



Analytical approaches to the modelling of multi-row turbine arrays

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I dedicate this thesis to Kevin Juniper, my Dad; your unending love and support
has given me the opportunity to pursue what I love.

With all my love,

Marcus

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Abstract

The increasing recognition of wind turbines as an effective source of renewable power generation that can deliver a net zero carbon future, has led to the construction of larger turbine arrays. Predicting the performance of large turbine arrays requires consideration of many physical factors, such as array geometry, turbine operation, inflow conditions and turbulent wake mixing. Due to the large parameter space that an array may be optimised over, low-order models with low computational cost are often employed. This thesis is concerned with the extension of one of those low-order models; the hybrid inviscid-viscous model of multi-row turbine arrays.

The first extension presented is to the inviscid portion of the model by generalising the linear momentum actuator disk theory, such that the system with non-uniform inflow may be analytically solved utilising a single cubic equation. The impact of shear on a single turbine is then explored, followed by an analysis of a multi-row turbine, which indicates possible performance gains due to varying turbine resistance at each row.

The second extension is in the construction of a novel wake mixing model based on the momentum deficit and self-similarity of Reynolds stresses. This self-similar mixing model is then compared with the results of a "uniform" mixing model. It is shown that the predictions made by each model are similar when the mixing rate parameterised in the uniform model is accurate for a given turbine resistance and array configuration.

The final extension presented in the thesis concerns itself with “anisotropic blockage”, in which a turbine is not equidistant from all nearby boundaries or lines of symmetry with other turbines. A range of Reynolds-averaged Navier-Stokes (RANS) simulations are conducted, followed by a new theoretical method of estimating the turbine efficiency based on scale-separation arguments. This method is benchmarked against an existing "isotropic blockage" model and is found to agree better with the RANS simulations.

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Nomenclature

Common Variables

(x, y)	Streamwise and spanwise coordinates
α	Non-dimensional flow speed through turbine
β	Non-dimensional downstream flow speed
Δp	Pressure difference across turbine
μ	Dynamic viscosity
ν	Kinematic viscosity
ν_T	Eddy viscosity
ϕ	Non-dimensional upstream flow speed
ρ	Density
σ	Characteristic wake width
A	Turbine swept area
B	Blockage ratio
B_{eff}	Effective blockage ratio
C_P	Power coefficient
C_T	Thrust coefficient
D	Turbine diameter
k	Turbine resistance coefficient
k^*	Wake growth rate

m	Uniform mixing rate parameter
P	Power generated
p	Pressure
p_1	Pressure upstream of the turbine
p_2	Pressure immediately upstream of the turbine
p_3	Pressure immediately downstream of the turbine
p_4	Pressure downstream of the turbine
R_i	Non-dimensional streamtube area
T	Turbine thrust
t	Time
u	Reference velocity
x	Streamwise coordinate
y	Spanwise coordinate

Superscript

$*$	Local/internal quantity
F	Quantity achieved from fixing resistance at each turbine row
G	Overall/global quantity
V	Quantity achieved from varying resistance at each turbine row

Subscript

∞	Far upstream or undisturbed value
av	Average

<i>Data</i>	Quantity calculated from RANS simulations
<i>l</i>	Largest sub-domain
<i>m</i>	Middle sub-domain
<i>max</i>	Maximum value
<i>opt</i>	Optimal value at which power coefficient is at its maximum
<i>Pred</i>	Quantity calculated from analytical model
<i>s</i>	Smallest sub-domain
<i>sim</i>	Quantity calculated from self-similar mixing model
<i>t</i>	Turbine
<i>total</i>	Total domain
<i>uni</i>	Quantity calculated from uniform mixing model

Chapter 2

$()'$	Fluctuating component
ΔU	Difference in wake and free stream velocity
$\delta(x)$	Characteristic scale
δ_{ij}	Kronecker delta
ϵ	Turbulence dissipation rate
ϵ'	Initial wake width
$\overline{()}$	Time-averaged component
ϕ	Physical property
f	External forces

$\mathbf{g}$ Gravitational acceleration

$\mathbf{s}$ Sinks/sources

$\mathbf{u}(u_1, u_2, u_3)$ Flow velocity

ξ Self-similarity variable

$C(x)$ Maximum velocity deficit

K Turbulence kinetic energy

S_{ij} Strain rate tensor

V Control volume

z Height

Chapter 3

δ Shear significance

r Core flow width

U_f Farm-layer averaged flow speed

U_{in} Inflow velocity

Chapter 4

$\Delta \mathbf{U}^2$ Momentum deficit per unit density

$\Delta p^*(x)$ Normalised pressure difference in wake

λ Turbine array density

$\mathbf{U}(U, V)$ Mean velocity vector

ζ Momentum response parameter

a Half initial wake width

b	Characteristic wake width
C_{f_0}	Bottom friction coefficient
$D(x)$	Maximum momentum deficit
d	Turbine diameter
F	Momentum gain in the wake
L	Lateral turbine spacing
T_i	Turbulence intensity
U_0	Bulk velocity
u_{in}	Velocity at start of viscous mixing zone
u_{out}	Velocity at end of viscous mixing zone
u_T	Average flow speed through turbine
X	Streamwise location of the end of the viscous mixing zone
x_{inv}	Inviscid wake expansion distance
X_i	Streamwise position of the i -th turbine
X_s	Streamwise row spacing

Chapter 5

γ	Inlet size scalar
A	Cross sectional area
H_A	Distance from hub to top of domain
H_B	Hub height
K	Turbine resistance coefficient

k Turbulence kinetic energy

W Domain width

1 | Introduction

*And we now find that it is not only right to strike
while the Iron is hot, but that it is very practicable
to heat it by continual Striking.*

Benjamin Franklin

1.1 Motivation

Over the past three centuries, human society has experienced four great industrial revolutions. The key theme that has driven these revolutions is our ability to produce and consume energy, starting with steam power in the first industrial revolution and progressing to the electricity production that has dominated the rapid sociological and technological advancements of the second (characterised by analogue technology), third (characterised by digital technology) and fourth (characterised by automation) revolutions (Groumpos, 2021). Our dependence on large energy production requires us to be aware of how we obtain our energy, where it comes from and its effect on the world around us. This is especially important as our energy demand is increasing rapidly and is expected to continue to do so; it is estimated to increase by 50% by 2050 when compared to 2019 (Energy Information Administration, 2019).

Due to the harmful environmental and ecological impacts of oil, coal and natural gas (IPCC, 2022) it is becoming increasingly important to fill our energy demand with clean and renewable energy sources such as wind, solar and hydroelectric. From an economic standpoint, it is beneficial for us to transfer to renewable energy sources as they are often much cheaper than their non-renewable counterparts (Macdonald, 2011; Department for Business, Energy & Industrial Strategy, 2016).

The United Kingdom (UK) is well positioned to take advantage of its geographical location in pursuit of net-zero emission targets, and we will explore these advantages in the following sections. The UK has begun to invest in the expansion of renewable energy sources and in 2019 the UK had 36.9% of its electricity generation filled by renewable sources; an increase of 3.9% from 2018. Where wind, both onshore and offshore, in 2019 made up roughly 60% of our renewable energy generation (Spry and Lucking, 2019), this is an increase from 2008 where only around 1.5% of our energy generation was from wind sources (Department for Business, Energy & Industrial Strategy, 2019a). This has been beneficial for reducing the UK's carbon emissions, as it has dropped from producing approximately 650 MtCO_{2e} in 2008 to producing an estimated 450 MtCO_{2e} in 2018 (Department for Business, Energy & Industrial Strategy, 2019b).

As wind takes up such a large portion of our renewable energy production it is important for us to understand how these turbines and, to a larger extent, wind farms work on a physical level in order to best extract energy from the wind and incorporate it into our energy system.

1.2 Origin of the wind

To understand how we may best extract energy from the wind it is important to understand the mechanisms that govern atmospheric flow, the amount of energy available from the wind, and current challenges and advances in atmospheric science.

The full atmospheric system is a highly complex system whose dynamics cannot be described as the analytic solution to a small set of equations. We will, therefore, first outline some of the key large-scale features of the atmosphere. The wind is primarily driven by solar radiation which supplies energy inhomogeneously to the

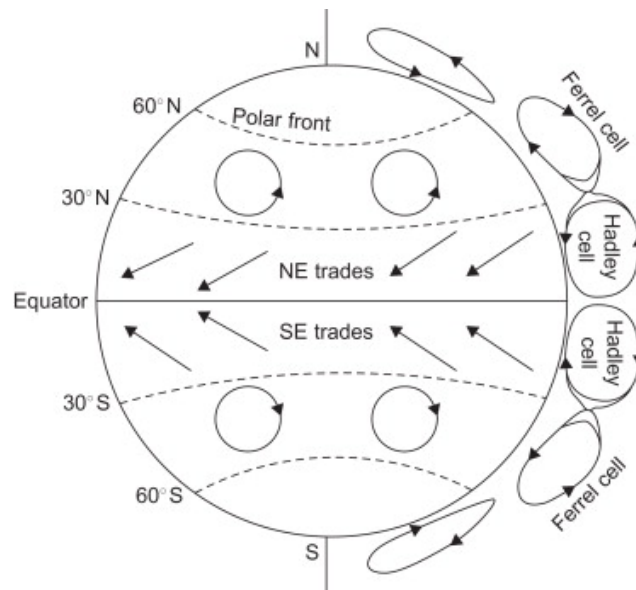


Figure 1.1: Sketch of the general three-celled structure of the atmosphere, reproduced from Cushman-Roisin and Beckers (2011)

Earth's atmosphere, with large variations depending on latitude, and day-night cycles. The atmosphere is often referred to as having a three-celled structure, as depicted in figure 1.1. The first cell, the Hadley Cell, appears due to a rising of warmer air at the equator and a sinking of cool air in the subtropics, with pressure gradients driving the flow back to the equator from the subtropics along the Earth's surface. This circulation due to rising heat and falling cold is referred to as a thermally direct cell. The Hadley cell appears in both hemispheres and is stronger in the winter hemisphere. The mid-latitudes, at around 60°, see a reversed overturning circulation with a rising motion in the high-mid-latitudes and a sinking in the subtropics. This cell is referred to as the Ferrel Cell. The third cell, the polar cell, is another thermally direct cell that appears in high latitudes; however, it is often weak or not present at all (Vallis, 2017).

To add further complexity, all these large-scale circulations are subject to the Coriolis force. Named after Gaspard-Gustave de Coriolis (Coriolis, 1835), the Corio-

lis force is an apparent force due to the Earth being a rotating frame of reference. This causes the wind to flow to the right of its induced direction from pressure gradients north of the equator and to the left in the south of the equator.

The potential vorticity gradient that the Coriolis force provides with change in latitude is a vital ingredient to what are perhaps the most important large-scale waves in the atmosphere: Rossby waves. When a fluid parcel is displaced it conserves its potential vorticity, and so the relative vorticity changes. This relative vorticity creates a velocity field which will displace neighbouring fluid parcels, repeating the process above (Holton, 2004).

A key climate feature that exists as a result of the three-celled structure and the Coriolis force are jet streams (depicted in figure 1.2), whose meandering motion is described as Rossby waves. Focusing on the polar jet that passes over the United Kingdom, this jet occurs where the cold dense air to the north meets the warm southerly air at the boundary between the Ferrel and polar cell. The large pressure gradient caused by the thermal variations in this transition zone causes wind; this wind is then deflected by the Coriolis force and travels along the boundary between the cells. This flow along the boundary of the cells is strengthened and maintained developing to a strong eastwardly jet. There are many ways to conceptualise the same mechanism that strengthens and maintains the jet; one of which is that the Rossby waves excited in the region can be proven to generate a momentum flux towards the source region of the waves. The convergence of momentum in this region drives a net eastwards flow (Vallis, 2017).

The UK experiences relatively high average wind speeds at 150m above ground level when compared to other European nations as seen in table 1.1, moreover, the coastal waters that surround the UK may provide even faster wind, as depicted in figure 1.3, that it can take advantage of for renewable energy extraction and

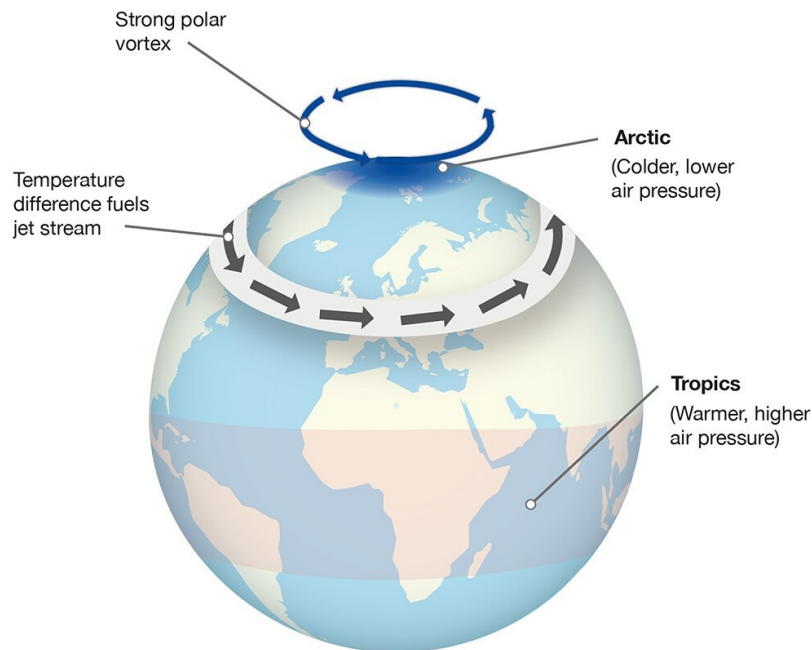


Figure 1.2: Depiction of a jet stream over the UK. Reproduced from Screen (2021).

fulfilling its net-zero obligations. These comparatively high wind speeds are due to the lack of landmass as wind approaches the UK from the west combined with the enhanced flow speed from the jetstream.

1.2.1 Capacity of the wind

Physics-based first-order estimates and numerical simulation models suggest that the available rate of kinetic energy generation supplied to the wind per meter squared of Earth's surface area is around 1.6 W m^{-2} up to around $2.1\text{-}2.5 \text{ W m}^{-2}$, respectively (Kleidon, 2021). This places the global available kinetic energy generation rate in the atmosphere at $800\text{-}1250 \text{ TW}$. This, however, is not the potential for wind turbines' electricity generation, as the basic limit of kinetic energy that a single turbine may extract from the surrounding fluid is approximately 59.25% (the Lanchester-Betz-Joukowski limit, discussed in Chapter 2). Currently

Country	Average Wind Speed (m/s)
United Kingdom	9.59
Denmark	9.55
Netherlands	9.12
Belgium	8.48
Norway	7.94
Germany	7.82
France	7.45
Portugal	6.62
Spain	6.33

Table 1.1: Mean onshore wind speed of various European countries (Davis et al., 2023).

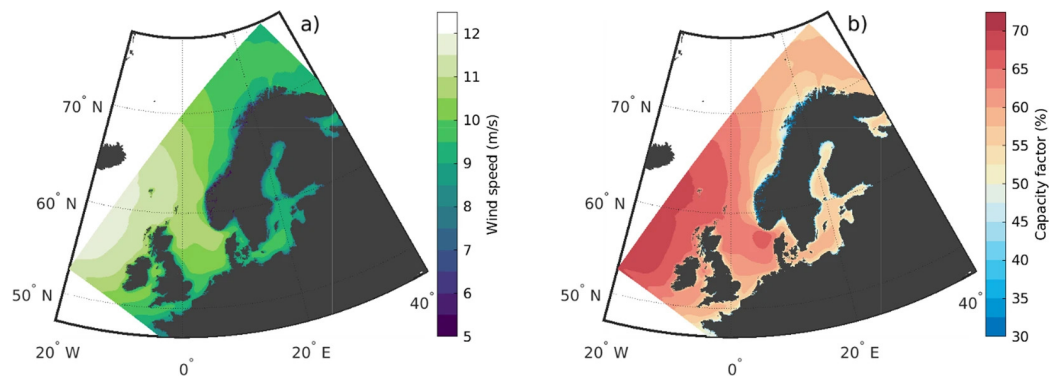


Figure 1.3: Climatology for the years 1996-2019 of wind speed at 150m above mean average sea level (a) and the corresponding capacity factor for the IEA-15-240-RWT reference turbine (b). Reproduced from Solbrekke and Sorteberg (2022)

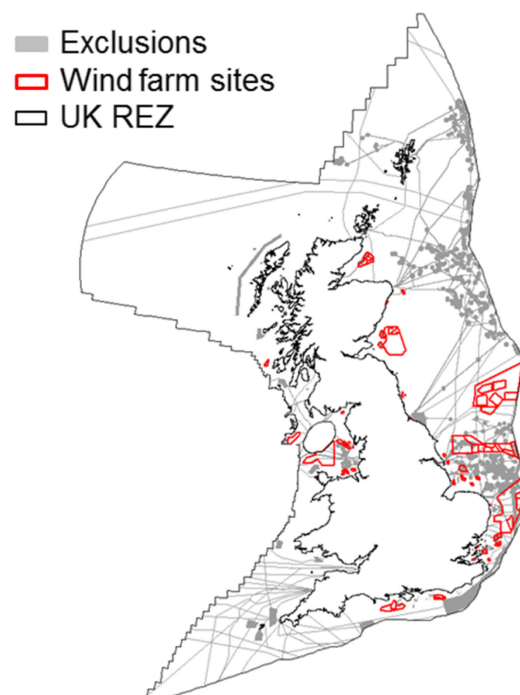


Figure 1.4: The UK Renewable Energy Zone along with operational wind farm sites (up until 2016) and excluded use zones. Reproduced from Cavazzi and Dutton (2016).

installed turbines operate at an efficiency of 40-50% (Carrillo et al., 2013), which leaves the current potential global generation rate of kinetic energy at 400-625 TW. If we wish to exceed or maximise the performance of current and future wind energy projects within this limit, it is vital for us to understand how leveraging local blockage effects (discussed in section 2.2.4), farm operational optimisation (discussed in chapter 4), and interaction with the atmospheric system (discussed in 2.4), impact a wind farm, to name a few key physical features explored in this thesis. For the long-term adaptation of large wind farm instalments, it will also be useful to understand the impact of the large-scale extraction of kinetic energy from the troposphere.

According to Cavazzi and Dutton (2016), the UK has a potential offshore wind capacity of between 2900 and 3000 GW, which accounts for 70% of the country's Renewable Energy Zone (shown in figure 1.4); the Renewable Energy Zone is the area of the sea that the UK has exclusive rights to utilise for energy production under section 84 of the Energy Act 2004. This presents a great opportunity for the UK to transition away from non-renewable energy sources. Despite this, the UK in 2023 has only installed 13.8 GW of its offshore wind energy potential (Pick, 2023). To best scale up our wind energy industry it is important to better understand the dynamics of large wind turbine arrays and how best to optimise them, including the selection of the geographic location of the farm.

1.2.2 Prediction and predictability of the wind

Understanding the optimal geographic placement of a large turbine array requires a deep understanding of the complex atmospheric system, in order to accurately model and predict the availability of wind resource. There have been great advances in both our theoretical and computational capabilities to model and understand the climate system in the past century, starting with Lewis Fry Richardson's

work at the Met Office in the 1920s to the first numerical weather prediction model of Jule Charney in 1950 (Lynch, 2008).

When attempting to create long-term forecasts of future weather patterns to evaluate the future performance of wind turbine arrays, we are limited by the chaotic dynamics that govern the climate system. In a deterministic autonomous dynamical system $\phi : R \times X \rightarrow X$, the impact of an error $\epsilon = \phi(t, x + \epsilon_0) - \phi(t, x)$ can be fully described by the initial error ϵ_0 , the lead time t , and the initial state of the system x , where the lead time is the time since the error has been introduced to the system. A measure of accuracy S (lower S implying greater accuracy) can be used to define an overall measure, $\bar{S}$, over some region D , where $\bar{S} = \int_{x \in D} S dx$. The range of predictability t_k is the point at which $\bar{S}(t, \epsilon_0) = \alpha$, for a given initial error ϵ_0 , where α is some accuracy threshold.

Lorenz's key works in 1969 (Lorenz, 1969) explored the incompressible two-dimensional Euler equations. Lorenz argued that the predictability horizon for Earth's atmosphere was inherently finite, and t_k cannot be made arbitrarily large by reducing ϵ_0 , unless $\epsilon_0 = 0$. Lorenz's analysis suggests that a typical 'weather' scale model would have an absolute predictability of about 1 day, whereas a planetary scale model would have an absolute predictability of about 10 days (Palmer et al., 2014).

As weather prediction models improved it became clear that Lorenz's predictability estimates were overly pessimistic, implying one or more of his assumptions was incorrect. Recent analysis into the predictability limit suggests a skilful forecast of mid-latitude weather is around 10 days, and reducing initial-condition uncertainty by an order of magnitude serves to extend the deterministic forecast lead times of day-to-day weather forecasts by up to 5 days (Zhang et al., 2019). The impact of large wind farms on our current meteorological projections and the pre-

Time-Scale	Application	Forecast Time Ahead
Very short-term	Grid stability operations	10s
	Voltage regulation actions	
Short-term	Economic load dispatch planning	1-hour
	Load increment or decrement decisions	3-hour
	Power reserve management	5-hour
	Operation security in day-ahead electricity market	6-hour
Medium-term	Generator Online/Offline decisions	24-hour
	Unit commitment decisions	72-hour
	Maintenance scheduling	
Long-term	Wind farm optimal design	30-days
	Restructured electricity markets	1-year to 4-years

Table 1.2: Various optimisation and decision-making opportunities and their associated time scales, reproduced from Santhosh et al. (2020)

dictability of weather is currently not well understood. Recent work, however, has highlighted that large wind farms can impact the local air-sea heat flux, cloud development, and precipitation (Akhtar et al., 2022; Lee et al., 2022).

The longer we are able to predict into the future the better we are able to optimise different aspects of wind farm operation. An example of the variety of optimisation and planning applications for accurately predicting flow at various time-scale is presented in table 1.2.

Although longer more accurate forecasts allow us to better plan, we will still require an understanding of how we optimise the performance of the turbine arrays for the predicted flow conditions. This optimisation problem occurs over a large parameter space which would be infeasible to simulate over with accurate computational fluid dynamics models. For this reason the use of computationally fast low order models are often employed, particularly at the farm scale. These models however have many simplifications that will need to be addressed or validated,

some of which will be tackled in this thesis. However, the farm scale optimisations do not always include the relationship with the atmospheric scale system; approaches to include this relationship will be discussed in the next subsection.

1.3 Wind energy: history and current challenges

Harnessing the wind for our benefit dates back thousands of years, with early vertical axis wind turbines (VAWTs) in use around the Persian-Afghan border starting from around 200 BC (Kaldellis and Zafirakis, 2011). The horizontal axis wind turbines (HAWTs) we are most used to seeing today for energy production were first seen around 1200-1300 AD in the Netherlands and the Mediterranean. These were primarily utilised to mill grains and pump water, famously aiding the Netherlands in reclaiming large amounts of land which resides below sea level (Hoeksema, 2007).

James Blyth, a Scottish professor, designed the first wind turbine used to produce electricity in 1887, which launched early studies into wind energy and aerodynamics. The rapid advances in airfoil design developed during the first world war saw them being applied to wind turbines in the post-war years. Many advances were made in the between-war years, primarily focused on the turbine scale dynamics and design of the turbine and its components. However, in the 1920's French engineers such as Louis Constantin already anticipated the modern wind turbine arrays being used together to provide power to a grid. During the second world war much wind energy research halted, the exception to that is in Denmark. F. L. Smidth produced three-bladed turbines with a diameter of 24m in approximately 1942. Although smaller than the 53.3m diameter two-bladed turbine installed by Smith-Putnam the year before, the F. L. Smidth had great longevity lasting up to 16 years and providing valuable services when fossil-fuels were in short supply.

These machines demonstrated their worth as commercial-scale power generators (Gipe and Möllerström, 2022). This would prime many nations to the notion of wind energy as a reliable alternative to fossil-fuel power generation and spark the Great Wind Revival of the 1980s, a response to the oil embargo that started in 1973 (Gipe and Möllerström, 2023).

1.3.1 Challenges in multi-scale modelling

For much of wind energy research's history, a key focus has been on the advancement of the device and turbine scale physics. However, current challenges are now arising regarding how we consider the larger system and include the mesoscale dynamics that drive the wind and have a great impact on the performance of large turbine arrays. The lack of historic focus on the mesoscale system has been driven by the lack of understanding we have had on the highly complex climate system, touched on in section 1.2, but also due to how wind turbines were utilised historically. Many early electricity-generating wind turbines were deployed to provide power to those who were not able to access the larger electricity grid and were not key parts of the energy mix of a nation's grid (Veers et al., 2019).

There are many challenges in considering the interaction between a turbine array and the larger atmospheric system. Much like the free surface effects observed in tidal systems (discussed in section 2.2.4), a large wind farm may perturb the atmospheric inversion layer above, and in doing so excite gravity waves. These gravity waves cause pressure perturbations which further impact the performance of the array (Allaerts and Meyers, 2018, 2019). Gravity wave excitation is directly linked to the structure of the atmospheric boundary layer. The structure of the atmospheric boundary layer is dependent on various meteorological factors, many of which the presence of a wind farm may impact. Wind farms can impact local meteorology through the changes caused in near-surface wind speed and momen-

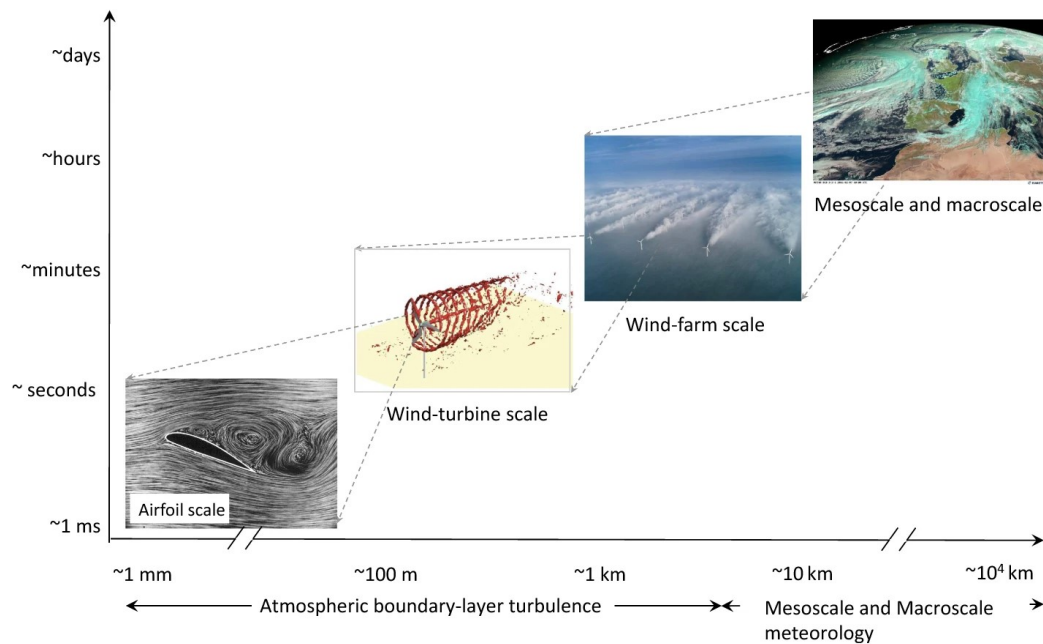


Figure 1.5: Schematic of the various flow scales relevant to wind energy. Reproduced from Porté-Agel et al. (2020)

tum fluxes (Porté-Agel et al., 2020). It has further been shown that extensive wind farm installation over vast areas can have an impact on the atmospheric flow at continental scales (Adams and Keith, 2013). It is clear that many of these flow features have cyclical impacts on each other, and to accurately model a wind farm we would need to account for all of these simultaneously due to this coupling.

A core challenge of modern wind energy science is how to accurately model across the wide range of fluid scales that impact a turbine, as depicted in figure 1.5. Resolving all spatial and temporal scales would be computationally infeasible, so we require other approaches to pair the turbine scale advances discussed in this section with the atmospheric modelling advances touched on in the previous section. An appealing method for incorporating the mesoscale and turbine scale fluid dynamics in a single model is Nishino and Dunstan’s two-scale momentum theory (Nishino and Dunstan, 2020). The theory splits the multi-scale system

into two separate, but coupled, fluid scales; the ‘internal’ turbine/array-scale and ‘external’ farm/atmospheric-scale problems.

The array-scale dynamics are dominated by operating conditions, turbine design and array layout; whereas the large-scale sub-system is dominated by atmospheric conditions, neighbouring wind farms, and terrain. Coupling the large-scale atmospheric system with an efficient array-scale solver will allow for the more accurate treatment of the turbine array optimisation challenges discussed in the previous section. It is the efficient computation of the array-scale sub-system and its optimisation that this thesis is concerned with.

1.4 Aim and objectives

Our understanding of the fluid dynamical phenomena that govern turbine and turbine array performance has expanded greatly over the past decades. A lot of our exploration of the physics that govern these systems has been utilising computational fluid dynamics (CFD) models which can be expensive to run, in both computational time and financial cost. When considering how to maximise the performance of turbine arrays there is a large parameter space to optimise over, which presents a problem when applying many CFD models. To address this, the capabilities of much cheaper computational codes based on analytical approaches are expanded, allowing them to model a wider range of flow problems at greater accuracy. By highlighting turbine configurations or operating controls that are most likely to be optimal, the output of these analytical models can be used to quickly shrink the parameter space that a turbine array is optimised over. The aim of this thesis is to advance the capabilities of an analytical, inviscid-viscous coupled turbine array model; firstly by improving the inviscid modelling, secondly by implementing a more realistic viscous mixing region, and finally by allowing

for the modelling of a turbine in an anisotropically confined flow domain (such as due to the atmospheric inversion layer).

1.4.1 Objective 1: Inviscid flow modelling

Linear Momentum Actuator Disk Theory (LMADT) has been a successful tool in analysing the performance of wind and tidal turbines. To analytically close the problem, LMADT assumes the flow around the turbine to be inviscid. It has been shown that inviscid flow phenomena dominate around the turbine (approximately 1-2 disk diameters downstream) (West and Lele, 2020) with viscous forces dominating in the wake region. LMADT has been extended to incorporate multiple different flow scenarios over the past couple of decades (discussed further in chapter 2); however, it has not been able to efficiently compute generalised flow profiles without additional assumptions. The handling of non-uniform or complex flow profiles is essential when we progress from analysing a single turbine to analysing multi-row arrays due to the flow complexities introduced in the wake of upstream turbines. In this thesis LMADT is extended to allow for the computationally efficient evaluation of a turbine in complex inviscid flow by solving a single cubic equation. This extension is utilised to explore the consequences of common fluid dynamical assumptions on the performance of the turbine.

1.4.2 Objective 2: Viscous wake modelling

The previous goal concentrated on the fluid dynamics immediately surrounding the turbine; however, for a multi-row array of turbines, we must also accurately compute the flow between the turbines, where the flow in the wake of a turbine is dominated by viscous forces.

Previous utilisations of the inviscid-viscous model (Nishino and Draper, 2019)

incorporated a uniform mixing model, where the flow in the wake was assumed to return to the average flow speed at a uniform rate. The inviscid-viscous model with the uniform mixing model has been able to capture trends observed in more accurate fluid dynamics simulations (Ouro and Nishino, 2021). Due to the computational efficiency of the uniform mixing model and the LMADT extension, an initial exploration into the performance of multi-row turbine arrays was conducted. The computational efficiency has allowed the exploration of a very large parameter space allowing for the comparison between the performance of maintaining turbine resistance throughout the array and varying it at each turbine row.

Although the uniform mixing model has been useful in capturing trends in previous utilisations, it does not predict the mixing rate as a function of streamwise distance or accurately describe the variation of wake recovery in the spanwise direction. These limitations have motivated the implementation of a new viscous mixing model to more accurately model the multi-row system. In order to preserve the computational efficiency of the inviscid-viscous model the use of a computationally efficient engineering wake model would be required. The current engineering wake models used in research are incompatible with the extended LMADT model without additional assumptions. Due to these limitations, in this thesis a new viscous mixing model is constructed to better link with the LMADT model. This extension has allowed for a more accurate exploration of multi-row turbine arrays and comparisons to the results obtained with the simplified uniform mixing model.

1.4.3 Objective 3: Anisotropic blockage modelling

The current extensions of the inviscid-viscous model have been for two-dimensional arrays or purely asymmetric systems. To better analyse the performance of turbines in realistic three-dimensional environments further extensions to LMADT

with local blockage (discussed further in chapter 2) are required. In most environments in which turbines operate they experience local blockage anisotropically, i.e., the blockage they experience from the ground, neighbouring turbines or any free surface above differs. No current models fully consider the anisotropic effects of blockage in full three-dimensional systems. Many previous estimations of anisotropically blocked turbines utilise an isotropically blocked model. This isotropic model is reasonably accurate when predicting the maximum power coefficient; however, the predicted resistance coefficient of the turbine at which it reaches the maximum power coefficient is not accurate. Therefore, the differences in the dynamics between the isotropic and anisotropic systems are significant, and require a better understanding for the accurate modelling of the full three-dimensional system.

The third main objective of this thesis is to explore avenues in which a fully three-dimensional inviscid-viscous flow model can be developed. To do this a new method of separating the scales at which blockage impacts the turbine is presented, in which a series of coupled LMADT computations are conducted. The predicted performance is then compared with a previous isotropic analytical method and three-dimensional computational fluid dynamics simulations.

1.5 Thesis layout

The format of the thesis closely follows the three key research objectives. The main part of the thesis (Chapters 3 to 5) begins with an analytic extension of the inviscid LMADT model, followed by the construction of a new engineering wake model, and finally the construction of a new model for anisotropically blocked turbines.

Chapter 2 of the thesis presents an overview of key methods and models required

to understand the following chapters. Firstly, it presents the origins and derivation of the Navier-Stokes equations and inviscid flow modelling through actuator disk theory along with key extensions to confined flow field scenarios. Secondly, a section on viscous modelling starts with the development of the Reynolds-Averaged Navier-Stokes equations along with the $k - \epsilon$ closure model, which is utilised for some of the computational work in the thesis. Finally, discussing analytical wake modelling approaches.

Following the methods section is the first chapter which contributes to turbine modelling research and addresses objective 1. Chapter 3 details the calculations and assumptions required to reduce the complex inviscid flow calculation down to a single cubic equation for a turbine, modelled as an actuator disk, exerting a uniform force on the flow. Utilising the method's capability to compute complex flow profiles an analysis of various inflow profiles and their specific impact on the power predictions of the model are presented, gaining insight into the possible limitations of various assumptions and simplifications employed in research.

Chapter 4 begins with an initial exploration of the multi-row turbine array. This is conducted with the inviscid-viscous model, utilising the improved inviscid model alongside the existing uniform mixing model. A key result of this initial analysis is the impact of varying the resistance of the turbines at each row to optimise the overall performance of the array. Following this an extension to the multi-row analysis by presenting an improved viscous mixing model that preserves the self-similar Gaussian/super-Gaussian shape of the wake (as seen experimentally) and predicts the mixing rate as a function of streamwise distance for the array. The model is a computationally fast engineering wake model which allows for the same investigations over large parameter spaces as presented with the uniform mixing model. A validation of the mixing model is presented against experimental data followed by a repeat analysis of the multi-row turbine arrays conducted with

the uniform mixing model. Comparisons between the two models are presented to validate previous utilisations of the uniform mixing model in inviscid-viscous investigations.

Chapter 5 explores the extension of the model to three dimensions and the limitations of how performance predictions of anisotropically blocked turbines are currently calculated. In this chapter, a new method for predicting the performance of turbines in anisotropically blocked environments is presented and compared against an existing method for isotropic blockage (Garrett and Cummins, 2007). The accuracy of both methods is determined by comparing predictions with three-dimensional Reynolds-Averaged Navier-Stokes (RANS) simulations. Following this, an analytical and numerical exploration of some key assumptions made during the construction of the model is conducted along with a discussion on some of the key limitations of current three-dimensional blocked turbine modelling approaches.

Chapter 6 presents the main conclusions and discussion on the primary contributions of the thesis followed by suggestions for future work.

2 | Theoretical background

If I were again beginning my studies, I would follow the advice of Plato and start with mathematics, a science which proceeds very cautiously and admits nothing as established until it has been rigidly demonstrated.

Galileo Galilei

In this chapter, a summary of the background theories which are required for the reader to fully understand the major thesis chapters is presented. The chapter begins with an overview and derivation of key fluid dynamics equations. Following this, a section focusing on the results of inviscid flow theory and some key analytical results for wind turbine aerodynamics is presented. This is followed by a viscous theory section where a CFD model, later utilised in the thesis, is outlined along with background on viscous mixing theory.

2.1 Navier-Stokes equations

To model and understand fluid flow we often start with the Navier-Stokes equations. Although named after Claude-Louis Navier (1823) and George Gabriel Stokes (1845), many researchers independently derived the equations at similar moments in the first half of the 19th century. The origin of the Navier-Stokes equations is with Euler (1757), who derived the equations of inviscid fluids dynamics.

The Navier-Stokes equations can be derived from a continuity equation. A continuity equation relates the rate of change of some property ϕ over a control volume V to what is gained or lost through the control volume's boundary along with any

gain or loss due to sinks or sources within the control volume. Mathematically this is expressed as

$$\frac{d}{dt} \int_V \phi dV = - \int_S \phi \mathbf{u} \cdot \mathbf{n} dS - \int_V \mathbf{s} dV, \quad (2.1)$$

where $\mathbf{u}$ is the flow velocity, $\mathbf{n}$ is the surface normal and s is any sinks or sources. By applying the divergence theorem ($\iiint_V (\nabla \cdot \mathbf{F}) dV = \iint_S (\mathbf{F} \cdot \mathbf{n}) dS$, where S is the surface which bounds volume V) to the surface integral we obtain

$$\frac{d}{dt} \int_V \phi dV = - \int_V \nabla \cdot (\phi \mathbf{u}) dV - \int_V \mathbf{s} dV. \quad (2.2)$$

Applying Reynolds transport theorem (Reynolds, 1903) (the three-dimensional generalised Leibniz integration rule) to the left-hand side term we then can collect all the volume integrals as

$$\int_V \left(\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) + \mathbf{s} \right) dV = 0. \quad (2.3)$$

This integral is zero for any control volume, for this to be true it is required that the integrand is zero, thus:

$$\frac{\partial \phi}{\partial t} + \nabla \cdot (\phi \mathbf{u}) + \mathbf{s} = 0. \quad (2.4)$$

From here we can consider values of ϕ to obtain our conservation of mass and momentum equations.

To obtain the conservation of mass equation we set our property of interest, ϕ , as mass density, ρ , and setting $s = 0$ (no mass is produced or destroyed in the control volume). We obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0. \quad (2.5)$$

For an incompressible fluid the mass density ρ is constant throughout the fluid in space and time, therefore equation 2.5 reduces down to

$$\nabla \cdot \mathbf{u} = 0, \quad (2.6)$$

our conservation of mass equation.

To obtain the conservation of momentum equation we set our property of interest, ϕ , as the mass flux, $\rho \mathbf{u}$, substituting this into equation 2.4, we obtain

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) = -\mathbf{s}, \quad (2.7)$$

where $\otimes$ is the tensor product. In this case $\mathbf{u} \otimes \mathbf{u}$ is a special case of the tensor product called a dyad. Using the divergence identity of a dyad we obtain

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + (\nabla \cdot \mathbf{u}) \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\mathbf{s}, \quad (2.8)$$

by equation 2.6 the second left hand term is zero which leaves us with

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\mathbf{s}. \quad (2.9)$$

Now all there is to consider is the sink/source term on the right-hand side. The sinks and sources for the momentum conservation equation are often split up into internal forces and external forces, $\mathbf{f}$. The internal forces are further split up into inviscid forces, pressure p , and viscous forces, stresses $\boldsymbol{\tau}$. Where the viscous stresses $\boldsymbol{\tau}$, for Newtonian fluids, is proportional to the local velocity gradient,

$\nabla \mathbf{u}$, via the scalar dynamic viscosity μ . This is often written as

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mu \nabla^2 \mathbf{u} + \mathbf{f}, \quad (2.10)$$

where μ is the dynamic viscosity, a measure of the frictional forces between neighbouring fluid parcels.

2.2 Inviscid theory

Much like how the sink and source term of the momentum conservation equation was split into an inviscid and viscous component, the theory underlying this thesis will also be split into inviscid theory and viscous theory. This section concerns itself with the inviscid theory.

2.2.1 Bernoulli's equation

Daniel Bernoulli's 1738 book *Hydrodynamica* (Bernoulli, 1738) described the phenomena in which pressure and velocity are linked, deducing that as pressure drops, velocity increases. Although stated by Bernoulli, it was first derived as an equation by Euler in 1752.

The Euler equations for fluid dynamics are our Navier-Stokes equations for conservation of mass and conservation of momentum (2.6 and 2.10), with no viscosity, $\mu = 0$. To derive Bernoulli's equation we will start with Euler's equations for incompressible flow and assume the only external force on the fluid is that of gravity, from this we obtain the conservation of momentum as

$$\rho \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p + \mathbf{g}, \quad (2.11)$$

to progress we examine flow that is steady and irrotational

$$\rho(\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla p + \mathbf{g}. \quad (2.12)$$

As the gravity term satisfies the condition that $\nabla \times \mathbf{g} = 0$ (irrotationallity), which by Poincaré's lemma (Poincaré, 1886) allows for the gravity term to be replaced by a scalar potential. Replacing the gravity term with its scalar potential $\mathbf{g} = -\nabla gz$, where z is the height of the fluid parcel, the above can be written as

$$\rho(\mathbf{u} \cdot \nabla \mathbf{u}) = -\nabla(gz + p), \quad (2.13)$$

due to linearity of ∇f . Now from here we may simplify the left hand side term by utilising the gradient vector identity

$$\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{B} \times (\nabla \times \mathbf{A}), \quad (2.14)$$

with $\mathbf{A} = \mathbf{B} = \mathbf{u}$ and $\nabla \times \mathbf{u} = 0$ (irrotational), equation 2.13 reduces to

$$\nabla\left(\frac{\rho}{2}|\mathbf{u}|^2 + p + gz\right) = 0, \quad (2.15)$$

for the gradient to be zero the internal portion must be constant, giving Bernoulli's equation as

$$\frac{\rho}{2}|\mathbf{u}|^2 + p + gz = C, \quad (2.16)$$

where C is some constant that is specific to the flow problem being considered. Although the steady and incompressible Bernoulli's equation was derived here it is possible to derive similar results for unsteady and compressible flow (Batchelor, 2000).

2.2.2 Linear momentum actuator disk theory

A key inviscid theory for the analysis of wind turbine aerodynamics is Linear Momentum Actuator Disk Theory (LMADT). The origins of LMADT begin with work by Rankine (1865) and Froude (1889). Their motivation was to better understand the flow field around the propeller of a ship, to do this they simplified the system by assuming steady incompressible flow and replacing the propeller with a porous disk which applies uniform thrust to the flow. In doing so they removed many of the complicated features of the system such as tip vortices and turbulent mixing.

In the early 20th century three aerodynamic researchers arrived at the same result, by considering the disk as a turbine that applies thrust against the flow direction instead of with it, that no more than $16/27$ of the kinetic energy in the upstream flow with the same cross-sectional area as the disk can be extracted by the disk.

2.2.3 Lanchester-Betz-Joukowski limit

The first of the three researchers to arrive at this result was Lanchester in 1915 (Lanchester, 1915), followed by Betz (1920) and Joukowski (1920) who independently arrived at the same result in 1920. Although all researchers arrived at the final efficiency limit with slightly different approaches, each derivation is a mix of mass, momentum and energy balance applied to the volume of the flow in the streamtube which passes through the actuator disk. However, all found that the induction in the disk is half that of the induction in the wake in the same manner (van Kuik, 2007). We will reproduce the results here that these three researchers derived over a century ago. Ignoring the tower and the nacelle for simplicity, the turbine is first replaced with an actuator disk of equal area, A , to the swept area of the turbine, as depicted in figure 2.1. It is assumed that the pressure at station

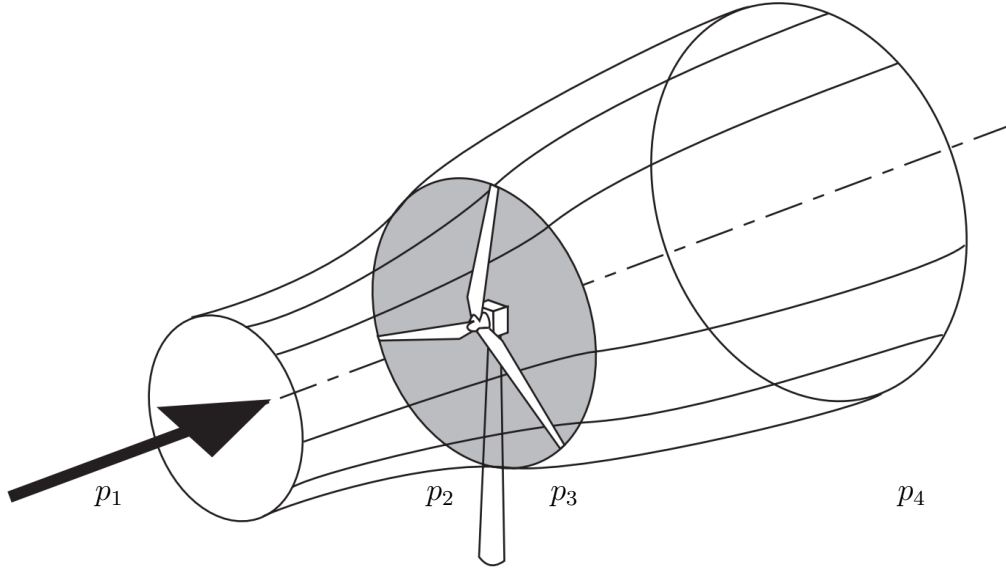


Figure 2.1: Diagram of a streamtube passing through an actuator disk (shaded grey). Reproduced from Wind Energy Handbook (Burton et al., 2011)

1 and 4, p_1 and p_4 , has equalised with atmospheric pressure, p_∞ , and the inflow is uniform, incompressible and steady.

Applying Bernoulli's equation 2.16 to the flow upstream of the turbine we obtain

$$p_1 + \frac{1}{2}\rho u_\infty^2 = p_2 + \frac{1}{2}\rho(\alpha u_\infty)^2, \quad (2.17)$$

likewise downstream of the turbine

$$p_3 + \frac{1}{2}\rho(\alpha u_\infty)^2 = p_4 + \frac{1}{2}\rho(\beta u_\infty)^2, \quad (2.18)$$

where α and β are scalars that represent the velocity at the disk and wake, respectively, as a fraction of the undisturbed flow speed. The pressures, p_2 and p_3 represent the static pressure upstream and downstream of the disk, respectively.

Equating equations 2.17 and 2.18, as $p_1 = p_4 = p_\infty$, we obtain

$$p_2 - p_3 = \frac{1}{2}\rho u_\infty^2(1 - \beta^2), \quad (2.19)$$

equalising forces across the disk gives $p_2 - p_3 = T/A$, where T is the thrust of the disk on the flow.

Now, considering conservation of momentum between stations 1 and 4, we require that

$$T = \rho(R_1 A u_\infty^2 - R_\beta A (\beta u_\infty)^2), \quad (2.20)$$

where $R_1 A$ is the upstream size of the streamtube and $R_\beta A$ is the downstream size of the streamtube. Applying conservation of mass we have that $R_1 A u_\infty = A \alpha u_\infty = R_\beta A \beta u_\infty$. Applying conservation of mass, equation 2.20 simplifies to

$$\frac{T}{A} = \rho \alpha u_\infty^2 (1 - \beta). \quad (2.21)$$

We may now equate 2.19 and 2.21 to obtain

$$\alpha = \frac{1}{2}(1 + \beta). \quad (2.22)$$

To obtain the power extracted by the actuator disk we multiply the thrust by the flow speed through the disk. Utilising equation 2.19 and replacing disk flow speed with equation 2.22 we obtain

$$P = T \alpha u_\infty = \frac{1}{4} \rho u_\infty^3 A (1 - \beta^2)(1 + \beta) = \frac{1}{2} \rho u_\infty^3 A C_P, \quad (2.23)$$

where C_P is the power coefficient. If we differentiate C_P with respect to β we obtain

$$\frac{dC_P}{d\beta} = \frac{1}{2}(-3\beta^2 - 2\beta + 1), \quad (2.24)$$

which has roots at $\beta = 1/3$ and $\beta = -1$. Only the former of these roots is physical for a turbine and corresponds to the maximum of the function $C_P(\beta)$, with $C_P(1/3) = \frac{16}{27} \approx 59.26\%$. This value is known as the Lanchester-Betz-Joukowski limit and states that no more than $\frac{16}{27}$ of the kinetic energy contained within the upstream flow with the same cross sectional area as the disk can be converted into useful work. This limit occurs because when the disk extracts kinetic energy from the upstream flow the resistance caused by the disk simultaneously diverts flow around itself. By increasing disk resistance above the optimal the disk would divert more of the upstream flow than it would gain due to increased thrust. Conversely, by decreasing from the optimal resistance the gains in flow speed through the turbine do not compensate for the reduction in thrust, reducing energy extracted.

This limit is the result of many simplifications to the flow, many of which would cause additional energy loss and further reduce the power available to the turbine. However, not all simplifications result in a loss of power, by considering a turbine's interaction with its local environment we may exceed the Lanchester-Betz-Joukowski limit.

2.2.4 Blockage effect

The above Lanchester-Betz-Joukowski limit was constructed assuming that any boundaries were far away from the actuator disk. Constructed for tidal turbines Garrett and Cummins (2007) considered the performance of a single turbine in a constrained flow, as depicted in figure 2.2.

They derived a new expression for the maximum power coefficient based off of the blockage ratio, B , which is the ratio of the cross-sectional area of the domain and the area of the actuator disk. Their new optimal power coefficient was ob-

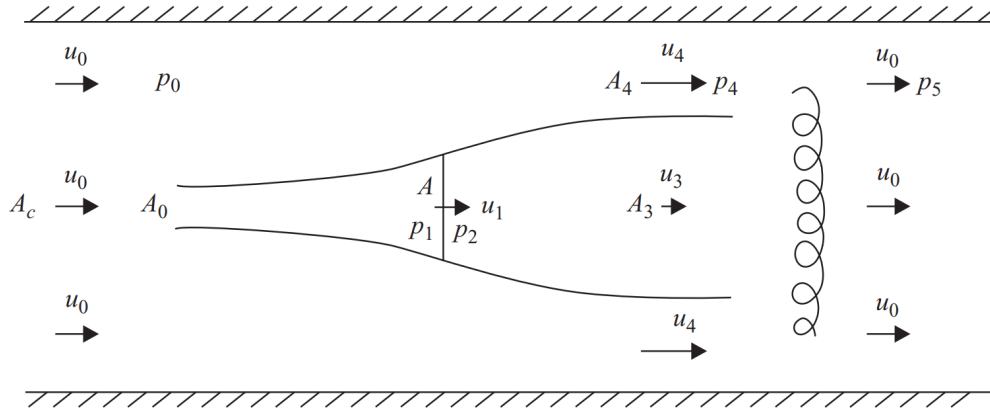


Figure 2.2: Diagram of a single turbine in a confined flow channel. Reproduced from (Garrett and Cummins, 2007)

tained as $C_{P_{max}} = \frac{16}{27} \frac{1}{(1-B)^2}$, with higher levels of blockage allowing for greater energy extraction. This occurs as the constriction of the flow limits its ability to divert around the turbine as the resistance of the disk increases, pushing more flow through the turbine and extracting more energy. In essence, the additional energy extracted by the turbine is from the pressure field and can be observed in the pressure drop from station 1 to 4. The boundaries of the turbine do not have to be the walls of a tidal channel like in the analysis of Garrett and Cummins (2007), it may also be lines of symmetry between neighbouring turbines with matching resistance. There have been multiple investigations into the performance of a finite and infinite row of turbines, often referred to as a turbine fence. This analysis, however, was limited to flow with a rigid free surface. Houlby et al. (2008) and Whelan et al. (2009) extended this theory further by allowing for a deformable free surface where the 'effective' channel blockage can change depending on the Froude number of the flow. These approaches are focused on the local efficiency of the turbines and do not account for the reduction in flow speed in the channel caused by the additional drag the devices produce (Vennell, 2010, 2011).

A further drawback in these models, when applied to a fence of turbines, is the assumption that the fence spans the entire channel; 'full' fences rather than 'partial' fences. Nishino and Willden (2012b, 2013b) explored the performance of a partial tidal fence by introducing the concept of scale separation between the flow around each turbine and the array itself. The model worked well in describing the trends of tidal fence performance limits. It was also shown that by increasing the number of disks in the array there is an alteration in the balance between the increased performance due to 'local' blockage (the blockage experienced by neighbouring turbines), and the flow reduction through the entire array. Much like we saw for a single actuator disk, as the resistance of the disk increases, it diverts more flow into the bypass, the same effect applies to the turbine array. In wind energy this is often referred to as "wind farm blockage" (Bleeg et al., 2018).

2.2.5 Further extensions to LMADT

The inclusion of blockage, deformable free surfaces, and partial fences are not the only key advances to LMADT. There have been many advances in the description of the actuator disk itself to better represent a turbine in different flow conditions. A commonly used extension has been through blade element momentum theory (Glauert, 1926). This approach combines the LMADT with Froude's blade element theory (Froude, 1920) to calculate the local forces on the blade by considering blade geometry and angle of attack along with flow characteristics (Ledoux et al., 2021).

Recent work has also incorporated the induction of yawed actuator disks (Heck et al., 2023). The key motivation for this type of study is to understand how we may deflect the wake of an upstream turbine to avoid interacting with a downstream turbine. These considerations advance our questioning from how to optimise a single turbine (or row of turbines) to the question of how we best optimise

a multi-row array of turbines.

There are two main features to understand the performance of a multi-row array. Firstly, how a turbine operates in a non-uniform flow profile created by being either fully or partially in the wake of another turbine. Secondly, the structure and recovery of a turbine's wake i.e. how the flow recovers back to the upstream flow profile with streamwise distance. For the first feature, there has been work by Draper et al. (2014, 2016) that has explored non-uniform flow through and around a single turbine in bounded flow. Further extensions to this have been presented in chapter 3. For the second feature, there have been multiple studies into the modelling of wind turbine wakes, discussed in section 2.3.2, but they are not often paired with a LMADT approach for modelling the flow through and around the turbine. One approach to explicitly pair the inviscid approach of LMADT with a wake model was that of Nishino and Draper (2019) who developed an inviscid-viscous model. The inviscid-viscous model assumes the flow around the turbine is inviscid and modelled using LMADT whereas in the wake the flow uniformly recovers to the mean flow speed based off of a single mixing rate parameter m . This approach is expanded upon when introduced in section 4.3.1. A full description of this approach, and an extension to that considers the transfer of momentum in the wake, is presented in section 4.4.1.

2.3 Viscous theory

The previous section concerned itself with inviscid theory. In this section, we will consider viscous flow modelling techniques.

2.3.1 Reynolds-Averaged Navier-Stokes equations

Due to the computational cost of solving the Navier-Stokes equations for most fluid flow problems, we often apply simplified, less computationally expensive methods. Here we will derive and outline the Reynolds-Averaged Navier-Stokes (RANS) equations. Although we present this as a less computationally expensive approach, its origins predate the modern computer.

2.3.1.1 Reynolds averaging

First proposed by Osborne Reynolds in 1895 (Reynolds, 1895) the Reynolds decomposition of the Navier-Stokes equations, which result in the RANS equations, allows us to explore turbulent flow systems by separating the mean flow and the fluctuations.

Starting with the Navier-Stokes equations in Einstein notation, with subscripts $i, j = 1, 2, 3$, where $(x_1, x_2, x_3) = (x, y, z)$.

$$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} \right) = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}. \quad (2.25)$$

The continuity equation is

$$\frac{\partial u_i}{\partial x_i} = 0. \quad (2.26)$$

We then apply a Reynolds decomposition on the velocity, in which we decouple the mean flow, $\bar{u}_i$, and the fluctuations, u'_i , such that $u_i = \bar{u}_i + u'_i$. The mean flow may be defined such that it is the time average of u_i over some period T , which is large relative to the characteristic time scale of the turbulent fluctuations, but short

compared to the characteristic time scale of the mean flow, such that

$$\bar{u}_i = \frac{1}{T} \int_{\tau}^{\tau+T} u_i dt. \quad (2.27)$$

Derivations for the final RANS equations can be found in the literature, e.g., (Pope, 2000). The continuity equation becomes

$$\frac{\partial u'_i}{\partial x_i} = \frac{\partial \bar{u}_i}{\partial x_i} = 0, \quad (2.28)$$

and the momentum equation becomes

$$\frac{\partial \bar{u}_i}{\partial t} + \bar{u}_j \frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \overline{u'_i u'_j}}{\partial x_j} = \bar{F}_i - \frac{1}{\rho} \frac{\partial \bar{p}}{\partial x_i} + \nu \frac{\partial^2 \bar{u}_i}{\partial x_j \partial x_j}, \quad (2.29)$$

here, for constant density, the $\overline{u'_i u'_j}$ term is called the Reynolds stress term (although the actual stress is this term multiplied by the density). The issue we have is that we have more unknowns than equations, therefore to solve the system we will have to employ some assumptions about the unknown quantities in the equation. In essence, we need to close this problem.

2.3.1.2 $K - \epsilon$ closure model

Introduced by Launder and Spalding (1974) the $K - \epsilon$ turbulence model is a two-equation model, meaning that two additional transport equations are introduced to represent the turbulent properties of the flow. Specifically, the turbulent kinetic energy K , and its dissipation rate, ϵ . The Boussinesq assumption (Boussinesq, 1877) is the basis for this model, which suggests that the turbulent viscosity is isotropic; meaning that the Reynolds stress is proportional to the mean strain rate

tensor S_{ij} , defined by

$$S_{ij} = \frac{1}{2} \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{1}{3} \frac{\partial u_k}{\partial x_k} \delta_{ij}, \quad (2.30)$$

where δ_{ij} is the Kronecker delta. The Reynolds stress term and mean strain rate follow the relationship

$$-\overline{u'_i u'_j} = 2\nu_T S_{ij} - \frac{2}{3} K \delta_{ij}, \quad (2.31)$$

where ν_T is the eddy viscosity. The eddy viscosity is a measure of the transfer of momentum caused by turbulent eddies. Therefore we can explicitly write the Reynolds stress term to be approximated as

$$-\overline{u'_i u'_j} = \nu_T \left(\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i} \right) - \frac{2}{3} K \delta_{ij}. \quad (2.32)$$

From here we need to model the eddy viscosity and define our additional turbulent quantities and their transport. Eddy viscosity is defined as

$$\nu_T = C_\nu \frac{K^2}{\epsilon}. \quad (2.33)$$

Turbulent kinetic energy (per unit density) is defined as

$$K = \frac{\overline{u'_i u'_i}}{2}. \quad (2.34)$$

The transport equations and definition for eddy viscosity rely on a set of model coefficients, such as C_ν . There is a standard set of coefficients that are often used but these constants may also be further tuned for the specific application. Some advances in the use of RANS for wind turbine applications have also seen the introduction of correction for blade-induced turbulence (Nishino and Willden,

2014) and machine learning closure models for wind turbine wake applications (Steiner et al., 2022).

2.3.2 Analytical wake modelling

The flow recovery in the wake is the main viscous process of interest in turbine aerodynamics. There are many iterative approaches that aim to resolve the full flow field such as Ainslie (1988), RANS (as above or adapted) and Large-eddy simulations. In this section, however, we will focus on so-called analytical wake models which estimate the wake velocity profile at a specific downstream position without the need to explicitly calculate the full flow profile before that downstream position.

2.3.2.1 Top-hat models

When considering the advancements in turbine wake modelling we often start with Jensen (1983), whose early but pioneering wake model is still used in literature to this day. Jensen’s model applies the conservation of mass to a control volume downstream the turbine and links the flow speed from LMADT to the turbine thrust coefficient C_T (Katić et al., 1987). It is assumed the flow speed remains uniform across the wake and that the wake expands linearly with downstream distance x .

Further advancements in wake modelling come from Frandsen et al. (2006) who considered the conservation of mass and momentum for a control volume around the turbine. Their work includes the inviscid expansion directly behind the turbine suggested by LMADT but remains a uniform wake velocity model.

These models are referred to as top-hat wake models due to their similar qualitative shape. In reality, the velocity deficit behind a turbine resembles a self-similar

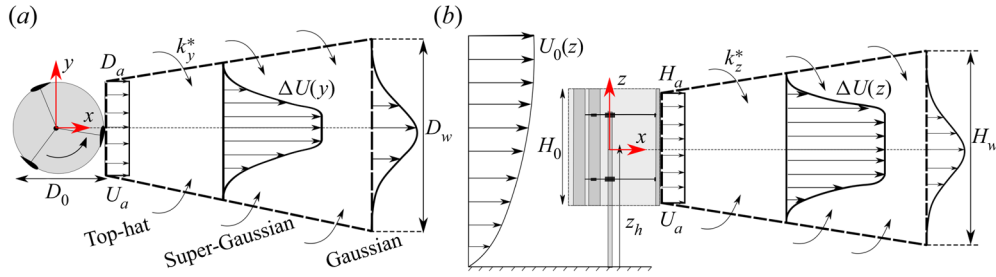


Figure 2.3: Schematic for the Top-hat, Super-Gaussian, and Gaussian velocity deficit shape in the wake of a turbine. Here depicted for VAWT, but identical formulation for HAWTs. Reproduced from Ouro and Lazennec (2021). Note, notation in the figure is inconsistent with that used in the rest of the thesis.

Gaussian flow profile rather than the top-hat shape. The Top-hat, Gaussian and super-Gaussian profiles are all depicted in figure 2.3, and formulations of the Gaussian and super-Gaussian are presented in the next section.

2.3.2.2 Self-similar wake models

The equalisation of two or more streams of fluid with differing flow speeds has been of interest to fluid dynamicists for various applications or theoretical insight. The most common studies of these turbulent free shear flows are jets, wakes and mixing layers (Pope, 2000).

A key early work in flow mixing was that of Görtler (1942) who studied the evolution of plane mixing layers. Plane mixing layers are wakes caused by the mixing of two fluid streams of differing velocities, as depicted in figure 2.4. Görtler, following Prandtl's assumption that turbulent viscosity is constant across the shear layer (Prandtl, 1942), derived self-similar solutions to the boundary-layer equations for a jet and plane mixing layer. Due to self-similarities importance in the study of turbulent flows, we will outline its definition.

Considering a quantity $Q(x, y)$ that depends on two independent variables, a self-similarity variable, ξ , is a variable which, by exploiting the symmetry of the sys-

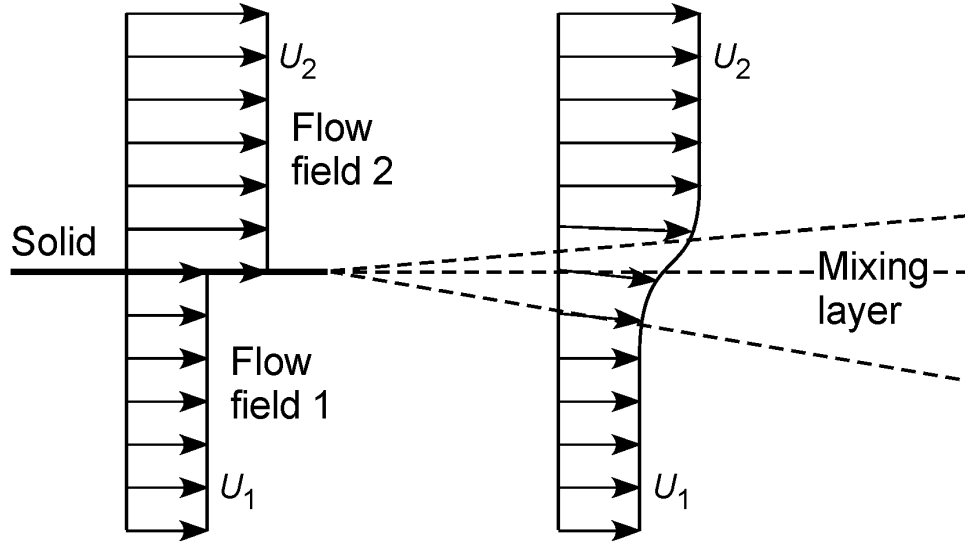


Figure 2.4: Schematic for the development of a shear mixing layer. Reproduced from Niven (2021)

tem, links the variables of the system to reduce the number of independent variables. In this example, defined as

$$\xi \equiv \frac{y}{\delta(x)}, \quad (2.35)$$

such that we can define the scaled dependent variable, Q' as

$$Q'(\xi, x) \equiv \frac{Q(x, y)}{Q_0(x)}, \quad (2.36)$$

where $\delta(x)$ and $Q_0(x)$ are characteristic scales. If the scaled dependent variable, Q' , is independent of x then $Q(x, y)$ is self-similar (Pope, 2000). Gortler's self-similar solution to the plane mixing layer profile was of the form

$$u^* = \frac{u_{av} - u_1}{u_2 - u_1} = \frac{1}{2} \left(1 + \operatorname{erf}\left(\frac{\sigma y}{x}\right) \right), \quad (2.37)$$

where "erf" is the error function ($\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$), u_{av} is the average flow

speed, u_2 is the flow speed of the faster layer, u_1 is the slower layers flow speed, and σ is an empirical mixing rate parameter. His results have been well benchmarked and validated against experimental results (Yoder et al., 2015).

When considering the wake of a turbine, early self-similar approaches were made by Bastankhah and Porté-Agel (2014) who showed through various numerical and experimental data that a self-similar Gaussian distribution can accurately represent the velocity deficit in a turbine's wake. Their normalised velocity at the turbine hub-height, in the y direction, is given as

$$\frac{\Delta U}{U_\infty} = C(x) \exp\left(-\frac{y^2}{2\sigma_y^2}\right), \quad (2.38)$$

where ΔU is the difference between free stream velocity U_∞ and wake velocity, $C(x)$ represents the maximum velocity deficit at downstream position x , and σ_y is the growth rate in the y direction. Again the wake width σ_y is assumed to grow linearly with downstream distance x , so is defined as

$$\frac{\sigma}{D} = k^* \frac{x}{D} + \epsilon', \quad (2.39)$$

where k^* is the wake growth rate and ϵ' is the initial wake width. Many extensions have been made to the Gaussian model, including but not limited to, the inclusion of variation in lateral and vertical wake growth rate (Xie and Archer, 2015), turbine yaw (Bastankhah and Porté-Agel, 2016), and variations due to favourable and adverse pressure gradients (Shamsoddin and Porté-Agel, 2018).

The use of a single Gaussian model for the wake has primarily been utilised for the far wake where the flow closely follows the Gaussian shape. However, in the near wake the flow is not Gaussian in shape. There are multiple approaches for considering the near wake region; one we will showcase here is the super-

Gaussian wake model presented by Blondel and Cathelain (2020). Their model allows for the more accurate consideration of the near wake region by maintaining the near top-hat shape in the near wake, then evolves into the Gaussian in the far wake. The profile is formulated as

$$\frac{\Delta U}{U_\infty} = C(x) \exp\left(-\frac{y^{n_y}}{2\sigma_y^2}\right), \quad (2.40)$$

where the key difference from the Gaussian model is the n_y term which controls the shape of the flow profile, with $n_y = 2$ returning to the Gaussian shape. The analytical expression for n_y suggested by Blondel is of the form

$$n_y = a_y \exp\left(-b_y \frac{x}{D}\right) + c_y, \quad (2.41)$$

where a_y , b_y and c_y are empirical parameters to be tuned.

Many popular models focus on the deficit of velocity in the wake of the turbine; an approach utilising the deficit of momentum in the wake is presented in chapter 4. The approach presented in this thesis aims to provide an alternative method to the super-Gaussian wake model format in modelling the near and far wake of the turbine.

2.3.2.3 Super position methods

The wake models discussed above only consider what happens behind a single turbine. To understand an array of turbines it is important to understand how these wakes interact and evolve together. In general, there are four main approaches to construct the superposition that result from the decision of how to define the velocity deficit and the physical property of interest. The physical property to superimpose is either the velocity deficit or the energy deficit in the wake of the

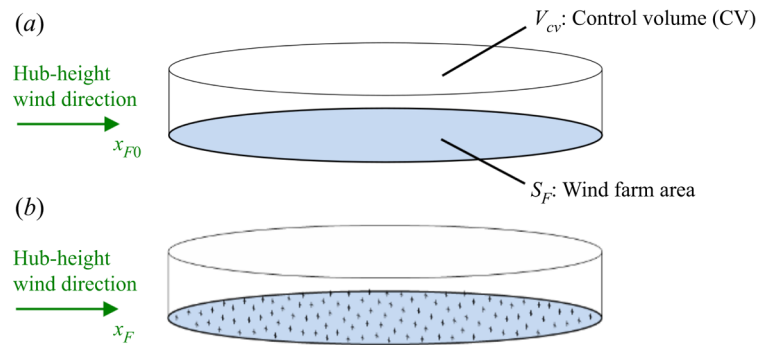


Figure 2.5: Control volume for an entire wind farm site: (a) without turbines and (b) with turbines. Reproduced from Kirby et al. (2023).

turbine. The deficit itself is then defined as the difference between the wake and the incoming boundary layer flow speed or between the wake and the incoming flow speed for that turbine. Katić et al. (1987) and Lissaman (1979) used the deficit compared to the boundary-layer flow, with Katić et al. deciding to examine the energy deficit and Lissaman choosing the velocity deficit. Voutsinas et al. (1990) and Niayifar and Porté-Agel (2016) used the deficit compared to the flow speed the turbine sees with Voutsinas considering the energy deficit and Niayifar and Porté-Agel the velocity deficit. Further details of these superposition methods can be found in Porté-Agel et al. (2020).

2.4 Two-scale momentum theory

A key motivation for the concentrated study on 'internal' wind farm aerodynamics is the two-scale momentum theory developed by Nishino and Dunstan (2020). Two-scale momentum theory starts by considering the conservation of momentum for a control volume with and without a wind farm present, as depicted in figure 2.5. Without a farm present the following equation can be derived for the

streamwise momentum conservation

$$\frac{\partial[\rho U_0]}{\partial t} = X_0 - C_0 - \left[\frac{\partial p_0}{\partial x_{F0}} \right] - \frac{\langle \tau_{w0} \rangle S_F}{V_{CV}}, \quad (2.42)$$

where the 0 subscript refers to the values when turbines are not present, $\langle \cdot \rangle$ represents the average over the farms horizontal surface area, $[\cdot]$ represents the control volume averaged quantity, X represents net streamwise momentum injection through the boundaries of the control volume, C is the streamwise component of the Coriolis force averaged over the the control volume, x_F is the streamwise direction at hub-height, τ_w is the bottom shear stress, S_F is the horizontal surface area of the wind farm control volume, V_{CV} is the volume of the control volume. If turbines are present then the following equation can be obtained

$$\frac{\partial[\rho U]}{\partial t} = X - C - \left[\frac{\partial p}{\partial x_F} \right] - \frac{\langle \tau_w \rangle S_F + \sum_{i=1}^n T_i}{V_{CV}}, \quad (2.43)$$

where T_i is the thrust of the i -th turbine in the farm for a farm consisting of n turbines. Rearranging equations 2.42 and 2.43, then dividing 2.43 by 2.42 we can obtain

$$\frac{\langle \tau_w \rangle S_F + \sum_{i=1}^n T_i}{\langle \tau_{w0} \rangle S_F} = \frac{X - C - \left[\frac{\partial p}{\partial x_F} \right] - \frac{\partial[\rho U]}{\partial t}}{X_0 - C_0 - \left[\frac{\partial p_0}{\partial x_{F0}} \right] - \frac{\partial[\rho U_0]}{\partial t}}. \quad (2.44)$$

We define the right-hand side of the equation as the momentum availability factor, M , which is a measure of the external atmospheric systems ability to maintain momentum available to the farm site in response to the presence of the turbines. Rewriting the left-hand side of equation 2.44 we can obtain

$$C_T^* \frac{\lambda}{C_{f0}} \beta^2 + \beta^\gamma = M, \quad (2.45)$$

where, $C_T^* \equiv \sum_{i=1}^n T_i / \frac{1}{2} \rho U_F^2 n A$ is the 'internal' thrust coefficient, $\lambda \equiv n A / S_F$ is the array density, $C_{f0} \equiv \langle \tau_{w0} \rangle / \frac{1}{2} \rho U_{F0}^2 n A$ is the natural friction coefficient of the surface below the farm, $\beta \equiv U_F / U_{F0}$ is the wind speed reduction factor for the farm (U_F is the average wind speed across the farm-layer height which is typically between 2 and 3 times hub height), and $\gamma \equiv \log_\beta(\langle \tau_w \rangle / \langle \tau_{w0} \rangle)$ is the bottom friction exponent. Note, the β used here is not the same as the β used in the LMADT analysis. Writing the momentum balances in the form of equation 2.45 allows us to separate the 'internal' conditions that impact the farm on the left-hand side and the 'external' conditions on the right-hand side. Using a numerical weather prediction model Patel et al. (2021) demonstrated that in general M varied almost linearly with β and thus can be approximated as

$$M = 1 + \zeta(1 - \beta), \quad (2.46)$$

where ζ is the momentum response factor which can vary from 5 to 60 for a typical large offshore wind farm (Kirby et al., 2023).

3 | Inviscid

*And soften'd sounds along the waters die: Smooth
flow the waves, the zephyrs gently play.*

Alexander Pope

Predicting the performance of large turbine arrays requires an understanding of many physical factors, such as array geometry, turbine operation, inflow conditions, and turbulent wake mixing. Due to the large parameter space that an array may be optimised over, low-order models with low computational cost are often employed. In this chapter, an extension to one of these models, the inviscid-viscous coupled model, for multi-row turbine modelling is presented. Firstly, an extension to the inviscid actuator disc theory is presented by removing the limit on the number of discrete streamtubes computed. The extended model allows for the quantification of the impact of shear in the bypass and core flows separately. In particular, it is shown that averaging a sheared bypass flow profile can result in a substantial over-prediction of the power of a turbine in a laterally bounded flow as the effective blockage of the flow increases.

3.1 Introduction

This segment of work concerns itself with the extension of previous LMADT models of a turbine in confined flow with non-uniform inflow. The interest in the flow confinement or ‘blockage’ effects encompasses two main applications, one being the performance of a turbine in a laboratory environment, and the other being for an array of turbines whose spanwise spacing may alter their performance due to ‘local’ blockage effects (Nishino and Willden, 2012b; Nishino and Draper, 2015). A number of blockage correction methods have been suggested over the years, and a recent experimental study by Ross and Polagye (2020) compared a range of popular methods. The methods found to have the greatest agreement with the experiments were those of Housby et al. (2008) and Barnsley and Wellicome (1990). Note, the report of Barnsley and Wellicome (1990) does not appear to be publicly available; however, it was introduced by Bahaj et al. (2007) whose outline of the method echoes the approach of Garrett and Cummins (2007), placing a 1D actuator disc in the middle of a rectangular channel (to consider additional streamtubes bypassing the disc as well as the ‘core’ streamtube passing through the disc). Barnsley & Wellicome’s method solves a set of conservation equations with a given turbine thrust, with Housby et al.’s extension allowing for a deformable lateral boundary. The merit of closing the LMADT system with turbine thrust (instead of other physical quantities, such as the wake width) is that it can be measured accurately when utilising these models to correct for blockage in laboratory settings. Extensions to the thrust closure method are presented in this chapter to address a wider range of flow features.

The work in this chapter firstly extends LMADT for a single turbine by analytically solving for a generalised number of bypass streamtubes, followed by a further extension for a generalised number of streamtubes passing through the turbine

as well as in the bypass. These extensions aim to expand the current capabilities of LMADT to handle non-uniform inflow profiles, along with variations in resistances across the turbine surface, allowing for a granular exploration of the impact of each of these components on the turbine performance in isolation. Although such an extension to consider non-uniform inflow has already been made by Draper et al. (2016), in this chapter we present a new formulation that leads to analytical solutions for any number of bypass streamtubes. This enables very fast calculations not only for a single turbine but also for a large number of turbines in a multi-row system.

3.2 Extension to LMADT

We consider many rows of spanwise equally spaced turbines as depicted in figure 3.1. The rows are considered to be infinitely wide (or sufficiently wide to ignore the spanwise end effect) and evenly spaced; therefore we can exploit the symmetry in the system to only consider a local flow domain that is laterally bounded as shown in figure 3.2. Note, the local domain here is constructed for the context of a turbine array, therefore y represents horizontal direction. However, y can also be defined as the vertical direction when considering a single turbine in a channel with a rigid lid. The local flow domain is primarily defined by the blockage ratio B , where the cross-sectional area of the domain is A/B , with A as the turbines swept area. Note, for uniform inflow when $B \rightarrow \infty$ the system limits to the Lanchester-Betz-Joukowski case. As the LMADT only considers the local flow domain, it is assumed that the blockage ratio and inflow profile are both known.

The main assumptions employed in this analysis are that the flow is inviscid, incompressible, and steady. The flow is also assumed to be symmetric about the centre of the turbine in 2D applications of the method and axisymmetric in 3D

applications. The actuator itself is modelled to apply a uniform force on the fluid; this will be discussed further in section 3.2.2.

For the 2D case presented in this chapter, A is our reference length defined as the length of the 1D actuator strip and u is our reference velocity (e.g. $u = 10$ m/s). The inflow profile is defined by $N + 1$ segments with velocity $\phi_i u$ and cross-sectional area $R_i A$, where $0 < R_i \leq 1/B$ and $\phi_i > 0$. All velocity and area values are defined as a real positive scalar times u and A , respectively, which act as dimensional parameters. At the downstream position the flow profile is defined by $N + 2$ segments with velocity $\beta_i u$ and cross-sectional area $R_{\beta_i} A$, where $0 < R_{\beta_i} \leq 1/B$ and $\beta_i > 0$. The velocity through the turbine is $\alpha_{av} u$, with $\alpha_{av} > 0$. We obtain an additional streamtube in the downstream as one of the upstream streamtubes is split by the turbine. For example, with one upstream streamtube (uniform inflow) we obtain two downstream streamtubes, i.e. the bypass and the wake. Although ϕ_0 is expressed explicitly in the formulations below all computations will be for normalised flow profiles such that $\phi_0 = 1$. Discrete streamtubes were chosen as a natural continuation of previous works (Draper et al., 2014; Juniper and Nishino, 2020). Although continuous functions may be utilised throughout the mathematics presented in this chapter, the benefits of the discrete method are in the simplicity of implementation for complex flow profiles, particularly ones in which a region of the flow has been averaged (as explored in section 3.4), and that by increasing the number of streamtubes we will limit to the continuous function case. Thus, we get ease of implementation and can match the trends observed in the continuous function formulation when a sufficient number of streamtubes is selected.

The problem can be split up into two cases that will be solved slightly differently. The first case is when the flow passing through the turbine has uniform velocity, as depicted in figure 3.2. The second case is when the flow velocity is non-uniform

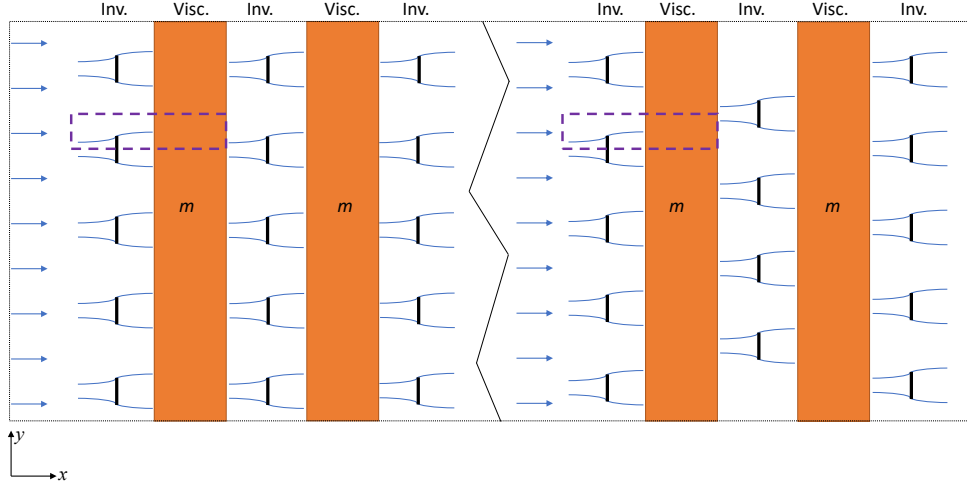


Figure 3.1: Schematic of the flow past three rows of turbines arranged as perfectly aligned (left) and perfectly staggered (right) arrays, divided into inviscid (inv.) and viscous (visc.) flow zones. The rectangular region enclosed by the dashed line corresponds to the local flow domain depicted in figure 2.

across the turbine, as depicted in figure 3.3.

3.2.1 Uniform flow through turbine

For uniform flow through a turbine Bernoulli's equation is first applied to the streamtubes which bypass the turbine. This gives

$$p_1 + \frac{1}{2}\rho u^2 \phi_i^2 = p_4 + \frac{1}{2}\rho u^2 \beta_{i+1}^2, \quad (3.1)$$

for $i \geq 1$, where ρ is the fluid density, p_1 is the static pressure upstream of the turbine, p_2 and p_3 are the static pressures immediately upstream and downstream of the turbine, respectively, and p_4 is the static pressure downstream of the turbine. It is assumed for p_1 and p_4 that the pressure has equalised in the spanwise direction.

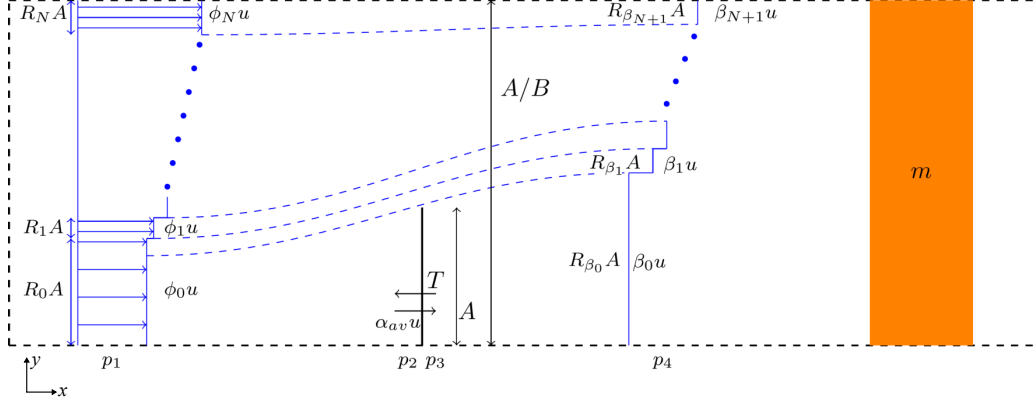


Figure 3.2: Diagram of non-dimensionalised local flow domain with a single streamtube passing through the turbine.

This can be rewritten as

$$p_1 - p_4 = \frac{1}{2} \rho u^2 (\beta_{i+1}^2 - \phi_i^2). \quad (3.2)$$

As the pressure drop $p_1 - p_4$ is constant for all bypass streamtubes we can equate the right hand side of 3.2 for two different bypass streamtubes (indexed i and j) to obtain a relationship between the velocities:

$$\beta_j = \sqrt{\beta_i^2 - \phi_{i-1}^2 + \phi_{j-1}^2}, \quad (3.3)$$

for $i, j \geq 1$. We can fix $i = 1$ so only the downstream bypass flow speed for the streamtube that immediately bypasses the turbine (β_1) is required to know all the other downstream bypass flow speeds (β_2 to β_{N+1}). Furthermore, if the bypass flow speeds are known then all downstream areas of bypass streamtubes are known via the conservation of mass:

$$R_i \phi_i = R_{\beta_{i+1}} \beta_{i+1}, \quad (3.4)$$

for $i \geq 1$. Now considering the momentum balance for the entire domain between the upstream and downstream locations the thrust of the turbine is obtained as

$$\frac{T}{A} = \sum_{i=0}^n R_i \phi_i^2 u^2 \rho - \sum_{i=0}^{n+1} R_{\beta_i} \beta_i^2 u^2 \rho + \frac{1}{B} (p_1 - p_4). \quad (3.5)$$

Using 3.2 with $i = 0$ to remove the pressure terms in the equation:

$$\frac{T}{A} = \left(\sum_{i=0}^n R_i \phi_i^2 - \sum_{i=0}^{n+1} R_{\beta_i} \beta_i^2 + \frac{1}{2B} (\beta_1^2 - \phi_0^2) \right) u^2 \rho. \quad (3.6)$$

Applying the Bernoulli equation upstream and downstream of the disc to obtain a relation between the pressure drop across the turbine and the velocity of the flow

$$(p_1 - p_2) + (p_3 - p_4) = \frac{1}{2} \rho u^2 ((\alpha_{av}^2 - \phi_0^2) + (\beta_0^2 - \alpha_{av}^2)). \quad (3.7)$$

Using the static equilibrium across the turbine the thrust is just the pressure drop across the turbine so from 3.7 we simplify to

$$\frac{T}{A} = p_2 - p_3 = \frac{1}{2} \rho u^2 (\beta_1^2 - \beta_0^2). \quad (3.8)$$

We will set the flow speed u , turbine area A , and density ρ equal to 1 for simplicity in the rest of the calculation. Now equating 3.6 and 3.8 gives

$$\frac{1}{2} (\beta_1^2 - \beta_0^2) = \sum_{i=0}^n R_i \phi_i^2 - \sum_{i=0}^{n+1} R_{\beta_i} \beta_i^2 + \frac{1}{2B} (\beta_1^2 - \phi_0^2). \quad (3.9)$$

From the conservation of mass applied to the streamtube which encompasses the turbine, the following expression is obtained:

$$R_0 \phi_0 = R_{\beta_1} \beta_1 + R_{\beta_0} \beta_0$$

$$\begin{aligned}
&= \beta_1 \left(\frac{1}{B} - \sum_{i=2}^{n+1} R_{\beta_i} - R_{\beta_0} \right) + \alpha_{av} \\
&= \beta_1 \left(\frac{1}{B} - \sum_{i=2}^{n+1} R_{\beta_i} - \frac{\alpha_{av}}{\beta_0} \right) + \alpha_{av}.
\end{aligned} \tag{3.10}$$

This can be rewritten in terms of α_{av} as

$$\alpha_{av} = \frac{\beta_0 \left(R_0 \phi_0 - \beta_1 \left(\frac{1}{B} - \sum_{i=2}^{n+1} R_{\beta_i} \right) \right)}{(\beta_0 - \beta_1)}. \tag{3.11}$$

Re-arranging 3.9 can give

$$\beta_0^2 + 2 \left(R_0 \phi_0^2 - R_{\beta_0} \beta_0^2 - R_{\beta_1} \beta_1^2 + \sum_{i=1}^n R_i \phi_i (\phi_i - \beta_{i+1}) \right) + \frac{1}{B} (\beta_1^2 - \phi_0^2) - \beta_1^2 = 0. \tag{3.12}$$

By the conservation of mass of the core flow, $R_{\beta_0} \beta_0 = \alpha_{av}$, and along with $R_{\beta_1} \beta_1 = R_0 \phi_0 - \alpha_{av}$ equation 3.12 can be rewritten as:

$$\begin{aligned}
&\beta_0^2 - 2\alpha_{av} \beta_0 + 2\beta_1 \alpha_{av} + 2 \left(R_0 \phi_0 (\phi_0 - \beta_1) + \sum_{i=1}^n R_i \phi_i (\phi_i - \beta_{i+1}) \right) + \\
&\quad \frac{1}{B} (\beta_1^2 - \phi_0^2) - \beta_1^2 = 0.
\end{aligned} \tag{3.13}$$

Let

$$C_1 = 2 \left(R_0 \phi_0 (\phi_0 - \beta_1) + \sum_{i=1}^n R_i \phi_i (\phi_i - \beta_{i+1}) \right) + \frac{1}{B} (\beta_1^2 - \phi_0^2) - \beta_1^2, \tag{3.14}$$

and

$$C_2 = \left(R_0 \phi_0 - \beta_1 \left(\frac{1}{B} - \sum_{i=2}^{n+1} R_{\beta_i} \right) \right). \tag{3.15}$$

Now substitute 3.11 in for α_{av} to obtain

$$\beta_0^3 - (2C_2 + \beta_1) \beta_0^2 + (2C_2 \beta_1 + C_1) \beta_0 - C_1 \beta_1 = 0, \tag{3.16}$$

which is a cubic equation of β_0 that can be solved analytically if we fix β_1 , as C_1 and C_2 are functions of known terms and downstream bypass flow terms which can be obtained from β_1 using 3.3 and 3.4. The power generated by the turbine assuming no internal losses is

$$P = T\alpha_{av}u = \alpha_{av}(\beta_1^2 - \beta_0^2)\left(\frac{1}{2}\rho Au^3\right) = C_P\left(\frac{1}{2}\rho Au^3\right), \quad (3.17)$$

reintroducing the area, density and velocity terms for clarity. Note that the power coefficient of the turbine, C_P , is a function of α_{av} (3.11), β_1 and β_0 , where only one of these needs to be given to set the operating condition of the turbine and thus solve the problem. In LMADT it is common to introduce a disc resistance (or local thrust) coefficient k to describe the operating condition of the turbine such that

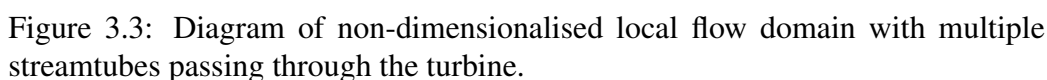
$$T = \frac{1}{2}k\rho A(\alpha_{av}u)^2, \quad (3.18)$$

therefore $k = (\beta_1^2 - \beta_0^2)/\alpha_{av}^2$, by equating 3.17 and 3.18. Hence, the value of C_P can be calculated analytically for a given set of k , B and a given inflow profile (R_i and ϕ_i).

The generalised approach means we are not constrained by the number of bypass streamtubes calculated. This removes, for example, the need to reduce the number of streamtubes through mixing assumptions used in previous multi-row studies which could result in substantial differences in the predicted power (Juniper and Nishino, 2020).

3.2.2 Non-uniform flow through turbine

When considering non-uniform flow through a turbine the calculations are much the same as the uniform case presented in section 3.2.1. For uniform flow through



The first method is often used as it gives a more accurate representation of an ideal porous disc, which is often used as a representation of a turbine in experimental work (e.g., Myers and Bahaj (2012)) and in computational studies (e.g., Nishino and Draper (2015)).

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near uniform resistance.

Chamorro and Arndt (2013) adopted the third method, uniform thrust, constructing a correction factor to the Lanchester-Betz-Joukowsky limit for a turbine in a neutrally stratified atmospheric boundary layer. This method will be adopted in our calculation for non-uniform flow through a turbine. In support of this approach (Revaz and Porté-Agel, 2021) found through LES that a model which computes non-uniform force distributions over the turbine and a model which assumes uniform forces over the turbine can both make accurate predictions for the flow in the far wake and power of the turbine. An exploration of the impact of non-uniform force for LMADT is presented in section 3.3.2.

For non-uniform flow through the turbine we start off as before by applying Bernoulli's equation to streamtubes that bypass the turbine:

$$p_1 - p_4 = \frac{1}{2}\rho u^2(\beta_{i+1}^2 - \phi_i^2), \quad (3.19)$$

which can be equated to obtain a relationship between all the bypass flow velocities:

$$\beta_{i+1} = \sqrt{\beta_{j+1}^2 + \phi_i^2 - \phi_j^2}, \quad (3.20)$$

for $i, j \geq M$, where M is the index of the streamtube that is split by the turbine, as depicted in figure 3.3. Applying Bernoulli's equation on streamtubes that pass through the turbine gives us

$$p_2 - p_3 = \Delta p = \frac{1}{2}\rho u^2(\phi_a^2 - \beta_a^2 - \phi_M^2 + \beta_{M+1}^2), \quad (3.21)$$

and, as Δp is held constant,

$$\beta_a = \sqrt{\beta_b^2 + \phi_a^2 - \phi_b^2}, \quad (3.22)$$

for $a, b \leq M$. So, all streamtube velocities are related once one downstream velocity in the bypass and one downstream velocity in the core flow is known.

Likewise the conservation of momentum is defined as

$$0 = (\beta_M^2 - \beta_{M+1}^2) + \frac{1}{B}(\beta_{M+1}^2 - \phi_M^2) + 2 \left(\sum_{i=0}^{M-1} R_i \phi_i (\phi_i - \beta_i) + \sum_{i=M+1}^n R_i \phi_i (\phi_i - \beta_{i+1}) + R' \phi_M (\phi_M - \beta_M) + R^* \phi_M (\phi_M - \beta_{M+1}) \right), \quad (3.23)$$

where $R_M = R' + R^*$ with R' as the upstream area of the streamtube which passes through the turbine and R^* being the upstream area of the streamtube which directly bypasses the turbine.

The power P is calculated as

$$P = \sum_{i=0}^M R_{\alpha_i} A \alpha_i u \Delta p, \quad (3.24)$$

where $R_{\alpha_i} A$ is the size of the i -th streamtube at the turbine plane; however, R_{α_i} is not calculated in this analysis. Therefore, this is rewritten using the conservation of mass to obtain an equivalent expression based on known upstream values:

$$P = Au \Delta p (R' \phi_M + \sum_{i=0}^{M-1} R_i \phi_i). \quad (3.25)$$

We define the power coefficient as

$$C_P = \frac{P}{\frac{1}{2} \rho A \bar{u}^3} = \frac{P}{\frac{1}{2} \rho A u^3 (R' \phi_M^3 + \sum_{i=0}^{M-1} R_i \phi_i^3)}, \quad (3.26)$$

which is the power normalised using the cube of the upstream velocity of the flow

that passes through the turbine, $\overline{u^3}$. To solve this system we need to know what portion of upstream flow passes through the turbine, allowing M to be determined. For this reason α_{av} , the average velocity through the turbine, is used as the input parameter, thus allowing M to be determined by conservation of mass. After α_{av} is selected a range of β_{M+1} and β_M values are iterated over to find the relevant root.

3.3 Actuator disc with non-uniform thrust

In this section we consider an actuator disc with non-uniform thrust and compare analytical solutions with 3D Reynolds Averaged Navier-Stokes (RANS) simulations.

The 3D incompressible RANS equations are solved, modelling the Reynolds stress terms using the standard $K - \epsilon$ model of Launder and Spalding (1974). The computations are conducted using ANSYS FLUENT 2021 R2 and performed as steady state. This RANS actuator disc method has been utilised successfully in the past (Nishino and Willden, 2012a; Nishino and Draper, 2015); in particular, it has been shown to agree well with the classical 1D inviscid actuator disc theory when used with low levels of ambient turbulence. The simulations are carried out with a uniform inflow profile with streamwise velocity of $U_{in} = 10\text{m/s}$. For the low level of ambient turbulence, we set the turbulence quantities as $K = 0.00015\text{m}^2/\text{s}^2$ and $\epsilon = 3.02 \times 10^{-7}\text{m}^2/\text{s}^3$ at the inlet, which is located $25D$ upstream of the disc (where $D = 100\text{m}$ is the disc diameter). The air density and viscosity are held constant at $1.225\text{kg}/\text{m}^3$ and $1.78 \times 10^{-5}\text{kg}/\text{ms}$, respectively. Flow speed was chosen to be consistent with average UK offshore wind speeds. Turbine diameter is consistent with previous studies and the size of early offshore wind turbine in the UK (for example, Walney Wind Farm's phase 1 107m diameter turbines, com-

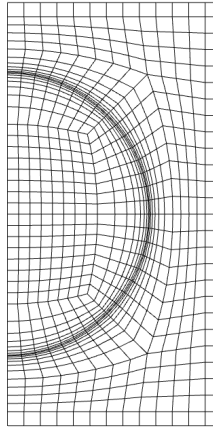


Figure 3.4: Cross-sectional segment of the computational mesh around the actuator half-disc.

pleted in 2011). Whilst modern turbines have larger diameters, the quantities of interest in this and future sections are non-dimensionalised so results should remain consistent across a range of diameters and flow speeds. Although a wind turbine in a real environment may experience blockage effects from an atmospheric capping layer above, we do not consider the stratification or vertical shear of the inflow for simplicity. Symmetry conditions are applied to the top, bottom and side boundaries to mimic the configuration considered by the LMADT, which represents (part of) an infinitely wide row of turbines.

3.3.1 Computational mesh

Figure 3.4 shows an example of a 2D slice of the computational mesh at the turbine plane. Only a half of the disc is simulated as the flow field around the disc is symmetric. This 2D slice is propagated in the streamwise direction (both upstream and downstream) to make a 3D mesh consisting of exclusively hexahedral cells. The slices are spaced non-uniformly, with a minimum spacing of $0.003D$ nearest to the turbine plane.

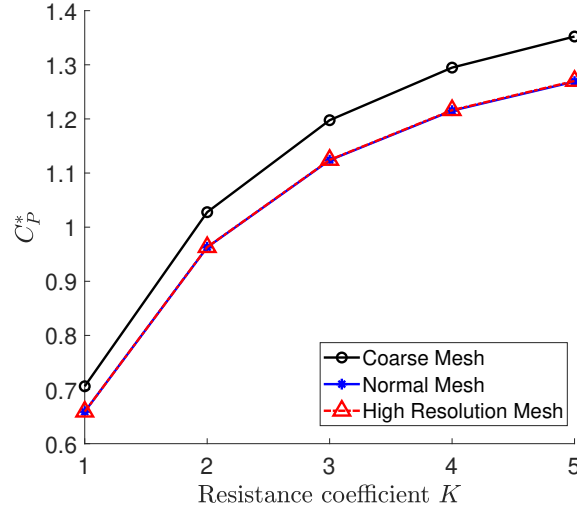


Figure 3.5: Impact of grid resolution on C_P^* for $B = 0.349$.

A mesh sensitivity study was conducted for the domain of width $0.75D$ (corresponding to $1.5D$ for the domain for a full disc) and height $1.5D$. The coarse mesh had a total of 86,200 cells, the normal resolution mesh had a total of 157,200, and the finer resolution mesh had 1,063,200 cells. This domain was chosen as the high shear present in a highly confined domain would be sensitive to mesh resolution. The normal resolution mesh was considered suitably fine to accurately capture the flow physics. As can be seen in figure 3.5, the normal and high resolution cases are near identical for all values of the resistance coefficient k , with a difference in C_P^* of less than 0.1% for all k values tested, suggesting that the normal resolution mesh is sufficient for this study.

3.3.2 LMADT with non-uniform thrust

In section 3.2.2, Δp was held constant across the disc surface. However, if we allow Δp to vary we can then create non-uniform thrust. To do this, we set the pressure jump across the disc in each streamtube to $c_i \Delta p$ where c_i is a scalar. From Bernoulli's equation for streamtubes passing through the disc we get

$$c_i \Delta p = \frac{1}{2} \rho u^2 (\phi_i^2 - \beta_i^2 - \phi_M^2 + \beta_{M+1}^2), \quad (3.27)$$

from which we can obtain an updated version of (3.22) which defines the relationship between the streamtubes passing through the turbine

$$\beta_j = \sqrt{\frac{c_j}{c_i} (\phi_M^2 + \beta_i^2 - \phi_i^2 - \beta_{M+1}^2) + \phi_j^2 - \phi_M^2 + \beta_{M+1}^2}. \quad (3.28)$$

To accurately obtain the varying resistance coefficient k_i we require α_i or R_{α_i} , which was not required for the previous analysis. To obtain these an additional assumption is required. The assumption used here is that the rate of expansion for each streamtube passing through the disc is constant between the upstream and wake positions. This allows us to obtain the area and velocity terms over the disc from the already obtained downstream flow conditions.

3.3.3 Results

Figure 3.6 shows RANS results of the variation of local C_T^* across an actuator disc with resistance coefficient $k = 3$ in the middle of a domain sized $2D \times 2D$, for various inflow profiles. For uniform inflow (figure 3.6a), the local C_T^* is greatest in the middle of the actuator disc and reduces as we move towards the tip. If we vary upstream flow, however, the variation of C_T^* further changes in relation to the local flow speed at the disk. An accelerated core as seen in figure 3.6b creates a greater peak in local thrust coefficient at the turbine centre, whereas a decelerated core as seen in figure 3.6c (reminiscent of a wake flow profile) causes a reduction in C_T^* at the centre, moving the peak in local thrust closer to the edge of the disk. By adding the capability of LMADT to handle variations of thrust across the disc surface, now we are able to examine the impact of this variation on the theoretical prediction of (disc-averaged) C_P^* .

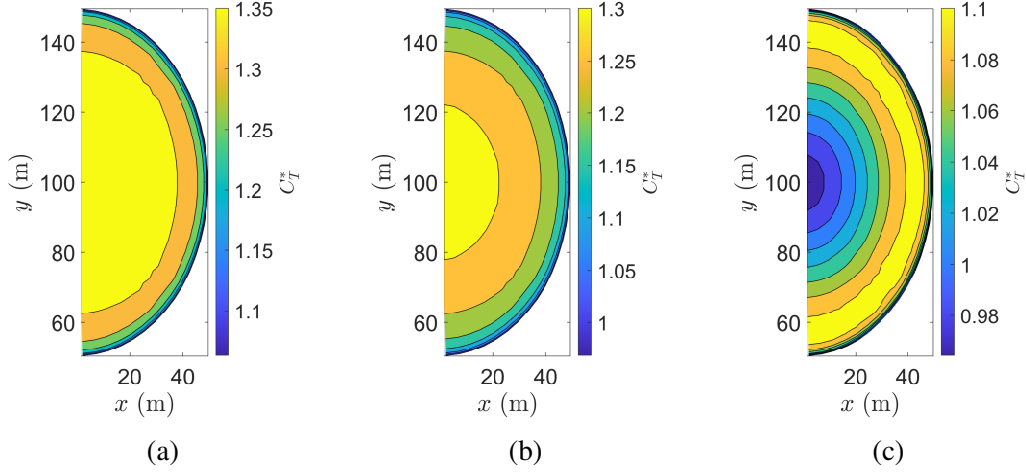


Figure 3.6: Variation of local C_T^* across an actuator disc placed in the middle of a square cross-section of $2D \times 2D$ for inflow profiles of: (a) uniform inflow $U_{in} = 10\text{m/s}$, (b) channel flow $U_{in} = 10(1 - 0.7\sqrt{(y - 100)^2 + x^2}/100)^{0.1}\text{m/s}$, and (c) wake flow $U_{in} = 10(1 - 0.2\exp(-((y - 100)^2 + x^2)/2000))\text{m/s}$.

To accurately compare the uniform and varying local C_T^* we take the thrust profile from the uniform inflow 3D RANS simulations (figure 3.6a) for a range of resistance coefficient k from 1 to 7. We use two different domains for this analysis, one in which the turbine is in the middle of a square domain as described above, and another in which the turbine is in the middle of a cylinder (axisymmetric) of the same blockage ratio. We then use these thrust profiles in the extended LMADT calculations to see if the differences in C_P^* between LMADT and the 3D RANS are accounted for by the non-uniform thrust profile.

As can be observed in figure 3.7 the inclusion of thrust variations aids the LMADT results in better capturing the values from axisymmetric 3D RANS. The results here also highlight that although the LMADT seems an accurate predictor for $C_{P\max}^*$ for a square domain, the disc resistance value predicted to achieve that $C_{P\max}^*$ for the square domain is greater than the value predicted from 3D RANS simulations. To further improve upon the modelling capabilities of LMADT we

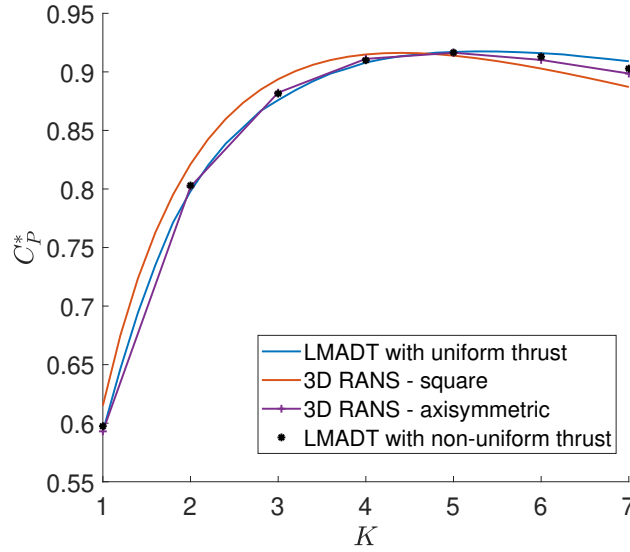


Figure 3.7: Theoretical C_p^* vs k for uniform and varied thrust across the actuator disc surface calculated with LMADT compared with 3D RANS, for $B = 0.196$ in a square and axisymmetric domain.

would need to consider other physical features impacting an actuator disc in three dimensional flow. The discrepancy observed in figure 3.7 is the geometric differences between the quasi-1D (and axisymmetric 3D RANS) and the square domain. This geometric difference in the variation of radial blockage (or ‘anisotropy’ of the blockage) is discussed further in chapter 5.

3.4 Single-row results

Now we present some example solutions of the extended LMADT for a single turbine, or a single row of turbines. It is worth noting that common wake models such as Jensen (1983) assume a ‘top-hat’ profile for turbine wakes with locally uniform velocity, instead of the more realistic sheared profile. The main aim of this section is therefore to investigate the physical implications of such a simplification of the inflow profile on the performance of a turbine in a laterally bounded domain.

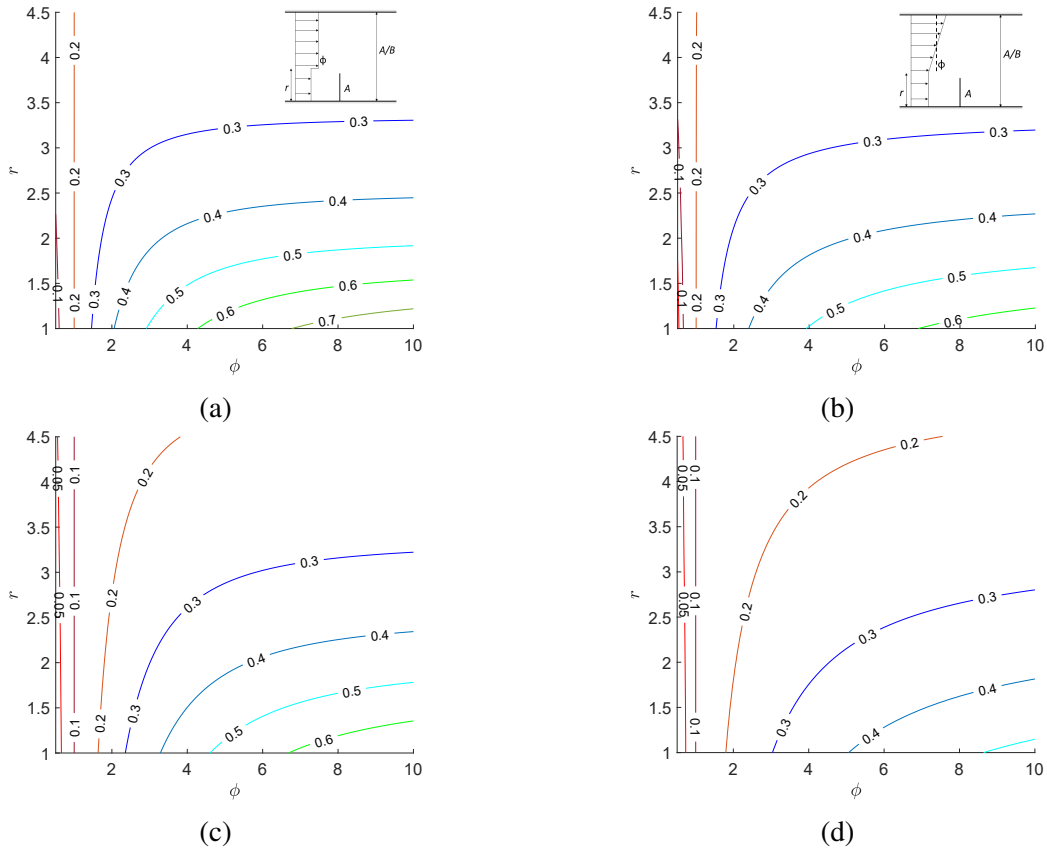


Figure 3.8: Contours of effective blockage ratio, B_{eff} , for (a,c) uniform bypass and (b,d) sheared bypass cases, for a fixed geometric blockage ratios (a,b) $B = 0.2$ and (c,d) $B = 0.1$.

3.4.1 Effective blockage with sheared bypass flow

A study into the impact of non-uniform inflow on the power of a turbine has been conducted by Draper et al. (2014), in which they found that a faster (or slower) bypass flow, compared to the core flow, could increase (or decrease) the power extracted by a turbine. This was explained by a change in the ‘effective blockage’ of the turbine, i.e. the faster (or slower) bypass flow inhibits (or enhances) the expansion of the core flow for a given geometrical blockage.

The effective blockage of a turbine is the blockage ratio required to attain the

same $C_{P\max}$ with a uniform inflow. Since $C_{P\max}$ for the case with uniform inflow is known to be $16/(27(1 - B)^2)$ (Garrett and Cummins, 2007; Dehtyriov et al., 2021), the effective blockage, B_{eff} , is calculated as

$$B_{eff} = 1 - \left(\frac{16}{27C_{P\max}} \right)^{\frac{1}{2}}. \quad (3.29)$$

The results of this work by Draper et al. (2014) was limited by the top-hat inflow profile shape as shown in figure 3.8a, where r is the width of the core flow and ϕu is the velocity of the bypass. It was unclear whether this top-hat inflow profile would accurately predict the flow induced blockage of sheared bypass flows. Draper et al. (2016) investigated the effective blockage of a turbine experiencing a variety of sheared inflow profiles. However, their study considers shear flow across the whole domain, so cannot fully clarify the impact of a sheared bypass compared to a uniform bypass on the power extraction of turbine.

In this chapter a comparison of the effective blockage was conducted between the cases with a top hat inflow profile and a flow profile with a uniform core and linearly sheared bypass flow, as shown in 3.8b. For the sheared case the upstream number of streamtubes is set at 201, which was found to be sufficient to accurately obtain the values of $C_{P\max}$ and thus B_{eff} (see figure 3.9). To make a fair comparison, the mass flux in the bypass is held constant between the uniform bypass and linearly sheared cases. A linear shear was selected here as the shear could be controlled with a single parameter δ , and all regions in the sheared flow experience the same amount of shear. In reality a turbine would experience non-linearly sheared and asymmetric inflows, particularly in the vertical direction. The treatment of these flows are not considered in this thesis, and work to extend the approach presented in this chapter would be required to make predictions for real world flows. However, trends observed for sheared axisymmetric flows would be

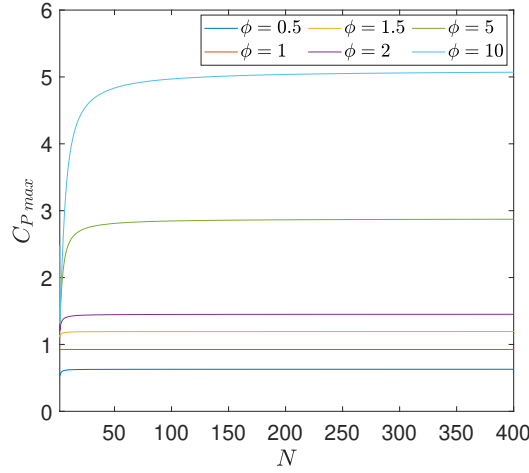


Figure 3.9: $C_{P_{max}}$ plotted against number of bypass streamtubes N for $B = 0.2$, $r = 1$ and six different ϕ values.

expected to carry over to asymmetric flows.

The effective blockage for various bypass speeds and bypass sizes is plotted in figure 3.8 for two different blockage ratios $B = 0.2$ and 0.1 . When $\phi = 1$ we obtain the Garrett and Cummins (2007) result for a blocked turbine experiencing uniform inflow. Across all bypass sizes and bypass speeds the uniform bypass scenarios predict a higher effective blockage than the sheared bypass flow cases. This is more prominent for the lower blockage case of $B = 0.1$. The over prediction by the uniform bypass case occurs as for a given background pressure gradient the uniform bypass case restricts the expansion of the core flow more than the sheared bypass case; thus less flow is diverted around the turbine. The additional constraint from the uniform bypass flow can be illustrated from the increased effective blockage ratio B_{eff} in the uniform case compared to the sheared bypass in figure 3.8.

The reduction in B_{eff} across all r and ϕ values for the linear shear, in comparison to the uniform bypass, implies that the performance of a turbine placed in another turbine's wake may be consistently over-predicted when the upstream bypass flow

is modelled as uniform instead of sheared. This result has implication for the use and benchmarking of wake models which maintain a top-hat flow profile in downstream, like that of Jensen (1983) and Frandsen et al. (2006), when applied to the analysis of multi-row turbine arrays. These top hat wake models are popular analytical models for turbine wakes, which are still used today for turbine array optimisation (e.g., Duc et al. (2019)).

3.4.2 Effective blockage with sheared core flow

In the previous section the impact of resolution/flow averaging on the power coefficient of a laterally bounded turbine with a sheared bypass flow was identified. In this section the impact of flow averaging for a shear flow passing through a turbine will be explored by comparing two flow profiles: sheared core and uniform core.

The base inflow profile which spans the domain will be a linear shear profile of the form

$$\phi = (1 - \delta) + (2\delta)\frac{y}{A}, \quad (3.30)$$

where y/A is the non-dimensional spanwise distance from the centre of the turbine and δ is a parameter to define the significance of the shear. The portion of the flow which passes through the turbine will either be kept as a shear profile or be averaged to become a uniform flow profile, as illustrated in figure 3.10b. All results presented below are for a geometric blockage ratios of $B = 0.1$ and $B = 0.2$. The spanwise resolution of the flow was defined by fixing the number of upstream streamtubes which pass through the turbine at 20, as it was found to be adequate to ensure convergence with a difference of less than 0.2% compared to calculating 500 streamtubes through the turbine.

The effective blockage is plotted against δ in figure 3.10 for the case of sheared flow through the turbine and uniform flow through the turbine. Across all δ values

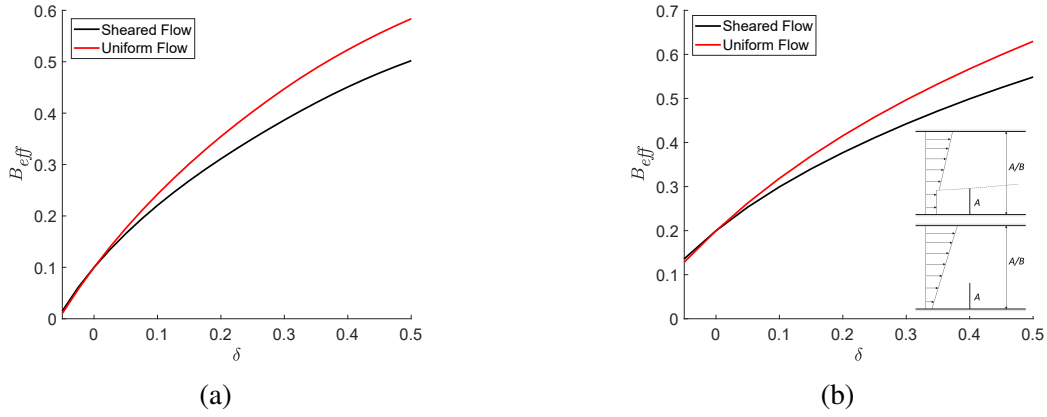


Figure 3.10: Difference in the effective blockage between the uniform flow through a turbine and the shear flow through a turbine, for a fixed geometric blockage ratios (a) $B = 0.1$ and (b) $B = 0.2$.

the uniform core flow case exaggerates the impact of the faster (if δ is positive) or slower (if δ is negative) bypass flow on the effective blockage. For negative δ values where the slower bypass leads to effective blockages lower than the geometric blockage ratio, a uniform core flow will under-predict the power extracted when compared with the sheared core flow case. Conversely, for positive δ values where the faster bypass increases the effective blockage ratio, a uniform core flow will over-predict the power extracted when compared with the sheared core flow case.

The theoretical approach detailed in section 3.2.2 only needs to consider which streamtube passes through the turbine and which enters the bypass. Hence, the spanwise position of each streamtube within the core or bypass flow does not impact the results, as long as the streamtube with velocity $\phi_M u$ is still split by the turbine. Therefore, although simple linear shears were considered for this analysis, the trends observed here can be expected in cases with more complicated flow profiles as long as the above condition is satisfied.

4 | Viscous

*Big whorls have little whorls
Which feed on their velocity
And little whorls have lesser whorls,
And so on to viscosity.*

Lewis Fry Richardson

The previous chapter concerned itself with the inviscid modelling of flow around a turbine. In this chapter we utilise the hybrid inviscid-viscous approach to include the flow recovery in the wake of a turbine, allowing the investigation of multi-row systems. Firstly, the model is used to confirm that an approximation using a limited number of streamtubes in some previous applications of the inviscid-viscous approach has a negligible impact on the results. Secondly, an investigation is conducted on the farm-averaged performance of a turbine array with uniform turbine resistance at each row, compared with the performance of a turbine array with turbine resistance varying at each row. Finally, we present a novel theoretical mixing model for turbine arrays, based on the momentum transfer in the wake.

4.1 Introduction

To best understand the performance of wind and tidal turbines we require accurate models of their flow dynamics in a wide range of conditions. However, explicitly resolving flow features across the wide range of scales which impact the performance of a large number of turbines is computationally infeasible. Therefore, we usually need to make choices about which fluid scales we wish to resolve, which scales we wish to model, and what assumptions we wish to make to more efficiently model the system. The choices we make about how we model the problem are informed by the physical processes we wish to investigate and the parameter space over which we explore the system. For the optimisation of wind and tidal power generation systems involving very large parameter spaces, we often wish to employ computationally fast analytic models of flow through and around a turbine.

The earliest of these is the linear momentum actuator disk theory (LMADT) approach of Lanchester-Betz-Joukowski (van Kuik, 2007). This model, however, was developed for a single turbine/rotor and did not consider multi-row arrays, which are also characterised by the viscous mixing that occurs between turbine rows. To include these viscous effects and retain the computational efficiency of the LMADT model, Nishino and Draper (2019) proposed a hybrid inviscid-viscous model, in which the flow directly around the turbine is dominated by inviscid mechanisms and modelled as inviscid flow (with LMADT) and, following this inviscid zone, a viscous mixing region was introduced. This approach is in line with the results of Large-Eddy Simulations (LES) of a turbine in which inviscid processes are found to dominate around the turbine (West and Lele, 2020). Early utilisations of this hybrid inviscid-viscous model employed a ‘uniform’ mixing model in the viscous region in which flow recovered spanwisely at a uniform

rate towards the average flow speed based on a single mixing parameter m (Ouro and Nishino, 2021). We will expand upon and utilise this model for the first half of this chapter.

4.2 Local and global power coefficients

A distinction is made here between what we define as the ‘internal’ and ‘external’ performance of a multi-row array. To demonstrate this distinction in a simple manner, we first consider a special case of multi-row arrays where the streamwise distance between each row is large enough for the wake of each turbine to be fully recovered. This special case has been studied in a tidal turbine array context by Vennell (2010) who combined the LMADT of Garrett and Cummins (2007) with an analytical tidal channel model (Garrett and Cummins, 2005). We present an equivalent special case in the wind turbine array context by combining results from the LMADT (with uniform inflow for each turbine) and an analytical ‘external’ model derived from the two-scale momentum theory for large wind farms (Nishino and Dunstan, 2020).

We define the array-average ‘local’ or ‘internal’ power coefficient as

$$C_P^* = \frac{\sum_{i=1}^n P_i}{\frac{1}{2}\rho u_{av}^3 n A}, \quad (4.1)$$

where P_i is the power at the i -th turbine and n the number of turbines in the array. This represents the total power of turbines for a given (fixed) mass flux through the local flow domain. Here we assume that the average velocity, u_{av} , corresponds to the ‘farm-layer average’ velocity, U_F , in the two-scale momentum theory (as the inflow profile is uniform for all turbines and thus we cannot define the farm layer in this simple example). By keeping the mass flux constant through the local

flow domain we restrict our attention to the ‘internal’ performance of the array, which can increase due to local blockage effects. This is in contrast to ‘global’ performance which can be impacted by wind-farm blockage effects (Bleeg et al., 2018; Kirby et al., 2022) reducing the power of a wind farm as the mass flux through the whole array decreases. Alongside the local power coefficient C_P^* , we also define the array-averaged ‘global’ power coefficient as

$$C_P^G = \frac{\sum_{i=1}^n P_i}{\frac{1}{2}\rho U_0^3 n A}, \quad (4.2)$$

where U_0 is undisturbed or ‘natural’ wind speed observed when the whole wind farm does not exist. Therefore, C_P^G is a measure of the ‘overall’ efficiency of the farm, represented by the ratio of power extracted by the turbines and the power of the undisturbed flow through the total turbine swept area. Here $C_P^G = \beta^3 C_P^*$ where $\beta = u_{av}/U_0$. To illustrate the impact of local blockage on C_P^G and C_P^* , we compare how these power coefficients change as we vary the lateral turbine spacing in a multi-row turbine array.

We calculate C_P^* directly from LMADT, whereas C_P^G is estimated by combining LMADT results with the two-scale momentum theory of Nishino and Dunstan (2020). To estimate the unknown velocity ratio, u_{av}/U_0 , from the theory, we consider that there are multiple rows of turbines placed equidistantly with a large streamwise spacing of $20D$, where D is the turbine diameter, and assume that this distance is large enough for each turbine wake to be fully recovered (see, e.g., Sedaghatizadeh et al. (2019)). This allows us to calculate the array density λ (which is the ratio of the total turbine swept area to the farm area) as a function of the lateral turbine spacing L , i.e., $\lambda = \pi D/80L$. We further assume a simple relationship between B and L as $B = \pi D/20L$ for convenience, meaning that the height of the local flow domain (which may be defined when there is a strong

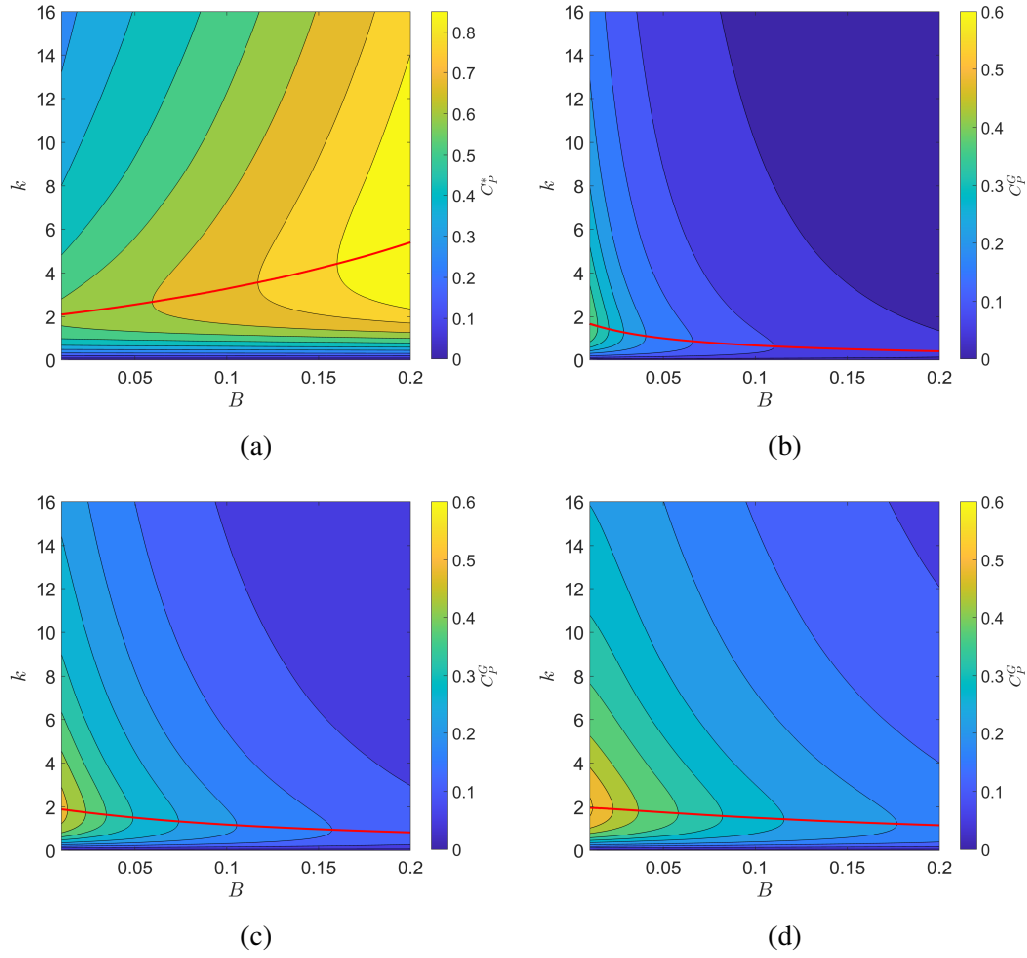


Figure 4.1: Contour plots of (a) C_P^* and (b, c, d) C_P^G of a multi-row turbine array without wake interactions between rows, for (b) $\zeta = 5$, (c) $\zeta = 15$, and (d) $\zeta = 25$ for various blockage ratios (B) and resistance coefficients (k). Optimal resistance k_{opt} is shown by a red line.

capping layer acting like a rigid lid) is $5D$. This means that we assume $\lambda = 0.25B$ in the following analysis.

To solve the two-scale coupled problem, we also need to calculate from LMADT the array-averaged ‘local’ or ‘internal’ thrust coefficient, defined as

$$C_T^* = \frac{\sum_{i=1}^n T_i}{\frac{1}{2}\rho u_{av}^2 n A}, \quad (4.3)$$

where T_i is the thrust at the i -th turbine. The two-scale coupled momentum balance equation from Nishino and Dunstan (2020) is

$$C_T^* \frac{\lambda}{C_{f0}} \beta^2 + \beta^\gamma = 1 + \zeta(1 - \beta), \quad (4.4)$$

where the momentum availability factor (right hand side) has been approximated using a linear wind extractability model (Patel et al., 2021). Here we also set the natural bottom friction coefficient as $C_{f0} = 0.002$ to be typical of an offshore wind farm scenario. We set the bottom friction exponent as $\gamma = 2$ which is a reasonable value and allows for the momentum equation 4.4 to be solved analytically. Therefore, β is reduced to a function of an external (mesoscale) flow parameter ζ . Here ζ is the momentum response parameter for the atmosphere, which represents the ability of the atmosphere to sustain wind speed against the resistance caused by the wind farm, i.e., the extractability of wind. Typical values of which are expected to be between 5 and 25 for a realistic large offshore wind farm (Patel et al., 2021), although a recent theoretical study (Kirby et al., 2023) suggests values could realistically range from 5 to 60.

Figure 4.1 shows the values of C_P^* and C_P^G (for three different atmospheric response strengths, $\zeta = 5, 15$ and 25) across a range of turbine resistance coefficients and blockage ratios. In each plot the red line shows the optimal resistance

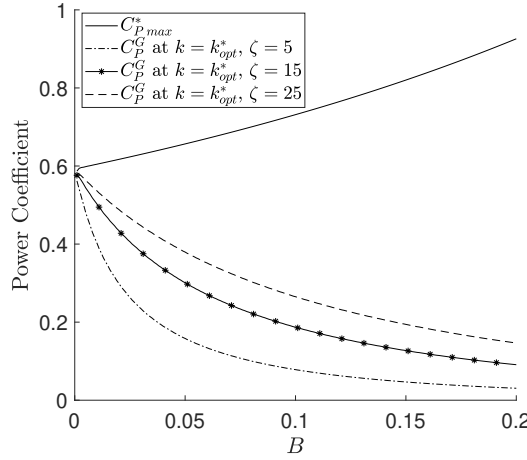


Figure 4.2: Maximum local power coefficient and corresponding global power coefficients of a multi-row turbine array without wake interactions between rows, plotted against the blockage ratio (B) for a range of momentum response parameters (ζ) (with the assumption that $\lambda = 0.25B$, $C_{f0} = 0.002$ and $\gamma = 2$).

coefficient, k_{opt} , which is defined as the value of k at which the maximum power coefficient is achieved at each blockage ratio B . If k is fixed the local power coefficient C_P^* always increases with B , whereas the global power coefficient C_P^G tends to decrease at different rates depending on ζ . Importantly, the value of k_{opt} for C_P^* increases rapidly with B (from 2 at $B = 0$ to 5.4 at $B = 0.2$) but k_{opt} for C_P^G tends to decrease, especially when ζ is small.

Figure 4.2 shows the maximum C_P^* and corresponding C_P^G values obtained with $k = k_{opt}^*$ at each blockage ratio B , where k_{opt}^* is the k_{opt} value for C_P^* . Increasing B (or decreasing the lateral turbine spacing L) causes $C_{P\ max}^*$ to increase, as observed in figure 4.1a, however, C_P^G at $k = k_{opt}^*$ decreases monotonically for all values of ζ between 5 and 25 (as the increase in C_P^* is not large enough to compensate for the decrease in β^3). Note that $B = \pi D/20L$ has been assumed in this example and hence $B = 0.157$ corresponds to $L/D = 1$, meaning that blockage ratios of higher than 0.157 are unphysical in this example (unless we allow for variable turbine hub-heights in each row).

Note, while a streamwise spacing of $20D$ was selected for the above analysis, larger spacing may be required for each turbine wake to be fully recovered in a real wind farm depending on the turbulence intensity. However, the trends in the relationship between the local and global power coefficients observed in this exploration remain consistent across choices of spacing.

In the following sections we consider the more general case where streamwise distance between rows is not large enough for the wake to be fully recovered. To model this we use an inviscid-viscous approach, outlined in the next sub-section, but for simplicity no longer consider the external momentum balance, therefore only focus on the ‘internal’ performance of the array represented by C_P^* .

4.3 Inviscid-viscous approach

4.3.1 Uniform mixing model

For the following multi-row analysis we utilise the inviscid-viscous approach as described in Ouro and Nishino (2021). As illustrated earlier in figure 3.1, this approach treats the region around the turbine as an inviscid flow region which is modelled with LMADT, followed by a viscous mixing zone downstream which is modelled separately. This approach is in line with the results of West and Lele (2020) who, in a large-eddy simulation (LES) study of flow around an actuator disc, showed that inviscid flow processes dominate around the turbine and viscous processes dominate further downstream. This result can also be theoretically argued due to dominant terms in the streamwise momentum equation. However, the point at which this transition from laminar to turbulent happens is difficult to

predict. Considering momentum in the x direction we have

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right). \quad (4.5)$$

Thus, the evolution of flow velocity, (LHS), is driven by either the inviscid pressure forces or the viscous forces, (RHS). Around the turbine pressure forces dominate due to the large pressure perturbation caused by the actuator disk, therefore this is modelled as inviscid flow. In the wake we can consider the turbulent boundary layer equations for wakes (Pope, 2000). We will obtain these by imposing the condition that the pressure is assumed to have equalised in the spanwise direction. If we consider the momentum in the y direction, with uniform spanwise pressure, we have that

$$u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = \nu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right). \quad (4.6)$$

Conducting a scale analysis on 4.6, such that $u \sim U$, $v \sim V$, $x \sim L$, and $y \sim \delta$, where U , V , L , and δ are representative velocity and length scales, we obtain,

$$\frac{UV}{L} + \frac{V^2}{\delta} \sim \nu \left(\frac{V}{L^2} + \frac{V}{\delta^2} \right). \quad (4.7)$$

From conservation of mass

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \quad (4.8)$$

we have that,

$$\frac{U}{L} \sim \frac{V}{\delta}. \quad (4.9)$$

Using equation 4.7 and 4.9 we can obtain

$$\frac{V^2}{\delta} \sim \frac{\nu}{LU} \left(\frac{VU}{L} + \frac{VUL}{\delta^2} \right), \quad (4.10)$$

which, including Reynolds number, Re , can be written as

$$\frac{V}{\delta} = \frac{1}{Re} \left(\frac{V}{\delta} + \frac{UL}{\delta^2} \right), \quad (4.11)$$

this can be further simplified to

$$\frac{V^2}{U^2} \left(1 + \frac{1}{Re} \right) \sim \frac{1}{Re}. \quad (4.12)$$

Thus, when Re is large $\frac{V^2}{U^2}$ must be small, giving us that for large Re

$$V \ll U. \quad (4.13)$$

By 4.13 we have that $\delta \ll L$, which allows us to reduce 4.5 to

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -\frac{1}{\rho} \frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 u}{\partial y^2} \right), \quad (4.14)$$

now in a form recognisable from boundary layer flows. If pressure in the wake has approximately equalised with the free stream then as

$$\frac{\partial p_\infty}{\partial x} = 0, \quad (4.15)$$

and $\partial p / \partial y = 0$, we have that our pressure in the wake must be constant in x .

Therefore, equation 4.14 becomes

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \nu \left(\frac{\partial^2 u}{\partial y^2} \right), \quad (4.16)$$

in which evolution of the flow velocity is driven by viscous shear stresses. The exact point at which this transition between laminar and turbulent flow occurs is outside the scope of this thesis.

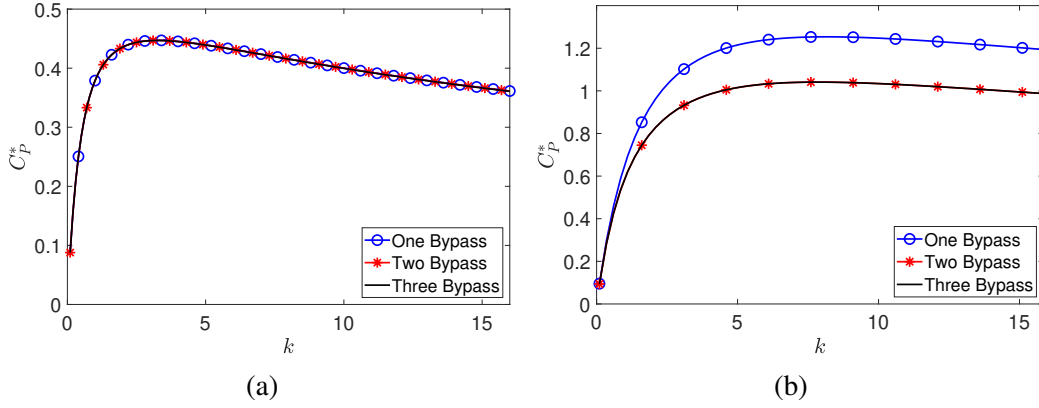


Figure 4.3: C_P^* predictions for four rows of (a) aligned and (b) staggered turbines for various numbers of upstream bypass streamtubes. Blockage and mixing rate are fixed at $B = 0.2$ and $m = 0.6$.

We adopt the same simplification in this analysis as in Ouro and Nishino (2021) that mixing takes place uniformly across the viscous mixing zone and will be defined by a single parameter $0 \leq m \leq 1$ such that $u_{out} = mu_{av} + (1 - m)u_{in}$, where u_{in} is a streamtubes velocity entering the viscous mixing zone, u_{av} is the average velocity across the whole domain and u_{out} is the velocity as it exits the viscous mixing zone. For example when $m = 1$ the whole flow becomes uniform and when $m = 0$ no mixing occurs. The inviscid-viscous approach with uniform mixing was shown by Ouro and Nishino (2021) to capture trends in the performance of multi-row turbine arrays by comparing the results to LES.

4.3.1.1 Validating previous uses

Previous utilisations of the inviscid-viscous approach for an infinitely large array of turbines (Ouro and Nishino, 2021; Nishino and Draper, 2019) only calculate a fixed number of bypass streamtubes. As the number of streamtubes in the system increases at each turbine row (meaning that an infinite number of streamtubes would be required to model an infinitely large array), an assumption was made to keep the number of bypass streamtubes fixed. The assumption is that the bypass

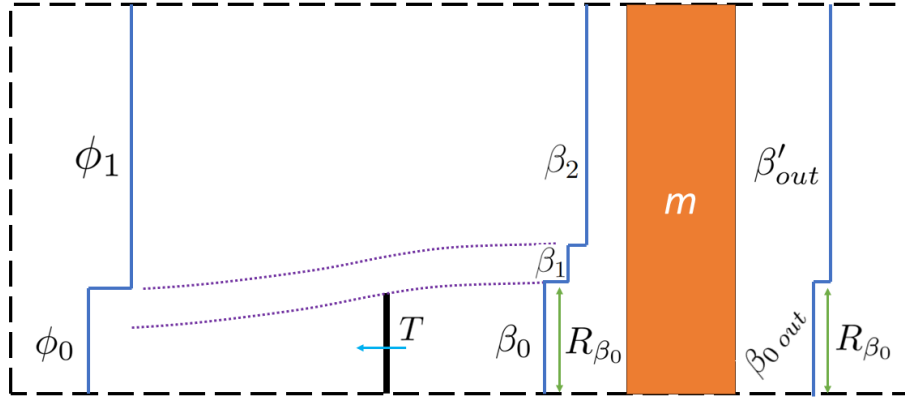


Figure 4.4: Diagram of the instant mixing assumption, where $\beta_{0\ out}$ and β'_{out} represent the velocity scalar after mixing of the streamtubes with velocity scalars β_0 and β' , respectively. The value of β' and the cross sectional area of the streamtube are given by equations 4.17 and 4.18

flow in the streamtube adjacent to the turbine (the streamtube with velocity β_1 in figure 4.4) is always fully mixed in the viscous mixing zone (regardless of the value of the mixing parameter m) with the bypass flow in the streamtube immediately outside (the streamtube with the velocity β_2). In the case with uniform flow through the turbine the streamtubes with velocity $\beta_1 u$ and $\beta_2 u$ are mixed to a single streamtube with the new velocity coefficient

$$\beta' = \frac{R_{\beta_1}\beta_1 + R_{\beta_2}\beta_2}{R_{\beta_1} + R_{\beta_2}}, \quad (4.17)$$

and cross sectional area of

$$R_{\beta'} = R_{\beta_1} + R_{\beta_2}, \quad (4.18)$$

therefore the number of bypass streamtubes remains constant in the upstream and downstream.

The extended LMADT for a generalised number of bypass streamtubes outlined in chapter 3 allows for the impact of this instant mixing assumption to be accurately quantified. For this investigation, both an aligned and a staggered turbine arrangement are considered with the first turbine row experiencing a fully uniform inflow. For a fixed blockage ratio and mixing rate, $B = 0.2$ and $m = 0.6$, the array-averaged power coefficient C_P^* of four rows of turbines is plotted in figure 4.3 against resistance coefficient k for three different levels of instant mixing. Firstly, the ‘one bypass’ case is for instant mixing such that there are only up to two upstream streamtubes (i.e. the number of streamtubes increases from two to three in the inviscid zone but decreases from three to two due to the instant mixing between the two bypass flows in the viscous zone), the ‘two bypass’ case allows for three upstream streamtubes, and the ‘three bypass’ case allows for four upstream streamtubes. The three bypass case is therefore the case where no instant mixing occurs for any of the four rows. The plots show that the only significant difference in predictions occurs in the staggered case when the upstream number of streamtubes is fixed at two (which is understandable as we need at least three upstream streamtubes to represent the wake of a staggered turbine upstream). Therefore as previous studies utilising this method (Ouro and Nishino, 2021; Nishino and Draper, 2019) have a minimum of two upstream streamtubes for the aligned case and at least three for the staggered, their predictions and conclusions are not significantly impacted by the instant mixing assumption.

4.3.2 Multi-row results for a uniform mixing model

4.3.2.1 Efficiency of aligned and staggered arrays

Figures 4.5 and 4.6 show the maximum C_P^* and the corresponding optimal local thrust coefficient, $C_{T_{opt}}^*$, for an aligned and staggered array of 12 turbine rows, as

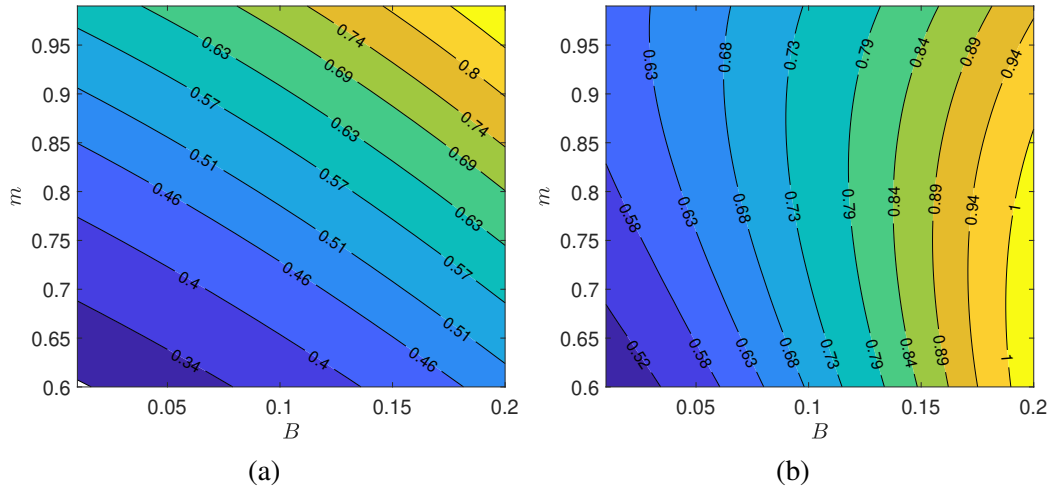


Figure 4.5: Contour plots of the maximum C_P^* for a given mixing rate m and blockage B across 12 turbine rows for (a) aligned and (b) staggered turbine arrangements.

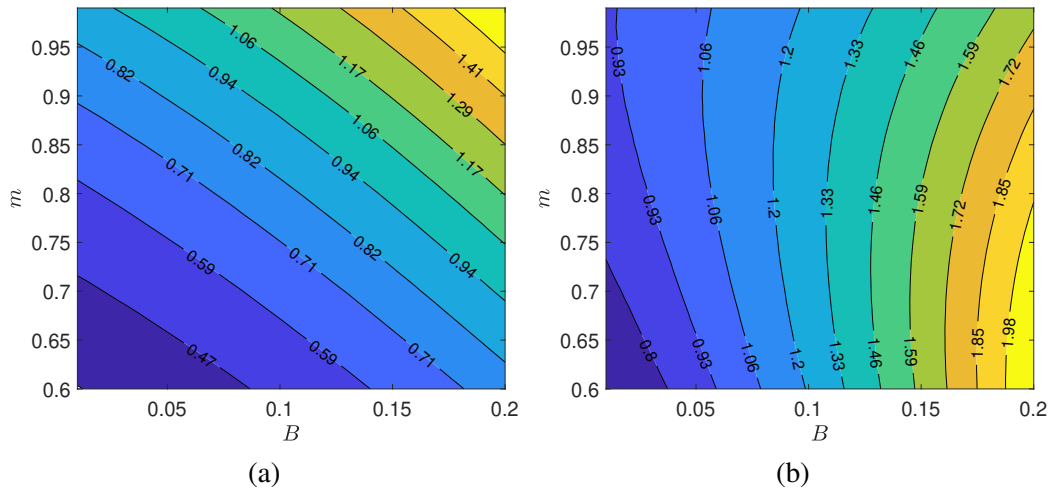


Figure 4.6: Contour plots of the thrust coefficient, C_T^* , required for obtaining maximum C_P^* for a given mixing rate m and blockage B across 12 turbine rows for (a) aligned and (b) staggered turbine arrangements.

an example of a multi-row analysis. The analysis was conducted without employing the instant mixing assumption. The optimal local thrust coefficient, $C_{T\ opt}^*$, is the C_T^* value at which the array obtains the maximum C_P^* . These example cases are with a uniform inflow profile at the first turbine row, a constant k for all 12 rows and it is assumed that mixing rate m also remains constant. For the aligned case in figure 4.5a we observe that the array achieves a maximum C_P^* at higher blockage and mixing rates, with $m = 1$ being optimal for all blockage ratios. For the staggered case the efficiency of the array can slightly improve when the mixing rate is less than one. This is because a downstream turbine can take advantage of the increased speed of flow bypassing upstream turbines. These trends agree well with the theoretical and computational results of Ouro and Nishino (2021) for infinitely large arrays of turbines.

In both turbine arrangements the regions of higher $C_{P\ max}^*$ are accompanied by higher $C_{T\ opt}^*$. However, the gradient of increasing $C_{T\ opt}^*$ does not exactly match that for increasing $C_{P\ max}^*$, so there are opportunities for improving C_P^* of an array without increasing C_T^* (meaning that the global power coefficient C_P^G could be improved by finding an optimal balance between increasing C_P^* and decreasing C_T^* for a given ‘external’ condition, as discussed earlier in section 4.2). For example, in a staggered arrangement, by using an array layout with blockage $B = 0.2$ and $m = 0.86$, we would obtain a near identical power coefficient as the $B = 0.19$ and $m = 0.6$ case with a $\sim 5.5\%$ reduction in C_T^* . As the blockage ratio B is inversely proportional to the spanwise distance between turbines whereas the mixing rate m is expected to increase with the streamwise distance between turbines, the above results imply that the optimal staggered array layout (to maximise C_P^G) would depend on a complex interaction between the local blockage, wake mixing and ‘external’ flow conditions (such as the momentum response parameter ζ for the case of wind farms).

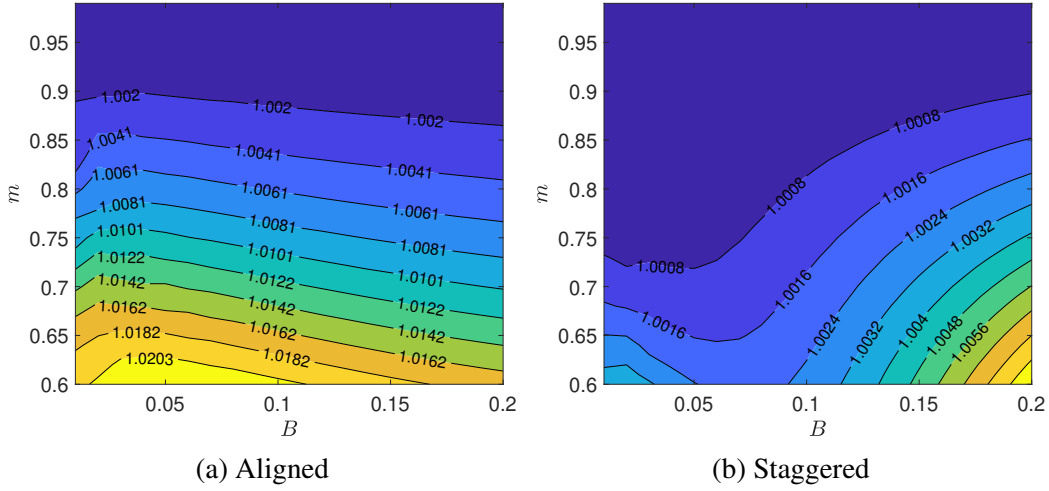


Figure 4.7: Contour plots of $C_{P_{max}}^{*V}/C_{P_{max}}^{*F}$, the ratio of the farm-averaged $C_{P_{max}}^*$ achieved by varying resistance coefficient, k , across three rows of turbines, to that obtained from the uniform k scenario.

4.3.2.2 Varying turbine resistance across multiple rows

The analytic model presented in section 3.2 allows for more complex explorations of multi-row turbine array optimisation such as the investigation of varying turbine resistance, k , across multiple rows, an investigation that would become too computationally expensive with other modelling techniques. Varying k across multiple rows may provide improvements in the maximum farm averaged C_P^* . Let $C_{P_{max}}^{*V}$ be the maximum C_P^* for varying k across multiple rows, and $C_{P_{max}}^{*F}$ be the optimal C_P^* when k is uniform for all rows. Figure 4.7 depicts $C_{P_{max}}^{*V}/C_{P_{max}}^{*F}$ for a range of blockage ratios and mixing rates, for the case of three rows of aligned or staggered turbines. As can be seen from figure 4.7, the magnitude of the improvements and where they are achieved differ between the aligned and staggered arrangements. The improvements over the uniform k scenario for the aligned case can be up to 2% in this example, and these improvements can be best obtained at low mixing and blockage ratios. For the staggered arrangement the improvements are minor (of less than 1% in this example) and are best seen for high blockage

ratios and low mixing rates; scenarios in which the impact of an upstream turbine on the core flow of a downstream turbine is at its greatest.

For the aligned case we see a peak in potential improvement over the uniform k case when m is low and $B \approx 0.05$. This peak occurs due to the interaction between the flow speed through the turbine, flow speed in the wake, and blockage effects. At low mixing rates, the interaction between upstream and downstream turbines is at its greatest, allowing for more opportunities to optimise their combined performance. The basic mechanism is that we can increase the average performance by reducing the resistance of upstream turbines to obtain a more favourable inflow for downstream turbines. At very low B this mechanism does not allow for much optimisation as changing k (from its optimal value for the uniform k scenario) has a large negative impact on C_P^* of that turbine and the downstream turbines can not compensate for this. As B increases to $B \approx 0.05$ a small reduction in k (when k is less than k_{opt} for the uniform k case) has a much smaller negative impact on α_{av} than a positive impact it has on β_0 , allowing us to better optimise the inflow for a downstream turbine without sacrificing much upstream turbine performance. As B increases further, varying k has less impact on the flow, meaning that a large change in k is required to alter the inflow for a downstream turbine in a significant manner, and this would not offset the great performance reduction in the upstream turbine experiences due to large k variations.

All these results are obtained under the assumption that the mixing rate, m , is held constant across the entire array. In reality, the mixing rate would change as we alter the resistance coefficient of the turbine, for example. Implementing a more physically realistic mixing model would provide greater insight into how we should design and operate multi-row turbine arrays. Nevertheless, the analysis presented above highlights the opportunity to explore large parameter space using

this computationally fast analytic model, with a single simulation taking less than 1×10^{-6} CPU seconds.

4.4 Momentum mixing model

The uniform mixing model was able to capture general trends observed in higher-fidelity LES; however, it was unable to accurately capture the power coefficient or the rate of decrease in power coefficient as the streamwise spacing between the rows of turbines increased; presumably due to the simplified relationship assumed between the streamwise spacing and the mixing rate parameter, m (Ouro and Nishino, 2021).

In the following sections, we present a novel theoretical mixing model for turbine arrays, based on the momentum transfer in the wake. The model is constructed to remain consistent with the set-up of the uniform thrust LMADT model presented in chapter 3, whilst additionally being able to capture non-uniform wake recovery by tuning new empirical model parameters based on the LES results of Vahidi and Porté-Agel (2022). Although a novel approach to model the momentum recovery in the wake is presented in this section, the core motivation of this section is to derive a model that reasonably approximates the wake recovery between rows of turbines and to investigate inaccuracies in the uniform mixing model utilised in previous studies.

4.4.1 Mixing model theory

The uniform thrust LMADT model presented in chapter 3 produces a top-hat velocity deficit behind a turbine only when the inflow is uniform. This prevents the use of existing engineering wake models in the viscous region of a multi-row turbine model (with non-uniform inflow) without employing additional assumptions or simplifications to the flow, as they often assume an initially uniform wake velocity deficit. To remedy this, a new mixing model is presented based upon the

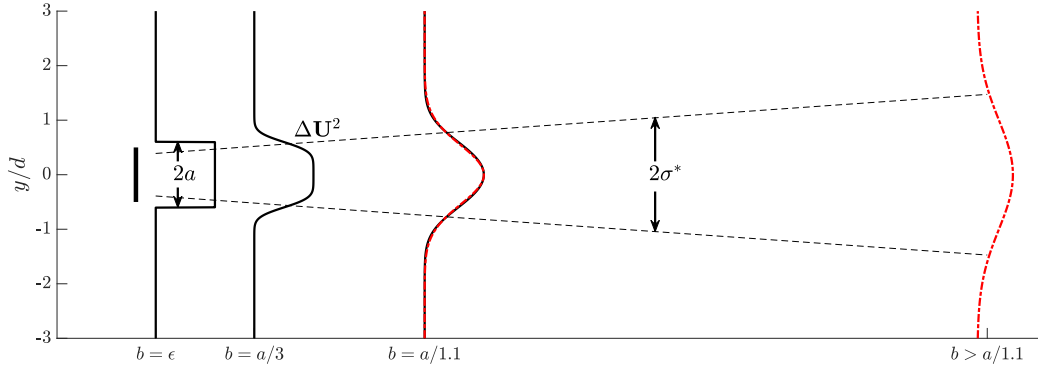


Figure 4.8: Black solid lines show the momentum deficit per unit density, ΔU^2 , calculated with the near wake model, the red dashed lines show the far wake model, and the thick black line represents the turbine. The dashed black lines represent the growth of the characteristic wake width.

uniform momentum deficit predicted by the LMADT model in the wake of a turbine. As the momentum deficit (behind a disk with a fixed uniform thrust) is not dependent on the inflow profile, the momentum transfer from the bypass flow into the wake may be modelled with a simple analytical function. This approach has the additional benefit of focusing our attention on the physical process that causes the wake to recover, i.e. momentum transfer. The model will be presented here as purely two-dimensional for simplicity and to match previous utilisations of the inviscid-viscous model; however, it may be extended to three dimensions. Note, in this analysis we have normalised the momentum deficit by density for ease of calculation, references to the "momentum deficit per unit density of fluid" will often be referred to as just "momentum deficit" for brevity.

We assume the top-hat momentum deficit will recover self-similarly, like the self-similar velocity deficit models (e.g. Bastankhah and Porté-Agel (2014)). We make this assumption due to the self-similarity found in Reynolds stresses, thus momentum flux, in axisymmetric wakes (Pope, 2000). In the near wake it is assumed that the mixing in the wake closely follows the form of free shear mixing layers (Görtler, 1942), separated by a spanwise distance of $2a$, which gives

$$\frac{\Delta \mathbf{U}^2(x, y)}{U_0^2} = \operatorname{erf}\left(\frac{y+a}{b(x)}\right) - \operatorname{erf}\left(\frac{y-a}{b(x)}\right), \quad (4.19)$$

where x and y are the streamwise and spanwise coordinates, respectively, $b(x)$ is a characteristic mixing layer width normalised by turbine diameter, d , and $\Delta \mathbf{U}^2$ is the momentum deficit (per unit density of fluid), defined as $\Delta \mathbf{U}^2 = |\mathbf{U}(x, y)|^2 - |\mathbf{U}(x^*, y)|^2$, where $x = x^*$ is a position upstream of each row, and $\mathbf{U} = (U, V)$ is the mean velocity vector. Here, U_0 is the bulk velocity (i.e. velocity averaged across the whole array span) which is assumed to remain constant. In reality, U_0 may also change depending on flow conditions outside of the whole array, causing a farm-scale power loss (Kirby et al., 2022); but this can be predicted by coupling the array model with an external momentum balance model, as discussed in section 4.2.

At a certain distance downstream, the two mixing layers interact and the momentum deficit is assumed to have a Gaussian profile of the form

$$\frac{\Delta \mathbf{U}^2(x, y)}{U_0^2} = \exp\left(-\frac{y^2}{\sigma^2(x)}\right), \quad (4.20)$$

with $\sigma^2(x) = 2\sigma^{*2}(x)$, where σ^* is our characteristic wake width normalised by turbine diameter, d . A schematic of the momentum deficit and its self-similar recovery is presented in figure 4.8, with the growth of the wake illustrated by an increase in $2\sigma^*$.

Although the effect of blockage is not explicitly considered in the wake mixing model, it still implicitly impacts our calculations of multi-row array performance in two parts. Firstly, in the inviscid flow calculations detailed in chapter 3 in which blockage impacts our thrust coefficient, C_T , and thus our momentum deficit (calculation of C_T is presented in section 4.4.2.2). Secondly, in the calculation of

the pressure drop Δp^* (introduced later in section 4.4.2), that is calculated by conservation of mass flux.

To examine the consequences of modelling the momentum recovery with the above self-similar profiles, we begin with the steady 2D Reynolds-averaged Navier-Stokes equation for the streamwise momentum balance:

$$U \frac{\partial U}{\partial x} + V \frac{\partial U}{\partial y} = \frac{-1}{\rho} \frac{dp}{dx} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'u'}}{\partial x}. \quad (4.21)$$

We will assume that the mean flow is approximately irrotational, therefore, $\frac{\partial U}{\partial y} \approx \frac{\partial V}{\partial x}$, and equation 4.21 can now be written as

$$U \frac{\partial U}{\partial x} + V \frac{\partial V}{\partial x} = \frac{-1}{\rho} \frac{dp}{dx} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'u'}}{\partial x}, \quad (4.22)$$

which simplifies to

$$\frac{1}{2} \frac{\partial |\mathbf{U}|^2}{\partial x} = \frac{-1}{\rho} \frac{dp}{dx} - \frac{\partial \overline{u'v'}}{\partial y} - \frac{\partial \overline{u'u'}}{\partial x}. \quad (4.23)$$

Following the approach of van der Laan et al. (2023) who omit the normal Reynolds stress, as the shear stress divergence is the main mechanism responsible for wake recovery, we obtain

$$\frac{\partial |\mathbf{U}|^2}{\partial x} = 2 \left(\frac{-1}{\rho} \frac{dp}{dx} - \frac{\partial \overline{u'v'}}{\partial y} \right). \quad (4.24)$$

Now we omit the pressure term in equation 4.24 (this will be reintroduced later), and then integrate the equation with respect to x between x_0 and x_1 . Finally, the term $\int_{x_0}^{x_1} \frac{\partial |\mathbf{U}|^2}{\partial x} dx$ is replaced by $(\Delta \mathbf{U}^2(x_1) - \Delta \mathbf{U}^2(x_0))/U_0^2$, which, in the case of

near wake, is defined as

$$\frac{|\mathbf{U}(x_1, y)|^2 - |\mathbf{U}(x_0, y)|^2}{U_0^2} = D(x_1) \left(\operatorname{erf} \left(\frac{y+a}{b(x_1)} \right) - \operatorname{erf} \left(\frac{y-a}{b(x_1)} \right) \right) - C_T \left(\operatorname{erf} \left(\frac{y+a}{\epsilon} \right) - \operatorname{erf} \left(\frac{y-a}{\epsilon} \right) \right), \quad (4.25)$$

where $D(x)$ is the maximum momentum deficit (non-dimensionalised by U_0^2), with $b(0) = \epsilon \ll 1$ and $D(0) = C_T$. The streamwise positions x_1 and x_0 represent two points in the wake across which we wish to calculate the gain/loss of momentum. For our purposes, x_1 is the location in the wake at which we wish to calculate the velocity profile, and $x_0 = 0$ is the location at which the wake has expanded to width $2a$. The location of x_0 is not important for this analysis but will be discussed further in section 4.4.2 for model implementation. Where, with setting $U_0 = 1$ for simplicity in notation,

$$-2 \int_{x_0}^{x_1} \frac{\partial \overline{u'v'}}{\partial y} dx = D(x_1) \left(\operatorname{erf} \left(\frac{y+a}{b(x_1)} \right) - \operatorname{erf} \left(\frac{y-a}{b(x_1)} \right) \right) - C_T \left(\operatorname{erf} \left(\frac{y+a}{\epsilon} \right) - \operatorname{erf} \left(\frac{y-a}{\epsilon} \right) \right), \quad (4.26)$$

therefore,

$$\frac{\partial \overline{u'v'}}{\partial y} = \frac{D(x)b'(x)}{\sqrt{\pi}b^2(x)} \left((y+a) \exp \left(\frac{-(y+a)^2}{b^2(x)} \right) - (y-a) \exp \left(\frac{-(y-a)^2}{b^2(x)} \right) \right) + \frac{D'(x)}{2} \left(\operatorname{erf} \left(\frac{y+a}{b(x)} \right) - \operatorname{erf} \left(\frac{y-a}{b(x)} \right) \right), \quad (4.27)$$

where $f'(x)$ represents the derivative of $f(x)$ with respect to x . Integrating 4.27

with respect to y , the Reynolds shear stress is recovered as

$$-\overline{u'v'} = \frac{D(x)b'(x) + D'(x)b(x)}{2\sqrt{\pi}} \left(\exp\left(\frac{-(y+a)^2}{b^2(x)}\right) - \exp\left(\frac{-(y-a)^2}{b^2(x)}\right) \right) + \frac{D'(x)}{2} \left((y+a)\operatorname{erf}\left(\frac{(y+a)}{b(x)}\right) - (y-a)\operatorname{erf}\left(\frac{(y-a)}{b(x)}\right) \right) + C_1. \quad (4.28)$$

At $y = 0$ we want $\overline{u'v'} = 0$ due to assumed symmetry in the wake, which gives us $C_1 = 0$. To obtain the streamwise velocity from the Reynolds shear stress, we will use the Boussinesq eddy viscosity hypothesis with x derivatives omitted (validity of this omission is explored in the next section),

$$-\overline{u'v'} = \nu_T \frac{\partial U}{\partial y}. \quad (4.29)$$

Now substituting equation 4.28 into equation 4.29 and solving for U we obtain

$$U = \frac{D(x)b'(x)b(x) + D'(x)b^2(x)}{4\nu_T} \left(\operatorname{erf}\left(\frac{y+a}{b(x)}\right) - \operatorname{erf}\left(\frac{y-a}{b(x)}\right) \right) + \frac{D'(x)}{8\nu_T} \left((2(y+a)^2 - b^2(x))\operatorname{erf}\left(\frac{y+a}{b(x)}\right) + \frac{2b(x)(y+a)}{\sqrt{\pi}} \exp\left(\frac{-(y+a)^2}{b^2(x)}\right) \right) - \frac{D'(x)}{8\nu_T} \left((2(y-a)^2 - b^2(x))\operatorname{erf}\left(\frac{y-a}{b(x)}\right) - \frac{2b(x)(y-a)}{\sqrt{\pi}} \exp\left(\frac{-(y-a)^2}{b^2(x)}\right) \right) + C_2, \quad (4.30)$$

where, as $y \rightarrow \infty$ we wish for $U \rightarrow U_\infty$, which is only possible when $D'(x) = 0$ therefore $D(x) = C$, where C is some constant, and $C_2 = U_\infty$. Here U_∞ is an undisturbed flow speed that would be observed without the turbines (for the case without blockage). This means that the use of equation 4.19 for self-similar near-wake mixing is only valid before the two mixing layers interact and begin to reduce the maximum momentum deficit, $D(x)$, in the centre line. This explains why an additional model will need to be employed for the far wake stage.

We will now consider the use of a self-similar Gaussian function for the recovery of momentum in the far wake. Using a function of the form

$$\frac{|\mathbf{U}(x_3, y)|^2 - |\mathbf{U}(x_2, y)|^2}{U_0^2} = D(x_3) \left(\exp \left(-\frac{y^2}{\sigma^2(x_3)} \right) \right) - D(x_2) \left(\exp \left(-\frac{y^2}{\sigma^2(x_2)} \right) \right), \quad (4.31)$$

to calculate the added momentum in the far wake stage, where x_2 is the start of the far wake. We again omit pressure and integrate between x_2 and x_3 (x_3 being analogous to x_1 in the near wake case), again setting $U_0 = 1$ for simplicity we obtain,

$$-2 \int_{x_2}^{x_3} \frac{\partial \overline{u'v'}}{\partial y} dx = D(x_3) \left(\exp \left(-\frac{y^2}{\sigma^2(x_3)} \right) \right) - D(x_2) \left(\exp \left(-\frac{y^2}{\sigma^2(x_2)} \right) \right), \quad (4.32)$$

therefore,

$$-\frac{\partial \overline{u'v'}}{\partial y} = \frac{1}{2} \left(D'(x) + \frac{2y^2 \sigma'(x) D(x)}{\sigma^2(x)} \right). \quad (4.33)$$

Integrating with respect to y we obtain the Reynolds shear stress as

$$-\overline{u'v'} = \frac{D'(x) \sqrt{\pi} \sigma(x)}{4} \operatorname{erf} \left(\frac{y}{\sigma(x)} \right) + \frac{D(x) \sigma'(x)}{4 \sigma(x)} \left(\sqrt{\pi} \sigma \operatorname{erf} \left(\frac{y}{\sigma(x)} \right) - 2y \exp \left(-\frac{y^2}{\sigma^2(x)} \right) \right) + C_1(x), \quad (4.34)$$

at $y = 0$ we want $\overline{u'v'} = 0$, again due to assumed symmetry in the wake, which gives us $C_1 = 0$. Using the eddy viscosity hypothesis again by substituting equa-

tion 4.34 into 4.29 and solving for U , we obtain

$$U = \frac{1}{2\nu_T} (D'(x)\sigma(x) + D(x)\sigma'(x)) \left(\operatorname{yerf} \left(\frac{y}{\sigma(x)} \right) + \frac{\sigma(x)}{\sqrt{\pi}} \exp \left(-\frac{y^2}{\sigma^2(x)} \right) \right) + \frac{D(x)\sigma(x)\sigma'(x)}{4\nu_T} \exp \left(-\frac{y^2}{\sigma^2(x)} \right) + C_2. \quad (4.35)$$

We again apply the boundary condition that as $y \rightarrow \infty$ we wish for $U \rightarrow U_\infty$. In applying this limit we find that the term $\operatorname{yerf}(y/\sigma) \rightarrow \infty$, which would imply an infinite flow speed. To prevent this we require the whole term to cancel out, and this occurs when the condition $D'(x)\sigma(x) + D(x)\sigma'(x) = 0$ is met. We have that

$$D'(x)\sigma(x) + D(x)\sigma'(x) = \frac{d}{dx}(D(x)\sigma(x)) = 0. \quad (4.36)$$

This implies that the maximum momentum deficit and the characteristic growth rate need to be inversely proportional, i.e.

$$D(x) = \frac{C_{def}}{\sigma(x)}, \quad (4.37)$$

where C_{def} is a constant relating the growth rate to the maximum deficit.

4.4.1.1 Validity of $\frac{\partial V}{\partial x}$ omission in eddy viscosity hypothesis

The calculations made in the previous section neglect the $\frac{\partial V}{\partial x}$ term in the eddy viscosity hypothesis for simplicity; here we will examine its validity. Starting with the far wake and assuming the velocity profile approximately follows a Gaussian deficit we have

$$\frac{U}{U_0} = 1 - C(x) \exp \left(-\frac{y^2}{\sigma^2} \right), \quad (4.38)$$

where $C(x)$ is the maximum velocity deficit in the wake at position x . From here we find that

$$\frac{\partial U}{\partial y} = \frac{2yC(x)U_0}{\sigma^2(x)} \exp\left(-\frac{y^2}{\sigma^2}\right), \quad (4.39)$$

using the conservation of mass

$$\frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} = 0, \quad (4.40)$$

we may obtain V from our prescribed function $U(x, y)$. We find that

$$V = \int \frac{\partial U}{\partial x} dy = U_0 \int \left(C' + \frac{2y^2 \sigma' C}{\sigma^3} \right) \exp\left(-\frac{y^2}{\sigma^2}\right) dy. \quad (4.41)$$

This becomes

$$\frac{V}{U_0} = \frac{\sqrt{\pi}}{2} (\sigma' C + C' \sigma) \operatorname{erf}\left(\frac{y}{\sigma}\right) - \frac{\sigma' C y}{\sigma} \exp\left(-\frac{y^2}{\sigma^2}\right) + C_1(x). \quad (4.42)$$

At $y = 0$ we wish $V = 0$ therefore $C_1(x) = 0$ and as $y \rightarrow \infty$ we wish for $V = 0$ which gives us the condition that $\sigma' C + C' \sigma = 0$, like before. This leaves us with just the second term on the right-hand side. From here we differentiate with respect to x to gain the other term in the eddy viscosity hypothesis

$$\frac{\partial V}{\partial x} = -\frac{2yC(x)U_0}{\sigma^2} \left(\frac{y^2 \sigma'^2}{\sigma^2} + \frac{\sigma'' \sigma}{2} + \frac{C' \sigma' \sigma}{2C} - \frac{\sigma'^2}{2} \right) \exp\left(-\frac{y^2}{\sigma^2}\right), \quad (4.43)$$

which becomes,

$$\frac{\partial V}{\partial x} = -\left(\frac{y^2 \sigma'^2}{\sigma^2} + \frac{\sigma'' \sigma}{2} + \frac{C' \sigma' \sigma}{2C} - \frac{\sigma'^2}{2} \right) \frac{\partial U}{\partial y}. \quad (4.44)$$

Again setting $\sigma'' = 0$ and $C'/C = -\sigma'/\sigma$ we obtain

$$\frac{\partial V}{\partial x} = - \left(\frac{y^2 \sigma'^2}{\sigma^2} - \sigma'^2 \right) \frac{\partial U}{\partial y}. \quad (4.45)$$

Comparing the size of both shear components gives us

$$\left| \frac{\partial V}{\partial x} \right| / \left| \frac{\partial U}{\partial y} \right| = \left| \left(\frac{y^2 \sigma'^2}{\sigma^2} - \sigma'^2 \right) \right| < \left| \left(\frac{y^2 \sigma'^2}{\sigma^2} \right) \right| \propto \frac{y^2}{x^2}, \quad (4.46)$$

as $\sigma \propto \sigma' x$. Therefore when $x > y$ the $\frac{\partial U}{\partial y}$ term dominates, which for our stage of interest in the far wake will hold as the boundary $y^2 = x^2$ grows with slope 1 whereas the wake (our stage of interest) grows with slope $\sigma' \ll 1$. For the case where $y \rightarrow \infty$ we have that $\frac{\partial U}{\partial y} \rightarrow 0$, therefore when $\frac{\partial V}{\partial x}$ is not dominated by $\frac{\partial U}{\partial y}$, both terms will be negligibly small. Thus omitting $\frac{\partial V}{\partial x}$ should not have a significant impact on results.

A similar calculation can be done for the near wake. However, $\frac{\partial U}{\partial y}$ term only dominates when downstream distance x grows large, which is not true in the near wake thus the assumption is only valid for the far wake. Despite this we will maintain the assumption in the near wake as well for simplicity.

4.4.2 Implementation

This section outlines the implementation of the above theoretical mixing model in the calculation of multi-row turbine wakes, using the hybrid inviscid-viscous approach. Firstly, we reintroduce pressure into the model as it was omitted from the derivation of the mixing model in section 3.2. To include pressure recovery, we will consider the conservation of mass flux and assume that spanwise equalisation of pressure is maintained in the viscous mixing region. We also reintroduce the characteristic wake width, σ^* , in the far wake function to match the commonly

used form. By setting

$$\Omega = \frac{|\mathbf{U}(x, y)|^2 - |\mathbf{U}(x_0, y)|^2}{U_0^2}, \quad (4.47)$$

we obtain the full momentum recovery equation as

$$\Omega = \begin{cases} C_T \left(\operatorname{erf} \left(\frac{y+A(x)}{b(x)} \right) - \operatorname{erf} \left(\frac{y-A(x)}{b(x)} \right) - \operatorname{erf} \left(\frac{y+A(x)}{\epsilon} \right) \right. \\ \quad \left. + \operatorname{erf} \left(\frac{y-A(x)}{\epsilon} \right) \right) + \Delta p^*(x), & \text{if } b(x) < \frac{A(x)}{3} \\ D(x) \left(\operatorname{erf} \left(\frac{y+A(x)}{b(x)} \right) - \operatorname{erf} \left(\frac{y-A(x)}{b(x)} \right) \right) \\ \quad - C_T \left(\operatorname{erf} \left(\frac{y+A(x)}{\epsilon} \right) - \operatorname{erf} \left(\frac{y-A(x)}{\epsilon} \right) \right) \\ \quad + \Delta p^*(x), & \frac{A(x)}{3} \leq b(x) < \frac{A(x)}{1.1} \\ D(x) \left(\exp \left(-\frac{y^2}{2\sigma^{*2}(x)} \right) \right) - C_T \left(\operatorname{erf} \left(\frac{y+A(x)}{\epsilon} \right) \right. \\ \quad \left. - \operatorname{erf} \left(\frac{y-A(x)}{\epsilon} \right) \right) + \Delta p^*(x), & b(x) \geq \frac{A(x)}{1.1} \end{cases} \quad (4.48)$$

where $\Delta p^*(x)$ is the difference in pressure between x_0 and x , normalised by density and characteristic velocity squared ($p^* = p/\rho U_0^2$). The replacement of a with $A(x)$ is detailed in section 4.4.2.2. To calculate Δp^* we consider the conservation of mass flux. For the case presented in figure 4.9, where at the end of the inviscid section there are $N + 2$ streamtubes with velocity $\beta_i U_0$ and cross-sectional area $R_{\beta_i} d/2$, we will define the average velocity in the same cross-sectional area, at a given position $x = X$, as $\xi_i U_0$, which is calculated from equation 4.48. For the conservation of mass flux to hold we require that

$$\sum_{i=0}^{N+1} R_{\beta_i} \beta_i = \sum_{i=0}^{N+1} R_{\beta_i} \xi_i. \quad (4.49)$$

To achieve this we root find for a Δp^* in equation 4.48, such that equation 4.49 is satisfied. In this section we consider the scenario in which the mass flux through the whole array is fixed. For real wind farms, this is not true, therefore, this analysis can be viewed as an analysis of the ‘internal’ flow problem (i.e. ignoring farm-scale effects) of a two-scale coupled problem (Nishino and Dunstan, 2020). Depending on how we model the ‘external’ flow problem, a different condition for the calculation of Δp^* may be employed and equation 4.49 may be adjusted accordingly. The other unknown in the model is σ^* , which we will tune against LES results of Vahidi and Porté-Agel (2022) later in section 4.4.3.

4.4.2.1 Definition of near and far wake stages

The near and far wake models proposed will be utilised in the following manner. The near wake model starts immediately downstream of the actuator disk ($x = x_0$) and progresses till $b(x) \approx \frac{A(x)}{3}$ where the two mixing layers begin to interact, i.e. this will signal the end of a ‘near wake’ stage and the start of an ‘intermediate wake’ stage. The intermediate stage will still be characterised by the two error functions, but will be normalised by the maximum momentum deficit $D(x)$, as the mixing is now impacting the centre of the wake. We maintain this form until $b(x) = \frac{A(x)}{1.1}$, at which point the functions for the near wake and far wake are approximately equal (as depicted in figure 4.8 at $b(x) = a/1.1$). Calculating the difference between the two plots for a in the range $0.5d \leq a \leq 400d$, there is a maximum difference of approximately 2.5%. Therefore, due to the small difference between the two modelling functions, we will use this point to transition between the near and far wake models.

4.4.2.2 Multi-row superposition

We need to consider how to model the interaction of multiple wakes from different (n) rows of turbines. To do this we apply a superposition approach, in which the momentum recovery at a distance X behind the i -th turbine, located at position $x = X_i$, is equal to the sum of momentum recovery behind all individual turbines located upstream of our position of interest ($x = X_i + X$), with an adjustment for the inviscid expansion each wake would experience as it passes through or around a downstream turbine, as detailed below.

Let us define $F_T(x)$ as the total momentum gain (or recovery of momentum deficit) at location $x = X_i + X$. We will define $F_k(X_i, x)$ as the momentum gain between $x = X_i$ and $x = X_i + X$ for the scenario where the k -th turbine is the only turbine in the flow (but still experiences an inflow that matches the case where all turbines located upstream of our position of interest, including the k -th turbine, are present).

$$F_T(x) = \sum_{k=1}^i \left(F_k(X_i, x) \prod_{j=k}^i Y_j^* \right), \quad (4.50)$$

where Y_j^* describes the mapping of the spanwise position (y) of streamlines in the inviscid region from upstream to downstream of the j -th turbine. The change in spanwise streamline location is illustrated in figure 4.9, which depicts the flow in the inviscid region of the hybrid inviscid-viscous model. For example, the streamline with spanwise position $y = R_0 d/2$ at the upstream position is deflected to the position $y = (R_{\beta_0} + R_{\beta_1})d/2$ in the downstream. Full calculations for solving this system can be found in chapter 3, however, we note that the thrust

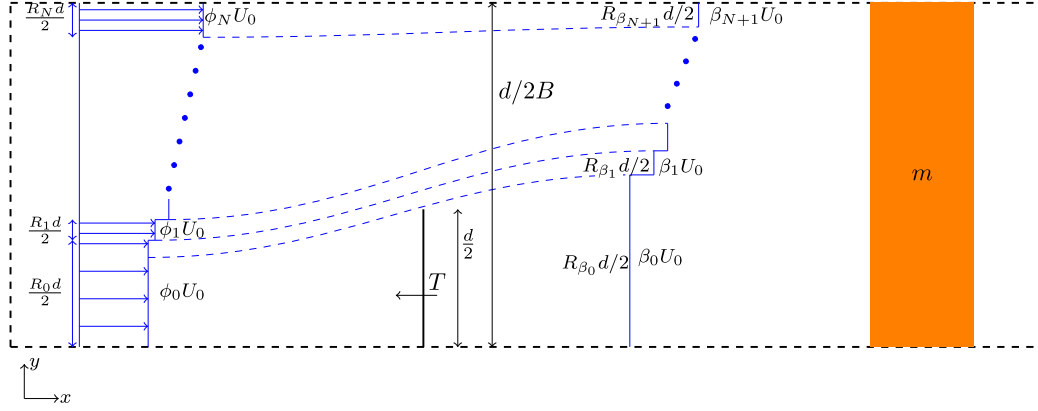


Figure 4.9: Schematic of the inviscid flow region with local blockage ratio B , reproduced with modifications from chapter 3

coefficient, C_T , used in the prior analysis is calculated as

$$C_T = \frac{T}{\frac{1}{2}\rho U_0^2 d}, \quad (4.51)$$

where T is the turbine thrust, applied uniformly across the disk. For a streamline starting at position (x_0, y_0) and ending at (x_1, y_1) we have that

$$Y^* = \frac{y_1}{y_0}, \quad (4.52)$$

where we can calculate y_1 and y_0 from the results of LMADT (chapter 3).

In addition to the calculation of Y^* , we also need to consider the impact of inviscid flow expansion on the width of our near wake model profile. This consideration is required because we use the wake model to predict the flow immediately downstream of the turbine, whilst the ‘initial’ width of our wake model, $2a$, corresponds to the width of the streamtube after the inviscid expansion predicted by the LMADT. The LMADT model, however, does not provide any information about the distance over which the streamtubes expand. We therefore arbitrarily introduce an inviscid expansion distance, $x_{inv} = 3d$, over which the core flow

downstream of the turbine expands. The expansion at location x is described approximately by

$$A(x) = 1 + (a - 1) \left(\tanh \left(\frac{3x}{x_{inv}} \right) \right), \quad (4.53)$$

such that at $x = x_{inv}$ the inviscid expansion has completed. The inviscid expansion region is considered such that the half initial wake width a utilised in the wake model is now replaced by $A(x)$, varying with x . This additional step is employed to make meaningful direct comparisons to LES data where mixing occurs immediately behind the turbine (as detailed in section 4.4.3). However, we still assume that there is no mixing $1d$ upstream of each turbine, and the velocity profile at $1d$ upstream is used as the inflow condition for the LMADT for each row of turbines.

Figure 4.10 depicts the difference between the self-similar mixing model presented in this chapter and the uniform mixing model used in chapter 3. The example case is of three turbine rows in unblocked flow with a streamwise spacing of $7d$. The self-similar mixing model is presented in figure 4.10a, whereas figure 4.10b depicts the wake calculated by the uniform mixing model with the mixing rate m being calculated from the results of the self-similar mixing model. Note that the inviscid region is assumed to be only from $1d$ upstream to the turbine position, i.e. we apply the mixing model immediately behind the disk. Each turbine operates at a uniform resistance coefficient $k = 2$. The resistance coefficient k is calculated as

$$k = \frac{T}{\frac{1}{2}\rho d(U_T^2)}, \quad (4.54)$$

where U_T is the average flow speed through the turbine.

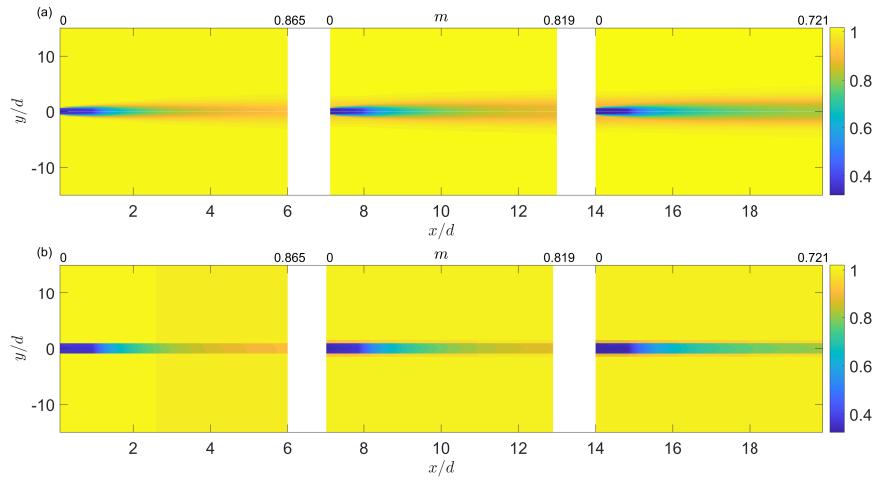


Figure 4.10: Normalised velocity contour U/U_0 of flow past three turbines at fixed resistance coefficient of $k = 2$ with streamwise spacing of $X_s/d = 7$. The white gaps in the figure indicate the inviscid flow regions in the hybrid inviscid-viscous approach. Plot (a) shows results utilising the self-similar mixing model proposed in this chapter, and plot (b) shows the equivalent uniform mixing model used in chapter 3. The corresponding uniform mixing rate, m , at the start and end of the viscous mixing zone, is plotted on the top x-axis.

4.4.2.3 Characteristic wake width

It is common to model $\sigma^*(x)$ as a linear function of x , in the form of $\sigma^* = k^*x + c$. However, the characteristic wake width does not always grow linearly with distance x (Tian et al., 2022). Therefore we model $\sigma^*(x)$ as evolving non-linearly with x , which allows us to also better capture the maximum momentum deficit which evolves non-linearly. With the condition from 4.37, we only have to model σ^* to model the evolution of the momentum deficit. We model $\sigma^*(x)$ as a linear function with a variable exponent

$$\sigma^*(x) = (k^*x + c)^q, \quad (4.55)$$

where k^* is the wake growth rate and dependent on the turbulence intensity T_i (defined such that $T_i = 0.1$ at 10% turbulence intensity), c is a constant such that $\sigma^*(x) = b(x)$ at $b(x) = a/1.1$, and q is a constant.

For the near wake, the characteristic wake width is modelled as

$$b(x) = \eta(k^*x)^q. \quad (4.56)$$

Here we have introduced another scalar, η , which may allow us to model additional effects in the near wake; such as the dynamics of blade tip vortices and blade-induced turbulence (however, these effects will not be considered explicitly in this chapter) (Nishino and Willden, 2014; Chamorro and Porté-Agel, 2009). Ideally, the impact of varying turbulence levels in the streamwise direction should also be incorporated into the model as a function of x , such as those proposed by Crespo and Hernández (1996); this, however, is out of the scope of the thesis.

4.4.3 Tuning the model

The model is tuned against existing LES data (Vahidi and Porté-Agel, 2022), in which the turbine is modelled as a rotating actuator disk. We are tuning against the results from the horizontal plane at the hub height. Two key differences to note are that the rotating actuator disk model does not produce uniform thrust, and has immediate mixing behind the disk as opposed to a region of tip vortices that prevent mixing between the bypass and core flows in the near wake (Lignarolo et al., 2015). In this chapter, all properties of the turbine such as thrust and power are still derived from the uniform-thrust LMADT (chapter 3) which does not consider any mixing in the region immediately downstream of the turbine. It is worth noting that the power and thrust coefficients will differ from the LES data of a rotating actuator disk in which mixing occurs immediately, as this will affect the momentum balance around the turbine (Nishino and Willden, 2013a); however, such effects of near-wake mixing on the power and thrust coefficients are not considered in this study for simplicity.

The model is tuned in two separate stages. Firstly, the characteristic wake width σ^* is obtained directly from the LES data and fed into the model to find the constant C_{def} from equation 4.37. Following this the parameters k^* , η , and q , are tuned to obtain a reasonable fit with σ^* . For lower turbulence intensities the differences between the current model and LES increase, particularly in the near wake, meaning the characteristic wake width is difficult to replicate. For this reason, our tuning will focus on medium to high turbulence intensity cases (cases 1 to 3 in Vahidi and Porté-Agel (2022)).

We can observe, in figures 4.11, that the agreement between the model and data is best for higher turbulence intensities. This is likely due to the differences between the uniform thrust LMADT and the rotating actuator disk model used in the LES

Case	k^*	q	η	T_i
1	0.34	1	1.8	14.13%
2	0.395	0.925	1.75	10.15%
3	0.58	0.91	1.45	7.89%

Table 4.1: Tuned parameter values for cases 1-3 from Vahidi and Porté-Agel (2022) with their respective ambient streamwise turbulence intensities.

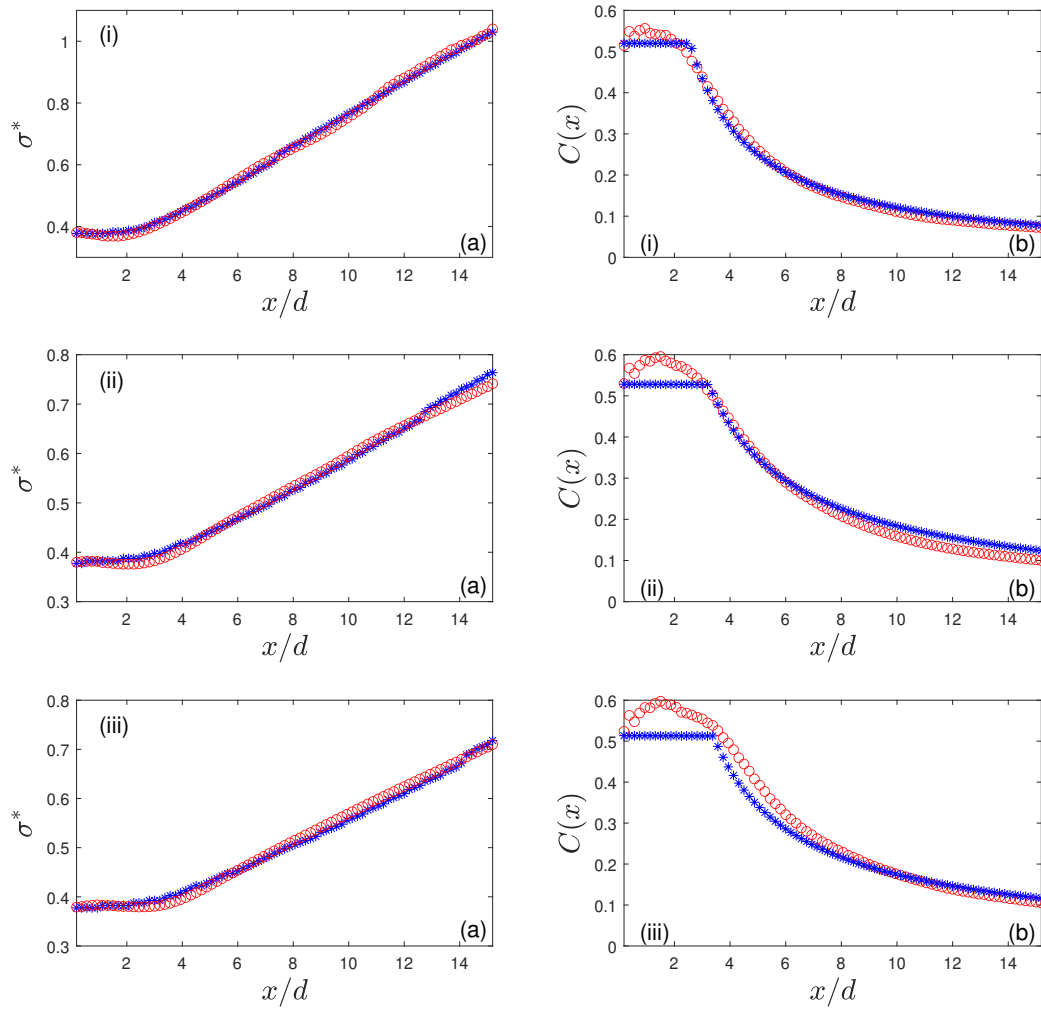


Figure 4.11: The characteristic wake width (a) and maximum velocity deficit in the wake (b) for cases 1 (i), 2 (ii), and 3 (iii). Red circles show data obtained from LES and blue stars show the results obtained from the momentum deficit model with the tuned parameter values shown in table 4.1.

which prescribes a thrust profile to the disk. The thrust profile prescribed by the rotating actuator disk model causes a greater maximum momentum deficit in the near wake, when compared with the uniform thrust model for the same C_T , and this difference in maximum momentum deficit is greatest in the near wake when the turbulence intensity (and thus the mixing rate) is small. This can be seen in figures 4.11 (ii,b) and (iii,b). However, at higher levels of wake mixing the inhomogeneity in the wake is reduced, leading to better agreement as seen in figure 4.11(i,b).

Presented below are the three empirical model functions fitted against the three cases summarised in table 4.1, for defining the parameters impacting the wake growth rate

$$k^* = 0.335 + 32.98\exp(-62T_i), \quad (4.57)$$

$$q = 1.43T_i + 0.79, \quad (4.58)$$

$$\eta = 1.806 - 214.7\exp(-81T_i). \quad (4.59)$$

Note, these were only fitted against minimal data and would only be valid for a range of ambient turbulence intensity of $7.89\% \leq T_i \leq 14.13\%$. These were also only for a single C_T value of ≈ 0.8 , and we expect these parameters may change further with varying C_T .

4.4.4 Comparison with uniform mixing model

Now we will use the proposed mixing model to predict the power output of multi-row turbine arrays. Our aim here is to make a comparison between the proposed model and the uniform mixing model used earlier in this chapter and in previous studies (Ouro and Nishino, 2021) to assess their impact on the prediction of array power. Since the uniform mixing model simply describes the completion

rate of mixing between adjacent rows without considering the streamwise position of each row explicitly, the analysis presented below is not concerned with predicting the wake profile at a given streamwise position x . Instead, the analysis only requires the relationship between $\sigma^*(x)$ and $C(x)$ to be accurate. For this reason, all self-similar wake computations in the following analysis will use a turbulence intensity of $T_i = 14.13\%$, matching case 1 from Vahidi and Porté-Agel (2022). Note, although the wake of an upstream turbine would be expected to vary the thrust profile across a turbines blade (as discussed in section 3.3.3), the downstream distances considered in this analysis are great enough to avoid any significant shear across the disk. Therefore, we expect maintaining the uniform thrust description of the turbine to well represent the system being analysed.

To compare the two approaches we need to relate the flow recovery in the self-similar model to the mixing rate parameter m in the uniform model. In this chapter, we consider aligned rows of turbines, and we define the mixing rate, m , as the average recovery rate of the flow with a cross-sectional area matching the area of the turbine. The mixing rate for the uniform mixing model can be written as

$$m = \frac{u_{out} - u_{in}}{U_0 - u_{in}}, \quad (4.60)$$

where u_{in} (u_{out}) is the flow speed at the start (end) of the viscous mixing zone. To estimate m from the self-similar model we take u_{in} as the average velocity of the flow for $y \leq d$ at $x = X_i$ and u_{out} is the average velocity for $y \leq d$ at the end of the viscous mixing zone. A visualisation of the flow field predicted by the uniform mixing model is presented in figure 4.10b, using the value of m calculated from the flow field shown in figure 4.10a.

4.4.4.1 Two turbine rows

For a perfectly aligned array of turbines consisting of two rows, we compare the power of the downstream turbine predicted using the self-similar model and the uniform mixing model. We conduct this comparison across a range of cross-sectional blockage ratios B , streamwise turbine spacings X_s/d , and resistance coefficients for the upstream and downstream turbines, k_1 and k_2 , respectively. Figure 4.12 shows the ratio of the two power coefficients predicted by the LMADT with the uniform mixing and self-similar mixing models, respectively, for the second turbine row (defined as $C_{P_{Uni}}/C_{P_{Sim}}$, where *Sim* refers to self-similar mixing and *Uni* refers to uniform mixing). The power coefficient C_P is defined as

$$C_P = \frac{P}{\frac{1}{2}\rho d \overline{u^3}}, \quad (4.61)$$

where P is the turbine power calculated by the LMADT (chapter 3) and $\overline{u^3}$ is the average of the cube of the upstream velocity (at the end of the upstream viscous region, defined as $1d$ upstream of the turbine) of the fluid passing through the turbine (Draper et al., 2016).

The subfigures in figure 4.12 are for different blockage ratios and resistance coefficients of the first turbine row. A higher value of k_1 implies a greater maximum velocity deficit behind the first-row turbine and thus greater shear in the inflow profile for the second-row turbine. Note, the range of mixing rate m varies between different subfigures because, although the range of streamwise inter-turbine spacings X_s/d is fixed at $5 \leq X_s/d \leq 25$, m is dependent on k_1 and B (due to the dependence of mixing on the thrust coefficient of the first-row turbine). A general trend is that the higher the resistance coefficients, the more the uniform mixing model overpredicts the power compared with the self-similar mixing model.

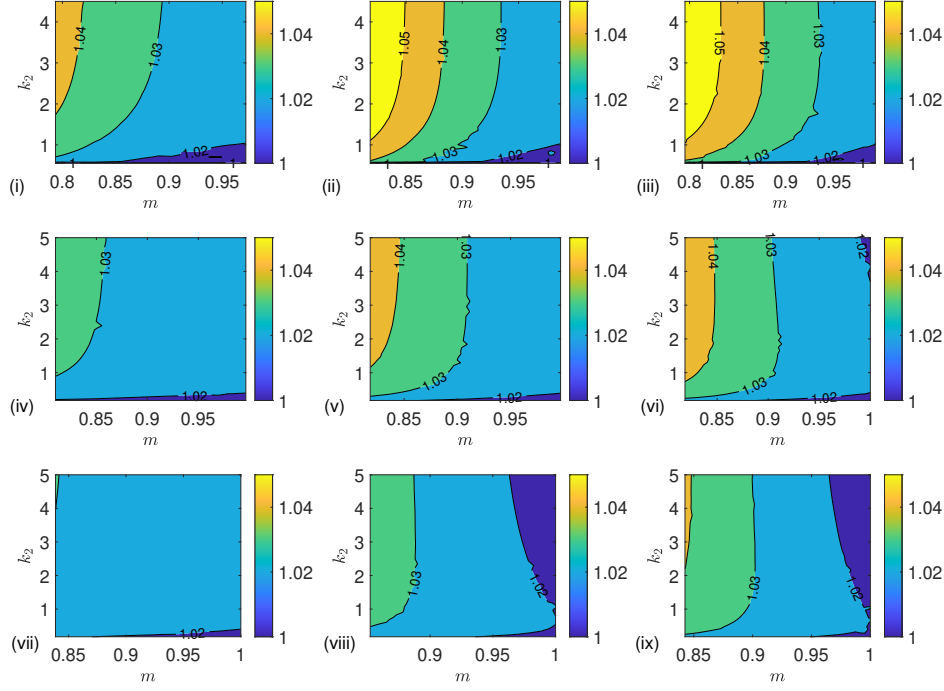


Figure 4.12: Contour plots of the ratio between the uniform mixing and self-similar mixing model predictions of the power coefficient, $C_{P_{Uni}}/C_{P_{Sim}}$, for the downstream turbine in two rows, for three different blockage ratios: $B = 0.03$ (i,ii,iii), 0.11 (iv,v,vi), and 0.2 (vii,viii,ix). Each plot is for a different upstream turbine resistance: $k_1 = 0.51$ (i), 1.9 (ii), 4.35 (iii), 0.57 (iv), 2.2 (v), 5.45 (vi), 0.64 (vii), 2.6 (viii), 6.9 (ix).

Overall the two models agree reasonably well with a difference in predictions only up to around 5%, and this difference decreases as we increase the blockage ratio. A greater difference at higher values of k_1 is expected as the higher the value of k_1 the greater the extent of shear across the turbine which is averaged across; this is consistent with the findings in chapter 3 for shear flow through a turbine.

4.4.4.2 Three turbine rows

The previous section compared the two different wake modelling approaches in terms of the prediction of downstream turbine power in two aligned rows. In this section, we focus on the total power of three aligned rows of turbines with

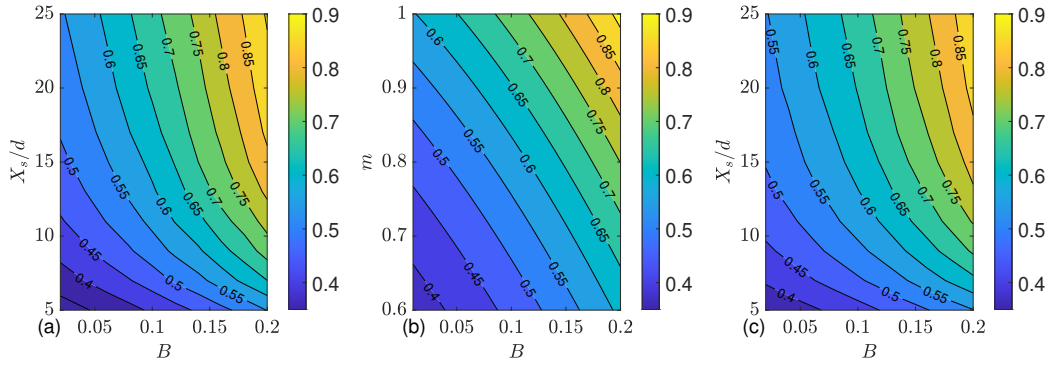


Figure 4.13: Contour plots of the maximum value of three-row-average C_P achievable with a single optimal value of k for a given array, predicted using the self-similar mixing model (a), the uniform mixing model (b), and the uniform mixing model with m being calculated from the self-similar model predictions for a given combination of X_s/d and B (c).

optimised turbine resistances. To visualise if the general trends predicted by the two mixing models are comparable to each other, we compare the maximum array averaged power achieved by optimising the turbine resistance k over the three rows, for various mixing rates (or row spacings) and blockage ratios.

To make a meaningful comparison between the two models we consider three different types of calculations: (i) using the self-similar model for a range of row spacing, (ii) using the uniform mixing model for a range of mixing rate m , and (iii) using the uniform mixing model with m given as a function of the turbine resistance, k , and distance, X_s , from the flow field predicted by the self-similar model. The aim of this analysis is to demonstrate how well the uniform mixing model may capture the trends of optimal array performance when the correct mixing rate (for a given array configuration) is known, with respect to the self-similar model predictions.

Figure 4.13 shows the maximum value of three-row-average C_P for a range of array configurations (i.e. streamwise row spacing, represented by either X_s/d or m , and spanwise turbine spacing, represented by the blockage ratio B) for the

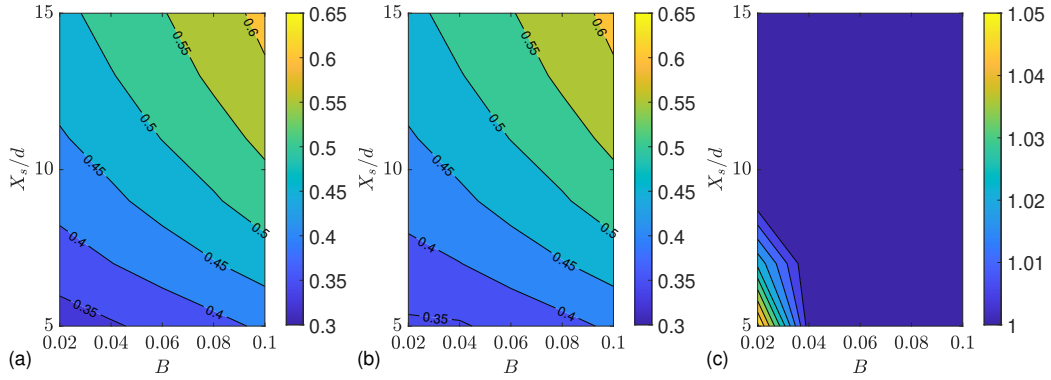


Figure 4.14: Contour plots of the maximum value of three-row-average C_P predicted using the self-similar mixing model for two different scenarios, where k is not allowed to vary across different rows (a) and k is allowed to vary across different rows (b). The ratio between the results of (b) and (a) are presented in figure (c).

scenario where a single optimal value of turbine resistance k is used for a given array. Overall we see very good agreement between the self-similar mixing model in figure 4.13a and the uniform mixing model, with m being calculated from the self-similar model for a given combination of X_s/d and B in figure 4.13c. These results suggest that the uniform mixing model used in chapter 3 and previous studies (Ouro and Nishino, 2021) is indeed sufficient to predict the maximum performance of multi-row turbines when the mixing rate m is modelled accurately as a function of X_s/d and B .

Figures 4.14a and 4.14b show the maximum value of three-row-average C_P for the scenarios where a single optimal value of k is used for the whole array and where the value of k is allowed to vary across the three rows, respectively, predicted using the self-similar model. To quantify the performance gains achieved by varying k , figure 4.14c shows the ratio $C_{P_{max}}^V / C_{P_{max}}^F$, where $C_{P_{max}}^V$ is the maximum C_P shown in figure 4.14b and $C_{P_{max}}^F$ is maximum C_P shown in figure 4.14a. Echoing the findings in section 4.4 there is a small performance increase of up to about 5% due to varying k for lower blockage ratios and smaller row spac-

ing. However, as we can observe in figures 4.14a and 4.14b, the value of C_P is low in this region of parameter space. For example, an increase in the row spacing by $2d$, with fixed resistance coefficient, would increase the average power coefficient by $\sim 15\%$; a much larger improvement over the benefits gained by varying k .

5 | Anisotropic

Symmetry is what we see at a glance; based on the fact that there is no reason for any difference.

Blaise Pascal

In previous chapters, we have explored the impact of blockage on the performance of a turbine or turbine array. The turbine has, so far, been placed in the centre of the channel. However, wind turbines in a wind farm do not experience blockage equally, i.e. there is a difference between the blockage effects experienced in each direction, from the ground, its sides (due to neighbouring turbines), and above (due to an atmospheric capping layer). This anisotropic local blockage effect results in the predictions of commonly used isotropic blockage correction models, for the optimal performance of a blocked turbine, to become inaccurate, as discussed later in this chapter. Although there are experiments and simulations exploring the impact of anisotropic blockage, there is little theoretical work to accurately predict its impact on wind turbines in a wind farm. The aim of this chapter is to create a theoretical framework, informed by data, for predicting the impact of anisotropic local blockage effects.

5.1 Introduction

As detailed in chapter 3 a range of blockage correction methods have been developed and applied to laboratory experiments or to arrays of turbines (Ross and Polagye, 2020). The methods that were found to most closely capture the experimental results were those of Houlby et al. (2008), Barnsley and Wellicome (1990), and Garrett and Cummins (2007). These methods, however, do not consider the variation in blockage a turbine experiences in each azimuthal direction from variations in physical boundaries or lines of symmetry. This causes inaccurate predictions when utilising them for anisotropically blocked systems like a lateral array of wind turbines.

In this section we present a novel model based on the separation of scales between flow which is constrained by the closest boundary, the flow which is constrained by the next closest, and the flow which is most constrained by the furthest boundary. Each of these scales are computed utilising the LMADT methods presented in section 3.2.

5.2 RANS simulations

In this section we detail the 3D RANS simulations of an infinitely wide array of (non-rotational) actuator disks used to explore the impact of anisotropic local blockage effects. We first present the flow and turbine set-up, followed by a description of the computational grid, along with a mesh validation study. The computational set-up for these simulations is the same as that in section 3.3.

5.2.1 Computational grid: mesh test

Figure 5.1 shows an example of a 2D slice of the computational mesh at the turbine plane. To make the 3D mesh, this 2D slice is propagated in the streamwise direction (both upstream and downstream) to make a computational mesh consisting of exclusively hexahedral cells. The slices are spaced non-uniformly with minimum spacing of $0.003D$, where $D = 100\text{m}$ is the disc diameter, nearest to the turbine plane. The mesh inside the $0.75D \times 1.5D$ rectangle around the half turbine is held constant across all different blockage cases considered in this chapter. Different sized domains are defined by three values, H_B , H_A and W , which are the hub height, distance from the hub to the top of the domain, and the domain width, respectively. H_A and H_B take 17 non-equidistant values between $0.75D$ and $20.5D$, and W takes 13 non-equidistant values between $0.75D$ and $10.5D$. For convenience, the three parameters which dictate the size and configuration of the simulated domain will be written as the triple (H_B, H_A, W) . Not all combinations of values are considered as due to the symmetry of the system for example $(0.75D, 10.5D, 2D) = (10.5D, 0.75D, 2D)$, and the total height ($H_B + H_A$) is limited to $21.5D$, resulting in 1989 different domains simulated for this chapter.

A grid sensitivity study was conducted for the domain sized $(0.75D, 0.75D, 0.75D)$. The lower resolution grid had a total of 157200 cells and was compared against a fine resolution grid of 1063200 cells. The domain was chosen as the high shear present in these simulations would be sensitive to grid resolution. We define a ‘local’ or ‘internal’ power coefficient C_P^* as

$$C_P^* = \frac{P}{\frac{1}{2}\rho U_{in}^3 A}, \quad (5.1)$$

where P is the power of the turbine, A is the turbine swept area, and ρ the fluid density. As can be seen in figure 5.2, the two cases are near identical for all values

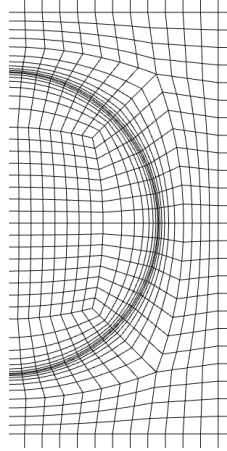


Figure 5.1: Cross-sectional segment of the computational mesh around the actuator half-disk at lower resolution.

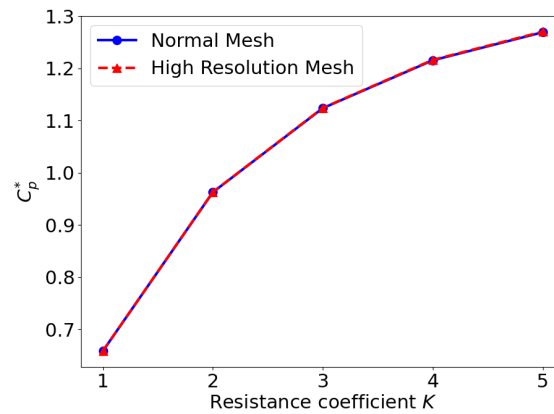


Figure 5.2: Impact of grid resolution on C_p^* for the highest blockage case.

of the disc resistance coefficient K , with a difference in C_p^* of less than 0.1% for all K values tested, suggesting that the lower resolution mesh is sufficient for this study.

Complexity	MSE	Equation
1	2.46749	1.847
2	1.69148	$\exp^B$
3	0.15287	$5.712 * B$
4	0.13487	$10.4 * B^2$
5	0.01157	$\frac{1}{(1.13-B)^2}$
6	0.00188	$\frac{1}{1.69*(1-B)}$
7	0.00003	$\frac{0.589}{(1-0.986*B)^2}$

Table 5.1: Suggested analytical functions for the isotropic blockage effect on $C_{P\max}^*$ found by symbolic regression along with their complexity and mean-squared error (MSE).

5.3 RANS simulation results

5.3.1 Symbolic regression for isotropic local blockage

Before presenting the results of 3D RANS simulations, we first present 2D RANS results, analysed using symbolic regression to showcase the technique for wind energy. The regression is run using an open source code called PySr (Cranmer, 2020). The motivation is that many physical systems can be accurately modelled with relatively concise analytic expressions. Symbolic regression works by testing some set of given functions against the data and rewarding functions which best fit the data. The first set of functions are randomly generated expressions for varying levels of complexity, where complexity is a measure of the number of operations in the expression. The code holds onto an n -sized set of ‘best’ equations for each level of complexity then mutates the best one for each complexity and replaces the weakest equation from the population with the new equation. By continually mutating and comparing with data the code attempts to “discover” an analytic expression which describes the data accurately. The code was given a range of operations to use in the search for an analytic expression (addition,

subtraction, multiplication, division, exponents, the exponential function, the logarithmic function, trigonometric functions, and hyperbolic functions).

An actuator disk (or line segment in the 2D system) of diameter 100m was placed in the middle of a 2D rectangular channel with 19 different heights to simulate a range of blockage ratios, B , from 2% to 66%. Where $B = A_t/A_{total}$ where A_t is the area of the turbine and A_{total} is the cross-sectional area of the domain. The other conditions remain identical to those detailed for the 3D RANS simulations. The symbolic regression method applied to the 2D RANS results suggested $C_{P\ max}^* = \frac{0.589}{(1-0.986*B)^2}$ for the analytical function to describe the isotropic blockage effect. This is promising, as not only does it find the format of the function it also closely approximates the true analytical equation $C_{P\ max}^* = \frac{16}{27(1-B)^2}$. The raw 2D RANS data differed from the analytical solution by as much as 3.5%, so although more accurate data is beneficial, the method can find promising results on relatively few and non-exact data points. The algorithm was fed many test functions to ensure that the result is not a product of guiding the algorithm to the known solution. A list of 7 equations, of varying complexity, suggested by the symbolic regression method are summarised in table 5.1. Now it is clear that symbolic regression can be a useful tool in the 'discovery' of analytical expressions to describe physical phenomena we will attempt to apply it to a scenario in which the expression is not already known, anisotropic local blockage.

5.3.2 Anisotropic local blockage

A selection of the results from the 3D simulations are presented in figure 5.3. The maximum C_P^* values predicted from the GC07 model are also plotted for comparison. We can observe that GC07 consistently under-predicts the power coefficient in an anisotropically blocked domain. This is understandable as GC07 assumes that the turbine is at the centre of the domain, away from any bound-

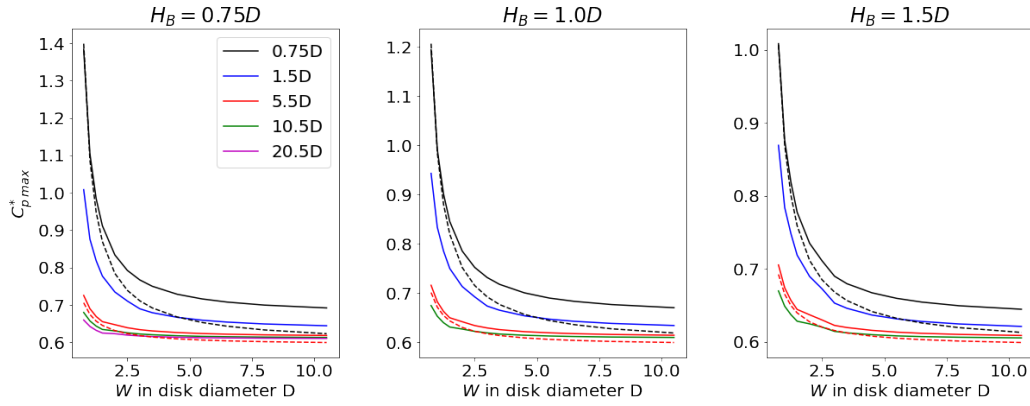


Figure 5.3: Effects of anisotropic local blockage on $C_{P,max}^*$. Solid lines are 3D RANS results. Dashed lines are GC07 prediction based on isotropic blockage. Different colours represent different values of H_A .

ary which would constrain the expansion of flow around the turbine, allowing the turbine extract more power from the flow, the same mechanism from which isotropically blocked flow can attain greater power coefficients. Although a large number of simulations were conducted for this particular study these simulations require a decent amount of computational resources. Unlike the isotropic case, the anisotropic case does not have a known analytical expression for optimal power coefficient such as the one developed by Garrett and Cummins (2007). Due to the above mentioned computational cost of simulating the anisotropic case we will explore the development of a new analytical model for anisotropic flow.

5.4 New method

The creation of a new method to estimate C_P^* for a turbine with an anisotropic local blockage has been attempted using data-driven methods. First, a series of refined analytical equations were obtained for the 3D problem from symbolic regression. The analytic equations obtained from this exploration improved on GC07, but they appeared as over-fitted equations with large exponents rather than

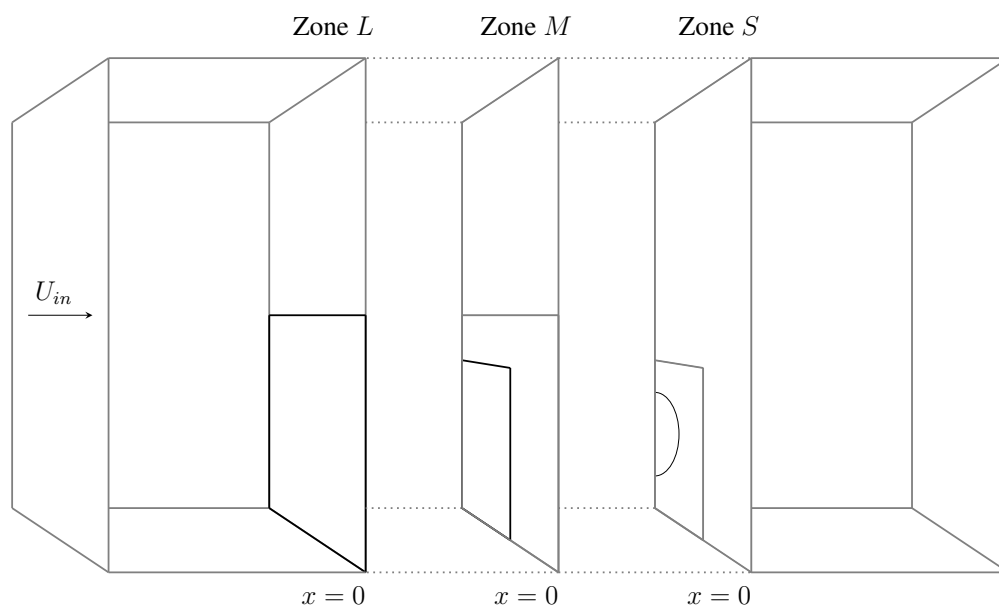


Figure 5.4: Schematic of the new method. Each black outlined shape is an actuator disk/surface, the grey lined zones are ignored when we focus on the actuator disk/surface in that domain.

concise equations describing interpretable physics. Hence, in this section, we will aim to create a new method in a different manner. This suggests that an analytic correction to the equation of GC07 is unlikely to exist in a manner which can be extracted from physical principals. That is not to say one is unattainable to derive, there may be other ways to define the domain which would provide fruitful results, it is just with the parameters and variables provided, an analytical equation with a clear functional form has not been found.

The inspiration for this new method is rooted in the two-scale actuator disk model set out by Nishino and Willden (2013b). The key idea used to develop this model is the deflection of upstream flow into a turbine's bypass; this first occurs at the array scale, then of the flow that passes through the array a portion will be deflected into the bypass at the turbine scale, as depicted in figure 5.5. In this formulation the scale at which the array and turbine impact the flow can be considered

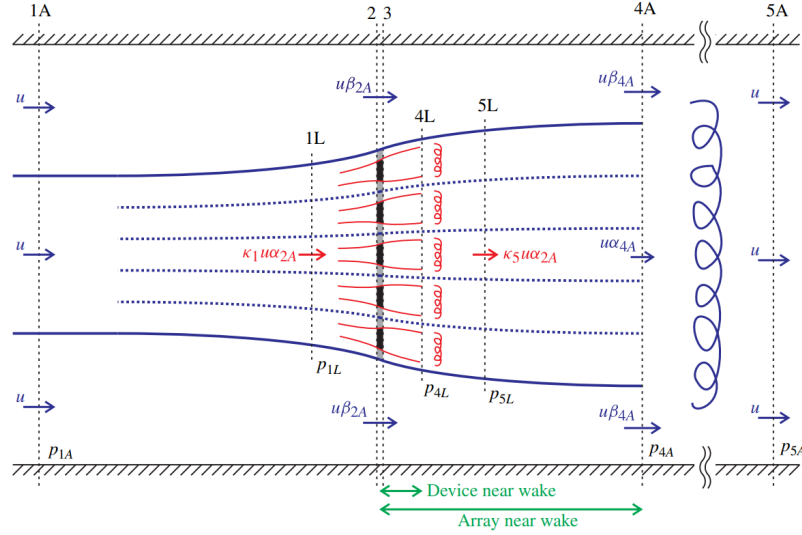


Figure 5.5: Schematic of the flow past a tidal fence showing flow at the array scale and turbine scale, reproduced from Nishino and Willden (2013b).

separately. In the same way that the array and turbines can be defined as separate but coupled regions of resistance, we aim to construct regions of resistance, in the anisotropic blockage case, to be coupled but separate that will sufficiently capture the deflection of upstream flow into the bypass for different turbine operating conditions and domain geometries.

Although this two-scale model was originally proposed for a short array of tidal turbines with isotropic local blockages, Nishino and Draper (2015) found that the model could be applied to this anisotropic case as well, where 3D RANS predicts $C_{P\max} = 0.65$ (for domain, $H_B = 1D$, $W = 0.75D$ and $H_A = 24D$) and the two-scale model predicts $C_{P\max} = 0.71$. The method presented below improves upon that with a prediction of $C_{P\max} = 0.66$.

The basic assumption made here for the new method is that the system can be approximately described using linear momentum actuator disk theory (LMADT) (chapter 3) by decomposing the expansion of flow through the disk into three

distinct but coupled sub-expansion processes. The new method begins by considering three sub-domains alongside the actuator disk. One is around the turbine itself, defined as the smallest box centred on the turbine which has an edge meeting a boundary (see figure 5.4); we will refer to this box as the small sub-domain, zone S . The middle sub-domain is defined as the smallest box centred on the turbine which meets at least one additional boundary than the small sub-domain, zone M . The largest sub-domain is defined as the next smallest box that meets an additional boundary to the middle sub-domain and consequently will always be the whole cross-sectional domain, zone L .

The turbine, smallest, and middle sub-domains are treated as actuator surfaces at the turbine plane with distinct blockage ratios and resistance coefficients. The smallest sub-domain has total area A_s , middle A_m , and largest A_l . The subscripts t , s , m , l will hereafter refer to the turbine, smallest sub-domain, middle sub-domain, and largest sub-domain, respectively. Blockage ratios between these three created domains and the size of the turbine A_t may be defined as $B_S = A_t/A_s$, $B_m = A_s/A_m$ and $B_l = A_m/A_l$.

We first assume the actuator surface being considered is contained within a uniform channel with the cross-sectional area that is equal to the maximum size of the streamtube passing through the sub-domain actuator that most immediately surrounds the considered surface (e.g. the small sub-domain when the considered surface is the turbine). This allows us to use the LMADT of Garrett and Cummins (2007) to calculate the expansion of flow through the considered surface. However, the true region in which the considered disk/surface operates is not uniform as assumed in the LMADT. This problem is overcome in this new method by introducing a key assumption that the proportion of upstream flow that passes through the turbine remains constant between the ‘uniform’ and ‘true’ cases.

To illustrate the above assumptions we discuss a simplified 2D system containing only two actuator surfaces instead of three, i.e. the turbine and the small sub-domain actuators, as depicted in figure 5.6. The LMADT equations are solved for the small sub-domain as an actuator disk in a domain sized A_t/B_{tot} with resistance coefficient K_s . The streamtube which passes through the small sub-domain actuator starts with size $\alpha_s A_t/B_S$ in the upstream, flows through the surface of area A_t/B_S with speed $\alpha_s U_{in}$, and then expands to the size of $R_s A_t/B_S$ in the downstream position. The blockage ratio for the turbine (for the hypothetical ‘uniform’ case) is now set as the ratio of the turbine area, A_t , to the largest size of the streamtube which bounds it, $R_s A_t/B_S$. Then applying the LMADT to the turbine with resistance coefficient K_t , a flow speed through the turbine (non-dimensionalised by the inflow speed U_{in}) α_t is obtained. This means that in this hypothetical ‘uniform’ case, the proportion α_t of the inflow passes through the turbine. By assuming that the same proportion of the upstream flow passes through the turbine area in the ‘true’ case, we eventually obtain $A_t \alpha_t \alpha_s / R_s$ as the true upstream domain width (instead of $A_t \alpha_t$) for the streamtube passing through the turbine area. Therefore, through conservation of mass, the true flow speed through the turbine (non-dimensionalised by U_{in}) is obtained as $\alpha_t \alpha_s / R_s$ instead of α_t .

The final step to construct this model is describing a relationship between the resistance coefficients for each actuator surface given a resistance coefficient for the turbine K_t . For the simplified 2D case discussed above, the relationship we adopt is described as

$$K_s = K_t \alpha_t^2 \alpha_s \frac{B_S}{R_s^4}. \quad (5.2)$$

Using the relation of K to the thrust T from LMADT, i.e. $T = \frac{1}{2} \rho A K (\alpha u)^2$, again

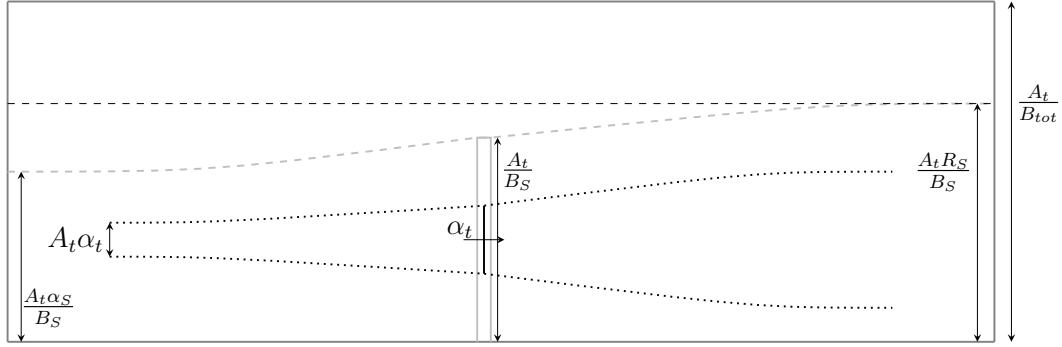


Figure 5.6: Schematic of the '2D' case for illustration of 'uniform' and 'true' flow passages. The solid black line is the turbine, the grey box encompassing it is the small sub-domain actuator, the grey dashed line is the 'true' bounding region, the black dashed line is the 'uniform' bounding region.

with appropriate subscripts, and ρ the density of the fluid, 5.2 can be rewritten as:

$$T_s = T_t \frac{\alpha_s}{R_s^2}. \quad (5.3)$$

This is equivalent to saying the thrust of the small sub-domain actuator (T_s) scaled to the size of the downstream recovery (R_s) is equal to the thrust of the turbine (T_t) scaled by the expansion rate of the streamtube passing through the small sub-domain (α_s/R_s). By applying the same approach to the original 3D system (containing three actuator surfaces instead of two), we have now obtained all three resistance coefficients and three blockage ratios for all three sub-domains. Finally, we numerically solve the LMADT equations for these three coupled sub-domains and calculate the value of C_P^* as

$$C_P^* = K_t \left(\frac{\alpha_t \alpha_s \alpha_m}{R_s R_m} \right)^3. \quad (5.4)$$

This method has the advantage of very low computational costs, taking about 0.1s to make a prediction (for a given blockage configuration) running on a single core, compared to the RANS simulations which can take several minutes to make

a prediction when running on multiple cores.

5.4.1 Assumption validation

In this section, we will explore the validity of the assumptions made in the analytical model. Here we will address the first of these assumptions: the proportion of upstream flow between the ‘uniform’ and ‘true’ cases remains constant. If we consider a turbine in a control volume with a cross-sectional inlet area of $\gamma A/B$ and outlet of A/B where $0 < \gamma \leq 1$, i.e. setting $\gamma = \alpha_S$, $B = B_S$, $A = A_t$, and $R_S = 1$ in figure 5.6. For the following calculations, we will write αu as the flow speed through the turbine, $\beta_1 u$ as the bypass flow speed, and $\beta_0 u$ as the wake flow speed. We follow a similar process to chapter 3. Starting by applying Bernoulli’s equation to the bypass flow we obtain

$$p_1 - p_4 = \frac{1}{2} \rho u^2 (\beta_1^2 - 1). \quad (5.5)$$

By applying Bernoulli’s equation to the flow through the turbine we obtain

$$(p_1 - p_2) + (p_3 - p_4) = \frac{1}{2} \rho u^2 ((\alpha^2 - 1) + (\beta_0^2 - \alpha^2)), \quad (5.6)$$

with $T = A(p_2 - p_3)$. This simplifies to

$$\frac{T}{A} = \frac{1}{2} \rho u^2 (\beta_1^2 - \beta_0^2). \quad (5.7)$$

Now considering the conservation of momentum balance for the control volume (with neglecting the net streamwise force due to pressure on the non-parallel streamtube surface) we get

$$\frac{1}{2} (\beta_1^2 - \beta_0^2) = R_0(1 - \beta_0) + R_1(1 - \beta_1) + \frac{1}{2B}(\beta_1^2 - 1) - \frac{(1 - \gamma)p_1}{\rho u^2 B}. \quad (5.8)$$

Where $R_0 A$ is the upstream cross sectional area of the streamtube which passes through the turbine, and $R_1 A$ is the upstream cross sectional area of the streamtube which bypasses the turbine. From the conservation of mass, we have that

$$R_0 = \alpha, \quad (5.9)$$

$$R_1 = \frac{\gamma}{B} - \alpha. \quad (5.10)$$

Equation 5.8 can now be rewritten as

$$0 = \beta_0^2 - \beta_1^2 - 2\alpha(\beta_0 - \beta_1) + \frac{2\gamma}{B}(1 - \beta_1) + \frac{1}{B}(\beta_1^2 - 1) - \frac{2(1 - \gamma)p_1}{\rho u^2 B}. \quad (5.11)$$

Again by applying the conservation of mass to the streamtube which encompasses the turbine the following is obtained:

$$\alpha = \frac{\beta_0(\gamma - \beta_1)}{B(\beta_0 - \beta_1)}. \quad (5.12)$$

The α terms in equation 5.11 are now replaced using equation 5.12, and we write equation 5.11 as a quadratic in terms of β_0 to get

$$0 = \beta_0^2 + \beta_0 \left(\frac{2(\beta_1 - \gamma)}{B} \right) + C, \quad (5.13)$$

where

$$C = \frac{2p_1(\gamma - 1)}{\rho u^2 B} + \frac{2\gamma(1 - \beta_1) + (\beta_1^2 - 1)}{B} - \beta_1^2. \quad (5.14)$$

When $\beta_1 = \gamma$ we wish for $\beta_0 = \gamma$ (i.e. the scenario in which the turbine applies no thrust to the flow), therefore from equation 5.13 we require $C = -\gamma^2$. To achieve this we need $p_1 = (\gamma - 1)/(2\rho u^2)$. This gives us our final equation for β_0 in terms

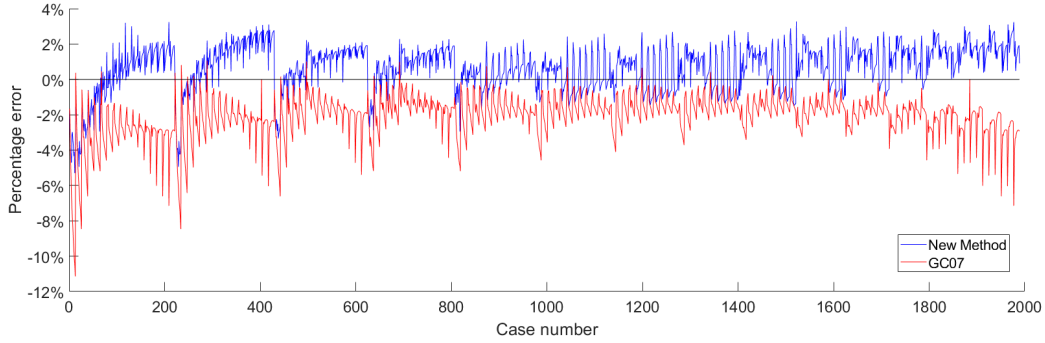


Figure 5.7: Percentage error of C_P^* between the two analytical predictions and RANS data.

of β_1 as

$$0 = \beta_0^2 + \beta_0 \left(\frac{2(\beta_1 - \gamma)}{B} \right) + \frac{(\gamma - 1)^2}{B} + \frac{2\gamma(1 - \beta_1) + (\beta_1^2 - 1)}{B} - \beta_1^2, \quad (5.15)$$

which may be simplified to

$$0 = \beta_0^2 + \beta_0 \left(\frac{2(\beta_1 - \gamma)}{B} \right) + \frac{(\beta_1 - \gamma)^2}{B} - \beta_1^2. \quad (5.16)$$

By taking out a γ^2 term this can be rewritten as

$$0 = \gamma^2 \left((\beta_0^*)^2 + \beta_0^* \left(\frac{2(\beta_1^* - 1)}{B} \right) + \frac{(\beta_1^* - 1)^2}{B} - (\beta_1^*)^2 \right), \quad (5.17)$$

where $\beta_i^* = \beta_i/\gamma$. We may also write equation 5.12 in terms of β_i^* and $\alpha^* = \alpha/\gamma$ as

$$\alpha^* = \frac{\beta_0^*(1 - \beta_1^*)}{B(\beta_0^* - \beta_1^*)}. \quad (5.18)$$

By comparing equations 5.12 and 5.16 to equations 5.18 and 5.17, we can observe that the solution to the problem for a given γ is equal to the solution when $\gamma = 1$ and we scale each velocity scalar by γ , validating our assumption.

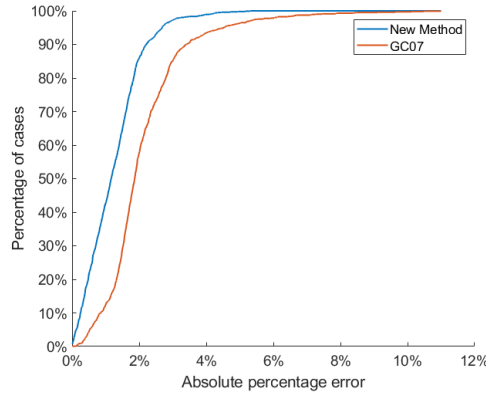


Figure 5.8: Cumulative distribution function for the absolute percentage error of C_P^* between the two analytic methods and the RANS simulation data.

5.4.2 Results

The percentage error, defined as one minus the power coefficient obtained from data (3D RANS) divided by the predicted coefficient ($1 - C_{P\text{ Data}}^*/C_{P\text{ Pred}}^*$), is plotted for both methods in figure 5.7, where the x-axis is the case number. The new method not only reduces the absolute mean error from 2.1% of GC07 down to 1.2%, it also reduces the maximum error from 11% to 5%. Note, some of the GC07 cases presented in figure 5.7 produce positive errors, whereas GC07 should always underestimate the peak power coefficient. This error occurs primarily in isotropic cases with low blockage in which the RANS simulations do not capture the 'true' peak power coefficient. Thus, GC07 predicts the isotropic maximum well but the RANS simulations slightly underestimate, causing the positive error. It should be noted that the relationship between the thrust at each sub-domain (given in equation 5.3 for the 2D case) has been given arbitrarily in this model (based partly on physical intuition, but also on the close agreement of the outcome with the RANS data; in this sense, the model may be referred to as a 'data-informed' analytic model). Hence, it is possible to further improve the agreement with the data by tuning the thrust relationship, although this is outside the scope

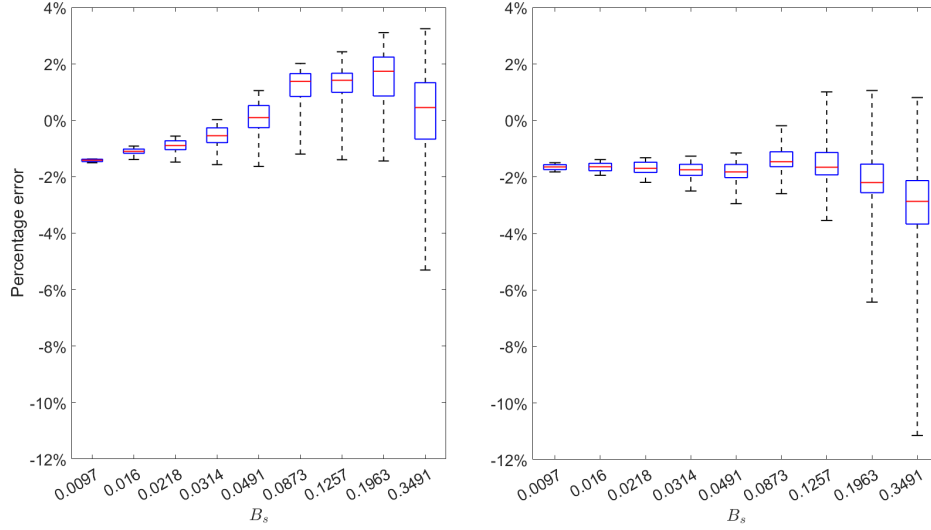


Figure 5.9: Percentage error of C_P^* with data for the new method (left) and GC07 (right) at various B_S values.

of this thesis.

The cumulative distribution function for the absolute error in the new method and GC07 is presented in figure 5.8. From this analysis, we can observe that 50% of the new method predictions have up to a 1% error when compared to RANS data, whereas 50% of GC07 predictions have up to a 1.9% error. The new method, however, has up to 85% of its predictions existing within that same 1.9% error range, compared to the GC07 predictions where 85% of cases have up to a 3% error. Figure 5.9 shows the percentage error for the new method and GC07 predictions at various values of B_S . For each box, the central mark indicates the median, and the bottom and top edges of the box indicate the 25th and 75th percentiles, respectively. The whiskers extend to the most extreme data points. Across all values of B_S the new method closely matched or improves upon GC07, suggesting the gains noted previously are across the whole parameter space. For low values of B_S both methods tend to give similar predictions, which is understandable as

anisotropic local blockage effects diminish when $B_S \rightarrow 0$. However, even at the lowest B_S tested, the new method maintains a slightly increased precision. The new method maintains a better precision at higher values of B_S , where there are more cases in which anisotropic local blockage effects have a significant impact; with the range of error for the presented method as almost half that of GC07.

As mentioned in section 5.4 the new method works on the assumption of scale separation and that square domains are reasonable representations of axisymmetric domains, the main issue with the later assumption is with the small sub-domain, where anisotropic effects are still large when considering a square domain. Thus, we get the greatest difference in our predictions in the cases in which these assumptions do not hold well, particularly when B_S is large. We saw in section 3.3.3 that the difference between the k at which peak C_P^* occurs can vary greatly between the square and axisymmetric domains and thus lead to inaccuracies in the new model.

6 | Conclusions and future work

*When old age shall this generation waste,
Thou shalt remain, in midst of other woe
Than ours, a friend to man, to whom thou say'st,
"Beauty is truth, truth beauty, -that is all
Ye know on earth, and all ye need to know."*
John Keats

This thesis has explored the optimal performance of multi-row turbine arrays through the advancement of computationally quick analytical models. A novel LMADT formulation is presented in which the system may be solved with a single cubic equation. This is utilised to examine the impact of sheared inflow on the power of a single blocked turbine. A self-similar mixing model based on momentum deficit is then developed to relate the extent of wake mixing, in the inviscid-viscous model, with downstream distance and ambient turbulence intensity. The results of this work are used to showcase the accuracy of a simplified uniform mixing model when compared with a self-similar model tuned on LES data. These improvements allowed for a parameter space study of a multi-row array, which suggested there are opportunities for improved power extraction from the flow by altering the resistance of each turbine row, as opposed to keeping turbine resistance fixed. A new model is constructed to improve performance estimates of turbines which experience blockage anisotropically. A range of numerical simulations are conducted to benchmark the accuracy of the new approach.

This chapter outlines the conclusions of the thesis in three parts: summarising the inviscid contributions, summarising the viscous and multi-row contributions, and finally summarising the anisotropic contributions. The chapter concludes with a section on suggestions for future work.

6.1 Conclusions

6.1.1 Inviscid model

A novel formulation of LMADT for a turbine in laterally confined flow is constructed. The model assumes that the thrust the turbine applied to the flow is uniform across the disk and that the turbine sat in the middle of the channel. The model is able to solve the flow problem for any general inflow profile. In the scenario in which the flow through the turbine is of uniform velocity, the system is able to be solved with a single cubic equation.

We first investigate the implication of ‘averaging’ either the bypass or the core part of a sheared inflow profile for the prediction of single turbine power. The results show that such an approximation of a sheared bypass inflow profile by a uniform profile will over-predict the effective blockage ratio, and thus the turbine power. This is essentially because a top-hat inflow profile inhibits the expansion of the core flow more than the corresponding inflow profile with shear. This suggests that using top-hat wake profiles (such as Jensen (1983)) as an inflow condition for a laterally blocked turbine may result in a substantial error in the prediction of the blockage effect on the turbine power. When considering the impact of shear on the core flow we find that the effect of a faster or slower bypass on a turbine’s power is exaggerated. Averaging the core flow resulted in over-predictions in the power extracted by the turbine when the bypass flow is faster than the core flow, and under-predictions when the bypass flow is slower. These trends are consistent across a range of local blockage ratios, B .

6.1.2 Viscous model and multi-row modelling

Initial investigations of multi-row turbine arrays modelled with the inviscid-viscous approach, utilising the ‘uniform’ mixing model, found that there is an opportunity, primarily in the aligned array layout, for farm-averaged performance increases due to varying the resistance of turbines at each row, as opposed to maintaining fixed resistance.

In this study, a novel low-order turbine wake mixing model based on self-similar momentum transfer is presented to extend the previously reported analysis of multi-row turbine array performance using the hybrid inviscid-viscous approach. It is worth noting that some existing wake models (e.g. Bastankhah et al. (2021)) consider momentum conservation to impose a correct maximum velocity deficit in their self-similar velocity deficit model. The self-similar momentum transfer model presented here does the inverse; it considers a self-similar momentum deficit and utilises the velocity in the free stream to impose a correct maximum momentum deficit. The momentum deficit model benefits from the pairing with the hybrid inviscid-viscous approach, allowing for an accelerated bypass flow and local blockage effects to be captured. In addition to this, the inclusion of pressure recovery allows for the pairing with an external farm-scale model, which would allow us to predict the effect of decreasing mass flux through the farm, the so-called farm-scale loss (Kirby et al., 2022), although the mass flux is fixed in the present study for simplicity.

The results obtained with the new self-similar model are compared with those of a simplified uniform mixing model. For two aligned rows of turbines, the uniform mixing model is found to overpredict the power of the downstream turbines when compared to the self-similar model. The uniform mixing model, however, still captures the trends of multi-row turbine performance correctly. In particular, we

show that the uniform mixing model could predict the maximum power of three rows of turbines (achieved with a single optimal turbine resistance K for a whole array) almost as accurately as the self-similar model when the mixing rate m is given correctly. This suggests that the inaccuracy of uniform mixing model predictions observed in previous studies such as Ouro and Nishino (2021) is largely due to the inaccurate relationship assumed between the mixing rate m and the turbine row spacing.

We also find, using the new self-similar mixing model, that the total power of three aligned rows of turbines could be slightly improved by varying K across different rows, with benefits of up to $\sim 5\%$, when the row spacing is as small as $5D$. These benefits by varying K are small compared to those achieved by increasing the row spacing. It is worth noting that these benefits are for a scenario with a high turbulence intensity of 14.13% . For cases with a lower turbulence level, we expect a greater benefit from varying K for a wider range of row spacings.

6.1.3 Anisotropic model

A series of 3D RANS simulations of a turbine represented as an actuator disk are conducted for various domain sizes and configurations. In total 1989 different domain sizes are simulated in this study, with the distance from the hub to the nearest boundary ranging from $0.75D$ to $20.5D$. In cases where the turbine is placed in the centre of a domain with a square cross-section, there is good agreement between the maximum power coefficient calculated by the 3D RANS simulation and the isotropic blockage model of Garrett and Cummins (2007). However, there is a discrepancy in the resistance coefficient at which that maximum power is achieved. Only in scenarios in which the turbine is in the middle of a cylindrical domain do the maximum power and optimal resistance values match between the isotropic model predictions and the 3D simulations. Therefore, the use of isotropic models

for real-world blockage would always introduce differences from the actual flow physics.

In this thesis, we propose a novel approach based on decomposing the expansion of flow through the actuator disk into a series of coupled sub-expansion processes. The approach is compared against the predictions of an isotropic blockage model, and both models are benchmarked against 3D RANS simulations of an actuator disk. The proposed anisotropic model consistently predicts the optimal performance of the turbine more accurately than the isotropic blockage model of Garrett and Cummins (2007), with 85% of the cases tested against sitting within a 1.9% error margin compared with only 50% of the isotropic model's predictions existing within that same error margin. The improvements can further be seen as it approximately halved the percentage error in the absolute mean error and the maximum error.

6.2 Future work

6.2.1 Inviscid model

The LMADT model presented here assumes no mixing occurs before pressure is equalised downstream of the turbine. An extension of the model to include near-wake mixing into the calculations of power and thrust would allow for greater tuning opportunities with data from a real turbine. Furthermore, the present model requires geometric symmetries between turbines; realistically this will not always be maintained in a turbine array. Future work to remedy this may link with advancements in our anisotropic blockage modelling. In many turbine arrays pressure may not equalise in the spanwise direction before it encounters the next turbine row. Hence, exploring ways to accurately include variation in spanwise pressure

recovery would allow for a greater range of turbine arrangements to be considered.

6.2.2 Viscous model

Future work should look into including the impact of local blockage on the wake growth rate, as explored by Shamsoddin and Porté-Agel (2018), and how to extend those results to a 3D domain in which a turbine experiences anisotropic blockage, as explored in chapter 5, to quantify the azimuthal variation in the wake growth rate. Additionally, the self-similar model's empirical parameters should be tuned against a larger set of data with a wider range of flow conditions. Further work should also look into pairing the model with an external flow model so that pressure recovery in the wake can be adjusted due to the reduction in mass flux through the farm, as discussed briefly in section 4.2.

6.2.3 Anisotropic model

The current anisotropic blockage model assumes that a turbine in a square domain behaves identically to the Garrett and Cummins (2007) LMADT case. This, however, as previously noted, is not the case. Although the maximum power coefficient is similar in both cases, the resistance coefficient at which that maximum is achieved can vary greatly. It would therefore be useful in future work to reformulate the anisotropic model presented here to consider the azimuthal variation in the radial distance to a boundary from the hub, as opposed to the square sub-domains currently utilised. This formulation would better match the axisymmetric condition needed to match the LMADT and CFD results. However, there will be complexities in handling the asymmetries in the domain that would have to be carefully considered. Another option for progress in anisotropic modelling would be to consider the pressure perturbations from the incoming flow much like the analytic global blockage approaches of Segalini (2021).

6.3 Published and planned papers

The work detailed in chapter 3 and chapter 4 up to section 4.4 has been published in Juniper and Nishino (2023). Another paper has been submitted based on the work completed in section 4.5, titled: *A self-similar momentum deficit model for mixing in multi-row turbine arrays*. The work presented in chapter 5 has been published in Juniper and Nishino (2022).

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