

High-Field Fractional Quantum Hall Effect in Optical Lattices

R. N. Palmer¹ and D. Jaksch¹

¹*Clarendon Laboratory, University of Oxford, Parks Road, Oxford OX1 3PU, United Kingdom*
(Received 26 December 2005; published 12 May 2006)

We consider interacting bosonic atoms in an optical lattice subject to a large simulated magnetic field. We develop a model similar to a bilayer fractional quantum Hall system valid near simple rational numbers of magnetic flux quanta per lattice cell. Then we calculate its ground state, magnetic lengths, fractional fillings, and find unexpected sign changes in the Hall current. Finally we study methods for detecting these novel features via shot noise and Hall current measurements.

DOI: 10.1103/PhysRevLett.96.180407

PACS numbers: 03.75.Lm, 32.80.Pj, 73.43.-f

The achievement of strongly correlated atomic systems [1] which are accurately described by simple condensed matter physics (CMP) models [2] has opened many new perspectives for research on strongly correlated systems. Such atomic ensembles share many properties with conventional CMP systems. Several of their features, however, distinguish them from CMP setups like, e.g., their extremely long coherence times and the ability to adjust Hamiltonian parameters over a wide range and in times short compared to the coherence time. These properties enable experiments of controlled coherent quantum dynamics [1,3], and the exploration of otherwise inaccessible parameter regimes. Because of their highly nonlinear behavior these studies may lead to striking new features.

In this Letter we consider such a novel situation arising if interacting ultracold bosonic atoms in a two dimensional optical lattice are subjected to a simulated huge magnetic field. A variety of proposals for realizing magnetic fields in optical lattices exist. One can, e.g., rotate the lattice as suggested in Refs. [4,5], use laser-induced hopping as proposed in Ref. [6], or employ an oscillating quadrupole potential as shown in Ref. [7]. All of these methods allow the creation of arbitrarily large fields with flux quanta per lattice cell α between $0 \leq \alpha \leq 1$ and can be realized with current or near future experimental technology. Optical lattice setups are virtually free of imperfections and can be cooled to almost zero temperature [1]. Furthermore, their large degree of flexibility [8] allows for the introduction of well controlled amounts of disorder [9]. This is in contrast to most CMP systems where only values $\alpha \ll 1$ are believed to be accessible and disorder is difficult to determine and control. Also, conventional CMP systems have fermions with a screened $1/r$ Coulomb interaction while we consider bosonic atoms interacting via a contact potential. Finally, dynamically adjustable smooth trapping potentials are used to confine atoms in an optical lattice to a certain region in space [8] while CMP systems are often bounded by sharp edges.

For a single particle in a lattice placed into a huge magnetic field with $\alpha \sim 1$ a fractal energy band structure was predicted by Hofstadter [10]. However, this regime was not thought to be experimentally accessible until

recently and is thus not explored in detail. Many studies on interaction effects in magnetic fields centered around the fractional quantum Hall (FQH) effect for $\alpha \ll 1$ [11] where it is essentially unaffected by the presence of a lattice. As $\alpha \sim 1$ the discrete periodic structure of the lattice becomes important and we find novel effects arising from the presence of the lattice. We show that the system is described by a bilayer FQH-like model near simple rational values $\alpha_c \equiv l/n$ where n is a small integer. The ground state of this system is an FQH state with a mean filling factor $\tilde{\nu} = n/(n+1)$ and a modified magnetic length. We will find striking features of the associated Hall current which changes its sign at $\alpha = \alpha_c$.

We consider ultracold bosonic atoms in the lowest motional band of a 3D optical lattice with lattice period d . The motion of the atoms along the z direction is suppressed by high optical potential barriers [8]. The system is thus decoupled into 2D planes and we assume that a large effective magnetic field along the z axis is simulated by one of the methods introduced in Refs. [4,6,7]. The Hamiltonian in one of the planes is given by

$$H = -J \sum_{p,q} (e^{2\pi i \alpha q} a_{p,q}^\dagger a_{p-1,q} + a_{p,q}^\dagger a_{p,q-1} + \text{h.c.}) \\ + V(p, q) a_{p,q}^\dagger a_{p,q} + \frac{U}{2} a_{p,q}^\dagger a_{p,q}^\dagger a_{p,q} a_{p,q}. \quad (1)$$

Here $a_{p,q}^\dagger$ ($a_{p,q}$) are bosonic creation (annihilation) operators, fulfilling standard bosonic commutation relations, for particles at a lattice site labeled by its position in the xy plane (p, q) . The on site interaction strength is U and J is the hopping amplitude. The parameter α is a measure for the strength of the simulated magnetic field [6]. The phase shift from hopping around one lattice cell is $2\pi\alpha$, and since α is only defined mod 1 we restrict $-1/2 \leq \alpha < 1/2$. Finally, $V(p, q)$ is a trapping potential varying slowly on the length scale d . We will mostly deal with linear geometries $V(p, q) \equiv V(q)$, where the x dependence of the atomic wave functions are plane waves, and thus work in Landau gauge. All parameters in the Hamiltonian can dynamically be varied via lasers [8].

We first discuss the limit $|\alpha| \ll 1$, in which the length scale of the wave functions is much larger than d and we

can hence use a continuum approximation. We approximate a one particle state $|\psi\rangle = \sum_{p,q} \psi(p, q) a_{p,q}^\dagger |\text{vac}\rangle$ with $|\text{vac}\rangle$ the vacuum state by a continuous wave function $\phi(p, q) = \psi(p, q)/d^2$. The Hamiltonian acting on ϕ is

$$H_0 = -\frac{1}{2m} \left[\frac{\partial^2}{\partial y^2} + \left(2m\Omega y - i \frac{\partial}{\partial x} \right)^2 \right] + V(x, y) - \frac{2}{md^2}, \quad (2)$$

where we have defined the effective mass $m = 1/2Jd^2$ and the cyclotron frequency $\Omega = \pi\alpha/md^2$. The last term has no dynamical effect and is left out in the following. For interacting particles at density ρ , and interparticle spacing $\rho^{-1/2} \gg d$, the continuum approximation is

$$H \approx \sum_i H_0(x_i, y_i) + \frac{u}{2} \sum_{i \neq j} \delta(x_i - x_j) \delta(y_i - y_j), \quad (3)$$

where (x_i, y_i) are the coordinates of particle i and $u = Ud^2$. By analogy with the solid state FQH we define the filling factor $\nu = \rho\pi/m\Omega$. The 2D Hamiltonian equation (3) has been studied, e.g., in the context of rotating ultracold atomic gases for the case of a weak trap [5,7,12,13] and it is well established that for an isotropic trap $V(x, y) = m\omega^2(x^2 + y^2)/2$ the $\nu = 1/2$ Laughlin state with a magnetic length ℓ given by $\ell_c = (m^2\Omega^2 + m^2\omega^2)^{-1/4}$ is an exact eigenstate. Its energy is $N(\Omega^2 + \omega^2)^{1/2}/2 + N(N-1)\Delta E$, where $\Delta E \equiv (\Omega^2 + \omega^2)^{1/2} - \Omega \approx \omega^2/2\Omega$ and N is the number of particles. It is the ground state of the system for $\omega \ll \Omega$, $u\Omega$. Since this state already has exactly zero interaction energy, the rest of the bosonic Laughlin sequence ($\nu = 1/4, 1/6, \dots$) does not occur. It is also possible to construct noncircular Laughlin-like states for other trap geometries. In general, these are only exact eigenstates for zero trap strength, but as they are always lowest Landau level (LLL) and zero interaction energy, they are reasonable trial states for weak traps. For instance, in linear geometry with a trap $V(y) = m\omega^2 y^2/2$ the variational energy is minimized for a Laughlin-like state with magnetic length $\ell_l = (m^2\Omega^2 + m^2\omega^2/4)^{-1/4}$.

These Laughlin states cannot remain the ground states if N gets too large as for $N \geq u/4\pi\ell^2\Delta E$ the trap potential energy $2N\Delta E$ to add a particle to the edge of the Laughlin state exceeds the interaction energy $\sim u/2\pi\ell^2$ required to add it in the center of the trap. For larger N , denser states are more favorable, and it is believed [14] that a series of incompressible states with half-integer filling $\nu = 1, 3/2, 2, \dots$, well approximated by symmetrized products of Laughlin states called Read-Rezayi states [15], forms. For $\nu \geq 6$ a transition to a compressible vortex lattice is predicted [14].

We now turn to the case of larger values of α which can be reached in optical lattice setups [6]. We restrict our considerations to values of α close to simple rationals $\alpha_c \equiv l/n$, where l, n are small integers. For a single particle we find from numerical calculation shown in Fig. 1 that when

$\alpha \approx \alpha_c$ the single particle ground state wave function has an approximate n -site periodicity, suggesting the representation $\psi(np + i, nq + j) = d^2 \chi_k(npd, nqd) \mathbf{B}_{ij}^{(k)}$ where χ_k is a continuous function and $\mathbf{B}^{(k)}$ an $n \times n$ matrix. We find by expansion about α_c that there are n linearly independent matrices $\mathbf{B}^{(k)}$ with degenerate energies, of the form $\mathbf{B}_{pq}^{(k)} = e^{2\pi i p k l / n} \mathbf{v}_{q-k}$ where \mathbf{v} is a fixed n component vector for each l, n ; e.g., for $\alpha_c = 1/2$ we have $\mathbf{v} = (\sqrt{1 - 1/\sqrt{2}}, \sqrt{1 + 1/\sqrt{2}})$ for the minimum energy eigenvector. The subscript $q - k$ wraps around mod n , and the χ_k obey

$$-\frac{C}{2m} \left[\frac{\partial^2}{\partial y^2} + \left(2m\tilde{\Omega}y - i \frac{\partial}{\partial x} \right)^2 \right] \chi_k + V(x, y) \chi_k = E \chi_k, \quad (4)$$

where $\tilde{\Omega} \equiv (\alpha - \alpha_c)\pi/md^2$ and C depends only on l, n . This formula reduces to Eq. (2) for $\alpha_c = 0/1$. The comparison in Fig. 1 shows excellent agreement of the wave functions for the cases $\alpha_c = 1/2$ and $\alpha_c = 1/3$.

For this procedure to be consistent, the magnetic length $\tilde{\ell}$ on which χ_k varies must be large compared to the small scale periodicity nd , but small enough that $m\tilde{\Omega}y d \ll 1$ for $y \sim \tilde{\ell}$. For a 1D harmonic trap $\tilde{\ell}_l = (m^2\tilde{\Omega}^2 + m^2\omega^2/4C)^{-1/4}$ and thus the analytic approximation is valid for

$$\frac{1}{n^4} \gg (\alpha - \alpha_c)^2 + \frac{\beta^2}{4C} \gg (\alpha - \alpha_c)^4, \quad (5)$$

where $\beta \equiv md^2\omega/\pi$ is the dimensionless trap strength. In Fig. 2 we compare approximate analytical and exact numerical results for values of $n \leq 8$ as a function of α and β . The plot shows large regions of validity of our calculation around $\alpha_c = 0$ (conventional FQH region) but also around $\alpha_c = 1/2$ and $\alpha_c = 1/3$. As expected for increasing n the regions of validity get narrower and are more easily destroyed by an external potential. The regions also

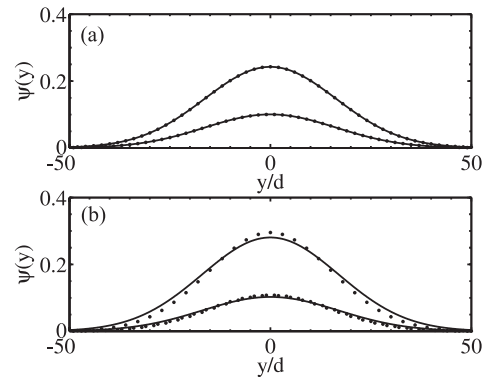


FIG. 1. Wave function for (a) $\alpha = 1/2$ and (b) $\alpha = 1/3$, in linear geometry with trap $V(y) = m\omega^2 y^2/2$ and $\omega = 10^{-3}\pi/md^2$. The dots show the numerical wave function and the solid curves the analytic approximations for $y/d \equiv 0 \pmod n$ (upper curve) and $y/d \not\equiv 0 \pmod n$ (lower curve).

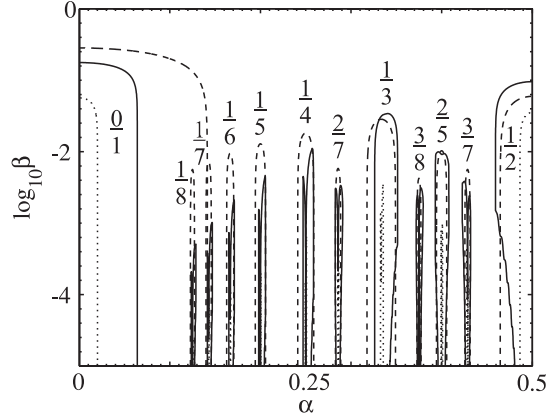


FIG. 2. Areas of validity of different fractions α_c , in a one-dimensional harmonic trap. The solid (dotted) curves show regions with overlap between exact numerical wave functions and their analytical approximations larger than 99% (99.9%) and resulting energy difference smaller than $0.01/md^2$ ($0.001/md^2$). The dashed lines are areas of validity obtained from Eq. (5) taking \gg to mean a ratio larger than 50.

get narrower for decreasing values of β with at the same time an increasing number of n 's fulfilling Eq. (5) at $\alpha = \alpha_c$. Our model is thus only valid in a finite neighborhood of an α_c if a suitable trapping potential is present.

We extend the Hamiltonian to the many particle case by including an on site interaction term into Eq. (4) giving

$$H \approx \int dx dy \sum_k \chi_k^\dagger \left\{ \frac{C}{2m} \left[-\frac{\partial^2}{\partial y^2} + \left(2m\tilde{\Omega}y - i\frac{\partial}{\partial x} \right)^2 \right] + V(x, y) \right\} \chi_k + u \sum_{k_1, k_2, k_3, k_4} G_{k_1, k_2, k_3, k_4} \chi_{k_1}^\dagger \chi_{k_2}^\dagger \chi_{k_3} \chi_{k_4}, \quad (6)$$

where $\chi_k(x, y)$ is now a bosonic field operator and the constants $G_{k_1, k_2, k_3, k_4} \equiv \sum_j \mathbf{v}_{j-k_1} \mathbf{v}_{j-k_2} \mathbf{v}_{j-k_3} \mathbf{v}_{j-k_4} / n$ if $k_1 + k_2 \equiv k_3 + k_4 \pmod{n}$ and 0 otherwise; e.g., for $\alpha_c = 1/2$

$$\langle a_i^\dagger a_j^\dagger a_{j'} a_{i'} \rangle = \sum_{k_i, k_j=1}^n d^2 \mathbf{B}_i^{(k_i)} \mathbf{B}_j^{*(k_j)} \mathbf{B}_{i'}^{*(k_i)} \mathbf{B}_{j'}^{(k_j)} \rho_2(i, j; i', j')_{k_i k_j; k_i k_j}, \quad (7)$$

where $\rho_2(i, j; j', i')_{k_i k_j; k_i k_j}$ is the continuum 2-particle density matrix. For LLL constant density states $\rho_2(i, j; i', j')_{k_i k_j; k_i k_j} \propto g_{k_i k_j} ((w_i - w_j)^* (w_{i'} - w_{j'}))$ [19] and is thus determined by analytic continuation of the two-point correlation functions $g_{k_i k_j}(|w_i - w_j|^2)$. We have calculated $g_{k_i k_j}$ for the Laughlin and 221 states by Monte Carlo methods [19]. The results in Fig. 3 show clear spatial antibunching up to distances of order $\tilde{\ell}_0$ which is very distinct from behavior of superfluid and MI states.

Finally we calculate the Hall currents which can be used to characterize FQH states near α_c . Unlike in a Galilean-invariant continuum FQH system the application of a linear potential $V(x, y) = -may$ does not induce a particle current with velocity of exactly $v_x = a/2\Omega$ since the lattice defines a rest frame. Instead, when Eq. (6) is valid, the

$1/2$, $G_{1111} = G_{2222} = 3/2$ and $G_{1212} = G_{2121} = G_{1122} = G_{2211} = 1/2$. Particles with the same k interact more strongly since their $\mathbf{v}^{(k)}$ are peaked on the same sites.

For $\alpha_c = 1/2$ this effective Hamiltonian is equivalent to a bilayer FQH system with $\chi_\pm = \chi_1 \pm i\chi_2$ being the two “layers,” and hence for weak potentials V the highest density zero interaction energy state is the 221 state using the notation of Ref. [11]. This bosonic analogue of the fermion 331 state [11] with magnetic length $\tilde{\ell}_0 = (m\tilde{\Omega})^{-1/2}$ has filling factor $\tilde{\nu} \equiv \rho\pi\tilde{\ell}_0^2 = 2/3$; it can be extended to larger n with $\tilde{\nu} = n/(n+1)$ by multiplication with $(z_i - z_j)^2$ if particles i, j are in the same layer and with $(z_i - z_j)$ if they are not. We expect a density step profile eventually breaking down into a vortex lattice, qualitatively similar to the small α regime, with the 221 state being the lowest step. As long as Eq. (6) is a valid approximation (i.e., the particle spacing is much larger than nd) every step would occur at fixed $\tilde{\nu}$ and not fixed ν . From Fig. 2, the range of validity for $\alpha_c = 1/2$ is $\alpha - \alpha_c \lesssim 0.04$ and hence for the $\tilde{\nu} = 2/3$ state, lattice filling $\rho d^2 \lesssim 0.03$. For $U \lesssim J$ the energy gap is $E_g \sim U\rho d^2$ which is about 10–100 Hz for typical optical lattice parameters consistent with Ref. [7] and will thus be measurable with near future technology. Measurement of the stepped density profile would confirm the existence of a series of incompressible states near $\alpha = \alpha_c$ of densities proportional to $\tilde{\Omega}$ but would not confirm that they are FQH states. We will thus next investigate two methods feasible in optical lattices to identify and characterize these states further.

We first study how shot noise measurements of the energy correlations [16,17] can reveal the two particle correlation functions $g(|w|^2)$ with $w = x + iy$ and thus distinguish the FQH states from Mott insulating (MI) states for which such measurements were recently carried out [18]. As shown in Ref. [16] results of these measurements are directly related to the expectation values

particle velocity is given by $\tilde{v}_x = 2\tilde{\Omega}a/(4\tilde{\Omega}^2 + \omega^2/C)$ if the linear potential is superimposed to a 1D harmonic trap. The Hall current will thus cease for $\alpha = \alpha_c$ and for $\alpha \lesssim \alpha_c$ a counterintuitive negative Hall current flows. In a trapped system at equilibrium these currents will not be visible since they flow along equipotentials and steps in the density profile also lie along equipotentials. They can be made visible by putting the system out of equilibrium, e.g., by suddenly changing the trap from V_1 to V_2 as shown in Fig. 4.

When sufficiently mild disorder is added to the optical lattice [9], some of the particles become localized and cannot carry current, but the average velocity is still \tilde{v}_x [11]. A simple model of this, valid for weak potentials, is to describe the disorder by a density of states $\rho_d(E)$ giving the

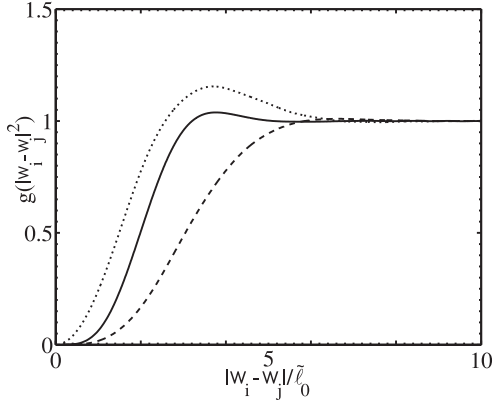


FIG. 3. Two-point functions: $\nu = 1/2$ Laughlin state (solid line); $\tilde{\nu} = 2/3$ 221 state $g_{11} = g_{22}$ (dashed line) and $g_{12} = g_{21}$ (dotted line).

number of LLL states at energy E per unit energy interval per unit area and hence $\int \rho_d(E) dE = m\tilde{\Omega}/\pi$. For a linear geometry we have

$$I = \frac{m}{2\pi} \int dy \frac{dV_2}{dy} \int^{\mu - V_1(y)} dE \sum_{j \geq 0} (\tilde{\nu}_{j-1} - \tilde{\nu}_j) \delta(E - E_j) \quad (8)$$

$$N = L \int dy \int^{\mu - V_1(y)} dE \sum_{j \geq 0} (\tilde{\nu}_j - \tilde{\nu}_{j-1}) \rho_d(E - E_j),$$

with L the system's length, I the net current (both in the x direction), and μ the chemical potential. The FQH states are at filling factors $\tilde{\nu}_j$ and energies per particle E_j [for $\alpha \approx 0$, $\nu_j = (j+1)/2$, and $E_j \approx (um\Omega/2\pi)j$].

In a harmonic trap the disorder will not lead to FQH plateaus, only corners each time a new extended level begins to fill (c.f. Fig. 4), but unlike the square-well case,

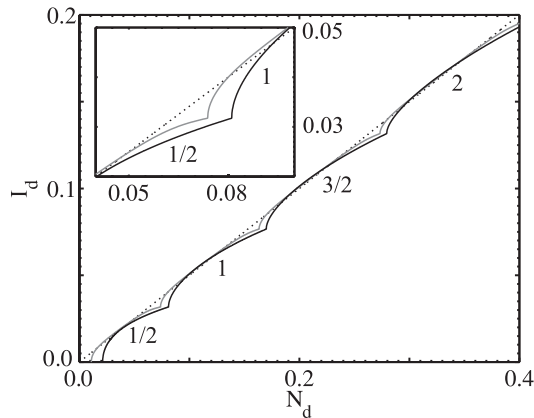


FIG. 4. Dimensionless Hall current $I_d = I/(2^{3/2}ma\sqrt{u\Omega}/\omega)$ against dimensionless number of atoms per unit length $N_d = N/(2^{3/2}L\Omega^{3/2}m\sqrt{u}/\omega)$ for $V_1(y) = m\omega^2 y^2/2$, $V_2(y) = V_1(y) - may$ at small α and a , for no disorder (dotted straight line), maximal disorder (constant density of states) (black curve), and Lorentzian disorder of width $um\Omega/10\pi$ (gray curve). Numbers under the curve are filling factors at trap center. The inset shows the $\nu = 1/2$ to $\nu = 1$ corner.

it is possible to obtain the complete distribution $\rho_d(E)$ by measuring I against N (or Ω) and using Eq. (8). The quasiparticles of our system carry fractional particle number [20] and fractional statistics [5] as in the conventional FQH system, but detecting this directly is likely to be more experimentally challenging.

In summary we have investigated bosonic optical lattice FQH systems at large magnetic fields. We have worked out their ground states, magnetic lengths, and the resulting fractional fillings, and found the Hall current to change its sign at α_c . Furthermore, we have studied ways for detecting these properties using recently developed experimental methods.

This work is supported by EPSRC through Project No. EP/C51933/1 and QIP IRC (GR/S82176/01), and by the EU through OLAQUI.

- [1] M. Greiner, O. Mandel, T. Esslinger, T. W. Haensch, and I. Bloch, *Nature (London)* **415**, 39 (2002).
- [2] D. Jaksch, C. Bruder, J. I. Cirac, C. W. Gardiner, and P. Zoller, *Phys. Rev. Lett.* **81**, 3108 (1998).
- [3] M. Köhl, H. Moritz, T. Stöferle, K. Günter, and T. Esslinger, *Phys. Rev. Lett.* **94**, 080403 (2005).
- [4] M. Polini, R. Fazio, A. H. MacDonald, and M. P. Tosi, *Phys. Rev. Lett.* **95**, 010401 (2005).
- [5] B. Paredes, P. Fedichev, J. I. Cirac, and P. Zoller, *Phys. Rev. Lett.* **87**, 010402 (2001).
- [6] D. Jaksch and P. Zoller, *New J. Phys.* **5**, 56 (2003).
- [7] A. S. Sørensen, E. Demler, and M. D. Lukin, *Phys. Rev. Lett.* **94**, 086803 (2005).
- [8] D. Jaksch and P. Zoller, *Ann. Phys. (N.Y.)* **315**, 52 (2005).
- [9] B. Damski, J. Zakrzewski, L. Santos, P. Zoller, and M. Lewenstein, *Phys. Rev. Lett.* **91**, 080403 (2003).
- [10] D. R. Hofstadter, *Phys. Rev. B* **14**, 2239 (1976).
- [11] Z. F. Ezawa, *Quantum Hall Effects: Field Theoretical Approach and Related Topics* (World Scientific, Singapore, 2000).
- [12] N. K. Wilkin, J. M. F. Gunn, and R. A. Smith, *Phys. Rev. Lett.* **80**, 2265 (1998).
- [13] V. Bretin, S. Stock, Y. Seurin, and J. Dalibard, *Phys. Rev. Lett.* **92**, 050403 (2004); V. Schweikhard, I. Coddington, P. Engels, V. P. Mogendorff, and E. A. Cornell, *Phys. Rev. Lett.* **92**, 040404 (2004).
- [14] N. R. Cooper, F. J. M. van Lankvelt, J. W. Reijnders, and K. Schoutens, *cond-mat/0508031*.
- [15] N. Read and E. Rezayi, *Phys. Rev. B* **59**, 8084 (1999).
- [16] E. Altman, E. Demler, and M. D. Lukin, *Phys. Rev. A* **70**, 013603 (2004).
- [17] N. Read and N. R. Cooper, *Phys. Rev. A* **68**, 035601 (2003).
- [18] S. Fölling, F. Gerbier, A. Widera, O. Mandel, T. Gericke, and I. Bloch, *Nature (London)* **434**, 481 (2005).
- [19] A. H. MacDonald and S. M. Girvin, *Phys. Rev. B* **38**, 6295 (1988); S. M. Girvin, *Phys. Rev. B* **30**, 558 (1984); S. M. Girvin, A. H. MacDonald, and P. M. Platzman, *Phys. Rev. B* **33**, 2481 (1986).
- [20] L. Saminadayar, D. C. Glatthli, Y. Jin, and B. Etienne, *Phys. Rev. Lett.* **79**, 2526 (1997); R. de Picciotto *et al.*, *Nature (London)* **389**, 162 (1997).