm{op}} & \end{array}$$

which preserves cocartesian lifts of morphisms in $\mathcal{F}\mathrm{in} \times \mathbb{\Delta}^{\mathrm{op}}_{\mathrm{in}}$ and sends the object $\underline{1}$ to $\Delta[1,1]$. In Section 4.2.2 we show how the object $\mathcal{X}_{1,1}$ inherits a totally lax bialgebra structure from $\Delta[1,1]$.

As outlined above, the construction of β proceeds by first defining, for a given $((f,\phi),\theta,\gamma) \in \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\sqcup})_k$, the master diagram of Eq. 4.2.2 and cone functor of Eq. 4.2.3. The image of $((f,\phi),\theta,\gamma)$ under β is then defined by a sequence of right Kan extensions and restrictions according to the diagram in Eq. 4.2.4.

Before going further we must define the categories Ω_ϕ which appear crucially at all stages of the construction. For a given $\phi \in N_k(\Delta^{\mathrm{op}})$, define Ω_ϕ to be the poset having object set

$$\left\{([i;j],b) \mid b \in [k]^{\mathrm{op}},\ [i;j] \in \Sigma^{\phi(b)}\right\},$$

and ordering defined by declaring $([i;j],b) \leq ([i';j'],b')$ if and only if $b \leq b' \in [k]^{\mathrm{op}}$ and $\phi_{b,b'}([i;j]) \leq [i';j'] \in \Sigma^{\phi(b')}$. In fact, Ω_ϕ is the ordinary Grothendieck construction of the functor

$$[k]^{\mathrm{op}} \xrightarrow{\ \phi\ } \Delta \xrightarrow{\ \Sigma^\bullet\ } \mathrm{Cat}\ .$$

Hence, for a natural transformation $\eta : \phi' \Rightarrow \phi$ one has a functor

$$\Omega(\eta) : \Omega_{\phi'} \to \Omega_\phi,\ ([i;j],b) \mapsto ([\eta_b(i);\eta_b(j)],b)$$

and for a morphism $\gamma : [n] \to [k]$ there is a functor

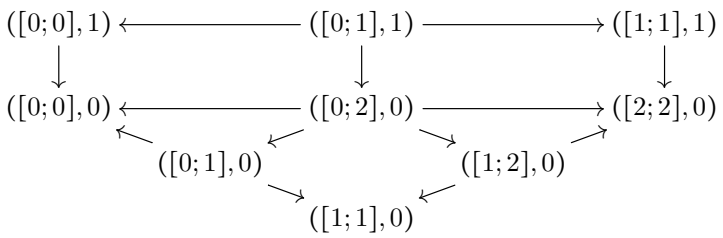
$$\Omega(\gamma) : \Omega_{\phi\gamma} \to \Omega_\phi,\ ([i;j],b) \mapsto ([i;j],\gamma(b)).$$

Finally, observe that V_ϕ is a subposet of Ω_ϕ according to the functor

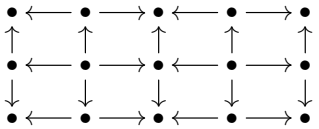
$$\nu_\phi : (a,b) \mapsto ([a;a],b).$$

Example 4.2.2. For $\phi \in N_k(\Delta^{\mathrm{op}})$ the constant map on $[n]$, the poset Ω_ϕ is $\Sigma^n \times [k]^{\mathrm{op}}$.

Example 4.2.3. For the unique active morphism $\phi = ([2] \twoheadleftarrow [1]) \in N_1(\Delta^{\mathrm{op}})$, the poset Ω_ϕ is



Example 4.2.4. For $\phi \in N_1(\Delta^{\mathrm{op}})$ the constant map on $[1]$, the poset Σ^{Ω_ϕ} is



We are now ready to proceed with the step-by-step construction of the symmetric monoidal lax functor $\beta : \mathcal{B}\mathrm{ialg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times}$.

Construction of the master diagram. Fix $((f,\phi),\theta,\gamma) \in \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg})_k$. We shall construct, according to Theorem 2.3.3, the master diagram $\overline{\beta} := \overline{\beta}(\theta,\gamma)$ of Eq. 4.2.2 as a normal oplax functor

$$\overline{\beta} : P(f(0))^{\mathrm{op}} \times \Omega_\phi \nrightarrow \mathrm{sp}\left((\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right).$$

Recall from Section 1.3 that ∇ is the category of spans of the form $\langle n\rangle \leftarrow \langle k\rangle \rightarrowtail \langle m\rangle$ in Δ_+. The category of *levelwise finite bi-nabla sets*, denoted Fin_{∇^2} is the category of functors $(\nabla^{\mathrm{op}})^{\times 2} \to \mathrm{Fin}$. From a pseudo-pullback square in Alg

$$\begin{array}{ccc} X' & \xrightarrow{q_2} & Y' \\ {\scriptstyle q_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ X & \xrightarrow[p_1]{} & Y \end{array}$$

one can define the following diagram of levelwise finite bi-nabla sets:

$$\begin{array}{ccccc} & & \coprod\limits_{y\in Y} \nabla\left[p_1^{-1}(y), p_2^{-1}(y)\right] & & \\ & \nearrow & & \nwarrow & \\ X' & & & & Y \end{array}$$

where X' and Y are constant bi-nabla sets and $\nabla[n,m]$ is the functor represented by $(\langle n\rangle, \langle m\rangle)$. The first map arises from the following morphism in $\nabla^{\times 2}$

$$\begin{array}{ccc} & (\{x'\},\{x'\}) & \\ \swarrow & & \searrow \\ (\{x'\},\{x'\}) & & (p_1^{-1}(p_1 q_1(x')), p_2^{-1}(p_2 q_2(x'))), \end{array}$$

and the second morphism from

$$\begin{array}{ccc} & (p_1^{-1}(y), p_2^{-1}(y)) & \\ \swarrow & & \searrow \\ (\{y\},\{y\}) & & (p_1^{-1}(y), p_2^{-1}(y)). \end{array}$$

We can then apply the functor $\mathfrak{G}^*$ of Eq. 1.3.1 to obtain a span in $(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}$.

Now let $f : (s,[i;j],b) \to (s,[i';j'],b')$ be a morphism in $P(f(0))^{\mathrm{op}} \times \Omega_\phi$. Then one has the following diagram in $\mathbb{\Lambda}^{\phi(b')}$

$$\begin{array}{ccc} [\phi_{b,b'}(i); \phi_{b,b'}(j)] & \longrightarrow & [i'; \phi_{b,b'}(j)] \\ \downarrow & & \downarrow \\ [\phi_{b,b'}(i); j'] & \longrightarrow & [i'; j'] \end{array} \tag{4.2.5}$$

The image of this square under $\theta_{b'}(f_{b',0}^{-1}(s),-)$ is, by assumption, a pseudo-pullback square in Alg. Define $\overline{\beta}$ on objects to be

$$\overline{\beta}_b(s,[i;j]) = \mathfrak{G}^*\left(\theta_b(f_{b,0}^{-1}(s),[i;j])\right) = \theta_b(f_{b,0}^{-1}(s),[i;j])\cdot\Delta\,[1,1]\,.$$

We can apply the preceding construction to produce a span in $(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}$

$$\begin{array}{ccc} & & \overline{\beta}(f) \\ & \nearrow & \nwarrow \\ \overline{\beta}_{b'}(s,[\phi_{b,b'}(i);\phi_{b,b'}(j)]) & & \overline{\beta}_{b'}(s,[i';j'])\,. \end{array}$$

The image of f under $\overline{\beta}$ is the span obtained by precomposing the left leg with the natural isomorphism $\gamma_{b,b'}$

$$\begin{array}{ccc} & & \overline{\beta}(f) \\ & \nearrow & \nwarrow \\ \overline{\beta}_b(s,[i;j]) & & \overline{\beta}_{b'}(s,[i';j']) \end{array}$$

Next, consider a diagram in Alg of the form

$$\begin{array}{ccccc} X'' & \longrightarrow & Y'' & \xrightarrow{s_2} & Z'' \\ \downarrow & & \downarrow{\scriptstyle r_2} & & \downarrow{\scriptstyle q_2} \\ X' & \xrightarrow[r_1]{} & Y' & \xrightarrow{t_2} & Z' \\ {\scriptstyle s_1}\downarrow & & \downarrow{\scriptstyle t_1} & & \downarrow{\scriptstyle p_2} \\ X & \xrightarrow[q_1]{} & Y & \xrightarrow[p_1]{} & Z \end{array}$$

where each square is a pseudo-pullback. From this one defines the following commutative diagram bi-nabla sets

$$\begin{array}{ccccccc} & & \coprod_{z\in Z}\nabla\left[(p_1q_1)^{-1}(z),(p_2q_2)^{-1}(z)\right] & & & & \\ & \nearrow & \uparrow{\scriptstyle a} & & & \nwarrow & \\ X'' & \longrightarrow & \coprod_{y\in Y'}\nabla\left[r_1^{-1}(y),r_2^{-1}(y)\right] & \longleftarrow & Y' \xrightarrow{b} & \coprod_{z\in Z}\nabla\left[p_1^{-1}(z),p_2^{-1}(z)\right] & \leftarrow Z\,, \end{array}$$

where the morphism a is induced by s_1 and s_2 and the morphism b is induced by q_1 and q_2. Applying the functor $\mathfrak{G}^*$ of Eq. 1.3.1 yields a diagram in Fin_{Δ^2}.

We will now define the oplax structure on $\overline{\beta}$. Let $(s,[i;j],b) \xrightarrow{f} (s,[i';j'],b') \xrightarrow{g} (s,[i'';j''],b'')$ be a pair of composable morphisms in $P(f(0))^{\mathrm{op}}\times\Omega_\phi$. Then one has the following diagram in $\mathbb{m}^{\phi(b'')}$

$$\begin{array}{ccccc} [\phi_{b,b''}(i);\phi_{b,b''}(j)] & \longrightarrow & [\phi_{b',b''}(i');\phi_{b,b''}(j)] & \longrightarrow & [i'';\phi_{b,b''}(j)] \\ \downarrow & & \downarrow & & \downarrow \\ [\phi_{b,b''}(i);\phi_{b,b''}(j')] & \longrightarrow & [\phi_{b',b''}(i');\phi_{b',b''}(j')] & \longrightarrow & [i'';\phi_{b',b''}(j')] \\ \downarrow & & \downarrow & & \downarrow \\ [\phi_{b,b''}(i);j''] & \longrightarrow & [\phi_{b',b''}(i');j''] & \longrightarrow & [i'';j''] \end{array} \tag{4.2.6}$$

Using the preceding discussion and the natural isomorphisms $\gamma_{b,b''}$ and $\gamma_{b',b''}$ we obtain a commutative diagram

$$\begin{array}{ccc} \overline{\beta}_{b'}(s',[i';j']) & \longrightarrow & \overline{\beta}(g) \\ \downarrow & & \downarrow \\ \overline{\beta}(f) & \longrightarrow & \overline{\beta}(g\circ f)\,. \end{array} \tag{4.2.7}$$

We define the corresponding component of the oplax structure,

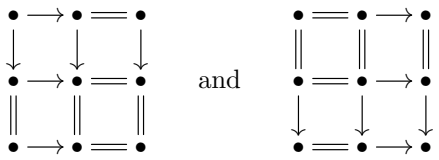
$$\overline{\mathfrak{B}}_{g,f} : \overline{\beta}(g) \coprod_{\overline{\beta}_{b'}(s',[i';j'])} \overline{\beta}(f) \to \overline{\beta}(g \circ f),$$

to be the universal morphism induced by Eq. 4.2.7.

Lemma 4.2.5. *For each $((f,\phi),\theta,\gamma) \in \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg})_k$, the data given above defines a normal oplax functor and hence, by Theorem 2.3.3, a functor*

$$\overline{\beta}(\theta,\gamma) : P(f(0))^{\mathrm{op}} \times \Sigma^{\Omega_\phi} \to (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}.$$

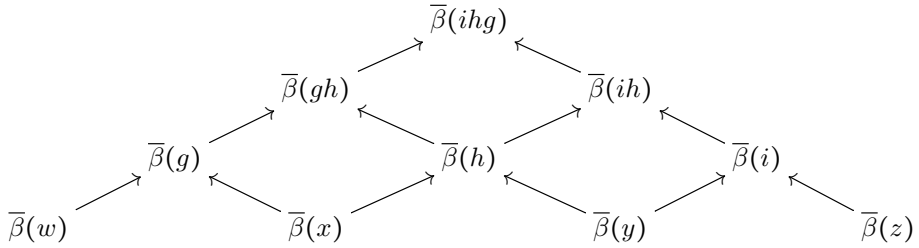
Proof. To see that $\overline{\mathfrak{B}}_{\mathrm{Id},g} = \mathrm{Id}_{\overline{\beta}(g)} = \overline{\mathfrak{B}}_{g,\mathrm{Id}}$ one need only note that the 2×2 grids of pseudo-pullbacks in Alg defining these components take, respectively, the forms



$$\begin{tikzcd}
\bullet \arrow[r] \arrow[d] & \bullet \arrow[r, equal] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[r, equal] \arrow[d, equal] & \bullet \arrow[d, equal] \\
\bullet \arrow[r] & \bullet \arrow[r, equal] & \bullet
\end{tikzcd}
\quad \text{and} \quad
\begin{tikzcd}
\bullet \arrow[r, equal] \arrow[d, equal] & \bullet \arrow[r] \arrow[d, equal] & \bullet \arrow[d, equal] \\
\bullet \arrow[r, equal] \arrow[d] & \bullet \arrow[r] \arrow[d] & \bullet \arrow[d] \\
\bullet \arrow[r, equal] & \bullet \arrow[r] & \bullet
\end{tikzcd}$$

Given a triple $w \xrightarrow{g} x \xrightarrow{h} y \xrightarrow{i} z$ of composable morphisms in $P(f(0))^{\mathrm{op}} \times \Omega_\phi$ one has, in the same manner as above, a 3×3 grid of pseudo-pullbacks in Alg. From this one obtains a diagram Fin_{Δ^2}

$$\begin{tikzcd}
 & & & \overline{\beta}(ihg) & & & \\
 & & \overline{\beta}(gh) \arrow[ur] & & \overline{\beta}(ih) \arrow[ul] & & \\
 & \overline{\beta}(g) \arrow[ur] & & \overline{\beta}(h) \arrow[ul] \arrow[ur] & & \overline{\beta}(i) \arrow[ul] & \\
\overline{\beta}(w) \arrow[ur] & & \overline{\beta}(x) \arrow[ul] \arrow[ur] & & \overline{\beta}(y) \arrow[ul] \arrow[ur] & & \overline{\beta}(z) \arrow[ul]
\end{tikzcd}$$



Then one has that

$$\overline{\mathfrak{B}}_{i,hg} \circ \left(\mathrm{Id}_{\overline{\beta}(i)} \coprod_{\overline{\beta}(y)} \overline{\mathfrak{B}}_{h,g} \right) = \overline{\mathfrak{B}}_{ih,g} \circ \left(\overline{\mathfrak{B}}_{i,h} \coprod_{\overline{\beta}(x)} \mathrm{Id}_{\overline{\beta}(g)} \right)$$

since both the left and right hand side are universal morphisms for the same colimit. Namely, the colimit over the bottom two rows of the preceding diagram. □

The functor $\overline{\beta}$ is vertically constant in the following sense.

Lemma 4.2.6. *For each $((f,\phi),\theta,\gamma) \in \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg})_k$, the image of every morphism in the composite functor*

$$P(f(0))^{\mathrm{op}} \times \Sigma^{V_\phi} \xrightarrow{\Sigma^{\nu_\phi}} P(f(0))^{\mathrm{op}} \times \Sigma^{\Omega_\phi} \xrightarrow{\overline{\beta}(\theta,\gamma)} (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}$$

is an isomorphism.

Proof. By Theorem 2.3.3 it suffices to show that for the normal oplax functor

$$P(f(0))^{\mathrm{op}} \times V_\phi \xrightarrow{\nu_\phi} P(f(0))^{\mathrm{op}} \times \Omega_\phi \xrightarrow{\overline{\beta}(\theta,\gamma)} \mathrm{sp}\left((\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right)$$

both the images of morphisms and the components of the oplax structure are isomorphisms. This follows from observing that when restricted to V_ϕ the diagrams 4.2.5 and 4.2.6 defining, respectively, the images of morphisms and components of the oplax structure, are constant. □

Before moving on to the construction of the cone functor, let us consider a simple example of the master diagram that will reappear later in this section.

Example 4.2.7. Consider the case of $(f,\phi) \in N_1(\mathrm{Fin}_* \times \Delta^{\mathrm{op}})$ with f the constant map on $\underline{1}_*$ and ϕ the constant map on $[1]$. Then an element $((f,\phi),\theta,\gamma) \in \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\sqcup})_1$ is the data of a diagram

$$\begin{array}{ccccc} X_0 & \xleftarrow{p_0} & Y & \xrightarrow{p_1} & X_1 \\ {\scriptstyle\gamma_0}\downarrow\wr & & \wr\downarrow{\scriptstyle\gamma} & & \wr\downarrow{\scriptstyle\gamma_1} \\ X_0' & \xleftarrow{p_0'} & Y' & \xrightarrow{p_1'} & X_1', \end{array}$$

in Alg and $\overline{\beta}$ is

$$\begin{array}{ccccccccc} \overline{\beta}(X_0) & \longrightarrow & \overline{\beta}(p_0) & \longleftarrow & \overline{\beta}(Y) & \longrightarrow & \overline{\beta}(p_1) & \longleftarrow & \overline{\beta}(X_1) \\ \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr & & \downarrow\wr \\ \overline{\beta}(\gamma_0) & \longrightarrow & \overline{\beta}(\gamma_0 p_0) & \longleftarrow & \overline{\beta}(\gamma) & \longrightarrow & \overline{\beta}(\gamma_1 p_1) & \longleftarrow & \overline{\beta}(\gamma_1) \\ \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr & & \uparrow\wr \\ \overline{\beta}(X_0') & \longrightarrow & \overline{\beta}(p_0') & \longleftarrow & \overline{\beta}(Y') & \longrightarrow & \overline{\beta}(p_1') & \longleftarrow & \overline{\beta}(X_1') \end{array}$$

Construction of the cone functor. Fix $\phi \in N_k(\Delta^{\mathrm{op}})$. We shall construct, as per Theorem 2.3.3, the cone functor of Eq. 4.2.3 as a normal oplax functor

$$\kappa_\phi : M_\phi \nrightarrow \mathrm{sp}\left(\widehat{\Sigma^{\Omega_\phi}}\right),$$

where $\widehat{\Sigma^{\Omega_\phi}}$ is the free completion of Σ^{Ω_ϕ}, which we now define.

The *free completion* of a poset Q, denoted $\widehat{Q}$, is the opposite of its poset of upwards-closed subsets, that is,

$$\widehat{P} = \mathrm{Fun}(P,[1])^{\mathrm{op}}.$$

The free completion $\widehat{Q}$ satisfies the following universal property: any functor $F : Q \to D$, for D complete, uniquely factors as

$$\begin{array}{ccc} \hat{Q} & \xrightarrow{\hat{F}} & D, \\ {\scriptstyle\uparrow}\uparrow & \nearrow_{F} & \\ Q & & \end{array}$$

where

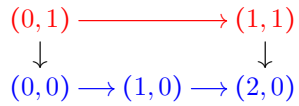
$$\uparrow\colon Q \hookrightarrow \widehat{Q}\,, \quad q \mapsto \{q' \in Q \mid q' \geq q\}.$$

The definition of the cone functor κ_ϕ is based on the following observation. To each interval $[(a,b);(a',b')]$ in M_ϕ one can associate the subposet of Ω_ϕ,

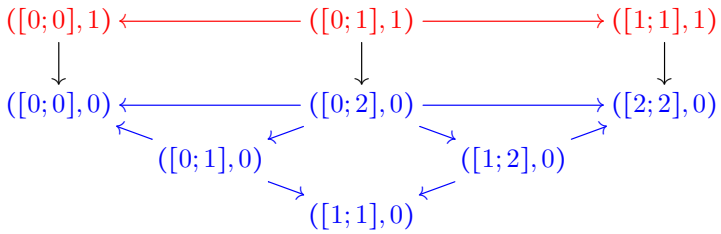
$$\overline{\kappa}_\phi[(a,b);(a',b')] = \{([i;j],c) \mid b \leq c \leq b',\, \phi_{b,c}(a) \leq i,\, \phi_{c,b'}(j) \leq a'\}\,.$$

These subposets are such that for $[x;x'] \subset [y;y']$ one has that $\overline{\kappa}_\phi[x;x'] \subset \overline{\kappa}_\phi[y;y']$.

Example 4.2.8. Let $\phi : [2] \twoheadleftarrow [1]$ be the unique active morphism and consider the following two intervals, coloured in red and blue, in M_ϕ



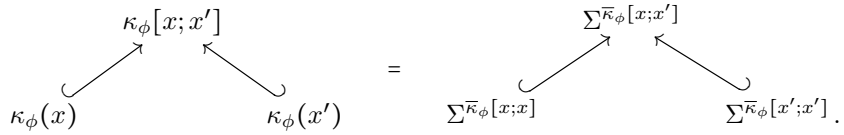
The associated subposets of Ω_ϕ defined by $\overline{\kappa}_\phi$ are the following



Informally, each interval in M_ϕ determines a region in Ω_ϕ. The cone functor maps each interval to the formal limit over all intervals in its associated region. To make this rigorous, we define the cone functor κ_ϕ on objects to be

$$\kappa_\phi : x \mapsto \Sigma^{\overline{\kappa}_\phi[x;x]} \in \widehat{\Sigma^{\Omega_\phi}},$$

and define it to send a morphism $x \leq x'$ to the following diagram defining a span in $\widehat{\Sigma^{\Omega_\phi}}$



Given a pair of composable morphisms $x \leq x' \leq x''$ in M_ϕ, one has the following commutative diagram in $\widehat{\Sigma^{\Omega_\phi}}^{\mathrm{op}}$

$$\begin{array}{ccc} \kappa_\phi(x') & \hookrightarrow & \kappa_\phi[x';x''] \\ \downarrow & & \downarrow \\ \kappa_\phi[x;x'] & \hookrightarrow & \kappa_\phi[x;x'']\,. \end{array} \tag{4.2.8}$$

We define the corresponding component of the oplax structure,

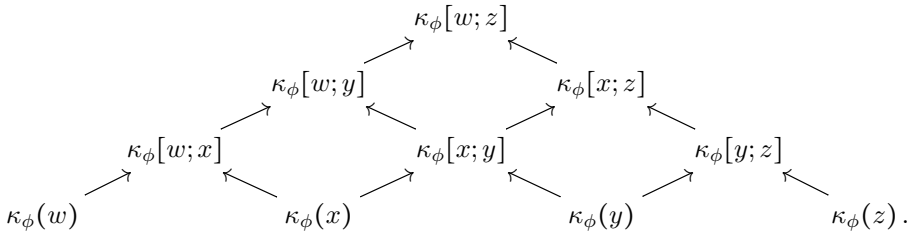
$$K_\phi[x;x';x''] : \kappa_\phi[x';x''] \coprod_{\kappa_\phi(x')} \kappa_\phi[x;x'] \to \kappa_\phi[x;x''],$$

to be the universal morphism induced by the diagram in Eq. 4.2.8.

Lemma 4.2.9. *For each $\phi \in N_k(\Delta^{\mathrm{op}})$ the above data defines a normal oplax functor and hence, by Theorem 2.3.3, a functor*

$$\kappa_\phi : \Sigma^{M_\phi} \to \widehat{\Sigma^{\Omega_\phi}}.$$

Proof. From the diagram in Eq. 4.2.8 it is clear that $K_\phi[x;x;x'] = \mathrm{Id}_{\kappa_\phi[x;x']} = K_\phi[x;x';x']$. Next, consider a triple of composable morphisms $w \le x \le y \le z$ in M_ϕ. One has the following diagram



Then the following equation holds

$$K_\phi[w;y;z] \circ \left(\mathrm{Id}_{\kappa_\phi[y;z]} \coprod_{\kappa_\phi(y)} K_\phi[w;x;y] \right) = K_\phi[w;x;z] \circ \left(K_\phi[x;y;z] \coprod_{\kappa_\phi(x)} \mathrm{Id}_{\kappa_\phi[w;x]} \right),$$

since both sides of the equation are the universal morphism coming from the colimit of the lower two rows of the preceding diagram. □

The cone functor κ_ϕ is natural in the following sense.

Lemma 4.2.10. *For each $\phi \in N_k(\Delta^{\mathrm{op}})$ and $\psi : [n] \to [k]$ the following diagram commutes*

$$\begin{array}{ccc} \Sigma^{M_{\phi\psi}} & \xrightarrow{\kappa_{\phi\psi}} & \widehat{\Sigma^{\Omega_{\phi\psi}}} \\ {\scriptstyle \Sigma^{M(\psi)}}\downarrow & & \downarrow{\scriptstyle \widehat{\Sigma^{\Omega(\psi)}}} \\ \Sigma^{M_\phi} & \xrightarrow[\kappa_\phi]{} & \widehat{\Sigma^{\Omega_\phi}}. \end{array}$$

Proof. It suffices to show that for each interval $[x;x']$ in $M_{\phi\psi}$,

$$\uparrow \Sigma^{\Omega(\psi)\overline{\kappa}_{\phi\psi}[x;x']} = \Sigma^{\overline{\kappa}_\phi[M(\psi)x;M(\psi)x']}.$$

This is straightforward, but notationally cumbersome, to verify. □

Finally, it is trivial to verify that κ_ϕ is vertically constant in the following sense.

Lemma 4.2.11. *For each $\phi \in N_k(\Delta^{\mathrm{op}})$ the following diagram commutes*

$$\begin{array}{ccc} \Sigma^{M_\phi} & \xrightarrow{\kappa_\phi} & \widehat{\Sigma^{\Omega_\phi}} \\ \uparrow & & \uparrow \\ \Sigma^{V_\phi} & \xrightarrow[\Sigma^{\nu}\phi]{} & \Sigma^{\Omega_\phi}. \end{array}$$

The morphism β is a symmetric monoidal lax functor. The diagram in Eq. 4.2.4 defines a functor

$$R'_{(f,\phi)} : \mathrm{Fun}\left(P(f(0))^{\mathrm{op}} \times \Sigma^{\Omega_\phi}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right) \to \mathrm{Fun}^{\mathrm{cart}}\left(\Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right).$$

We define $\beta(\theta,\gamma)$, for $((f,\phi),\theta,\gamma) \in \mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg})_k$, to be

$$\beta(\theta,\gamma) := R'_{(f,\phi)}\overline{\beta}(\theta,\gamma) \in \langle(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\rangle_k\,,$$

where $\langle(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\rangle$ is the sub simplicial set of the unstraightening of $\mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times}$ from Proposition 2.4.25.

Lemma 4.2.12. *The assignment $((f,\phi),\theta,\gamma) \mapsto \beta(\theta,\gamma)$ defines a morphism of simplicial sets*

$$\beta : \mathrm{Un}\left(\mathcal{B}\mathrm{ialg}^{\amalg}\right) \to \langle(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\rangle\,.$$

Proof. Fix $((f,\phi),\theta,\gamma) \in \mathrm{Un}\left(\mathcal{B}\mathrm{ialg}^{\amalg}\right)$ and $\psi : [n] \to [k]$. The functor $\beta(\theta,\gamma)$ is cartesian by construction and Lemmas 4.2.11 and 4.2.6 imply that it is vertically constant.

It remains to show that the functors $\psi^*\beta(\theta,\gamma)$ and $\beta(\psi^*\theta,\psi^*\gamma)$ agree, where

$$(\psi^*\theta)_i = (f_{\psi(i),i}, \phi_{\psi(i),i})_*\theta_i \quad \text{and} \quad (\psi^*\gamma)_i = (f_{\psi(i+1),i+1}, \phi_{\psi(i+1),i+1})_*\gamma_i.$$

Let $\overline{P(f(0))^{\mathrm{op}}}$ denote the full subcategory of $\Pi(f(0))^{\mathrm{op}}$ containing $P(f(0))^{\mathrm{op}}$ as well as those $U \subset f(0)$ such that $U \subset f^{-1}_{\psi(0),0}(s)$ for some $s \in f\psi(0)$. Then by Lemma 4.2.10 the following diagram commutes

$$\begin{array}{ccc}
\mathrm{Fun}^{\mathrm{cart}}\left(\Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right) & \xrightarrow{\psi^*} & \mathrm{Fun}^{\mathrm{cart}}\left(\Pi(f\psi(0))^{\mathrm{op}} \times \Sigma^{M_{\phi\psi}}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right) \\
{\scriptstyle R'_{(f,\phi)}}\uparrow & & \uparrow{\scriptstyle R'_{(f,\phi)}} \\
\mathrm{Fun}\left(\overline{P(f(0))^{\mathrm{op}}} \times \Sigma^{\Omega_\phi}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right) & \xrightarrow[\psi^*]{} & \mathrm{Fun}\left(P(f\psi(0))^{\mathrm{op}} \times \Sigma^{\Omega_{\phi\psi}}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right)
\end{array}$$

To prove the Lemma it suffices to show that for each $s \in f\psi(0)$,

$$\overline{\beta}(\psi^*\theta,\psi^*\gamma)(s,-) = \coprod_{t \in f^{-1}_{\psi(0),0}(s)} \psi^*\overline{\beta}(\theta,\gamma)(t,-)$$

as normal oplax functors $\Omega_{\phi\psi} \twoheadrightarrow \mathrm{sp}\left((\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right)$. This is the case as, by definition, the functors θ_i are cocartesian. □

We have therefore constructed a morphism of fibrations

$$\begin{array}{ccc}
\mathrm{Un}(\mathcal{B}\mathrm{ialg}^{\amalg}) & \xrightarrow{\quad\beta\quad} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times} \\
& \searrow \quad \swarrow & \\
& \mathcal{F}\mathrm{in}_* \times \mathbb{\Delta}^{\mathrm{op}} &
\end{array}$$

such that the image of $\underline{1}$ is $\Delta[1,1]$. To show that β endows the standard $(1,1)$-simplex $\Delta[1,1]$ with the structure of a totally lax bialgebra it remains only to show that β is a symmetric monoidal lax functor.

Proposition 4.2.13. *The morphism of simplicial sets β is a symmetric monoidal lax functor*

$$\beta : \mathcal{B}\mathrm{ialg}^{\amalg} \rightsquigarrow \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times}.$$

Proof. Fix $((f,\phi),\theta,\gamma)$ with $\phi : [m] \twoheadleftarrow [n]$ inert. We must show that the induced functor

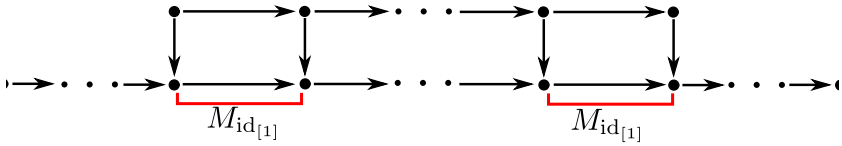
$$\beta(\theta,\gamma)_{1,[1]} : \Pi(f(1))^{\mathrm{op}} \times \Sigma^{\phi(1),1} \longrightarrow \Pi(f(0))^{\mathrm{op}} \times \Sigma^{M_\phi} \xrightarrow{\beta} (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}$$

is an equivalence in the quasicategory $\mathcal{S}\mathrm{p}_{f(1),\phi(1)}((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}})$ ([46] 3.2.5.2).

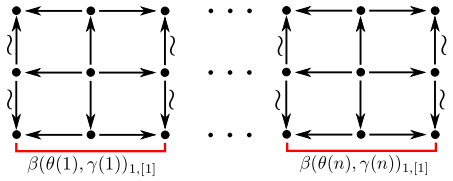
As every inert morphism can be written as

$$\phi : \ [a] \vee \left(\bigvee_{i=1}^{n} [1]\right) \vee [b] \longleftarrow\!\prec \bigvee_{i=1}^{n} [1],$$

the poset M_ϕ is of the form



Let $(\theta(i),\gamma(i))$ denote the restriction of (θ,γ) to the i'th summand of $[n]$. Then for each $U \in \Pi(f(1))^{\mathrm{op}}$, the functor $\beta(\theta,\gamma)_{1,[1]}(U,-)$ is the right Kan extension of a diagram of the following form in $(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}$:



We conclude that $\beta(\theta,\gamma)_{1,[1]}$ is an equivalence if and only if $\beta(\theta(i),\gamma(i))_{1,[1]}$ is an equivalence for each i.

It therefore suffices to consider the case when ϕ is the identity morphism. This is exactly the case considered above in Example 4.2.7. For each $U \in \Pi(f(1))^{\mathrm{op}}$, the functor $\beta(\theta,\gamma)_{1,[1]}(U,-)$ is

$$\begin{array}{ccccc}
\beta(X_0) & \longrightarrow & \beta(p_0) \coprod_{\beta(Y)} \beta(p_1) & \longleftarrow & \beta(X_1) \\
\downarrow\wr & & \downarrow\wr & & \downarrow\wr \\
\beta(\gamma_0) & \longrightarrow & \beta(\gamma_0 p_0) \coprod_{\beta(\gamma)} \beta(\gamma_1 p_1) & \longleftarrow & \beta(\gamma_1) \\
\uparrow\wr & & \uparrow\wr & & \uparrow\wr \\
\beta(X_0') & \longrightarrow & \beta(p_0') \coprod_{\beta(Y')} \beta(p_1') & \longleftarrow & \beta(X_1')
\end{array}$$

and so $\beta(\theta,\gamma)_{1,[1]}$ is an equivalence. □

4.2.2 The totally lax bialgebra structure inherited from $\Delta[1,1]$

Let $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ be a bisimplicial object in an ∞-category having finite limits. We can now show how the object $\mathcal{X}_{1,1}$ inherits the structure of a totally lax bialgebra from the one endowed upon $\Delta[1,1]$ in Proposition 4.2.13.

Recall that any bisimplicial object $\mathcal{X}$ defines a finite limit preserving functor $\mathcal{X} : (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}} \to \mathcal{C}$ by right Kan extension,

$$\begin{array}{ccc} (\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}} & \xrightarrow{\mathcal{X}} & \mathcal{C} \\ \uparrow & \nearrow_{\mathcal{X}} & \\ (\mathbb{\Delta}^{\mathrm{op}})^2 & & \end{array}$$

By Theorem 3.2.8 this induces a symmetric monoidal functor

$$\mathcal{X} : \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times} \to \mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}.$$

Then the following is a direct corollary of Proposition 4.2.13.

Theorem 4.2.14. *Let $\mathcal{C}$ be an ∞-category with finite limits and $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ be a bisimplicial object. Then the composite*

$$\beta_{\mathcal{X}} : \ \mathcal{B}\mathrm{ialg}^{\amalg} \xrightarrow{\beta} \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times} \xrightarrow{\mathcal{X}} \mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}$$

is a symmetric monoidal lax functor which endows $\mathcal{X}_{1,1}$ with the structure of a totally lax bialgebra.

4.3 Double 2-Segal spaces define lax bialgebras

Having equipped the object of $(1,1)$-simplices $\mathcal{X}_{1,1}$ in a bisimplicial object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ with a totally lax bialgebra structure in Theorem 4.2.14, it is now time to show that the double 2-Segal condition enforces the (co)associativity of the (co)product. This will provide our definition of the universal Hall bialgebra of a double 2-Segal object.

Recall that in Theorem 3.3.1 we proved that a simplicial object $\mathcal{Y} \in \mathcal{C}_{\mathbb{\Delta}}$ is 2-Segal if and only if the composite

$$\alpha_{\mathcal{Y}} : \ \mathcal{A}\mathrm{lg}^{\amalg} \xrightarrow{\alpha} \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times} \xrightarrow{\mathcal{Y}} \mathcal{S}\mathrm{pan}_2\left(\mathcal{C}\right)^{\times}$$

is a symmetric monoidal functor. We leverage this to deduce the relation between the double 2-Segal and (co)associativity by proving that the lax algebra structure on $\mathcal{X}_{1,1}$ is induced by the simplicial object $\mathcal{X}_{\bullet,1}$ and the lax coalgebra structure is induced by $\mathcal{X}_{1,\bullet}$.

Lemma 4.3.1. *The totally lax bialgebra structure on $\Delta[1,1]$ is compatible with the lax algebra and coalgebra structures on $\Delta[1]$. That is, the following diagram commutes*

$$\begin{array}{ccc} \mathcal{A}\mathrm{lg}^{\amalg} & \xrightarrow{\alpha} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times} \\ {\scriptstyle \iota_a}\downarrow & & \downarrow{\scriptstyle (-)^h} \\ \mathcal{B}\mathrm{ialg}^{\amalg} & \xrightarrow[\beta]{} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}^2})^{\mathrm{op}}\right)^{\times} \\ {\scriptstyle \iota_c}\uparrow & & \uparrow{\scriptstyle (-)^v} \\ \mathcal{C}\mathrm{oalg}^{\amalg} & \xrightarrow[\chi]{} & \mathcal{S}\mathrm{pan}_2\left((\mathcal{F}\mathrm{in}_{\mathbb{\Delta}})^{\mathrm{op}}\right)^{\times} \end{array}$$

Proof. We shall only prove that the top square commutes, as the proof for the bottom square is essentially identical.

Fix $((f,\phi),\theta) \in \mathrm{Un}(\mathcal{A}\mathrm{lg}^{\sqcup})_k$. We must show that $\alpha(\theta)^h = \beta(\iota_a(\theta))$, where $\iota_a(\theta)$ is composed of the functors

$$\Pi(f(i)) \times \mathbb{\Lambda}^{\phi(i)} \longrightarrow \Pi(f(i)) \times \phi(i) \longrightarrow \Pi(f(0)) \times \phi(0) \xrightarrow{\theta} \mathrm{Alg}$$

and identity natural transformations between them.

From the natural transformation $\Sigma^\bullet \Rightarrow [\bullet]$ one has a functor $\Omega_\phi \to M_\phi$, and hence a functor

$$\rho_\phi : \Sigma^{\Omega_\phi} \longrightarrow \Sigma^{M_\phi} \xrightarrow{\Sigma^{p_\phi}} \Sigma^{\phi(0)} .$$

Consider the diagram

$$\begin{array}{ccc} \Sigma^{\phi(0)} & \xleftarrow{\Sigma^{p_\phi}} & \Lambda^{M_\phi} \\ {\scriptstyle\rho_\phi}\uparrow & \nwarrow^{\hat\rho_\phi} & \downarrow{\scriptstyle\kappa_\phi} \\ \Sigma^{\Omega_\phi} & \hookrightarrow & \widehat{\Sigma^{\Omega_\phi}} \\ & & \uparrow \end{array} .$$

The bottom triangle commutes by definition, and the upper triangle commutes since for any interval $[(a,b);(a',b')]$ in M_ϕ,

$$[\phi_{b,0}(a);\phi_{b',0}(a')] = \min\{[\phi_{c,0}(i);\phi_{c',0}(i')] \,|\, b \le c \le c' \le b',\, \phi_{b,c}(a) \le i,\, \phi_{b,c'}(a') \le i'\} .$$

It follows that the following diagram commutes

$$\begin{array}{ccc} \mathrm{Fun}\left(P(f(0))^{\mathrm{op}} \times \Sigma^{\phi(0)}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right) & \xrightarrow{(\Sigma^{p_\phi})^*} & \mathrm{Fun}\left(P(f(0))^{\mathrm{op}} \times \Lambda^{M_\phi}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right) \\ {\scriptstyle\rho_\phi^*}\downarrow & & \uparrow{\scriptstyle\kappa_\phi^*} \\ \mathrm{Fun}\left(P(f(0))^{\mathrm{op}} \times \Sigma^{\Omega_\phi}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right) & \xrightarrow{(\hat{-})} & \mathrm{Fun}\left(P(f(0))^{\mathrm{op}} \times \widehat{\Sigma^{\Omega_\phi}}, (\mathrm{Fin}_{\Delta^2})^{\mathrm{op}}\right) \end{array}$$

since

$$\kappa_\phi^*(\hat{-})\rho_\phi^* = \kappa_\phi^*(\hat{-}) \uparrow^* \hat\rho_\phi^* = \kappa_\phi^*\hat\rho_\phi^* = (\Sigma^{p_\phi})^* .$$

To show that $\alpha(\theta)^h = \beta(\iota_a(\theta))$ it therefore suffices to show that

$$\begin{array}{ccc} P(f(0))^{\mathrm{op}} \times \Omega_\phi & \xrightarrow{\overline\beta(\iota_a(\theta))} & \mathcal{S}\mathrm{p}(\mathrm{Fin}_{\Delta^2})^{\mathrm{op}} \\ \downarrow & & \uparrow{\scriptstyle(-)^h} \\ P(f(0))^{\mathrm{op}} \times \phi(0) & \xrightarrow[\overline\alpha(\theta)]{} & \mathcal{S}\mathrm{p}(\mathrm{Fin}_{\Delta})^{\mathrm{op}} \end{array}$$

where the horizontal maps are normal oplax functors. The diagram commutes on objects, as given an object $(s,[i;j],b)$ one has

$$\iota_a(\theta)_b(s,[i;j]) \cdot \Delta[1,1] = \theta(\phi_{b,0}(i)) \cdot \Delta[1,1] .$$

They agree on morphisms as for a morphism $(s,[i;j],b) \to (s,[i';j'],b')$, the image under $\iota_a(\theta)_{b'}$ of the diagram in Eq. 4.2.5 is

$$\begin{array}{ccc} \theta(\phi_{b,0}(i)) & \longrightarrow & \theta(\phi_{b',0}(i')) \\ \| & & \| \\ \theta(\phi_{b,0}(i)) & \longrightarrow & \theta(\phi_{b',0}(i')) . \end{array}$$

Similarly, for a pair of composable morphisms $(s,[i;j],b) \to (s,[i';j'],b') \to (s,[i'';j''],b'')$ the image under $\iota_a(\theta)_{b''}$ of the diagram in Eq. 4.2.6 is

$$\begin{array}{ccccc}
\theta(\phi_{b,0}(i)) & \longrightarrow & \theta(\phi_{b',0}(i')) & \longrightarrow & \theta(\phi_{b'',0}(i'')) \\
\| & & \| & & \| \\
\theta(\phi_{b,0}(i)) & \longrightarrow & \theta(\phi_{b',0}(i')) & \longrightarrow & \theta(\phi_{b'',0}(i'')) \\
\| & & \| & & \| \\
\theta(\phi_{b,0}(i)) & \longrightarrow & \theta(\phi_{b',0}(i')) & \longrightarrow & \theta(\phi_{b'',0}(i''))
\end{array}$$

showing that the diagram commutes on the level of oplax structures. □

Theorem 4.3.2. *Let $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ be a bisimplicial object in an ∞-category having finite limits. Then $\mathcal{X}$ is a double 2-Segal object if and only if the symmetric monoidal lax functor $\beta_{\mathcal{X}}$ endows $\mathcal{X}_{1,1}$ with a lax bialgebra structure.*

Definition 4.3.3. The *universal Hall bialgebra* of a double 2-Segal object $\mathcal{X}$ is $\mathcal{X}_{1,1}$ equipped with the lax bialgebra structure of Theorem 4.3.2

Proof. By Lemma 4.3.1 the following diagram commutes for any bisimplicial object $\mathcal{X}$

$$\begin{array}{ccc}
\mathcal{A}\mathrm{lg}^{\amalg} & \overset{\alpha_{\mathcal{X}_{\bullet,1}}}{\rightsquigarrow} & \\
{\scriptstyle \iota_a}\downarrow & & \\
\mathcal{B}\mathrm{ialg}^{\amalg} & \overset{\beta_{\mathcal{X}}}{\rightsquigarrow} & \mathcal{S}\mathrm{pan}_2(\mathcal{C})^{\times} \\
{\scriptstyle \iota_c}\uparrow & & \\
\mathcal{C}\mathrm{oalg}^{\amalg} & \overset{\chi_{\mathcal{X}_{1,\bullet}}}{\rightsquigarrow} &
\end{array}$$

By Theorem 3.3.1, a simplicial object $\mathcal{Y} \in \mathcal{C}_{\mathbb{\Delta}}$ is 2-Segal if and only if $\alpha_{\mathcal{Y}}$ and $\chi_{\mathcal{Y}}$ are symmetric monoidal functors. Therefore $\beta_{\mathcal{X}}$ endows $\mathcal{X}_{1,1}$ with a lax bialgebra structure if and only if $\mathcal{X}_{\bullet,1}$ and $\mathcal{X}_{1,\bullet}$ are 2-objects. □

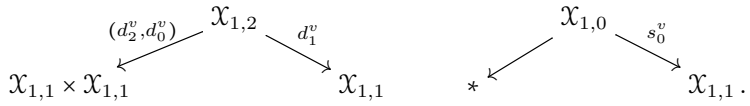
4.4 The universal Hall bialgebra in the homotopy bicategory

Upon passing to homotopy bicategories, the universal Hall bialgebra of a double 2-Segal object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ gives, by Remark 2.2.5, a lax bialgebra object in $\mathrm{Span}_2^{\times}(\mathcal{C})$. By Proposition 2.5.11 this is equivalently a coalgebra object in the symmetric monoidal bicategory $\mathrm{Alg}^{\mathrm{olax}}(\mathrm{Span}_2^{\times}(\mathcal{C}))$. We conclude this section with an explicit description of this lax bialgebra object for a general double 2-Segal object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ and the specialise to the case $\mathcal{C} = \mathcal{G}\mathrm{rpd}$ and $\mathcal{X} = S(A)$, the iterated Waldhausen S-construction of a proto-exact category A.

One can directly apply our analysis in Section 3.4 to describe the algebra and coalgebra structures on $\mathcal{X}_{1,1}$ for a general 2-Segal object $\mathcal{X} \in \mathcal{C}_{\mathbb{\Delta}^2}$ as these structures are, respectively, inherited from the 2-Segal objects $\mathcal{X}_{\bullet,1}$ and $\mathcal{X}_{1,\bullet}$. In particular, the product and unit are given by the spans

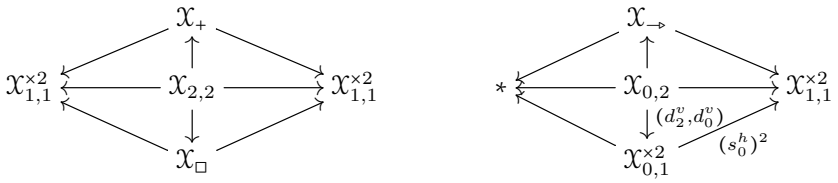
$$\begin{array}{ccccc}
 & & \mathcal{X}_{2,1} & & \\
 & {\scriptstyle (d_2^h,d_0^h)}\swarrow & & \searrow{\scriptstyle d_1^h} & \\
\mathcal{X}_{1,1}\times\mathcal{X}_{1,1} & & & & \mathcal{X}_{1,1}
\end{array}
\qquad\qquad
\begin{array}{ccccc}
 & & \mathcal{X}_{0,1} & & \\
 & \swarrow & & \searrow{\scriptstyle s_0^h} & \\
* & & & & \mathcal{X}_{1,1}\,,
\end{array}
\tag{4.4.1}$$

and the coproduct and counit are given by the spans



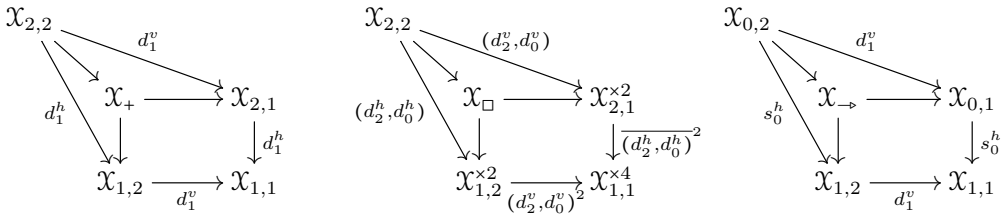
(4.4.2)

It remains only to discuss the 2-morphisms defining the oplax structure on the coproduct and counit as per the diagrams in Eq. 2.5.2 and 2.5.3, respectively. The 2-morphisms defining the oplax structure on the coproduct are the diagrams



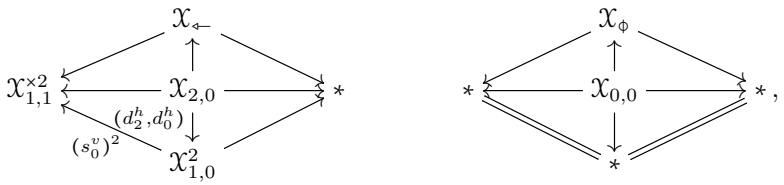
(4.4.3)

where $\mathcal{X}_+$, $\mathcal{X}_\square$ and $\mathcal{X}_{\twoheadrightarrow}$ are the following pullbacks in $\mathcal{C}$



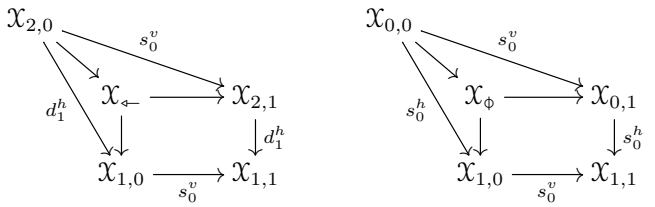
(4.4.4)

where $\overline{(d_2^h,d_0^h)}^2$ is $(d_2^h,d_0^h)^2$ followed by functor interchanging the second and third factor. The 2-morphisms defining the oplax structure on the counit are the diagrams



(4.4.5)

where $\mathcal{X}_{\twoheadleftarrow}$ and $\mathcal{X}_\phi$ are the following pullbacks in $\mathcal{C}$



(4.4.6)

For the special case of $\mathcal{X} = S = S(A)$, the iterated Waldhausen S-construction of a proto-exact category A, recall that $S_{n,1} = S_n = S_{1,n}$. Therefore the (co)product and (co)unit are exactly those given in Eq. 3.4.1. The groupoid $\mathcal{X}_\phi$ is trivial and the groupoids S_+, $S_\square$, $S_{\twoheadrightarrow} \simeq S_{\twoheadleftarrow}$ are, respectively,

the groupoids of diagrams of the forms

$$\begin{array}{ccc} & b & \\ & \downarrow & \\ a' \longrightarrow & b' & \longrightarrow c' \\ & \downarrow & \\ & b'' & \end{array} \qquad \begin{array}{ccc} a & \longrightarrow b \longrightarrow & c \\ \downarrow & & \downarrow \\ a' & & c' \\ \downarrow & & \downarrow \\ a'' & \longrightarrow b'' \longrightarrow & c'' \end{array} \qquad a' \longrightarrow 0 \longrightarrow c' \,,$$

where each row and column is a admissible short exact sequence.

Recalling that $S_{2,2}$ is the groupoid of 3x3 grids of admissible short exact sequences, the functors from $S_{2,2}$ to S_+ and $S_\square$ are apparent from the diagrams. The product and coproduct in the universal Hall bialgebra of a proto-exact category are only laxly compatible precisely because there is, a priori, no reason for the functors $S_{2,2} \to S_+$ and $S_{2,2} \to S_\square$ to be equivalences. Indeed, for an abelian category A the latter functor is an equivalence if and only if A is the trivial category ([16] 2.38).

Chapter 5

The Hall bimonoidal category

Finally, in this last chapter we present our second main contribution: The definition of the Hall bimonoidal category $\mathfrak{H}(\mathcal{X})$ of a locally bifinite double 2-Segal groupoid $\mathcal{X}$. In general, $\mathfrak{H}(\mathcal{X})$ is a lax bimonoidal category. We, nonetheless, identify two interesting families of bifinitary proto-exact categories A for which $\mathfrak{H}(A)$ is a bimonoidal category.

Theorem. *The Hall bimonoidal categories of the categories* $\mathrm{Vect}_{\mathbb{F}_q}$ *and* Γ_0-$\mathrm{Mod}_{\mathbb{F}_1}$ *are categorifications of Zelevinsky's positive self-adjoint Hopf algebras.*

Section 5.1 reviews Morton's categorical linearisation. Section 5.2 introduces locally bifinite double 2-Segal spaces and defines their Hall bimonoidal categories The thesis concludes with the proof of the above Theorem in Section 5.3.

5.1 Categorical linearisation

Recall that an abelian category has finite length if, for each object, every strictly decreasing chain of subobjects has finite length. Throughout the remainder of this thesis, a *linear category* will mean a finite length abelian category compatibly enriched in Vect, the category of finite dimensional complex vector spaces. Linear categories are naturally the objects of the following 2-category.

Definition 5.1.1. The 2-category of linear categories, Lin, has as objects linear categories, 1-morphisms additive, right exact Vect-enriched functors and 2-morphisms Vect-enriched natural transformations.

Remark 5.1.2. By a functor between linear categories we shall always mean an additive, right exact Vect-enriched functor, and by a natural transformation between two such functors we shall always mean a Vect-enriched natural transformation.

The 2-category of linear categories can be endowed with a symmetric monoidal structure using a categorification of the tensor product of vector spaces known as the *Deligne product* [15]. For linear

categories A and B, their Deligne product is a linear category $A \boxtimes B$ which corepresents bilinear functors out of $A \times B$.

Remark 5.1.3. Note that the Deligne product may not exist for arbitrary abelian categories. The assumptions we have placed on our linear categories suffice to ensure its existence ([21] 3).

We will primarily be interested in a class of linear categories arising from *representations of locally finite groupoids*. Recall that a category is said to be *locally finite* if it has finite Hom-sets and that a *finite functor* $F : \mathbb{G} \to \mathbb{H}$ between locally finite groupoids is a functor which is finite-to-one on π_0. The ∞-category $\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}$ is the subcategory of $\mathcal{G}\mathrm{rpd}$ of locally finite groupoids and finite functors.

Every locally finite groupoid is equivalent to one which is specified by two pieces of data: namely, a set I indexing the isomorphism classes of objects and, for each $i \in I$, a finite group G_i of automorphisms of i. We write such a groupoid as

$$\mathbb{G} \simeq \coprod_{i \in I} G_i.$$

For $\mathbb{G} = \coprod_{i \in I} G_i$ and $\mathbb{H} = \coprod_{j \in J} H_j$, a finite functor $F : \mathbb{G} \to \mathbb{H}$ is given by

$$f : I \to J \text{ and } \forall i \in I,\, F_i : G_i \to H_{f(i)},$$

where f is finite-to-one.

Definition 5.1.4. The category of representations of a locally finite groupoid $\mathbb{G} \in \mathrm{Grpd}$, denoted $\mathfrak{R}(\mathbb{G})$, is the category of finitely supported functors $\mathbb{G} \to \mathrm{Vect}$. That is, $\mathfrak{R}(\mathbb{G})$ is the full subcategory of $\mathrm{Fun}(\mathbb{G}, \mathrm{Vect})$ consisting of those functors which send all but finitely many isomorphism classes of objects in $\mathbb{G}$ to the 0 vector space.

The category of representations of a locally finite groupoid $\mathbb{G} \simeq \coprod_{i \in I} G_i$ is

$$\mathfrak{R}(\mathbb{G}) \simeq \bigoplus_{i \in I} \mathrm{Rep}(G_i),$$

and so is semisimple with simple objects the irreducible representations of the finite groups G_i. One can readily verify ([21] 19) that

$$\mathfrak{R}(\mathbb{G}) \boxtimes \mathfrak{R}(\mathbb{G}') \simeq \mathfrak{R}(\mathbb{G} \times \mathbb{G}').$$

Every finite functor $F : \mathbb{G} \to \mathbb{H}$ in Grpd defines, by restriction, a functor $F^* : \mathfrak{R}(\mathbb{H}) \to \mathfrak{R}(\mathbb{G})$ which admits of an *ambidextrous adjoint* ([53] 2). That is, there is a functor $F_* : \mathfrak{R}(\mathbb{G}) \to \mathfrak{R}(\mathbb{H})$ which is both left and right adjoint to F^*. For each $\rho \in \mathfrak{R}(\mathbb{G})$ and $j \in J$, the functor F_* is given by

$$F_*(\rho)(j) = \bigoplus_{i \,:\, f(i) = j} \mathbb{C}[H_j] \times_{\mathbb{C}[G_i]} \rho(j).$$

The units and counits which witness F_* being both left and right adjoint to F^* can be readily computed. In particular, the unit for the right adjunction and the counit for the left adjunction,

$$\eta_F^R : \mathrm{Id}_{\mathfrak{R}(\mathbb{H})} \Rightarrow F_* F^* \text{ and } \epsilon_F^L : F_* F^* \Rightarrow \mathrm{Id}_{\mathfrak{R}(\mathbb{H})},$$

are described as follows. For each $\rho \in \mathfrak{R}(\mathbb{H})$ and object j of $\mathbb{H}$ these natural transformations have components

$$\begin{aligned} \eta_F^R(\rho)(j) : \rho(j) &\to \bigoplus_{i\,:\,f(i)=j} \mathbb{C}[H_j] \otimes_{\mathbb{C}[G_i]} \rho(j) \\ v &\mapsto \bigoplus_{i\,:\,f(i)=j} \frac{1}{|G_i|} \sum_{h \in H_j} h \otimes h^{-1} \cdot v \end{aligned} \tag{5.1.1}$$

and

$$\begin{aligned} \epsilon_F^L(\rho)(j) : \bigoplus_{i\,:\,f(i)=j} \mathbb{C}[H_j] \otimes_{\mathbb{C}[G_i]} \rho(j) &\to \rho(j) \\ \bigoplus_{i\,:\,f(i)=j} h_j \otimes v &\mapsto \sum_{i\,:\,f(i)=j} h_j \cdot v. \end{aligned} \tag{5.1.2}$$

Remark 5.1.5. The finiteness assumptions in $\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}$ are essential for these constructions to go through as stated. To ensure the existence of the Deligne tensor product we must consider only finite dimensional representations. We restrict to finite functors so that the left and right adjoints to the restriction F^* are again finite dimensional representations. Finally, ambidexterity of these adjoints follows from the statement that, for finite dimensional representations of finite groups, induction is an ambidextrous adjoint to restriction.

Putting all of this together, one can define Morton's categorical linearisation ([53] 5, [52] 4)

$$\mathfrak{R} : \mathrm{Span}_2^{\times}(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}) \to \mathrm{Lin}^{\boxtimes}. \tag{5.1.3}$$

On 1-morphisms the linearisation is given by

$$\begin{array}{ccccc} & & \mathbb{H} & & \\ & {\scriptstyle F}\swarrow & & \searrow{\scriptstyle G} & \\ \mathbb{G} & & & & \mathbb{G}' \end{array} \qquad \mapsto \qquad \mathfrak{R}(\mathbb{G}) \xrightarrow{G_* F^*} \mathfrak{R}(\mathbb{G}')\,,$$

while on 2-morphisms it is given by

$$\begin{array}{ccccc} & & \mathbb{H} & & \\ & {\scriptstyle F}\swarrow & {\scriptstyle K}\uparrow & \searrow{\scriptstyle F'} & \\ \mathbb{G} & & \Downarrow & & \mathbb{G}' \\ & {\scriptstyle G}\nwarrow & {\scriptstyle K'}\downarrow & \nearrow{\scriptstyle G'} & \\ & & \mathbb{H}' & & \end{array} \qquad \mapsto \qquad \begin{array}{ccc} & {\scriptstyle F'_* F^*} & \\ \mathfrak{R}(\mathbb{G}) & {\scriptstyle \epsilon^L_{K'} \eta^R_K}\Downarrow & \mathfrak{R}(\mathbb{G}')\,. \\ & {\scriptstyle G'_* G^*} & \end{array}$$

We would like to apply this categorical linearisation to the universal Hall bialgebra of a proto-exact category, or more generally, a double 2-Segal groupoid. However, this will only be a lax bialgebra in $\mathcal{S}\mathrm{pan}_2(\mathcal{G}\mathrm{rpd})^{\times}$, not $\mathcal{S}\mathrm{pan}_2(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}})^{\times}$. We now turn to the study of which properties must be placed on a double 2-Segal groupoid such that its universal Hall bialgebra is in $\mathcal{S}\mathrm{pan}_2(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}})^{\times}$.

5.2 The Hall bimonoidal category of a locally bifinite 2-Segal groupoid

Naively, the universal Hall bimonoidal category of a double 2-Segal groupoid $\mathcal{X} \in \mathcal{G}\mathrm{rpd}_{\Delta^2}$ will be defined to be the image of its universal Hall bialgebra of Definition 4.3.3 under the categorical linearisation $\mathfrak{R}$ of Eq. 5.1.3. This is a naive definition as the functor $\mathfrak{R}$ has domain $\mathrm{Span}_2^{\times}(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}})$ rather than $\mathrm{Span}_2^{\times}(\mathcal{G}\mathrm{rpd})$ wherein the universal Hall bialgebra of $\mathcal{X}$ lives. One might therefore be tempted to consider only double 2-Segal objects in $\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}$. Unfortunately, enforcing that the iterated Waldhausen S-construction $S(A)$ of a proto-exact category A lie within $\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}$ places overly strong finiteness conditions upon A: The fibre of $d_2^h : S_{2,1}(A) \to S_{1,1}(A)$ over the zero object is equivalent to the groupoid of objects in A. Therefore d_2^h being finite implies, in particular, that A has only finitely many objects.

Fortunately, to apply the categorical linearisation to the universal Hall bialgebra of $\mathcal{X}$ one needs only ensure that the data defining the universal Hall bialgebra, as detailed in Section 4.4, lies within $\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}$. We shall now describe the correct conditions to enforce on $\mathcal{X}$ so that this occurs.

Definition 5.2.1. A simplicial groupoid $\mathcal{X} \in \mathcal{G}\mathrm{rpd}_{\Delta^2}$ is said to be:

- *locally finite* if the groupoid $\mathcal{X}_1$ is locally finite and the functors $s_0 : \mathcal{X}_0 \to \mathcal{X}_1$ and $d_1 : \mathcal{X}_2 \to \mathcal{X}_1$ are finite ([24] 7.4); or,
- *locally cofinite* if the groupoid $\mathcal{X}_0$ is finite and the functor $(d_2, d_0) : \mathcal{X}_2 \to \mathcal{X}_1^{\times 2}$ is finite.

A bisimplicial groupoid $\mathcal{X} \in \mathcal{G}\mathrm{rpd}_{\Delta}$ is said to be *locally bifinite* if the simplicial groupoids $\mathcal{X}_{k,\bullet}$ and $\mathcal{X}_{\bullet,k}$ for $k = 1, 2$ are both locally finite and locally cofinite.

Remark 5.2.2. When $\mathcal{X}$ is a 2-Segal groupoid, the condition of being locally finite is equivalent to requiring that each groupoid $\mathcal{X}_n$ is locally finite and that the functors $\mathcal{X}_m \to \mathcal{X}_n$ induced by active maps $[n] \twoheadrightarrow [m]$ are finite ([24] 7.5).

Lemma 5.2.3. *Let $\mathcal{X} \in \mathcal{G}\mathrm{rpd}_{\Delta^2}$ be a double 2-Segal groupoid. Then $\mathcal{X}$ is locally bifinite if and only if the universal Hall bialgebra of $\mathcal{X}$ is a lax bialgebra in $\mathrm{Span}_2^{\times}(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}})$, that is, one has the factorisation*

$$\begin{array}{ccc} \mathrm{Bialg}^{\sqcup} & \rightsquigarrow & \mathrm{Span}_2^{\times}(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}) \\ & \overset{\beta_{\mathcal{X}}}{\searrow} & \downarrow \\ & & \mathrm{Span}_2^{\times}(\mathcal{G}\mathrm{rpd}) \,. \end{array}$$

Proof. The assumption that $\mathcal{X}$ is locally bifinite implies that, in Eqs. 4.4.1, 4.4.2, 4.4.3, 4.4.4, 4.4.5 and 4.4.6, all groupoids are locally finite and all functors are finite. One need only use Remark 5.2.2, the fact that any groupoid with a finite functor to a locally finite groupoid is locally finite, and that for composable functors F and G between locally finite groupoids, if G and $G \circ F$ are finite than so is F. The converse statement is trivial. □

Definition 5.2.4. The *Hall bimonoidal category* of a locally bifinite double 2-Segal groupoid $\mathcal{X}$, denoted $\mathfrak{H}(\mathcal{X})$, is the lax bimonoidal category obtained as the composite

$$\mathrm{Bialg}^{\sqcup} \xrightarrow{\beta_{\mathcal{X}}} \mathrm{Span}_2^{\times}(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}) \xrightarrow{\mathfrak{R}} \mathrm{Lin}\ .$$

Having identified that being locally bifinite is a necessary and sufficient condition to apply the categorical linearisation we shall now determine what conditions must be placed on a proto-exact category A so that $S_{\bullet,\bullet}(A)$ is locally bifinite. To do so we must first introduce some terminology.

First, given a pair of objects $a, c \in A$, the groupoid $\mathrm{Ext}(c,a)$ is the groupoid of pushout diagrams of the form

$$\begin{array}{ccc} a & \xrightarrow{f} & b \\ \downarrow & & \downarrow{\scriptstyle g} \\ 0 & \longrightarrow & c \end{array}$$

with $f \in \mathfrak{M}$ and $g \in \mathfrak{E}$. Second, given an object $a \in A$, the groupoid of *admissible subobjects* of a, denoted $\mathrm{Sub}(a)$, is the groupoid underlying $\mathfrak{M}_{/a}$.

Remark 5.2.5. When A is abelian, $\pi_0(\mathrm{Ext}(c,a))$ is the set of isomorphism classes of extensions of c by a and $\mathrm{Sub}(a)$ is the poset of subobjects of a.

Remark 5.2.6. For A a locally finite proto-exact category $\mathrm{Ext}(c,a)$ and $\mathrm{Sub}(a)$ are finite groupoids if and only if they have finitely isomorphism classes of objects.

Definition 5.2.7. A proto-exact category A is said to be:

- *finitary* if it is locally finite and $\mathrm{Ext}(c,a)$ is a finite groupoid for each pair of objects $a, c \in A$; or,
- *cofinitary* if $\mathrm{Sub}(a)$ is a finite groupoid for each object $a \in A$; or,
- *bifinitary* if it is both finitary and cofinitary.

Remark 5.2.8. Dyckerhoff–Kapranov have also used the terms finitary and cofinitary to describe finiteness conditions on Quillen exact categories ([17] 8.2.9). While the notions introduced above do not exactly match theirs, a Quillen exact category which is both finitary and cofinitary in the Dyckerhoff–Kapranov sense is bifinitary.

Lemma 5.2.9. *A proto-exact category A is bifinitary if and only if the iterated S-construction $S_{\bullet,\bullet}(A)$ is locally bifinite.*

Proof. To begin, observe for any pair of objects $a, c \in A$ one has the following pullback diagrams in $\mathcal{G}\mathrm{rpd}$

$$\begin{array}{ccc} \mathrm{Ext}(c,a) & \longrightarrow & S_1 \\ \downarrow & & \downarrow{\scriptstyle (d_2,d_0)} \\ * & \xrightarrow[(c,a)]{} & S_1^{\times 2} \end{array} \qquad \begin{array}{ccc} \mathrm{Sub}(a) & \longrightarrow & S_2 \\ \downarrow & & \downarrow{\scriptstyle d_1} \\ * & \xrightarrow[a]{} & S_1 \end{array}$$

Since $S_0 \simeq *$ the proto-exact category A is bifinitary if and only if $S_\bullet$ is both locally finitary and locally cofinitary. It remains therefore to show that A bifinitary implies that $S_{2,\bullet} \simeq S_{\bullet,2}$ is both locally finitary and locally cofinitary.

We must show that the functors $d_1 : S_{2,2} \to S_{2,1}$ and $(d_2, d_0) : S_{2,2} \to S_{2,1}^{\times 2}$ are finite whenever A is bifinitary. These functors are given by

$$\begin{array}{ccc} a \\ \downarrow \\ a' \\ \downarrow \\ a'' \end{array} \qquad \begin{array}{c} c \\ \downarrow \\ c' \\ \downarrow \\ c'' \end{array} \quad \xleftarrow{(d_2,d_0)} \quad \begin{array}{ccc} a & \longrightarrow & b & \longrightarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ a' & \longrightarrow & b' & \longrightarrow & c' \\ \downarrow & & \downarrow & & \downarrow \\ a'' & \longrightarrow & b'' & \longrightarrow & c'' \end{array} \quad \xmapsto{d_1} \quad \begin{array}{c} b \\ \downarrow \\ b' \\ \downarrow \\ b'' \end{array}$$

The finiteness of $\mathrm{Ext}(c, a)$ implies the finiteness of (d_2, d_0) while the finiteness of $\mathrm{Sub}(b)$ implies that of d_1. □

Remark 5.2.10. Another source of locally bifinite double 2-Segal spaces are the monoidal decomposition spaces coming from combinatorics [26] and restriction species [28]. The Hall bimonoidal categories of these monoidal decomposition spaces will be the subject of future work.

We can therefore define, as a special case of Definition 5.2.4, the Hall bimonoidal category of a bifinitary proto-exact category A, denoted $\mathfrak{H}(A)$, to be the composite

$$\mathrm{Bialg}^{\amalg} \xrightsquigarrow{\beta_A} \mathrm{Span}_2^{\times}(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}) \xrightarrow{\mathfrak{R}} \mathrm{Lin} \,.$$

In the next section we will analyse the Hall bimonoidal category of a bifinitary proto-exact category A.

5.3 Categorification of positive, self-adjoint Hopf algebras

It is worth emphasising that, a priori, the Hall bimonoidal category of a bifinitary proto-exact category is only a *lax* bimonoidal category. Based on our analysis in Section 4.4 this stems from the failure of the functors $S_{2,2} \to S_+$ and $\mathfrak{b} : S_{2,2} \to S_\square$ to be equivalences. Therefore, there are two possible avenues towards examples of bifinitary proto-exact categories A so that $\mathfrak{H}(A)$ is a bimonoidal category. The first is the obvious route of seeking out categories wherein these functors are equivalences. The second is to find categories where the categorical linearisation $\mathfrak{R}$ inverts the first 2-morphism in Eq. 4.4.3.

For A an abelian category one has that $S_{2,2} \simeq S_+$ as any cross of short exact sequences can be completed to a 3x3 grid of short exact sequences in an essentially unique way using pullbacks, pushouts, and (co)kernels. The functor $\mathfrak{b}$ is significantly more subtle. From an object $f \in S_\square$ given by

$$\begin{array}{ccccc} a & \longrightarrow & b & \longrightarrow & c \\ \downarrow & & \downarrow & & \downarrow \\ a' & \longrightarrow & b' & \longrightarrow & c' \\ \downarrow & & \downarrow & & \downarrow \\ a'' & \longrightarrow & b'' & \longrightarrow & c'' \end{array}$$

one has a Yoneda 2-extension $\xi_f \in \mathrm{Ext}^2(c'', a)$

$$\xi_f : \ 0 \longrightarrow a \longrightarrow a' \coprod_a b \longrightarrow b'' \times_{c''} c' \longrightarrow c'' \longrightarrow 0 \ .$$

Lemma 5.3.1 ([16] 2.38)**.** *For A an abelian category having enough projectives, the homotopy fibre of $\mathfrak{b}$ over f is empty unless $\xi_f = 0$, in which case it is equivalent to*

$$\coprod_{\mathrm{Ext}^1(c'',a)} \mathrm{Hom}(c'', a).$$

In particular, $\mathfrak{b}$ is an equivalence if and only if A is the trivial category.

Therefore our first route spectacularly fails in the abelian setting. As for the second route, one has the following.

Lemma 5.3.2. *Let A be a bifinitary abelian category having enough projectives. Then the counit $\epsilon^L_{\mathfrak{b}} : \mathfrak{b}_*\mathfrak{b}^* \Rightarrow \mathrm{Id}_{\mathfrak{R}(S_\square)}$ is a natural isomorphism if and only if A is semisimple.*

Proof. We begin by recalling two facts. First, for a group homomorphism $G \to H$, thought of as a functor between groupoids having only one object, the homotopy fibre over the unique object is the action groupoid for the induced action of G on H. In particular, the homotopy fibre has objects parametrised by the orbits of the G-action and automorphisms given by the kernel of the homomorphism. Second, for $G \to H$ a homomorphism of finite groups and V a finite dimensional H-representation,

$$\dim\left(\mathbb{C}[H] \otimes_{\mathbb{C}[G]} V\right) = \dim V \cdot [H : G].$$

Then by Proposition 5.3.1, for $\rho \in \mathfrak{R}(S_\square)$ and $f \in S_\square$,

$$\dim\left(\mathfrak{b}_*\mathfrak{b}^*\rho(f)\right) = \dim(\rho(f)) \sum_{\hat{f} \in \mathrm{Ext}^1(c'',a)} [\mathrm{Aut}(f) : \mathrm{Aut}(\hat{f})]. \tag{5.3.1}$$

Assume that A is semisimple. Then by Lemma 5.3.1 the homotopy fibre of $\mathfrak{b}$ over an $f \in S_\square$ is the one object groupoid having automorphisms $\mathrm{Hom}(c'', a)$. Since the homotopy fibre of $\mathfrak{b}$ over f has only one object $\hat{f} \in S_{2,2}$ the group homomorphism $\mathfrak{b}_{\hat{f}} : \mathrm{Aut}(\hat{f}) \to \mathrm{Aut}(f)$ is surjective. It follows that for any $\rho \in \mathfrak{R}(S_\square)$, the component of the counit

$$\epsilon^L_{\mathfrak{b}}(\rho)(f) : \mathbb{C}[\mathrm{Aut}(f)] \otimes_{\mathbb{C}[\mathrm{Aut}(\hat{f})]} \rho(f) \to \rho(f)$$

is surjective, and both the source and target are vector spaces of the same dimension. We therefore conclude that the counit is a natural isomorphism.

Next, assume that the counit $\epsilon^L_{\mathfrak{b}}$ is a natural isomorphism. In particular one must have that $\dim\left(\mathfrak{b}_*\mathfrak{b}^*\rho(f)\right) = \dim(\rho(f))$. So long as f is not the 0 diagram, for any $\hat{f} \in S_{2,2}$ in the fibre over f the homomorphism $\mathrm{Aut}(\hat{f}) \to \mathrm{Aut}(f)$ is non-zero and so $[\mathrm{Aut}(f) : \mathrm{Aut}(\hat{f})] \geq 1$. Therefore by the dimension formula in Eq. 5.3.1 the abelian category A is semisimple. □

As an immediate corollary of Lemma 5.3.2 and the explicit description of the universal Hall bialgebra of a proto-exact category presented in Sections 3.4 and 4.4 we have the following.

Theorem 5.3.3. *The Hall bimonoidal category of a bifinitary abelian category A having enough projectives is a bimonoidal category if and only if A is semisimple.*

Semisimple abelian categories have enough projectives as every epimorphism admits a section. Letting $i \in I$ index the simple objects X_i of A, for a semisimple A one has that

$$A \simeq \bigoplus_{i \in I} \mathrm{End}(X_i) - \mathrm{Mod},$$

where, by Schur's Lemma, $\mathrm{End}(X_i)$ is a division algebra. When A is also bifinitary, each $\mathrm{End}(X_i)$ is a finite division algebra and hence a finite field. Therefore the ur-examples of semisimple bifinitary abelian categories are the categories of finite dimensional vector spaces over finite fields, $\mathrm{Vect}_{\mathbb{F}_q}$. Their Hall bimonoidal categories are

$$\mathfrak{H}\left(\mathrm{Vect}_{\mathbb{F}_q}\right) = \bigoplus_{n \geq 0} \mathrm{Rep}\left(\mathrm{GL}_n(\mathbb{F}_q)\right),$$

with product and coproduct given by parabolic induction and restriction. These categories, along with their products given by parabolic induction, were introduced by Joyal–Street [41]. We have therefore extended the monoidal structure of Joyal–Street to a bimonoidal structure.

Moving away from the abelian context we lose the tools from homological algebra that were used to characterise those bifinitary abelian categories A having $\mathfrak{H}(A)$ bimonoidal. We do, however, know of a family of non-abelian, bifinitary proto-exact categories satisfying this condition. Namely, the categories Γ_0-$\mathrm{Mod}_{\mathbb{F}_1}$, introduced in Ex. 3.1.7. Not only is every short exact sequence of the form $\underline{n} \to \underline{n+m} \to \underline{m}$, but they satisfy the even stronger condition that the automorphism group of such a sequence is isomorphic to $\mathrm{Aut}(\underline{n}) \times \mathrm{Aut}(\underline{m})$. This should be compared to the category $\mathrm{Vect}_{\mathbb{F}_q}$ wherein the automorphism group of such a short exact sequence is the parabolic subgroup of block upper triangular matrices.

We leverage this special property of the automorphisms of short exact sequences in Γ_0-$\mathrm{Mod}_{\mathbb{F}_1}$ to show that its universal Hall bialgebra is a lax bialgebra.

Theorem 5.3.4. *For each finite group Γ the universal Hall bialgebra of Γ_0-$\mathrm{Mod}_{\mathbb{F}_1}$ is a bialgebra in $\mathcal{S}\mathrm{pan}_2\left(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}\right)^{\times}$, and hence $\mathfrak{H}(\Gamma_0$-$\mathrm{Mod}_{\mathbb{F}_1})$ is a bimonoidal category.*

Proof. We need to verify that the first 2-morphism in Eq. 4.4.3 is an equivalence in $\mathrm{Span}_2^{\times}\left(\mathcal{G}\mathrm{rpd}_{\mathrm{fin}}\right)$. For the same reasons as for an abelian category the $S_{2,2} \to S_+$ is an equivalence. Every object in $S_{2,2}$ and $S_{\square}$ is, respectively, equivalent to one of the form

$$\begin{array}{ccccc} \underline{n_1} & \longrightarrow & \underline{n_1+n_2} & \longrightarrow & \underline{n_2} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{n_1+n_3} & \longrightarrow & \underline{n_1+n_2+n_3+n_4} & \longrightarrow & \underline{n_2+n_4} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{n_3} & \longrightarrow & \underline{n_3+n_4} & \longrightarrow & \underline{n_4} \end{array} \qquad \begin{array}{ccccc} \underline{n_1} & \longrightarrow & \underline{n_1+n_2} & \longrightarrow & \underline{n_2} \\ \downarrow & & & & \downarrow \\ \underline{n_1+n_3} & & & & \underline{n_2+n_4} \\ \downarrow & & & & \downarrow \\ \underline{n_3} & \longrightarrow & \underline{n_3+n_4} & \longrightarrow & \underline{n_4} \end{array}$$

Furthermore, the automorphism groups of both objects are

$$(\Gamma \wr \Sigma_{n_1}) \times (\Gamma \wr \Sigma_{n_2}) \times (\Gamma \wr \Sigma_{n_3}) \times (\Gamma \wr \Sigma_{n_4}),$$

hence the functor $\mathfrak{b} : S_{2,2} \to S_{\square}$ is an equivalence. □

Remark 5.3.5. The product on the universal Hall bialgebra of Γ_0-$\mathrm{Mod}_{\mathbb{F}_1}$ is given by disjoint union. As such, the double 2-Segal space $S_{\bullet,\bullet}(\Gamma_0\text{-}\mathrm{Mod}_{\mathbb{F}_1})$ is an example of a monoidal decomposition space ([23] 9).

These Hall bimonoidal categories are

$$\mathfrak{H}(\Gamma_0\text{-}\mathrm{Mod}_{\mathbb{F}_1}) = \bigoplus_{n\geq 0} \mathrm{Rep}(\Gamma \wr \Sigma_n),$$

with product and coproduct given by induction and restriction. By a categorification of Schur-Weyl duality for wreath products due to Walde ([66] 6.4.1), the bimonoidal categories $\mathfrak{H}(\Gamma\text{-}\mathrm{Mod}_{\mathbb{F}_1})$ are equivalent to the bimonoidal categories of Γ-*equivariant polynomial functors* introduced by Gal–Gal ([22] 7.1).

Recall that the *Grothendieck group* of a semisimple linear category L, denoted $K_0(L)$, is the free abelian group on the isomorphism classes of simple objects ([20] 1.5.8). Since the Deligne product $L \boxtimes L'$ of two semisimple linear categories is semisimple with simple objects the products of the simple objects in L and L' ([21] 19), one has that $K_0(L \boxtimes L') \simeq K_0(L) \otimes_{\mathbb{Z}} K_0(L')$. Finally, every functor $L \to L'$ induces a group homomorphism $K_0(L) \to K_0(L')$. We can therefore conclude that the Grothendick group of a semisimple bimonoidal category with simple unit object is a bialgebra over $\mathbb{Z}$. Furthermore, this bialgebra is canonically equipped with a $\mathbb{Z}$-basis given by the isomorphism classes of simple objects.

The bimonoidal categories $\mathfrak{H}(\mathrm{Vect}_{\mathbb{F}_q})$ and $\mathfrak{H}(\Gamma_0\text{-}\mathrm{Mod}_{\mathbb{F}_1})$ are semisimple with simple objects the irreducible representations. The unit objects of these bimonoidal categories are, respectively, the trivial representations of $\mathrm{GL}_0(\mathbb{F}_q)$ and $\Gamma \wr \Sigma_0$ and so are simple. As such, their Grothendieck groups,

$$H(q) \coloneqq K_0\left(\mathfrak{H}(\mathrm{Vect}_{\mathbb{F}_q})\right), \text{ and } \mathrm{H}(\Gamma) \coloneqq \mathrm{K}_0\left(\mathfrak{H}(\Gamma_0\text{-}\mathrm{Mod}_{\mathbb{F}_1})\right), \tag{5.3.2}$$

are bialgebras over $\mathbb{Z}$ with basis given by the irreducible representations. Furthermore, since the monoidal and comonoidal structures on $\mathfrak{H}(\mathrm{Vect}_{\mathbb{F}_q})$ and $\mathfrak{H}(\Gamma_0\text{-}\mathrm{Mod}_{\mathbb{F}_1})$ are adjoint functors by construction, the structure constants for the product and coproduct on the bialgebras $H(q)$ and $H(\Gamma)$ with respect to the basis of irreducible representations coincide. Furthermore, both bialgebras are graded by dimension and the degree 0 components are isomorphic to $\mathbb{Z}$.

Summarising, the Grothendieck groups $H(q)$ and $H(\Gamma)$ are connected, graded Hopf algebras equipped with a basis having the property that the structure constants for the product and coproduct are all non-negative integers and coincide. These are known as *positive, self-adjoint Hopf (PSH) algebras*, and were introduced by Zelevinsky [69] to study the representation theory of the groups

$\mathrm{GL}_n(\mathbb{F}_q)$ and $\Gamma \wr \Sigma_n$ using Hopf algebraic methods. We have therefore, as an application of the universal Hall bialgebra of a proto-exact category, constructed categorifications of Zelevinsky's PSH algebras.

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