A Topic in Functional Analysis

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Abstract

We introduce the class *AUMD* of Banach spaces X for which X-valued analytic martingales converge unconditionally. We shew that various possible definitions of this class are equivalent by methods of martingale decomposition. We shew that such X have finite cotype and are q-complex uniformly convex in the sense of Garling. Using multipliers we shew that analytic martingales valued in L^1 converge unconditionally and that *AUMD* spaces have the analytic Radon-Nikodym property.

We shew that X has the *AUMD* property if and only if strong Hörmander-Mihlin multipliers are bounded on the Hardy space $H^1_X(\mathbf{T})$. We achieve this by representing multipliers as martingale transforms. It is shewn that if X is in *AUMD* and is of cotype two then X has the Paley Theorem property.

Using an isomorphism result we shew that if $\mathcal{A}$ is an injective operator system on a separable Hilbert space and P a completely bounded projection on $\mathcal{A}$, then either $P\mathcal{A}$ or $(I-P)\mathcal{A}$ is completely boundedly isomorphic to $\mathcal{A}$. The finite-dimensional version of this result is deduced from Ramsey's Theorem. It is shewn that $B(\ell^2)$ is primary.

It is shewn that weakly compact homomorphisms T from the disc algebra into $B(\ell^2)$ are necessarily compact. An explicit form for such T is obtained using spectral projections and it is deduced that such T are absolutely summing.

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Introduction

In this dissertation we present results in the area of analysis lying between geometry of Banach spaces, classical analysis and operator theory. We have approached the theory via the study of classical examples since these are interesting in their own right and allow one to interpret abstract results in a concrete setting.

The dissertation divides naturally into four chapters which have been written with the intention that they could be read independently. Thus brief abstracts and technical introductions precede each chapter; in this introduction we confine ourselves to more general observations in which we seek to relate this to others' work. Inevitably, in a subject still undergoing investigation, matters of formulation can obscure the connections between various results and hence we comment on the presentation of our results below.

2. Analytic *UMD* spaces

One of the most fruitful areas of Banach space theory in recent years has been local theory, that is to say the study of properties defined in terms of finite-dimensional subspaces of the Banach space. The success of this approach is partly due to the way in which one can use the methods and results of probability theory to investigate generic finite-dimensional subspaces.

For local theory to distinguish between various spaces it is necessary to introduce several subtly differing local properties and to investigate carefully the relationships between them. The three most basic sorts of local properties are convexity inequalities, which are related to the parallelogram law in Hilbert space, conditions on the boundedness of certain operators on spaces of vector-valued functions and the convergence criteria for sequences of vector-valued random variables. In the first two sections of this dissertation we introduce a new local property defined in terms of convergence of random variables and shew how it is related to convexity properties and behaviour of certain operators. This new property is particularly helpful in describing some classical Banach spaces.

It has been recognised for some time that the convergence properties of martingales taking values in a Banach space X can convey useful information about the geometry of X. For general martingales we can identify several notions of good behaviour, the weakest of which is the Radon Nikodym Property (*RNP*) and the strongest is the Unconditional Martingale Difference (*UMD*) property. These correspond to the vector-valued cases of Doob's and Burkholder's Theorems on martingale convergence respectively [18, chapter 5]. For X to be a *RNP* space, we demand that all L^1-bounded X-valued martingales converge almost surely, while *UMD* spaces Y are characterized by the property that L^2-bounded martingales converge unconditionally. Between these properties one has various notions of convexity defined in terms of growth conditions on L^2 norms of martingales.

The properties of *UMD* spaces have been investigated in detail, a notable result being that X has the *UMD* property if and only if the Hilbert transform is bounded on the Bochner-Lebesgue space $L^2_X(\mathrm{T})$ of square-integrable X-valued functions on the circle [6,11]. Not only did this result permit the development of a theory of abstract spectral decompositions of well-bounded operators on *UMD* spaces [2], it also provided new proofs of the the classical theorem of M.Riesz on the boundedness of the Hilbert transform on $L^p(\mathrm{T})$ $(1<p<\infty)$. The analogy between martingale transforms and translation-invariant operators was thus made precise, with Brownian motion providing the link between functions on the circle and martingales.

In the course of investigations into analytic functions taking values in Banach spaces it emerged that several classical Banach spaces have a complex convexity structure [14]. This means that the X-valued analytic martingales of the form

$$\Sigma_{n=0}^{\infty} d_n(\theta_1,\ldots,\theta_{n-1})e^{i\theta_n} \qquad d_n \in L^2_X(\mathrm{T}^{n-1})$$

can have quite different properties from general martingales. A significant example of this is the result that L^1-bounded analytic martingales valued in L^1 converge almost surely, despite the fact that L^1 does not have the Radon Nikodym Property [10,16].

The author observed that by applying a multiplier theorem of Stein [43] one could show that analytic martingales valued in L^1 are unconditionally convergent. This suggested that one could use analytic martingales to develop a theory parallel to that of *UMD* spaces including an analogue of the Bourgain-Burkholder characterization of *UMD* spaces in terms of multipliers on $L^2_X(T)$.

In the first chapter of this dissertation we introduce the notions of analytic martingales and the analytic *UMD* property. We provide a very detailed proof of the equivalence of various definitions of analytic *UMD* so as to convince the reader that the Burkholder martingale decompositions can be carried out with analytic martingales. We then establish the main result of this chapter, namely that L^1-bounded analytic martingales valued in analytic *UMD* spaces converge almost surely. Our approach recalls the original martingale convergence proof of Doob [18, p 87] which involved counting the upcrossings of a martingale.

After completing the work presented here in Chapter 1, the author learned that similar results had been obtained by Garling and that the paper [20] had already been submitted for publication. Our proof that L^1 has the analytic *UMD* property is not the simplest, but is presented here because it emphasizes the link between transforms of analytic martingales and multipliers on the vector-valued Hardy spaces. A referee for *Studia Mathematica* provided the author with a proof of our Proposition 8 of chapter 1 which is based upon the remark at the end of section 1.2 and which avoids the technical lemma 7.

Before embarking on a discussion of the connection between vector-valued analytic functions and martingales it is appropriate to review how they are involved in the scalar valued theory. The study of $H^p(T)$ spaces frequently made use of such devices as dividing out the zeros of functions by Blaschke products which are special to the single-variable case [24]. The achievement of Fefferman and Stein [19] was to introduce techniques such as smooth maximal functions, martingales and atomic decomposition which work for functions of several variables. It is these which are most appropriate in the vector-valued theory, but since the arguments of [19] do not generally work for vector-valued functions one must modify them. Martingale theory has proved to be a particularly flexible method in these investigations. It turns out that the structure of finite-dimensional subspaces of the Banach space is of crucial importance in determining which results actually hold.

The relationship between multipliers and analytic martingales is investigated in the second chapter where we prove that *X* has the analytic *UMD* property if and only if multipliers obeying strong Hörmander-Mihlin conditions are bounded on the Hardy spaces $H^1_X(T)$. The difficult part of his theorem is the forward implication and we make use of the fact that Brownian motion in the plane is transformed into time-changed Brownian motion by the operation of analytic functions. In detail our methods are adaptated from [16] and [30] but the conceptual basis was provided by the elegant work of Maurey [31]. Chapter two is concluded by some remarks on the connection between analytic *UMD* and certain other multiplier properties.

The formulation of the main theorem of chapter two deserves a little comment. We have presented our characterization in terms of strong Hörmander-Mihlin multipliers, which are not quite the multipliers for which Stein [43] established his $H^1(T)$ square-function theorem. The essential difference is that strong Hörmander-Mihlin multipliers are smoother than Stein's and we require this in our main lemma. The situation is analagous to the proof of the Fefferman-Stein Theorem in [19] where it is necessary to introduce a class of smooth functions to form maximal functions, Dini conditions not being sufficient to deal with all cases of interest.

One apparent defect of results in the theory of complex convexity is that they can involve replacing the norm of a Banach space by an equivalent quasi-norm. Indeed, we have only been able to assert in Proposition 6 of chapter 1 that analytic *UMD* spaces have an equivalent quasi-norm under which they are q-complex uniformly convex. One could argue that in the study of complex convexity it is natural to work with quasi-normed spaces since the Hardy spaces $H^p(\mathbb{T})$ $(0<p<1)$ have tractable properties. This is not the point of view taken in this thesis, however, and the reader should note that *I do not claim that the proof of theorem 3 of chapter 2 works for quasi-normed spaces*. A similar remark must be made about Corollary 7 concerning the Paley Theorem property.

In the course of investigations into the factorization of vector-valued analytic functions [37] another class of Banach spaces with good complex convexity properties has been identified. Kouba [27] has shewn that the Banach spaces X for which the canonical map $H^1\hat{\otimes}X \to H^1\check{\otimes}X$ is a metric surjection have the analytic *UMD* property and are of cotype two. It would be of interest to obtain more examples of analytic *UMD* spaces to determine which of these various classes are the same.

Using factorization methods, Haagerup and Pisier [23] have obtained a proof that the space c_1 of trace-class operators has the analytic Radon-Nikodym property. The crux of their argument is an application of the inequality

$$\|P_r*f\|^2 + c\|f-P_r*f\|^2 \le \|f\|^2 \qquad (f\in H^1_{c_1}(\mathbb{T}))$$

strongly reminiscent of the definition of 2-complex uniform convexity in the sense of [14]. It is not known whether q-complex uniform convexity implies the analytic Radon-Nikodym property and hence it is not clear how to eliminate arguments such as the proof of corollary 4 in chapter 1 from the theory.

3. Primary spaces and completely bounded maps

The third section of this dissertation is concerned with global isomorphic theory of certain noncommutative spaces rather than with their local theory. Haagerup and Pisier [23] also proved that the space c_1 of trace class operators does not have the analytic *UMD* property. This provided weak evidence that $B(\ell^2)$, the dual of c_1, was primary *i.e.* if $B(\ell^2)$ is isomorphic to $X \oplus Y$ then $B(\ell^2)$ is isomorphic to X or to Y. The author established that this is indeed the case, but he prefers to formulate his results in the language of completely bounded maps.

The first examples of primary spaces included spaces with symmetric bases such as ℓ^p $(1<p<\infty)$ and also the space ℓ^∞, where special arguments could be used [29, p 131]. The proof that ℓ^∞ is primary can be reduced to a diagonalization and factorization argument much as in our section 3.3 . Bourgain [5] shewed how to establish that H^∞ is primary by representing it as an ℓ^∞ sum of finite-dimensional spaces. Our proof uses the same idea.

The basic result of Robertson and Wassermann [39] is that $B(\ell^2)$ is isomorphic as a Banach space to $M = (\Sigma_1^\infty M_n)_\infty$. This representation as an ℓ^∞ sum of finite-dimensional spaces is the starting point for our combinatorial approach. We prefer to work with finite-dimensional operator algebras as we have an a *priori* relationship between the operator norm and the Hilbert-Schmidt norm of a finite matrix which allows us to estimate the norms of certain maps on the algebra.

In Banach space theory one usually classifies spaces up to isomorphic equivalence. One of the fundamental results is the Hahn-Banach Theorem which allows one to extend linearly operators valued in ℓ^∞. When dealing with the noncommutative spaces arising in operator theory one considers operators valued in $B(\ell^2)$ and to obtain an analogue of the Hahn-Banach Theorem one restricts attention to those operators which are completely bounded. Hence one aims to classify such spaces up to completely bounded isomorphism.

The introduction of completely bounded maps provides us with few fresh difficulties. Most of the technical arguments of chapter three are concerned with controlling the norms of operators on subspaces of the finite-dimensional matrix algebras M_n. Our main tool is Ramsey's Theorem with which we successively select subsets of the usual unit basis of M_n. As in most proofs that a space is primary, we rely on the decomposition method of Pelczynski to establish that certain spaces are isomorphic. A version of the Pelczynski method for completely bounded maps also lies at the heart of the proof in [39]. Hence our arguments do not provide us with a useful explicit form of the isomorphism between $B(\ell^2)$ and the complemented subspace.

The study of the classical spaces has had many implications for the development of the abstract theory, partly because the classical spaces exhibit unexpected properties [36]. The properties of the commutative spaces considered by Banach are now much more fully understood and more recent work [8,38,31] has been concerned with spaces of analytic functions and spaces of operators. These have natural analogues amongst the Lebesgue spaces. The technique of argument by analogy is of particular importance when we come to consider the noncommutative spaces. For example, in the study of the Banach space properties of von Neumann algebras it is helpful to consider the commutative space ℓ^∞. This suggested the main result of the third chapter, but the concluding proposition 6 notes that there is a limit to how far the analogy can be pursued.

The Banach space structures of ℓ^∞ and $B(\ell^2)$ exhibit several differences. A crucial feature of ℓ^∞ is that it is prime *i.e.* if X and Y are infinite-dimensional spaces such that ℓ^∞ is isomorphic to $X \oplus Y$ then ℓ^∞ is isomorphic to X and to Y. We shew that $B(\ell^2)$ has a subspace isomorphic to ℓ^2 which is the range of a completely contractive projection. This shows is a strong sense that $B(\ell^2)$ is not prime, as the range of this projection is separable and reflexive. The existence of such projections is the origin of the difficulties involved in formulating a noncommutative version of Grothendieck's Theorem [38].

On the basis of Theorem 5 of chapter 3 it is tempting to conjecture that the space K of compact operators on separable Hilbert space is primary. It should be noted, however, that K cannot be represented as a c_0 sum of finite-dimensional spaces. This may be seen fron the fact that c_1, the dual of K, contains ℓ^2 copies and hence does not have the Schur property that weak sequential convergence and norm sequential convergence coincide. Thus it appears that to study the Banach structure of K one must employ techniques which are different from those deployed here.

4. Weakly compact homomorphisms

In the fourth chapter we turn to the theory of Banach algebras and consider a question related to work of Ransford on the nature of weakly compact homomorphisms between Banach algebras. The particular question considered is to determine the form of weakly compact homomorphisms T from the disc algebra A into the space $B(\ell^2)$ of bounded linear operators on separable Hilbert space. Homomorphisms from A into algebras of operators arise naturally via von Neumann's Inequality when one considers analytic trigonometric polynomials of a contraction.

We obtain an explicit formula for such T and establish that they are all compact. Our conclusions are weaker than those of Ransford for the case of weakly compact homomorphisms from the space $C(T)$ of continuous functions, but it is easy to construct homomorphisms from A to $B(\ell^2)$ which do not extend to homomorphisms on $C(\mathbb{T})$.

Our methods are based on the functional calculus for elements in a von Neumann algebra and make use of certain classical interpolation theorems from the theory of functions on the unit disc. It is natural to ask whether our methods extend to show that all weakly compact homomorphisms from the disc algebra A to any Banach algebra B are compact. The chief difficulty is that our proof uses similarity theorems concerning unitary elements which are not intelligible outside the context of C^*-algebras. On a more positive note, it is almost obvious that any homomorphism from A to a commutative radical Banach algebra is compact.

I Unconditional Analytic Martingale Convergence

Abstract: We introduce a class of Banach spaces for which analytic martingales converge unconditionally. We show this class contains L^1 and that its members have the Analytic Radon-Nikodym Property.

1.1 Introduction

The purpose of this chapter is to develop a theory of unconditional convergence for analytic martingales in a general class of Banach spaces. Burkholder [11] and Bourgain [6] have considered the class of *UMD* spaces for which Walsh-Paley martingales converge unconditionally and identified this class as those for which the Hilbert transform is bounded on $L^p_X(\mathbb{T})$ $(1<p<\infty)$. For such Banach spaces an extensive theory of singular integral operators has already been developed.

In this chapter we consider the natural complex analogue of Walsh-Paley martingales, termed analytic martingales by Edgar [16]. It is known ([10], [16]) that the Banach spaces for which L^1-bounded analytic martingales converge almost surely are precisely those for which functions in the Hardy spaces $H^1_X(D)$ on the unit disc have radial limits almost everywhere. This defines the Analytic Radon-Nikodym Property (*ARNP*). A noteworthy example is the Banach space L^1, which does not enjoy the Radon-Nikodym Property. We shall prove that analytic martingales in this space even converge unconditionally.

In the second section of this chapter we introduce necessary definitions and establish the main result, namely that *AUMD* spaces have the Analytic Radon-Nikodym Property. Our basic method is decomposition of analytic martingales, following Burkholder [12] where possible.

The third section of this chapter concerns itself with the complex uniform convexity property. We will show that if X is an *AUMD* space, then there is an equivalent continuous quasi-norm under which X is q-complex uniformly convex for some finite q.

1.2 The main result

We introduce several basic definitions.

Let X be a complex Banach space. Our standard probability space P is the infinite torus $\mathbb{T}^{\infty}$, endowed with its Haar measure. We shall also have occasion to consider the Rademacher functions r_n on the unit interval [0,1] with Lebesgue measure m.

Definition 1:(i) An L^p-bounded analytic martingale f is a sequence (f_n) of functions in the Bochner-Lebesgue space $L^p_X(\mathbb{T}^{\infty})$ having the form:

$$f_n(\theta) = \Sigma_{k=1}^{n} d_k(\theta_1,\ldots,\theta_{k-1})e^{i\theta_k}, \qquad (1.1)$$

with $\|f\|_p = sup_n\|f_n\|_p < \infty$.

(ii) Let $v = (v_n)$ be a predictable sequence of measurable functions enjoying the following property:

$$\|v\|_\infty = \sup_n \sup_\theta |v_n(\theta_1, \dots \theta_{n-1})| < \infty.$$

Then we define the martingale transform operator K_v associated with v by

$$K_v : \Sigma_{k=1}^n d_k e^{i\theta_k} \to \Sigma_{k=1}^n v_k d_k e^{i\theta_k}. \qquad (1.2)$$

(iii) We say that X is an analytic unconditional martingale difference (*AUMD*) space if there is a constant C, depending only on the space X, such that, for all such v,

$$\sup_{\lambda>0} \lambda P(g^* > \lambda) \le C\|v\|_\infty \|f\|_1, \qquad (1.3)$$

where here $g = K_v f$ is the transform of the L^1 analytic martingale f by v and g^* denotes the usual martingale maximal function

$$g^*(\theta) = \sup_n \|g_n(\theta)\| \qquad (\theta \in \mathrm{T}^\infty).$$

It is clear that (1.3) is satisfied whenever X is a *UMD* space. We shall give other examples of *AUMD* spaces later. Firstly, we establish the following:

LEMMA 2: *Let X be an AUMD space. Then there is a finite q such that X is of cotype q.*

Proof: It is evident that *AUMD* is a local property and hence by the Maurey-Pisier Theorem [32], it suffices to show that c_0 is not an *AUMD* space. Suppose otherwise, and let C be the constant of (1.3). We use an argument suggested by [1].

We consider a small positive δ and a large number L, to be chosen later. For this δ we subdivide the unit circle into congruent arcs of length δ. We introduce a sequence of random times T_n defined by $T_1 = 1$,

$$T_{n+1}(\Theta) = \inf\{k > T_n(\Theta): \exp(i\theta_k) \text{ is in a different arc from } \exp(i\theta_{T_n})\}\wedge\{T_n(\Theta)+L+1\} \quad (n\geq 1).$$

Let e_j be the j^{th} element of the usual basis of c_0 and define $d_k = (-1)^k e_n$ for $T_{n-1}(\Theta) < k \leq T_n(\Theta)$. Let $v_k=(-1)^k$ $(k=1,2,\ldots)$.

Then, forming $f = \Sigma_k d_k e^{i\theta_k}$, we have that $\|f\|_1 \leq 2(L+1)\sin\delta + 1$. We consider $g=K_v f$, the transform of f by v. It is easy to see that

$$P(g^* > L\cos\delta+1) \geq P\{(T_{n+1}-T_n) = L+1 \text{ for some } n\} = 1, \qquad (1.4)$$

by independence of the random variables $(T_{n+1}-T_n)$ on $\mathbf{T}^\infty$. It follows from (1.3) and (1.4) that

$$-1 + L\cos\delta = (L\cos\delta+1)P(g^* > L\cos\delta+1) \leq C(2(L+1)\sin\delta + 1). \qquad (1.5)$$

Choosing δ small and then L large, we see that (1.5) cannot hold for all choices of δ and L. This contradiction concludes the proof.

THEOREM 3: *The following properties of the complex Banach space X are equivalent:*

(i) The space X belongs to AUMD;

(ii) If f is an L^1-bounded analytic martingale and $g = K_v f$ its transform by a predictable sequence v, then g converges almost everywhere;

(iii) For some, equivalently each, p $(1<p<\infty)$ there is a constant C_p, depending only on X and p, such that if f is an

L^p-bounded analytic martingale and $g = K_v f$ its transform by a predictable sequence v, then $\|g\|_p \le c_p \|v\|_\infty \|f\|_p$.

The implication (i)⇒(ii) gives us the following result.

COROLLARY 4: *If X belongs to AUMD, then X has the Analytic Radon-Nikodym Property.*

Proof of the Theorem: By standard approximation arguments, we can assume whenever it is convenient that the d_k in the definition of f are step functions. We can further assume that the analytic martingales f in the statements (i) and (iii) have only finitely many non-zero martingale differences $d_k e^{i\theta_k}$. The implications (ii)⇒(i) and (i)⇒(iii) were carried out for the *UMD* case in Burkholder's paper [12] and only simple modifications are needed here.

To see that (i)⇒(ii), we suppose that there is an L^1-bounded analytic martingale f with transform g and a positive ϵ such that $\limsup_{m,n\to\infty} \|g_m(\theta) - g_n(\theta)\| > \epsilon$ for all θ in a set E of positive Haar measure. We introduce a sequence of random times $S_1(\theta) = 1$ $(\theta \in \mathbb{T}^\infty)$,

$$S_{n+1}(\theta) = \min\{k > S_n : \|g_k(\theta) - g_{S_n(\theta)}(\theta)\| > \epsilon/4\} \quad (\theta \in \mathbb{T}^\infty,\ n \ge 1) \qquad (1.6)$$

and the family of predictable sequences $v^t = (v_k(\theta, t))$, parametrized by $t \in [0,1]$, where $v_1(\theta,t) = 1$, $v_k(\theta,t) = r_n(t)$ $(k \ge 2)$ for $S_{n-1}(\theta) < k \le S_n(\theta)$.

Let $g^t = (g_k(\theta,t))$ denote the transform of g by v^t, and let r be an arbitrary natural number. We claim that

$$\liminf_{k\to\infty} m\{t: \|g_k(\theta,t)\| > 2^r\} \geq 1/16 \quad (\theta\in E). \tag{1.7}$$

Indeed, if $m\{t: \|g_k(\theta,t)\| > 2^r\} < 1/16$, then by Kahane's Theorem [26, p 18 Theorem 3] we can estimate

$$\int\|g_k(\theta,t)\|dm(t) < 1+ \Sigma_{n=1}^{\infty} 2^{n+1} m\{t: \|g_k(\theta,t)\|>2^n\} \leq 2^{2r+1}.$$

By Lemma 2, X has cotype q for some finite q. Hence if c_q denotes the cotype q constant of X, it follows by definition of S_n that

$$\int\|g_k(\theta,t)\|dm(t) \geq \int\|\Sigma_n r_{n+1}(t)(f_{S_{n+1}(\theta)}(\theta)-f_{S_n(\theta)}(\theta))\|dm(t) \geq$$

$$\geq c_q(\Sigma_n\|f_{S_{n+1}(\theta)}(\theta)-f_{S_n(\theta)}(\theta)\|^q)^{1/q} \to\infty \text{ as } k\to\infty \text{ for } \theta\in E.$$

This proves (1.7). Now we apply Fatou's Lemma to (1.7) to deduce that

$$\liminf_{k\to\infty} \int m\{t: \|g_k(\theta,t)\| > 2^r\}dP \geq P(E)/16. \tag{1.8}$$

By Fubini's Theorem it follows from (1.8) that for some k and t

$$2^r P\{\|g_k(\ ,t)\| > 2^r\} \geq 2^{r-5}P(E).$$

By construction, g^t is a martingale transform of f and r is arbitrary, so this last inequality contradicts (1.3). This completes the proof of (ii)⇒(i).

To prove that (iii)⇒(i), we employ an argument based on a martingale decomposition of Davis [13]. Let f be an L^1-bounded analytic martingale and let $\lambda>0$ be given. Let S be

Derivation of Equation (1.10)

Equation (1.10) follows from the observation that

$$\begin{aligned} 2(d_k^* - d_{k-1}^*) &\geq 2\aleph_{A_k}(\|d_k\| - d_{k-1}^*) = \\ &= \aleph_{A_k}\|d_k\| + \aleph_{A_k}(\|d_k\| - 2d_{k-1}^*) \geq \\ &\geq \aleph_{A_k}\|d_k\| \end{aligned}$$

by definition of A_k. Now to obtain the second term in (1.10) we require to estimate the maximal function of

$$K_v b = \Sigma_k v_k \aleph_{A_k} \aleph_{[k \leq S]} d_k e^{i\theta_k}.$$

Using the above inequality we have

$$\begin{aligned} (K_v b)^* &\leq \Sigma_k \|v_k\| \aleph_{A_k} \|d_k\| \leq \\ &\leq \|v\|_\infty \Sigma_k 2(d_k^* - d_{k-1}^*) \leq \\ &\leq 2\sup_k \|d_k\| \leq 4f^* , \end{aligned}$$

since here $\|v\|_\infty \leq 1$.

Hence we have the required estimate

$$P((K_v b)^* > \lambda/3) \leq P(4f^* > \lambda/3).$$

the stopping time given by $S = min\{k: \|f_k\| > \lambda/3\}$. Introduce $d_k^* = sup_{r\le k}\|d_r\|$ $(k\ge 1)$ and let

$$A_k = \{\theta\in T^\infty: \|d_k(\theta)\| \ge 2d_{k-1}^*(\theta)\}.$$

We decompose the stopped martingale $f^S = (f_{k\wedge S})$ into the 'bad part'

$$b(\theta) = \Sigma_k \chi_{A_k}(\theta)\chi_{[k\le S]}(\theta)d_k(\theta)e^{i\theta_k} \tag{1.9}$$

and the 'good part' $f^S - b$. We note that for θ in A_k $\|d_k\| \le 2(d_k^* - d_{k-1}^*)$, so $b^* \le 4f^*$ and $(f^S-b)^* \le 5f^*\wedge 3\lambda$.

Let $h = K_v f$ be a transform of f by a predictable sequence v. Then we have

$$P(h^*>\lambda) \le$$

$$\le P\{(K(f-f^S))^* > \lambda/3\} + P\{(Kb)^* > \lambda/3\} + P\{(K(f^S-b))^* > \lambda/3\}.$$

so that if $\|v\|_\infty \le 1$ then

$$P\{h^*>\lambda\} \le \tag{1.10}$$

$$\le P\{(f-f^S)^*>0\} + P\{4f^* > \lambda/3\} + P\{(K(f^S-b))^* > \lambda/3\}.$$

To deal with the final term, we note that by Doob's Inequality and (iii),

$$P\{(K(f^S-b))^* > \lambda/3\} \le (\lambda/3)^{-p}\|(K(f^S-b))^*\|_p \le C_p\lambda^{-p}\|(f^S-b)^*\|_p =$$

$$= C_p\lambda^{-p}\int p\mu^{p-1}P\{(f^S-b)^*>\mu\}d\mu \le$$

$$\le C_p\lambda^{-p}\int_0^{3\lambda} p\mu^{p-1}P\{5f^*>\mu\}d\mu \le$$

$$\le C_p\lambda^{-p}\int_0^{3\lambda} p\mu^{p-2}\|f\|_1 d\mu \le C_p(\lambda/3)^{-1}\|f\|_1. \tag{1.11}$$

It is easy to see that by Doob's Maximal Inequality

$$P\{(f-f^S)^*>0\} \le P\{f^* \ge \lambda/3\} \le (\lambda/3)^{-1}\|f\|_1.$$

Hence we have shewn that there is a constant C, depending only on X (and p), such that $P\{h^*>\lambda\} \le C\lambda^{-1}\|f\|_1$ and this completes the proof.

REMARK: We remark that Bourgain [7] has shewn that there is a numerical constant c such that if f is any analytic martingale, then $\|f^*\|_1 \le c\|f\|_1$. Hence in the above decomposition we can ensure that $\|b\|_1 \le 4c\|f\|_1$ and so we have obtained a decomposition of our analytic martingale which resembles closely that achieved for positive scalar-valued L^1 martingales [22],[42]. It also shews that we can improve condition (iii) of Theorem 3 to read $1\le p<\infty$. Another consequence is that almost everywhere convergence of an L^1 analytic martingale implies L^1 convergence.

1.3 Complex uniform convexity

We continue to study the general theory by recalling following definition of Davis, Garling and Tomczak-Jaegermann [14].

Definition 5: Let $0<p<\infty$ and $2\le r<\infty$. Then the continuously quasi-normed space $(Y,\|.\|)$ is r-uniformly PL-convex if there is a $\lambda>0$ such that

$$\{(2\pi)^{-1}\int\|x+e^{i\theta}y\|^p d\theta\}^{1/p} \ge (\|x\|^r + \lambda\|y\|^r)^{1/r} \quad (x,y\in Y).$$

PROPOSITION 6: *Let X be an AUMD space of finite cotype q. Then there is an equivalent continuous quasi-norm on X under which X is q-uniformly PL-convex.*

Proof: By [14, Theorem 5.2], it suffices to show that there is a constant α such that if $f = \Sigma_k d_k e^{i\theta_k}$ is an analytic martingale valued in X, then

$$\alpha \Sigma_k \|d_k\|_q^q \le \|f\|_q^q. \tag{1.12}$$

Let f be any L^q-bounded analytic martingale in X and let $r_n(t)$ denote the n^{th} Rademacher function. By (iii) above, there is a constant C_q, independent of t, such that

$$\|f\|_q^q \ge C_q \int \|\Sigma_k r_k(t) d_k e^{i\theta_k}\|^q dP. \tag{1.13}$$

We now integrate (1.13) with respect to t, using Fubini's Theorem and the cotype q property of X. We have $\|f\|_q^q \ge C_q' \Sigma_k \|d_k\|_q^q$ and (1.12) follows.

1.4 Technical Results

In order to exhibit a non-trivial example of an *AUMD* space we first establish the following

LEMMA 7: *Let X be a complex Banach space for which there exists a constant C(X) such that the following inequality holds*

$$sup_{\lambda>0} \lambda P(g^* > \lambda) \le C(X) \|v\|_\infty \|f\|_1$$

whenever $g = K_v f$ *is the transform of* f *by a bounded sequence of constants v. Then X belongs to AUMD.*

Proof: Let f be an L^1-bounded analytic martingale and g its transform by a predictable sequence of simple functions $v=(v_n)$. We shall construct a new analytic martingale $F = \Sigma D_k e^{i\phi_k}$ and a sequence of integers $N(n)$ such that the f_n have the same distribution as $F_{N(n)}$ $(n\geq 1)$ and the g_n have the same distribution as the $G_{N(n)}$, where $G=(G_n)$ is the transform of F by a bounded sequence of constants (μ_n).

We suppose that only finitely many v_n are non-zero. The first step is to let $D_1=d_1$ and $\mu_1=v_1$. We introduce ϕ_1, which is uniformly distributed on $[0,2\pi]$. Now decompose the simple function v_2 as

$$v_2(\theta_1) = \Sigma_{j=2}^{N(2)}\mu_j\chi_{I_j}(\theta_1)$$

where the I_j are disjoint subsets of T whose union is T. Now introduce independent random variables $\phi_2,\ldots,\phi_{N(2)}$, uniformly distributed on $[0,2\pi]$, and form the analytic martingale

$$F_{N(2)} = D_1 e^{i\phi_1} + D_2(\phi_1)e^{i\phi_2} + \ldots + D_{N(2)}(\phi_1)e^{i\phi_{N(2)}}$$

where $D_j(\phi_1) = d_2(\phi_1)\chi_{I_j}(\phi_1) \quad (j=2,\ldots,N(2))$.

Clearly, f_2 has the same distribution as $F_{N(2)}$ and g_2 has the same distribution as $G_{N(2)}$, where

$$G_{N(2)} = \mu_1 D_1 e^{i\phi_1} + \mu_2 D_2(\phi_1)e^{i\phi_2} + \ldots + \mu_{N(2)}D(\phi_1)e^{i\phi_{N(2)}}.$$

The next step in the construction of F should make it clear how the induction argument is carried out. We decompose

v_3 as

$$v_3(\theta_1,\theta_2) = \Sigma_{j=N(2)+1}^{N(3)}\mu_j\chi_{I_j}(\theta_1,\theta_2) .$$

We suppose without loss that the I_j $(j=N(2)+1,...,N(3))$ are disjoint and that each I_j may be written as $J_j\times K_j$ where $J_j\subseteq I_k$ for some unique $k=k(j)\in\{2,...,N(2)\}$. We introduce new independent random variables $\phi_{N(2)+1},...,\phi_{N(3)}$, uniformly distributed on $[0,2\pi]$, and define for $j=N(2)+1,...,N(3)$ the functions

$$D_j(\phi_1,...,\phi_{N(2)}) = \chi_{I_j}(\phi_1,\phi_k)d_3(\phi_1,\phi_k) \text{ where here } k=k(j).$$

Then f_3 has the same distribution as $F_{N(3)}$, where

$$F_{N(3)} = D_1e^{i\phi_1} + D_2(\phi_1)e^{i\phi_2} + ...+ D_{N(3)}(\phi_1,...,\phi_{N(2)})e^{i\phi_{N(3)}}$$

and g_3 has the same distribution as $G_{N(3)}$, where

$$G_{N(3)} = \mu_1D_1e^{i\phi_1} + ... + \mu_{N(3)}D_{N(3)}(\phi_1,...,\phi_{N(2)})e^{i\phi_{N(3)}}.$$

Since $(\|F_n\|)_{n\geq 1}$ is a submartingale, $(\|F_n\|_1)$ is an increasing sequence. Hence $\|F\|_1=\|f\|_1$. We deduce that $\lambda P\{g^*>\lambda\} = \lambda P\{G_{N(k)}>\lambda \text{ for some } k\} \leq \lambda P\{G^*>\lambda\} \leq C\|F\|_1= C\|f\|_1$, using the hypotheses of the lemma. This completes the proof for the case in which the v_n are simple functions and the general case may be dealt with by a standard approximation argument.

We now consider a convolution operator we will make use of later. Introduce for each sequence of signs $\epsilon=(\nu_n)$ the kernel

$$T_{\epsilon,r}= \Sigma_{j=1}^{r}\nu_j M_j$$

where

$$M_j(e^{i\theta}) = \exp(3.2^{j-2}i\theta)(2F_{2^{j-1}-1}-F_{2^{j-2}-1})$$

and F_n denotes the n^{th} Fejer kernel. It can be shewn [19] that $T_{\epsilon,r}$ satisfies the following Hörmander condition

$$\int_{\pi\geq|t|\geq 2\tau}|T_{\epsilon,r}(e^{i(t-\tau)})-T_{\epsilon,r}(e^{it})|dt \leq 4 \qquad (1.14)$$

By well-known results of multiplier theory [19], condition (1.14) ensures that the Stein operator $T_\epsilon : f \to \lim_{r\to\infty}T_{\epsilon,r}*f$, defined initially on the Schwartz class on T, extends to a bounded linear operator from $H^1(T)\to H^1(T)$, $L^p(T)\to L^p(T)$ and $BMO(T)\to BMO(T)$. By a Littlewood-Paley Theorem of Bourgain [7], T_ϵ is bounded on $L^2_X(T)$ for any *UMD* space X. Our aim is to show that T_ϵ is bounded from $H^1_X(T)$ to itself whenever X is *UMD*. To do this we require an atomic decomposition result of Bourgain [7], which shews that it suffices to obtain a strong $(1,1)$ estimate for a, when a is an $L^\infty_X(T)$ function supported on an arc $I\subseteq T$ with $\int a = 0$, $\int\|a\|_X = 1$ and $\|a\|_X \leq 1/|I|$.

To establish this we let J be an arc concentric with I and of twice its length. Then the estimate (1.14) gives

$$\|T_\epsilon * a\|_1 = \|(T_\epsilon * a)\chi_J\|_1 + \|(T_\epsilon * a)\chi_{T-J}\|_1 \leq$$

$$\leq |J|^{1/2}\|T_\epsilon * a\|_2 + \int_I \int_{T-J} |T_\epsilon(e^{i(t-s)}) - T_\epsilon(e^{it})| dt \|a(s)\| ds \leq$$

$$\leq C|J|^{1/2}\|a\|_2 + 4 \leq C\|a\|_1.$$

1.5 Examples of *AUMD* spaces

The results of the preceding section are used in the proof of the following proposition, which we shall improve upon later.

PROPOSITION 8: *Let X be an UMD space. Then $L^1_X[0,1]$ is in AUMD.*

Proof: We employ a now-familiar transfer argument. Let $f = \Sigma d_n e^{i\theta_n}$ be an L^1 analytic martingale. We suppose without loss that the d_n are trigonometric polynomials and choose recursively integers N_n $(n\geq 1)$ such that four times the greatest member of the spectrum of

$$\zeta \to \Sigma_{k=1}^n d_k(\theta_1+N_1\zeta,\ldots,\theta_{k-1}+N_{k-1}\zeta)e^{i(\theta_k+N_k\zeta)}$$

is less than four times the least member of the spectrum of

$$\zeta \to d_{n+1}(\theta_1+N_1\zeta,\ldots,\theta_n+N_n\zeta)e^{i(\theta_{n+1}+N_{n+1}\zeta)}$$

(where we suppose that this is non-empty). As the transformation $(\theta_j)\to(\theta_j+N_j\zeta)$ of $\mathbb{T}^\infty$ is measure-preserving, it follows from the boundedness of T_ϵ that

$$2\pi\int(\int\|\Sigma d_k e^{i\theta_k}\|_X dm(t))dP =$$

$$= \int\int\int\|\Sigma d_k(\theta_1+N_1\zeta,\ldots,\theta_{k-1}+N_{k-1}\zeta)e^{i(\theta_k+N_k\zeta)}\|dm(t)dPd\zeta =$$

$$= \int\int(\int\|\Sigma d_k e^{i(\theta_k+N_k\zeta)}\|d\zeta)dm(t)dP \geq$$

$$\geq C\int\int(\int\|\Sigma v_k d_k e^{i(\theta_k+N_k\zeta)}\|d\zeta)dm(t)dP = C\int\int\|\Sigma_k v_k d_k e^{i\theta_k}\|dm(t)dP.$$

REMARKS (i) The particular case of $X=\mathbb{C}$ in Proposition 8 shows that L^1-bounded analytic martingales valued in $L^1(\mathbb{T})$ converge almost everywhere. This also follows from work of Bukhalov & Danivelich [10] and Edgar [17]. This example shows that *AUMD* spaces need not have type greater than one, need not have the Radon-Nikodym Property and their duals need not have *AUMD*. This contrasts with known properties of *UMD* spaces.

(ii) The converse to Lemma 1 is false, because L^1/H^1_0 has cotype two [8] but not the *ARNP* [10]. Results of [23] concerning the space c_1 of trace-class operators shew that the converse to proposition 6 is false.

(iii) Blasco kindly brought to the author's attention the significance of an example of Montgomery-Smith to the theory of *AUMD* spaces. One can constuct an interpolation couple whose endpoint spaces are isomorphic to ℓ^1 and which contains c_0 an a complemented copy in the midpoint space. This shows that the *AUMD* proprerty is not stable under complex interpolation. In this respect the *AUMD* property differs from the *UMD* property, although Kalton has used an interpolation construction to show that *UMD* is not a three-space property. It also suggests that it may be difficult to construct new examples of *AUMD* spaces.

II A Multiplier Characterization of Analytic UMD Spaces

Abstract: We prove that the Banach spaces X for which analytic martingales converge unconditionally are precisely those for which certain multipliers are bounded on the Hardy space $H^1_X(T)$.

2.1 Introduction

The purpose of this chapter is to characterize the *AUMD* spaces of Chapter 1 in terms of boundedness of certain translation-invariant operators on the vector-valued Hardy spaces $H^1_X(T)$.

Bourgain [6] and Burkholder [11] have shewn that the so-called *UMD* Banach spaces X, defined to be those in which Walsh-Paley martingales converge unconditionally, are precisely those for which the Hilbert transform is bounded from $L^2_X(T)$ to itself. Their methods are based on transference and we use a refinement of such arguments here.

Results of the previous chapter show that the class *AUMD* is strictly larger than than the class *UMD* and includes such spaces as $L^1(T)$, which do not even enjoy the Radon-Nikodym property [20].

The rest of this chapter is arranged as follows. In the second section we introduce some basic definitions and provide a formal statement of the result given in the abstract. We also sketch the proof of the easy half of the theorem.

In the next section we reformulate the problem in probabilistic terms, following where possible an argument of McConnell [30]. In the penultimate section we establish the multiplier theorem. Our argument uses a result of Edgar [16] which allows us to approximate certain Brownian martingales by discrete-parameter analytic martingales. In the final section of this paper we mention some other properties of analytic *UMD* spaces.

Garling has introduced a more general class of martingales, termed Hardy martingales, which may be used to prove renorming theorems [20]. It is known that the Banach spaces for which analytic martingales converge unconditionally are those for which Hardy martingales converge unconditionally. Indeed, this follows from the techniques of this paper.

2.2 The main result

We remind the reader of several basic definitions.

Let X be a complex Banach space, T^{∞} the infinite torus endowed with Haar measure P.

Definition 1: (i) An L^1-bounded analytic martingale f is a sequence (f_n) of functions in the Bochner-Lebesgue space $L^1_X(\mathrm{T}^{\infty})$ having the form

$$f_n(\theta) = \Sigma_{k=1}^{n}\beta_k(\theta_1,\ldots,\theta_{k-1})e^{i\theta_k} \qquad (2.1)$$

with $\|f\|_1 = sup_n\|f_n\|_1 < \infty$.

(ii) We say that X is an analytic unconditional martingale difference (*AUMD*) space if there is a constant C, depending only on the space X, such that if $\tilde{f}=(\tilde{f}_n)$, where

$$\tilde{f}_n(\theta) = \Sigma_{k=1}^{n} \epsilon_k \beta_k(\theta_1,\dots,\theta_{k-1}) e^{i\theta_k} \tag{2.2}$$

is the transform of $f=(f_n)$ by a sequence of constants ϵ_k bounded in modulus by 1, then $\|\tilde{f}\|_1 \le C\|f\|_1$.

The equivalence of this definition with the one given in earlier was proved in Lemma 7 of Chapter 1. To state the theorem we need the following:

Definition 2: We say that the multiplier operator T_m associated with the distribution $m = \Sigma m_n e^{in\theta}$ satisfies strong Hörmander-Mihlin conditions if there is a constant C such that

(i) $|m_n| \le C$, (ii) $n|\Delta m_n| = n|m_n - m_{n-1}| \le C$,

$$\text{(iii) } n^2|\Delta^2 m_n| = n^2|m_{n+1} - 2m_n + m_{n-1}| \le C . \tag{2.3}$$

We remark that in Definition 2 condition (ii) is a consequence of (i) and (iii), but we include it since it is needed in the proof of lemma 4.

Our main result is the following:

THEOREM 3: *The complex Banach space X belongs to AUMD if and only if T gives a bounded operator on $H^1_X(\mathbb{T})$ whenever T is a multiplier satisfying strong Hörmander-Mihlin conditions.*

Proof: The backward implication follows from an application of a well-known method of transference [6]. By an approximation argument one reduces to the case of

$$f = \Sigma_{k=1}^{n} \beta_k(\theta_1, \ldots, \theta_{k-1}) e^{i\theta_k}, \tag{2.4}$$

where the β_k are trigonometric polynomials. Consider the family of measure-preserving transformations of T^{∞} given by $(\theta_k) \to (\theta_k + n_k\zeta)$, where ζ is a parameter and the n_k are positive integers chosen recursively so that four times the greatest member of the spectrum of

$$\zeta \to \Sigma_{j=0}^{k-1} \beta_j(\theta_1 + n_1\zeta, \ldots, \theta_{j-1} + n_{j-1}\zeta) e^{i\theta_j + in_j\zeta}$$

is less than the least member of the spectrum of

$$\gamma_k(\zeta) = \beta_k(\theta_1 + n_1\zeta, \ldots, \theta_{k-1} + n_{k-1}\zeta) e^{i\theta_k + in_k\zeta}. \tag{2.5}$$

By considering smooth partitions of unity on [0,1], one can find multipliers T given by $\Sigma m_n e^{in\theta}$, satisfying strong Hörmander-Mihlin conditions, where the m_n are constant on long stretches of integers. For a given sequence of constants ϵ_n, bounded in modulus by 1, we can find such a T so that it multiplies γ_k by ϵ_k $(k=1, \ldots, n)$. Since T is bounded on $H_X^1(T)$, one can complete the proof as follows:

$$\|f\|_1 = \int\int|\Sigma\beta_k(\theta_1+n_1\zeta,\ldots,\theta_{k-1}+n_{k-1}\zeta)e^{i\theta_k+in_k\zeta}|d\zeta dP \geq$$

$$\geq C\int\int|\Sigma\epsilon_k\beta_k(\theta_1+n_1\zeta,\ldots,\theta_{k-1}+n_{k-1}\zeta)e^{i\theta_k+in_k\zeta}|d\zeta dP =$$

$$= C\|\tilde{f}\|_1. \tag{2.6}$$

2.3 <u>Probabilistic formulation of the problem</u>

In this section it will be convenient to use the following notation. Let f be an analytic trigonometric polynomial valued in an *AUMD* space X, and let T be the convolution operator associated with the distribution $m = \Sigma_n m_n e^{in\theta}$. Let u,v,w denote the Poisson integrals of f,m,Tf respectively. We may suppose without loss that the first few derivatives of u and of v vanish at the origin and that v is a trigonometric polynomial.

By the semigroup property of the Poisson integrals $(p_r)_{0<r<1}$, we have

$$w(r_1r_2e^{i\theta}) = \int v(r_1e^{i\phi})u(r_2e^{i(\theta-\phi)})d\phi(2\pi)^{-1}. \tag{2.7}$$

We differentiate (2.7) twice with respect to r_1, once with respect to r_2 and set $r_1 = s$, $r_2 = s^3$. This gives us, after a little reduction,

$$s^7w_{,rrr}(s^4e^{i\theta}) =$$

$$= \int(v_{,rr}(se^{i\phi})-2s^{-1}v_{,r}(se^{i\phi}))u_{,r}(s^3e^{i(\theta-\phi)})d\phi(2\pi)^{-1} +$$
$$+ 2\int s^{-4}v_{,r}(se^{i\phi})u(s^3e^{i(\theta-\phi)})d\phi(2\pi)^{-1}. \tag{2.8}$$

We now integrate the following expression twice, and change variables. This gives us

$$Tf(e^{i\theta}) = \int_0^1 w_{,r}(re^{i\theta})dr = 2^{-1}\int_0^1 (1-r)^2 w_{,rrr}(re^{i\theta})dr$$

$$= 2\int_0^1 (1-s^4)^2 w_{,rrr}(s^4e^{i\theta})s^3 ds =$$

$$= 2\int\int(1-s^4)^2(v_{,rr}(se^{i\phi})-2s^{-1}v_{,r}(se^{i\phi}))$$

$$u_{,r}(s^3e^{i(\theta-\phi)})s^{-4}d\phi(2\pi)^{-1}ds +$$

$$+ 4\int_0^1\int(1-s^4)^2 v_{,r}(se^{i\phi})u(s^3e^{i(\theta-\phi)})s^{-8}d\phi(2\pi)^{-1}ds \qquad (2.9)$$

where we have used (2.8) at the last step.

The $L_X^1(T)$ norm of the last term in equation (2.9) is estimated as follows. Since $s^{-8}(1-s^4)^2 v_{,r}(se^{i\phi})$ is bounded we have

$$\int|\int_0^1\int(1-s^4)^2 v_{,r}(se^{i\phi})u(s^3e^{i(\theta-\phi)})s^{-8}d\phi ds|d\theta \le$$

$$\le \int_0^1\int(1-s^4)^2|v_{,r}(se^{i\phi})|s^{-8}\int|u(s^3e^{i(\theta-\phi)})|d\theta ds d\phi \le$$

$$\le C\int|f(\theta)|d\theta. \qquad (2.10)$$

By the smoothness assumptions on f, we can approximate the integral with respect to s in the other term in (2.9) by a Riemann sum. The following approximants converge boundedly to Tf:

$$g_N(e^{i\theta}) = \tag{2.11}$$

$$=\Sigma_{j=0}^{N-1}\int(1-r_j^4)^2 r_j^{-6}(v_{,rr}(r_j e^{i\phi})-2r_j^{-1}v_{,r}(r_j e^{i\phi}))[u(r_{j+1}^2 r_j e^{i(\phi-\theta)})- -u(r_j^3 e^{i(\phi-\theta)})]d\phi/2\pi$$

where $0<A=r_0<r_1<\ldots<r_N=1$ and A is chosen sufficiently small. We need the following elementary lemma, which can be proved using Abel summation.

LEMMA 4: *The function*

$$h(r,\phi) = (1-r^4)^2(v_{,rr}(re^{i\phi})-2r^{-1}v_{,r}(re^{i\phi}))p_r^{-1}(\phi)r^{-6} \tag{2.12}$$

is bounded $(1/2<r<1,\ \phi\in\mathbb{R})$ *when* T *is a multiplier of strong Hörmander-Mihlin type.*

The basic idea of the following proof is that the image of a Brownian motion in the unit disc under an analytic vector-valued trigonometric polynomial function behaves like an analytic martingale. The situation is complicated by the fact that it seems necessary to introduce auxiliary Brownian motions.

We let z_t be Brownian motion in D and consider various conditional probability measures on the Wiener space. Let P^θ denote the probability measure given by conditioning z_t to begin at the origin and to exit the unit disc at θ, and let P_z denote the measure given by conditioning the motion to begin at z. The following result may be obtained from Durrett's book [15, section 3.2].

LEMMA 5: *Let z_t be Brownian motion conditioned to begin at $z_0 = 0$ and to exit D at θ. Then there is a strong Markov process $X(r)$ $(0<r\leq 1)$ valued in* T *with sample paths continuous a.s. such that $z_{S_r} = re^{iX(r)}$ where $S_r = \inf\{t: |z_t| = r\}$. Further, this process has independent increments.*

We introduce a Brownian motion $\tilde{z}_t$ independent of z_t and denote its expectations, probabilities and associated X-process with a tilde. For convenience we denote

$$d_j = u(r_{j+1}r_j, X(r_{j+1})-\tilde{X}(r_j))-u(r_j^2, X(r_j)-\tilde{X}(r_j)). \qquad (2.13)$$

LEMMA 6: *We have the following representation for g_N,*

$$g_N(e^{i\theta}) = \tilde{E}^0 \; E^\theta \Sigma_{i=0}^{N-1} \; h(r_j, \tilde{X}(r_j))d_j. \qquad (2.14)$$

Proof: We recall that the function u is harmonic on D and that the transition densities of the Markov process $X(r)$ are given by Poisson kernels.

Hence, applying Fubini's Theorem, we have

$$E^\theta \tilde{E}^0 (h(r_j, \tilde{X}(r_j))d_j) = \qquad (2.15)$$

$$= E^\theta \int p_{r_j}(\phi) h(r_j,\phi)[u(r_{j+1}r_j, X(r_{j+1})-\phi)-u(r_j^2, X(r_j) - \phi)]\, d\phi/2\pi =$$

$$= E^\theta \int (1-r_j^4)^2 r_j^{-6} (v_{,rr}(r_j e^{i\phi})-2r_j^{-1} v_{,r}(r_j e^{i\phi}))$$
$$[u(r_{j+1}r_j, X(r_{j+1})-\phi)-u(r_j^2, X(r_j)-\phi)]d\phi/2\pi =$$

$$= \int (1-r_j^4)^2 r_j^{-6} (v_{,rr}(r_j e^{i\phi}) - 2r_j^{-1} v_{,r}(r_j e^{i\phi}))$$
$$E^{\theta}[u(r_{j+1} r_j, X(r_{j+1}) - \phi) - u(r_j^2, X(r_j) - \phi)] d\phi/2\pi =$$

$$= \int (1-r_j^4)^2 r_j^{-6} (v_{,rr}(r_j e^{i\phi}) - 2r_j^{-1} v_{,r}(r_j e^{i\phi}))$$
$$[u(r_{j+1}^2 r_j, \theta - \phi) - u(r_j^3, \theta - \phi)] d\phi/2\pi.$$

On summing over j we obtain the desired representation.

2.4 An approximation argument

In the following section we wish to apply martingale transforms to martingales constructed by applying analytic functions to Brownian motion. In order to justify this, we now show how to approximate such a martingale by an analytic martingale.

There is no loss in generality in supposing that X is finite-dimensional, since *AUMD* is evidently a local property. This serves to simplify the construction of martingales. We note that under the conditional probability $P_{Ae^{i\phi}}$, Σd_j is a martingale for fixed $\tilde{\omega}$. We aim to approximate this by an analytic martingale.

At the j^{th} stage of construction we consider the process

$$u((z_t + r_j e^{iX(r_j)})(r_j e^{-i\tilde{X}(r_j)})) \quad (t \geq 0), \tag{2.16}$$

where z_t is a Brownian motion starting at the origin. By a result of Edgar [16], we can find an analytic martingale

$$\Sigma_{n_j+1}^{n_{j+1}} \beta_k(\theta_{n_j+1},\ldots,\theta_{k-1},X(r_j),\tilde{\omega})e^{i\theta_k} \tag{2.17}$$

depending measurably on $\tilde{\omega}$, $X(r_j)$ and such that

$$E|u((z_{\tau_{j+1}}+r_je^{iX(r_j)})r_je^{-i\tilde{X}(r_j)})-u(r_j^2e^{i(X(r_j)-\tilde{X}(r_j))})-$$
$$\Sigma_{n_j+1}^{n_{j+1}} \beta_k(\theta_{n_j+1},\ldots,\theta_{k-1},X(r_j),\tilde{\omega})e^{i\theta_k}| < \epsilon/N \quad \forall\tilde{\omega}, \tag{2.18}$$

where $\tau_{j+1} = inf\{t:\ |z_t+r_je^{iX(r_j)}| = r_{j+1}\}$.

In this way we construct a discrete-parameter analytic martingale

$$\Sigma\beta_k(\theta_1,\ldots,\theta_{k-1},\tilde{\omega})e^{i\theta_k}$$

which approximates Σd_k in the sense of (2.18).

2.5 Conclusion of the proof of the theorem

We are now in a position to apply the previous observations to estimate the $L_X^1(T)$ norm of g_N. Lemma 6 gives

$$\int|g_N(e^{i\theta})|d\theta \le \int\tilde{E}^0E^{\theta}|\Sigma_{j=0}^{N-1}h(r_j,\tilde{X}(r_j))d_j|d\theta =$$

$$= \int\tilde{E}^0\int p_A(\theta-\phi)E^{\theta}_{Ae^{i\phi}}|\Sigma_{j=0}^{N-1}h(r_j,\tilde{X}(r_j))d_j|d\theta d\phi$$

$$= \int \tilde{E}^0 E_{Ae^{i\phi}} |\Sigma_{j=0}^{N-1} h(r_j, \tilde{X}(r_j)) d_j| d\phi. \tag{2.19}$$

By the approximation argument of the previous section and the hypothesis that X is in *AUMD*, this is bounded by

$$C \int \tilde{E}^0 E_{Ae^{i\phi}} |\Sigma_{j=0}^{N-1} d_j| d\phi + \epsilon \leq C \int \tilde{E}^0 E^\theta |\Sigma_{j=0}^{N-1} d_j| d\theta + \epsilon. \tag{2.20}$$

We now let $\eta_{2j} = r_j^2$, $\eta_{2j+1} = r_j r_{j+1}$; $\alpha_{2j} = 1$, $\alpha_{2j+1} = 0$ for $j=0,1,\ldots,N-1$.

By the properties of $X(r)$ mentioned in Lemma 5 of section 2.3, the sequence of random variables $X(r_0)-\tilde{X}(r_0)$, $X(r_1)-\tilde{X}(r_0),\ldots,X(r_{N-1})-\tilde{X}(r_{N-1})$ has the same joint distributions under $P^\theta \otimes \tilde{P}^0$ as $X(\eta_0), X(\eta_1),\ldots,X(\eta_{2N-1})$ under P^θ. Hence we have

$$\int \tilde{E}^0 E^\theta |\Sigma_{j=0}^{N-1} d_j| d\theta = \tag{2.21}$$

$$= \int E^\theta |\Sigma_{j=0}^{2N-2} \alpha_j (u(\eta_j, X(\eta_j)) - u(\eta_{j+1}, X(\eta_{j+1}))| d\theta =$$

$$= \int\int p_A(\phi-\theta) E_{Ae^{i\phi}}^\theta |\Sigma_{j=0}^{2N-2} \alpha_j (u(\eta_j, X(\eta_j)) - u(\eta_{j+1}, X(\eta_{j+1})))| d\phi d\theta.$$

We are now able to use the fact that this latest expression is the L_X^1 norm of a martingale which we can approximate by an analytic martingale. This gives us

$$\int |g_N(e^{i\theta})|d\theta \le$$

$$\le C\int E_{Ae^{i\phi}}|\Sigma_{j=0}^{2N-2}(u(\eta_j,X(\eta_j))-u(\eta_{j+1},X(\eta_{j+1})))|d\phi. \qquad (2.22)$$

Since the norm on the Banach space X is subharmonic and $X(r)$ has density $p_A(\ -\phi)$ under $P_{Ae^{i\phi}}$, we can complete the calculation as follows:

$$\int |g_N(e^{i\theta})|d\theta \le C\int E_{Ae^{i\phi}}|u(\eta_{2N-2},X(\eta_{2N-2}))|d\phi \le$$

$$\le C\int E_{Ae^{i\phi}}|u(1,X(1))|d\phi = C\int\int p_A(\phi-\theta)|f(e^{i\theta})|d\theta d\phi =$$

$$= C\int |f(e^{i\theta})|d\theta. \qquad (2.23)$$

2.6 Concluding remarks

Given the characterization of *AUMD* spaces contained in Theorem 3 of this chapter we can improve upon Proposition 8 of chapter 1.

COROLLARY 6: *Let X be an AUMD space. Then $L^1_X(\mathrm{T})$ is also an AUMD space.*

Proof: A straightforward Fubini Theorem argument shews that if strong Hörmander-Mihlin multipliers are bounded on $H^1_X(\mathrm{T})$ then they are also bounded on $H^1_Y(\mathrm{T})$ where $Y = L^1_X(\mathrm{T})$.

Another consequence of Theorem 3 concerns the Paley Theorem property [3].

COROLLARY 7: *Let X be an analytic UMD space with cotype 2. Then X satisfies Paley's Theorem i.e. there is a positive constant C, depending only on X, such that*

$$\int |\Sigma e^{in\theta} x_n| d\theta \geq C(\Sigma |x_{2^n}|^2)^{1/2} \tag{2.24}$$

for all analytic trigonometric polynomials $\Sigma e^{in\theta} x_n$.

Proof: To see this, note that there exists a sequence (T_k) of strong Hörmander-Mihlin multipliers such that $T_k(e^{i2^k\theta}) = e^{i2^k\theta}$ and for every sequence of signs $\epsilon=(\epsilon_k)$, $T_\epsilon = \Sigma \epsilon_k T_k$ is a strong Hörmander-Mihlin multiplier with norm bounded by a constant independent of ϵ. Then the stated properties of X give positive constants C such that

$$\int |\Sigma_n e^{in\theta} x_n| d\theta \geq C E_\epsilon \int |T_\epsilon {}^* \Sigma_n e^{in\cdot} x_n| d\theta \geq$$

$$\geq C \int E_\epsilon |\Sigma_k \epsilon_k T_k {}^* (\Sigma_n e^{in\cdot} x_n)| d\theta \geq$$

$$\geq C \int (\Sigma_k |T_k {}^* \Sigma_n e^{in\cdot} x_n|^2)^{1/2} d\theta \geq$$

$$\geq C(\Sigma_k (\int |T_k {}^* \Sigma_n e^{in\cdot} x_n| d\theta)^2)^{1/2} \geq$$

$$\geq C(\Sigma_k |x_{2^k}|^2)^{1/2}. \tag{2.25}$$

REMARKS: (i) As the authors of [3] observe, the hypothesis that X has cotype two is necessary. This can be seen from a transfer argument between the Sidon set $(2^n)_{n\geq 0}$ in $\mathbb{Z}$ and the subset of the Cantor group on which the Rademacher functions 'live'. Altrnatively, one can adapt the method of [28, lemma 2.8 & proposition 3.1 (ii) ⇒ (iii)] to shew that *PT* spaces have Gaussian cotype 2. The crucial point is that lacunary trigonometric series obey a central limit theorem [41].

(ii) The validity of the inequality (2.24) does not imply that X is in *AUMD*, however. In [23] the authors shew that the space c_1 of trace-class operators does not belong to *AUMD*, whereas c_1 satisfies Paley's Theorem [3].

(iii) Corollary 7 is contained in a paper of Pisier where it is proved by Brownian motion techniques [37, Theorem 7.8 & proposition 7.5].

III Injective Operator Systems are Primary

Abstract: We prove that if $\mathcal{A}$ is an injective operator system on ℓ^2 and P is a completely bounded projection on $\mathcal{A}$ then either $P\mathcal{A}$ or $(I-P)\mathcal{A}$ is completely boundedly isomorphic to $\mathcal{A}$. We also prove that if $B(\ell^2)$ is linearly homeomorphic to $X \oplus Y$ then either X or Y is linearly homeomorphic to $B(\ell^2)$.

3.1 Introduction

A Banach space Z is said to be primary if whenever Z is linearly homeomorphic to $X \oplus Y$ either X or Y is linearly homeomorphic to Z. The purpose of this paper is to generalize a known result about the space ℓ^∞ by establishing that infinite-dimensional injective operator systems on a separable Hilbert space are primary. An operator system $\mathcal{A}$ on ℓ^2 is a self-adjoint subspace of $B(\ell^2)$ containing the identity; $\mathcal{A}$ is said to be injective if there is a completely positive projection from $B(\ell^2)$ onto $\mathcal{A}$. It is natural to classify operator systems up to completely bounded isomorphism instead of linear homeomorphism, where operator systems $\mathcal{A}, \mathcal{B}$ are said to be completely boundedly isomorphic if there is a completely bounded map $T:\mathcal{A}\to\mathcal{B}$ with completely bounded inverse $T^{-1}:\mathcal{B}\to\mathcal{A}$. A recent result of Robertson and Wassermann [39] shows that if $\mathcal{A}$ is an infinite-dimensional injective operator system, then $\mathcal{A}$ is completely boundedly isomorphic either to ℓ^∞ or to M, the sum of finite-dimensional matrix algebras

$$M = (\Sigma_{n=1}^{\infty} M_n)_\infty .$$

As we shall see later, this allows us to reduce the proof of our main theorem to a proposition concerning completely bounded maps on the finite-dimensional matrix algebras M_n. This we establish using Ramsey theory.

For basic properties of completely bounded maps we refer the reader to [34] and for the necessary combinatorial results to [4, chapter 20].

The rest of this chapter is arranged as follows. In the second section we state and prove the finite-dimensional analogue of our main theorem. In the third section we adapt an argument of Bourgain [5] which allows us to deduce the main result from the finite-dimensional version. In the final section we use the results of the previous sections to obtain the theorems stated in the abstract.

3.2 The finite-dimensional case

In this section we prove the finite-dimensional version of the fact that M is primary. The basic combinatorial principle used is that if $f: S \times T \to \{0,1\}$ is any function on the product of finite sets S,T, then we can find a large subset S' of S, and a large subset T' of T, such that $f|S' \times T'$ is constant. To state our result we require the following:

Definition: Let $\sigma=\{\sigma_1,\dots,\sigma_n\}$, $\tau=\{\tau_1,\dots,\tau_n\}$ be two subsets of $\{1,2,\dots,N\}$ of order n. Let $A=[a_{ij}]$, $B=[b_{ij}]$ be matrices in M_n. Define $J_{\sigma,\tau}: M_n \to M_N$ by

$$(J_{\sigma,\tau}(A))_{kl} = \begin{cases} a_{ij} & \text{if } k=\sigma_i, l=\tau_j \\ 0 & \text{else.} \end{cases} \tag{3.1}$$

Further, let $K_{\sigma,\tau}:M_N \to M_n$ be defined by

$$(K_{\sigma,\tau}(B))_{ij} = b_{\sigma(i),\tau(j)}.$$

We note that $K_{\sigma,\tau}J_{\sigma,\tau} = I_n$, the identity map on M_n, and that $J_{\sigma,\tau}K_{\sigma,\tau} = P_{\sigma,\tau}$ is a projection onto $J_{\sigma,\tau}(M_n)$. We call $P_{\sigma,\tau}$ a block projection of order n. Note that $J_{\sigma,\tau}$, $K_{\sigma,\tau}$ and $P_{\sigma,\tau}$ are all completely bounded with completely bounded norm 1.

PROPOSITION 1: *Given n, ϵ with $0<\epsilon<1/4$ and $K<\infty$ there exists N_0 such that if $N \geq N_0$ and $T \in L(M_N,M_N)$, with $\|T\| \leq K$, then there exist (disjoint) subsets σ,τ of $\{1,2,\ldots,N\}$ of order n and a constant c such that*

$$\|K_{\sigma,\tau}TJ_{\sigma,\tau} - cI_n\|_{cb} < \epsilon. \tag{3.2}$$

Consequently, one of $K_{\sigma,\tau}TJ_{\sigma,\tau}$ and $K_{\sigma,\tau}(I_N-T)J_{\sigma,\tau}$ is invertible, with inverse of completely bounded norm at most 4.

Proof: We consider T,K,n as in the statement and let $0<\epsilon<1/4$. We divide the disc $\{z \in \mathbb{C}: |z| \leq K\}$ into finitely many disjoint subsets V_k of diameter $\epsilon n^{-4}/4$. We define the colouring on pairs of $\{1,2,\ldots,N\}$ by

$$\{i,j\} \to k \quad \text{if} \quad (Te_{ij},e_{ij}) \in V_k \quad \text{where } i<j. \tag{3.3}$$

By Ramsey's Theorem we can find an index k' and a large monochromatic subset σ_1 of $\{1,2,\ldots,N\}$ whose colour is k'. Let c be any point in $V_{k'}$.

Our next task is to show that there is a subset σ_5 of σ_1 such that $|(Te_{ij},e_{kl})|$ is small whenever $(i,j)\neq(k,l)$ and i,k belong to the first n elements of σ_5 and j,l belong to the last n elements of σ_5.

We introduce a $\delta=\delta(\epsilon,n)>0$, to be specified later. Consider the 2-colouring of the quadruples of σ_1 given by

$$\{i,j,k,l\} \text{ is bad if } i<k<j<l \text{ and } |(Te_{ij},e_{kl})| \geq \delta. \qquad (3.4)$$

Ramsey's Theorem guarantees the existence of a large monochromatic subset σ_2 of σ_1. Suppose that σ_2 is a bad monochromatic subset of σ_2. We let $k<j<l$ be the three largest elements of σ_2 and let i range from the smallest element of σ_2 to the largest element of σ_2 smaller than k. For a suitable choice of unimodular complex numbers ϵ_i

$$\|T\Sigma_i\epsilon_i e_{ij}\| \geq \Sigma_i\epsilon_i(Te_{ij},e_{kl}) \geq \delta(|\sigma_2|-3), \qquad (3.5)$$

whereas $\|\Sigma_i\epsilon_i e_{ij}\| \leq (|\sigma_2|-3)^{1/2}$. Hence $K \geq \delta(|\sigma_2|-3)^{1/2}$.

By a suitable choice of N we ensure that σ_2 is good. Proceeding in this way we obtain a large subset σ_3 of σ_2 such that

$$|(Te_{ij},e_{kl})| \leq \delta \qquad (3.6)$$

whenever i,j,k and l are distinct elements of σ_3 with $\max\{i,k\}<\min\{j,l\}$.

The other off-diagonal terms we need to consider are dealt with as follows. For the case $i=k<j<l$, we consider the 2-colouring of triples of σ_3 given by

$$\{i,j,l\} \text{ is bad if } i<j<l \text{ and } |(Te_{ij},e_{il})| \geq \delta. \qquad (3.7)$$

Again Ramsey's Theorem grants us a large monochromatic subset. If ρ is a bad monochromatic subset we fix i as small as we can, l as large as we can and let j range through the bad subset so that $i<j<l$. Here, for suitable unimodular complex numbers ϵ_j, we have that

$$\|\Sigma_j \epsilon_j e_{ij}\| = (|\rho|-2)^{1/2}$$

while $$\|\Sigma_j T\epsilon_j e_{ij}\| \geq \Sigma_j \epsilon_j (Te_{ij}, e_{il}) = \delta(|\rho|-2). \tag{3.8}$$

This ensures that we have a large good monochromatic subset σ_4 of σ_3. The other cases of this type are when $i=k<l<j$, $i<k<j=l$ and $k<i<j=l$. These are dealt with in a similar way.

The conlusion is that there is a large subset σ_5 of σ_4 such that when (i,j) and (k,l) are distinct pairs in σ_5 with $\max\{i,j\} < \min\{k,l\}$, we have

$$|(Te_{ij}, e_{kl})| \leq \delta. \tag{3.9}$$

We split up σ_5 by taking the first n elements to form σ and the last n to form τ. Our construction of σ,τ gives

$$|(Te_{ij}, e_{ij})-c| \leq \epsilon n^{-4}/4 \qquad ((i,j)\epsilon\sigma\times\tau). \tag{3.10}$$

The above estimates show that $K_{\sigma,\tau}TJ_{\sigma,\tau}$ operates like cI_n, since we can write using (3.10)

$$\|(K_{\sigma,\tau}TJ_{\sigma,\tau}-cI_n)\Sigma_{(i,j)}a_{ij}e_{ij}\| \leq$$

$$\leq \|\Sigma_{(i,j)}a_{ij}((Te_{\sigma_i,\tau_j}, e_{\sigma_i,\tau_j})-c)e_{ij}\| +$$
$$+ \|\Sigma_{(i,j)}a_{ij}\Sigma_{(k,l):(k,l)\neq(i,j)}(Te_{\sigma_i,\tau_j}, e_{\sigma_k,\tau_l})e_{kl}\| \leq$$

$$\leq n^2 \max|a_{ij}|\epsilon n^{-4}/4 + n^4\delta\max|a_{ij}| \leq$$

$$\leq (n^2\epsilon n^{-4}/4 + \delta n^4)\|\Sigma_{(i,j)}a_{ij}e_{ij}\|. \tag{3.11}$$

From this latest inequality it is clear that the choice of $\delta = \epsilon n^{-6}/4$ ensures that

$$\|K_{\sigma,\tau}TJ_{\sigma,\tau}-cI_n\| < \epsilon/n^2. \tag{3.12}$$

Using the fact that all linear functionals are completely bounded, one can easily show by calculation that if $\mathcal{A}$ is an operator system and $\phi:\mathcal{A}\to M_n$ a bounded linear map, then $\|\phi\|_{cb} \leq n^2\|\phi\|$. Hence we have

$$\|K_{\sigma,\tau}TJ_{\sigma,\tau}-cI_n\|_{cb} < \epsilon. \tag{3.13}$$

The final statement of the proposition follows by considering Neumann series to obtain inverses of operators which are close to multiples of the identity.

3.3 Localization

In this section we show how the proof that M is primary can be reduced to a question concerning finite-dimensional matrix algebras.

Proposition 1 deals almost at once with the case of completely bounded projections P on $M=(\Sigma_{n=1}^{\infty}M_n)_\infty$ of the form $P=\oplus_n Q_n$ where $Q_n:M_n\to M_n$ is a projection. We show that the general case can be reduced to analysis of diagonal operators *i.e.* operators of the form $T=\oplus T_n$, where $T_n\in L(M_n,M_n)$. This is a consequence of the following lemma.

LEMMA 2: *Given a completely bounded operator $T:M\to M$ and $\epsilon>0$ there is a completely bounded operator T' on M such that*
(1) one can factor T' through T using completely bounded maps and $I-T'$ through $I-T$ using completely bounded maps;
(2) T' is almost diagonal (in the sense that $\|T'-D\|_{cb} < \epsilon$ where D is diagonal).

This is in turn deduced from the following lemma:

LEMMA 3: *Given $n\in\mathbb{N}$, $\epsilon>0$ there is a $N'(n,\epsilon)$ such that if $N \geq N'(n,\epsilon)$ and E is an n-dimensional subspace of M_N there is a subspace F of M_N and a block projection q from M_N onto F such that q is of order n and*

$$\|qx\| \leq \epsilon\|x\| \quad for\ x\in E. \tag{3.14}$$

Proof of Lemma 3: Let $\epsilon>0$ and $n\in\mathbb{N}$ be given. Let E be an n-dimensional subspace of M_{nm}, where m is sufficiently large (see below). By a simple volume argument [33, lemma 2.6] we can find an $\epsilon/2$ net $x_1,x_2,\ldots,x_L$ of the unit sphere of E, where L depends only on ϵ and n. We identify M_{nm} with $M_n\otimes M_m$ and consider the natural projections Q_r $(r=1,2,\ldots,m^2)$ onto the $n\times n$ blocks. By an easy estimate we have

$$L = \Sigma_{k=1}^{L} \|x_k\|^2 \geq (mn)^{-1} \Sigma_{k=1}^{L} \|x_k\|_{HS}^2 =$$

$$= (mn)^{-1} \Sigma_{k=1}^{L} \Sigma_{r=1}^{m^2} \|Q_r x_k\|_{HS}^2 \geq$$

$$\geq (mn)^{-1} \Sigma_{r=1}^{m^2} \Sigma_{k=1}^{L} \|Q_r x_k\|^2. \qquad (3.15)$$

Now we let m be sufficiently large that $Ln < m(\epsilon/2)^2$. Then equation (3.15) shows that for some r

$$(\epsilon/2)^2 \geq \Sigma_{k=1}^{L} \|Q_r x_k\|^2. \qquad (3.16)$$

Hence we can choose Q_r to be our q of the lemma, for our selection argument has given us (3.16) which implies (3.14).

Although Bourgain [5] sketches how to find T' from T, we shall give a little more detail since complete boundedness is also at issue. The following notation will be used in the proof.

Definition: Let I be any subset of $\mathbb{N}$. We introduce the following projections on M

$$(P_I(x))_j = \begin{cases} x_j & j \in I \\ 0 & \text{else}, \qquad (x=(x_j)\in M) \end{cases}$$

$$P_k(x_1, x_2, \ldots) = (x_1, x_2, \ldots, x_k, 0, 0, \ldots);$$

$$R_k(x_1, x_2, \ldots) = (0, \ldots, 0, x_{k+1}, x_{k+2}, \ldots)$$

and $p_k : M \to M_k$, the n^{th} coordinate projection.

Proof of lemma 2: We wish to select T' such that the following conditions are satisfied

$$\|p_n T' P_{n-1}\|_{cb} < \epsilon/2^n, \quad \|p_n T' R_n\|_{cb} < \epsilon/2^n. \qquad (3.17)$$

Now $p_n T'$ takes values in M_n and hence as in the proof of (3.13) it suffices to obtain the following estimates in operator norm

$$\|p_n T' P_{n-1}\| < \epsilon/(n^2 2^n), \quad \|p_n T' R_n\| < \epsilon/(n^2 2^n). \tag{3.18}$$

To satisfy the second condition we use the fact that if F is a finite-dimensional space, $S \in L(M,F)$ and $\epsilon > 0$, then there is an infinite subset I of $\mathbb{N}$ such that $\|SP_I\| < \epsilon$. To see this, it suffices to consider $S \in M^*$ and to remark that $\mathbb{N}$ can be partitioned into infinitely many disjoint infinite subsets S_ξ.

We define recursively an increasing sequence of integers m_k, a decreasing sequence (I_k) of infinite subsets of $\mathbb{N}$ and block projections q_k on M_{m_k} such that

(i) q_k is of order k,

(ii) if $x \in p_{m_k} T(\oplus_{s=1}^{k-1} q_s(M_{m_s}))$ then $\|q_k(x)\| \le (\epsilon/k^2 2^k)\|x\|$

(iii) $m_k \in I_k - I_{k+1}$ and

(iv) $\|p_{m_k} T P_{I_{k+1}}\| \le \epsilon/k^2 2^k$. (3.19)

We use lemma 3 to satisfy (i) and (ii), while the fact stated above allows us to fulfil conditions (iii) and (iv). We let $r_k : M_k \to q_k(M_{m_k})$ be the natural isometry and introduce

$$T' = (\oplus r_k^{-1} q_k) T (\oplus r_k). \tag{3.20}$$

It is clear that $\|T'\|_{cb} \le \|T\|_{cb}$ and that $I = (\oplus r_k^{-1} q_k)(\oplus r_k)$. Hence T' factors through T with completely bounded maps, and $I - T'$ factors through $I - T$. We now show that

T' is close to a diagonal operator in completely bounded norm. More precisely, we have that $\|p_n T' P_{n-1}\| < \epsilon/n^2 2^n$ by (ii) and hence $\|p_n T' P_{n-1}\|_{cb} < \epsilon/2^n$. Further, (iv) gives that

$$\|p_n T' R_n\| < \epsilon/n^2 2^n \quad \text{and so} \quad \|p_n T' R_n\|_{cb} < \epsilon/2^n.$$

From these estimates it is easy to see that $\|T'-D\|_{cb} < \epsilon$, where $D = \oplus p_n T' p_n$ is diagonal. This completes the proof of lemma 2.

3.4 The main result

Combining the results of the previous sections with the theorem of Robertson and Wassermann stated in the introduction gives us our main theorem.

THEOREM 4: *Let A be an infinite-dimensional injective operator system on a separable Hilbert space and P a completely bounded projection on A. Then A is completely boundedly isomorphic either to PA or to $(I-P)A$.*

Proof: By [39, Corollary 7] there are only two cases to consider, namely when A is completely boundedly isomorphic to M and to ℓ^∞. In the second case one simply carries out an almost diagonalization as in the third section of this chapter.

Using the same notation as before, we apply our arguments to $T=P$ and $M=A$. By Proposition 1 we can find an increasing sequence of integers n_j and completely bounded maps U_{n_j}, V_{n_j}

with $\|U_{n_j}\|_{cb} \leq 4$ and $\|V_{n_j}\|_{cb} \leq 4$ such that the following diagram commutes, where $Q_{n_j} = p_{n_j} T' p_{n_j}$ for all j or $Q_{n_j} = p_{n_j}(I-T')p_{n_j}$ for all j.

$$\begin{array}{ccc} M_{N(n_j)} & \xrightarrow{Q_{n_j}} & M_{N(n_j)} \\ U_{n_j} \uparrow & & \downarrow V_{n_j} \\ M_{n_j} & \xrightarrow{I} & M_{n_j} \end{array}$$

We conclude that there are completely bounded maps U', V' such that the diagram below commutes

$$\begin{array}{ccc} M & \xrightarrow{D'} & M \\ U' \uparrow & & \downarrow V' \\ (\Sigma_j M_{n_j})_\infty & \xrightarrow{I} & (\Sigma_j M_{n_j})_\infty \end{array}$$

where $D'=D$ or $D'=I-D$. An easy perturbation argument combined with lemma 2 gives us completely bounded maps U, V such that

$$\begin{array}{ccc} M & \xrightarrow{T''} & M \\ U \uparrow & & \downarrow V \\ (\Sigma_j M_{n_j})_\infty & \xrightarrow{I} & (\Sigma_j M_{n_j})_\infty \end{array} \qquad (3.21)$$

commutes, where $T''=T$ or $T''= I-T$.

Hence we have that $(\Sigma_j M_{n_j})_\infty$ is completely boundedly isomorphic to the subspace $T''U(\Sigma_j M_{n_j})_\infty$ of $T''M$ complemented by the completely bounded projection $T''UV$. But $(\Sigma_j M_{n_j})_\infty$ is completely isomorphic to M and so when T is a completely bounded projection, the Pelczynski decomposition method for completely bounded maps [39,Theorem 3; 29 p54] shows that $T''M$ is completely boundedly isomorphic to M.

Although we have considered completely bounded maps throughout, the only place we have used the complete boundedness property is in ensuring that the isomorphism between M and either PM or $(I-P)M$ is completely bounded with completely bounded inverse. It is easy to see that our proof of the above theorem leads to the following result.

THEOREM 5: *The Banach space $B(\ell^2)$ is primary.*

Proof: The proof of [39, Theorem 6] shows in particular that $B(\ell^2)$ is linearly homeomorphic to M. By similar, but easier arguments to those above we have that if P is any bounded projection on M, then M is linearly homeomorphic either to PM or to $(I-P)M$.

We close this chapter by stating a basic fact about the ranges of certain completely bounded projections on $B(\ell^2)$.

PROPOSITION 6: *Hilbert space ℓ^2 is linearly homeomorphic to the range of a completely bounded projection on $B(\ell^2)$.*

Proof: If we introduce the usual orthonormal basis $(e_j)_{j\in\mathbb{N}}$ of ℓ^2, it is easily seen that the subspace S of $B(\ell^2)$ spanned by $(e_j \otimes e_1)_{j\in\mathbb{N}}$ is isomorphic to ℓ^2. Further, the map defined by right multiplication

$$a \to a(e_1 \otimes e_1) \qquad (a \in B(\ell^2)) \tag{3.22}$$

is a completely bounded projection with completely bounded norm one whose range is S.

IV Weakly Compact Homomorphisms

Abstract: We obtain an explicit formula for weakly compact homomorphisms from the disc algebra A to $B(\ell^2)$ and shew that such maps are compact. It is shewn that they are also 1-absolutely summing.

4.1 Introduction

The work of this chapter was motivated by questions considered by Ransford on the nature of weakly compact homomorphisms on Banach algebras. Ransford shewed that weakly compact homomorphisms from the space $C(\mathrm{T})$ of continuous functions on the circle to a (commutative) Banach algebra have ranges which are finite-dimensional and semisimple. His methods apply if one replaces $C(\mathrm{T})$ by any amenable commutative Banach algebra, but do not appear to work for algebras such as the disc algebra A which have bounded point derivations. In operator theory it is quite natural to consider analytic polynomials of a single operator and so we were led to consider weakly compact homomorphisms taking values in $B(\ell^2)$. Simple examples given below shew that we cannot hope to obtain a result as precise as that of Ransford. Indeed, von Neumann's Inequality shews that analytic polynomials operate boundedly on all contractions, not only on normal contractions [34].

Our method of proof is to use spectral theory to obtain an explicit representation for the most general weakly compact homomorphism from A to $B(\ell^2)$. A novelty of our arguments lies in our use of the interpolation theorems of Carleson-Rudin and Carleson-Newman in the context of spectral theory. Although the disc algebra is commutative, one must do analysis in a nonabelian C^*-algebra to obtain the desired representation.

The scope of our main theorem is increased by the observation that if the Banach algebra $\mathfrak{B}$ is an operator algebra *i.e.* is algebraically isomorphic to a closed subalgebra of $B(\ell^2)$ then, as an immediate corollary of Theorem 1, any weakly compact homomorphism from A to $\mathfrak{B}$ is compact. By considering L^∞ as a space of multiplication operators on L^2, it is evident that uniform algebras are operator algebras.

Varopoulos [46] gives necessary and sufficient conditions for a Banach algebra $\mathfrak{B}$ to be an operator algebra. The hypotheses of Varopoulos' Theorem follow from certain Banach space properties of $\mathfrak{B}$ and one obtains as a consequence that some Banach spaces are operator algebras no matter what Banach algebra multiplication is introduced. For example, from a difficult result of Bourgain [9] one deduces that any Banach algebra structure on H^∞ gives an operator algebra.

As Varopoulos observes [46, Prop. 4.2], one can even introduce a commutative multiplication on $B(\ell^2)$ which makes it into an operator algebra. We do not know whether every Banach algebra structure on $B(\ell^2)$ gives an operator algebra. The route by which one arrives at Bourgain's result on H^∞ is obstructed by the fact that examples like the one presented at the end of chapter 3 show that not all maps from $B(\ell^2)$ to its dual are 2-absolutely summing [38, Remark 1.2]. Consequently Tonge's method of fulfilling the hypotheses of Varopoulos' Criterion fails [45]. Since the arguments here are concerned with factorization of operators and tensor products, the fact that $B(\ell^2)$ does not have the approximation property threatens to introduce further complications [44].

The rest of this chapter is arranged as follows. We first introduce some notation and state more formally the result stated in the abstract. We then consider the spectrum of a particular element which generates the range of our homomorphism. The connection with the classical interpolation problems is explained in the subsequent section. In the course of a functional calculus argument in the following section a similarity theorem of Sz.Nagy is used to show that certain operators are projections. In the final section we consider the absolutely summing norms of the weakly compact homomorphisms.

4.2 The main result

We begin with a little notation. As usual, A will denote the disc algebra, here regarded as a space of functions on $\bar{D}$. The identity function on $\bar{D}$ is denoted by z and by homomorphism we mean Banach algebra homomorphism. Of particular interest here are those homomorphisms for which the image of the unit ball is relatively weakly compact. For such we establish the following:

THEOREM 1: *Let* $T:A\to B(\ell^2)$ *be a weakly compact homomorphism. Then* T *is compact.*

The proof of this theorem is contained in sections 4.3 and 4.4. Notice that it is easy to construct (weakly) compact homomorphisms from A to $B(\ell^2)$. The most obvious examples are:
(i) $f \to f(\zeta)P \quad (f\in A)$ where $\zeta\in\bar{D}$ is fixed and P is any projection,
(ii) $f \to \Sigma_{n=0}^{\infty} \hat{f}(n)V^n \quad (f\in A)$ where V has spectral radius $r(V)$ less than one. (4.1)

Our aim is to show that any weakly compact homomorphism T is built from these two examples. This is done by analysing the spectrum of Tz.

4.3 Two classical interpolation problems

The basic decomposition of the spectrum of Tz is given by the following:

LEMMA 2: *Let* B *be a Banach algebra and* $T:A\to B$ *be a weakly compact homomorphism. Then the spectrum* $\sigma(Tz)$ *of* Tz *is contained in* $\bar{D}$ *and there is an* $\alpha<1$ *such that* $\{\zeta\in\sigma(Tz):|\zeta|>\alpha\}$ *is finite.*

Proof: The first statement follows easily from the spectral radius formula. Indeed we have that

$$r(Tz) = \lim_n \|(Tz)^n\|^{1/n} =$$
$$= \lim_n \|Tz^n\|^{1/n} \le \lim_n (\|T\|\|z^n\|)^{1/n} = 1. \qquad (4.2)$$

We now consider the set $\sigma(Tz) \cap \{|\zeta|=1\}$. This set is finite, for if not we can select from it a sequence $(\xi_j)_{j\ge 1}$ of distinct points converging to a point ξ_0. The set $\{\xi_j : j\ge 0\}$ is of linear Lebesgue measure zero and so by the theorem of Rudin-Carleson [25] we can find a sequence of analytic trigonometric polynomials p_j $(j\ge 1)$ such that

$$\|p_j\|_\infty \le K, \quad p_j(\xi_j) = 1, \quad \text{and } |p_j(\xi_k)| < \epsilon 2^{-j} \quad (j\ne k). \qquad (4.3)$$

The closed subalgebra $\mathfrak{B}$ of B generated by $T(A)$ is commutative and so can be mapped into a subalgebra of $C(\sigma(Tz))$ by the Gelfand transform $\hat{}$. It follows that if we introduce the restriction homomorphism *rest* the range of the composite

$$A \xrightarrow{T} \mathfrak{B} \xrightarrow{\hat{}} C(\sigma(Tz)) \xrightarrow{rest} C(\{\xi_j : j\ge 0\}) \qquad (4.4)$$

contains a c_0 copy. This clearly contradicts the weak compactness of T.

It remains to shew that all but a finite subset of $\sigma(Tz)$ lies in a disc of radius $\alpha<1$. If not, one could choose a sequence (ζ_j) in $\sigma(Tz)$ such that $\zeta_j \to \zeta$ where ζ lies on the unit cicle. By passing to a subsequence if necessary we can suppose that the sequence (ζ_j) satisfies the Carleson-Newman separation condition [21]

$$\prod_{j:j\neq k}\{|\zeta_j-\zeta_k||1-\bar{\zeta}_j\zeta_k|^{-1}\} \geq \delta \qquad (k\in\mathbb{N}) \tag{4.5}$$

for some $\delta>0$. Now, by an interpolation theorem of Hoffman [25], we can find for each k an analytic trigonometric polynomial p_k such that

$$p_k(\zeta_k) = 1, \quad |p_k(\zeta_j)| < 2^{-k}\epsilon \quad (j\neq k), \quad \|p_k\| \leq c(\delta). \tag{4.6}$$

Arguing as above, we contradict the hypothesis that T is weakly compact.

4.4 Spectral subsets

Lemma 2 gives us sufficient information concerning the spectrum of Tz. In this section we restrict attention to the case considered in the theorem, namely when T takes values in $B(\ell^2)$, and use functional calculus on the various spectral subsets.

We write using lemma 2 the spectral decomposition

$$\sigma(Tz) = \{\xi_1\}\cup\{\xi_2\}\cup\ldots\cup\{\xi_n\}\cup\sigma_{n+1} \tag{4.7}$$

where the ξ_j have unit modulus and the elements of σ_{n+1} have modulus at most $\alpha<1$. By the usual functional calculus argument we can introduce spectral projections P_j $(j=1,\ldots,n+1)$ corresponding to the decomposition (4.7). We recall that the P_j are orthogonal and commute with Tz.

By the choice of σ_{n+1} we have that the spectral radius of $P_{n+1}(Tz)$ is at most α. An easy calculation, similar to that in section 4.5, shews that the map

$$f \to P_{n+1}(Tf) \qquad (f\in A) \tag{4.8}$$

is compact. It remains to shew that the other spectral projections give compact maps $f \to P_j(Tf)$ $(j=1,\ldots,n)$. This follows from the technical result below.

PROPOSITION 3: *Let S be an operator from ℓ^2 to itself with the following properties*

(i) $\sigma(S)=\{1\}$

(ii) $\|S^n\| \leq K$ for some $K<\infty$ and all positive integers n

(iii) the sequence (S^n) contains a subsequence weakly convergent in $B(\ell^2)$.

Then S is the identity operator on ℓ^2.

Proof: The idea of the proof is to use a theorem of Sz.Nagy to show that S is similar to a unitary operator, for then condition (i) will give the desired conclusion. The hypothesis for Sz.Nagy theorem [34] is that there be a constant K' such that $\|S^n\| \leq K'$ for all integers n. In light of condition (ii), we need to estimate the norms of the negative powers of S.

By condition (iii), we can pick a subsequence (S^{n_k}) such that $S^{n_k} \to Y$ weakly, for some Y in $B(\ell^2)$. By elementary arguments $\|Y\| \leq K$. We claim also that Y is invertible.

Indeed, by Mazur's Theorem we can choose for each $\epsilon>0$ a sequence of nonnegative numbers λ_{n_k} such that $\Sigma_k\lambda_{n_k}=1$ and

$$\|Y-\Sigma_k\lambda_{n_k}S^{n_k}\| \le \epsilon. \tag{4.9}$$

Since we are in a commutative subalgebra of $B(\ell^2)$ we have that

$$r(Y-I) \le r(\Sigma_k\lambda_{n_k}S^{n_k}-I) + r(Y-\Sigma_k\lambda_{n_k}S^{n_k}). \tag{4.10}$$

By the spectral mapping theorem and hypothesis (i) we have that $r(\Sigma_k\lambda_{n_k}S^{n_k}-I) = 0$, while the spectral radius formula together with (4.9) gives $r(Y-\Sigma_k\lambda_{n_k}S^{n_k}) \le \epsilon$. By (4.10) we conclude that $\sigma(Y)=\{1\}$ and so, in particular, Y is invertible.

We now claim that there is a $\delta>0$ such that

$$\|S^n x\| \ge \delta\|x\| \qquad (x\in\ell^2,\ n\ge 0) \tag{4.11}$$

which, taken with (ii), is equivalent to power-boundedness of S. We begin with the subsequence (S^{n_k}) and set $\delta = \|Y^{-1}\|^{-2}K^{-2}/2$. Suppose that for some k and some x, $\|S^{n_k}x\| < \delta\|x\|$. We introduce the functional on $B(\ell^2)$ defined by $C \to (Cx,Yx)$ $(C\in B(\ell^2))$. Now since Y is invertible it follows that

$$(Yx,Yx) = \|Yx\|^2 \ge \|Y^{-1}\|^{-2}\|x\|^2 , \tag{4.12}$$

but for $l\ge k$ the choice of k gives

$$|(S^{n_l}x,Yx)| \le \|S^{n_l}x\|\|Yx\| \le$$
$$\le \|S^{n_l-n_k}\|\|S^{n_k}x\|\|Yx\| \le K^2\delta\|x\|^2. \tag{4.13}$$

Letting $l\to\infty$ we conclude that

$$|(Yx,Yx)| = lim_l|(S^{n_l}x,Yx)| \le K^2\delta\|x\|^2 = \|Y^{-1}\|^{-2}\|x\|^2/2,$$

which contradicts (4.12).

We now consider general n. If for some n and some x the following inequality were valid

$$\|S^n x\| \le \delta\|x\|/(2K), \qquad (4.13)$$

then we could pick an n_k greater than n so that

$$\|S^{n_k}x\| \le \|S^{n_k-n}\|\|S^n x\| \le K\delta\|x\|/(2K)\ ,$$

which contradicts the above argument.

In [34] Sz.Nagy's Theorem is deduced from general properties of amenable groups and so for the reader's greater convenience we sketch an elementary argument. Let Lim be a Banach limit on $\mathbb{N}$ and consider the the inner product defined on ℓ^2 by

$$\langle\langle x,x\rangle\rangle = Lim_n(S^n x,S^n x) \qquad (x\in\ell^2) \qquad (4.14)$$

which gives an equivalent norm on ℓ^2 since S is power-bounded. By basic properties of Banach limits we have that

$$\langle\langle Sx,Sy\rangle\rangle = Lim_n(S^{n+1}x,S^{n+1}y) =$$
$$= Lim_n(S^n x,S^n y) = \langle\langle x,y\rangle\rangle \quad (x,y\in\ell^2) \qquad (4.15)$$

and so S is similar to a unitary operator on the Hilbert space $(\ell^2,\langle\langle .,.\rangle\rangle)$.

Proof of the Theorem: From the remark before Proposition 3 it remains only to observe that the spaces $P_j(\ell^2)$ are invariant under the operation of Tz and that the hypotheses of Proposition 3 are satisfied by the operators $S = \bar{\xi}_j P_j(Tz)$

regarded as operators from $P_j(\ell^2)$ to itself. Indeed, since T is a homomorphism we have that

$$\|(P_j(Tz))^n\| \le \|P_j(Tz)^n\| \le$$
$$\le \|P_j\|\|T(z^n)\| \le \|P_j\|\|T\|\|z^n\| \le \|P_j\|\|T\| \quad (4.16)$$

from whence (ii) follows, while the spectral projection argument gives (i). Condition (iii) is a consequence of the weak compactness of T.

4.5 Absolutely summing norms

We conclude this chapter by establishing the following easy result.

COROLLARY 4: *Let T be a weakly compact homomorphism from A to $B(\ell^2)$. Then T is 1-absolutely summing.*

Proof: In the course of the proof of Theorem 1 we established the form of such a map T. Since the sum of absolutely summing operators is absolutely summing we need only check the cases (i) and (ii) of (4.1). It is trivially true that operators of rank one are absolutely summing, while in case (ii) we can calculate as follows:

$$\|Tf\| = \|\Sigma_n \hat{f}(n)V^n\| \le$$
$$\le \Sigma_n|\hat{f}(n)|\|V^n\| \le (\Sigma_n\|V^n\|)\int|f|d\theta \le K_T\int|f|d\theta \quad (4.17)$$

Hence for each $m\in\mathbb{N}$ and $f_1,f_2,\ldots,f_m$ belonging to A we have that

$$\Sigma_{n=1}^{m}\|Tf_n\| \le K_T\int\Sigma_{n=1}^{m}|f_n|d\theta \le$$

$$\le K_T sup\{\Sigma_{n=1}^{m}|(f_n,\mu)|:\mu\in Ball(A^*)\}. \qquad (4.18)$$

By definition T is 1-absolutely summing and equation (4.18) gives us explicitly a Radon-Pietsch measure on Ball(A^*) [29, p 64].

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