

DIRAC COHOMOLOGY OF ONE- W -TYPE REPRESENTATIONS

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ABSTRACT. The smooth hermitian representations of a split reductive p -adic group whose restriction to a maximal hyperspecial compact subgroup contain a single K -type with Iwahori fixed vectors have been studied in [BM3] in the more general setting of modules for graded affine Hecke algebras with parameters. We show that every such one K -type module has nonzero Dirac cohomology (in the sense of [BCT]), and use Dirac operator techniques to determine the semisimple part of the Langlands parameter for these modules, thus completing their classification.

1. INTRODUCTION

The category of smooth representations of a reductive p -adic group generated by their vectors fixed under an Iwahori subgroup is equivalent to the category of modules over the Iwahori-Hecke algebra [Bo]. Furthermore, the category of Iwahori-Hecke algebra modules is equivalent to a product of categories of certain graded affine Hecke algebra modules [Lu1]. It is known that these equivalences induce bijections between the unitary representations in the corresponding categories [BM1, BM2].

An interesting class of unitary representations are those which have a single K -type with Iwahori fixed vectors, or in terms of Borel's equivalence of categories, the Iwahori-Hecke algebra modules whose restrictions to the finite Hecke algebra are irreducible. These representations are expected to be automorphic, for example, the Speh representations for $GL(n, \mathbb{Q}_p)$ ([Ta]) are such one- K -type representations.

In terms of the corresponding graded affine Hecke algebra \mathbb{H} (Definition 2.5.1), we are interested in the one- W -type modules, i.e., unitary modules of \mathbb{H} with real central character whose restrictions to the group algebra $\mathbb{C}[W]$ of the Weyl group are irreducible. In [BM3], it was determined which irreducible Weyl group representations support an action of hermitian (hence necessarily unitary) \mathbb{H} -module with respect to the natural $*$ -operation (2.5.2). In particular, it is shown that whenever this is possible, there is a unique way to define the action of \mathbb{H} . In this paper, we first provide a simpler, uniform proof for this fact, Proposition 3.1.1; the idea is similar in spirit with the classical argument from real groups used to show that there are no nontrivial finite-dimensional unitary representations of a simple real group, e.g. [Kn, Corollary 2.3] for $SL(2, \mathbb{R})$.

In [BCT], a Dirac operator for graded Hecke algebras was introduced and a theory of Dirac cohomology for \mathbb{H} -modules was proposed. We show here that every one- W -type \mathbb{H} -module has nonzero Dirac cohomology, in fact the Dirac operator

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is identically zero on such modules, Proposition 3.2.1. This fact, combined with certain calculations of tensor products of representations of the pin double cover of the Weyl group \widetilde{W} (defined in section 2.2), allows us to compute explicitly the Dirac cohomology of these modules in all cases. Vogan's conjecture, in the setting of Hecke algebras proved in [BCT], essentially says that the Dirac cohomology determines the central character of the \mathbb{H} -module. Thus, as a corollary of our calculations, we find the central character of the one- W -type \mathbb{H} -modules, which completes their identification in the Langlands classification. The explicit results are listed in sections 3.3-3.5.

2. DIRAC OPERATOR FOR GRADED HECKE ALGEBRAS

In this section, we review the construction and properties of the Dirac operator from [BCT] and the relation between spin representations of the Weyl group and nilpotent orbits from [Ci].

2.1. The root system. Let V be a finite dimensional Euclidean vector space, with inner product $(\cdot, \cdot) : V \times V \rightarrow \mathbb{R}$. Let Φ be a semisimple reduced root system in V , with finite Weyl group $W \subset O(V)$. Let $s_\alpha \in W$ denote the reflection through the hyperplane perpendicular to α .

Choose a system $\Phi^+ \subset \Phi$ of positive roots, and let Π be a basis of Φ^+ , the set of simple roots. The group W admits a Coxeter presentation

$$W = \langle s_\alpha, \alpha \in \Pi : (s_\alpha s_\beta)^{m(\alpha, \beta)} = 1 \rangle. \quad (2.1.1)$$

2.2. The pin cover of W . Let $C(V)$ be the Clifford algebra of $V, (\cdot, \cdot)$. This is the real associative algebra generated by $\{v \in V\}$ subject to the relations

$$v_1 v_2 + v_2 v_1 = -2(v_1, v_2), \quad v_1, v_2 \in V. \quad (2.2.1)$$

If one assigns degree one to the elements $v \in V$, then $C(V)$ is naturally a filtered algebra whose associated graded algebra is $\wedge V$. In particular, $C(V)$ has a $\mathbb{Z}/2\mathbb{Z}$ -grading $C(V) = C(V)_{\text{even}} + C(V)_{\text{odd}}$ given by the degree mod 2. Let ϵ be the automorphism of $C(V)$ which is 1 on $C(V)_{\text{even}}$ and -1 on $C(V)_{\text{odd}}$. Let ${}^t : C(V) \rightarrow C(V)$ be the anti-involution given by $v^t = -v$ for $v \in V$. Define the pin group:

$$\text{Pin}(V) = \{a \in C(V) : \epsilon(a)Va^{-1} \subset V, a^t = a^{-1}\}, \quad (2.2.2)$$

a central $\mathbb{Z}/2\mathbb{Z}$ -extension of $O(V)$ with the projection map $p : \text{Pin}(V) \rightarrow O(V)$, $p(a)(v) = \epsilon(a)va^{-1}$, $a \in C(V)$, $v \in V$. Since $W \subset O(V)$, define

$$\widetilde{W} = p^{-1}(W) \subset \text{Pin}(V). \quad (2.2.3)$$

Analogous to (2.1.1), \widetilde{W} also admits a Coxeter-like presentation:

$$\widetilde{W} = \langle z, \tilde{s}_\alpha, \alpha \in \Pi : z^2 = 1, \tilde{s}_\alpha^2 = z, (\tilde{s}_\alpha \tilde{s}_\beta)^{m(\alpha, \beta)} = z \rangle. \quad (2.2.4)$$

The embedding $\widetilde{W} \subset \text{Pin}(V)$ is given by

$$z \mapsto -1, \quad \tilde{s}_\alpha \mapsto \frac{1}{|\alpha|} \alpha, \quad \alpha \in \Phi^+, \quad (2.2.5)$$

where $|\alpha| = \sqrt{(\alpha, \alpha)}$.

When $\dim V$ is even, the Clifford algebra $C(V)$ is a central simple algebra, and therefore has a unique simple complex module S^\square of dimension $2^{\dim V/2}$. When $\dim V$ is odd, the center of $C(V)$ is two dimensional. The subalgebra $C(V)_{\text{even}}$ is

central simple and has a unique simple complex module which can be extended in two nonisomorphic ways to $C(V)$. Thus $C(V)$ has two simple modules S^+ and S^- of dimension $2^{(\dim V - 1)/2}$.

In the sequel, we refer to any one of S^\square, S^+, S^- as a spin module. Notice that since \widetilde{W} generates $C(V)$, the restriction of a spin module S to \widetilde{W} remains irreducible.

2.3. Adjoint nilpotent orbits. We assume now that the root system Φ is crystallographic. Let $\Phi^\vee \subset V^*$ be the dual root system to $\Phi \subset V$. More precisely, Φ^\vee is the set of coroots α^\vee , where $\alpha^\vee(v) = \frac{2}{(\alpha, \alpha)}(\alpha, v)$, $v \in V$.

Let \mathfrak{g} be the complex semisimple Lie algebra with root system Φ^\vee . We identify a Cartan subalgebra \mathfrak{h} of \mathfrak{g} with V . Let \mathcal{N} denote the nilpotent cone in \mathfrak{g} , i.e., the set of ad-nilpotent elements of \mathfrak{g} . The adjoint group G whose Lie algebra is \mathfrak{g} acts via the adjoint action on \mathcal{N} with finitely many orbits. A classical result of Kostant is that there is a bijection between the G -orbits in \mathcal{N} and the G -conjugacy classes of Lie triples

$$\{e, h, f\} : [h, e] = 2e, [h, f] = -2f, [e, f] = h. \quad (2.3.1)$$

Let \mathcal{B} be the flag variety, the variety of all Borel subalgebras of \mathfrak{g} . For $e \in \mathcal{N}$, let \mathcal{B}_e be the subvariety of Borel subalgebras which contain e . Let $A(e)$ be the component group of the centralizer of e in G .

Springer theory defines an action of $W \times A(e)$ on the cohomology groups $H^\bullet(\mathcal{B}_e, \mathbb{C})$ such that:

- (1) for every $e \in \mathcal{N}$ and ϕ an irreducible $A(e)$ -representation, the ϕ -isotypic component of the top cohomology group $H^{2d_e}(\mathcal{B}_e, \mathbb{C})^\phi$ is an irreducible W -representation or zero; denote $\widehat{A(e)}_0$, the set of irreducible $A(e)$ -representation for which $\sigma(e, \phi) := H^{2d_e}(\mathcal{B}_e, \mathbb{C})^\phi \neq 0$;
- (2) the set $\{\sigma(e, \phi) : e \in G \backslash \mathcal{N}, \phi \in \widehat{A(e)}_0\}$ equals the set of irreducible representations of W .

2.4. Genuine \widetilde{W} -representations and nilpotent orbits. Suppose $k : \Phi^+ \rightarrow \mathbb{R}$, $k(\alpha) = k_\alpha$ is a W -invariant function. Define ([BCT])

$$\Omega_{\widetilde{W}, k} = z \left(\sum_{\alpha \in \Phi^+} k_\alpha |\alpha| \widetilde{s}_\alpha \right)^2. \quad (2.4.1)$$

This is an element in the center of $\mathbb{C}[\widetilde{W}]$. Set

$$\mathcal{N}_{\text{sol}} = \{e \in \mathcal{N} : Z_G(e)^0 \text{ is solvable}\}, \quad (2.4.2)$$

a G -invariant subset of \mathcal{N} .

Call an irreducible \widetilde{W} -representation $\tilde{\sigma}$ genuine if $\tilde{\sigma}(z) = -1$, i.e., if $\tilde{\sigma}$ does not descend to a representation of W . Denote by $\text{Irr}_{\text{gen}} \widetilde{W}$ the set of irreducible genuine \widetilde{W} -representations.

Theorem 2.4.1 ([Ci]). *Suppose the parameter function is $k = 1$. There exists a surjective map*

$$\Psi : \text{Irr}_{\text{gen}} \widetilde{W} \rightarrow G \backslash \mathcal{N}_{\text{sol}}, \quad (2.4.3)$$

with the following properties:

- (1) If for $\tilde{\sigma} \in \text{Irr}_{\text{gen}} \widetilde{W}$, $\Psi(\tilde{\sigma}) = G \cdot e$, where $e \in \mathcal{N}_{\text{sol}}$, then $\tilde{\sigma}(\Omega_{\widetilde{W}, 1}) = (h, h)$, where $\{e, h, f\}$ is a Lie triple for e with $h \in V$.

- (2) Given $e \in \mathcal{N}_{sol}$, if $\tilde{\sigma} \in \Psi^{-1}(G \cdot e)$, then there exists $\phi \in \widehat{A(e)}_0$ and a spin module S such that $\tilde{\sigma}$ occurs in $\sigma(e, \phi) \otimes S$.
- (3) Given $e \in \mathcal{N}_{sol}$, $\phi \in \widehat{A(e)}_0$, and S a spin module, there exists $\tilde{\sigma} \in \Psi^{-1}(G \cdot e)$ such that $\tilde{\sigma}$ occurs in $\sigma(e, \phi) \otimes S$.

There is an explicit description of the map Ψ for every simple root system [Ci].

2.5. Graded affine Hecke algebra. Let $V_{\mathbb{C}}$ denote the complexification of V and $S(V_{\mathbb{C}})$ the symmetric algebra of $V_{\mathbb{C}}$.

Definition 2.5.1 (Lusztig[Lu2, 0.1]). *The graded affine Hecke algebra \mathbb{H} associated to the root system $\Phi \subset V$ and parameter function $k : \Phi^+ \rightarrow \mathbb{R}$ is the \mathbb{C} -vector space $S(V_{\mathbb{C}}) \otimes_{\mathbb{C}} \mathbb{C}[W]$, endowed with a structure of associative complex algebra with identity defined by the relations:*

- (1) $\mathbb{C}[W] \rightarrow \mathbb{H}$, $w \mapsto 1 \otimes w$ is an algebra homomorphism;
- (2) $S(V_{\mathbb{C}}) \rightarrow \mathbb{H}$, $\omega \mapsto \omega \otimes 1$ is an algebra homomorphism;
- (3) $(\xi \otimes 1)(1 \otimes w) = \xi \otimes w$, $\xi \in S(V_{\mathbb{C}})$, $w \in W$;
- (4) $(1 \otimes s_{\alpha})(\xi \otimes 1) - (s_{\alpha}(\xi) \otimes 1)(1 \otimes s_{\alpha}) = k_{\alpha} \frac{\xi - s_{\alpha}(\xi)}{\alpha}$, $\xi \in S(V_{\mathbb{C}})$, $\alpha \in \Pi$.

For simplicity, in what follows, we will denote $\omega = \omega \otimes 1$ and $t_w = 1 \otimes w$ in \mathbb{H} . In particular, the last relation becomes

$$t_{s_{\alpha}} \xi - s_{\alpha}(\xi) t_{s_{\alpha}} = k_{\alpha} \frac{\xi - s_{\alpha}(\xi)}{\alpha}.$$

The center of \mathbb{H} is $Z(\mathbb{H}) = S(V_{\mathbb{C}})^W$, where $S(V_{\mathbb{C}})$ denotes the symmetric algebra of $V_{\mathbb{C}}$. Consequently, \mathbb{H} is finite over its center, hence every irreducible \mathbb{H} -module is finite dimensional, and the center of \mathbb{H} acts by characters in the irreducible modules. The central characters are parameterized by $W \backslash V_{\mathbb{C}}^*$. If $\nu \in V_{\mathbb{C}}^*$, write χ_{ν} for the central character parameterized by $W\nu$. We say that a central character χ_{ν} is real if $W\nu \subset V^*$.

In [BCT], the Casimir element Ω of \mathbb{H} was introduced: if $\{\omega_i\}$ is an orthonormal basis of V , set

$$\Omega = \sum_i \omega_i^2 \in S(V_{\mathbb{C}})^W. \quad (2.5.1)$$

If (π, X) is an irreducible \mathbb{H} -module with central character χ_{ν} , then $\pi(\Omega) = (\nu, \nu)$.

Let $*$ be the conjugate linear anti-involution of \mathbb{H} defined on generators by [BM2]:

$$t_w^* = t_{w^{-1}}, \quad \omega^* = -t_{w_0} w_0(\omega) t_{w_0}, \quad w \in W, \quad \omega \in V, \quad (2.5.2)$$

where w_0 is the longest Weyl group element. We say that an \mathbb{H} -module (π, X) is $*$ -hermitian if X has a hermitian form $\langle \cdot, \cdot \rangle$ which is $*$ -invariant, i.e.,

$$\langle \pi(h)x, y \rangle = \langle x, \pi(h^*)y \rangle. \quad (2.5.3)$$

If the form is positive definite, we say that (π, X) is $*$ -unitary.

2.6. The Dirac operator. For every $\omega \in V$, define

$$\tilde{\omega} = \frac{1}{2}(\omega - \omega^*) = \omega - p_{\omega}, \quad \text{where } p_{\omega} = \frac{1}{2} \sum_{\alpha \in \Phi^+} k_{\alpha} \frac{\omega - s_{\alpha}(\omega)}{\alpha} t_{s_{\alpha}}. \quad (2.6.1)$$

By definition, $\tilde{\omega}^* = -\tilde{\omega}$. In addition, these elements have the known properties (e.g., [BCT]):

$$t_w \tilde{\omega} t_{w^{-1}} = \widetilde{w(\omega)}, \quad [\tilde{\omega}_1, \tilde{\omega}_2] = -[p_{\omega_1}, p_{\omega_2}]. \quad (2.6.2)$$

Let $\{\omega_i\}$ be an orthonormal basis of V . The Dirac element defined in [BCT] is

$$\mathcal{D} = \sum_i \tilde{\omega}_i \otimes \omega_i \in \mathbb{H} \otimes C(V). \quad (2.6.3)$$

It is independent of the choice of basis and its square is

$$\mathcal{D}^2 = -\Omega \otimes 1 + \frac{1}{4} \Delta_{\widetilde{W}}(\Omega_{\widetilde{W},k}), \quad (2.6.4)$$

where $\Delta_{\widetilde{W}}$ is the diagonal embedding of $\mathbb{C}[\widetilde{W}]$ into $\mathbb{H} \otimes C(V)$ extending $\Delta_{\widetilde{W}}(\tilde{w}) = p(\tilde{w}) \otimes \tilde{w}$.

If (π, X) is an \mathbb{H} -module, and S is a spin $C(V)$ -module, left action by \mathcal{D} defines a Dirac operator

$$D_X : X \otimes S \rightarrow X \otimes S. \quad (2.6.5)$$

When X is $*$ -hermitian, D_X is a self-adjoint operator with respect to the tensor product form (the hermitian form on S being the natural one). Moreover, D_X is $\text{sgn } \widetilde{W}$ -invariant; here sgn is the sign (non-genuine) character of \widetilde{W} .

Definition 2.6.1. *The Dirac cohomology of X (with respect to S) is $H^D(X) = \ker D_X / \ker D_X \cap \text{im } D_X$, a \widetilde{W} -representation.*

An analogue of the algebraic version of Vogan's conjecture from real groups, proved in this setting in [BCT], says that there exists an algebra homomorphism

$$\zeta : Z(\mathbb{H}) \rightarrow \mathbb{C}[\widetilde{W}]^{\widetilde{W}}, \quad (2.6.6)$$

such that for every $z \in Z(\mathbb{H})$, there exists a unique $a \in \mathbb{H} \otimes C(V)$ with the property

$$z \otimes 1 = \Delta_{\widetilde{W}}(\zeta(z)) + \mathcal{D}a + a\mathcal{D}. \quad (2.6.7)$$

For every $\tilde{\sigma} \in \text{Irr}_{\text{gen}} \widetilde{W}$, define a central character $\chi^{\tilde{\sigma}} : Z(\mathbb{H}) \rightarrow \mathbb{C}$ by $\chi^{\tilde{\sigma}}(z) = \tilde{\sigma}(\zeta(z))$, $z \in Z(\mathbb{H})$. As explained in [BCT], (2.6.7) implies the following result.

Theorem 2.6.1 ([BCT]). *Let X be an irreducible \mathbb{H} -module, and suppose the irreducible \widetilde{W} -representation $\tilde{\sigma}$ occurs in $H^D(X)$ (in particular $H^D(X) \neq 0$). Then the central character of X is $\chi^{\tilde{\sigma}}$.*

In light of this theorem, it is necessary to describe the central characters $\chi^{\tilde{\sigma}}$ more explicitly. When the parameter function k is identically 1, it turns out that

$$\chi^{\tilde{\sigma}} = \frac{1}{2}h, \quad (2.6.8)$$

where h is the semisimple element of an \mathfrak{sl}_2 -triple $\{e, h, f\}$ with $\Psi(\tilde{\sigma}) = G \cdot e$.

When W is of type B_n , every irreducible genuine \widetilde{W} -representation is of the form $\tilde{\sigma} = (\sigma \times 0) \otimes S$, where σ is a partition of n , $\sigma \times 0$ labels the irreducible W -representation in the bipartition notation, and S is a spin $C(V)$ -module. For such a $\tilde{\sigma}$, the character $\chi^{\tilde{\sigma}}$ is described in [Ci] by the following combinatorial procedure. Suppose the parameter function k takes the value k_s on the short simple root, and k_ℓ on the long simple roots of type B_n . In the left justified decreasing Young diagram of shape σ , label each box starting with k_s in the upper left corner, then increasing by k_ℓ to the right and decreasing by k_ℓ down. The entries of the resulting Young tableau, viewed in $\mathbb{R}^n \cong V$ form the central character $\chi^{\tilde{\sigma}}$.

When \mathbb{H} is of type G_2 or F_4 with arbitrary parameters k , the lists of central characters $\chi^{\tilde{\sigma}}$ are given in [COT].

3. ONE- W -TYPE MODULES

Recall from the introduction that a simple \mathbb{H} -module with real central character is called a one- W -type module if it is $*$ -hermitian and its restriction to $\mathbb{C}[W]$ is irreducible. In [BM3], a classification of one- W -type \mathbb{H} -modules is obtained. We simplify one of the arguments in [BM3] and show that every one- W -type module has nonzero Dirac cohomology.

3.1. Action of $\tilde{\omega}$. The following proposition is implicitly proved in [BM3], under an assumption about the restriction of the W -type to maximal parabolic subgroups. We give here a different proof that works in general and can be regarded as an analogue of the well-known argument which proves that the only irreducible finite dimensional unitary representation of $SL(2, \mathbb{R})$ is trivial, e.g., [Kn].

Proposition 3.1.1. *Let (π, X) be a one- W -type \mathbb{H} -module. Then $\pi(\tilde{\omega}) = 0$ for all $\omega \in V$. In particular, an irreducible W -representation σ can be extended to a one- W -type \mathbb{H} module if and only if $[\sigma(p_{\omega_1}), \sigma(p_{\omega_2})] = 0$, for every $\omega_1, \omega_2 \in V$.*

Proof. By definition, (π, X) has a $*$ -invariant hermitian form $\langle \cdot, \cdot \rangle_X^*$. Since X is a one- W -type module, this form can be normalized so that it is positive definite. Since $\tilde{\omega}^* = -\tilde{\omega}$ for $\omega \in V$, it follows that $\pi(\tilde{\omega})$ is a skew-symmetric operator on $X, \langle \cdot, \cdot \rangle_X^*$. In particular, $\pi(\tilde{\omega})$ is diagonalizable and acts with purely imaginary eigenvalues on X .

Consider now the conjugate linear anti-involution \bullet of \mathbb{H} studied in [BC1], defined on generators by

$$t_w^\bullet = t_{w^{-1}}, \quad \omega^\bullet = \omega, \quad w \in W, \omega \in V. \quad (3.1.1)$$

From (2.6.1), it is immediate that $\tilde{\omega}^\bullet = \tilde{\omega}$ for every $\omega \in V$. The two anti-involutions $*$ and \bullet are related via

$$h^\bullet = t_{w_0} \mathcal{A}(h^*) t_{w_0}, \quad h \in \mathbb{H}, \quad (3.1.2)$$

where \mathcal{A} is the automorphism of \mathbb{H} defined by $\mathcal{A}(t_w) = t_{w_0 w w_0}$ and $\mathcal{A}(\omega) = -w_0(\omega)$, $w \in W, \omega \in V$. Let $(\pi^\mathcal{A}, X)$ be the \mathcal{A} -twist of the module (π, X) : $\pi^\mathcal{A}(h)x = \pi(\mathcal{A}(h))x$, $h \in \mathbb{H}, x \in X$.

By the classification of $*$ -hermitian irreducible modules [BM2], $(\pi, X) \cong (\pi^\mathcal{A}, X)$. Let $\kappa : X \rightarrow X$ be an intertwiner of the two actions. Using the $*$ -invariant form $\langle \cdot, \cdot \rangle_X^*$ on X , one defines a \bullet -invariant form $\langle \cdot, \cdot \rangle_X^\bullet$ on X by

$$\langle x, y \rangle_X^\bullet = \langle x, \pi(t_{w_0})\kappa(y) \rangle_X^*, \quad x, y \in X. \quad (3.1.3)$$

We verify that this form is indeed \bullet -invariant:

$$\begin{aligned} \langle \pi(h)x, y \rangle_X^\bullet &= \langle \pi(h)x, \pi(t_{w_0})\kappa(y) \rangle_X^* = \langle x, \pi(h^*)\pi(t_{w_0})\kappa(y) \rangle_X^* \\ &= \langle x, \pi(t_{w_0})\pi(\mathcal{A}(h^\bullet))\kappa(y) \rangle_X^* = \langle x, \pi(t_{w_0})\kappa(\pi(h^\bullet)y) \rangle_X^* \\ &= \langle x, \pi(h^\bullet)y \rangle_X^\bullet. \end{aligned}$$

Since $X|_W$ is irreducible, the form $\langle \cdot, \cdot \rangle_X^\bullet$ may also be normalized so that it is positive definite. Since $\tilde{\omega}^\bullet = \tilde{\omega}$, $\omega \in V$, it follows that every $\pi(\tilde{\omega})$ is symmetric, and thus it acts with real eigenvalues.

In conclusion, every $\pi(\tilde{\omega}) = 0$. The second claim is immediate because the only relation between $\tilde{\omega}_1$ and $\tilde{\omega}_2$ is $[\tilde{\omega}_1, \tilde{\omega}_2] = -[p_{\omega_1}, p_{\omega_2}]$. \square

Remark 3.1.1. *Proposition 3.1.1 implies that there is a one-to-one correspondence between (hermitian) one- W -type \mathbb{H} -modules and simple modules of the quotient algebra $\mathbb{C}[W]/J$, where J is the two-sided ideal generated by $\{[p_{\omega_1}, p_{\omega_2}] : \omega_1, \omega_2 \in V\}$. These simple modules were identified for each irreducible group W in [BM3] and those results are used in the sequel. For example, for $W = S_n$, they are parameterized by rectangular Young diagrams of shape $d \times k$, $dk = n$. It may be of independent interest to study the ideal J in more detail.*

3.2. Dirac cohomology.

Proposition 3.2.1. *Let (π, X) be a one- W -type \mathbb{H} -module. Then $H^D(X) = X \otimes S$.*

Proof. Since X is a unitary module, $\ker D \cap \operatorname{im} D = 0$, and $H^D(X) = \ker D_X$. But by Proposition 3.1.1 and the definition of D , D is identically zero on $X \otimes S$. \square

Corollary 3.2.1. *Let X be a one- W -type module and suppose $X|_W = \sigma$. If $\tilde{\sigma}$ is an irreducible \widetilde{W} -representation that occurs in $\sigma \otimes S$, for some spin $C(V)$ -module S , then the central character of X is $\chi^{\tilde{\sigma}}$.*

Proof. This is immediate from Proposition 3.2.1 and Theorem 2.6.1. \square

We use Corollary 3.2.1 to determine the central characters for all one- W -type modules. We refer to the case by case classification in [BM3].

3.3. Exceptional root systems. For the exceptional root systems, we use the computer algebra system GAP/chevie together with the character tables for \widetilde{W} -representations [Mo] to decompose the tensor products $\sigma \otimes S$, where σ is a W -type which can be extended to a one- W -type representation (the explicit list is in [BM3]), and S a spin module. The results are summarized in the tables below. The notation for W -representations is as in [Ca], while the notation for \widetilde{W} -representations is as in [Mo].

When the root system admits unequal parameters, we write k_s and k_ℓ for the value of the parameters of the short and long roots, respectively. In the tables for F_4 and G_2 , ω_i , $i = 1, 4$ and $i = 1, 2$, respectively, denote the fundamental weights in V .

TABLE 1. E_6

W -type σ	$\sigma \otimes S$	$G \backslash \mathcal{N}_{\text{sol}}$
$(1, 0), (1, 36)$	8_s	E_6
$(10, 9)$	$40_{ss} + 20_s + 20_{ss}$	$D_4(a_1)$
$(15, 4), (15, 16)$	120_s	$E_6(a_3)$
$(24, 6), (24, 12)$	$120_s + 72_s$	$E_6(a_3)$

3.4. Type A_{n-1} . For A_{n-1} , the only S_n -types that can be extended to a (\ast -hermitian) one- W -type module correspond to partitions of rectangular shape [BM3]. Let $\sigma_{d \times k}$ be the irreducible S_n -representation parameterized by the rectangular partition $\underbrace{(d, d, \dots, d)}_k$, where $dk = n$.

One says that a partition λ of n is strict if all the parts of λ are distinct. One says that λ is even or odd if $n - |\lambda|$ is even or odd, respectively; here $|\lambda|$ denotes

TABLE 2. E_7

W -type σ	$\sigma \otimes S$	$G \backslash \mathcal{N}_{\text{sol}}$
$(1, 0), (1, 63)$	8_s	E_7
$(15, 7), (15, 28)$	120_s	$E_7(a_4)$
$(21, 3), (21, 36)$	168_s	$E_7(a_2)$
$(35, 4), (35, 31)$	280_s	$E_7(a_3)$
$(70, 9), (70, 18)$	560_s	$E_7(a_5)$
$(84, 12), (84, 15)$	$560_s + 112_{ss}$	$E_7(a_5)$
$(105, 6), (105, 15), (105, 12), (105, 21)$	$120_s + 720_s$	$E_7(a_4)$
$(210, 10), (210, 21)$	$112_{ss} + 2 \cdot 560_s + 448_s$	$E_7(a_5)$

TABLE 3. E_8

W -type σ	$\sigma \otimes S$	$G \backslash \mathcal{N}_{\text{sol}}$
$(1, 0), (1, 120)$	16_s	E_8
$(50, 8), (50, 56)$	800_s	$E_8(b_4)$
$(84, 4), (84, 64)$	1344_{ss}	$E_8(a_3)$
$(175, 12), (175, 36)$	2800_{ss}	$E_8(b_6)$
$(525, 12), (525, 36)$	$5600_{sss} + 2800_s$	$E_8(a_5)$
$(700, 16), (700, 28)$	$8400_s + 2800_{ss}$	$E_8(b_6)$
$(972, 12), (972, 32)$	$9072_s + 6480_s$	$E_8(a_6)$
$(168, 24)$	$1120_s + 224_s + 1344_s$	$E_8(a_7)$
$(420, 20)$	$1120_s + 2016_{ss} + 2016_{sss} + 224_s + 1344_s$	$E_8(a_7)$
$(448, 9)$	7168_s	$E_8(b_5)$

TABLE 4. F_4

W -type σ	$\sigma \otimes S$	$G \backslash \mathcal{N}_{\text{sol}}$	$\chi^{\tilde{\sigma}}, k \neq 1$
$(1, 0), (1, 24)$	4_s	F_4	$k_\ell \omega_1 + k_\ell \omega_2 + k_s \omega_3 + k_s \omega_4$
$(1, 12)', (1, 12)''$	4_{ss}	$F_4(a_3)$	$k_\ell \omega_1 + k_\ell \omega_2 + (-2k_\ell + k_s) \omega_3 + k_s \omega_4$
$(2, 4)'', (2, 16)'$	8_{sss}	$F_4(a_1)$	$k_\ell \omega_1 + k_\ell \omega_2 + (-k_\ell + k_s) \omega_3 + k_s \omega_4$
$(2, 4)', (2, 16)''$	8_{ssss}	$F_4(a_2)$	$k_\ell \omega_1 + k_\ell \omega_2 + (-2k_\ell + k_s) \omega_3 + (3k_\ell - k_s) \omega_4$
$(4, 8)$	$8_s + 8_{ss}$	$F_4(a_3)$	$k_\ell \omega_2 + (-k_\ell + k_s) \omega_4$
$(4, 7), (4, 7)'$	$12_{ss} + 4_{ss}$	$F_4(a_3)$	no
$(6, 6)'$	$12_{ss} + 8_s + 4_{ss}$	$F_4(a_3)$	no
$(8, 3)', (8, 9)''$	$8_{ssss} + 24_s$	$F_4(a_2)$	no

TABLE 5. G_2

W -type σ	$\sigma \otimes S$	$G \backslash \mathcal{N}_{\text{sol}}$	$\chi^{\tilde{\sigma}}, k \neq 1$
$(1, 0), (1, 6)$	2_s	G_2	$k_\ell \omega_1 + k_s \omega_2$
$(1, 3)', (1, 3)''$	2_{ss}	$G_2(a_1)$	$k_\ell \omega_1 + (-k_\ell + k_s) \omega_2$
$(2, 2)$	$2_{ss} + 2_{sss}$	$G_2(a_1)$	no

the number of parts in λ . The classification of irreducible \tilde{S}_n -representations goes back to Schur. To every strict partition λ of n , one constructs one irreducible \tilde{S}_n -representation $\tilde{\sigma}_\lambda$, when λ is even, and two irreducible \tilde{S}_n -representations $\tilde{\sigma}_\lambda^\pm$

when λ is odd. Moreover, $\tilde{\sigma}_\lambda^+ \cong \tilde{\sigma}_\lambda^- \otimes \text{sgn}$. These representations are pairwise nonisomorphic and exhaust the dual of \tilde{S}_n .

As explained in [BC2, Lemma 3.6.2], the tensor product rules from [St] imply that

$$\sigma_{d \times k} \otimes S^\epsilon = \begin{cases} 2^{(k-1)/2}(\tilde{\sigma}_{\text{hook}(d \times k)}^+ + \tilde{\sigma}_{\text{hook}(d \times k)}^-), & k \text{ odd, } d \text{ even} \\ 2^{[k/2]} \tilde{\sigma}_{\text{hook}(d \times k)}^\epsilon, & \text{otherwise.} \end{cases} \quad (3.4.1)$$

Here $\text{hook}(d \times k) = (d + k - 1, d + k - 3, \dots, |d - k| + 1)$, and the symbol ϵ stands for $+$ or $-$ if there exist two associate representations of \tilde{S}_n or “empty” if there is only one.

Finally, the central character of the one- W -type representation supported on $\sigma_{d \times k}$ is one half the middle element of the nilpotent orbit given in Jordan form by the strict partition $\text{hook}(d \times k)$.

3.5. Classical types. We treat the case of the graded Hecke algebra of type B_n with parameters k_ℓ (on the long roots) and k_s (on the short roots). There is an obvious isomorphism with the graded Hecke algebra of type C_n by changing the parameter function appropriately. Also, as is well known, the graded Hecke algebra of type B_n with $k_s = 0$ is isomorphic with the graded Hecke algebra of type D_n extended by $\mathbb{Z}/2\mathbb{Z}$.

Set $\delta = 2k_s/k_\ell$. By [BM3, Proposition 3.24, Theorem 3.28], the $W(B_n)$ -representations that can be extended to a one- W -type module are:

- (T1) $\lambda_L = 0$ or $\lambda_R = 0$ (arbitrary parameters k);
- (T2) $\lambda_L = d_1 \times m_1$ and $\lambda_R = d_2 \times m_2$ when $m_1 - d_1 = m_2 - d_2 + \delta$.

The classification of irreducible $\widetilde{W(B_n)}$ -representations was obtained in [Re]: for every partition λ of n , the representations

$$(\lambda \times 0) \otimes S, \text{ when } n \text{ is even, } (\lambda \times 0) \otimes S^\pm, \text{ when } n \text{ is odd,} \quad (3.5.1)$$

are irreducible, pairwise nonisomorphic, and exhaust the dual of $\widetilde{W(B_n)}$.

For one- W -type modules of type (T1), i.e., the form $\lambda_L \times 0$ or $0 \times \lambda_R$, it is thus clear that the Dirac cohomology equals

$$(\lambda_L \times 0) \otimes S^\epsilon, \quad (\lambda_R^t \times 0) \otimes S^{-\epsilon}, \quad (3.5.2)$$

respectively. In particular, the central character of the one- W -type module is obtained by the combinatorial rule from [Ci] explained at the end of section 2.6.

Thus, it remains to treat the case of one- W -type modules of type (T2). To determine the central character, it is sufficient, in light of Corollary 3.2.1, to find one partition λ on n such that

$$\text{Hom}_{\widetilde{W}}[(\lambda \times 0) \otimes S, (\lambda_L \times \lambda_R) \otimes S] \neq 0, \text{ equivalently, } \text{Hom}_W[(\lambda \times 0) \otimes (S \otimes S^*), (\lambda_L \times \lambda_R)] \neq 0.$$

Since $S \otimes S \cong \wedge V_{\mathbb{C}}$, when n is even, $S^+ \otimes S^+ \cong \wedge^{\text{even}} V_{\mathbb{C}}$ and $S^+ \otimes S^- \cong \wedge^{\text{odd}} V_{\mathbb{C}}$, when n is odd, it is sufficient to find λ and s such that

$$\text{Hom}_W[(\lambda \times 0) \otimes \wedge^s V_{\mathbb{C}}, \lambda_L \times \lambda_R] \neq 0. \quad (3.5.3)$$

Let s equal the sum of parts of the partition λ_R . Using Clifford theory for the semidirect product $W(B_n) = S_n \ltimes (\mathbb{Z}/2\mathbb{Z})^n$, we see that

$$\lambda_L \times \lambda_R = \text{Ind}_{S_{n-s} \times S_s \times (\mathbb{Z}/2\mathbb{Z})^{n-s} \times (\mathbb{Z}/2\mathbb{Z})^s}^{S_n} (\lambda_L \boxtimes \lambda_R \boxtimes \text{triv}^{n-s} \boxtimes \text{sgn}^s). \quad (3.5.4)$$

On the other hand, in bipartition notation, $\wedge^s V_{\mathbb{C}} = (n-s) \times (1^s)$, and

$$(\lambda \times 0) \otimes ((n-s) \times (1^s)) = \text{Ind}_{S_{n-s} \times S_s \times (\mathbb{Z}/2\mathbb{Z})^{n-s} \times (\mathbb{Z}/2\mathbb{Z})^s}^{S_n} ((\lambda|_{S_{n-s} \times S_s} \otimes (\text{triv}_{S_{n-s}} \boxtimes \text{sgn}_{S_s})) \boxtimes \text{triv}^{n-s} \boxtimes \text{sgn}^s). \quad (3.5.5)$$

Comparing (3.5.4) and (3.5.5), we obtain:

Lemma 3.5.1.

$$(\lambda_L \times \lambda_R) \otimes S^\epsilon = \sum_{\lambda} c_{\lambda_L, \lambda_R}^{\lambda} (\lambda \times 0) \otimes S^{\epsilon'}, \quad (3.5.6)$$

where $c_{\lambda_L, \lambda_R}^{\lambda} = \dim \text{Hom}_{S_n}[\lambda, \text{Ind}_{S_{n-s} \times S_s}^{S_n}(\lambda_L \boxtimes \lambda_R^t)]$ is the Littlewood-Richardson coefficient, s is the size of λ_R . Here ϵ and ϵ' are “empty” if n is even, and when n is odd, $\epsilon' = \epsilon$ if s is odd and $\epsilon' = -\epsilon$ if s is even.

Every partition λ that appears in the right hand side of (3.5.6) determines the central character of the one- W -type representation.

Since in our case (T2), λ_L and λ_R are rectangular partitions, the Littlewood-Richardson coefficients can be explicitly described; see [Ok]. Recall that $\lambda_L = d_1 \times m_1$ and $\lambda_R = d_2 \times m_2$. Then $\lambda_R^t = m_2 \times d_2$. Without loss of generality suppose that $m_1 \geq d_2$. Then a partition λ occurs in (3.5.6) if and only if λ has length at most $m_1 + d_2$, and if $\lambda = (\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_{m_1+d_2} \geq 0)$, then

- (i) $\lambda_j + \lambda_{m_1+d_2-j+1} = d_1 + m_2$, for all $j = 1, d_2$;
- (ii) $\lambda_{d_2} \geq \max(d_1, m_2)$;
- (iii) $\lambda_{d_2+1} = \lambda_{d_2+2} = \dots = \lambda_{m_1} = d_1$;

moreover, for all these λ , the coefficient $c_{\lambda_L, \lambda_R}^{\lambda} = 1$. For example, if λ_v and λ_h are obtained by “gluing” the two rectangular shapes λ_L and λ_R^t vertically and horizontally, respectively, then λ_v and λ_h both appear in (3.5.6). Consequently, any one determines the central character by the procedure outlined at the end of section 2.6. It is easy to check that they give the same central character if and only if the condition $m_1 - d_1 = m_2 - d_2 + \delta$ from (T2) is satisfied. In fact, by Corollary 3.2.1, we know that every λ satisfying the rules (i)-(iii) above must yield the same central character!

APPENDIX A. TENSORING WITH THE REFLECTION REPRESENTATION

Let (σ, X) be an irreducible W -representation. One may also ask if it is possible to extend σ to a simple \mathbb{H} -module, not necessarily $*$ -hermitian. Suppose there exists such a module (π, X) . Then one can consider the vector subspace

$$Y = \{\pi(\tilde{\omega})x : \omega \in V_{\mathbb{C}}, x \in X\} \subset X.$$

Using the commutation relation (2.6.2), one sees immediately that Y is W -stable. If $Y = 0$, we are in the same setting as in Proposition 3.1.1, and thus (π, X) is a (hermitian) one- W -type module.

Suppose now that $Y \neq 0$. Since (σ, X) is irreducible, it follows that $Y = X$. Moreover, there is a natural surjective map

$$\tau : X \otimes V_{\mathbb{C}} \rightarrow Y, \quad \tau(x \otimes \omega) = \pi(\tilde{\omega})x, \quad (A.0.7)$$

which is W -equivariant with respect to the reflection representation action on $V_{\mathbb{C}}$:

$$\pi(w)\tau(x \otimes \omega) = \pi(w)\pi(\tilde{\omega})x = \pi(\widetilde{w(\omega)})\sigma(w)x = \tau(\sigma(w)x \otimes w(\omega)) = \tau((\sigma \otimes \text{refl})(w)(x \otimes \omega)).$$

By the assumptions, this map is nonzero, which implies the following lemma.

Lemma A.0.2. *A necessary condition for a W -type σ to extend to a non-hermitian \mathbb{H} -module is that $\text{Hom}_W[\sigma \otimes \text{refl}, \sigma] \neq 0$.*

We now list the irreducible W -representations σ with this property. Recall that in type A_{n-1} , the S_n -types are parameterized by partitions of n .

The irreducible $W(B_n)$ -representations are parameterized by bipartitions (λ_L, λ_R) . Denote the corresponding representation by $\lambda_L \times \lambda_R$; see [Ca]. Our convention is that $n \times 0$ is the trivial representation, the reflection representation is $(n-1) \times 1$, and 0×1^n is the sign representation.

When $\lambda_L \neq \lambda_R$, the restriction $\lambda_L \times \lambda_R|_{W(D_n)} \cong \lambda_R \times \lambda_L|_{W(D_n)}$ is irreducible. If $\lambda_L = \lambda_R$, then the restriction splits into two nonisomorphic irreducible $W(D_n)$ -representations, $(\lambda_L \times \lambda_L)^\pm$.

For exceptional Weyl groups, we use the notation of [Ca] for W -types.

Proposition A.0.1. *Suppose the root system is irreducible.*

- (1) *If w_0 is central in W , then $\text{Hom}_W[\sigma \otimes \text{refl}, \sigma] = 0$ for every W -type σ .*
- (2) *In type A_{n-1} , $\dim \text{Hom}_W[\sigma_\lambda \otimes \text{refl}, \sigma_\lambda] = d - 1$, where d is the number of distinct parts in the partition λ . In particular, $\text{Hom}_W[\sigma_\lambda \otimes \text{refl}, \sigma_\lambda] = 0$ if and only if λ is a partition of rectangular shape.*
- (3) *In type D_n , n odd, $\dim \text{Hom}_W[(\lambda_L \times \lambda_R) \otimes \text{refl}, \lambda_L \times \lambda_R] = 1$ if and only if λ_R (viewed as a Young diagram) is obtained from λ_L by removing one box; otherwise $\text{Hom}_W[(\lambda_L \times \lambda_R) \otimes \text{refl}, \lambda_L \times \lambda_R] = 0$.*
- (4) *In type E_6 , $\dim \text{Hom}_W[\sigma \otimes \text{refl}, \sigma] = 1$ if and only if σ is one of the representations:*

$$(20, 2), (20, 20), (60, 5), (60, 8), (60, 11), (64, 4), (64, 13), (81, 6), (81, 10), (90, 8);$$

$$\text{otherwise } \text{Hom}_W[\sigma \otimes \text{refl}, \sigma] = 0.$$

Proof. (1) When w_0 is central, it acts by a scalar multiple of the identity in each irreducible W -representation. For refl , this scalar is -1 . (More generally, for an irreducible W -representation σ the scalar is $(-1)^d$, where d is the lowest harmonic degree of σ .) The claim follows.

It is worth recalling that for type B_n , we also have the following known rule for tensoring with the reflection representation:

$$(\lambda_L \times \lambda_R) \otimes (n-1) \times 1 = \sum_{(\lambda'_L, \lambda'_R)} \lambda'_L \times \lambda'_R, \quad (\text{A.0.8})$$

where (λ'_L, λ'_R) are all possible bipartitions obtained from (λ_L, λ_R) by removing a box from one partition (diagram) and adding it to the other.

- (2) This is an easy application of the Littlewood-Richardson rule. More precisely,

$$\sigma_\lambda \otimes \text{refl} + \sigma_\lambda = \sigma_\lambda \otimes \text{Ind}_{S_{n-1}}^{S_n}(\text{triv}) = \text{Ind}_{S_{n-1}}^{S_n}(\sigma_\lambda|_{S_{n-1}}). \quad (\text{A.0.9})$$

- (3) This follows immediately from (A.0.8) and the restriction rule from $W(B_n)$ to $W(D_n)$.

- (4) We verified the statement directly using GAP 3.4.4 and the package 'chevie'.

□

Remark A.0.1. *In type A_{n-1} , it is well known that every S_n -type can be lifted to a simple \mathbb{H} -module. This is a consequence of the existence of a surjective \mathbb{C} -algebra homomorphism $\mathbb{H} \rightarrow \mathbb{C}[S_n]$, see for example [BC2, Lemma 3.2.1].*

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