COBORDISMS OF SUTURED MANIFOLDS AND THE FUNCTORIALITY OF LINK FLOER HOMOLOGY

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Abstract. It has been a central open problem in Heegaard Floer theory whether cobordisms of links induce homomorphisms on the associated link Floer homology groups. We provide an affirmative answer by introducing a natural notion of cobordism between sutured manifolds, and showing that such a cobordism induces a map on sutured Floer homology. This map is a common generalization of the hat version of the closed 3-manifold cobordism map in Heegaard Floer theory, and the contact gluing map defined by Honda, Kazez, and Matić. We show that sutured Floer homology, together with the above cobordism maps, forms a type of TQFT in the sense of Atiyah. Applied to the sutured manifold cobordism complementary to a decorated link cobordism, our theory gives rise to the desired map on link Floer homology. Hence, link Floer homology is a categorification of the multi-variable Alexander polynomial. We outline an alternative definition of the contact gluing map using only the contact element and handle maps. Finally, we show that a Weinstein sutured manifold cobordism preserves the contact element.

1. Introduction

Knot Floer homology, introduced independently by Ozsváth and Szabó [35], and Rasmussen [41], and later generalized to links by Ozsváth and Szabó [39], has proven to be a very sensitive invariant of knots and links. The graded Euler characteristic of link Floer homology is the multi-variable Alexander polynomial, and it completely determines the Thurston norm of the link complement [40]. Furthermore, it determines whether a knot is fibered according to the work of Ghiggini [10], Ni [33, 32], and the author [19, 20]. The simplest proofs of these results uses sutured Floer homology, an invariant of sutured manifolds defined by the author.

Since the introduction of knot Floer homology, it has been a natural question whether knot cobordisms induce maps on knot Floer homology, exhibiting it as a categorification of the Alexander polynomial. Compare this with the results of Jacobsson [18] and Khovanov [26] that Khovanov homology is a categorification of the Jones polynomial, also see [5]. As link Floer homology turns out to be an invariant of based links [25], and in many cases moving a basepoint around the link induces a non-trivial automorphism of link Floer homology due to the work of Sarkar [43], it is necessary to endow the link cobordisms with some decorations. One of the main results of this paper is that decorated link cobordisms induce functorial maps on link Floer homology. We are going to study the properties of these maps in a forthcoming paper.

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Note that, via grid diagrams and grid movies, Sarkar [42] has presented a candidate for link cobordism maps induced on HFK^- by surfaces in $S^3 \times I$. However, to date, we only know that this map is invariant under eight of the fifteen marked movie moves by the work of Graham [11]. So, while it is relatively simple to define link cobordism maps via a Morse-theoretic approach, showing independence of the Morse function seems to be impractical. Instead, we approach the problem via sutured manifold theory, as this is much more general and also provides maps induced by cobordisms of 3-manifolds with boundary, not just link complements.

Hence, we show that cobordisms of sutured manifolds induce maps on sutured Floer homology, a Heegaard Floer type invariant of 3-manifolds with boundary introduced by the author [19]. Cobordism maps in Heegaard Floer homology were first outlined by Ozsváth and Szabó [38] for cobordisms between closed 3-manifolds, but their work did not address two fundamental questions. The first was the issue of assigning a well-defined Heegaard Floer group – not just an isomorphism class – to a 3-manifold, and the functoriality of the construction under diffeomorphisms. We addressed this with Dylan Thurston [25]. The second issue was exhibiting the independence of their cobordism maps of the surgery description of the cobordism. They did check invariance under Kirby moves, but did not address how this gives rise to a well-defined map without running into naturality issues. I gave a general framework for constructing cobordism maps – and TQFTs in particular – via surgery in [22], and the present work is the first application of that framework in the Heegaard Floer setting. The first version of this paper was posted online in 2009. Shortly thereafter, I discovered the above-mentioned naturality issues that we fixed in [25] and [22]. Hence, the completion of this work has been considerably delayed, and should be viewed as the culmination of all that foundational work of the past six years.

Sutured manifolds, introduced by Gabai [8], have been of great use in 3-manifold topology, and especially in knot theory. A sutured manifold (M, γ) is a compact oriented 3-manifold M with boundary, together with a decomposition of the boundary ∂M into a positive part $R_+(\gamma)$ and a negative part $R_-(\gamma)$ that meet along a "thickened" oriented 1-manifold $\gamma \subset \partial M$ called the suture. Honda, Kazez, and Matić [15] drew a parallel between convex surface theory and sutured manifold theory, making apparent the usefulness of sutured manifolds in contact topology. The author [19, 23] defined an invariant called sutured Floer homology, in short SFH, for balanced sutured manifolds. SFH can be viewed as a common generalization of the hat version of Heegaard Floer homology and link Floer homology, both defined by Ozsváth and Szabó [36, 39]. The author [20] showed that SFH behaves nicely under sutured manifold decompositions, which has several important consequences, such as the above-mentioned detection of the genus and fibredness by knot Floer homology.

In the present paper, we define a notion of cobordism between sutured manifolds (M_0, γ_0) and (M_1, γ_1). It consists of a triple $\mathcal{W} = (W, Z, [\xi])$, where W is a 4-manifold with boundary and corners. The horizontal part of ∂W is $-M_0 \cup M_1$. The vertical part $Z \subset \partial W$ is a cobordism from $-\partial M_0$ to $-\partial M_1$, and carries a positive cooriented contact structure ξ such that ∂M_i is a convex surface with dividing set γ_i for $i \in \{0, 1\}$. We say that two such contact structures on Z are equivalent if they are homotopic through such contact structures, and denote the equivalence class of ξ by $[\xi]$. Throughout this paper, all contact structures are considered to be cooriented. Balanced sutured manifolds, together with certain equivalence classes of cobordisms between them, form a category. We extend *SFH* to a functor from this category to finite dimensional $\mathbb{Z}_2$-vector spaces, giving a type of TQFT.

There might seem to be an alternative definition for cobordisms between sutured manifolds. One could consider triples (W, Z, F), where W is a 4-manifold with corners, and has horizontal boundary $-M_0 \cup M_1$. Furthermore, Z is a cobordism between $-\partial M_0$ and $-\partial M_1$, and $F \subset Z$ is a cobordism between γ_0 and γ_1. A Morse-theoretic approach to define cobordism maps for such objects would require that every non-singular level set is a balanced sutured manifold. For this, the pair (Z, F) has to be built up from pairs of 3-dimensional and 2-dimensional handles that are cut into two equal halves by the 2-dimensional handle, or equivalently, contact handles. A contact handle decomposition of Z gives rise to a contact structure ξ up to equivalence, and we arrive at the previous definition.

The construction of the map $F_{\mathcal{W}}$, assigned to a cobordism $\mathcal{W}$, goes as follows. The sutured manifold $(-M_0, -\gamma_0)$ is a sutured submanifold of $(-N, -\gamma_1)$ in the sense of Honda, Kazez, and Matić [14], where $N = M_0 \cup -Z$. The contact structure $-\xi$ on Z – which is ξ with the opposite coorientation – has dividing set $-\gamma_0$ on ∂M_0 and $-\gamma_1$ on ∂M_1. It induces a gluing map

$$\Phi_{-\xi} \colon SFH(M_0, \gamma_0) \to SFH(N, \gamma_1),$$

as described in [14]. Then one can view W as a cobordism $\mathcal{W}_1 = (W, Z_1, [\xi_1])$ from (N, γ_1) to (M_1, γ_1) such that $Z_1 = \partial M_1 \times I$, and ξ_1 is an I-invariant contact structure such that $\partial M_1 \times \{t\}$ is a convex surface with dividing set $\gamma_1 \times \{t\}$ for every $t \in I$. We call such a cobordism special, and it can be described using 1-, 2-, and 3-handle attachments along the interior of N. Generalizing the hat version of the cobordism maps on Heegaard-Floer homology, one gets a map $F_{\mathcal{W}_1}$ induced by a special cobordisms $\mathcal{W}_1$. Finally, we set $F_{\mathcal{W}} = F_{\mathcal{W}_1} \circ \Phi_{-\xi}$. This map is functorial; i.e., $F_{\mathcal{W} \circ \mathcal{W}'} = F_{\mathcal{W}} \circ F_{\mathcal{W}'}$. Note that, in Remark 11.15, we outline a definition of the contact gluing maps, and more generally, the sutured manifold cobordism maps, purely in terms of special cobordism maps and the *EH* class in sutured Floer homology [16].

We showed in [25] that $\widehat{HF}$ is an invariant of based 3-manifolds. Given a based 3-manifold (Y, p), moving p around a loop induces an automorphism of $\widehat{HF}$, giving rise to an action of $\pi_1(Y, p)$ on $\widehat{HF}(Y, p)$. There are examples when this action is non-trivial, but we conjectured that it always factors through $H_1(Y)$. For a more precise formulation of this conjecture, see page 4 of [23]. This has recently been settled by Zemke [47], building on methods of this paper. Consequently, given a connected cobordism X between the closed connected 3-manifolds Y_0 and Y_1, the construction of cobordism maps $\widehat{F}_X$ have to take into consideration the choice of basepoints. Given basepoints $p_0 \in Y_0$ and $p_1 \in Y_1$, we have to fix an embedded arc $a \colon I \to X$ from p_0 to p_1. Then we define the cobordism map

$$\widehat{F}_{X,a} \colon \widehat{HF}(Y_0, p_0) \to \widehat{HF}(Y_1, p_1)$$

to be $F_{\mathcal{W}}$ for $\mathcal{W} = (W, Z, \xi)$, where $W = X \setminus N(a)$, the vertical boundary $Z = \overline{\partial N(a) \setminus \partial X}$, and ξ is obtained from the unique tight contact structure on $\partial N \approx S^3$ by removing two standard contact balls. Note that $\mathcal{W}$ is a cobordism between the sutured manifolds $Y_0(p_0)$ and $Y_1(p_1)$ for some framings of p_0 and p_1. Hence $F_{\mathcal{W}}$ maps $SFH(Y_0(p_0)) = \widehat{HF}(Y_0, p_0)$ to $SFH(Y_1(p_1)) = \widehat{HF}(Y_1, p_1)$.

Suppose that in the based 3-manifold (Y, p), moving p around the loop $\eta(t)$ induces a non-trivial automorphism η_* of $\widehat{HF}(Y, p)$. Consider the product cobordism $X = Y \times I$ from Y to itself, together with the arc $a(t) = (\eta(t), t)$ for $t \in I$. Then the cobordism map

$$\widehat{F}_{X,a} \colon \widehat{HF}(Y, p) \to \widehat{HF}(Y, p)$$

agrees with η_* and hence is not the identity. On the other hand, for the arc $a(t) = (p, t)$ for $t \in I$, the map $\widehat{F}_{X,a}$ is the identity.

As a special case of cobordism maps on SFH, we also get maps on link Floer homology, induced by decorated link cobordisms. More precisely, we consider decorated links (Y, L, P), where P consists of a positive even number of points on each component of the link L, together with a decomposition of L into compact 1-dimensional submanifolds $R_+(P)$ and $R_-(P)$ such that

$$R_+(P) \cap R_-(P) = P.$$

Sarkar [43] showed that knot Floer homology is only an invariant of based knots, as moving the basepoint around the knot induces a non-trivial automorphism of knot Floer homology for most knots.

A cobordism from the decorated link (Y_0, L_0, P_0) to (Y_1, L_1, P_1) consists of a triple (X, F, σ), where X is an oriented cobordism from Y_0 to Y_1, the surface $F \subset X$ is orientable with boundary $\partial F = L_0 \cup L_1$, and $\sigma \subset F$ is a properly embedded 1-manifold such that the map

$$\pi_0(\partial\sigma) \to \pi_0((L_0 \setminus P_0) \cup (L_1 \setminus P_1))$$

is a bijection. Furthermore, σ divides F into two compact subsurfaces that meet along σ, and we can orient each component R of $F \setminus \sigma$ such that whenever $\partial\overline{R}$ crosses a point of P_0, it goes from $R_+(P_0)$ to $R_-(P_0)$, and whenever it crosses a point of P_1, it goes from $R_-(P_1)$ to $R_+(P_1)$. Finally, for every closed component F_0 of F, we have $\sigma \cap F_0 \neq \emptyset$.

According to Lutz [30], the decoration σ uniquely defines an S^1-invariant contact structure ξ on the total space $Z = \overline{\partial N(F) \setminus (Y_0 \cup Y_1)}$ of the normal S^1-bundle of F in X up to equivalence, making $(X \setminus N(F), Z, [\xi])$ into a cobordism $\mathcal{W}$ between the sutured manifolds $(Y_i \setminus N(L_i), P_i \times S^1)$ for $i \in \{0, 1\}$ complementary to the decorated links. By [21, Proposition 9.2],

$$SFH(Y_i \setminus N(L_i), P_i \times S^1) \cong \widehat{HFL}(Y, L) \otimes V^{d_i},$$

where $V \cong \mathbb{Z}_2^2$, and d_i depends on the distribution of the marked points on L_i. Hence, the cobordism map $F_{\mathcal{W}}$ maps between certain link Floer homology groups. For computations of some elementary link cobordism maps, and a relationship of these with the reduced Khovanov TQFT, see our paper with Marengon [24]. Also see the work of Kronheimer and Mrowka [29], where they define maps induced by knot cobordisms on their singular instanton knot homology, which is then used to prove that Khovanov homology detects the unknot.

Finally, we extend the notion of Weinstein cobordisms to cobordisms between contact manifolds with convex boundary. If $\mathcal{W}$ is a Weinstein cobordism from the contact manifold (M_0, γ_0, ζ_0) to (M_1, γ_1, ζ_1), then we can view $\mathcal{W}$ as a cobordism $\overline{\mathcal{W}}$

from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$. We prove that

$$F_{\overline{\mathcal{W}}}(EH(M_1, \gamma_1, \zeta_1)) = EH(M_0, \gamma_0, \zeta_0),$$

where EH is the contact element in sutured Floer homology introduced by Honda, Kazez, and Matić [16].

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2. The cobordism category of sutured manifolds

Sutured manifolds were introduced by Gabai [8]. The following definition is slightly less general, in that it excludes toroidal sutures.

Definition 2.1. A *sutured manifold* (M, γ) is a compact oriented 3-manifold M with boundary, together with a set $\gamma \subset \partial M$ of pairwise disjoint annuli. Furthermore, the interior of each component of γ contains a *suture*; i.e., a homologically nontrivial oriented simple closed curve. We denote the union of the sutures by $s(\gamma)$. Finally, every component of

$$R(\gamma) = \partial M \setminus \mathrm{Int}(\gamma)$$

is oriented. Define $R_+(\gamma)$ (or $R_-(\gamma)$) to be those components of $\partial M \setminus \mathrm{Int}(\gamma)$ whose normal vectors point out of (into) M. The orientation on $R(\gamma)$ must be coherent with respect to $s(\gamma)$; i.e., if δ is a component of $\partial R(\gamma)$ and is given the boundary orientation, then δ must represent the same homology class in $H_1(\gamma)$ as some suture.

Remark 2.2. In this paper, we are not going to make a distinction between γ and $s(\gamma)$, as it is usually clear from the context which one we mean. One can think of γ as a thickened oriented 1-manifold.

We now review some fundamental notions and results about contact structures and set our orientation conventions; for more details see the notes of Etnyre [6] and the book of Geiges [9]. Let M be an oriented 3-manifold. A *contact structure* ξ on M is a nowhere integrable 2-plane field. This is equivalent to the condition that each point of M has a neighborhood U together with a 1-form α such that $\xi|_U = \ker(\alpha)$ and $\alpha \wedge d\alpha$ is nowhere zero. If ξ is *coorientable*, then one can choose α globally. We say that ξ is *positive* if the 3-form $\alpha \wedge d\alpha$ is coherent with the orientation of M. This is independent of the choice of contact form α. In this paper, all contact structures are positive and cooriented, unless otherwise stated. Given such a contact structure ξ, we write $-\xi$ for the same 2-plane field with the opposite coorientation; this is also a positive contact structure.

A vector field v on M is a *contact vector field* if its flow preserves ξ. In terms of the contact form α, this means that $\mathcal{L}_v\alpha = f\alpha$ for some function $f\colon M \to \mathbb{R}$. If F is a properly embedded surface in M, then F is *convex* if there exists a contact vector field v transverse to F. Every surface F is C^∞-close to a convex surface, and every convex surface has a product neighborhood in which the contact structure is invariant in the normal direction.

The contact structure ξ defines a singular foliation ξF on F called the *characteristic foliation* of ξ on F. Given an orientation of F, we can orient the leaves of ξF as follows. Pick a point $p \in F$ where ξF is non-singular, and let l_p be the line tangent to the leaf of ξF through p. Then l_p is oriented as $\xi_p \cap T_pF$. More concretely, a vector $v \in l_p$ defines the positive orientation of l_p if when we choose vectors $v_\xi \in \xi_p$ and $v_F \in T_pF$ such that (v, v_ξ) orients ξ and (v, v_F) orients T_pF, then the triple (v, v_ξ, v_F) orients T_pM. Given a singular point $p \in F$ of the foliation ξF, we can associate to it a sign depending on whether ξ_p is positively or negatively tangent to F.

Given a convex surface F and a contact vector field v transverse to it, we can define the *dividing set*

$$\Gamma = \{\, p \in F : v(p) \in \xi_p \,\}.$$

This is always a 1-manifold transverse to the characteristic foliation ξF, and given another contact vector field, the resulting dividing set will be isotopic. We orient Γ so that it is positively transverse to ξ. If F is oriented, then Γ splits F into two subsurfaces F_+ and F_- that meet along Γ. The leaves of ξF go from F_+ to F_-, and F_+ contains the positive singularities of ξF, while F_- contains the negative ones. Observe that no matter how we orient F, the dividing set Γ is always oriented as the boundary of F_+. If v is positively transverse to F, then F_+ is the set of points p of F where v_p is positively transverse or tangent to ξ_p, while F_- is where v_p is negatively transverse or tangent to ξ. A surprising result of Giroux states that the dividing set determines ξ uniquely up to isotopy in a neighborhood of F.

Definition 2.3. Let (M, γ) be a sutured manifold, and suppose that ξ_0 and ξ_1 are contact structures on M such that ∂M is a convex surface with dividing set γ with respect to both ξ_0 and ξ_1. Then we say that ξ_0 and ξ_1 are *equivalent* if there is a one-parameter family $\{\, \xi_t : t \in I \,\}$ of contact structures such that ∂M is convex with dividing set γ with respect to ξ_t for every $t \in I$. In this case, we write $\xi_0 \sim \xi_1$, and we denote by $[\xi]$ the equivalence class of a contact structure ξ.

Definition 2.4. Let (M_0, γ_0) and (M_1, γ_1) be sutured manifolds. A *cobordism from* (M_0, γ_0) *to* (M_1, γ_1) is a triple $\mathcal{W} = (W, Z, [\xi])$, where

1. W is a compact, oriented 4-manifold with boundary,
2. Z is a compact, codimension-0 submanifold with boundary of ∂W, and $\partial W \setminus \text{Int}(Z) = -M_0 \sqcup M_1$,
3. ξ is a positive contact structure on Z, such that ∂Z is a convex surface with dividing set γ_i on ∂M_i for $i \in \{0, 1\}$.

Remark 2.5. For orienting the boundary of a manifold, we always use the "outward normal first" convention. We think of W as a 4-manifold with corners along ∂Z. Furthermore, we say that $-M_0 \sqcup M_1$ is the horizontal and Z is the vertical part of the boundary of W. Note that the orientation of the dividing set of ξ on ∂Z only depends on the coorientation of ξ, and is independent of how we orient the convex surface ∂Z.

Lemma 2.6. *If the sutured manifolds* (M_0, γ_0) *and* (M_1, γ_1) *are cobordant, then*

$$\chi(R_+(\gamma_0)) - \chi(R_-(\gamma_0)) = \chi(R_+(\gamma_1)) - \chi(R_-(\gamma_1)).$$

Furthermore, the map $\pi_0(\gamma_i) \to \pi_0(\partial M_i)$ *is surjective for* $i \in \{0, 1\}$.

Proof. Recall that (Z, ξ) is a contact manifold with convex boundary $\partial M_0 \cup -\partial M_1$ and dividing set $\gamma = \gamma_0 \cup \gamma_1$. We denote by $R_+(\gamma)$ and $R_-(\gamma)$, respectively, the positive and negative subsurfaces of Z induced by ξ. Then

$$\chi(R_+(\gamma)) - \chi(R_-(\gamma)) = \langle\, e(\xi), [\partial Z]\,\rangle = 0.$$

Moreover,

$$\chi(R_+(\gamma)) = \chi(R_+(\gamma_0)) + \chi(R_-(\gamma_1)) \text{ and } \chi(R_-(\gamma)) = \chi(R_-(\gamma_0)) + \chi(R_+(\gamma_1)),$$

as on $-\partial M_1$ the positive subsurface induced by ξ is $R_-(\gamma_1)$ and the negative subsurface is $R_+(\gamma_1)$, while on ∂M_0 the positive subsurface is $R_+(\gamma_0)$ and the negative subsurface is $R_-(\gamma_0)$. The second claim follows from the fact that the dividing set on a closed convex surface is never empty, see [9, p.230]. □

Definition 2.7. The cobordisms $\mathcal{W} = (W, Z, [\xi])$ and $\mathcal{W}' = (W', Z', [\xi'])$ between the same sutured manifolds (M_0, γ_0) and (M_1, γ_1) are called *equivalent* if there is an orientation preserving diffeomorphism $d \colon W \to W'$ such that $d(Z) = Z'$ and $d_*(\xi) \sim \xi'$; furthermore, $d(x) = x$ for every $x \in M_0 \cup M_1$. Such a map d is called an *equivalence*.

If $\mathcal{W} = (W, Z, [\xi])$ is a cobordism from the sutured manifold (M_0, γ_0) to (M_1, γ_1) and $\mathcal{W}' = (W', Z', [\xi'])$ is a cobordism from (M_0', γ_0') to (M_1', γ_1'), then $\mathcal{W}$ and $\mathcal{W}'$ are called *diffeomorphic* if there is an orientation preserving diffeomorphism $d \colon W \to W'$ such that $d(Z) = Z'$, $d(\gamma_0) = \gamma_0'$, $d(\gamma_1) = \gamma_1'$, and $d_*(\xi) \sim \xi'$. Such a map d is called a *diffeomorphism* from $\mathcal{W}$ to $\mathcal{W}'$.

Definition 2.8. Let (M, γ) be a sutured manifold such that there is at least one suture on each component of ∂M. The *trivial cobordism from* (M, γ) *to* (M, γ) is the triple $\mathcal{W} = (W, Z, [\xi])$, where

1. $W = M \times I$,
2. $Z = \partial M \times I$,
3. ξ is an I-invariant contact structure on Z such that $\partial M \times \{t\}$ is a convex surface with dividing set $\gamma \times \{t\}$ for every $t \in I$. Note that such a ξ is well-defined up to equivalence.

Remark 2.9. To be completely precise, just as in Milnor [31], one should define a cobordism from (N_0, ν_0) to (N_1, ν_1) as a 5-tuple

$$\mathcal{W} = ((W, Z, [\xi]), (M_0, \gamma_0), (M_1, \gamma_1), h_0, h_1),$$

where $(W, Z, [\xi])$ is a cobordism from (M_0, γ_0) to (M_1, γ_1) in the sense of Definition 2.4, and for $i \in \{0, 1\}$, the map $h_i \colon M_i \to N_i$ is an orientation preserving diffeomorphism such that $h_i(\gamma_i) = \nu_i$. If we have two such cobordisms $\mathcal{W}$ and $\mathcal{W}'$ from (N_0, ν_0) to (N_1, ν_1), then an equivalence between them is a diffeomorphism g from $\mathcal{W}$ to $\mathcal{W}'$ such that $g|_{M_i} = (h_i')^{-1} \circ h_i$ for $i \in \{0, 1\}$.

If (N_0, ν_0) and (N_1, ν_1) are disjoint, then we can safely restrict ourselves to cobordisms between them where $M_i = N_i$ and $h_i = \mathrm{Id}_{N_i}$ for $i \in \{0, 1\}$, in which case the above notion of equivalence coincides with the one in Definition 2.7. However, to define the identity morphism from (N, ν) to itself, one does need the above more precise approach to cobordisms. To keep the notation simple, we will use our previous less rigorous terminology, which should not cause much confusion.

Definition 2.10. Suppose that $\mathcal{W}_0 = (W_0, Z_0, [\xi_0])$ is a cobordism from (M_0, γ_0) to (M_1, γ_1) and $\mathcal{W}_1 = (W_1, Z_1, [\xi_1])$ is a cobordism from (M_1, γ_1) to (M_2, γ_2).

Since ∂M_1 is a convex surface with dividing set γ_1 in both (Z_0, ξ_0) and (Z_1, ξ_1), we can glue the contact structures ξ_0 and ξ_1 together along ∂M_1 to obtain a cooriented contact structure $\xi_1 \cup \xi_2$ on $Z_0 \cup_{\partial M_1} Z_1$, well-defined up to equivalence. Then the *composition* $\mathcal{W}_1 \circ \mathcal{W}_0$ is the cobordism from (M_0, γ_0) to (M_2, γ_2) given by the triple

$$(W_0 \cup_{M_1} W_1, Z_0 \cup_{\partial M_1} Z_1, [\xi_0 \cup \xi_1]).$$

Definition 2.11. The *cobordism category of sutured manifolds*, **Sut**, is given as follows. Its objects are sutured manifolds (M, γ) that have at least one suture on each boundary component. The set of morphisms from (M_0, γ_0) to (M_1, γ_1) is the set of equivalence classes of cobordisms from (M_0, γ_0) to (M_1, γ_1). Composition is given by Definition 2.10. The identity morphism from (M, γ) to itself is the equivalence class of the trivial cobordism introduced in Definition 2.8.

By Lemma 2.6, for a given integer $k \in \mathbb{Z}$, those sutured manifolds that satisfy

$$\chi(R_+(\gamma)) - \chi(R_-(\gamma)) = k$$

form a full subcategory of **Sut** called $\mathbf{Sut}_k$, and the sum category of $\{\, \mathbf{Sut}_k : k \in \mathbb{Z} \,\}$ is exactly **Sut**.

Note that, in order to have a unique identity morphism for each sutured manifold and to be able to define the composition of cobordisms, it was necessary to work with equivalence classes of contact structures. It is not possible to set up a cobordism category using contact structures without factoring out by this equivalence relation in addition to taking equivalence classes of cobordisms. Indeed, an equivalence from (W, Z, ξ) to (W', Z', ξ') would have to map the characteristic foliation of ξ on ∂Z to that of ξ' on $\partial Z' = \partial Z$. Hence, the equivalence classes of trivial cobordisms for a given sutured manifold (M, γ) would decompose along the set of possible characteristic foliations on ∂Z. Furthermore, to be able to compose the cobordism (W_0, Z_0, ξ_0) from (M_0, γ_0) to (M_1, γ_1) with the cobordism (W_1, Z_1, ξ_1) from (M_1, γ_1) to (M_2, γ_2), the characteristic foliation of ξ_0 on ∂M_1 has to agree with the characteristic foliation of ξ_1 on ∂M_1. If we are working with equivalence classes of contact structures, we can always homotope ξ_0 and ξ_1 until the two characteristic foliations line up and we can perform the gluing.

The following notion was introduced by the author [19].

Definition 2.12. A sutured manifold (M, γ) is *balanced* if

(1) $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$,
(2) the map $\pi_0(\gamma) \to \pi_0(\partial M)$ is surjective, and
(3) M has no closed components.

Remark 2.13. The objects of $\mathbf{Sut}_0$ are precisely those sutured manifolds that can be written as finite disjoint unions of balanced sutured manifolds and closed oriented 3-manifolds.

It is also worth noting that if $\mathcal{W} = (W, Z, [\xi])$ is a cobordism from (M_0, γ_0) to (M_1, γ_1), then $(-W, -Z, [-\xi])$ is not a cobordism from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$ since $-\xi$ is a negative contact structure on $-Z$. But we can view $(W, Z, [-\xi])$ as a cobordism $\overline{\mathcal{W}}$ from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$ by writing

$$\partial W = -(-M_1) \cup Z \cup -M_0.$$

Loosely speaking, this is turning the cobordism $\mathcal{W}$ upside down.

Definition 2.14. We say that a cobordism from (M_0, γ_0) to (M_1, γ_1) is *balanced* if both (M_0, γ_0) and (M_1, γ_1) are balanced sutured manifolds. The balanced sutured manifolds and equivalence classes of balanced cobordisms form a full subcategory of $\mathbf{Sut}_0$ that we denote by $\mathbf{BSut}$.

Sutured Floer homology was introduced by the author [19]. Over $\mathbb{Z}_2$, it assigns a finite-dimensional $\mathbb{Z}_2$ vector space $SFH(M, \gamma)$ to every balanced sutured manifold (M, γ). The main goal of the present paper is to promote SFH to a functor from $\mathbf{BSut}$ to $\mathbf{Vect}_{\mathbb{Z}_2}$. That is, for every balanced cobordism $\mathcal{W}$ from (M_0, γ_0) to (M_1, γ_1), we are going to define a linear map

$$F_{\mathcal{W}} \colon SFH(M_0, \gamma_0) \to SFH(M_1, \gamma_1)$$

such that $F_{\mathcal{W}_1 \circ \mathcal{W}_0} = F_{\mathcal{W}_1} \circ F_{\mathcal{W}_0}$, and if $\mathcal{W}$ is a trivial cobordism, then $F_{\mathcal{W}} = \text{Id}$. In Theorem 11.11, we will show that this is an instance of a $(3 + 1)$-dimensional TQFT, as defined by Atiyah [2, 3].

3. Relative Spin^c structures

First, we briefly review the definition of relative Spin^c structures on sutured manifolds as defined by the author [19]. The definition given here requires a slightly less restrictive but equivalent boundary condition in order to be able to talk about Spin^c structures represented by contact structures with convex boundary.

Definition 3.1. Given a sutured manifold (M, γ), we say that a vector field v defined on a subset of M containing ∂M is *admissible* if it is nowhere vanishing, it points into M along $R_-(\gamma)$, it points out of M along $R_+(\gamma)$, and $v|_\gamma$ is tangent to ∂M and either points into $R_+(\gamma)$ or is positively tangent to γ (as before, we think of ∂M as a smooth surface, and of γ as a 1-manifold).

Let v and w be admissible vector fields on M. We say that v and w are homologous, and we write $v \sim w$, if there is a collection of balls $B \subset M$, one in each component of M, such that v and w are homotopic on $M \setminus B$ through admissible vector fields. Then $\text{Spin}^c(M, \gamma)$ is the set of homology classes of admissible vector fields on M.

According to [21, Proposition 3.5], $\text{Spin}^c(M, \gamma) \neq \emptyset$ if and only if for every component M_0 of M, we have

$$\chi(M_0 \cap R_+(\gamma)) = \chi(M_0 \cap R_-(\gamma)).$$

The space of vector fields arising as $v|_{\partial M}$ for v admissible is convex, hence contractible. Suppose that (M, γ) is a sutured submanifold of (N, ν); i.e., $M \subset \text{Int}(N)$. If v is an admissible vector field on (M, γ) and w is an admissible vector field on $(N \setminus \text{Int}(M), \gamma \cup \nu)$, then there is a homotopically unique deformation of v through admissible vector fields such that $v|_{\partial M} = w|_{\partial M}$. This gives a unique way of gluing the Spin^c structures represented by v and w to obtain a Spin^c structure on (N, ν).

Definition 3.2. Let (M, γ) be a sutured manifold. We say that an oriented 2-plane field ξ defined on a subset of M containing ∂M is *admissible* if there exists a Riemannian metric g on M such that $\xi^{\perp_g}$ is an admissible vector field. If v is defined on the whole manifold M, we write

$$\mathfrak{s}_\xi = [\xi^{\perp_g}] \in \text{Spin}^c(M, \gamma).$$

This is independent of the choice of g since the space of metrics g for which $\xi^{\perp_g}$ is an admissible vector field is convex.

Lemma 3.3. *If ξ is a contact structure on M such that ∂M is a convex surface with dividing set γ, then ξ is admissible.*

Proof. Let w be a contact vector field on M transverse to S such that

$$\{\, p \in S : \; w(p) \in \xi_p \,\} = \gamma.$$

Then we choose a Riemannian metric g on M such that $w(p) \perp T_pS$ for every $p \in S$, and such that $\xi_p \perp T_p\gamma$ for every $p \in \gamma$ (the latter is possible since ξ is transverse to γ). Then the vector field $\xi^{\perp_g}$ is admissible. So we can talk about the induced relative Spin^c-structure $\mathfrak{s}_\xi$. □

Next, we recall a standard result from complex geometry.

Lemma 3.4. *Let V be a 4-dimensional real vector space, together with an endomorphism J such that $J^2 = -I$. Then every 3-dimensional subspace $U < V$ contains a unique J-invariant plane.*

Proof. Think of (V, J) as a complex vector space. Since two different complex lines span V over $\mathbb{R}$, they cannot both lie in U. Thus $U \cap J(U)$ is the unique J-invariant 2-plane in U. □

So if J is an almost complex structure on a 4-manifold W and H is a 3-dimensional submanifold, then there is a 2-plane field induced on H called the field of *complex tangencies* along H. The following definition generalizes the one given by Ozsváth and Szabó [36, Section 8.1.3], also see [28, Lemma 2.1].

Definition 3.5. Suppose that $\mathcal{W} = (W, Z, [\xi])$ is a cobordism from the sutured manifold (M_0, γ_0) to (M_1, γ_1). We say that an almost complex structure J defined on a subset of W containing ∂Z is *admissible* if the field of complex tangencies in $TM_i|_{\partial M_i}$ is admissible in (M_i, γ_i) for $i \in \{0, 1\}$, and the field of complex tangencies in $TZ|_{\partial Z}$ is admissible in $(Z, \gamma_0 \cup \gamma_1)$.

A *relative Spin^c structure* on $\mathcal{W}$ is a homology class of pairs (J, P), where

- $P \subset \mathrm{Int}(W)$ is a finite collection of points,
- J is an admissible almost complex structure defined over $W \setminus P$, and
- if ξ_J is the field of complex tangencies along Z, then $\mathfrak{s}_\xi = \mathfrak{s}_{\xi_J}$.

We say that (J, P) and (J', P') are *homologous* if there exists a compact 1-manifold $C \subset W \setminus \partial Z$ such that P, $P' \subset C$; furthermore, $J|_{W \setminus C}$ and $J'|_{W \setminus C}$ are isotopic through admissible almost complex structures. Denote by $\mathrm{Spin}^c(\mathcal{W})$ the set of relative Spin^c structures over $\mathcal{W}$.

Given any Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(\mathcal{W})$ and $i \in \{1, 2\}$, we can define

$$\mathfrak{s}_i = \mathfrak{s}|_{(M_i, \gamma_i)} \in \mathrm{Spin}^c(M_i, \gamma_i)$$

as the Spin^c structure of the field of complex tangencies of J along M_i for an arbitrary representative (J, P) of $\mathfrak{s}$. By definition, $\mathfrak{s}|_{(Z, \gamma_0 \cup \gamma_1)} = \mathfrak{s}_\xi$.

Let i be the embedding of the pair $(Z, \partial Z)$ into $(W, \partial Z)$, and consider the induced restriction map

$$i^* \colon H^2(W, \partial Z) \to H^2(Z, \partial Z).$$

Then $\mathrm{Spin}^c(\mathcal{W})$ is an affine space over $\ker(i^*)$. Indeed, homology classes of admissible almost complex structures on W form an affine space over $H^2(W, \partial Z)$ as the space of admissible almost complex structures on $TW|_{\partial Z}$ is contractible. Two such almost complex structures restrict to the same element of $\text{Spin}^c(Z, \gamma_0 \cup \gamma_1)$ if and only if their difference lies in $\ker(i^*)$. We now define a related space of relative Spin^c structures.

Definition 3.6. Suppose that we are given an admissible almost complex structure J' on $TW|_Z$ such that $\mathfrak{s}_\xi = \mathfrak{s}_{\xi_{J'}}$, where $\xi_{J'}$ is the filed of complex tangencies of J' along Z. Then $\text{Spin}^c(\mathcal{W}, J')$ is the set of homology classes of pairs (J, P) such that J is an almost complex structure on $W \setminus P$ and $J|_Z = J'$.

By obstruction theory, $\text{Spin}^c(\mathcal{W}, J')$ is an affine space over $H^2(W, Z)$. Note that we mainly focus on $\text{Spin}^c(\mathcal{W})$ instead of $\text{Spin}^c(\mathcal{W}, J')$ in this paper because in the definition of $\mathcal{W}$ we only fix the equivalence class $[\xi]$ of a contact structure, so there is no (homotopically) unique almost complex structure along Z that we could use as a boundary condition. Had we fixed a concrete contact structure ξ along Z, equivalent balanced cobordisms would induce the same characteristic foliations on ∂M_i, making it impossible to compose cobordisms, or to define the identity morphism from (M, γ) to itself.

Lemma 3.7. *Suppose that for the balanced cobordism $\mathcal{W} = (W, Z, [\xi])$ we have $H^k(Z, \partial M_1) = 0$ for $k \in \{1, 2\}$. Then, given an almost complex structure J' on Z as above, the restriction map*

$$q\colon \mathit{Spin}^c(\mathcal{W}, J') \to \mathit{Spin}^c(\mathcal{W})$$

is a bijection.

Proof. Consider the sequence of embeddings

$$(W, \partial M_1) \overset{e}{\hookrightarrow} (W, \partial Z) \overset{f}{\hookrightarrow} (W, Z).$$

Then, on second cohomology, $(f \circ e)^* = e^* \circ f^*$. The restriction map q is an affine map modeled on

$$f^*\colon H^2(W, Z) \to \ker(i^*).$$

From the long exact sequence of the triple $(W, Z, \partial Z)$, we have $\text{im}(f^*) = \ker(i^*)$. Furthermore, by the long exact sequence of the triple $(W, Z, \partial M_1)$ and our assumptions on $H^k(Z, \partial M_1)$, we see that the map

$$(f \circ e)^*\colon H^2(W, Z) \to H^2(W, \partial M_1)$$

is an isomorphism. Hence f^* is injective. By the above, f^* is also surjective onto $\ker(i^*)$. This shows that f^* is a bijection, and so is q. □

Remark 3.8. As we shall see in Section 5, the space $\text{Spin}^c(\mathcal{W}, J)$ naturally appears when parameterizing homotopy classes of pseudo-holomorphic polygons. In Definition 5.1, we will introduce *special* cobordisms, these satisfy $Z = \partial M_0 \times I$. To define maps induced by special cobordisms, we will count pseudo-holomorphic triangles. Lemma 3.7 implies that, for special cobordisms, the spaces $\text{Spin}^c(\mathcal{W})$ and $\text{Spin}^c(\mathcal{W}, J')$ are isomorphic.

4. Link cobordisms

Definition 4.1. For $i \in \{0,1\}$, let Y_i be a connected, oriented 3-manifold, and let L_i be a non-empty link in Y_i. Then a *link cobordism* from (Y_0, L_0) to (Y_1, L_1) is a pair (X, F), where

1. X is a connected, oriented cobordism from Y_0 to Y_1,
2. F is a properly embedded, compact, orientable surface in X,
3. $\partial F = L_0 \cup L_1$.

We would like to associate to a link cobordism (X, F) a balanced cobordism $(W, Z, [\xi])$. However, to define the contact structure ξ, we need more information, namely a set of dividing curves on F. For this, let us recall the notion of a surface with divides from Honda et al. [15, Definition 4.1], with the difference that we drop the orientation of the surface.

Definition 4.2. A *surface with divides* (S, σ) is a compact orientable surface S, possibly with boundary, together with a properly embedded 1-manifold σ that divides S into two compact subsurfaces that meet along σ.

Link Floer homology of a link L is isomorphic to the *SFH* of the sutured manifold complementary to L. Together with Dylan Thurston [25], we constructed link Floer homology in a functorial way by first defining sutured Floer homology functorially, and then applying a real blowup construction to L to obtain a unique link complement, without having to make a choice of tubular neighborhood. We now review this blowup procedure.

Definition 4.3. Suppose that M is a smooth manifold, and let $L \subset M$ be a properly embedded submanifold. For every $p \in L$, let $N_pL = T_pM/T_pL$ be the fibre of the normal bundle of L over p, and let $UN_pL = (N_pL \setminus \{0\})/\mathbb{R}_+$ be the fibre of the unit normal bundle of L over p. Then the *(spherical) blowup* of M along L, denoted by $\mathrm{Bl}_L(M)$, is a manifold with boundary obtained from M by replacing each point $p \in L$ by UN_pL. There is a natural projection $\mathrm{Bl}_L(M) \to M$. For further details, see Arone and Kankaanrinta [1].

Definition 4.4. A *decorated link* is a triple (Y, L, P), where L is a non-empty link in the connected oriented 3-manifold Y, and $P \subset L$ is a finite set of points. We require that for every component L_0 of L, the number $|L_0 \cap P|$ is positive and even. Furthermore, we are given a decomposition of L into compact 1-manifolds $R_+(P)$ and $R_-(P)$ such that $R_+(P) \cap R_-(P) = P$.

We can canonically assign a balanced sutured manifold $\mathcal{W}(Y, L, P) = (M, \gamma)$ to every decorated link (Y, L, P), as follows. Let $M = \mathrm{Bl}_L(Y)$ and $\gamma = \bigcup_{p \in P} UN_pL$. Furthermore,

$$R_\pm(\gamma) := \bigcup_{x \in R_\pm(P)} UN_xL,$$

oriented as $\pm\partial M$, and we orient γ as $\partial R_+(\gamma)$.

Definition 4.5. We say that the triple $\mathcal{X} = (X, F, \sigma)$ is a *decorated link cobordism* from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) if

1. (X, F) is a link cobordism from (Y_0, L_0) to (Y_1, L_1),
2. (F, σ) is a surface with divides such that the map

$$\pi_0(\partial\sigma) \to \pi_0((L_0 \setminus P_0) \cup (L_1 \setminus P_1))$$

is a bijection,

(3) we can orient each component R of $F \setminus \sigma$ such that whenever $\partial \overline{R}$ crosses a point of P_0, it goes from $R_+(P_0)$ to $R_-(P_0)$, and whenever it crosses a point of P_1, it goes from $R_-(P_1)$ to $R_+(P_1)$,

(4) if F_0 is a closed component of F, then $\sigma \cap F_0 \neq \emptyset$.

Two decorated link cobordisms $\mathcal{X} = (X, F, \sigma)$ and $\mathcal{X}' = (X', F', \sigma')$ between the same decorated links (Y_0, L_0, P_0) and (Y_1, L_1, P_1) are said to be *equivalent* if there is an orientation preserving diffeomorphism $d \colon X \to X'$ such that $d(F) = F'$ and $d(\sigma) = \sigma'$; moreover, $d(y) = y$ for every $y \in Y_0 \cup Y_1$.

Suppose that $\mathcal{X} = (X, F, \sigma)$ is a cobordism from (Y_0, L_0, P_0) to (Y_1, L_1, P_1) and let $\mathcal{X}' = (X', F', \sigma')$ be a cobordism from (Y_0', L_0', P_0') to (Y_1', L_1', P_1'). We say that $\mathcal{X}$ and $\mathcal{X}'$ are *diffeomorphic* if there exists an orientation preserving diffeomorphism d from X to X' such that $d(F) = F'$ and $d(\sigma) = \sigma'$, and $d(R_\pm(P_i)) = R_\pm(P_i')$ for $i \in \{0, 1\}$.

Decorated links and equivalence classes of decorated link cobordisms form a category **DLink** with the obvious composition and identity morphisms. As each link component has at least two marked points, when composing two decorated link cobordisms, we do not create undecorated closed components of the surface.

Note that in the above definition, neither the links L_0 and L_1, nor the surface F are required to be oriented.

Proposition 4.6. *Condition* (3) *of Definition 4.5 implies that every non-closed component s of σ connects either $R_+(P_0)$ and $R_+(P_1)$, or $R_-(P_0)$ and $R_-(P_1)$, or $R_+(P_i)$ and $R_-(P_i)$ for $i \in \{0, 1\}$.*

Proof. Let R be the closure of a component of $F \setminus \sigma$ such that $s \subset \partial R$. Then ∂R is a collection of polygonal curves with edges alternatingly in σ and $\partial F = L_0 \cup L_1$ along each component. Each edge in ∂F contains exactly one point of $P_0 \cup P_1$. If we can orient R such that (3) is satisfied, then if s has both endpoints in L_0 then it starts in $R_-(P_0)$ and ends in $R_+(P_0)$, and if it has both endpoints in L_1, then it starts in $R_+(P_1)$ and ends in $R_-(P_1)$. As we go from $+$ to $-$ along L_0 and from $-$ to $+$ along L_1, if s goes from L_0 to L_1, it start in $R_-(P_0)$ and ends in $R_-(P_1)$, and if it goes from L_1 to L_0, then it starts in $R_+(P_1)$ and ends in $R_+(P_0)$. ☐

Remark 4.7. Note that the converse of the above statement is not true. For this end, take the product cobordism from the two-component unlink with two marked points on each component to itself, where the dividing set σ consists of four vertical lines, then connect the two cylinders with a tube. If chosen appropriately, it is not possible to orient the component of $F \setminus \sigma$ containing the tube correctly as for each orientation exactly one of the two boundary components will go from $R_+(P_0)$ to $R_-(P_0)$.

Let $\pi \colon M \to F$ be a principal circle bundle over the compact oriented surface F, where the orientation of M is determined by the orientation of the base and the fibre. If ξ is an S^1-invariant contact structure on M, then it defines a dividing set σ on F as follows. A point $x \in F$ lies in σ if and only if ξ is tangent to the fibre $M_x = \pi^{-1}(x)$. Let $R_+(\sigma)$ consist of those $x \in F$ for which M_x is positively transverse or tangent to ξ. Similarly, $R_-(\sigma)$ is the set of those $x \in F$ for which M_x is negatively transverse or tangent to ξ. Then $R_+(\sigma)$ and $R_-(\sigma)$ are compact subsurfaces of F that meet along σ. The S^1 action defines a contact vector field v

on M tangent to the fibres. The image of any (local) section of π is hence a convex surface with dividing set projecting onto σ.

The converse of the above is also true, in the following sense. Let $\pi\colon M \to F$ be as above, and let σ be a dividing set on F that intersects each component of F non-trivially and divides F into the subsurfaces $R_+(\sigma)$ and $R_-(\sigma)$. According to Lutz [30] and Honda [13, Theorem 2.11 and Section 4], up to isotopy, there is a unique S^1-invariant contact structure ξ_σ on M such that the dividing set associated to ξ_σ is exactly σ, the coorientation of ξ_σ induces the splitting $R_\pm(\sigma)$, and the boundary ∂M is a convex. Furthermore, if F has no S^2 or T^2 components, this correspondence is bijective between the isotopy classes of those dividing sets σ that have no homotopically trivial components, and the isotopy classes of universally tight contact structures on M.

The dividing set of ξ_σ on ∂M, which we denote by γ_σ, is S^1-invariant. In other words, each component of γ_σ projects to a single point in ∂M under π. By [6, Lemma 6.6], between any two adjacent points of $\partial\sigma$, there is exactly one point of $P = \pi(\gamma_\sigma)$ and vice versa; i.e., the map

$$\pi_0(\partial\sigma) \to \pi_0(\partial F \setminus P)$$

is a bijection. The coorientation of ξ_σ determines a splitting of ∂M into compact subsurfaces $R_+(\gamma_\sigma)$ and $R_-(\gamma_\sigma)$ that meet along γ_σ. Let $R_\pm(P) = \pi(R_\pm(\gamma_\sigma))$.

Lemma 4.8. *Whenever $\partial R_+(\sigma)$ crosses a point of P, it goes from $R_+(P)$ to $R_-(P)$.*

Proof. Let $p \in P \cap R_+(\sigma)$. Since $p \in P$, the fiber M_p is a component of γ_σ. As $p \in R_+(\sigma)$, the orientation of M_p is positively transverse to ξ_σ, and hence on M_p the fiber orientation coincides with the orientation of the dividing set γ_σ. On the other hand, γ_σ is oriented as the boundary of $R_+(\gamma_\sigma)$. Given an arbitrary point $x \in M_p$, a vector $v_+ \in T_x\partial M$ pointing out of $R_+(\gamma_\sigma)$, and a vector $v_p \in T_xM_p$ orienting M_p, the pair (v_+, v_p) orients ∂M as $\partial R_+(\gamma_\sigma)$ is oriented via the "outward normal first" rule. If $w \in T_xM$ is an outward normal of M, then the basis (w, v_+, v_p) orients M. But M is oriented via taking the orientation of the base F, followed by the orientation of the fibre M_p. Since v_p orients M_p, it follows that $(d\pi(w), d\pi(v_+))$ is a positive basis of T_pF. Since $d\pi(w)$ is an outward normal of F, we get that $d\pi(v_+)$ orients ∂F. This proves that if p lies in $R_+(\sigma)$, then ∂F is oriented from $R_+(P)$ to $R_-(P)$. □

Definition 4.9. Let (X, F, σ) be a decorated link cobordism from (Y_0, L_0, P_0) to (Y_1, L_1, P_1). Then we define the cobordism $\mathcal{W} = \mathcal{W}(X, F, \sigma)$ as follows. Choose an arbitrary splitting of F into $R_+(\sigma)$ and $R_-(\sigma)$, and orient F such that $\partial R_+(\sigma)$ crosses P_0 from $R_+(P_0)$ to $R_-(P_0)$ and P_1 from $R_-(P_1)$ to $R_+(P_1)$. Then $\mathcal{W}$ is defined to be the triple $(W, Z, [\xi])$, where $W = \mathrm{Bl}_F(X)$ and $Z = UNF$, oriented as a submanifold of ∂W, finally $\xi = \xi_\sigma$.

Note that $\mathcal{W}$ is a cobordism from $\mathcal{W}(Y_0, L_0, P_0) = (M_0, \gamma_0)$ to $\mathcal{W}(Y_1, L_1, P_1) = (M_1, \gamma_1)$. Indeed, let $\pi\colon Z \to F$ be the natural projection, then we can assume that $\pi(\gamma_\sigma) = P_0 \cup P_1$. By Lemma 4.8 and our assumptions above,

$$\pi(R_\pm(\gamma_\sigma)) \cap L_0 = R_\pm(P_0) \text{ and } \pi(R_\pm(\gamma_\sigma)) \cap L_1 = R_\mp(P_1).$$

This implies that $R_\pm(\gamma_\sigma) \cap \partial M_0 = R_\pm(\gamma_0)$ and $R_\pm(\gamma_\sigma) \cap \partial M_1 = R_\mp(\gamma_1)$. Since $\partial Z = \partial M_0 \cup (-\partial M_1)$, we obtain that $\gamma_\sigma \cap \partial M_0 = \gamma_0$ and $\gamma_\sigma \cap \partial M_1 = \gamma_1$.

The above definition is independent of the choice of splitting of F into $R_+(\sigma)$ and $R_-(\sigma)$. Indeed, if we swap the splitting on a component F_0 of F, then on F_0 the orientation of F is swapped as well. Hence, the orientation of each fiber of UNF_0 is also reversed. The same contact structure ξ_σ with the same coorientation is still S^1-invariant, and induces the reversed R_+ and R_- on F_0 as the fiber orientation is reversed.

The following proposition is straightforward to verify using the definitions. Recall that for an object (Y, L, P) of **DLink**, the sutured manifold $\mathcal{W}(Y, L, P)$ was introduced in Definition 4.4, and for a morphism $\mathcal{X}$ in **DLink**, the cobordism $\mathcal{W}(\mathcal{X})$ was defined in Definition 4.9.

Proposition 4.10. *The map $\mathcal{W}$ is a* functor *from* **DLink** *to* **BSut**. *Furthermore, if the decorated link cobordisms $\mathcal{X}$ and $\mathcal{X}'$ are equivalent or diffeomorhpic, then $\mathcal{W}(\mathcal{X})$ and $\mathcal{W}(\mathcal{X}')$ are also equivalent or diffeomorhpic, respectively.*

Hence, the composition $SFH \circ \mathcal{W}$ gives a functor from **DLink** to $\mathbf{Vect}_{\mathbb{Z}_2}$. If the triple (Y, L, P) is an object of **DLink** and the components of L are $L_1, \dots, L_k$, set

$$d = d(Y, L, P) = \sum_{i=1}^{k} (|L_i \cap P|/2 - 1).$$

Then, by work of the author [21, Proposition 9.2],

$$SFH(\mathcal{W}(Y, L, P)) \cong \widehat{HFL}(Y, L) \otimes V^{\otimes d},$$

where $V = \mathbb{Z}_2^2$.

5. Special cobordisms, sutured multi-diagrams, and naturality

Our first goal is to extend the hat version of the cobordism maps introduced by Ozsváth and Szabó [38] to the class of sutured manifold cobordisms that are trivial along the boundary.

Definition 5.1. We say that a cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0, γ_0) to (M_1, γ_1) is *special* if

(1) $\mathcal{W}$ is balanced,
(2) $\partial M_0 = \partial M_1$, and $Z = \partial M_0 \times I$ is the trivial cobordism between them,
(3) ξ is an I-invariant contact structure on Z such that each $\partial M_0 \times \{t\}$ is a convex surface with dividing set $\gamma_0 \times \{t\}$ for every $t \in I$ with respect to the contact vector field $\partial/\partial t$.

In particular, it follows from (3) that $\gamma_0 = \gamma_1$.

Recall that we introduced the notion of equivalence and diffeomorphism of sutured manifold cobordisms in Definition 2.7. Balanced sutured manifolds and equivalence classes of special cobordisms form a subcategory $\mathbf{BSut}'$ of **BSut**.

For a special cobordism $\mathcal{W} = (W, Z, [\xi])$, we have $H^k(Z, \partial M_1) = 0$ for $k \in \{1, 2\}$. Hence Lemma 3.7 implies that, given an admissible almost complex structure J' on Z such that $\mathfrak{s}_\xi = \mathfrak{s}_{\xi_{J'}}$, the restriction map $q \colon \mathrm{Spin}^c(\mathcal{W}, J') \to \mathrm{Spin}^c(\mathcal{W})$ is a bijection.

We now make SFH into a functor from $\mathbf{BSut}'$ to $\mathbf{Vect}_{\mathbb{Z}_2}$; i.e., we define the map $\Phi_{\mathcal{W}}$ if $\mathcal{W}$ is a special cobordism. For this, we generalize the work of Ozsváth and Szabó [38] on cobordism maps induced on the Heegaard Floer homology of closed 3-manifolds.

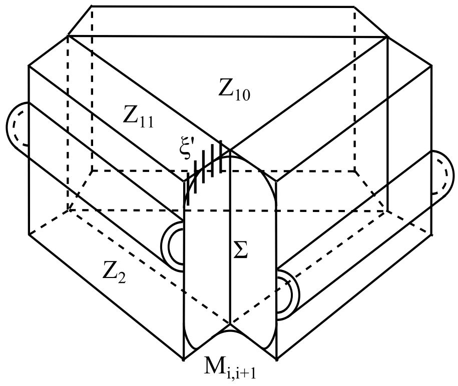

FIGURE 1. A schematic picture of the cobordism $W_{\eta^0,\dots,\eta^n}$. Here $n = 2$, and $P_{n+1} \times \Sigma$ is represented by a triangle times a vertical interval. Note how the corners are rounded. We also illustrate the 2-plane field ξ' on Z.

5.1. **Sutured multi-diagrams and pseudo-holomorphic polygons.** Some of the necessary steps have already been done by Grigsby and Wehrli [12], we review and extend their results first. In particular, we include the contact structure ξ on Z into the theory.

Definition 5.2. A *balanced sutured multi-diagram* is a tuple $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$, where Σ is a compact, oriented, surface without closed components, and there is a non-negative integer d such that, for every $0 \le i \le n$, the set $\boldsymbol{\eta}^i$ consists of d pairwise disjoint simple closed curves $\eta_1^i, \dots, \eta_d^i \subset \text{Int}(\Sigma)$ that are linearly independent in $H_1(\Sigma)$.

Remark 5.3. By a slight abuse of notation, we will also write $\boldsymbol{\eta}^i$ for the 1-dimensional submanifold $\bigcup \boldsymbol{\eta}^i$ of $\text{Int}(\Sigma)$.

Suppose that we are given a balanced sutured multi-diagram $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$. Then we associate to it a balanced cobordism

$$\mathcal{W}_{\eta^0,\dots,\eta^n} = (W_{\eta^0,\dots,\eta^n}, Z_{\eta^0,\dots,\eta^n}, [\xi_{\eta^0,\dots,\eta^n}]).$$

For an illustration of the construction, see Figure 1.

For $0 \le i \le n$, let U_i be the sutured compression body obtained from $\Sigma \times I$ by attaching 2-handles along $\boldsymbol{\eta}^i \times \{0\} \subset \Sigma \times \{0\}$, and rounding the corners along $\partial\Sigma \times \{0\}$. Then

$$\partial U_i = \Sigma^1 \cup (\partial\Sigma \times I) \cup \Sigma_{\eta^i},$$

where $\Sigma^1 = \Sigma \times \{1\}$ and Σ_{η^i} is obtained from $\Sigma \times \{0\}$ by performing surgery along each component of $\boldsymbol{\eta}^i \times \{0\}$.

Let P_{n+1} denote a regular $(n+1)$-gon, with vertices v_i for $i \in \mathbb{Z}_{n+1}$, labeled in a clockwise fashion. Denote the edge connecting v_i and v_{i+1} by e_i. Then let

$$W_{\eta^0,\dots,\eta^n} = \frac{(P_{n+1} \times \Sigma) \sqcup \coprod_{i=0}^n (e_i \times U_i)}{(e_i \times \Sigma) \sim (e_i \times \Sigma^1)},$$

where we round the corners along each $\{v_i\} \times \Sigma$ for $i \in \mathbb{Z}_{n+1}$.

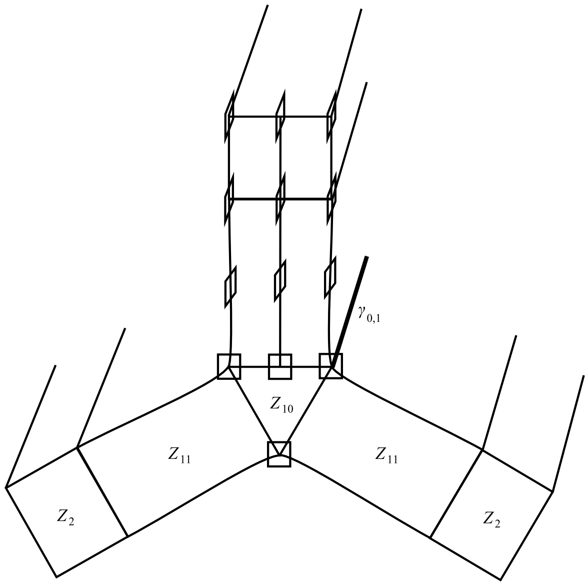

FIGURE 2. A schematic picture of the sutured manifold (Z, γ) and the 2-plane field $\xi_{J'}$ isotopic to the contact structure ξ.

Denote by $(M_{i,j}, \gamma_{i,j})$ the balanced sutured manifold defined by the diagram $(\Sigma, \boldsymbol{\eta}^i, \boldsymbol{\eta}^j)$. Then

$$M' = M_{0,1} \sqcup \cdots \sqcup M_{n-1,n} \sqcup -M_{0,n} \subset \partial W,$$

and we write $Z_{\eta^0,\dots,\eta^n}$ for $\partial W \setminus \mathrm{Int}(M')$.

Finally, we define the contact structure $\xi = \xi_{\eta^0,\dots,\eta^n}$ on the balanced sutured manifold

$$(Z, \gamma) = (Z_{\eta^0,\dots,\eta^n}, \gamma_{0,1} \cup \cdots \cup \gamma_{n-1,n} \cup \gamma_{n,0})$$

by giving a sutured manifold hierarchy of (Z, γ). A sutured manifold hierarchy is a special case of a convex hierarchy by Honda et al. [15], hence it gives rise to a contact structure ξ, well-defined up to equivalence.

Note that Z consists of three parts: $Z_{10} = P_{n+1} \times \partial\Sigma$, $Z_{11} = \bigcup_{i=0}^n (e_i \times \partial\Sigma \times I)$, and $Z_2 = \bigcup_{i=0}^n (e_i \times \Sigma_{\eta^i})$, see Figure 2. We put $Z_1 = Z_{10} \cup Z_{11}$; then

$$Z_1 = \left(P_{n+1} \cup \bigcup_{i=0}^{n} (e_i \times I) \right) \times \partial\Sigma = P_{2n+2} \times \partial\Sigma.$$

Here, we get P_{2n+2} by gluing the rectangle $e_i \times I$ to P_{n+1} by identifying $e_i \times \{1\}$ with e_i for each $i \in \mathbb{Z}_{n+1}$, then rounding the corners at $v_0, \dots, v_n$. We still label the edge $e_i \times \{0\}$ of P_{2n+2} by e_i, and the edge containing v_i is called g_i. Observe that $\gamma_{i,i+1} = g_{i+1} \times \partial\Sigma$. Recall that when we defined the compression body U_i, we rounded its corners along $\partial\Sigma \times \{0\}$. When we glue Z_1 and Z_2, this corresponds to rounding the corners along $e_i \times \partial\Sigma$, so for every $x \in e_i$, the surfaces $\{x\} \times \partial\Sigma \times I \subset Z_{11}$ and $\{x\} \times \Sigma_{\eta^i} \subset Z_2$ match smoothly, cf. Figure 1.

Let s_i be a 1-manifold parallel to $\partial\Sigma_{\eta^i}$ inside $\mathrm{Int}(\Sigma_{\eta^i})$, and let $A_i = e_i \times s_i$. Then $A = A_0 \cup \cdots \cup A_n$ is a decomposing surface, a union of product annuli, inside (Z, γ). Consider the sutured manifold decomposition $(Z, \gamma) \rightsquigarrow^A (Z', \gamma')$. Then (Z', γ') is the disjoint union of the product sutured manifolds $(\Sigma_{\eta^i} \times I, \partial\Sigma_{\eta^i} \times I)$ for $i \in \mathbb{Z}_{n+1}$, and $S^1 \times D^2$ components with $2n+2$ longitudinal sutures on each. Every product piece has a product disk decomposable contact structure, unique up to equivalence. We further decompose each $S^1 \times D^2$ along $\{\text{pt}\} \times D^2$ to get a ball with a single suture, which carries a unique tight contact structure. Note that we orient $\{\text{pt}\} \times D^2$ such that it is positively transverse to the $\partial\Sigma$ factor. Our sequence of decompositions terminates in a product sutured manifold. Hence, by the work of Honda et al. [15], we obtain a tight contact structure ξ on (Z, γ), which is well-defined up to equivalence.

We will use the following lemma to show that the relative Spin^c structures used by Grigsby and Wehrli [12, Proposition 3.7] and defined by a 2-plane field ξ' along Z also define Spin^c structures relative to ξ.

Lemma 5.4. *Consider $\mathcal{W}_{\eta^0,\dots,\eta^n} = (W, Z, [\xi])$. Let $\xi' = \xi'_{\eta^0,\dots,\eta^n}$ be the 2-plane field in $TW|_Z$ such that on $Z_1 = P_{2n+2} \times \partial\Sigma$ it is tangent to Σ, and on Z_2 it is tangent to Σ_{η^i}. (This is smooth on Z since we rounded the corners along $e_i \times \partial\Sigma$, cf. Figure 1.) Choose an arbitrary almost complex structure J' on $TW|_Z$ such that ξ' consists of complex lines, and let $\xi_{J'}$ denote the 2-plane field of complex tangencies along Z. Then*

$$\mathfrak{s}_\xi = \mathfrak{s}_{\xi_{J'}} \in \mathit{Spin}^c(Z, \gamma).$$

Proof. It suffices to check that the 2-plane field $\xi_{J'}$ in TZ never agrees with $-\xi$, for some representative ξ of the equivalence class $[\xi]$. For an illustration of the following argument, see Figure 2. Since $\xi'|_{Z_1}$ is tangent to Σ and is J'-invariant, the 2-planes $\xi_{J'}$ on $Z_1 = P_{n+1} \times \partial\Sigma$ must be positively transverse to the $\partial\Sigma$ factor. On Z_2, the planes $\xi_{J'}$ agree with ξ', which are tangent to Σ_{η^i}. The contact structure ξ is a perturbation of the horizontal foliation on each $\Sigma_{\eta^i} \times I$. On Z_1, it is a perturbation of the foliation by multi-saddles, so it is also transverse to the $\partial\Sigma$ factor. In particular, ξ and $\xi_{J'}$ are never opposite. □

The almost complex structures J' in Lemma 5.4 form a contractible space, so homotopically it is unique, we denote it by $J'_{\eta^0,\dots,\eta^n}$. Since $\mathfrak{s}_\xi = \mathfrak{s}_{\xi_{J'}}$ and it is admissible, we can talk about the set $\text{Spin}^c(\mathcal{W}_{\eta^0,\dots,\eta^n}, J'_{\eta^0,\dots,\eta^n})$ of Spin^c structures restricting to $J'_{\eta^0,\dots,\eta^n}$ along Z. We are going to use the notation

$$\underline{\text{Spin}}^c\left(\mathcal{W}_{\eta^0,\dots,\eta^n}\right) := \text{Spin}^c\left(\mathcal{W}_{\eta^0,\dots,\eta^n}, J'_{\eta^0,\dots,\eta^n}\right).$$

Furthermore, just as in Lemma 3.7, we have a restriction map

$$q\colon \underline{\text{Spin}}^c\left(\mathcal{W}_{\eta^0,\dots,\eta^n}\right) \to \text{Spin}^c\left(\mathcal{W}_{\eta^0,\dots,\eta^n}\right).$$

As usual, we denote by $\mathbb{T}_{\eta^i}$ the d-torus $\eta^i_1 \times \cdots \times \eta^i_d$ inside $\text{Sym}^d(\Sigma)$. For $i \in \mathbb{Z}_{n+1}$, let $\mathbf{x}_{i+1} \in \mathbb{T}_{\eta^i} \cap \mathbb{T}_{\eta^{i+1}}$. Then we write $\pi_2(\mathbf{x}_0, \dots, \mathbf{x}_n)$ for the set of homotopy classes of Whitney $(n+1)$-gons inside $\text{Sym}^d(\Sigma)$ connecting $\mathbf{x}_0, \dots, \mathbf{x}_n$. We now recall a result of Grigsby and Wehrli [12, Proposition 3.7], which relates Whitney $(n+1)$-gons and Spin^c structures.

Proposition 5.5. *Suppose that $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$ is a sutured multi-diagram, and $\mathcal{W} = \mathcal{W}_{\eta^0,\dots,\eta^n}$ is the associated cobordism. Then there is a well-defined map*

$$\underline{\mathfrak{s}}\colon \pi_2(\mathbf{x}_0, \dots, \mathbf{x}_n) \to \underline{\mathit{Spin}}^c(\mathcal{W})$$

such that $\underline{\mathfrak{s}}(\Psi)|_{M_{i,i+1}} = \mathfrak{s}(\mathbf{x}_i)$ for every $i \in \mathbb{Z}_{n+1}$.

Proof. The construction of Grigsby and Wehrli [12], based on the work of Ozsváth and Szabó [36, Section 8], associates to any Whitney $(n+1)$-gon u a 2-plane field ξ_u on W minus a contractible 1-complex c that agrees with $\xi' = \xi'_{\eta^0,\dots,\eta^n}$ along Z. Let J be an almost complex-structure on $W \setminus c$ such that ξ_u consists of complex lines. By construction, $J|_Z = J'_{\eta^0,\dots,\eta^n}$. The relative homology class of J gives an element $\underline{\mathfrak{s}}(\Psi)$ of $\underline{\text{Spin}}^c(\mathcal{W})$ that satisfies the property $\underline{\mathfrak{s}}(\Psi)|_{M_{i,i+1}} = \mathfrak{s}(\mathbf{x}_i)$. $\square$

Definition 5.6. Let $\underline{\mathfrak{s}}$ be the map introduced in Proposition 5.5, and let q be the restriction map from $\underline{\text{Spin}}^c(\mathcal{W})$ to $\text{Spin}^c(\mathcal{W})$. Then we define

$$\mathfrak{s}\colon \pi_2(\mathbf{x}_0,\dots,\mathbf{x}_n) \to \text{Spin}^c(\mathcal{W})$$

to be the composition $q \circ \underline{\mathfrak{s}}$.

It follows from Proposition 5.5 that $\mathfrak{s}(\Psi)|_{M_{i,i+1}} = \mathfrak{s}(\mathbf{x}_i)$ for every $i \in \mathbb{Z}_{n+1}$.

Definition 5.7. Let $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$ be a balanced sutured multi-diagram. Let $D_1, \dots, D_l$ denote the closures of the components of $\Sigma \setminus (\boldsymbol{\eta}^0 \cup \dots \cup \boldsymbol{\eta}^n)$ disjoint from $\partial\Sigma$. Then the set of *domains* in $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$ is

$$D(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n) = \mathbb{Z}\langle\, D_1, \dots, D_l \,\rangle.$$

For a domain $\mathcal{D} \in D(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$, we write $\mathcal{D} \geq 0$ if $\mathcal{D} \in \mathbb{Z}_{\geq 0}\langle\, D_1, \dots, D_l \,\rangle$. As usual, if

$$(\mathbf{x}_0, \dots, \mathbf{x}_n) \in (\mathbb{T}_{\eta^n} \cap \mathbb{T}_{\eta^0}) \times \dots \times (\mathbb{T}_{\eta^{n-1}} \cap \mathbb{T}_{\eta^n}),$$

then $D(\mathbf{x}_0, \dots, \mathbf{x}_n)$ denotes the set of domains connecting $\mathbf{x}_0, \dots, \mathbf{x}_n$.

Finally, an $(n+1)$-*periodic domain* is an element $\mathcal{P} \in D(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$ such that $\partial\mathcal{P}$ is a $\mathbb{Z}$-linear combination of curves in $\boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n$.

The following proposition implies that any two Whitney $(n+1)$-gons in the affine set $\pi_2(\mathbf{x}_0, \dots, \mathbf{x}_n)$ differ by an $(n+1)$-periodic domain.

Proposition 5.8. *If* $\pi_2(\mathbf{x}_0, \dots, \mathbf{x}_n) \neq \emptyset$*, then*

$$\pi_2(\mathbf{x}_0, \dots, \mathbf{x}_n) \cong \ker\left(\bigoplus_{i=0}^{n} H_1(\boldsymbol{\eta}^i) \to H_1(\Sigma)\right) \cong H_2(W_{\eta^0,\dots,\eta^n}).$$

Furthermore,

$$\text{coker}\left(\bigoplus_{i=0}^{n} H_1(\boldsymbol{\eta}^i) \to H_1(\Sigma)\right) \cong H_1(W_{\eta^0,\dots,\eta^n}).$$

Proof. This was shown by Grigsby and Wehrli [12, Proposition 3.3 and 3.4]. $\square$

The correspondence in Proposition 5.8 can be made explicit by associating to each periodic domain $\mathcal{P}$ an element $H(\mathcal{P})$ of $H_2(W_{\eta^0,\dots,\eta^n})$, as follows. Pick an interior point $x \in P_{n+1}$, and connect x to every e_i by a straight arc a_i. For each curve $\eta^i_j \in \boldsymbol{\eta}^i$, let $E^i_j \subset W_{\eta^0,\dots,\eta^n}$ denote the union of the annulus $a_i \times \eta^i_j \subset P_{n+1} \times \Sigma$, the annulus $(a_i \cap e_i) \times \eta^i_j \times I$ in Z_{11}, and the core disk of the 2-handle attached to $(a_i \cap e_i) \times \eta^i_j \times \{0\}$ in Z_2. Suppose that

$$\partial\mathcal{P} = \sum_{i=0}^{n} \sum_{j=1}^{d} e^j_i \eta^i_j.$$

Then

$$H(\mathcal{P}) = \{x\} \times \mathcal{P} + \sum_{i=0}^{n} \sum_{j=1}^{d} e_i^j E_j^i \in H_2(W_{\eta^0,\dots,\eta^n}).$$

The following is a corrected version of [12, Proposition 3.9]. Recall from Proposition 5.5 that $\underline{\mathfrak{s}}$ is the relative Spin^c map defined by Grigsby and Wehrli, which is different from the map $\mathfrak{s}$.

Proposition 5.9. *Let Ψ, $\Psi' \in \pi_2(\mathbf{x}_0, \dots, \mathbf{x}_n)$. Then $\underline{\mathfrak{s}}(\Psi) = \underline{\mathfrak{s}}(\Psi')$ if and only if $\Psi - \Psi'$ can be written as a $\mathbb{Z}$-linear combination of doubly-periodic domains.*

Definition 5.10. A balanced sutured multi-diagram is *admissible* if every non-trivial $(n+1)$-periodic domain has both positive and negative coefficients.

The following statement is [12, Lemma 3.12].

Lemma 5.11. *Every balanced sutured multi-diagram is isotopic to an admissible one.*

This weak form of admissibility is specific to the hat version of Heegaard Floer homology. It enables us to talk about various polygon counts without restricting to a particular Spin^c structure, making all sums finite. For the following, see Grigsby and Wehrli [12, Proposition 3.14].

Proposition 5.12. *If $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$ is admissible, then for every*

$$(\mathbf{x}_0, \dots, \mathbf{x}_n) \in (\mathbb{T}_{\eta^n} \cap \mathbb{T}_{\eta^0}) \times \cdots \times (\mathbb{T}_{\eta^{n-1}} \cap \mathbb{T}_{\eta^n}),$$

the set $\{\, \mathcal{D} \in D(\mathbf{x}_0, \dots \mathbf{x}_n) : \mathcal{D} \geq 0 \,\}$ is finite.

Let $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$ be an admissible sutured multi-diagram, and for every $i \in \{1, \dots, n\}$, let $\mathbf{x}_i \in \mathbb{T}_{\eta^{i-1}} \cap \mathbb{T}_{\eta^i}$ and $\mathbf{y} \in \mathbb{T}_{\eta^0} \cap \mathbb{T}_{\eta^n}$. Fix a complex structure on Σ, and a 1-parameter variation of the induced almost complex structure on $\mathrm{Sym}^d(\Sigma)$. As usual, we denote by $\mathcal{M}(\phi)$ the moduli-space of pseudo-holomorphic representatives of Whitney $(n+1)$-gons lying in the homotopy class $\phi \in \pi_2(\mathbf{x}_1, \dots, \mathbf{x}_n, \mathbf{y})$. The Maslov index; i.e., the expected dimension, of $\mathcal{M}(\phi)$ is denoted by $\mu(\phi)$. If $n = 1$ and $\mu(\phi) = 1$, then there is a natural $\mathbb{R}$-action on $\mathcal{M}(\phi)$, we let $\widehat{\mathcal{M}}(\phi) = \mathcal{M}(\phi)/\mathbb{R}$. When $n > 1$ and $\mu(\phi) = 0$, the moduli space $\mathcal{M}(\phi)$ is compact. Then $\#\mathcal{M}(\phi)$ denotes the number of points in $\mathcal{M}(\phi)$ modulo two. In the case $n = 1$ and $\mu(\phi) = 1$, the reduced moduli space $\widehat{\mathcal{M}}(\phi)$ is compact.

For $0 \leq i < j \leq n$, we let

$$CF(\Sigma, \boldsymbol{\eta}^i, \boldsymbol{\eta}^j) = \mathbb{Z}_2\langle\, \mathbb{T}_{\eta^i} \cap \mathbb{T}_{\eta^j} \,\rangle.$$

This becomes a chain complexes when endowed with the differential that counts points in $\widehat{\mathcal{M}}(\phi)$ modulo two, where ϕ is a homotopy class of Whitney bigons with boundary on $\mathbb{T}_{\eta^i}$ and $\mathbb{T}_{\eta^j}$ and having $\mu(\phi) = 1$. Its homology is the sutured Floer homology group $SFH(\Sigma, \boldsymbol{\eta}^i, \boldsymbol{\eta}^j)$.

Definition 5.13. Let $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^n)$ be an admissible sutured multi-diagram such that $n \geq 2$, and fix a relative Spin^c structure $\underline{\mathfrak{s}} \in \underline{\mathrm{Spin}}^c(\mathcal{W}_{\eta^0,\dots,\eta^n})$. Then we have chain maps

$$f_{\eta^0,\dots,\eta^n} \colon \bigotimes_{i=1}^{n} CF(\Sigma, \boldsymbol{\eta}^{i-1}, \boldsymbol{\eta}^i) \to CF(\Sigma, \boldsymbol{\eta}^0, \boldsymbol{\eta}^n)$$

and

$$f_{\eta^0,\dots,\eta^n}(\,\cdot\,,\underline{\mathfrak{s}})\colon \bigotimes_{i=1}^{n} CF(\Sigma,\boldsymbol{\eta}^{i-1},\boldsymbol{\eta}^{i},\underline{\mathfrak{s}}|_{M_{i-1,i}}) \to CF(\Sigma,\boldsymbol{\eta}^0,\boldsymbol{\eta}^n,\underline{\mathfrak{s}}|_{M_{0,n}}),$$

defined by the formulas

$$f_{\eta^0,\dots,\eta^n}(x_1\otimes\cdots\otimes x_n) = \sum_{\mathbf{y}\in\mathbb{T}_{\eta^0}\cap\mathbb{T}_{\eta^n}}\ \sum_{\{\,\phi\in\pi_2(\mathbf{x}_1,\dots,\mathbf{x}_n,\mathbf{y})\,:\ \mu(\phi)=0\,\}} \#\mathcal{M}(\phi)\cdot\mathbf{y}$$

and

$$f_{\eta^0,\dots,\eta^n}(x_1\otimes\cdots\otimes x_n,\underline{\mathfrak{s}}) = \sum_{\mathbf{y}\in\mathbb{T}_{\eta^0}\cap\mathbb{T}_{\eta^n}}\ \sum_{\{\,\phi\in\pi_2(\mathbf{x}_1,\dots,\mathbf{x}_n,\mathbf{y})\,:\ \mu(\phi)=0,\ \underline{\mathfrak{s}}(\phi)=\underline{\mathfrak{s}}\,\}} \#\mathcal{M}(\phi)\cdot\mathbf{y}.$$

We denote by $F_{\eta^0,\dots,\eta^n}$ and $F_{\eta^0,\dots,\eta^n}(\,\cdot\,,\underline{\mathfrak{s}})$ the maps induced on the homology. Analogous maps can be defined for $\mathfrak{s}\in\mathrm{Spin}^c(\mathcal{W})$, replacing $\underline{\mathfrak{s}}(\phi)$ by $\mathfrak{s}(\phi)$.

The finiteness of the above sums is ensured by Proposition 5.12, since if ϕ supports a pseudo-holomorphic representative, then its domain $\mathcal{D}(\phi)\geq 0$.

Proposition 5.14. *Let $(\Sigma,\boldsymbol{\eta}^0,\dots,\boldsymbol{\eta}^n)$ be an admissible multi-diagram, and set $\mathcal{W}=\mathcal{W}_{\eta^0,\dots,\eta^n}$. Then there are only finitely many $\underline{\mathfrak{s}}\in\underline{Spin}^c(\mathcal{W})$ for which the map $f_{\eta^0,\dots,\eta^n}(\,\cdot\,,\underline{\mathfrak{s}})$ is non-zero, and*

$$f_{\eta^0,\dots,\eta^n} = \sum_{\underline{\mathfrak{s}}\in\underline{Spin}^c(\mathcal{W})} f_{\eta^0,\dots,\eta^n}(\,\cdot\,,\underline{\mathfrak{s}}).$$

An analogous statement holds for $F_{\eta^0,\dots,\eta^n}$ and $F_{\eta^0,\dots,\eta^n}(\,\cdot\,,\underline{\mathfrak{s}})$, and also for Spin^c structures $\mathfrak{s}\in Spin^c(\mathcal{W})$.

Proof. Each $\mathbb{T}_{\eta^i}\cap\mathbb{T}_{\eta^j}$ is finite, so there are only finitely many choices for $\mathbf{x}_1,\dots,\mathbf{x}_n$ and $\mathbf{y}$, and for each choice, there are only finitely many $\phi\in\pi_2(\mathbf{x}_1,\dots,\mathbf{x}_n,\mathbf{y})$ such that $\mathcal{M}(\phi)\neq\emptyset$ by Proposition 5.12. Finally, such a ϕ can only appear in the formula defining $f_{\eta^0,\dots,\eta^n}(\,\cdot\,,\underline{\mathfrak{s}}(\phi))$. The result follows. □

We now generalize the associativity theorem of the triangle maps due to Ozsváth and Szabó [36, Theorem 8.16] to the sutured setting. Fix a sutured quadruple diagram $(\Sigma,\boldsymbol{\eta}^0,\dots,\boldsymbol{\eta}^3)$, and let $\mathcal{W}_{\eta^0,\dots,\eta^3}$ be the corresponding cobordism. Then we have a restriction map

$$\underline{\mathrm{Spin}}^c(\mathcal{W}_{\eta^0,\dots,\eta^3}) \to \underline{\mathrm{Spin}}^c(\mathcal{W}_{\eta^0,\eta^1,\eta^2})\times\underline{\mathrm{Spin}}^c(\mathcal{W}_{\eta^0,\eta^2,\eta^3}),$$

which corresponds to splitting the cobordism $\mathcal{W}_{\eta^0,\dots,\eta^3}$ along an embedded copy of M_{02}. There is a subgroup

$$\delta H^1(M_{02},\partial M_{02}) < H^2(W_{\eta^0,\dots,\eta^3},Z_{\eta^0,\dots,\eta^3})$$

whose orbits on $\underline{\mathrm{Spin}}^c(\mathcal{W}_{\eta^0,\dots,\eta^3})$ are the fibers of the restriction map, where δ is the coboundary map in the corresponding relative Mayer-Vietoris sequence. Similarly, we have a restriction map

$$\underline{\mathrm{Spin}}^c(\mathcal{W}_{\eta^0,\dots,\eta^3}) \to \underline{\mathrm{Spin}}^c(\mathcal{W}_{\eta^0,\eta^1,\eta^3})\times\underline{\mathrm{Spin}}^c(\mathcal{W}_{\eta^1,\eta^2,\eta^3}),$$

which corresponds to splitting along M_{13}, and a subgroup

$$\delta H^1(M_{13},\partial M_{13}) < H^2(W_{\eta^0,\dots,\eta^3},Z_{\eta^0,\dots,\eta^3}).$$

Theorem 5.15. *Let* $(\Sigma, \boldsymbol{\eta}^0, \boldsymbol{\eta}^1, \boldsymbol{\eta}^2, \boldsymbol{\eta}^3)$ *be an admissible sutured quadruple diagram, and fix a*

$$\delta H^1(M_{02}, \partial M_{02}) + \delta H^1(M_{13}, \partial M_{13})$$

orbit $\mathfrak{S}$ *in* $\underline{\mathrm{Spin}}^c(W_{\eta^0,\dots,\eta^3})$. *For any* $\underline{\mathfrak{s}} \in \mathfrak{S}$ *and* $i \in \{0,1,2\}$, *the restriction* $\mathfrak{s}_{i,i+1} = \underline{\mathfrak{s}}|_{M_{i,i+1}}$ *is independent of the choice of* $\underline{\mathfrak{s}}$*; pick an element*

$$x_{i,i+1} \in SFH(\Sigma, \boldsymbol{\eta}^i, \boldsymbol{\eta}^{i+1}, \mathfrak{s}_{i,i+1}).$$

Furthermore, for $0 \le i < j < k \le 4$, *let* $F_{ijk} = F_{\eta^i,\eta^j,\eta^k}$ *and* $\mathfrak{s}_{ijk} = \underline{\mathfrak{s}}|_{\mathcal{W}_{\eta^i,\eta^j,\eta^k}}$. *Then*

$$\sum_{\underline{\mathfrak{s}} \in \mathfrak{S}} F_{023}(F_{012}(x_{01} \otimes x_{12}, \mathfrak{s}_{012}) \otimes x_{23}, \mathfrak{s}_{023}) =$$

$$= \sum_{\underline{\mathfrak{s}} \in \mathfrak{S}} F_{013}(x_{01} \otimes F_{123}(x_{12} \otimes x_{23}, \mathfrak{s}_{123}), \mathfrak{s}_{013}).$$

Proof. Every subdiagram of an admissible sutured multi-diagram is also admissible. Hence, the proof of Ozsváth and Szabó [36, Theorem 8.16] works in this setting too, since the admissibility of $(\Sigma, \boldsymbol{\eta}^0, \dots, \boldsymbol{\eta}^3)$ ensures the finiteness of all the counts of pseudo-holomorphic bigons, triangles, and rectangles that appear in the formula for the chain homotopy connecting the two sides. □

In a similar manner, one can prove an associativity result without fixing an orbit $\mathfrak{S}$ of Spin^c structures on $\mathcal{W}_{\eta^0,\dots,\eta^3}$.

Theorem 5.16. *Let* $(\Sigma, \boldsymbol{\eta}^0, \boldsymbol{\eta}^1, \boldsymbol{\eta}^2, \boldsymbol{\eta}^3)$ *be an admissible sutured quadruple diagram, and for every* $i \in \{0,1,2\}$, *pick an element* $x_{i,i+1} \in SFH(\Sigma, \boldsymbol{\eta}^i, \boldsymbol{\eta}^{i+1})$. *Then*

$$F_{023}(F_{012}(x_{01} \otimes x_{12}) \otimes x_{23}) = F_{013}(x_{01} \otimes F_{123}(x_{12} \otimes x_{23})).$$

Note that writing associativity using Spin$^c(\mathcal{W}_{\eta^0,\dots,\eta^3})$ would be cumbersome as such Spin^c structures do not restrict to the cobordisms $\mathcal{W}_{\eta^0,\eta^1,\eta^2}$ etc.

5.2. **Naturality of sutured Floer homology.** When the first draft of this paper appeared, the assignment of a Heegaard Floer group to a 3-manifold was not functorial. Ozsváth and Szabó [36] showed that different admissible Heegaard diagrams of the same manifold give rise to isomorphic Floer homology groups. However, this is not sufficient to define maps induced by cobordisms, or to talk about the diffeomorphism action. It had also caused some confusion in the case of the contact invariant, as it is unclear what it means to be an element of Heegaard Floer homology. Being aware of this issue, Ozsváth and Szabó [38, Theorem 2.1] constructed "canonical isomorphisms" for equivalent Heegaard diagrams. However, as the author noticed, they did not check that these maps were indeed isomorphisms, or that the composition of two canonical isomorphisms was a canonical isomorphism. Furthermore, it turns out it is not enough to say that a Heegaard diagram of a manifold is an abstract triple $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, one has to take into account how Σ is embedded in the manifold. Together with Dylan Thurston, we settled the above issues in [25]. This was a prerequisite for defining our cobordism maps.

In this section, we review the functorial construction of sutured Floer homology, which includes $\widehat{HF}$. The other flavors of Heegaard Floer homology can be addressed in an analogous manner. In particular, we explain how to canonically associate a group $SFH(M, \gamma)$ to every balanced sutured manifold (M, γ) via a simple limiting procedure. This will enable us to precisely define how diffeomorphisms act on sutured Floer homology. For more details, we refer the reader to [25].

The following definition is a refinement of [19, Definition 2.7]. By a balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, we mean a balanced sutured double-diagram in the sense of Definition 5.2. Recall that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defines a sutured manifold $(M_{\alpha,\beta}, \gamma_{\alpha,\beta})$ as in [19, Definition 2.8], which is unique up to diffeomorphism relative to Σ. The following is [25, Definition 2.14].

Definition 5.17. Let (M, γ) be a sutured manifold. Then we say that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a *diagram of* (M, γ) if

(1) $\Sigma \subset M$ is an oriented surface with $\partial\Sigma = s(\gamma)$ as oriented 1-manifolds,
(2) the components of $\boldsymbol{\alpha}$ bound disjoint disks to the negative side of Σ, and the components of $\boldsymbol{\beta}$ bound disjoint disks to the positive side of Σ,
(3) if we compress Σ along $\boldsymbol{\alpha}$, we get a surface isotopic to $R_-(\gamma)$ relative to γ,
(4) if we compress Σ along $\boldsymbol{\beta}$, we get a surface isotopic to $R_+(\gamma)$ relative to γ.

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be an *admissible* diagram of the balanced sutured manifold (M, γ). Since $\chi(R_+(\gamma)) = \chi(R_-(\gamma))$, we have $|\boldsymbol{\alpha}| = |\boldsymbol{\beta}|$, we denote this number by k. We also choose a complex structure $\mathfrak{j}$ on Σ and a generic perturbation $J_s \subset \mathcal{U}$ of the induced complex structure on $\mathrm{Sym}^g(\Sigma)$, where $\mathcal{U}$ is a contractible set of almost complex structures. The sutured Floer homology $SFH_{J_s}(\mathcal{H})$ is the Lagrangian intersection Floer homology of the tori $\mathbb{T}_\alpha$ and $\mathbb{T}_\beta$ inside $\mathrm{Sym}^k(\Sigma)$ endowed with a particular symplectic structure compatible with J_s. Recall that we are using $\mathbb{Z}_2$-coefficients. Furthermore, this splits along relative Spin^c structures:

$$SFH_{J_s}(\mathcal{H}) = \bigoplus_{\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)} SFH_{J_s}(\mathcal{H}, \mathfrak{s}).$$

Given two different choices $(\mathfrak{j}, J_s)$ and $(\mathfrak{j}', J_s')$ of complex structures and perturbations, Ozsváth and Szabó [38, Lemma 2.11] constructed isomorphisms

$$\Phi_{J_s \to J_s'} \colon SFH_{J_s}(\mathcal{H}, \mathfrak{s}) \to SFH_{J_s'}(\mathcal{H}, \mathfrak{s}).$$

These isomorphisms are natural in the sense that

$$\Phi_{J_s \to J_s''} = \Phi_{J_s' \to J_s''} \circ \Phi_{J_s \to J_s'}$$

and $\Phi_{J_s \to J_s}$ is the identity. Then we define $SFH(\mathcal{H}, \mathfrak{s})$ to be the set of those elements g of $\prod_{J_s} SFH_{J_s}(\mathcal{H}, \mathfrak{s})$ for which $\Phi_{J_s \to J_s'}(g(J_s)) = g(J_s')$ for any pair of generic perturbations J_s and J_s' in $\mathcal{U}$.

Let $\mathcal{H}(M, \gamma)$ be the set of all admissible diagrams of (M, γ). (Note that this is indeed a set, not a proper class, as $\Sigma \subset M$.) Our goal is to construct an isomorphism

$$F_{\mathcal{H},\mathcal{H}'} \colon SFH(\mathcal{H}) \to SFH(\mathcal{H}')$$

for any pair $(\mathcal{H}, \mathcal{H}') \in \mathcal{H}(M, \gamma) \times \mathcal{H}(M, \gamma)$ such that these isomorphisms respect the splitting along $\mathrm{Spin}^c(M, \gamma)$. We require that the groups $SFH(\mathcal{H})$ and isomorphisms $F_{\mathcal{H},\mathcal{H}'}$ form a *transitive system*; i.e., that $F_{\mathcal{H},\mathcal{H}} = \mathrm{Id}_{\mathcal{H}}$ and

$$F_{\mathcal{H},\mathcal{H}''} = F_{\mathcal{H}',\mathcal{H}''} \circ F_{\mathcal{H},\mathcal{H}'}$$

for every $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}'' \in \mathcal{H}(M, \gamma)$. Then $SFH(M, \gamma)$ is the set of elements g of $\prod_{\mathcal{H} \in \mathcal{H}(M,\gamma)} SFH(\mathcal{H})$ for which $F_{\mathcal{H},\mathcal{H}'}(g(\mathcal{H})) = g(\mathcal{H}')$ for every $\mathcal{H}$, $\mathcal{H}' \in \mathcal{H}(M, \gamma)$. Projection onto the factor corresponding to the diagram $\mathcal{H}$ gives an isomorphism

$$P_{\mathcal{H}} \colon SFH(M, \gamma) \to SFH(\mathcal{H}).$$

Any pair of diagrams $\mathcal{H}$, $\mathcal{H}' \in \mathcal{H}(M,\gamma)$ can be connected by a sequence of moves that we describe next. After that, we define an elementary isomorphism $F_{\mathcal{H},\mathcal{H}'}$ for each such move. One of the main results of [25] is that, given any two diagrams $\mathcal{H}$, $\mathcal{H}' \in \mathcal{H}(M,\gamma)$, no matter how we get from $\mathcal{H}$ to $\mathcal{H}'$, the composition of the corresponding elementary isomorphisms is always the same.

Given two 1-manifolds $\boldsymbol{\delta}$ and $\boldsymbol{\delta}'$ on a surface Σ, we say that they are *equivalent*, and write $\boldsymbol{\delta} \sim \boldsymbol{\delta}'$, if one can obtain one from the other via a sequence of isotopies and handleslides. Suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}')$ are both diagrams of the sutured manifold (M,γ). Then, by [25, Lemma 2.11], we have $\boldsymbol{\alpha} \sim \boldsymbol{\alpha}'$ and $\boldsymbol{\beta} \sim \boldsymbol{\beta}'$.

Definition 5.18. We say that the diagrams $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ and $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$ are *strongly equivalent* if $\Sigma_1 = \Sigma_2$, $\boldsymbol{\alpha}_1 \sim \boldsymbol{\alpha}_2$, and $\boldsymbol{\beta}_1 \sim \boldsymbol{\beta}_2$.

Definition 5.19. The sutured diagram $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$ is obtained from $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ by a *stabilization* if

- there is a disk $D \subset \Sigma_1$ and a punctured torus $T \subset \Sigma_2$ such that we have $\Sigma_1 \setminus D = \Sigma_2 \setminus T$,
- $\boldsymbol{\alpha}_1 = \boldsymbol{\alpha}_2 \cap (\Sigma_2 \setminus T)$ and $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2 \cap (\Sigma_2 \setminus T)$,
- $\boldsymbol{\alpha}_2 \cap T$ and $\boldsymbol{\beta}_2 \cap T$ are simple closed curves that intersect each other transversely in a single point.

In this case, we also say that $(\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ is obtained from $(\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$ by a *destabilization*.

Definition 5.20. Given diagrams $\mathcal{H}_1 = (\Sigma_1, \boldsymbol{\alpha}_1, \boldsymbol{\beta}_1)$ and $\mathcal{H}_2 = (\Sigma_2, \boldsymbol{\alpha}_2, \boldsymbol{\beta}_2)$, a diffeomorphism $d\colon \mathcal{H}_1 \to \mathcal{H}_2$ is an orientation preserving diffeomorphism $d\colon \Sigma_1 \to \Sigma_2$ such that $d(\boldsymbol{\alpha}_1) = \boldsymbol{\alpha}_2$ and $d(\boldsymbol{\beta}_1) = \boldsymbol{\beta}_2$.

Suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are both diagrams for (M,γ), and let $\iota_i\colon \Sigma_i \to M$ be the inclusion for $i \in \{1,2\}$. Then a diffeomorphism $d\colon \mathcal{H}_1 \to \mathcal{H}_2$ is *isotopic to the identity in* M if $\iota_2 \circ d\colon \Sigma_1 \to M$ is isotopic to $\iota_1\colon \Sigma_1 \to M$ relative to $s(\gamma)$.

For any two diagrams $\mathcal{H}$, $\mathcal{H}' \in \mathcal{H}(M,\gamma)$, there exists a sequence

$$\mathcal{H}_1, \dots, \mathcal{H}_n \in \mathcal{H}(M,\gamma)$$

such that for every $i \in \{\,1, \dots, n-1\,\}$, the diagrams $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ are related by a strong equivalence, a (de)stabilization, or a diffeomorphism isotopic to the identity in M.

We now review [25, Lemma 9.2]. If $\mathcal{H} = (\Sigma, \boldsymbol{\delta}, \boldsymbol{\delta}')$ is an admissible diagram and $\boldsymbol{\delta} \sim \boldsymbol{\delta}'$, then this defines the connected sum of a product sutured manifold and k copies of $S^1 \times S^2$. (Note that if Σ is disconnected, then we might be taking connected sums along different components of a disconnected product sutured manifold diffeomorphic to $(\Sigma(\boldsymbol{\delta}) \times I, \partial\Sigma \times I)$, where $\Sigma(\boldsymbol{\delta})$ is Σ compressed along $\boldsymbol{\delta}$.) There is a unique Spin^c structure $\mathfrak{s}_0$ on this manifold with $c_1(\mathfrak{s}_0) = 0$ and which can be represented by a vertical vector field on the product summand. Then

$$SFH(\mathcal{H}, \mathfrak{s}_0) \cong \Lambda^* H_1(S^1 \times S^2; \mathbb{Z}_2)$$

as relative $\mathbb{Z}$-graded groups. Hence, the "top" non-zero grading is isomorphic to $\mathbb{Z}_2$, we denote its generator by $\Theta_{\delta,\delta'}$.

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\beta}')$ is an admissible triple, then we write $\Psi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\to\boldsymbol{\beta}'}$ for the map

$$F_{\alpha,\beta,\beta'}(- \otimes \Theta_{\beta,\beta'})\colon SFH(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to SFH(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}').$$

Similarly, if the triple $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\alpha}, \boldsymbol{\beta})$ is admissible and $\boldsymbol{\alpha} \sim \boldsymbol{\alpha}'$, then we write $\Psi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}_{\boldsymbol{\beta}}$ for the map

$$F_{\alpha',\alpha,\beta}(\Theta_{\alpha',\alpha} \otimes -)\colon SFH(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to SFH(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}).$$

According to [25, Proposition 9.8], these maps are isomorphisms.

Suppose that the diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}')$ are admissible and strongly equivalent. If the quadruple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is admissible, then let

$$\Psi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}_{\boldsymbol{\beta}\to\boldsymbol{\beta}'} := \Psi^{\boldsymbol{\alpha}'}_{\boldsymbol{\beta}\to\boldsymbol{\beta}'} \circ \Psi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}_{\boldsymbol{\beta}}.$$

If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is not necessarily admissible, by [25, Lemma 9.3], there exist attaching sets $\overline{\boldsymbol{\alpha}}$, $\overline{\boldsymbol{\beta}} \subset \Sigma$ such that the quadruple diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ and $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ are both admissible. Then we obtain an isomorphism

$$\Phi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}_{\boldsymbol{\beta}\to\boldsymbol{\beta}'} := \Psi^{\overline{\boldsymbol{\alpha}}\to\boldsymbol{\alpha}'}_{\overline{\boldsymbol{\beta}}\to\boldsymbol{\beta}'} \circ \Psi^{\boldsymbol{\alpha}\to\overline{\boldsymbol{\alpha}}}_{\boldsymbol{\beta}\to\overline{\boldsymbol{\beta}}}$$

from $SFH(\mathcal{H})$ to $SFH(\mathcal{H}')$ that is independent of the choice of $\overline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\beta}}$. Furthermore, when $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is admissible, then $\Phi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}_{\boldsymbol{\beta}\to\boldsymbol{\beta}'} = \Psi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}_{\boldsymbol{\beta}\to\boldsymbol{\beta}'}$.

We show in [25, Section 9] that all the isomorphisms above are functorial in the obvious way. For example, if the diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}')$, and $(\Sigma, \boldsymbol{\alpha}'', \boldsymbol{\beta}'')$ are all admissible, then

$$\Phi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}''}_{\boldsymbol{\beta}\to\boldsymbol{\beta}''} = \Phi^{\boldsymbol{\alpha}'\to\boldsymbol{\alpha}''}_{\boldsymbol{\beta}'\to\boldsymbol{\beta}''} \circ \Phi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}'}_{\boldsymbol{\beta}\to\boldsymbol{\beta}'},$$

and $\Phi^{\boldsymbol{\alpha}\to\boldsymbol{\alpha}}_{\boldsymbol{\beta}\to\boldsymbol{\beta}}$ is the identity.

Given a diffeomorphism $d\colon \mathcal{H} \to \mathcal{H}'$ between the diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$, we defined an isomorphism

$$d_*\colon SFH(\mathcal{H}) \to SFH(\mathcal{H}')$$

in [25, Definition 9.19]. For this end, choose a perturbation $(\mathfrak{j}, J_s)$ for $\mathcal{H}$. We can push this forward along d to obtain a perturbation $(\mathfrak{j}', J_s')$ for $\mathcal{H}'$. Then d induces a tautological isomorphism

$$d_{J_s,J_s'}\colon SFH_{J_s}(\mathcal{H}) \to SFH_{J_s'}(\mathcal{H}').$$

These isomorphisms then descend to the limits along the different perturbations, giving rise to d_*. When $\mathcal{H}$ and $\mathcal{H}'$ are both diagrams of (M, γ) and d is isotopic to the identity in M, we let $F_{\mathcal{H},\mathcal{H}'}$ be d_*.

Finally, suppose that the diagram $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is obtained from $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ via a stabilization. Then $\boldsymbol{\alpha}' = \boldsymbol{\alpha} \cup \{\alpha\}$ and $\boldsymbol{\beta}' = \boldsymbol{\beta} \cup \{\beta\}$, where $\alpha \cap \beta$ consists of a single point c. The map

$$\sigma_{\mathcal{H}\to\mathcal{H}'}\colon CF(\mathcal{H}) \to CF(\mathcal{H}')$$

mapping $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ to $\mathbf{x} \times \{c\} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ is a chain map for a suitable choice of complex structures on Σ and Σ', and induces an isomorphism $F_{\mathcal{H},\mathcal{H}'}$ on homology. When $\mathcal{H}'$ is obtained from $\mathcal{H}$ via a destabilization, we let $F_{\mathcal{H},\mathcal{H}'} = F^{-1}_{\mathcal{H}',\mathcal{H}}$.

As explained above, given any pair of admissible diagrams $\mathcal{H}$, $\mathcal{H}' \in \mathcal{H}(M, \gamma)$, we obtain a "canonical" isomorphism

$$F_{\mathcal{H},\mathcal{H}'}\colon SFH(\mathcal{H}) \to SFH(\mathcal{H}')$$

by composing the elementary isomorphisms associated to an arbitrary sequence of Heegaard moves connecting $\mathcal{H}$ and $\mathcal{H}'$.

Now we define the diffeomorphism action on sutured Floer homology. Consider a diffeomorphism $\phi\colon (M,\gamma) \to (M',\gamma')$, and pick an admissible diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for (M,γ). Let $d = \phi|_\Sigma$, then $\mathcal{H}' = d(\mathcal{H})$ is an admissible diagram of (M',γ'). We define $\phi_*\colon SFH(M,\gamma) \to SFH(M',\gamma')$ to be $(P_{\mathcal{H}'})^{-1} \circ d_* \circ P_{\mathcal{H}}$. In other words, the following diagram is commutative:

$$\begin{array}{ccc} SFH(M,\gamma) & \xrightarrow{\phi_*} & SFH(M',\gamma') \\ \downarrow{\scriptstyle P_{\mathcal{H}}} & & \downarrow{\scriptstyle P_{\mathcal{H}'}} \\ SFH(\mathcal{H}) & \xrightarrow{d_*} & SFH(\mathcal{H}'). \end{array}$$

We conclude this section with a lemma that will be useful later on.

Lemma 5.21. *Let $d\colon \Sigma \to \Sigma'$ be a diffeomorphism mapping the admissible sutured multi-diagram $(\Sigma, \eta^0, \dots, \eta^n)$ to $(\Sigma', \nu^0, \dots, \nu^n)$. For every $i \in \mathbb{Z}_n$, this induces an isomorphism*

$$d^i_*\colon SFH(\Sigma, \eta^i, \eta^{i+1}) \to SFH(\Sigma', \nu^i, \nu^{i+1}).$$

Then, for every $x_i \in SFH(\Sigma, \eta^i, \eta^{i+1})$ for $i \in \{\,0, \dots, n-1\,\}$, we have

$$d^n_*\left(F_{\eta^0,\dots,\eta^n}(x_0 \otimes \cdots \otimes x_{n-1})\right) = F_{\nu^0,\dots,\nu^n}\left(d^0_*(x_0) \otimes \cdots \otimes d^{n-1}_*(x_{n-1})\right).$$

Proof. Choose a complex structure $\mathfrak{j}$ on Σ, and let $\mathfrak{j}' = d_*(\mathfrak{j})$, together with corresponding 1-parameter perturbations J_s on $\mathrm{Sym}^k(\Sigma)$ and J'_s on $\mathrm{Sym}^k(\Sigma')$. Then $\mathrm{Sym}^k(d)$ is a symplectomorphism from $\mathrm{Sym}^k(\Sigma)$ to $\mathrm{Sym}^k(\Sigma')$ that maps the Lagrangians $\mathbb{T}_{\eta^0}, \dots, \mathbb{T}_{\eta^n}$ to $\mathbb{T}_{\nu^0}, \dots, \mathbb{T}_{\nu^n}$, respectively, and intertwines the almost complex structures J_s and J'_s. Hence, the statement becomes a tautology. □

6. The map associated with a framed link

Here, we generalize the work of Ozsváth and Szabó [38, Section 4]. Some of the following notions already appeared in the paper of Grigsby and Wehrli [12, Section 4] for framed links in product sutured manifolds, in the context of a link surgery spectral sequence.

Definition 6.1. A *framed link* $\mathbb{L}$ in a sutured manifold (M,γ) is a collection of n pairwise disjoint, smoothly embedded circles $K_1, \dots, K_n \subset \mathrm{Int}(M)$, together with a choice of homology classes $\ell_i \in H_1(\partial N(K_i))$ satisfying $m_i \cdot \ell_i = 1$, where m_i is the meridian of K_i.

By attaching 2-handles along the components of the framed link $\mathbb{L}$, we naturally get a special cobordism $W(\mathbb{L})$ from (M,γ) to a sutured manifold $(M(\mathbb{L}),\gamma)$. Note that $M(\mathbb{L})$ is obtained by surgery along $\mathbb{L}$, and γ is left unchanged. We call the cobordism $W(\mathbb{L})$ the *trace* of the surgery.

For every framed link $\mathbb{L}$ in (M,γ) and Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(W(\mathbb{L}))$, we are going to define maps

$$F_{M,\mathbb{L}}\colon SFH(M,\gamma) \to SFH(M(\mathbb{L}),\gamma) \text{ and}$$
$$F_{M,\mathbb{L},\mathfrak{s}}\colon SFH(M,\gamma,\mathfrak{s}|_M) \to SFH\left(M(\mathbb{L}),\gamma,\mathfrak{s}|_{M(\mathbb{L})}\right).$$

Definition 6.2. A *bouquet $B(\mathbb{L})$ for the link* $\mathbb{L}$ is a 1-complex embedded in M which is the union of $\mathbb{L}$ with a collection of arcs $a_1, \dots, a_n$, such that for every $i \in \{\,1, \dots, n\,\}$, the arc a_i connects K_i and $R_+(\gamma)$. We denote the punctured torus $\partial N(K_i) \setminus N(a_i)$ by F_i.

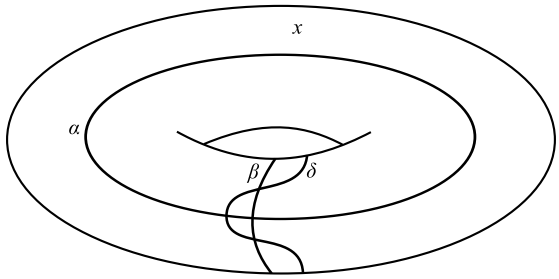

FIGURE 3. A proper stabilization of a triple diagram is obtained by taking the connected sum with the above diagram at the point marked by x.

Definition 6.3. A *sutured triple diagram subordinate to the bouquet* $B(\mathbb{L})$ is a triple diagram

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}) = (\Sigma, \{\, \alpha_1, \dots, \alpha_d \,\}, \{\, \beta_1, \dots, \beta_d \,\}, \{\, \delta_1, \dots, \delta_d \,\}),$$

such that

(1) the triple $(\Sigma, \{\, \alpha_1, \dots, \alpha_d \,\}, \{\, \beta_{n+1}, \dots, \beta_d \,\})$ is a diagram of the sutured manifold $(M', \gamma') = (M \setminus N(B(\mathbb{L})), \gamma)$ (in particular, $\Sigma \subset M'$),
(2) the curves $\delta_{n+1}, \dots, \delta_d$ are small isotopic translates of $\beta_{n+1}, \dots, \beta_d$,
(3) after compressing Σ along $\beta_{n+1}, \dots, \beta_d$, for $i \in \{\, 1, \dots, n \,\}$, the induced curves β_i and δ_i on $R_+(\gamma')$ lie in the punctured torus F_i,
(4) for $i \in \{\, 1, \dots, n \,\}$, the curve β_i represents a meridian of K_i that is disjoint from all the δ_j for $j \neq i$, and meets δ_i in a single transverse intersection point,
(5) for $i \in \{\, 1, \dots, n \,\}$, the homology class of δ_i corresponds to the framing ℓ_i.

Definition 6.4. By a *stabilization* of a triple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ subordinate to some bouquet, we mean the following. Take the connected sum of $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ with a diagram $(E, \alpha_{d+1}, \beta_{d+1}, \delta_{d+1})$, where $E \subset M'$ is a genus one surface, $|\alpha_{d+1} \cap \beta_{d+1}| = 1$, and δ_{d+1} is a small isotopic translate of β_{d+1} such that $|\beta_{d+1} \cap \delta_{d+1}| = 2$, see Figure 3. Furthermore, we say that a stabilization is *proper* if the connected sum tube joins a component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{\delta})$ that intersects $\partial\Sigma$ nontrivially with the component of $E \setminus (\alpha_{d+1} \cup \beta_{d+1} \cup \delta_{d+1})$ disjoint from the isotopy connecting β_{d+1} and δ_{d+1}. A *(proper) destabilization* is the reverse of a (proper) stabilization.

The following generalizes the corresponding result of Ozsváth and Szabó [38, Lemma 4.5].

Lemma 6.5. *Let (M, γ) be a balanced sutured manifold, together with a framed link $\mathbb{L} \subset M$ and associated bouquet $B(\mathbb{L})$. Then there is a sutured triple diagram subordinate to $B(\mathbb{L})$, and any two such triple diagrams can be connected by a sequence of the following moves:*

(1) isotopies and handleslides amongst $\{\, \alpha_1, \dots, \alpha_d \,\}$,
(2) isotopies and handleslides amongst $\{\, \beta_{n+1}, \dots, \beta_d \,\}$, while carrying along the curves $\delta_{n+1}, \dots, \delta_d$, as well,
(3) proper stabilizations and destabilizations,

(4) *for $i \in \{1, \dots, n\}$, an isotopy of β_i, or a handleslide of β_i across a β_j with $j \in \{n+1, \dots, d\}$,*
(5) *for $i \in \{1, \dots, n\}$, an isotopy of δ_i, or a handleslide of δ_i across a δ_j with $j \in \{n+1, \dots, d\}$*
(6) *a diffeomorphism isotopic to the identity in M'.*

Proof. By the work of the author [19, Proposition 2.13], there exists a sutured diagram

$$(\Sigma, \{\alpha_1, \dots \alpha_d\}, \{\beta_{n+1}, \dots, \beta_d\})$$

defining the sutured manifold $(M', \gamma') = (M \setminus N(B(\mathbb{L})), \gamma)$. Using [25, Proposition 2.37], any two sutured diagrams defining (M', γ') can be connected by isotopies and handleslides of the α- and β-curves, diffeomorphisms of the Heegaard surface isotopic to the identity in M', stabilizations, and destabilizations. To see that proper stabilizations suffice (i.e., stabilizations in a region intersecting $\partial\Sigma$), note that we can obtain an arbitrary stabilization by isotoping the α- and β-curves via a finger move along an arc a connecting $\partial\Sigma$ with the stabilization point, performing a proper stabilization, followed by a sequence of handleslides along the inverse of a. The curves $\delta_{n+1}, \dots, \delta_d$ are chosen to be small translates of $\beta_{n+1}, \dots, \beta_d$, respectively. It follows from the above discussion that any two such triple diagrams

$$(\Sigma, \{\alpha_1, \dots \alpha_d\}, \{\beta_{n+1}, \dots, \beta_d\}, \{\delta_{n+1}, \dots, \delta_d\})$$

can be connected by moves (1), (2), (3), and (6).

Since Σ surgered along $\beta_{n+1}, \dots, \beta_d$ is canonically diffeomorphic to $R_+(\gamma')$, parts (3)–(5) of Definition 6.3 prescribe how to choose $\beta_1, \dots, \beta_n$ and $\delta_1, \dots, \delta_n$. For $i \in \{1, \dots, n\}$, the framed link specifies the homology classes of β_i and δ_i in F_i. Different choices β_i and β_i' can be connected by an isotopy in F_i. It follows that in Σ they can be connected by a sequence of isotopies and handleslides across the curves $\beta_{n+1}, \dots, \beta_d$. The same argument works for $\delta_1, \dots, \delta_n$. These give rise to moves (4) and (5). □

The following proposition is a generalization of Ozsváth and Szabó [38, Proposition 4.3].

Proposition 6.6. *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ be a triple diagram subordinate to the bouquet $B(\mathbb{L})$ in (M, γ).*

(1) *$\mathcal{W}_{\alpha,\beta,\delta}$ is a cobordism from (M, γ) to the disjoint union of $(M(\mathbb{L}), \gamma)$ and*
$$(M_{\beta,\delta}, \gamma_{\beta,\delta}) \approx (R_+ \times I, \partial R_+ \times I)\#^{d-n}(S^2 \times S^1),$$
where $R_+ = R_+(\gamma)$, and different copies of $S^2 \times S^1$ might be summed along different components of $R_+ \times I$.
(2) *The Heegaard surface Σ divides (M, γ) into the sutured compression bodies U_α and U_β, such that the framed link $\mathbb{L} \subset U_\beta$. The sutured monodiagram $(\Sigma, \boldsymbol{\beta})$ defines a cobordism $\mathcal{W}_\beta$ from $U_\beta \cup_\Sigma (-U_\beta)$ to $\emptyset$. Let $\mathcal{W}_\beta(\mathbb{L})$ be the cobordism obtained from $\mathcal{W}_\beta$ by attaching 4-dimensional 2-handles to $\mathbb{L}$ along U_β. Then $\mathcal{W}_\beta(\mathbb{L})$ is a cobordism from $(M_{\beta,\delta}, \gamma_{\beta,\delta})$ to $\emptyset$. Then we can glue $\mathcal{W}_{\alpha,\beta,\delta}$ and $\mathcal{W}_\beta(\mathbb{L})$ along $(M_{\beta,\delta}, \gamma_{\beta,\delta})$ such that we obtain a cobordism equivalent to $W(\mathbb{L})$, and this equivalence is well-defined up to isotopy.*

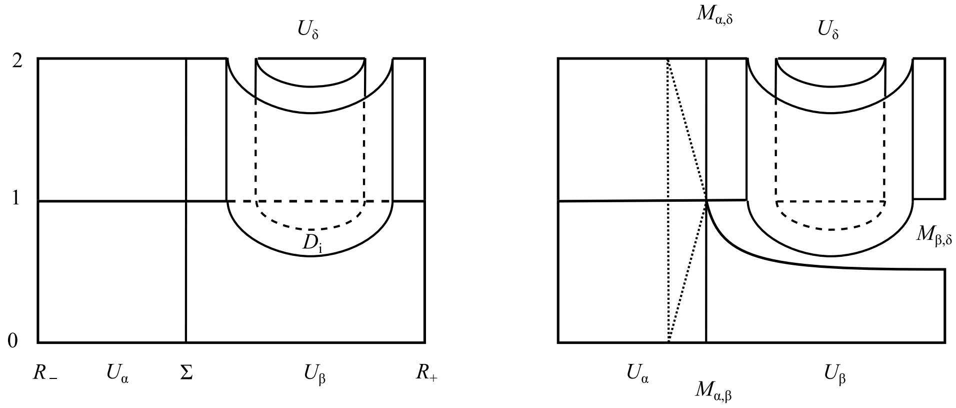

FIGURE 4. The cobordism $\mathcal{W}(\mathbb{L})'$ on the left, and $\mathcal{W}(\mathbb{L})''$ on the right.

Proof. It follows from part (4) of Definition 6.3 that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a diagram of the sutured manifold (M, γ). Furthermore, by part (5), the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta})$ defines the surgered manifold $(M(\mathbb{L}), \gamma)$, since we glue 3-dimensional 2-handles to the complement (M', γ') of the bouquet $B(\mathbb{L})$ along curves specified by the framing of $\mathbb{L}$. Finally, let Σ_β be Σ surgered along $\boldsymbol{\beta}$. Then the diagram $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\delta})$ defines

$$(\Sigma_\beta \times I, \partial\Sigma_\beta \times I) \,\#^n S^3 \#^{d-n} \left(S^2 \times S^1\right),$$

where β_i and δ_i give the S^3 components for $i \in \{1, \dots, n\}$, and the $S^2 \times S^1$ components for $i \in \{n+1, \dots, d\}$. Note that different copies of S^3 or $S^2 \times S^1$ might be added to different components of $\Sigma_\beta \times I$. However, $\Sigma_\beta \approx R_+(\gamma)$, which concludes the proof of (1).

We now prove (2); i.e., that we can glue $\mathcal{W}_\beta(\mathbb{L})$ to $\mathcal{W}_{\alpha,\beta,\delta}$ such that we obtain $W(\mathbb{L})$. Let $\mathcal{W}_{\alpha,\beta,\delta} = (W, Z, [\xi])$ and $\mathcal{W}_\beta(\mathbb{L}) = (W', Z', [\xi'])$. As usual, P_3 denotes a regular triangle, and we label its edges e_α, e_β, and e_δ in a clockwise fashion. Furthermore, U_α, U_β, and U_δ are the sutured compression bodies corresponding to $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\delta}$, respectively. Recall that W is obtained from $P_3 \times \Sigma$ by gluing $e_\alpha \times U_\alpha$, $e_\beta \times U_\beta$, and $e_\delta \times U_\delta$ along $e_\alpha \times \Sigma$, $e_\beta \times \Sigma$, and $e_\delta \times \Sigma$, respectively, and smoothing corners.

Let P_1 be a monogon with edge e. Then W_β is obtained from $\Sigma \times P_1$ by gluing $e \times U_\beta$ along $e \times \Sigma$, and smoothing corners. The only end of this cobordism is obtained by gluing two copies of U_β, namely the components of $\partial e \times U_\beta$, along Σ. Then we attach 4-dimensional 2-handles along $\mathbb{L}$ in one of the U_β components to obtain W'. An alternative way to describe W' is the following. Take $U_\beta \times I$, and attach 4-dimensional 2-handles to $U_\beta \times \{1\}$ along $\mathbb{L} \times \{1\}$, obtaining the 4-manifold $(U_\beta \times I)(\mathbb{L})$. The top boundary $U_\beta \times \{1\}$ becomes $U_\beta(\mathbb{L}) = U_\delta$. After smoothing corners,

$$(U_\beta \times \{0\}) \cup (\Sigma \times I) \cup U_\delta$$

becomes diffeomorphic to $M_{\beta,\delta}$.

Recall that the cobordism $W(\mathbb{L})$ is obtained from $M \times I$ by attaching 4-dimensional 2-handles $D_1, \dots, D_n$ to $M \times \{1\}$ along the components of the framed link $\mathbb{L} \times \{1\}$. Note that

$$W(\mathbb{L}) = (U_\alpha \times I) \cup_{\Sigma \times I} (U_\beta \times I)(\mathbb{L}).$$

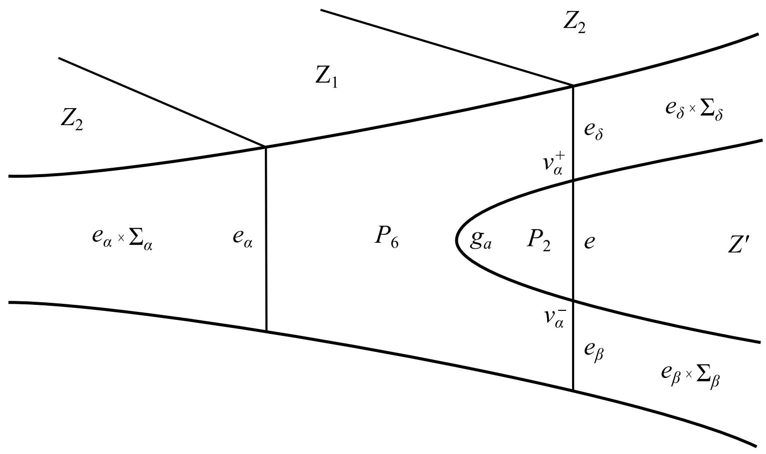

FIGURE 5. A section of the manifold $Z \cup Z'$.

There is an isotopically unique equivalence from $W(\mathbb{L})$ to the cobordism $\mathcal{W}(\mathbb{L})'$ obtained by gluing $W(\mathbb{L})$ and the trivial cobordism $M(\mathbb{L}) \times [1,2]$ along $M(\mathbb{L}) \times \{1\}$, see the left-hand side of Figure 4. Let $N(\mathbb{L})$ be the tubular neighborhood of $\mathbb{L}$ in U_β used for attaching $D_1, \dots, D_n$, and denote by $E(\mathbb{L})$ the link exterior $U_\beta \setminus \text{Int}(N(\mathbb{L}))$. If we remove the interiors of $D_1, \dots, D_n$ from $W(\mathbb{L})'$, then cut the resulting manifold along $E(\mathbb{L}) \times \{1\}$, and finally smooth the resulting corner at $\Sigma \times \{1\}$, we obtain a cobordism $\mathcal{W}(\mathbb{L})''$ that is isotopically uniquely equivalent to $\mathcal{W}_{\alpha,\beta,\delta}$. Indeed, $\mathcal{W}(\mathbb{L})''$ is obtained by taking $U_\alpha \times [0,2]$, and gluing $U_\beta \times [0,1]$ along $\Sigma \times [0,1]$ and $U_\delta \times [1,2]$ along $\Sigma \times [1,2]$. For an illustration of $\mathcal{W}(\mathbb{L})''$, see the right-hand side of Figure 4. The dotted triangle indicates the identification with $\mathcal{W}_{\alpha,\beta,\delta}$.

We also obtain $\mathcal{W}(\mathbb{L})''$ from $\mathcal{W}(\mathbb{L})'$ if we remove the interior of

$$(U_\beta \times [1-\varepsilon, 1]) \cup D_1 \cup \dots \cup D_n,$$

and the latter manifold is diffeomorphic to $(U_\beta \times I)(\mathbb{L})$, which we have shown to agree with W'. So indeed, we can glue W' to W along $M_{\beta,\delta}$ such that we obtain a manifold diffeomorphic to $W(\mathbb{L})$.

Next, we check that the diffeomorphism from $W \cup W'$ to $W(\mathbb{L})$ constructed above maps $(Z \cup Z', [\xi \cup \xi'])$ to $(\partial M \times I, [\zeta])$, where ζ is an I-invariant contact structure such that every $\partial M \times \{t\}$ is a convex surface with dividing set γ. This will conclude the proof of $\mathcal{W}_{\alpha,\beta,\delta} \cup \mathcal{W}_\beta(\mathbb{L}) = W(\mathbb{L})$.

Recall that $Z = Z_1 \cup Z_2$, where $Z_1 = P_6 \times \partial\Sigma$ and

$$Z_2 = (e_\alpha \times \Sigma_\alpha) \cup (e_\beta \times \Sigma_\beta) \cup (e_\delta \times \Sigma_\delta),$$

see Figure 5. The contact structure ξ is given by a hierarchy that starts with decomposing along a set of product annuli $A \subset Z_2$ parallel to

$$(e_\alpha \times \partial\Sigma_\alpha) \cup (e_\beta \times \partial\Sigma_\beta) \cup (e_\delta \times \partial\Sigma_\delta),$$

then along surfaces $P_6 \times \{q\}$, for one q in each component of Z_1.

Similarly, $Z' = Z_1' \cup Z_2'$, where $Z_1' = P_2 \times \partial\Sigma$ and $Z_2' = e \times \Sigma_\beta$. Furthermore, ξ' is defined by decomposing the sutured manifold $(Z', \gamma_{\beta,\delta})$ along product annuli A' parallel to $e \times \partial R_+$, after which we get Z_1', a union of tori with two longitudinal sutures on each, and Z_2', a product sutured manifold. Hence Z' is the manifold

$\Sigma_\beta \times I$ with ξ' being the canonical I-invariant contact structure. In particular, $(Z', \gamma_{\beta,\delta})$ is diffeomorphic to the product sutured manifold $(\Sigma_\beta \times I, \partial\Sigma_\beta \times I)$.

Let g_α be the edge of P_6 lying between e_β and e_δ, and let $e_\beta \cap g_\alpha = v_\alpha^-$ and $e_\delta \cap g_\alpha = v_\alpha^+$. The surface Σ_δ is naturally identified with Σ_β, and Z' is glued to Z along

$$(\{v_\alpha^-\} \times \Sigma_\beta) \cup (\{v_\alpha^+\} \times \Sigma_\delta) \cup (g_\alpha \times \partial\Sigma)$$

using this identification. More precisely, $R_-(Z', \gamma_{\beta,\delta}) = \Sigma_\beta \times \{0\}$ is glued to $\{v_\alpha^-\} \times \Sigma_\beta$ and $R_+(Z', \gamma_{\beta,\delta}) = \Sigma_\beta \times \{1\}$ is glued to $\{v_\alpha^+\} \times \Sigma_\delta$, whereas $\gamma_{\beta,\delta}$ is glued to $g_\alpha \times \partial\Sigma$. It follows that $Z \cup Z'$ is diffeomorphic to $(\Sigma_\alpha \cup \gamma_{\alpha,\beta} \cup \Sigma_\beta) \times I = \partial M \times I$.

The dividing set of $\xi \cup \xi'$ on $\partial M \times \{i\}$ is $s(\gamma_{\alpha,\beta}) \times \{i\}$ for both $i = 0$ and $i = 1$. The product annuli A and A' for (Z, γ) and $(Z', \gamma_{\beta,\delta})$ glue up to a set of product annuli $A \cup A'$ inside $\partial M \times I$. After decomposing $\partial M \times I$ along $A \cup A'$, we get a product sutured manifold diffeomorphic to

$$(\Sigma_\alpha \times I, \partial\Sigma_\alpha \times I) \cup (\gamma_{\alpha,\beta} \times I, \partial\gamma_{\alpha,\beta} \times I) \cup (\Sigma_\beta \times I, \partial\Sigma_\beta \times I).$$

The decomposing surfaces $P_6 \times \{q\}$ and $P_2 \times \{q\}$ glue together to give product disks inside $(\gamma_{\alpha,\beta} \times I, \partial\gamma_{\alpha,\beta} \times I)$. Hence $\xi \cup \xi'$ is given by a hierarchy which starts with the product annuli $A \cup A'$, and continues with decompositions along product disks, and is consequently equivalent to an I-invariant contact structure. Since the dividing set on $\partial M \times \{0\}$ is $s(\gamma_{\alpha,\beta}) \times \{0\}$, we have $\xi \cup \xi' = \zeta$.

Alternatively, using the description of ξ in the proof of Lemma 5.4, we see that the 2-pane field $\xi \cup \xi'$ is never opposite to the 2-plane field that is tangent to the product foliation on $\Sigma_\alpha \times I$ and $\Sigma_\beta \times I$, and rotates π as we traverse $\gamma_{\alpha,\beta} \times \{t\}$ from Σ_α to Σ_β. Hence $\xi \cup \xi'$ is equivalent to a contact structure such that $\partial M \times \{t\}$ is convex for every $t \in I$ with dividing set $s(\gamma_{\alpha,\beta}) \times \{t\}$. □

Using part (2) of Proposition 6.6, we get a restriction map

$$r\colon \mathrm{Spin}^c(W(\mathbb{L})) \to \underline{\mathrm{Spin}}^c(\mathcal{W}_{\alpha,\beta,\delta}).$$

To define $r(\mathfrak{s})$ for $\mathfrak{s} \in \mathrm{Spin}^c(W(\mathbb{L}))$, first choose an I-invariant contact structure η on $\partial M \times I \subset W(\mathbb{L})$ such that each $\partial M \times \{t\}$ is a convex surface with dividing set $\gamma \times \{t\}$. Then pick an almost complex structure J' on $T(M \times I)|_{\partial M \times I}$ such that the field of complex tangencies $\xi_{J'} = \eta$. Since $W(\mathbb{L})$ is a special cobordism, the map

$$q\colon \mathrm{Spin}^c(W(\mathbb{L}), J') \to \mathrm{Spin}^c(W(\mathbb{L}))$$

is an affine isomorphism. We take $q^{-1}(\mathfrak{s}) \in \mathrm{Spin}^c(W(\mathbb{L}), J')$, and restrict it to the complement of

$$(U_\beta \times [1-\varepsilon, 1]) \cup D_1 \cup \dots \cup D_n,$$

which we identified with $\mathcal{W}_{\alpha,\beta,\delta}$. The result in $\underline{\mathrm{Spin}}^c(\mathcal{W}_{\alpha,\beta,\delta})$ is independent of the choices made. This construction will enable us to define maps on *SFH* induced by cobordisms equipped with Spin^c structures.

By the connected sum formula [19, Proposition 9.15], we have

$$SFH\left((R_+ \times I, \partial R_+ \times I) \# \left(\#^{d-n}(S^2 \times S^1)\right)\right) \cong \Lambda^* H^1\left(\#^{d-n}(S^1 \times S^2)\right).$$

This is supported in the Spin^c structure $\mathfrak{s}_0$ that is characterized by

(1) the restriction of $\mathfrak{s}_0$ to $(R_+ \times I, \partial R_+ \times I)$ is homologous to the 2-plane field tangent to the horizontal foliation,
(2) the restriction of $\mathfrak{s}_0$ to $\#^{d-n}(S^1 \times S^2)$ extends to $\natural^{d-n}(S^1 \times D^3)$, or equivalently, its first Chern class vanishes.

We introduce the shorthand

$$M(R_+, d-n) = (R_+ \times I, \partial R_+ \times I) \# \left(\#^{d-n}(S^2 \times S^1)\right).$$

Then $SFH(M(R_+, d-n))$ with the relative Maslov grading is isomorphic to

$$\Lambda^* H^1\left(\#^{d-n}(S^1 \times S^2); \mathbb{Z}_2\right).$$

So the "top-dimensional" part of $SFH(M(R_+, d-n))$ is $\mathbb{Z}_2$, whose generator we denote by Θ.

Lemma 6.7. *Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ be a triple diagram subordinate to the bouquet $B(\mathbb{L})$ in (M, γ). Furthermore, let $\mathfrak{s} \in Spin^c(W(\mathbb{L}))$ be an arbitrary $Spin^c$ structure. Then, using the identification between $(M_{\beta,\delta}, \gamma_{\beta,\delta})$ and $M(R_+, d-n)$ in part (1) of Proposition 6.6, we have*

$$r(\mathfrak{s})|_{(M_{\beta,\delta},\gamma_{\beta,\delta})} = r(\mathfrak{s})|_{M(R_+,d-n)} = \mathfrak{s}_0.$$

Proof. To see that $r(\mathfrak{s})|_{M(R_+,d-n)}$ satisfies condition (1) characterizing $\mathfrak{s}_0$ above, notice that the $(R_+ \times I, \partial R_+ \times I)$ component of $M(R_+, d-n)$ is parallel to $(Z', \gamma_{\beta,\delta}) \subset \partial M \times I$ in the proof of Proposition 6.6, which carries the "horizontal" Spin^c structure.

Now we check property (2). Recall from the proof of Proposition 6.6 that $M_{\beta,\delta}$ is embedded in $U_\beta(\mathbb{L})$ as

$$(U_\beta \times \{0\}) \cup (\Sigma \times I) \cup U_\delta.$$

Observe that $r(\mathfrak{s})$ extends to $U_\beta(\mathbb{L})$, as we obtain $W(\mathbb{L})$ from $\mathcal{W}_{\alpha,\beta,\delta}$ by gluing $U_\beta(\mathbb{L})$ to $(M_{\beta,\delta}, \gamma_{\beta,\delta})$ along the above subset of its boundary. Let C_i be the core of the 3-dimensional 2-handle in U_β glued to $\Sigma \times I$ along $\beta_i \times \{0\}$, together with the annulus $\beta_i \times I$. Then $C_i \times I$ is a 3-ball in $U_\beta(\mathbb{L})$ with boundary a 2-sphere that is isotopic to $\{pt\} \times S^2$ in the i-th $S^1 \times S^2$ component of $(M_{\beta,\delta}, \gamma_{\beta,\delta}) \approx M(R_+, d-n)$. Since $r(\mathfrak{s})$ extends to $C_i \times I$, we see that $c_1(\mathfrak{s}_0)$ vanishes on this $S^1 \times S^2$ component. □

Definition 6.8. Let $\mathbb{L}$ be a framed link in (M, γ), and fix a Spin^c structure $\mathfrak{s} \in$ Spin$^c(W(\mathbb{L}))$. Then we define maps

$$F_{M,\mathbb{L}} \colon SFH(M, \gamma) \to SFH(M(\mathbb{L}), \gamma) \text{ and}$$
$$F_{M,\mathbb{L},\mathfrak{s}} \colon SFH\left(M, \gamma, \mathfrak{s}|_M\right) \to SFH\left(M(\mathbb{L}), \gamma, \mathfrak{s}|_{M(\mathbb{L})}\right),$$

as follows.

Pick a bouquet $B(\mathbb{L})$ for $\mathbb{L}$, and an admissible triple diagram $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ subordinate to this bouquet. As above, Θ is the generator of the top-dimensional part of

$$SFH(\Sigma, \boldsymbol{\beta}, \boldsymbol{\delta}, \mathfrak{s}_0) \cong SFH(M(R_+, d-n), \mathfrak{s}_0).$$

Let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, which is a diagram of (M, γ), and $\mathcal{H}_\mathbb{L} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta})$, which is a diagram of $(M(\mathbb{L}), \gamma)$. According to Definition 5.13, we have a triangle map

$$F_{\alpha,\beta,\delta} \colon SFH(\mathcal{H}) \otimes SFH(\Sigma, \boldsymbol{\beta}, \boldsymbol{\delta}) \to SFH(\mathcal{H}_\mathbb{L}).$$

We define the map $E_\mathcal{T} \colon SFH(\mathcal{H}) \to SFH(\mathcal{H}_\mathbb{L})$ via $E_\mathcal{T}(x) = F_{\alpha,\beta,\delta}(x \otimes \Theta)$. By Lemma 6.7, this can be refined using Spin$^c(W(\mathbb{L}))$, and we define

$$E_{\mathcal{T},\mathfrak{s}} \colon SFH\left(\mathcal{H}, \mathfrak{s}|_M\right) \to SFH\left(\mathcal{H}_\mathbb{L}, \mathfrak{s}|_{M(\mathbb{L})}\right)$$

via $E_{\mathcal{T},\mathfrak{s}}(x) = F_{\alpha,\beta,\delta}(x \otimes \Theta, r(\mathfrak{s}))$. Then

$$F_{M,\mathbb{L}} = P_{\mathcal{H}_\mathbb{L}}^{-1} \circ E_\mathcal{T} \circ P_\mathcal{H},$$

where $P_{\mathcal{H}}\colon SFH(M,\gamma) \to SFH(\mathcal{H})$ and $P_{\mathcal{H}_{\mathbb{L}}}\colon SFH(M(\mathbb{L}),\gamma) \to SFH(\mathcal{H}_{\mathbb{L}})$ are the natural isomorphisms introduced in Section 5.2. Similarly, we let

$$F_{M,\mathbb{L},\mathfrak{s}} = P_{\mathcal{H}_{\mathbb{L}}}^{-1} \circ E_{\mathcal{T},\mathfrak{s}} \circ P_{\mathcal{H}}.$$

The following theorem ensures that the above definition is independent of the choice of bouquet and subordinate triple diagram.

Theorem 6.9. *Let (M,γ) be a balanced sutured manifold equipped with a framed link $\mathbb{L}$, and let $\mathfrak{s} \in Spin^c(W(\mathbb{L}))$. Suppose $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ and $\mathcal{T}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$ are admissible triple diagrams subordinate to bouquets B and B' for $\mathbb{L}$, respectively, and let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, $\mathcal{H}_{\mathbb{L}} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta})$, $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$, and $\mathcal{H}'_{\mathbb{L}} = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\delta}')$. Then we have a commutative diagram*

$$\begin{array}{ccc} SFH\left(\mathcal{H}, \mathfrak{s}|_M\right) & \xrightarrow{E_{\mathcal{T},\mathfrak{s}}} & SFH\left(\mathcal{H}_{\mathbb{L}}, \mathfrak{s}|_{M(\mathbb{L})}\right) \\ \big\downarrow F_{\mathcal{H},\mathcal{H}'} & & \big\downarrow F_{\mathcal{H}_{\mathbb{L}},\mathcal{H}'_{\mathbb{L}}} \\ SFH\left(\mathcal{H}', \mathfrak{s}|_M\right) & \xrightarrow{E_{\mathcal{T}',\mathfrak{s}}} & SFH\left(\mathcal{H}'_{\mathbb{L}}, \mathfrak{s}|_{M(\mathbb{L})}\right), \end{array} \tag{6.1}$$

where $F_{\mathcal{H},\mathcal{H}'}$ and $F_{\mathcal{H}_{\mathbb{L}},\mathcal{H}'_{\mathbb{L}}}$ are the canonical isomorphisms defined in Section 5.2. An analogous statement holds for $E_{\mathcal{T}}$ and $E_{\mathcal{T}'}$.

Proof. We follow the proof of [38, Theorem 4.4]. The most important new point is that for naturality reasons we also consider the embeddings of Σ and Σ' in M.

First, assume that $B = B'$. Then $\mathcal{T}$ and $\mathcal{T}'$ can be connected by a sequence of moves (1)–(6) of Lemma 6.5. It suffices to check that diagram (6.1) is commutative when the two triples differ by exactly one of these moves. Indeed, the general case follows by writing down a ladder where each small rectangle corresponds to an elementary move and is hence commutative, while the composition of the maps along the two vertical sides give the canonical isomorphisms $F_{\mathcal{H},\mathcal{H}'}$ and $F_{\mathcal{H}_{\mathbb{L}},\mathcal{H}'_{\mathbb{L}}}$ as the composition of any sequence of canonical isomorphisms is again a canonical isomorphism.

First, we check invariance under move (3), a proper stabilization, cf. Definition 6.4. Just as in [38, Lemma 4.7], we obtain that the maps $E_{\mathcal{T},\mathfrak{s}}$ commute with the isomorphisms induced by proper stabilizations. The argument is somewhat simpler in our case, since we are stabilizing near $\partial\Sigma$ and holomorphic discs avoid the boundary, which make neck-stretching unnecessary. Indeed, we obtain $\mathcal{T}'$ by taking the connected sum of $\mathcal{T}$ in a region that intersects $\partial\Sigma$ with the diagram $(E, \alpha_{d+1}, \beta_{d+1}, \delta_{d+1})$ at the point $x \in E$, see Figure 3. Let $\alpha_{d+1} \cap \beta_{d+1} = \{x_{d+1}\}$, $\alpha_{d+1} \cap \delta_{d+1} = \{y_{d+1}\}$, and let w_{d+1} be the point of $\beta_{d+1} \cap \delta_{d+1}$ of higher relative grading. If $\theta \in \mathbb{T}_\beta \cap \mathbb{T}_\delta$ represents the top-dimensional generator $\Theta \in SFH(\Sigma, \boldsymbol{\beta}, \boldsymbol{\delta}, \mathfrak{s}_0)$, then $\theta \times \{w_{d+1}\}$ represents the top-dimensional generator $\Theta' \in SFH(\Sigma, \boldsymbol{\beta}', \boldsymbol{\delta}', \mathfrak{s}_0)$. For $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}(\mathbf{x}) = r(\mathfrak{s})$, the coefficient $n_{\mathbf{y}}$ of $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\delta$ in the chain $f_{\mathcal{T}}(\mathbf{x} \otimes \theta, r(\mathfrak{s}))$ is obtained by counting rigid pseudo-holomorphic triangles with corners at $\mathbf{x}$, θ, and $\mathbf{y}$ and inducing the Spin^c structure $\mathfrak{s}$. The stabilization map $F_{\mathcal{H}_{\mathbb{L}},\mathcal{H}'_{\mathbb{L}}}$ on the chain level is given by $\mathbf{y} \mapsto \mathbf{y} \otimes \{y_{d+1}\}$. On the other hand, the stabilization map $F_{\mathcal{H},\mathcal{H}'}$ on the chain level is given by $\mathbf{x} \mapsto \mathbf{x} \times \{x_{d+1}\}$. The coefficient of $\mathbf{y} \otimes \{y_{d+1}\}$ in the chain

$$f_{\mathcal{T}'}((\mathbf{x} \times \{x_{d+1}\}) \otimes (\theta \times \{w_{d+1}\}), r(\mathfrak{s}))$$

is also $n_{\mathbf{y}}$, as in E there is a unique rigid holomprphic triangle with corners x_{d+1}, y_{d+1}, and w_{d+1} having multiplicity zero at the connected sum point $x \in E$, giving a bijection between pseudo-holomorphic representatives of each homotopy class in $\pi_2(\mathbf{x}, \mathbf{y}, \theta)$ and of the corresponding class in

$$\pi_2(\mathbf{x} \times \{x_{d+1}\}, \mathbf{y} \times \{y_{d+1}\}, \theta \times \{w_{d+1}\}).$$

This shows that diagram 6.1 is commutative already on the chain level for proper stabilizations.

To check invariance under move (6), suppose that the two triple diagrams $\mathcal{T}$ and $\mathcal{T}'$ differ by a diffeomorphism isotopic to the identity in M'. Then, in particular, there is a diffeomorphism $\phi\colon \Sigma \to \Sigma'$ such that $\phi(\boldsymbol{\alpha}) = \boldsymbol{\alpha}'$, $\phi(\boldsymbol{\beta}) = \boldsymbol{\beta}'$, and $\phi(\boldsymbol{\delta}) = \boldsymbol{\delta}'$. Note that $\phi_*(\Theta_{\beta,\delta}) = \Theta_{\beta',\delta'}$, as ϕ_* is an isomorphism mapping the top-dimensional part of $SFH(\Sigma, \boldsymbol{\beta}, \boldsymbol{\delta}, \mathfrak{s}_0)$, generated by $\Theta_{\beta,\delta}$, to the top-dimensional part of $SFH(\Sigma', \boldsymbol{\beta}', \boldsymbol{\delta}', \mathfrak{s}_0)$, generated by $\Theta_{\beta',\delta'}$. Hence the diagram

$$\begin{array}{ccc} SFH\left(\mathcal{H}, \mathfrak{s}|_M\right) & \xrightarrow{F_{\mathcal{T}}(\,\cdot\,\otimes \Theta_{\beta,\delta}, r(\mathfrak{s}))} & SFH\left(\mathcal{H}_{\mathbb{L}}, \mathfrak{s}|_{M(\mathbb{L})}\right) \\ \big\downarrow \phi_* & & \big\downarrow \phi_* \\ SFH\left(\mathcal{H}', \mathfrak{s}|_M\right) & \xrightarrow{F_{\mathcal{T}'}(\,\cdot\,\otimes \Theta_{\beta',\delta'}, r(\mathfrak{s}))} & SFH\left(\mathcal{H}'_{\mathbb{L}}, \mathfrak{s}|_{M(\mathbb{L})}\right) \end{array}$$

commutes by Lemma 5.21.

Invariance under moves (1), (2), (4), and (5) follows in a way similar to the proof of [38, Proposition 4.6], but it is slightly simpler as our naturality isomorphisms do not involve the continuation maps Γ, cf. Section 5.2. More concretely, suppose that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ and $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$ are related by one of the moves above. Then, by [25, Lemma 9.3], we can take isotopic copies $\overline{\boldsymbol{\alpha}}$, $\overline{\boldsymbol{\beta}}$, and $\overline{\boldsymbol{\delta}}$ of $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and $\boldsymbol{\delta}$, respectively, such that the multi-diagrams

$$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\delta}}) \text{ and } (\Sigma, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\delta}}, \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$$

are both admissible. Let $\overline{\mathcal{T}} = (\Sigma, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}}, \overline{\boldsymbol{\delta}})$, $\overline{\mathcal{H}} = (\Sigma, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$, and $\overline{\mathcal{H}}_{\mathbb{L}} = (\Sigma, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\delta}})$. Then we claim that the following diagram is commutative:

$$\begin{array}{ccc} SFH\left(\mathcal{H}, \mathfrak{s}|_M\right) & \xrightarrow{F_{\mathcal{T}}(\,\cdot\,\otimes \Theta_{\beta,\delta}, r(\mathfrak{s}))} & SFH\left(\mathcal{H}_{\mathbb{L}}, \mathfrak{s}|_{M(\mathbb{L})}\right) \\ \big\downarrow \Psi^{\alpha\to\overline{\alpha}}_{\beta\to\overline{\beta}} & & \big\downarrow \Psi^{\alpha\to\overline{\alpha}}_{\delta\to\overline{\delta}} \\ SFH\left(\overline{\mathcal{H}}, \mathfrak{s}|_M\right) & \xrightarrow{F_{\overline{\mathcal{T}}}(\,\cdot\,\otimes \Theta_{\overline{\beta},\overline{\delta}}, r(\mathfrak{s}))} & SFH\left(\overline{\mathcal{H}}_{\mathbb{L}}, \mathfrak{s}|_{M(\mathbb{L})}\right). \end{array}$$

Indeed, for $x \in SFH(\mathcal{H}, \mathfrak{s})$, suppressing Spin^c structures in our notation, we have

$$\begin{aligned} \Psi^{\alpha\to\overline{\alpha}}_{\delta\to\overline{\delta}}\left(F_{\mathcal{T}}(x \otimes \Theta_{\beta,\delta})\right) &= F_{\overline{\alpha},\alpha,\overline{\delta}}\left(\Theta_{\overline{\alpha},\alpha} \otimes F_{\alpha,\delta,\overline{\delta}}\left(F_{\mathcal{T}}(x \otimes \Theta_{\beta,\delta}) \otimes \Theta_{\delta,\overline{\delta}}\right)\right) = \\ & F_{\overline{\alpha},\alpha,\overline{\delta}}\left(\Theta_{\overline{\alpha},\alpha} \otimes F_{\alpha,\beta,\overline{\delta}}\left(x \otimes F_{\beta,\delta,\overline{\delta}}\left(\Theta_{\beta,\delta} \otimes \Theta_{\delta,\overline{\delta}}\right)\right)\right) = \\ F_{\overline{\alpha},\alpha,\overline{\delta}}\left(\Theta_{\overline{\alpha},\alpha} \otimes F_{\alpha,\beta,\overline{\delta}}\left(x \otimes \Theta_{\beta,\overline{\delta}}\right)\right) &= F_{\overline{\alpha},\beta,\overline{\delta}}\left(F_{\overline{\alpha},\alpha,\beta}(\Theta_{\overline{\alpha},\alpha} \otimes x) \otimes \Theta_{\beta,\overline{\delta}}\right) = \\ & F_{\overline{\alpha},\beta,\overline{\delta}}\left(F_{\overline{\alpha},\alpha,\beta}(\Theta_{\overline{\alpha},\alpha} \otimes x) \otimes F_{\beta,\overline{\beta},\overline{\delta}}\left(\Theta_{\beta,\overline{\beta}} \otimes \Theta_{\overline{\beta},\overline{\delta}}\right)\right) = \\ F_{\overline{\mathcal{T}}}\left(F_{\overline{\alpha},\beta,\overline{\beta}}\left(F_{\overline{\alpha},\alpha,\beta}(\Theta_{\overline{\alpha},\alpha} \otimes x) \otimes \Theta_{\beta,\overline{\beta}}\right) \otimes \Theta_{\overline{\beta},\overline{\delta}}\right) &= F_{\overline{\mathcal{T}}}\left(\Psi^{\alpha\to\overline{\alpha}}_{\beta\to\overline{\beta}}(x) \otimes \Theta_{\overline{\beta},\overline{\delta}}\right), \end{aligned}$$

where we used Theorem 5.15 – the associativity of the triangle maps – in the second, fourth, and sixth step. The third step follows from the observation that

$$F_{\beta,\delta,\overline{\delta}}\left(\Theta_{\beta,\delta} \otimes \Theta_{\delta,\overline{\delta}}\right) = \Psi^{\beta}_{\delta\to\overline{\delta}}(\Theta_{\delta,\overline{\delta}}) = \Theta_{\beta,\overline{\delta}},$$

which holds as $\Psi^{\beta}_{\delta\to\overline{\delta}}\colon\ SFH_{\text{top}}(\Sigma,\beta,\delta,\mathfrak{s}_0)\to SFH_{\text{top}}(\Sigma,\beta,\overline{\delta},\mathfrak{s}_0)$ is an isomorphism, and so maps the generator $\Theta_{\beta,\delta}$ to the generator $\Theta_{\beta,\overline{\delta}}$. Similarly, we have

$$F_{\beta,\overline{\beta},\overline{\delta}}\left(\Theta_{\beta,\overline{\beta}}\otimes\Theta_{\overline{\beta},\overline{\delta}}\right)=\Theta_{\beta,\overline{\delta}},$$

which implies the fifth step. An analogous argument gives the commutativity of the following diagram:

$$\begin{array}{ccc} SFH\left(\overline{\mathcal{H}},\mathfrak{s}|_M\right) & \xrightarrow{F_{\overline{\mathcal{T}}}(\,\cdot\,\otimes\Theta_{\overline{\beta},\overline{\delta}},r(\mathfrak{s}))} & SFH\left(\overline{\mathcal{H}}_{\mathbb{L}},\mathfrak{s}|_{M(\mathbb{L})}\right) \\ \Big\downarrow \Psi^{\overline{\alpha}\to\alpha'}_{\overline{\beta}\to\beta'} & & \Big\downarrow \Psi^{\overline{\alpha}\to\alpha'}_{\overline{\delta}\to\delta'} \\ SFH\left(\mathcal{H}',\mathfrak{s}|_M\right) & \xrightarrow{F_{\mathcal{T}'}(\,\cdot\,\otimes\Theta_{\beta',\delta'},r(\mathfrak{s}))} & SFH\left(\mathcal{H}'_{\mathbb{L}},\mathfrak{s}|_{M(\mathbb{L})}\right), \end{array}$$

which, together with the previous commutative diagram, implies the commutativity of diagram (6.1). The claim in the case $B=B'$ follows.

We now show that $F_{M,\mathbb{L},\mathfrak{s}}$ is independent of the bouquet, in the spirit of [38, Lemma 4.8]. Suppose that B and B' are a pair of bouquets that differ in the choice of one arc $a_1\subset B$ and $a_1'\subset B'$. If a_1' is a small isotopic translate of a_1, then we can use the same triple diagram $\mathcal{T}=\mathcal{T}'$ subordinate to both B and B', in which case diagram (6.1) is obviously commutative. Hence, if $(B\setminus\mathbb{L})\cap a_1'\neq\emptyset$, we can perturb a_1' by a small isotopy such that it becomes disjoint from $B\setminus\mathbb{L}$. We construct two triples $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta})$ and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}',\boldsymbol{\delta}')$ subordinate to B and B', respectively, such that $\boldsymbol{\beta}'$ is obtained from $\boldsymbol{\beta}$ by a sequence of isotopies and handleslides, and $\boldsymbol{\delta}'$ is obtained from $\boldsymbol{\delta}$ by a sequence of isotopies and handleslides.

To this end, consider $(M'',\gamma'')=(M\setminus N(B\cup B'),\gamma)$. Note that

$$\chi(R_-(\gamma''))-\chi(R_+(\gamma''))=2(n+1).$$

By [19, Proposition 2.13], there exists a sutured diagram

$$(\Sigma,\{\alpha_1,\dots,\alpha_d\},\{\beta_{n+2},\dots,\beta_d\})$$

defining (M'',γ''). Let $\beta_1,\dots,\beta_n$ be meridians of the components $K_1,\dots,K_n$ of $\mathbb{L}$, respectively. Furthermore, β_{n+1} is a meridian of a_1. Similarly, β'_{n+1} is a meridian of a_1', and we set $\beta_i'=\beta_i$ if $i\in\{\,1,\dots,d\,\}\setminus\{n{+}1\}$. The curves $\delta_1,\dots,\delta_n$ correspond to the framing of $\mathbb{L}$, and for $i\in\{\,n+1,\dots,d\,\}$, the curve δ_i is a small isotopic translate of β_i. Finally, δ'_{n+1} is a small isotopic translate of β'_{n+1}, and we set $\delta_i'=\delta_i$ for $i\in\{\,1,\dots,d\,\}\setminus\{n{+}1\}$. Then $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta})$ is subordinate to B, while $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}',\boldsymbol{\delta}')$ is subordinate to B'.

If we surger Σ along $\beta_1,\dots,\beta_n,\beta_{n+2},\dots,\beta_d$, then we obtain $(R_+(\gamma)\setminus(B_1\cup B_2))\cup A$, where B_1, $B_2\subset R_+(\gamma)$ are disjoint disks and A is an annulus glued along $\partial B_1\cup\partial B_2$. Then β_{n+1} and β'_{n+1} induce two disjoint, embedded, homologically non-trivial curves lying in A. These curves must then be isotopic in A, thus β'_{n+1} can be obtained by handlesliding β_{n+1} over some collection of the curves $\beta_1,\dots,\beta_n,\beta_{n+2},\dots,\beta_d$. We can obtain δ'_{n+1} from δ_{n+1} in an analogous manner. Consequently, $\boldsymbol{\beta}'$ is obtained from $\boldsymbol{\beta}$ by a sequence of isotopies and handleslides, and $\boldsymbol{\delta}'$ is obtained from $\boldsymbol{\delta}$ by a sequence of isotopies and handleslides.

From here, commutativity of diagram (6.1) follow just like invariance under moves (1), (2), (4), and (5) in the case $B=B'$, applied to the triples $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta},\boldsymbol{\delta})$ and $(\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta}',\boldsymbol{\delta}')$. As one can always get from a bouquet B to any other bouquet B' by a sequence of small isotopies and changing one arc at a time, the map $F_{\mathbb{L},\mathfrak{s}}$ is independent of both the bouquet and the subordinate triple diagram. $\square$

We have the following analogue of [38, Proposition 4.9].

Proposition 6.10. *Let $\mathbb{L}$ be a framed link in (M,γ). Suppose that we are given a partition $\mathbb{L}_1 \cup \mathbb{L}_2$ of $\mathbb{L}$. Then we have cobordisms*

$$W(\mathbb{L}_1)\colon (M,\gamma) \to (M(\mathbb{L}_1),\gamma),$$

and if we view $\mathbb{L}_2$ as a link in $M(\mathbb{L}_1)$,

$$W(\mathbb{L}_2)\colon (M(\mathbb{L}_1),\gamma) \to (M(\mathbb{L}),\gamma).$$

Choose Spin^c structures $\mathfrak{s}_i \in \text{Spin}^c(W(\mathbb{L}_i))$ for $i \in \{1,2\}$. Then there is an isotopically unique diffeomorphism between $W(\mathbb{L})$ and $W(\mathbb{L}_2)\circ W(\mathbb{L}_1)$, and under this identification,

$$\sum_{\mathfrak{s}\in Spin^c(W(\mathbb{L}))\colon\, \mathfrak{s}|_{W(L_1)}=\mathfrak{s}_1,\ \mathfrak{s}|_{W(\mathbb{L}_2)}=\mathfrak{s}_2} F_{M,\mathbb{L},\mathfrak{s}} = F_{M(\mathbb{L}_1),\mathbb{L}_2,\mathfrak{s}_2} \circ F_{M,\mathbb{L}_1,\mathfrak{s}_1}.$$

Moreover, $F_{M,\mathbb{L}} = F_{M(\mathbb{L}_1),\mathbb{L}_2} \circ F_{M,\mathbb{L}_1}$.

Proof. The proof is completely analogous to the proof of [38, Proposition 4.9]. Note that one can disregard the last paragraph there, as one can always achieve admissibility for the associativity theorem (Theorem 5.15) in our case. The summation is necessary in the untwisted case due to the corresponding ambiguity of the gluing of Spin^c structures in the associativity theorem. □

7. One- and three-handles

Definition 7.1. Let (M,γ) be a sutured manifold. A *framed pair of points* $\mathbb{P}$ consists of two distinct points p_+ and p_- lying in the interior of M, together with a positive frame $\langle v_1^+, v_2^+, v_3^+ \rangle$ of $T_{p_+}M$ and a negative frame $\langle v_1^-, v_2^-, v_3^- \rangle$ of $T_{p_-}M$.

Given a sutured manifold (M,γ) and a framed pair of points $\mathbb{P}$, let

$$W(\mathbb{P}) = (W, Z, [\xi])$$

be the cobordism from (M,γ) to $(M(\mathbb{P}),\gamma)$ obtained by attaching a single 4-dimensional 1-handle $H = D^1 \times D^3$ to $M \times I$ along $\mathbb{P}$. I.e., $(-1,0) \in H$ is glued to p_- and $(1,0) \in H$ is glued to p_+, and $\{-1\} \times D^3$ induces the framing of p_-, while $\{1\} \times D^3$ induces the framing of p_+. This is also a special cobordism, with $Z = \partial M \times I$, and ξ being an I-invariant contact structure such that for every $t \in I$ the surface $\partial M \times \{t\}$ is convex with dividing set $\gamma \times \{t\}$. The manifold $M(\mathbb{P})$ is obtained from M by removing the interiors of the balls $\{-1\} \times D^3$ and $\{1\} \times D^3$, and gluing $D^1 \times S^2$. There are two possibilities for $M(\mathbb{P})$:

(1) If p_+ and p_- lie in the same component M_0 of M, then
$$M(\mathbb{P}) \approx M\#(S^1 \times S^2),$$
where the connected sum is taken between M_0 and $S^1 \times S^2$.

(2) Otherwise, there are distinct components M_1 and M_2 of M such that $M(\mathbb{P})$ has M_1 and M_2 replaced by $M_1\#M_2$.

In both cases, we have $SFH(M(\mathbb{P}),\gamma) \cong SFH(M,\gamma) \otimes \mathbb{Z}^2$ by [19, Proposition 9.15]. As the restriction map $H^2(W,Z) \to H^2(M,\partial M)$ is an isomorphism, we have $\text{Spin}^c(W(\mathbb{P})) \cong \text{Spin}^c(M,\gamma)$ (recall that for a special cobordism it does not matter whether we consider Spin^c structures relative to ∂Z or Z). In case (1), a Spin^c structure $\mathfrak{s}'$ on $(M(\mathbb{P}),\gamma)$ extends over $W(\mathbb{P})$ if and only if $\mathfrak{s}'|_{S^1\times S^2} = \mathfrak{s}_0$, where $\mathfrak{s}_0$ is characterized by $c_1(\mathfrak{s}_0) = 0$. In case (2), every Spin^c structure extends from $(M(\mathbb{P}),\gamma)$ over $W(\mathbb{P})$. Given a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(W(\mathbb{P}))$, we also write $\mathfrak{s}$ for the corresponding element of $\text{Spin}^c(M,\gamma)$ by a slight abuse of notation, and let $\mathfrak{s}_{\mathbb{P}} = \mathfrak{s}|_{(M(\mathbb{P}),\gamma)}$. Note that in case (1), we have $\mathfrak{s}_{\mathbb{P}} = \mathfrak{s}\#\mathfrak{s}_0$.

Definition 7.2. Let $\mathbb{P}$ be a framed pair of points in the sutured manifold (M,γ). A *bouquet for* $\mathbb{P}$ consists of a pair of disjoint embedded arcs η_+ and η_- in M and a framing of $\eta_\pm$ given by a normal vector field $v_\pm$ such that

(1) $\eta_\pm(0) = p_\pm$ and $\eta_\pm(1) \in s(\gamma)$,
(2) $\eta_\pm'(0) = v_1^\pm$ and $\eta_\pm'(1)$ is transverse to ∂M,
(3) $v_\pm(0) = v_2^\pm$.

Definition 7.3. Let (M,γ) be a sutured manifold, together with a framed pair of points $\mathbb{P}$, and let $B(\mathbb{P})$ be a bouquet for $\mathbb{P}$. We say that the diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is *subordinate to* $B(\mathbb{P})$ if

(1) $\eta_+ \cup \eta_- \subset \Sigma$,
(2) the framings v_+ and v_- are tangent to Σ, and
(3) $(\eta_+ \cup \eta_-) \cap (\boldsymbol{\alpha} \cup \boldsymbol{\beta}) = \emptyset$.

Lemma 7.4. *Let (M,γ) be a balanced sutured manifold, together with a framed pair of points $\mathbb{P}$ and bouquet $B(\mathbb{P})$. Then (M,γ) has an admissible diagram subordinate to $B(\mathbb{P})$.*

If $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\overline{\mathcal{H}} = (\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ are admissible diagrams of (M,γ) subordinate to $B(\mathbb{P})$, then there is a sequence of admissible diagrams $\mathcal{H}_0, \dots, \mathcal{H}_n$, each subordinate to $B(\mathbb{P})$, such that $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{H}_n = \overline{\mathcal{H}}$. Furthermore, for every $i \in \{0, \dots, n-1\}$, the diagrams $\mathcal{H}_i$ and $\mathcal{H}_{i+1}$ are related by an strong equivalence, a stabilization, a destabilization, or a diffeomorphism isotopic in (M,γ) to the identity through diagrams subordinate to $B(\mathbb{P})$.

Proof. Let $N_\pm$ be a regular neighborhood of $\eta_\pm$, and choose properly embedded disks $D_\pm \subset N_\pm$ such that $\eta_\pm \subset D_\pm$, $v_\pm$ is tangent to $D_\pm$, and

$$D_\pm \cap \partial M = s(\gamma) \cap N_\pm.$$

We define the suture manifold (M',γ') by taking $M' = \overline{M \setminus (N_+ \cup N_-)}$ and rounding the corners, and

$$s(\gamma') = s(\gamma) \triangle \partial(D_+ \cup D_-),$$

where $\triangle$ denotes the symmetric difference. By [19], the sutured manifold (M',γ') has an admissible diagram $(\Sigma', \boldsymbol{\alpha}, \boldsymbol{\beta})$. If we let $\Sigma = \Sigma' \cup D_+ \cup D_-$, then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible diagram of (M,γ) subordinate to the bouquet $B(\mathbb{P})$.

Now suppose we are given admissible diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\overline{\mathcal{H}} = (\overline{\Sigma}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ of (M,γ) subordinate to $B(\mathbb{P})$. Choose regular neighborhoods N_+ and N_- of η_+ and η_- so thin that they are disjoint from $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\alpha}'$, and $\boldsymbol{\beta}'$, and let $D_\pm = \Sigma \cap N_\pm$. After removing $D_+ \cup D_-$ from both Σ and $\overline{\Sigma}$, we obtain admissible diagrams $\mathcal{H}'$ and $\overline{\mathcal{H}'}$ of (M',γ'). By [25, Proposition 2.37], these can be connected by a sequence of admissible diagrams $\mathcal{H}_0', \dots, \mathcal{H}_n'$ such that for every $i \in \{0, \dots, n-1\}$, the diagrams $\mathcal{H}_i'$ and $\mathcal{H}_{i+1}'$ are related by an strong equivalence, a stabilization, a destabilization, or a diffeomorphism isotopic in (M',γ') to the identity. By adding $D_+ \cup D_-$ to each $\mathcal{H}_i'$, we obtain the desired sequence of diagrams $\mathcal{H}_0, \dots, \mathcal{H}_n$. □

We define the map for 1-handle addition as follows.

Definition 7.5. Let (M,γ) be a balanced sutured manifold together with a framed pair of points $\mathbb{P}$, and let $W(\mathbb{P})$ be the corresponding cobordism from (M,γ) to $(M(\mathbb{P}),\gamma)$. Let (A,α,β) be a sutured diagram, where $A = D^1 \times S^1$ is an annulus inside the 1-handle $H = D^1 \times D^3$, and α and β are homologically non-trivial, transverse simple closed curves such that $|\alpha \cap \beta| = 2$. Let $\theta \in \alpha \cap \beta$ be the intersection point with higher relative grading.

Choose a bouquet $B(\mathbb{P})$ for $\mathbb{P}$, and suppose that $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible balanced diagram of (M,γ) subordinate to $B(\mathbb{P})$. Consider the open disks $D_- = \{-1\} \times \operatorname{Int}(D^2)$ and $D_+ = \{1\} \times \operatorname{Int}(D^2)$ whose union is $\operatorname{int}(\Sigma \cap H)$ (these are neighborhoods of p_- and p_+ in Σ, respectively), and let $\Sigma^0 = \Sigma \setminus (D_- \cup D_+)$. We write $\Sigma_{\mathbb{P}}$ for the surface obtained from Σ^0 by gluing A along $\partial D_- \cup \partial D_+$, and smoothing the corners. Furthermore, we isotope $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ without crossing p_+ and p_- such that they become disjoint from D_+ and D_-. Then

$$\mathcal{H}_{\mathbb{P}} = (\Sigma_{\mathbb{P}}, \boldsymbol{\alpha}_{\mathbb{P}}, \boldsymbol{\beta}_{\mathbb{P}}) = (\Sigma^0 \cup A, \boldsymbol{\alpha} \cup \{\alpha\}, \boldsymbol{\beta} \cup \{\beta\})$$

is a balanced diagram of $(M(\mathbb{P}),\gamma)$.

For $\mathfrak{s} \in \operatorname{Spin}^c(W(\mathbb{P}))$, we define the map

$$f_{\mathcal{H},\mathbb{P},\mathfrak{s}} \colon CF(\mathcal{H},\mathfrak{s}) \to CF(\mathcal{H}_{\mathbb{P}},\mathfrak{s}_{\mathbb{P}})$$

by the formula $g_{\mathcal{H},\mathbb{P},\mathfrak{s}}(\mathbf{x}) = \mathbf{x} \times \{\theta\}$. This makes sense since $\mathfrak{s}(\mathbf{x} \times \{\theta\}) = \mathfrak{s}_{\mathbb{P}}$, and is a chain map since ∂D_+ and ∂D_- lie in components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ that intersect $\partial\Sigma$ non-trivially. The induced map on homology is

$$F_{\mathcal{H},\mathbb{P},\mathfrak{s}} \colon SFH(\mathcal{H},\mathfrak{s}) \to SFH(\mathcal{H}_{\mathbb{P}},\mathfrak{s}_{\mathbb{P}}).$$

Then the map associated to attaching a 1-handle along $\mathbb{P}$ is

$$F_{M,\mathbb{P},\mathfrak{s}} = P_{\mathcal{H}_{\mathbb{P}}}^{-1} \circ F_{\mathcal{H},\mathbb{P},\mathfrak{s}} \circ P_{\mathcal{H}} \colon SFH(M,\gamma,\mathfrak{s}) \to SFH(M(\mathbb{P}),\gamma,\mathfrak{s}).$$

Similarly, we define the map

$$F_{\mathcal{H},\mathbb{P}} \colon SFH(\mathcal{H}) \to SFH(\mathcal{H}_{\mathbb{P}})$$

to be the one induced by $f_{\mathcal{H},\mathbb{P}}(\mathbf{x}) = \mathbf{x} \times \{\theta\}$, and let

$$F_{M,\mathbb{P}} = P_{\mathcal{H}_{\mathbb{P}}}^{-1} \circ F_{\mathcal{H},\mathbb{P}} \circ P_{\mathcal{H}} \colon SFH(M,\gamma) \to SFH(M(\mathbb{P}),\gamma).$$

The following theorem is an analogue of [38, Theorem 4.10], and ensures that the maps $F_{M,\mathbb{P}}$ and $F_{M,\mathbb{P},\mathfrak{s}}$ are well-defined; i.e., independent of the choice of bouquet and subordinate diagram.

Theorem 7.6. *Let $\mathbb{P}$ be a framed pair of points in the balanced sutured manifold (M,γ). If $\mathcal{H} = (\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}')$ are admissible balanced diagrams for (M,γ) subordinate to the bouquets $B(\mathbb{P})$ and $B(\mathbb{P})'$, respectively, and if $\mathfrak{s} \in Spin^c(W(\mathbb{P}))$, then the following diagram is commutative:*

$$\begin{array}{ccc} SFH(\mathcal{H},\mathfrak{s}) & \xrightarrow{F_{\mathcal{H},\mathbb{P},\mathfrak{s}}} & SFH(\mathcal{H}_{\mathbb{P}},\mathfrak{s}_{\mathbb{P}}) \\ \big\downarrow F_{\mathcal{H},\mathcal{H}'} & & \big\downarrow F_{\mathcal{H}_{\mathbb{P}},\mathcal{H}'_{\mathbb{P}}} \\ SFH(\mathcal{H}',\mathfrak{s}) & \xrightarrow{F_{\mathcal{H}',\mathbb{P},\mathfrak{s}}} & SFH(\mathcal{H}'_{\mathbb{P}},\mathfrak{s}_{\mathbb{P}}), \end{array} \tag{7.1}$$

where the vertical maps are the canonical isomorphisms defined in Section 5.2. An analogous statement holds for $F_{\mathcal{H},\mathbb{P}}$ and $F_{\mathcal{H}',\mathbb{P}}$.

Proof. First, suppose that $B(\mathbb{P}) = B(\mathbb{P})'$. By Lemma 7.4, we can connect $\mathcal{H}$ and $\mathcal{H}'$ via a sequence of admissible diagram $\mathcal{H}_0, \dots, \mathcal{H}_n$ such that consecutive diagrams differ by a strong equivalence, a (de)stabilization, or a diffeomorphism isotopic to the identity through diagrams subordinate to $B(\mathbb{P})$. This gives rise to a sequence of diagrams $(\mathcal{H}_0)_\mathbb{P}, \dots, (\mathcal{H}_n)_\mathbb{P}$ of $(M(\mathbb{P}), \gamma)$ such that consecutive diagrams also differ by one of the above moves. It suffices to show that diagram (7.1) commutes for $\mathcal{H} = \mathcal{H}_i$ and $\mathcal{H}' = \mathcal{H}_{i+1}$ for every $i \in \{\,0, \dots, n-1\,\}$.

If $\mathcal{H}$ and $\mathcal{H}'$ differ by a strong equivalence, then the maps $F_{\mathcal{H},\mathcal{H}'}$ and $F_{\mathcal{H}_\mathbb{P},\mathcal{H}'_\mathbb{P}}$ are compositions of triangle maps where the relevant triple diagrams have multiplicity zero along η_+ and η_-, and there is a single small triangle in A that contributes, cf. [38, Theorem 4.10]. More precisely, recall that the map $F_{\mathcal{H},\mathcal{H}'}$ is constructed by taking attaching sets $\overline{\boldsymbol{\alpha}}$ and $\overline{\boldsymbol{\beta}}$ such that the quadruple diagrams $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ and $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}', \overline{\boldsymbol{\alpha}}, \overline{\boldsymbol{\beta}})$ are both admissible, and then

$$F_{\mathcal{H},\mathcal{H}'} = \Psi^{\overline{\boldsymbol{\alpha}} \to \boldsymbol{\alpha}'}_{\overline{\boldsymbol{\beta}} \to \boldsymbol{\beta}'} \circ \Psi^{\boldsymbol{\alpha} \to \overline{\boldsymbol{\alpha}}}_{\boldsymbol{\beta} \to \overline{\boldsymbol{\beta}}}.$$

Furthermore,

$$\Psi^{\overline{\boldsymbol{\alpha}} \to \boldsymbol{\alpha}'}_{\overline{\boldsymbol{\beta}} \to \boldsymbol{\beta}'} = \Psi^{\boldsymbol{\alpha}'}_{\overline{\boldsymbol{\beta}} \to \boldsymbol{\beta}'} \circ \Psi^{\overline{\boldsymbol{\alpha}} \to \boldsymbol{\alpha}'}_{\overline{\boldsymbol{\beta}}} \text{ and } \Psi^{\boldsymbol{\alpha} \to \overline{\boldsymbol{\alpha}}}_{\boldsymbol{\beta} \to \overline{\boldsymbol{\beta}}} = \Psi^{\overline{\boldsymbol{\alpha}}}_{\boldsymbol{\beta} \to \overline{\boldsymbol{\beta}}} \circ \Psi^{\boldsymbol{\alpha} \to \overline{\boldsymbol{\alpha}}}_{\boldsymbol{\beta}}.$$

By finger moves along $\eta_\pm$, we can also arrange that $(\eta_+ \cup \eta_-) \cap (\overline{\boldsymbol{\alpha}} \cup \overline{\boldsymbol{\beta}}) = \emptyset$. Hence, it suffices to show that diagram (7.1) is commutative either when $(\Sigma, \boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is an admissible triple, the diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma, \boldsymbol{\delta}, \boldsymbol{\beta})$ are subordinate to $B(\mathbb{P})$, and $F_{\mathcal{H},\mathcal{H}'}$ is given by $\Psi^{\boldsymbol{\alpha} \to \boldsymbol{\delta}}_{\boldsymbol{\beta}}$, or when $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ is admissible, the diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta})$ are subordinate to $B(\mathbb{P})$, and $F_{\mathcal{H},\mathcal{H}'}$ is given by $\Psi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta} \to \boldsymbol{\delta}}$. Since the second case is completely analogous, we will only prove the first one. The attaching sets $\boldsymbol{\alpha}_\mathbb{P} = \boldsymbol{\alpha} \cup \{\alpha\}$, $\boldsymbol{\beta}_\mathbb{P} = \boldsymbol{\beta} \cup \{\beta\}$, and $\boldsymbol{\delta}_\mathbb{P} = \boldsymbol{\delta} \cup \{\delta\}$ on $\Sigma_\mathbb{P}$ are obtained by adding the meridional curves α, β, δ in the annulus A that pairwise intersect in two points, and are arranged as shown in Figure 6. The "top" intersection point between α and β is denoted by $\theta_{\alpha,\beta}$, and we define $\theta_{\delta,\alpha}$ and $\theta_{\delta,\beta}$ similarly. Pick a generator $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, then commutativity of diagram (7.1) follows if we show that

$$F_{\delta_\mathbb{P},\alpha_\mathbb{P},\beta_\mathbb{P}}\left(\Theta_{\delta_\mathbb{P},\alpha_\mathbb{P}} \otimes (\mathbf{x} \times \{\theta_{\alpha,\beta}\})\right) = F_{\delta,\alpha,\beta}(\Theta_{\delta,\alpha} \otimes \mathbf{x}) \times \{\theta_{\delta,\beta}\}.$$

Note that $\Theta_{\delta_\mathbb{P},\alpha_\mathbb{P}} = \Theta_{\delta,\alpha} \times \{\theta_{\delta,\alpha}\}$. Now suppose that $\mathcal{D}_\mathbb{P}$ is a positive domain in $(\Sigma, \boldsymbol{\delta}_\mathbb{P}, \boldsymbol{\alpha}_\mathbb{P}, \boldsymbol{\beta}_\mathbb{P})$ connecting the intersection points $\Theta_{\delta,\alpha} \times \{\theta_{\delta,\alpha}\}$, $\mathbf{x} \times \{\theta_{\alpha,\beta}\}$, and $\mathbf{y} \times \{\theta_{\delta,\beta}\}$, where $\mathbf{y} \in \mathbb{T}_\delta \cap \mathbb{T}_\beta$. Then the coefficients of $\mathcal{D}_\mathbb{P}$ are zero along ∂A, and hence $\mathcal{D}_\mathbb{P} = \mathcal{D} + \mathcal{D}'$, where $\mathcal{D}$ is supported in Σ, while $\mathcal{D}'$ is supported in A.

Now we determine $\mathcal{D}'$. As in Figure 6, since $\mathcal{D}'$ connects $\theta_{\delta,\alpha}$, $\theta_{\alpha,\beta}$, and $\theta_{\delta,\beta}$, we must have $\partial(\partial\mathcal{D}' \cap \alpha) = \theta_{\alpha,\beta} - \theta_{\delta,\alpha}$, $\partial(\partial\mathcal{D}' \cap \beta) = \theta_{\delta,\beta} - \theta_{\alpha,\beta}$, and $\partial(\partial\mathcal{D}' \cap \delta) = \theta_{\delta,\alpha} - \theta_{\delta,\beta}$. Hence, there are integers a, b, and c such that $\partial\mathcal{D}' \cap \alpha$ has coefficients a and $a-1$, while $\partial\mathcal{D}' \cap \beta$ has coefficients b and $b-1$, and $\partial\mathcal{D}' \cap \delta$ has coefficients d and $d-1$, with the orientation as in Figure 6. From this, we can determine all the coefficients of $\mathcal{D}'$, they are as in Figure 6. As the coefficients of $\mathcal{D}'$ are zero along ∂A, it follows that $a - b - d + 1 = 0$. Since $\mathcal{D}' \geq 0$, we have both $b - 1 = a - d \geq 0$ and $1 - b \geq 0$, hence $b = 1$ and $a = d$. Furthermore, $a - 1 \geq 0$ and $1 - d = 1 - a \geq 0$, so $a = 1$. It follows that all the coefficients of $\mathcal{D}'$ are zero, except it is one in the shaded triangle. Hence, by the Riemann mapping theorem, the moduli space of pseudo-holomorphic triangles $\mathcal{M}(\mathcal{D})$ can be identified with $\mathcal{M}(\mathcal{D}_\mathbb{P})$ by pasting in the unique holomorphic representative of the shaded triangle.

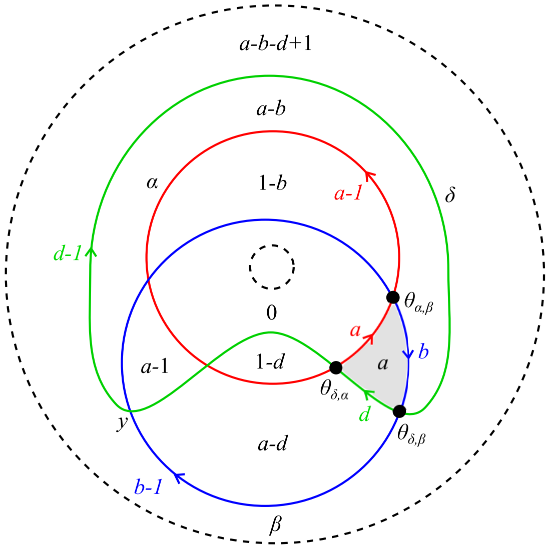

Figure 6. The annulus A is bounded by the two dashed circles. The diagram shows possible multiplicities of a domain of a triangle in the triple diagram (δ, α, β) connecting the "top" intersection points $\theta_{\delta,\alpha}$, $\theta_{\alpha,\beta}$, and $\theta_{\delta,\beta}$.

If $\{y\} = (\delta \cap \beta) \setminus \{\theta_{\delta,\beta}\}$, then $\mathbf{y} \times \{y\}$ does not appear in

$$F_{\delta_{\mathbb{P}},\alpha_{\mathbb{P}},\beta_{\mathbb{P}}}\left(\Theta_{\delta_{\mathbb{P}},\alpha_{\mathbb{P}}} \otimes (\mathbf{x} \times \{\theta_{\alpha,\beta}\})\right),$$

as $\mu\left(\mathbf{y} \times \{\theta_{\delta,\beta}\}, \mathbf{y} \times \{y\}\right) = 1$. In fact, a computation analogous to the above shows that in $(A, \delta, \alpha, \beta)$, there are three domains $\mathcal{D}'$ of triangles connecting $\theta_{\delta,\alpha}$, $\theta_{\alpha,\beta}$, and y, each one of which has Maslov index one. Hence, if $\mathcal{D}_{\mathbb{P}} = \mathcal{D} + \mathcal{D}'$ did contribute to the coefficient of $\mathbf{y} \times \{y\}$, then $\mu(\mathcal{D}) = \mu(\mathcal{D}_{\mathbb{P}}) - \mu(\mathcal{D}') = 0 - 1 = -1$, and so $\mathcal{D}$, and also $\mathcal{D}_{\mathbb{P}}$, would have no pseudo-holomorphic representative for a generic path of almost complex structures.

Stabilization invariance is straightforward. When $\mathcal{H}$ and $\mathcal{H}'$ differ by a diffeomorphism d isotopic to the identity, as this isotopy can be chosen to fix a neighborhood of $\eta_+ \cup \eta_-$, then d extends to a diffeomorphism $d_{\mathbb{P}} \colon \mathcal{H}_{\mathbb{P}} \to \mathcal{H}'_{\mathbb{P}}$ isotopic to the identity in $(M(\mathbb{P}), \gamma)$. Furthermore, $d(\mathbf{x} \times \{\theta\}) = d(\mathbf{x}) \times \{\theta\}$, and the diagram commutes. So we have shown that diagram (7.1) commutes when $B(\mathbb{P}) = B(\mathbb{P})'$.

Now we show commutativity of diagram (7.1) when the bouquets $B(\mathbb{P})$ and $B(\mathbb{P})'$ are ambient isotopic through bouquets of $\mathbb{P}$. More precisely, if $\phi_t \colon M \to M$ is the isotopy, then we require that

- $\phi_t(s(\gamma)) = s(\gamma)$,
- $\phi_t(\mathbb{P}) = \mathbb{P}$, and
- $\phi_1(B(\mathbb{P})) = B(\mathbb{P})'$.

By the previous part, it suffices to show commutativity of (7.1) for a single pair of admissible diagrams $\mathcal{H}$ and $\mathcal{H}'$ subordinate to $B(\mathbb{P})$ and $B(\mathbb{P})'$, respectively. Choose an arbitrary admissible diagram $\mathcal{H}$ subordinate to $\mathbb{P}$, then $\mathcal{H}' = \phi_1(\mathcal{H})$ is a diagram of (M, γ) subordinate to $B(\mathbb{P})'$. Since $d = \phi_1|_{\mathcal{H}} \colon \mathcal{H} \to \mathcal{H}'$ is a diffeomorphism isotopic to the identity,

$$F_{\mathcal{H},\mathcal{H}'} = d_* : SFH(\mathcal{H},\mathfrak{s}) \to SFH(\mathcal{H}',\mathfrak{s}).$$

As each ϕ_t fixes $\mathbb{P}$, the diffeomorphism d extends to a diffeomorphism $d_{\mathbb{P}} \colon \mathcal{H}_{\mathbb{P}} \to \mathcal{H}'_{\mathbb{P}}$ isotopic to the identity in $(M(\mathbb{P}),\gamma)$, and such that $d_{\mathbb{P}}$ fixes A pointwise. In particular, $F_{\mathcal{H}_{\mathbb{P}},\mathcal{H}'_{\mathbb{P}}} = (d_{\mathbb{P}})_*$. For any generator $\mathbf{x}$ of $CF(\mathcal{H},\mathfrak{s})$, we get that

$$f_{\mathcal{H}',\mathbb{P},\mathfrak{s}} \circ F_{\mathcal{H},\mathcal{H}'}(\mathbf{x}) = d(\mathbf{x}) \times \{\theta\} = d_{\mathbb{P}}(\mathbf{x} \times \{\theta\}) = F_{\mathcal{H}_{\mathbb{P}},\mathcal{H}'_{\mathbb{P}}} \circ f_{\mathcal{H},\mathbb{P},\mathfrak{s}}(\mathbf{x}).$$

This concludes the proof of invariance under ambient isotopy of the bouquet.

The final step is showing that diagram (7.1) is commutative for an arbitrary pair of bouquets $B(\mathbb{P})$ and $B(\mathbb{P})'$. We can make $B(\mathbb{P})$ and $B(\mathbb{P})'$ disjoint in the complement of $\mathbb{P}$ by an ambient isotopy, and by the previous paragraph, it suffices to prove the claim under this assumption.

Next, we construct a diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ subordinate to $B(\mathbb{P})$ and almost subordinate to $B(\mathbb{P})'$, except that $|\eta'_\pm \cap \boldsymbol{\alpha}| = 1$ and $|\eta'_\pm \cap \boldsymbol{\beta}| = 1$. This can be done similarly to the proof of Lemma 7.4. Indeed, pick regular neighborhoods $N_\pm$ of $\eta_\pm \cup \eta'_\pm$, and let $D_\pm \subset N_\pm$ be properly embedded disks containing $\eta_\pm \cup \eta'_\pm$ and tangent to the vector fields $v_\pm$ and $v'_\pm$. We also assume that $D_\pm \cap \partial M = s(\gamma) \cap N_\pm$. We define the sutured manifold (M',γ') by taking $M' = M \setminus (N_+ \cup N_-)$ and

$$s(\gamma') = s(\gamma) \triangle \partial(D_+ \cup D_-).$$

Pick an admissible diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ of (M',γ'). If we set $\Sigma = \Sigma' \cup D_+ \cup D_-$, then $(\Sigma, \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is an admissible diagram of (M,γ) with holes drilled above and below $D_\pm$. We can fill these holes by attaching 3-dimensional 2-handles, which translates to adding an α-curve α_+ and a β-curve β_+ intersecting $\eta_+ \cup \eta'_+$ in one point each, which we can then isotope to intersect only η'_+, and similarly, we add a new α-curve α_- and a β-curve β_- intersecting η'_- in one point each. It will be helpful later if we isotope $\alpha_\pm$ and $\beta_\pm$ such that they each intersect $D_\pm$ in a single arc, these arcs have two transverse intersection points on opposite sides of $\eta'_\pm$, and the point $\alpha_\pm \cap \eta'_\pm$ is closer to $p_\pm$ along $\eta'_\pm$ than $\beta_\pm \cap \eta'_\pm$.

So if we set $\boldsymbol{\alpha} = \boldsymbol{\alpha}' \cup \alpha_+ \cup \alpha_-$ and $\boldsymbol{\beta} = \boldsymbol{\beta}' \cup \beta_+ \cup \beta_-$, then the diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a diagram of (M,γ) that is subordinate to $B(\mathbb{P})$ and almost subordinate to $B(\mathbb{P})'$. If we perform a finger move on $\alpha_\pm$ and $\beta_\pm$ along $\eta'_\pm$, obtaining curves $\alpha'_\pm$ and $\beta'_\pm$, and applying a small Hamiltonian isotopy to $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$, we get a diagram $\mathcal{H}' = (\Sigma, \boldsymbol{\delta}, \boldsymbol{\eta})$ that is subordinate to $B(\mathbb{P})'$ and almost subordinate to $B(\mathbb{P})$. By a small isotopy, we also arrange that $\alpha_\pm$ and $\alpha'_\pm$ – and similarly, $\beta_\pm$ and $\beta'_\pm$ – intersect transversely in two points. The curves α_+, α'_+, and β_+ inside D_+ are depicted in Figure 7. For simplicity, we write $\alpha = \alpha_+$, $\beta = \beta_+$, and $\delta = \alpha'_+$. It suffices to show the commutativity of diagram (7.1) for these diagrams $\mathcal{H}$ and $\mathcal{H}'$. Note that the quadruple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta}, \boldsymbol{\eta})$ is admissible, so

$$F_{\mathcal{H},\mathcal{H}'} = \Psi^{\boldsymbol{\delta}}_{\boldsymbol{\beta}\to\boldsymbol{\eta}} \circ \Psi^{\boldsymbol{\alpha}\to\boldsymbol{\delta}}_{\boldsymbol{\beta}}.$$

We will explain why diagram (7.1) with vertical map $\Psi^{\boldsymbol{\alpha}\to\boldsymbol{\delta}}_{\boldsymbol{\beta}}$ and diagrams $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{H}' = (\Sigma, \boldsymbol{\delta}, \boldsymbol{\beta})$ is commutative, as commutativity for $\Psi^{\boldsymbol{\delta}}_{\boldsymbol{\beta}\to\boldsymbol{\eta}}$ is analogous.

We denote by $\Theta_{\delta,\alpha}$ the "top" intersection point between $\boldsymbol{\delta}$ and $\boldsymbol{\alpha}$, this has coordinate $\theta_{\delta,\alpha} \in \delta \cap \alpha$. Let $\mathcal{D}$ be a positive domain in $(\Sigma, \boldsymbol{\delta}, \boldsymbol{\alpha}, \boldsymbol{\beta})$ connecting the generators $\Theta_{\delta,\alpha}$, $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, and $\mathbf{y} \in \mathbb{T}_\beta \cap \mathbb{T}_\delta$ that has a rigid pseudo-holomorphic representative for a generic path of almost complex structures. We claim that the

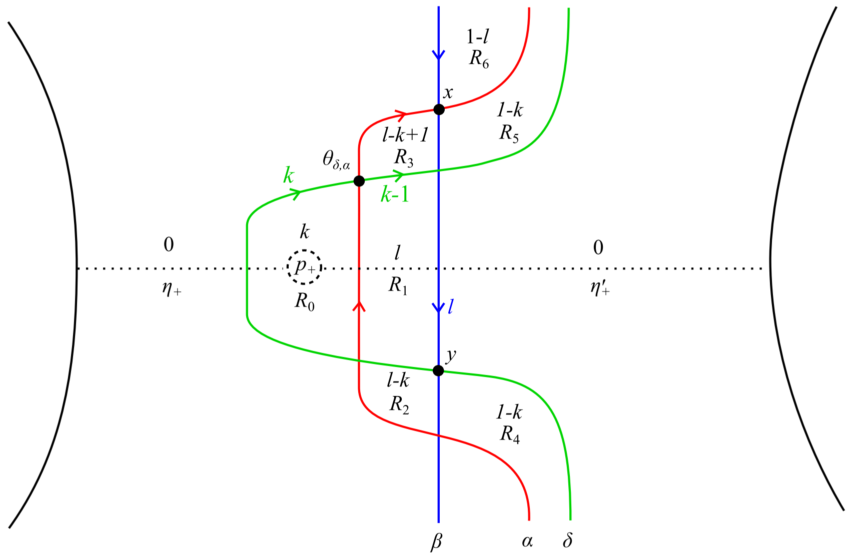

FIGURE 7. The curves $\alpha = \alpha_+$, $\beta = \beta_+$, and $\delta = \alpha'_+$ inside the disk D_+.

coefficients of $\mathcal{D}$ at p_+ and p_- are both zero. We label the closures of some of the components of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta} \cup \boldsymbol{\delta})$ by $R_0, \dots, R_6$, as in Figure 7. Let k and l denote the coefficients of $\mathcal{D}$ at R_0 and R_1, respectively. Since $\partial(\partial\mathcal{D} \cap \alpha) = x - \theta_{\delta,\alpha}$, where $x = \mathbf{x} \cap \alpha$, the multiplicities of $\partial\mathcal{D} \cap \alpha$ are $l-k$ and $l-k+1$ along the two components of $\alpha \setminus \{\theta_{\delta,\alpha}, x\}$. As $\mathcal{D}$ has multiplicity zero along $\eta_+ \setminus R_0$, the coefficient of $\mathcal{D}$ at R_2 has to be $l-k$, hence $l \geq k$. Furthermore, the multiplicity of $\partial\mathcal{D} \cap (\beta \cap R_1)$ is l, so if x were not the point $R_3 \cap R_6$ shown in Figure 7, then R_6 would have coefficient $-l$. As $\mathcal{D} \geq 0$, this would imply that $l = 0$, and hence also $k = 0$.

Now assume that $x = R_3 \cap R_6$, then the coefficient of R_6 is $1-l$. So if $k \geq 1$, then $\mathcal{D} \geq 0$ implies that $k = l = 1$. If $y = \mathbf{y} \cap \beta$ was not the point $R_1 \cap R_4$, then, by considering the multiplicities of $\partial\mathcal{D}$ along δ, we see that R_4 would have coefficient $-k = -1$. We conclude that $y = R_1 \cap R_4$, and inside D_+, the domain $\mathcal{D}$ has coefficients one in R_0, R_1, and R_3, and zero everywhere else. In particular, around the "triangle" $T = R_0 \cup R_1 \cup R_3$, all multiplicities are zero. Any pseudo-holomorphic representative of such a domain $\mathcal{D}$ would have to consist of a branched cover where one component is a triangle mapping to T. But one can make cuts along the arcs $R_0 \cap R_1$ and along $R_1 \cap R_3$, and so by the Riemann mapping theorem, the space of such mappings is one-dimensional, and so cannot be rigid. In fact, the Maslov index of T is one, so if $\mu(\mathcal{D}) = 0$, then $\mu(\mathcal{D} - T) < 0$, and hence generically $\mathcal{D}$ has no pseudo-holomorphic representative. An analogous argument shows that the coefficient of $\mathcal{D}$ at p_- is also zero.

Then the commutativity of diagram (7.1) follows just like in the case when we had $\mathcal{H}$ and $\mathcal{H}'$ strongly equivalent and both subordinate to the same bouquet. □

Now we define the map induced by attaching a 3-handle. Let (M', γ') be a balanced sutured manifold, together with a framed 2-sphere $S \subset M$. By this, we mean that a neighborhood of S is identified with $S^2 \times D^1$ such that $S = S^2 \times \{0\}$. Let $W(S)$ be the special cobordism from (M', γ') to (M, γ) obtained as the trace of attaching a three-handle $D^3 \times D^1$ to (M', γ') along the framed two-sphere S. Then we obtain (M, γ) from (M', γ') by removing $S \times (-1, 1)$ and capping off the two boundary components with $D^3 \times D^0$. Note that if S is non-separating, then (M, γ) is automatically balanced. When S is separating, then we make the additional assumption that (M, γ) is also balanced. Let $\mathbb{P}$ be the pair of points

$$p_\pm = (0, \pm 1) \in D^3 \times D^0 \subset M,$$

framed by $(\partial/\partial x, \partial/\partial y, \partial/\partial z)$ at $p_+ \in D^3 \times \{1\}$, and by the opposite vectors at $p_- \in D^3 \times \{-1\}$.

Definition 7.7. We say that the diagram $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ is adapted to the framed 2-sphere S if it is of the form $\mathcal{H}_\mathbb{P} = (\Sigma^0 \cup A, \boldsymbol{\alpha} \cup \{\alpha\}, \boldsymbol{\beta} \cup \{\beta\})$ for some admissible diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of (M, γ) subordinate to some bouquet $B(\mathbb{P})$ for $\mathbb{P}$ (recall that $\mathcal{H}_\mathbb{P}$ was introduced in Definition 7.5). In other words,

$$A := \Sigma' \cap (S \times D^1) = S^1 \times D^1,$$

while $\boldsymbol{\alpha}' \cap A = \{\alpha\}$ and $\boldsymbol{\beta}' \cap A = \{\beta\}$ are two meridians of A that intersect each other in a pair of points, and both components of ∂A can be connected with $\partial \Sigma'$ along an embedded arc disjoint from $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, and the interior of A.

Such a diagram always exists according to Lemma 7.4 applied to (M, γ) and an arbitrary bouquet $B(\mathbb{P})$ for $\mathbb{P}$.

Definition 7.8. Let $S \subset M'$ be a framed embedded sphere in the balanced sutured manifold (M', γ'), and take an adapted balanced diagram

$$\mathcal{H}' = (\Sigma^0 \cup A, \boldsymbol{\alpha} \cup \{\alpha\}, \boldsymbol{\beta} \cup \{\beta\}) = \mathcal{H}_\mathbb{P}.$$

For $\mathfrak{s} \in \mathrm{Spin}^c(W(S))$, let

$$f_{\mathcal{H}', S, \mathfrak{s}} \colon CF\,(\mathcal{H}', \mathfrak{s}|_{M'}) \to CF\,(\mathcal{H}, \mathfrak{s}|_M)$$

such that for $\mathbf{x} \in \mathbb{T}_{\boldsymbol{\alpha}} \cap \mathbb{T}_{\boldsymbol{\beta}}$ and $y \in \alpha \cap \beta$, we have $f_{\mathcal{H}, \mathfrak{s}}(\mathbf{x} \times \{y\}) = 0$ if $y = \theta$, and $f_{\mathcal{H}', S, \mathfrak{s}}(\mathbf{x} \times \{y\}) = \mathbf{x}$ if y is the intersection point of smaller relative grading. Then $f_{\mathcal{H}', S, \mathfrak{s}}$ is a chain map, since any domain in $\mathcal{H}'$ has coefficient zero along ∂A. It induces the map

$$F_{\mathcal{H}', S, \mathfrak{s}} \colon SFH\,(\mathcal{H}', \mathfrak{s}|_{M'}) \to SFH\,(\mathcal{H}, \mathfrak{s}|_M)\,.$$

on the homology. From this, we obtain the map

$$F_{M', S, \mathfrak{s}} = \mathcal{P}_\mathcal{H}^{-1} \circ f_{\mathcal{H}', S, \mathfrak{s}} \circ \mathcal{P}_{\mathcal{H}'} \colon SFH\,(M', \gamma', \mathfrak{s}|_{M'}) \to SFH\,(M, \gamma, \mathfrak{s}|_M)\,.$$

Similarly, we also have a map

$$F_{M', S} \colon SFH(M', \gamma') \to SFH(M, \gamma)$$

that does not refer to a particular Spin^c structure.

Theorem 7.9. *Let $S \subset M'$ be a framed embedded sphere in the balanced sutured manifold (M', γ'), and let (M, γ), together with the framed pair of points $\mathbb{P}$, be the result of surgering (M', γ') along S. Suppose that (M, γ) is balanced, and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be diagrams for (M, γ) adapted to bouquets $B_1(\mathbb{P})$ and $B_2(\mathbb{P})$, respectively. Then $\mathcal{H}_1' = (\mathcal{H}_1)_\mathbb{P}$ and $\mathcal{H}_2' = (\mathcal{H}_2)_\mathbb{P}$ are diagrams of (M', γ') adapted to S. If*

$\mathcal{V}(S)$ is the 3-handle cobordism corresponding to S and $\mathfrak{s} \in Spin^c(W(S))$, then the following diagram commutes:

$$\begin{array}{ccc} SFH\left(\mathcal{H}_1', \mathfrak{s}|_{M'}\right) & \xrightarrow{F_{\mathcal{H}_1',S,\mathfrak{s}}} & SFH\left(\mathcal{H}_1, \mathfrak{s}|_M\right) \\ \downarrow F_{\mathcal{H}_1',\mathcal{H}_2'} & & \downarrow F_{\mathcal{H}_1,\mathcal{H}_2} \\ SFH\left(\mathcal{H}_2', \mathfrak{s}|_{M'}\right) & \xrightarrow{F_{\mathcal{H}_2',S,\mathfrak{s}}} & SFH\left(\mathcal{H}_2, \mathfrak{s}|_M\right), \end{array}$$

where the vertical maps are the canonical isomorphisms. An analogous result holds for $F_{\mathcal{H}_1',S}$ and $F_{\mathcal{H}_2',S}$.

Proof. This is analogous to the proof of Theorem 7.6. □

8. The map associated to a special cobordism

In [25], we constructed a functor *SFH* from the category of balanced sutured manifolds and diffeomorphisms to $\mathbf{Vect}_{\mathbb{Z}_2}$. Furthermore, we have homomorphisms $F_{M,\mathbb{P},\mathfrak{s}}$ induced by surgery along a framed pair of points $\mathbb{P}$, homomorphisms $F_{M,\mathbb{L},\mathfrak{s}}$ induced by surgery along a framed link $\mathbb{L}$, and $F_{M,S,\mathfrak{s}}$ induced by surgery along a framed 2-sphere S. In [22, Theorem 1.2], we give a set of necessary and sufficient conditions for these to give rise to a TQFT $F\colon \mathbf{BSut}' \to \mathbf{Vect}_{\mathbb{Z}_2}$. Note that this result also states that we can avoid 0-handle and 4-handle attachments. We restate this theorem here for convenience specifically for $\mathbf{BSut}'$. We only enrich the theory with Spin^c structures later for clarity, as they introduce an additional layer of complexity.

Let (M,γ) be a balanced sutured manifold. A framed k-sphere in M is an embedding of $S^k \times D^{3-k}$ in the interior of M. We denote by $W(M,\mathbb{S})$ the elementary cobordism obtained by attaching a 4-dimensional k-handle to $M \times I$ along $M \times \{1\}$ with an I-invariant contact structure on $Z = \partial M \times I$ with dividing set $s(\gamma) \times \{t\}$ on $\partial M \times \{t\}$ for every $t \in I$. This is a cobordism from (M,γ) to $(M(\mathbb{S}),\gamma)$, where $M(\mathbb{S})$ is obtained by surgery along $\mathbb{S}$, and hence we call $W(M,\mathbb{S})$ the trace of the surgery. Usually the manifold M is unambiguous from the notation $\mathbb{S}$, in which case we only write $W(\mathbb{S})$ instead of $W(M,\mathbb{S})$. If $\mathbb{S}\colon S^k \times D^{3-k} \hookrightarrow M$ is a framed k-sphere for $k < 3$, let $\overline{\mathbb{S}}$ be the framed sphere defined by

$$\overline{\mathbb{S}}(\underline{x},\underline{y}) = \mathbb{S}\left(r_{k+1}(\underline{x}), r_{3-k}(\underline{y})\right),$$

where $\underline{x} \in \mathbb{R}^{k+1}$, $\underline{y} \in \mathbb{R}^{3-k}$, and

$$r_n(x_1, x_2, \dots, x_n) = (-x_1, x_2, \dots, x_n).$$

Theorem 8.1. *To define a functor $F\colon \mathbf{BSut}' \to \mathbf{Vect}_{\mathbb{Z}_2}$, it suffices to construct a functor F from the category of balanced sutured manifolds and diffeomorphisms to $\mathbf{Vect}_{\mathbb{Z}_2}$, and for every balanced sutured manifold (M,γ), framed 0-, 1- or 2-sphere $\mathbb{S} \subset Int(M)$, a linear map*

$$F_{M,\mathbb{S}}\colon F(M,\gamma) \to F(M(\mathbb{S}),\gamma)$$

that satisfy the following axioms:

(1) If $d \in Diff_0(M,\gamma)$, then $F(d) = Id_{F(M,\gamma)}$.

(2) *Given a diffeomorphism* $d\colon (M,\gamma) \to (M',\gamma')$ *between balanced sutured manifolds and a framed 0-, 1-, or 2-sphere* $\mathbb{S} \subset \mathit{Int}(M)$, *let* $\mathbb{S}' = d(\mathbb{S})$ *and*

$$d^{\mathbb{S}}\colon (M(\mathbb{S}),\gamma') \to (M'(\mathbb{S}'),\gamma')$$

be the induced diffeomorphisms. Then the following diagram is commutative:

$$\begin{array}{ccc} F(M,\gamma) & \xrightarrow{F_{M,\mathbb{S}}} & F(M(\mathbb{S}),\gamma) \\ \downarrow F(d) & & \downarrow F(d^{\mathbb{S}}) \\ F(M',\gamma') & \xrightarrow{F_{M',\mathbb{S}'}} & F(M'(\mathbb{S}'),\gamma'). \end{array}$$

(3) *If* (M,γ) *is a balanced sutured manifold and* $\mathbb{S}$ *and* $\mathbb{S}'$ *are* disjoint *framed 0-, 1-, or 2-spheres in* M, *then* $M(\mathbb{S})(\mathbb{S}') = M(\mathbb{S}')(\mathbb{S})$, *we denote this manifold by* $M(\mathbb{S},\mathbb{S}')$. *Then the following diagram is commutative:*

$$\begin{array}{ccc} F(M,\gamma) & \xrightarrow{F_{M,\mathbb{S}}} & F(M(\mathbb{S}),\gamma) \\ F_{M,\mathbb{S}'} \downarrow & & \downarrow F_{M(\mathbb{S}),\mathbb{S}'} \\ F(M(\mathbb{S}'),\gamma) & \xrightarrow{F_{M(\mathbb{S}'),\mathbb{S}}} & F(M(\mathbb{S},\mathbb{S}'),\gamma). \end{array}$$

(4) *If* $\mathbb{S}' \subset M(\mathbb{S})$ *intersects the belt sphere of the handle attached along* $\mathbb{S}$ *once transversely, then there is a diffeomorphism* $\varphi\colon M \to M(\mathbb{S})(\mathbb{S}')$ *(which is the identity on* $M \cap M(\mathbb{S})(\mathbb{S}')$*, and is unique up to isotopy), such that*

$$F_{M(\mathbb{S}),\mathbb{S}'} \circ F_{M,\mathbb{S}} = F(\varphi).$$

(5) *For every framed 0-, 1-, or 2-sphere* $\mathbb{S}$, *we have* $F_{M,\mathbb{S}} = F_{M,\overline{\mathbb{S}}}$.

The functor F *is a TQFT if and only if it is symmetric and monoidal. In the opposite direction, every functor* $F\colon \mathbf{BSut}' \to \mathbf{Vect}_{\mathbb{Z}_2}$ *arises in this way.*

Theorem 8.2. *The functor SFH together with the surgery maps* $F_{M,\mathbb{P}}$, $F_{M,\mathbb{L}}$, *and* $F_{M,S}$ *satisfy the axioms listed in Theorem 8.1, hence they give rise to a TQFT* $F\colon \mathbf{BSut}' \to \mathbf{Vect}_{\mathbb{Z}_2}$.

Proof. Axiom (1), isotopy invariance, is part of [25, Theorem 1.9]. Axiom (5) is straightforward as the construction of the map $F_{M,\mathbb{S}}$ is independent of the orientation of the framed sphere $\mathbb{S}$.

We now show axiom (2), naturality of the surgery maps. First, let $\mathbb{S}$ be a framed 1-sphere. Choose a bouquet $B(\mathbb{S})$ for $\mathbb{S}$; this is just an embedded arc connecting $\mathbb{S}$ and $R_+(\gamma)$. Furthermore, let $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ be a triple diagram subordinate to $B(\mathbb{S})$, and let $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ and $\mathcal{H}_{\mathbb{S}} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta})$. Then the triple diagram

$$\mathcal{T}' = d(\mathcal{T}) = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$$

is subordinate to the bouquet $B(\mathbb{S}') = d(B(\mathbb{S}))$. Let $\phi = d|_{\Sigma}\colon \Sigma \to \Sigma'$. If we write $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\delta}')$ and $\mathcal{H}'_{\mathbb{S}'} = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\delta}')$, then the following diagram commutes by Lemma 5.21:

$$\begin{array}{ccc} SFH(\mathcal{H}) & \xrightarrow{F_{\mathcal{T}}(\,\cdot\,\otimes\Theta_{\beta,\delta})} & SFH(\mathcal{H}_{\mathbb{S}}) \\ \downarrow \phi_* & & \downarrow \phi_* \\ SFH(\mathcal{H}') & \xrightarrow{F_{\mathcal{T}'}(\,\cdot\,\otimes\Theta_{\beta',\delta'})} & SFH(\mathcal{H}'_{\mathbb{S}'}). \end{array}$$

Commutativity of the diagram in axiom (2) follows from this by considering the commutative cube whose top face is the diagram above, its bottom face is the diagram in axiom (2), the maps along the vertical edges are given by $P_{\mathcal{H}}$, $P_{\mathcal{H}'_{\mathbb{S}'}}$, $P_{\mathcal{H}_{\mathbb{S}}}$, and $P_{\mathcal{H}'}$ described in Section 5.2, and whose vertical faces commute by the definition of the maps $F(d)$, $F(d^{\mathbb{S}})$, $F_{M,\mathbb{S}}$, and $F_{M',\mathbb{S}'}$. When $\mathbb{S}$ is a framed 0- or 2-sphere, axiom (2) follows similarly, by choosing a diagram adapted to $\mathbb{S}$, and pushing it forward along d.

Next, we verify axiom (3), commutativity of disjoint surgeries. There are six different cases depending on the dimensions s and s' of the framed spheres $\mathbb{S}$ and $\mathbb{S}'$. The case when $s = s' = 1$ follows immediately from Proposition 6.10.

Now suppose that $s = s' = 0$. Pick disjoint bouquets $B(\mathbb{S})$ and $B(\mathbb{S}')$ for $\mathbb{S}$ and $\mathbb{S}'$, respectively, as in Definition 7.2. The proof of Lemma 7.4 can be adapted to show the existence of an admissible diagram $\mathcal{H} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ of (M, γ) subordinate to both $B(\mathbb{S})$ and $B(\mathbb{S}')$. We obtain the diagram $\mathcal{H}_{\mathbb{S},\mathbb{S}'}$ of $M(\mathbb{S}, \mathbb{S}')$ by adding two annuli A and A' to Σ, together with new curves α, β in A and α', β' in A'. We get the diagram $\mathcal{H}_{\mathbb{S}}$ of $M(\mathbb{S})$ by only adding (A, α, β), and $\mathcal{H}_{\mathbb{S}'}$ of $M(\mathbb{S}')$ by adding (A', α', β'). Then the diagram in axiom (3) already commutes on the chain level. Indeed, for any $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we have

$$f_{\mathcal{H}_{\mathbb{S}},\mathbb{S}'} \circ f_{\mathcal{H},\mathbb{S}}(\mathbf{x}) = \mathbf{x} \times \{\theta_{\alpha,\beta}\} \times \{\theta_{\alpha',\beta'}\} =$$
$$\mathbf{x} \times \{\theta_{\alpha',\beta'}\} \times \{\theta_{\alpha,\beta}\} = f_{\mathcal{H}_{\mathbb{S}'},\mathbb{S}} \circ f_{\mathcal{H},\mathbb{S}'}(\mathbf{x}).$$

The case $s = s' = 2$ is similar. We consider the manifold $M(\mathbb{S}, \mathbb{S}')$, and if $\mathbb{P}$ and $\mathbb{P}'$ are the belt spheres of the handles attached along $\mathbb{S}$ and $\mathbb{S}'$, respectively, then we choose disjoint bouquets $B(\mathbb{P})$ and $B(\mathbb{P}')$ for $\mathbb{P}$ and $\mathbb{P}'$, respectively, and a diagram $\mathcal{H}$ subordinate to both. Then commutativity follows on the chain level in the diagram $\mathcal{H}_{\mathbb{P},\mathbb{P}'}$ adapted to both $\mathbb{S}$ and $\mathbb{S}'$.

When $s = 0$ and $s' = 2$, or vice versa, consider the manifold $M(\mathbb{S}')$, and let $\mathbb{P}'$ be the belt sphere of the handle attached along $\mathbb{S}'$. Then pick disjoint bouquets $B(\mathbb{S})$ and $B(\mathbb{P}')$ for $\mathbb{S}$ and $\mathbb{P}'$, respectively, and take a diagram $\mathcal{H}$ subordinate to both. The diagram $\mathcal{H}_{\mathbb{P}'}$ is then a diagram subordinate to both $\mathbb{S}$ and $\mathbb{S}'$, in which commutativity follows on the chain level.

In the case $s = 0$ and $s' = 1$, or vice versa, pick disjoint bouquets $B(\mathbb{S})$ and $B(\mathbb{S}')$. Combining the proofs of Lemma 7.4 and Lemma 6.5, we obtain a triple diagram $\mathcal{T} = (\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ subordinate to $B(\mathbb{S}')$ such that $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is subordinate to $B(\mathbb{S})$ and $\boldsymbol{\delta} \cap B(\mathbb{S}) = \emptyset$. Then commutativity follows from the small triangle argument in the annulus A attached along $\mathbb{S}$ illustrated by Figure 6.

Finally, consider the case $s = 1$ and $s' = 2$. Then, as above, we can obtain a diagram adapted to both $\mathbb{S}$ and $\mathbb{S}'$, in which commutativity follows from a small triangle argument.

Lemma 8.3. *SFH and the surgery maps $F_{M,\mathbb{S}}$ satisfy axiom* (4) *when $\mathbb{S}$ is a framed* 0*-sphere and $\mathbb{S}'$ is a framed* 1*-sphere.*

Proof. We obtain $M(\mathbb{S})$ from M by attaching the tube $D^1 \times S^2$ to $M \setminus N(\mathbb{S})$ along $\partial N(\mathbb{S})$. Since the curve $\mathbb{S}'$ intersects the core $\{0\} \times S^2$ of the tube $D^1 \times S^2$ once transversely, both points of $\mathbb{S}$ have to lie in the same component M_0 of M. Furthermore, we can isotope $\mathbb{S}'$ such that

$$\mathbb{S}' \cap (D^1 \times S^2) = D^1 \times \{(0, 0, 1)\}.$$

Pick a bouquet $B(\mathbb{S}')$ for $\mathbb{S}' \subset M(\mathbb{S})$ by connecting $\mathbb{S}'$ and $R_+(\gamma)$ via an arc disjoint from the tube $D^1 \times S^2$. Then extend $D^1 \times \{(0,0,-1)\}$ to an embedded arc s in $M(\mathbb{S})$ with both endpoints lying in $R_-(\gamma)$ and disjoint from $B(\mathbb{S}')$.

We are now going to construct a triple diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$ of $(M(\mathbb{S}), \gamma)$ subordinate to $B(\mathbb{S}')$ that has a specific form in $D^1 \times S^2$. Namely, we require that

$$\Sigma' \cap (D^1 \times S^2) = D^1 \times S^1, \tag{8.1}$$

and that $\boldsymbol{\alpha}' \cap (D^1 \times S^1) = \{0\} \times S^1$, call this curve α. Furthermore, $\boldsymbol{\beta}' \cap (D^1 \times S^1)$ is a small translate of α such that α and β intersect in two points transversely.

For this end, consider the sutured manifold (M', γ) obtained from $(M(\mathbb{S}), \gamma)$ by removing a regular neighborhood N of $B(\mathbb{S}')$ from $M(\mathbb{S})$ and adding $\partial N \setminus R_+(\gamma)$ to R_+. Note that $U := (\{0\} \times S^2) \setminus N$ is diffeomorphic to D^2. Let B be a regular neighborhood of $U \cup s$, properly embedded in M'; this is a manifold with corners. Recall that sutured functions and gradient-like vector fields for them were defined in [25, Definitions 5.12 and 5.13]. We can choose a Morse function $f \colon B \to [-1,1]$ and a gradient-like vector field v for f such that $f|_{B \cap \partial N} \equiv 1$, $f|_{B \cap R_-(\gamma)} \equiv -1$, it has an index one critical point at $o = (0,(0,0,-1))$ with stable manifold s and unstable manifold U, and no other critical points. Indeed, consider the Morse function

$$g(x,y,z) = -x^2 + y^2 + z^2$$

on $\mathbb{R}^3$. Then there is a diffeomorphism

$$d \colon B \to g^{-1}([-1,1]) \cap 2D^3,$$

and we let $f = g \circ d$. The vector filed v is obtained by pulling back the Euclidean gradient of g along d.

We extend f and v to the rest of (M', γ) as a generic sutured function and a gradient-like vector field, respectively. As in [25, Definition 6.14], choose a spanning tree $T_\pm$ of the graph $\Gamma_\pm(f,v)$ and a splitting surface $\Sigma' \in \Sigma(f,v)$. This data gives rise to a diagram $H(f,v,\Sigma',T_-,T_+)$ of (M',γ) that we denote $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta})$. Since $s = W^s(o)$ corresponds to a loop edge of T_-, it does not lie in the spanning tree T_-. Hence, $\alpha := U \cap \Sigma' \subset \boldsymbol{\alpha}'$ by definition.

By construction, $\Sigma' \cap U = \alpha$. As U is the unstable manifold of o and $\partial U \subset R_+(\gamma)$, there is no flow-line from o to another critical point, and hence $\alpha \cap \boldsymbol{\beta} = \emptyset$. If follows that we can isotope Σ' by "stretching out" a neighborhood of α in $(D^1 \times S^2) \setminus N$ such that it satisfies (8.1), $\boldsymbol{\alpha} \cap (D^1 \times S^1) = \alpha$, and $\boldsymbol{\beta} \cap (D^1 \times S^1) = \emptyset$. If β is a small isotopic translate of α, then $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta} \cup \{\beta\})$ is a diagram of $(M(\mathbb{S}), \gamma)$. We let $\boldsymbol{\beta}' := \boldsymbol{\beta} \cup \{\beta\}$. The framing of $\mathbb{S}'$ is given by a curve in $\partial N(\mathbb{S}')$, flowing this up along v we obtain a curve δ in Σ' that intersects both α and β in a single point. Hence $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta} \cup \{\delta\})$ is a diagram of $(M(\mathbb{S})(\mathbb{S}'), \gamma)$.

We write $A = D^1 \times S^1$, and let c be a component of $s(\gamma)$. Pick disjoint embedded arcs $\eta_\pm \subset \Sigma' \setminus \text{Int}(A)$ connecting $\{\pm 1\} \times S^1$ with c. We can achieve that

$$\eta_\pm \cap (\boldsymbol{\alpha} \cup \boldsymbol{\beta}) = \emptyset$$

by handlesliding each attaching curve over α or β along $\eta_\pm$. Let $\boldsymbol{\delta}$ be a small Hamiltonian translate of $\boldsymbol{\beta}$, and we write $\boldsymbol{\delta}' = \boldsymbol{\delta} \cup \{\delta\}$. By the proof of [12, Lemma 3.12], we can achieve that the triple $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$ is admissible by winding the attaching curves in the complement of A. We do not need to wind in A, since after connecting η_+ and η_- in A, we obtain a properly embedded arc η in Σ' dual to β that only intersects α and β, and these two curves already intersect each other in two points so no further winding is required along η.

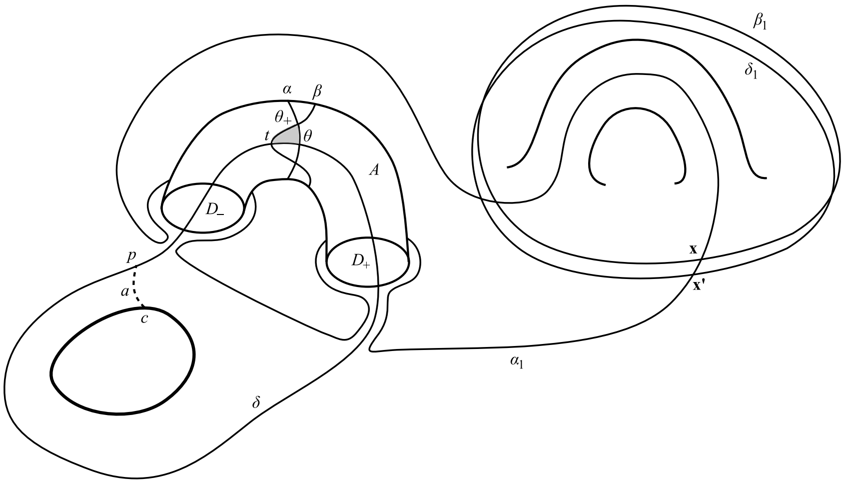

FIGURE 8. A triple diagram corresponding to a canceling pair of one and two-handles.

We set

$$\Sigma := (\Sigma' \setminus \mathrm{Int}(A)) \cup (\partial D^1 \times D^2),$$

where $\partial D^1 \times D^2$ is given by the framing of $\mathbb{S}$ that identifies a neighborhood of each point of $\mathbb{S}$ with D^3. Then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a diagram of (M, γ). We denote $\{\pm 1\} \times D^2$ by $D_\pm$. The surface Σ' gives a framing of the arcs η_+ and η_-, which then give rise to a bouquet $B(\mathbb{S})$ for $\mathbb{S}$. By construction, the diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is subordinate to $B(\mathbb{S})$.

In summary, we have obtained an admissible triple diagram

$$(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}') = (\Sigma^0 \cup A, \boldsymbol{\alpha} \cup \{\alpha\}, \boldsymbol{\beta} \cup \{\beta\}, \boldsymbol{\delta} \cup \{\delta\}), \text{ where}$$

- $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ is a balanced diagram of (M, γ) subordinate the bouquet $B(\mathbb{S})$,
- there is a component c of $\partial\Sigma$, and disks D_- and D_+ lying in the component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ containing c, such that $\Sigma^0 = \Sigma \setminus (D_- \cup D_+)$,
- $A = D^1 \times S^1 \subset M(\mathbb{S})$ is an annulus attached to Σ^0 along $\partial D_- \cup \partial D_+$,
- the curve $\alpha = \{0\} \times S^1 \subset A$, and β is a small Hamiltonian translate of α,
- the attaching set $\boldsymbol{\delta}$ is a small Hamiltonian translate of $\boldsymbol{\beta}$, and δ intersects both α and β transversally in a single point,
- the triple diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$ is subordinate to the bouquet $B(\mathbb{S}')$ for $\mathbb{S}'$, in such a way that β is a meridian of $\mathbb{S}'$, and δ represents the framing of $\mathbb{S}'$.

For an illustration, see Figure 8. Since D_- and D_+ lie in the same component of $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ as c, there is an arc a that connects a point p of δ with c, and whose interior lies in $\Sigma^0 \setminus (\boldsymbol{\alpha}' \cup \boldsymbol{\beta}' \cup \boldsymbol{\delta}')$. Indeed, connect ∂D_- to c with an arc a' inside $\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ that intersects δ transversely, and take a to be the closure of the last component of $a' \setminus \delta$. Since $a \cap \boldsymbol{\beta} = \emptyset$ and $\boldsymbol{\delta}$ is a small Hamiltonian translate of $\boldsymbol{\beta}$, we can also achieve that $a \cap \boldsymbol{\delta} = \emptyset$.

We can also assume that $\delta \cap \boldsymbol{\beta} = \emptyset$, since $\boldsymbol{\delta}$ is a small Hamiltonian translate of $\boldsymbol{\beta}$. Next, we achieve that $\delta \cap \boldsymbol{\alpha} = \emptyset$. Just push the curves in $\boldsymbol{\alpha}$ that intersect δ simultaneously using a finger move along the arcs $\delta \setminus (A \cup \{p\})$ towards $\partial D_- \cup \partial D_+$, then handleslide them over α. This process can be done away from the arc a.

Furthermore, since $\delta \cap \boldsymbol{\beta} = \emptyset$, the new $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ differs from the old one by isotoping the α curves inside $\Sigma \setminus \boldsymbol{\beta}$. So we can suppose that we started with a triple diagram where $\delta \cap \boldsymbol{\alpha} = \emptyset$.

We use the diagram

$$(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}') = (\Sigma^0 \cup A, \boldsymbol{\alpha} \cup \{\alpha\}, \boldsymbol{\beta} \cup \{\beta\})$$

to define the map $F_{M,\mathbb{S}}$, and we compute $F_{M(\mathbb{S}),\mathbb{S}'}$ using the triple $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$. If $\alpha \cap \beta = \{\theta_+, \theta_-\}$, where θ_+ has higher relative grading, then $F_{M,\mathbb{S}}(\mathbf{x}) = \mathbf{x} \times \{\theta_+\}$. The fact that the arc a connects δ and $\partial\Sigma$ and its interior avoids $\boldsymbol{\alpha}' \cup \boldsymbol{\beta}' \cup \boldsymbol{\delta}'$ ensures that every domain in the triple diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$ has multiplicity zero on one side of $\delta \setminus A$. Here, we can talk about the two sides of $\delta \setminus A$ since $\delta \cap (\boldsymbol{\alpha} \cup \boldsymbol{\beta}) = \emptyset$. Note that the only component of $\boldsymbol{\alpha}' \cup \boldsymbol{\beta}' \cup \boldsymbol{\delta}'$ that intersects $\partial D_- \cup \partial D_+$ is δ. Moreover, both $\partial D_- \setminus \delta$ and $\partial D_+ \setminus \delta$ are connected, so every domain in the triple diagram $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$ is the disjoint union of a domain supported in A and a domain supported in Σ^0; i.e., it has zero multiplicity along $\partial D_- \cup \partial D_+$. If $\alpha \cap \delta = \{\theta\}$ and $\beta \cap \delta = \{t\}$, then there is a unique Maslov index zero triangle connecting θ_+, t, and θ that contributes to the composite map in A, which is shaded in Figure 8. Hence the composite map induces the same map on homology as the map

$$CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}) \to CF(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\delta}')$$

given by $\mathbf{x} \mapsto f_{\alpha',\beta',\delta'}((\mathbf{x} \times \{\theta_+\}) \otimes \Theta_{\beta',\delta'}) = f_{\alpha,\beta,\delta}(\mathbf{x} \otimes \Theta_{\beta,\delta}) \times \{\theta\}$ for $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$.

We now compute the map $F(\varphi)$. Recall that φ is the identity on the intersection $M \cap M(\mathbb{S})(\mathbb{S}')$. As $\Sigma' \cap \mathbb{S}' = \emptyset$ and

$$\Sigma' \cap M = \Sigma' \setminus A = \Sigma \setminus (D_+ \cup D_-),$$

it follows that

$$\Sigma \cap (M \cap M(\mathbb{S})(\mathbb{S}')) = \Sigma \setminus (D_+ \cup D_-).$$

But $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are disjoint from $D_\pm$, so $\varphi(\boldsymbol{\alpha}) = \boldsymbol{\alpha}$, $\varphi(\boldsymbol{\beta}) = \boldsymbol{\beta}$, and

$$\varphi(\Sigma) \cap \Sigma \supset \Sigma \setminus (D_+ \cup D_-).$$

Let

$$\mathcal{H} = (\varphi(\Sigma), \varphi(\boldsymbol{\alpha}), \varphi(\boldsymbol{\beta})) = (\varphi(\Sigma), \boldsymbol{\alpha}, \boldsymbol{\beta})$$

and $\mathcal{H}' = (\Sigma', \boldsymbol{\alpha}', \boldsymbol{\delta}')$. For a homology class $x \in SFH(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ represented by an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$, we obtain $F(\varphi)(x)$ by taking the homology class represented by $\varphi(\mathbf{x})$ in $SFH(\mathcal{H})$, then finding its image under the natural isomorphism with $SFH(\mathcal{H}')$. As explained in Section 5.2, we obtain the canonical isomorphism $F_{\mathcal{H},\mathcal{H}'}$ by connecting $\mathcal{H}$ and $\mathcal{H}'$ by a sequence of elementary moves, and composing the isomorphisms induced by these. We can get from $\mathcal{H}$ to $\mathcal{H}'$ by first performing a small Hamiltonian isotopy of $\boldsymbol{\beta}$ to $\boldsymbol{\delta}$, then stabilizing by adding the tube A, both of whose feet are in the same component of $\varphi(\Sigma) \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\delta})$ containing $c \subset \partial\varphi(\Sigma)$, and adding the curves α and δ. As we can connect the feet of A in the complement of the attaching curves, this is the same as taking the connected sum of the diagram with a punctured torus. The isotopy induces the map $\Psi^{\boldsymbol{\alpha}}_{\boldsymbol{\beta}\to\boldsymbol{\delta}}(\mathbf{x}) = f_{\alpha,\beta,\delta}(\mathbf{x} \otimes \Theta_{\beta,\delta})$ given by the same triangle count as $F_{M(\mathbb{S}),\mathbb{S}'}$ in the complement of A. The stabilization then maps $f_{\alpha,\beta,\delta}(\mathbf{x} \otimes \Theta_{\beta,\delta})$ to $f_{\alpha,\beta,\delta}(\mathbf{x} \otimes \Theta_{\beta,\delta}) \times \{\theta\}$. This shows that

$$F_{M(\mathbb{S}),\mathbb{S}'} \circ F_{M,\mathbb{S}} = F(\varphi),$$

and concludes the proof of Lemma 8.3. □

Lemma 8.4. *SFH and the surgery maps $F_{M,\mathbb{S}}$ satisfy axiom* (4) *when* $\mathbb{S}$ *is a framed* 1*-sphere and* $\mathbb{S}'$ *is a framed* 2*-sphere.*

Proof. The proof follows from "turning around" the proof of Lemma 8.3. □

Having shown that *SFH* and the surgery maps $F_{M,\mathbb{S}}$ satisfy all the axioms, this concludes the proof of Theorem 8.2. □

8.1. **Spin^c structures.** To show that the the cobordism maps split along Spin^c structures, we need to be more careful, since there is no canonical way of composing two Spin^c cobordisms. As in the work of Ozsváth and Szabó [38], the solution is to attach all the 2-handles simultaneously, but then we also need to consider handleslides among them.

However, 1- and 3-handles cause no trouble, as we shall now see. Recall from Lemma 3.7 that, for a special cobordism $\mathcal{W}$, it does not matter whether we take the Spin^c structures relative to ∂Z or Z. Hence, in the latter case, if $\mathcal{W} = \mathcal{W}_1 \circ \mathcal{W}_2$, then we can restrict any Spin^c structure to $\mathcal{W}_1$ and $\mathcal{W}_2$.

Lemma 8.5. *Let $\mathcal{W} = (W, Z, \xi)$ be a special cobordism from (M_0, γ_0) to (M_1, γ_1) that is diffeomorphic to the trace of attaching 1-handles to (M_0, γ_0). Then the restriction map Spin$^c(\mathcal{W}) \to$ Spin$^c(M_1, \gamma_1)$ is injective. If $\mathcal{W}$ is a 3-handle cobordism, then the restriction map Spin$^c(\mathcal{W}) \to$ Spin$^c(M_0, \gamma_0)$ is injective.*

Proof. We only treat the case when $\mathcal{W}$ is diffeomorphic to the trace of attaching 1-handles to (M_0, γ_0), as the other case follows by "turning the cobordism upside down." This follows from the exact sequence of the triple $(W, M_1, \partial M_1)$:

$$H^2(W, M_1) \to H^2(W, \partial M_1) \cong H^2(W, Z) \to H^2(M_1, \partial M_1),$$

and the fact that $H^2(W, M_1) \cong H_2(M_0 \cup Z) \cong H_2(W, M_0) = 0$. □

Note that, in the first case, $\mathfrak{t} \in \text{Spin}^c(M_1, \gamma_1)$ extends to $\mathcal{W}$ if and only if $c_1(\mathfrak{t})$ vanishes on the belt spheres of all the 1-handles attached to (M_0, γ_0). In the second case, $\mathfrak{t}$ extends to $\mathcal{W}$ if and only if $c_1(\mathfrak{t})$ vanishes on the belt spheres of the 1-handles in the reversed cobordism $\overline{\mathcal{W}'}$.

Lemma 8.6. *Let $\mathcal{W} = (W, Z, \xi)$ be a special cobordism from (M_0, γ_0) to (M_1, γ_1), and let $\mathcal{W}' = (W', Z', \xi')$ be a special cobordism from (M_1, γ_1) to (M_2, γ_2). If $\mathcal{W}$ or $\mathcal{W}'$ is diffeomorphic to the trace of attaching 1- or 3-handles to (M_0, γ_0), and given Spin^c structures $\mathfrak{s} \in$ Spin$^c(\mathcal{W})$ and $\mathfrak{s}' \in$ Spin$^c(\mathcal{W}')$ such that $\mathfrak{s}|_{M_1} = \mathfrak{s}'|_{M_1}$, then there is a unique Spin^c structure $\mathfrak{s}''$ on the composite $\mathcal{W}' \circ \mathcal{W}$ such that $\mathfrak{s}''|_{\mathcal{W}} = \mathfrak{s}$ and $\mathfrak{s}''|_{\mathcal{W}'} = \mathfrak{s}'$.*

Proof. We only treat the case when $\mathcal{W}$ is the trace of attaching 1- or 3-handles to (M_0, γ_0), as the other case follows by "turning $\mathcal{W}$ upside down." Consider the following relative Mayer-Vietoris sequence:

$$\cdots \to H^1(W, Z) \oplus H^1(W', Z') \xrightarrow{j} H^1(M_1, \partial M_1) \xrightarrow{\delta}$$
$$H^2(W \cup W', Z \cup Z') \to H^2(W, Z) \oplus H^2(W', Z') \to \dots.$$

Then the gluing of $\mathfrak{s}$ and $\mathfrak{s}'$ is unique if and only if $\delta = 0$, or equivalently, if j is surjective. As $j(a, b) = a|_{M_1} - b|_{M_1}$, it suffices to show that the map $H^1(W, Z) \to H^1(M_1, \partial M_1)$ is onto. This again follows from the long exact sequence of the triple $(W, M_1, \partial M_1)$:

$$H^1(W, \partial M_1) \cong H^1(W, Z) \to H^1(M_1, \partial M_1) \to H^2(W, M_1),$$

and the fact that $H^2(W, M_1) \cong H_2(W, M_0) = 0$ using cellular homology, as W only consists of 1- or 3-handles attached to M_0. □

Definition 8.7. We say that a TQFT $F\colon \mathbf{BSut}' \to \mathbf{Vect}_{\mathbb{Z}_2}$ *splits along Spin^c structures* if, for every sutured manifold (M, γ) and $\mathfrak{t} \in \mathrm{Spin}^c(M, \gamma)$, there is a group $F(M, \gamma, \mathfrak{t})$, and for every special cobordism $\mathcal{W}$ from (M_0, γ_0) to (M_1, γ_1) and Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(\mathcal{W})$, there is a linear map

$$F_{\mathcal{W},\mathfrak{s}}\colon F(M, \gamma, \mathfrak{s}|_{M_0}) \to F(M, \gamma, \mathfrak{s}|_{M_1})$$

that satisfy the following properties:

(1) For every sutured manifold (M, γ),
$$F(M, \gamma) = \bigoplus_{\mathfrak{t} \in \mathrm{Spin}^c(M,\gamma)} F(M, \gamma, \mathfrak{t}),$$

(2) Given a diffeomorphism $d\colon (M, \gamma) \to (M', \gamma')$ and $\mathfrak{t} \in \mathrm{Spin}^c(M, \gamma)$, the image of
$$F(d, \mathfrak{t}) := F(d)|_{SFH(M,\gamma,\mathfrak{t})}$$
lies in $F(M, \gamma, d_*(\mathfrak{t}))$.

(3) For every special cobordism $\mathcal{W}$,
$$F_{\mathcal{W}} = \sum_{\mathfrak{s} \in \mathrm{Spin}^c(M,\gamma)} F_{\mathcal{W},\mathfrak{s}}.$$

(4) Given a diffeomorphism D from the Spin^c special cobordism
$$(\mathcal{W}, \mathfrak{s})\colon (M_0, \gamma_0, \mathfrak{t}_0) \to (M_1, \gamma_1, \mathfrak{t}_1)$$
to the Spin^c special cobordism
$$(\mathcal{W}', \mathfrak{s}')\colon (M_0', \gamma_0', \mathfrak{t}_0') \to (M_1', \gamma_1', \mathfrak{t}_1'),$$
the following diagram is commutative:
$$\begin{array}{ccc} F(M_0, \gamma_0, \mathfrak{t}_0) & \xrightarrow{F_{\mathcal{W},\mathfrak{s}}} & F(M_1, \gamma_1, \mathfrak{t}_1) \\ \downarrow{\scriptstyle F(D|_{M_0}, \mathfrak{t}_0)} & & \downarrow{\scriptstyle F(D|_{M_1}, \mathfrak{t}_1)} \\ F(M_0', \gamma_0', \mathfrak{t}_0') & \xrightarrow{F_{\mathcal{W}',\mathfrak{s}'}} & F(M_1', \gamma_1', \mathfrak{t}_1'). \end{array}$$

(5) Let $\mathcal{W}_1$ be a special cobordism from (M_0, γ_0) to (M_1, γ_1), and $\mathcal{W}_2$ a special cobordism from (M_1, γ_1) to (M_2, γ_2), and set $\mathcal{W} = \mathcal{W}_2 \circ \mathcal{W}_1$. Fix Spin^c structures $\mathfrak{s}_i \in \mathrm{Spin}^c(\mathcal{W}_i)$ for $i \in \{1, 2\}$ such that $\mathfrak{s}_1|_{M_1} = \mathfrak{s}_2|_{M_1}$. Then
$$F_{\mathcal{W}_2,\mathfrak{s}_2} \circ F_{\mathcal{W}_1,\mathfrak{s}_1} = \sum_{\{\mathfrak{s} \in \mathrm{Spin}^c(\mathcal{W})\colon\, \mathfrak{s}|_{\mathcal{W}_1} = \mathfrak{s}_1,\, \mathfrak{s}|_{\mathcal{W}_2} = \mathfrak{s}_2\}} F_{\mathcal{W},\mathfrak{s}}.$$

Definition 8.8. Let $\mathbb{L}$ be a framed link in the sutured manifold (M, γ). We are given an embedded framed arc a connecting two distinct components L_i and L_j of $\mathbb{L}$ whose interior is disjoint form $\mathbb{L}$, and whose framing at ∂a is tangent to $\mathbb{L}$. We say that $\mathbb{L}'$ is obtained form $\mathbb{L}$ by *handlesliding L_i over L_j along a* if $\mathbb{L}' = (\mathbb{L} \setminus L_i) \cup L_i'$, where L_i' is the band sum of L_i and a push-off of L_j in the direction of its framing, taken along a.

Corresponding to the above handleslide, there is an ambient isotopy $\{d_t\colon t \in I\}$ in $M(\mathbb{L} \setminus L_i)$ such that $d_1(L_i) = L_i'$. This consists of a finger move of L_i along a, continued by a radial isotopy across the center of the disk $\{a(1)\} \times D^2$ in the handle $D^2 \times D^2$ attached along L_j, and then radially "blowing out" the portion of the curve in $\{a(1)\} \times D^2$ until it reaches $\{a(1)\} \times S^1$. The space of such isotopies is contractible. Then d_t extends to an isotopy D_t of $W(\mathbb{L} \setminus L_i)$ that is the identity outside a collar neighborhood of $M(\mathbb{L} \setminus L_i)$. Furthermore, d_1 induces a diffeomorphism $d_1^{L_i}\colon M(\mathbb{L}) \to M(\mathbb{L}')$, and D_1 induces a diffeomorphism $D_1^{L_i}\colon W(\mathbb{L}) \to W(\mathbb{L}')$ in a natural manner.

We have the following refinement of Theorem 8.1. Note that here we also use the symbol $\mathbb{S}$ to denote a framed link.

Theorem 8.9. *To define a functor $F\colon \mathbf{BSut}' \to \mathbf{Vect}_{\mathbb{Z}_2}$ that splits along Spin^c structures, it suffices to construct a functor F from the category of Spin^c balanced sutured manifolds and diffeomorphisms to $\mathbf{Vect}_{\mathbb{Z}_2}$, and for every balanced sutured manifold (M,γ), framed 0-sphere, link, or 2-sphere $\mathbb{S} \subset Int(M)$, and Spin^c structure $\mathfrak{s} \in Spin^c(W(\mathbb{S}))$, a linear map*

$$F_{M,\mathbb{S},\mathfrak{s}}\colon F(M,\gamma,\mathfrak{s}|_M) \to F(M(\mathbb{S}),\gamma,\mathfrak{s}|_{M(\mathbb{S})})$$

that satisfy the following axioms:

1. *If $d \in Diff_0(M,\gamma)$, then $F(d,\mathfrak{s}) = Id_{F(M,\gamma,\mathfrak{s})}$.*
2. *Consider a diffeomorphism $d\colon (M,\gamma) \to (M',\gamma')$ between balanced sutured manifolds, a framed 0-sphere, link, or 2-sphere $\mathbb{S} \subset Int(M)$, and a Spin^c structure $\mathfrak{s} \in Spin^c(W(\mathbb{S}))$. Let $\mathbb{S}' = d(\mathbb{S})$, let*
$$d^{\mathbb{S}}\colon (M(\mathbb{S}),\gamma') \to (M'(\mathbb{S}'),\gamma') \text{ and } D^{\mathbb{S}}\colon W(\mathbb{S}) \to W(\mathbb{S}')$$
be the induced diffeomorphisms, and write $\mathfrak{s}' = D^{\mathbb{S}}(\mathfrak{s}) \in Spin^c(W(\mathbb{S}'))$. Then the following diagram is commutative:
$$\begin{array}{ccc} F(M,\gamma,\mathfrak{s}|_M) & \xrightarrow{F_{M,\mathbb{S},\mathfrak{s}}} & F(M(\mathbb{S}),\gamma,\mathfrak{s}|_{M(\mathbb{S})}) \\ \downarrow{\scriptstyle F(d)} & & \downarrow{\scriptstyle F(d^{\mathbb{S}})} \\ F(M',\gamma',\mathfrak{s}'|_{M'}) & \xrightarrow{F_{M',\mathbb{S}',\mathfrak{s}'}} & F(M'(\mathbb{S}'),\gamma',\mathfrak{s}'|_{M'(\mathbb{S}')}). \end{array}$$
3. *If (M,γ) is a balanced sutured manifold and $\mathbb{S}$ and $\mathbb{S}'$ are* disjoint *framed 0-spheres, links, or 2-spheres in M, then $M(\mathbb{S})(\mathbb{S}') = M(\mathbb{S}')(\mathbb{S})$, we denote this manifold by $M(\mathbb{S},\mathbb{S}')$. There is an isotopically unique equivalence*
$$\mathcal{W} := W(M(\mathbb{S}),\mathbb{S}') \circ W(M,\mathbb{S}) \cong W(M(\mathbb{S}'),\mathbb{S}) \circ W(M,\mathbb{S}').$$
Given $\mathfrak{s} \in Spin^c(\mathcal{W})$, this equivalence allows us to define $\mathfrak{s}_{\mathbb{S}} = \mathfrak{s}|_{W(M,\mathbb{S})}$, $\mathfrak{s}_{\mathbb{S}'} = \mathfrak{s}|_{W(\mathbb{S}')}$, $\mathfrak{s}_{\mathbb{S},\mathbb{S}'} = \mathfrak{s}|_{W(M(\mathbb{S}),\mathbb{S}')}$, and $\mathfrak{s}_{\mathbb{S}',\mathbb{S}} = \mathfrak{s}|_{W(M(\mathbb{S}'),\mathbb{S})}$. If at least one of $\mathbb{S}$ and $\mathbb{S}'$ is not 1-dimensional, then the following diagram is commutative:
$$\begin{array}{ccc} F(M,\gamma,\mathfrak{s}|_M) & \xrightarrow{F_{M,\mathbb{S},\mathfrak{s}_{\mathbb{S}}}} & F(M(\mathbb{S}),\gamma,\mathfrak{s}|_{M(\mathbb{S})}) \\ \downarrow{\scriptstyle F_{M,\mathbb{S}',\mathfrak{s}_{\mathbb{S}'}}} & & \downarrow{\scriptstyle F_{M(\mathbb{S}),\mathbb{S}',\mathfrak{s}_{\mathbb{S},\mathbb{S}'}}} \\ F(M(\mathbb{S}'),\gamma,\mathfrak{s}|_{M(\mathbb{S}')}) & \xrightarrow{F_{M(\mathbb{S}'),\mathbb{S},\mathfrak{s}_{\mathbb{S}',\mathbb{S}}}} & F(M(\mathbb{S},\mathbb{S}'),\gamma,\mathfrak{s}|_{M(\mathbb{S},\mathbb{S}')}). \end{array}$$

(4) *Let $\mathbb{L} = \mathbb{L}_1 \cup \mathbb{L}_2$ be a partition of a framed link in M, and write $\mathcal{W}_1 = W(M, \mathbb{L}_1)$, $\mathcal{W}_2 = W(M(\mathbb{L}_1), \mathbb{L}_2)$, and $\mathcal{W} = \mathcal{W}_2 \circ \mathcal{W}_1 \cong W(\mathbb{L})$. If $\mathfrak{s}_1 \in Spin^c(\mathcal{W}_1)$ and $\mathfrak{s}_2 \in Spin^c(\mathcal{W}_2)$, then*

$$F_{M(\mathbb{L}_1),\mathbb{L}_2,\mathfrak{s}_2} \circ F_{M,\mathbb{L}_1,\mathfrak{s}_1} = \sum_{\mathfrak{s} \in Spin^c(\mathcal{W}) \colon \, \mathfrak{s}|_{\mathcal{W}_1} = \mathfrak{s}_1, \; \mathfrak{s}|_{\mathcal{W}_2} = \mathfrak{s}_2} F_{M,\mathbb{L},\mathfrak{s}}.$$

(5) *If $\mathbb{S}' \subset M(\mathbb{S})$ intersects the belt sphere of the handle attached along $\mathbb{S}$ once transversely, then there is a diffeomorphism $\varphi \colon M \to M(\mathbb{S})(\mathbb{S}')$ (which is the identity on $M \cap M(\mathbb{S})(\mathbb{S}')$, and is unique up to isotopy), such that for every $\mathfrak{s} \in Spin^c(W(\mathbb{S}') \circ W(\mathbb{S}))$, if we write $\mathfrak{s}_{\mathbb{S}} = \mathfrak{s}|_{W(\mathbb{S})}$ and $\mathfrak{s}_{\mathbb{S}'} = \mathfrak{s}|_{W(\mathbb{S}')}$, then $\mathfrak{s}|_{M(\mathbb{S})(\mathbb{S}')} = \varphi_*(\mathfrak{s}|_M)$, and*

$$F_{M(\mathbb{S}),\mathbb{S}',\mathfrak{s}_{\mathbb{S}'}} \circ F_{M,\mathbb{S},\mathfrak{s}_{\mathbb{S}}} = F(\varphi, \mathfrak{s}|_M),$$

where $F(\varphi, \mathfrak{s}|_M)$ is the restriction of $F(\varphi)$ to $SFH(M, \gamma, \mathfrak{s}|_M)$.

(6) *The 2-handle maps are invariant under handleslides. More precisely, suppose that $\mathbb{L}$ is a framed link in (M, γ), and $\mathbb{L}'$ is obtained by sliding L_i over L_j along a framed arc a. Let $\mathfrak{s} \in Spin^c(W(\mathbb{L}))$, and let $\mathfrak{s}' = (D_1^{L_i})_*(\mathfrak{s})$, $\mathfrak{t} = \mathfrak{s}|_M$, $\mathfrak{t}_{\mathbb{L}} = \mathfrak{s}|_{M(\mathbb{L})}$, and $\mathfrak{t}_{\mathbb{L}'} = \mathfrak{s}'|_{M(\mathbb{L}')}$. Then the following diagram is commutative:*

$$\begin{array}{ccc} F(M,\gamma,\mathfrak{t}) & \xrightarrow{\quad Id \quad} & F(M,\gamma,\mathfrak{t}) \\ {\scriptstyle F_{M,\mathbb{L},\mathfrak{s}}} \downarrow & & \downarrow {\scriptstyle F_{M,\mathbb{L}',\mathfrak{s}'}} \\ F(M(\mathbb{L}),\gamma,\mathfrak{t}_{\mathbb{L}}) & \xrightarrow{F\left(d_1^{L_i}, \mathfrak{t}_{\mathbb{L}}\right)} & F(M(\mathbb{L}'),\gamma,\mathfrak{t}_{\mathbb{L}'}). \end{array}$$

(7) *For every framed 0-sphere, link, or 2-sphere $\mathbb{S}$ and $\mathfrak{s} \in Spin^c(W(\mathbb{S})) = Spin^c(W(\overline{\mathbb{S}}))$, we have $F_{M,\mathbb{S},\mathfrak{s}} = F_{M,\overline{\mathbb{S}},\mathfrak{s}}$.*

*The functor F is a TQFT if and only if it is symmetric and monoidal. In the opposite direction, every functor $F \colon$ **BSut**$' \to$ **Vect**$_{\mathbb{Z}_2}$ that splits along $Spin^c$ structures arises in this way.*

Proof. This is similar to the proof of [22, Theorem 1.2]. The additional idea is the following. Let $\mathcal{W} = (W, Z, \xi)$ be a special cobordism from (M_0, γ_0) to (M_1, γ_1), and fix a metric on W. We say that a Morse function f on W is *nice* if it is the projection $\partial M_0 \times I \to I$ on Z, has only critical points of index 1, 2, and 3, and such that all index 2 critical values lie between the index 1 and 3 critical values. Furthermore, we require that all index 1 and 3 critical points have distinct values, and there is no gradient flow-line between two index 2 critical points.

A nice Morse function gives rise to a type of parametrized Cerf decomposition (cf. [22]) where we first attach 1-handles, then attach all 2-handles simultaneously along a framed link, and finally attach the 3-handles. More specifically:

Definition 8.10. A *parameterized Kirby decomposition* of a special cobordism $\mathcal{W}$ from (M, γ) to (M', γ') consists of

- a decomposition
$$\mathcal{W} = \mathcal{W}_0 \circ \mathcal{W}_1 \circ \cdots \circ \mathcal{W}_m,$$
where each $\mathcal{W}_i$ is a cobordism from (M_i, γ_i) to (M_{i+1}, γ_{i+1}); furthermore, $(M_0, \gamma_0) = (M, \gamma)$ and $(M_{m+1}, \gamma_{m+1}) = (M', \gamma')$,

- there is a number $c \in \{1, \dots, m\}$ such that $\mathcal{W}_i$ is an elementary 1-handle cobordism for $i < c$ and is an elementary 3-handle cobordism for $i > c$, and $\mathcal{W}_c$ is diffeomorphic to $W(\mathbb{L})$ for some framed link $\mathbb{L} \subset M_c$,
- an attaching sphere $\mathbb{S}_i \subset M_i$ for W_i, where $\mathbb{S}_c = \mathbb{L}$,
- a diffeomorphism $D_i \colon W(\mathbb{S}_i) \to W_i$, well-defined up to isotopy, such that $D_i(x, 0) = x$ for $x \in M_i$. We write d_i for
$$D_i|_{M_i(\mathbb{S}_i)} \colon (M_i(\mathbb{S}_i), \gamma_i) \to (M_{i+1}, \gamma_{i+1}).$$

Let $\mathfrak{s} \in \text{Spin}^c(\mathcal{W})$, and consider $\mathfrak{t} = \mathfrak{s}|_M$ and $\mathfrak{t}' = \mathfrak{s}|_{M'}$. Given a parameterized Kirby decomposition $\mathcal{K}$ of $\mathcal{W}$, we write $\mathfrak{s}_i$ for $(D_i)_*^{-1}(\mathfrak{s}|_{\mathcal{W}_i}) \in \text{Spin}^c(W(\mathbb{S}_i))$ for $i \in \{1, \dots, m\}$. Then we define

$$F(\mathcal{W}, \mathfrak{s}, \mathcal{K}) = \prod_{i=0}^{m} (F(d_i) \circ F_{M_i, \mathbb{S}_i, \mathfrak{s}_i}) : F(M, \gamma, \mathfrak{t}) \to F(M', \gamma', \mathfrak{t}').$$

This is consistent with the composition rule (5) of Definition 8.7 since $\mathfrak{s}|_{\mathcal{W}_c}$ uniquely determines $\mathfrak{s}$ according to Lemmas 8.5 and 8.6. In particular, if F is a TQFT that splits along Spin^c structures, then

$$F_{\mathcal{W}, \mathfrak{s}} = \prod_{i=0}^{m} F_{\mathcal{W}_i, \mathfrak{s}|_{\mathcal{W}_i}}$$

since $\mathfrak{s}$ is the only Spin^c structure on $\mathcal{W}$ that restricts to $\mathfrak{s}|_{\mathcal{W}_i}$ for every $i \in \{1, \dots, m\}$. Furthermore, if we compose two special cobordisms endowed with parameterized Kirby decompositions, then we can move the 1-handles to the bottom and the 3-handles to the top using axiom (3) of Theorem 8.9, then compose the 2-handle cobordisms maps via axiom (4). Again, Lemma 8.6 ensures that gluing Spin^c structures is not unique only in the case of two 2-handle cobordisms.

It remains to show that $F(\mathcal{W}, \mathfrak{s}, \mathcal{K})$ is independent of $\mathcal{K}$. Let f_0 and f_1 be nice Morse functions on W, with associated parameterized Kirby decompositions $\mathcal{K}_0$ and $\mathcal{K}_1$, respectively. By part 3 of the paper of Kirby on the calculus [27], there is a path of smooth function $\{f_t \colon t \in I\}$ connecting f_0 and f_1 such that it is through nice Morse functions, except for a finite number of parameter values, when there is either an index 1-2 birth-death between the index 1 and 2 Morse critical values, or an index 2-3 birth-death between the index 2 and 3 Morse critical values. Furthermore, two index 1 or two index 3 critical points might have the same value, or there might be a gradient flow-line between two index 2 critical points.

These bifurcations give rise to a set of moves connecting $\mathcal{K}_0$ and $\mathcal{K}_1$, where one has to keep track of the diffeomorphisms D_i, not just d_i. Invariance of $F(\mathcal{W}, \mathfrak{s}, \mathcal{K})$ under birth-death bifurcations follows from axiom (5). Note that if $\mathcal{W} = \mathcal{W}_3 \circ \mathcal{W}_2 \circ \mathcal{W}_1$, and $\mathcal{W}_1$ is an elementary 1-handle cobordism canceling the elementary 2-handle cobordism $\mathcal{W}_2$, and $\mathcal{W}_3$ is a 2-handle cobordism, then every $\mathfrak{s}_3 \in \text{Spin}^c(\mathcal{W}_3)$ uniquely extends to a Spin^c structure $\mathfrak{s} \in \text{Spin}^c(\mathcal{W})$ as $\mathcal{W}_1 \circ \mathcal{W}_2$ is a product cobordism, and hence $\mathfrak{s}|_{\mathcal{W}_1}$ and $\mathfrak{s}|_{\mathcal{W}_2}$ are also uniquely determined by $\mathfrak{s}_3$. So there is no ambiguity in gluing Spin^c structures when checking handle cancelation invariance. An analogous argument applies for canceling a 2- and a 3-handle. Invariance under a 2-handle slide, corresponding to a flow-line between index 2 critical points, follows from axiom (6). Invariance under two index 1 or two index 3 critical points passing each other follows from axiom (3). If there are no bifurcations, the Kirby decompositions change by isotoping the diffeomorphisms d_i and the attaching spheres $\mathbb{S}$, and invariance under these follows from axioms (1) and (2). From here, we proceed as in [22, Theorem 1.2]. □

Theorem 8.11. *The functor SFH together with the surgery maps $F_{M,\mathbb{P},\mathfrak{s}}$, $F_{M,\mathbb{L},\mathfrak{s}}$, and $F_{M,S,\mathfrak{s}}$ satisfy the axioms listed in Theorem 8.9, hence they give rise to a TQFT $F\colon \mathbf{BSut}' \to \mathbf{Vect}_{\mathbb{Z}_2}$ that splits along Spin^c structures.*

Proof. We proved that the maps $F_{M,\mathbb{L},\mathfrak{s}}$ satisfy axiom (4) in Proposition 6.10. Every other axiom follows essentially as in the proof of Theorem 8.1, except for axiom (6), invariance of $F_{M,\mathbb{L},\mathfrak{s}}$ under 2-handles slides:

We generalize the proof of [38, Lemma 4.14]. Let $K_1, \dots, K_n$ be the components of $\mathbb{L}$, and $K_1', \dots, K_n'$ the components of $\mathbb{L}'$. Suppose that K_1' is obtained from K_1 by a handleslide over K_2 along a framed arc a, and $K_i' = K_i$ for $i \in \{2, \dots, n\}$. After the handleslide, there is a natural path a' joining K_1' and K_2.

We construct a bouquet $B(\mathbb{L})$ for $\mathbb{L}$ by picking arcs a_i connecting K_i to $R_+(\gamma)$ for $i \in \{1, \dots, n\}$, as follows. Isotope a fixing ∂a until it intersects $R_+(\gamma)$ in a single point p. Then choose the arcs a_1 and a_2 such that they run close to a, and they both end near p. We pick $a_3, \dots, a_n$ in an arbitrary way. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ be a triple diagram subordinate to $B(\mathbb{L})$, such that β_1 is dual to K_1 and β_2 is dual to K_2. Let $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', \boldsymbol{\delta}')$ be the triple diagram where β_2' is obtained by a handleslide of β_2 over β_1 along a, and γ_1' is obtained by a handleslide of γ_1 over γ_2 along a, while $\beta_i' = \beta_i$ and $\gamma_i' = \gamma_i$ for $i \in \{2, \dots, n\}$. Then $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', \boldsymbol{\delta}')$ is subordinate to a bouquet $B(\mathbb{L}')$ for $\mathbb{L}'$, constructed using a'.

We then have the commutative diagram

$$\begin{array}{ccc}
SFH(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{t}) & \xrightarrow{\otimes \Theta_{\beta,\beta'}} & SFH(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}', \mathfrak{t}) \\
\Big\downarrow{\scriptstyle \otimes \Theta_{\beta,\delta}} & & \Big\downarrow{\scriptstyle \otimes \Theta_{\beta',\delta'}} \\
SFH(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \mathfrak{t}_{\mathbb{L}}) & \xrightarrow{\otimes \Theta_{\delta,\delta'}} & SFH(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}', \mathfrak{t}_{\mathbb{L}'}),
\end{array}$$

where the vertical arrows only count triangles representing the Spin^c structures $r(\mathfrak{s})$ and $r(\mathfrak{s}')$, respectively. Commutativity follows from associativity, and the observation that

$$F_{\beta,\beta',\delta'}(\Theta_{\beta,\beta'} \otimes \Theta_{\beta',\delta'}, \mathfrak{s}_0) = \Theta_{\beta,\delta'} = F_{\beta,\delta,\delta'}(\Theta_{\beta,\delta} \otimes \Theta_{\delta,\delta'}, \mathfrak{s}_0),$$

according to the handleslide invariance of the homology groups. This concludes the proof of Theorem 8.11. □

Proposition 8.12. *Let $\mathcal{W}$ be a special cobordism. Then there are only finitely many $\mathfrak{s} \in Spin^c(\mathcal{W})$ for which $F_{\mathcal{W},\mathfrak{s}} \neq 0$, and*

$$F_{\mathcal{W}} = \sum_{\mathfrak{s} \in Spin^c(\mathcal{W})} F_{\mathcal{W},\mathfrak{s}}.$$

Proof. Write $\mathcal{W} = \mathcal{W}_3 \circ \mathcal{W}_2 \circ \mathcal{W}_1$, where $\mathcal{W}_i$ is a composition of i-handle cobordisms for $i \in \{1, 2, 3\}$. If $\mathfrak{s}$, $\mathfrak{s}' \in \text{Spin}^c(\mathcal{W})$ satisfy $\mathfrak{s}|_{\mathcal{W}_2} = \mathfrak{s}'|_{\mathcal{W}_2}$, then $\mathfrak{s} = \mathfrak{s}'$ according to Lemmas 8.5 and 8.6. So the claim follows from Proposition 5.14. □

In the following two results, we explicitly spell out two of the consequences of Theorem 8.9 stating that *SFH* is a TQFT that splits along Spin^c structures.

Proposition 8.13. *Let (M, γ) be a balanced sutured manifold. If $\mathcal{W} = (W, Z, [\xi])$ is the trivial cobordism from (M, γ) to (M, γ), then $F_{\mathcal{W}}$ is the identity of $SFH(M, \gamma)$. Furthermore, the restriction map from $Spin^c(\mathcal{W})$ to $Spin^c(M, \gamma)$ is an isomorphism, and for every $\mathfrak{s} \in Spin^c(\mathcal{W})$, the map $F_{\mathcal{W},\mathfrak{s}}$ is the identity of $SFH(M, \gamma, \mathfrak{s}|_M)$.*

Proof. Since $W = M \times I$, $Z = \partial M \times I$, and ξ is I-invariant, $\text{Spin}^c(\mathcal{W})$ and $\text{Spin}^c(M, \gamma)$ are obviously isomorphic. The rest follows from the fact that there is a relative handle decomposition of $\mathcal{W}$ with no handles at all. If $\mathbb{L} = \emptyset$, then both $F_{M,\mathbb{L}}$ and $F_{M,\mathbb{L},\mathfrak{s}}$ are identity maps, as they are defined via a triple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$, where δ_i is a small Hamiltonian translate of β_i for every $i \in \{1, \dots, d\}$. □

Theorem 8.14. *Let $\mathcal{W}_1$ be a special cobordism from (M_0, γ_0) to (M_1, γ_1), and $\mathcal{W}_2$ a special cobordism from (M_1, γ_1) to (M_2, γ_2), and set $\mathcal{W} = \mathcal{W}_2 \circ \mathcal{W}_1$. Fix $Spin^c$ structures $\mathfrak{s}_i \in Spin^c(\mathcal{W}_i)$ for $i \in \{1, 2\}$ such that $\mathfrak{s}_1|_{M_1} = \mathfrak{s}_2|_{M_1}$. Then*

$$F_{\mathcal{W}_2,\mathfrak{s}_2} \circ F_{\mathcal{W}_1,\mathfrak{s}_1} = \sum_{\{\mathfrak{s} \in Spin^c(\mathcal{W}) \colon\, \mathfrak{s}|_{\mathcal{W}_1} = \mathfrak{s}_1,\, \mathfrak{s}|_{\mathcal{W}_2} = \mathfrak{s}_2\}} F_{\mathcal{W},\mathfrak{s}}.$$

Moreover, $F_{\mathcal{W}} = F_{\mathcal{W}_2} \circ F_{\mathcal{W}_1}$.

9. The contact invariant EH and the gluing map Φ_ξ

Here, we review the necessary definitions and result from [16], [34], and [14], and enrich the theory with Spin^c structures. We will prove the necessary naturality results in a separate paper.

9.1. The contact invariant EH.

Suppose that (M, ξ) is a contact 3-manifold with convex boundary and dividing set γ on ∂M. We denote such a contact manifold by (M, γ, ξ). Honda, Kazez, and Matić [16] defined an invariant of (M, γ, ξ) which is an element $EH(M, \gamma, \xi)$ of $SFH(-M, -\gamma)$, also see [34]. We briefly review the construction.

Definition 9.1. A *partial open book decomposition* is a pair $(S, h\colon P \to S)$, where

- S is a compact oriented surface with $\partial S \neq \emptyset$, called the *page*,
- $P \subset S$ is a compact subsurface, such that each component of ∂P is polygonal with consecutive sides $r_1, \dots, r_{2n}$, and $r_i \subset \partial S$ for i even,
- $h\colon P \to S$ is a diffeomorphism such that $h|_{P \cap \partial S} = \text{Id}$.

The partial open book (S, h) *defines* a contact manifold (M, γ, ξ) as follows. Let $M = S \times I/_{\sim_h}$, where $\sim_h$ is the equivalence relation such that $(x, t) \sim_h (x, t')$ for all $x \in \partial S$ and $t \in I$, and $(x, 1) \sim_h (h(x), 0)$ for all $x \in P$. Furthermore, let $R_+(\gamma) = \text{Int}(S \setminus P) \times \{1\}$ and $R_-(\gamma) = \text{Int}(S \setminus h(P)) \times \{0\}$, whereas the sutures are $\gamma = \partial R_+(\gamma) = \partial R_-(\gamma)$. To define ξ, notice that one can obtain (M, γ) by gluing together the product sutured manifolds $(S \times I, \partial S \times I)$ and $(P \times I, -\partial P \times I)$. Each of them carries a unique product disc decomposable contact structure, ξ is obtained by gluing these together.

Given a contact manifold (M, γ, ξ), there exists a compatible partial open book decomposition $(S, h\colon P \to S)$ by [16, Theorem 1.3]. The closure of $S \setminus P$ is naturally identified with $R_+(\gamma)$. A set $\{b_1, \dots, b_d\}$ of properly embedded, pairwise disjoint arcs in P is called a *basis for* $(S, R_+(\gamma))$ if $S \setminus \left(\bigcup_{i=1}^d b_i\right)$ deformation retracts onto $R_+(\gamma)$. In fact, $\{b_1, \dots, b_d\}$ is a basis for $H_1(P, P \cap \partial S)$; fix such a basis.

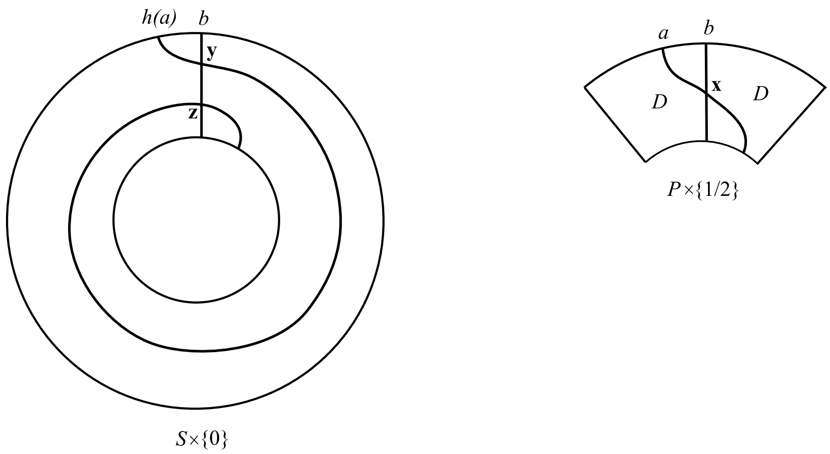

FIGURE 9. A balanced diagram obtained from a partial open book decomposition.

For $i \in \{1, \dots, d\}$, let a_i be an arc that is isotopic to b_i by a small isotopy such that the following hold:

- The endpoints of b_i are isotoped along ∂S in the direction given by the boundary orientation of S.
- The arcs a_i and b_i intersect transversely in one point in $\text{Int}(S)$.

Then we obtain a balanced diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defining (M, γ) by setting

$$\Sigma = (S \times \{0\}) \cup -(P \times \{1/2\}),$$

and taking

$$\alpha_i = (a_i \times \{1/2\}) \cup (h(a_i) \times \{0\})$$

and $\beta_i = \partial(b_i \times [0, 1/2])$ for $i \in \{1, \dots, d\}$, see Figure 9. To see that the orientations match up, note that $\partial\Sigma = \gamma$, and Σ is oriented as the boundary of the $\boldsymbol{\alpha}$ compression body.

Let $x_i = (a_i \cap b_i) \times \{1/2\}$ for $i = 1, \dots, d$. Then $\mathbf{x} = (x_1, \dots, x_d)$ is a cycle in $CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$. Indeed, if D is the closure of a component of $-\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ such that $x_i \in \partial D$ and $\partial D \cap \alpha_i$ points out of x_i, then $D \cap \partial\Sigma \neq \emptyset$. So every domain emanating from $\mathbf{x}$ is forced to be zero. Note that $(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defines $(-M, -\gamma)$. Hence $\mathbf{x}$ defines a class in $SFH(-M, -\gamma)$, which is shown to be an invariant of the contact manifold (M, γ, ξ) in [16, Theorem 3.1]. This is the contact invariant $EH(M, \gamma, \xi)$.

Remark 9.2. Note that we changed the notation of [16], where they labeled our α curves by β, our β curves by α, and our $-\Sigma$ by Σ. We did this to make the underlying orientation conventions more transparent. In fact, with the notation of [16], one has $\mathbf{x} \in CF(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$, but their $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defines $(M, -\gamma)$ instead of (M, γ). If $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defined (M, γ), then $(\Sigma, \boldsymbol{\beta}, \boldsymbol{\alpha})$ would define $(-M, \gamma)$. However, this is not a real issue as $SFH(-M, \gamma)$ and $SFH(-M, -\gamma)$ are canonically isomorphic by [7, Proposition 2.14].

Example 9.3. To illustrate the above construction, let us review [34, Example 2]. For an illustration, see Figure 9. In this example, the page S of our partial open

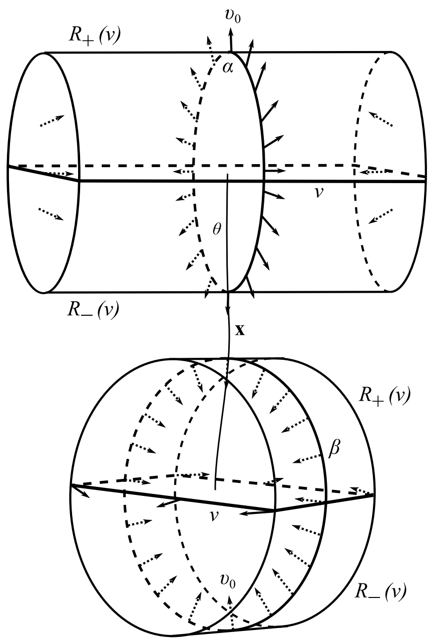

FIGURE 10. A contact 1-handle above a contact 2-handle, together with the vector field v_0 used for constructing the relative Spin^c structures.

book decomposition is an annulus, and P is a radial sector of S. The diffeomorphism $h\colon P \to S$ is the restriction of a left-handed Dehn twist to P. This partial open book decomposition $(S, h\colon P \to S)$ defines a contact manifold (M, γ, ξ). In the resulting chain complex $CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$, there are three generators; we denote them by $\mathbf{x}$, $\mathbf{y}$, and $\mathbf{z}$, as in Figure 9. Observe that $\partial \mathbf{x} = 0$, while $\partial \mathbf{y} = \mathbf{x}$ and $\partial \mathbf{z} = \mathbf{x}$. Thus $SFH(-M, -\gamma) \cong \mathbb{Z}_2$ and $EH(M, \gamma, \xi) = [\mathbf{x}] = 0$. The sutured manifold (M, γ) is the once punctured sphere $S^3(1)$, and ξ is an overtwisted contact structure on it.

As described above [7, Proposition 2.14], if the vector field v represents a Spin^c structure on (M, γ), then the same vector field also represents a Spin^c structure on $(-M, -\gamma)$. Hence there is a canonical identification between Spin$^c(M, \gamma)$ and Spin$^c(-M, -\gamma)$, and we will not distinguish between the two.

Proposition 9.4. *Suppose that (M, γ, ξ) is a contact manifold, and let $\mathfrak{s}_\xi \in$ Spin$^c(M, \gamma)$ be the homology class of the vector field $\xi^\perp$. Then*

$$EH(M, \gamma, \xi) \in SFH(-M, -\gamma, \mathfrak{s}_\xi).$$

Proof. The partial open book used to define the contact class $EH(M, \gamma, \xi)$ is constructed using a relative contact handle decomposition of (M, γ, ξ). One takes the product $R_-(\gamma) \times I$, and attaches d contact 1-handles, then d contact 2-handles. Suppose that H is a contact 1- or 2-handle. This means that $\xi|_H$ is contactomorphic to the unique tight contact structure on D^3 with convex boundary and connected dividing set. Denote by ν the dividing set on ∂H.

The handle decomposition gives a balanced diagram $(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $(-M, -\gamma)$, which agrees with the one arising from the partial open book decomposition. The Heegaard surface Σ is obtained from $R_-(\gamma) \times \{1\}$ by performing the 1-handle surgeries. If H is a 1-handle, then its belt circle is an α-curve that intersects ν in exactly two points. If H is a 2-handle, then its attaching circle is a β-curve that also intersects ν in two points. The contact element is represented by an intersection point $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ such that $\mathbf{x} \cap H$ lies on $R_-(\nu)$ if H is a 1-handle, and on $R_+(\nu)$ if H is a 2-handle.

The construction of the Spin^c structure $\mathfrak{s}(\mathbf{x})$ associated to an intersection point $\mathbf{x}$ coming from a relative handle decomposition is explained in Section 3.4 of [7]. A vector field v representing $\mathfrak{s}(\mathbf{x})$ is obtained as follows. One takes the vector field v_0 on M that agrees with $\partial/\partial t$ on $M \times I$. On a 1-handle $H = D^1 \times D^2$, we take

$$v_0 = -x \cdot \frac{\partial}{\partial x} + y \cdot \frac{\partial}{\partial y} + z \cdot \frac{\partial}{\partial z},$$

while on a 2-handle $H = D^2 \times D^1$ one considers

$$v_0 = -x \cdot \frac{\partial}{\partial x} - y \cdot \frac{\partial}{\partial y} + z \cdot \frac{\partial}{\partial z}.$$

We also choose a smooth 1-chain θ in M that connects the centers of the 1- and 2-handles, and passes through $\mathbf{x}$. Then one extends $v_0|_{M \setminus N(\theta)}$ to a nowhere zero vector field v on M. Suppose that $\theta_1, \dots, \theta_d$ are the components of θ, and θ_i connects the centers of the handles H_{α_i} and H_{β_i} corresponding to α_i and β_i, respectively, for $i \in \{1, \dots, d\}$. Along θ_i, the vectors v_0 point from the center of H_{α_i} towards the center of H_{β_i}, whereas v points in the opposite direction.

Now we show that v and $\xi^\perp$ are homotopic over the 2-skeleton of M; i.e., they are homologous. This will follow if we prove that for every 1- or 2-handle H, and for every $p \in \partial H$, we have

$$v_p \neq -(\xi^\perp)_p.$$

Indeed, on $R_-(\gamma) \times I$ both v and $\xi^\perp$ point up. If v is generic, the set where v and $\xi^\perp$ are opposite represent the difference of the corresponding Spin^c structures. If each component of this difference cycle lies in a handle, then it is obviously null-homologous.

First, observe that $\xi^\perp$ points into H along $R_-(\nu)$, and points out of H along $R_+(\nu)$. Hence, one has

$$(v_0)_p = -(\xi^\perp)_p$$

for exactly one point $p \in \partial H$, which we can assume to be $\mathbf{x} \cap H$. Indeed, if H is a 1-handle H_{α_i}, then along its belt circle α_i the field v_0 points out, and $\alpha_i \cap R_-(\nu)$ consists of an arc whose "midpoint" x_i is where v_0 and $\xi^\perp$ are opposite. Similarly, if H is a 2-handle H_{β_i} with attaching circle β_i, then the midpoint x_i of the arc $\beta_i \cap R_+(\gamma)$ is where v_0 and $\xi^\perp$ are opposite. However, as described above, v is opposite to v_0 along θ. In particular, v and $\xi^\perp$ point in the same direction at x_i, and are never opposite elsewhere along the 2-skeleton. □

9.2. **The gluing map Φ_ξ.**

Definition 9.5. We say that (M', γ') is a *sutured submanifold* of the sutured manifold (M, γ) if M' is a submanifold with boundary of M, and $M' \subset \text{Int}(M)$. A connected component C of $M \setminus \text{Int}(M')$ is called *isolated* if $C \cap \partial M = \emptyset$.

Next, we recall [14, Theorem 1.1].

Theorem 9.6. *Let (M', γ') be a sutured submanifold of (M, γ), and let ξ be a contact structure on $M \setminus Int(M')$ with convex boundary, and dividing set γ on ∂M and γ' on $\partial M'$. If $M \setminus Int(M')$ has m isolated components, then ξ induces a natural map*

$$\Phi_\xi \colon SFH(-M', -\gamma') \to SFH(-M, -\gamma) \otimes V^{\otimes m}.$$

Here $V = \widehat{HF}(S^1 \times S^2) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$ is a $\mathbb{Z}$-graded vector space, where the two summands have gradings that differ by one, say 0 and 1.

Moreover, if ξ' is any contact structure on M' with dividing set γ' on $\partial M'$, then

$$\Phi_\xi(EH(M', \gamma', \xi')) = EH(M, \gamma, \xi' \cup \xi) \otimes (x \otimes \cdots \otimes x),$$

where x is the contact class of the standard contact structure on $S^1 \times S^2$.

Honda, Kazez, and Matić call Φ_ξ the "gluing map." Its construction is rather involved, we only describe it briefly and highlight its properties that we need later.

Suppose for now that $m = 0$. First, we choose an arbitrary balanced diagram $(-\Sigma_0', \boldsymbol{\alpha}_0', \boldsymbol{\beta}_0')$ for $(-M', -\gamma')$. Let $T = \partial M'$, and denote by ζ the I-invariant contact structure on $T \times I$ such that for every $t \in I$ the surface $T \times \{t\}$ is convex with dividing set $\gamma' \times \{t\}$. Then we construct two special partial open book decompositions for $(T \times I, \zeta)$, and denote by $(-\Sigma_\zeta^1, \boldsymbol{\alpha}_\zeta^1, \boldsymbol{\beta}_\zeta^1)$ and $(-\Sigma_\zeta^2, \boldsymbol{\alpha}_\zeta^2, \boldsymbol{\beta}_\zeta^2)$ the corresponding balanced diagrams. We set $\Sigma' = \Sigma_0' \cup \Sigma_\zeta^1 \cup \Sigma_\zeta^2$, and extend $\boldsymbol{\alpha}_0' \cup \boldsymbol{\alpha}_\zeta^1 \cup \boldsymbol{\alpha}_\zeta^2$ and $\boldsymbol{\beta}_0' \cup \boldsymbol{\beta}_\zeta^1 \cup \boldsymbol{\beta}_\zeta^2$ to full sets of attaching circles $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$, respectively. We end up with a special kind of balanced diagram $(-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ for $(-M', -\gamma')$, that Honda, Kazez, and Matić [14] call *contact compatible with ζ near $\partial M'$.*

In the second step, using the contact structure ξ, one extends the balanced diagram $(-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ to a balanced diagram $(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ for $(-M, -\gamma)$ such that $\Sigma' \subset \Sigma$, while $\boldsymbol{\alpha} = \boldsymbol{\alpha}' \cup \boldsymbol{\alpha}''$ and $\boldsymbol{\beta} = \boldsymbol{\beta}' \cup \boldsymbol{\beta}''$. The curves $\boldsymbol{\alpha}''$ and $\boldsymbol{\beta}''$ come from a partial open book decomposition, hence there is a distinguished intersection point

$$\mathbf{x}'' = (x_1'', \dots, x_m'') \in \mathbb{T}_{\alpha''} \cap \mathbb{T}_{\beta''},$$

with the properties described above Remark 9.2. More precisely, if D is the closure of a component of $-\Sigma \setminus (\boldsymbol{\alpha} \cup \boldsymbol{\beta})$ such that $x_i'' \in \partial D$ and $\partial D \cap \alpha_i$ points out of x_i'', then $D \cap \partial\Sigma \neq \emptyset$.

We define the map

$$\phi_\xi \colon CF(-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}') \to CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$$

on each generator $\mathbf{y} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ of $CF(-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ by the formula $\phi_\xi(\mathbf{y}) = (\mathbf{y}, \mathbf{x}'')$, and extend it to $CF(-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ linearly. This is a chain map, since every domain emanating from $(\mathbf{y}, \mathbf{x}'')$ has zero multiplicities around $\mathbf{x}''$, hence must consist of a domain in $(\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ emanating from $\mathbf{y}$, plus the trivial domain from $\mathbf{x}''$ to itself. The induced map on the homology is

$$\Phi_\xi \colon SFH(-M', -\gamma') \to SFH(-M, -\gamma).$$

We point out that the definition of Φ_ξ also works when $(M', \gamma') = \emptyset$, so (M, γ, ξ) is a contact manifold. Then $SFH(M', \gamma') = \mathbb{Z}_2$, and the map Φ_ξ is given by

$$\Phi_\xi(1) = EH(M, \gamma, \xi).$$

The *naturality* of the map Φ_ξ means the following. Suppose that $\mathcal{H}_1'$ and $\mathcal{H}_2'$ are balanced diagrams for $(-M', -\gamma')$ that are contact compatible with ζ near $\partial M'$, and let $\mathcal{H}_1$ and $\mathcal{H}_2$ be their extensions to $(-M,-\gamma)$, respectively. Then the following diagram is commutative:

$$\begin{array}{ccc} SFH(\mathcal{H}_1') & \xrightarrow{\Phi_\xi^1} & SFH(\mathcal{H}_1) \\ \Big\downarrow F_{\mathcal{H}_1',\mathcal{H}_2'} & & \Big\downarrow F_{\mathcal{H}_1,\mathcal{H}_2} \\ SFH(\mathcal{H}_2') & \xrightarrow{\Phi_\xi^2} & SFH(\mathcal{H}_2), \end{array}$$

where $F_{\mathcal{H}_1',\mathcal{H}_2'}$ and $F_{\mathcal{H}_1,\mathcal{H}_2}$ are canonical isomorphisms induced by equivalences between the sutured diagrams, cf. Section 5.2. Furthermore, if (M',γ') is a sutured submanifold of (M,γ) with contact structure ξ on $Z = M \setminus \mathrm{Int}(M')$ and dividing set γ on ∂M and γ' on $\partial M'$, and $(\overline{M'},\overline{\gamma'})$ is a sutured submanifold of $(\overline{M},\overline{\gamma})$ with contact structure $\overline{\xi}$ on $\overline{Z} = \overline{M} \setminus \mathrm{Int}(\overline{M'})$ with dividing set $\overline{\gamma}$ on $\partial\overline{M}$ and $\overline{\gamma'}$ on $\partial\overline{M'}$, then an orientation preserving diffeomorphism $d\colon M \to \overline{M}$ such that $d(Z) = \overline{Z}$, $d(\gamma') = \overline{\gamma'}$, $d(\gamma) = \overline{\gamma}$, and $d_*(\xi) = \overline{\xi}$ gives rise to a commutative diagram

$$\begin{array}{ccc} SFH(-M',-\gamma') & \xrightarrow{\Phi_\xi} & SFH(-M,-\gamma) \\ \Big\downarrow (d|_{M'})_* & & \Big\downarrow d_* \\ SFH(-\overline{M'},-\overline{\gamma'}) & \xrightarrow{\Phi_{\overline{\xi}}} & SFH(-\overline{M},-\overline{\gamma}). \end{array}$$

For the case $m > 0$, see the discussion on page 7 of [14]. We now list some basic properties of this gluing map. The first is [14, Theorem 6.1].

Theorem 9.7. *Let (M,γ) be a balanced sutured manifold, and let ξ be an I-invariant contact structure on $\partial M \times I$ with dividing set $\gamma \times \{t\}$ on $\partial M \times \{t\}$. Then the gluing map*

$$\Phi_\xi\colon SFH(-M,-\gamma) \to SFH(-M,-\gamma)$$

obtained by attaching $(\partial M \times I, \xi)$ onto (M,γ) along $\partial M \times \{0\}$ is the identity map.

The following statement is [14, Proposition 6.2].

Proposition 9.8. *Consider the inclusions $(M_0,\gamma_0) \subset (M_1,\gamma_1) \subset (M_2,\gamma_2)$ of sutured submanifolds. For $i \in \{0,1\}$, let ξ_i be a contact structure on $M_{i+1} \setminus Int(M_i)$ that has convex boundary and dividing set γ_j on ∂M_j for $j \in \{i, i+1\}$. Then*

$$\Phi_{\xi_1} \circ \Phi_{\xi_0} = \Phi_{\xi_0 \cup \xi_1}\colon SFH(-M_0,-\gamma_0) \to SFH(-M_2,-\gamma_2).$$

Definition 9.9. Let (M',γ') be a sutured submanifold of (M,γ), and let ξ be a contact structure on $M \setminus \mathrm{Int}(M')$ with convex boundary, and dividing set γ on ∂M and γ' on $\partial M'$. As in the proof of Lemma 3.3, choose a Riemannian metric such that $\xi^\perp$ is admissible. If $\mathfrak{s}' \in \mathrm{Spin}^c(M',\gamma')$ and v' is a vector field representing $\mathfrak{s}'$, then the homology class of the vector field $v = v' \cup \xi^\perp$ is obviously independent of the choice of v'. This defines a map

$$f_\xi\colon \mathrm{Spin}^c(M',\gamma') \to \mathrm{Spin}^c(M,\gamma),$$

by setting $f_\xi(\mathfrak{s}')$ to be the homology class of v.

Given $\mathfrak{s}_1$, $\mathfrak{s}_2 \in \mathrm{Spin}^c(M,\gamma)$, their difference $\mathfrak{s}_1 - \mathfrak{s}_2$ is an element of $H^2(M,\partial M)$, which we identify with $H_1(M)$ using Poincaré duality.

Lemma 9.10. *If* $\mathfrak{s}_1'$, $\mathfrak{s}_2' \in \mathit{Spin}^c(M',\gamma')$, *then*

$$f_\xi(\mathfrak{s}_1') - f_\xi(\mathfrak{s}_2') = e_*(\mathfrak{s}_1' - \mathfrak{s}_2'),$$

where $e_*\colon H_1(M') \to H_1(M)$ *is the map induced by the embedding* $e\colon M' \hookrightarrow M$.

Proof. Let v_i' be a vector field representing $\mathfrak{s}_i'$ for $i \in \{1,2\}$. After fixing a trivialization of TM, we can view $v_1 = v_1' \cup \xi^\perp$ and $v_2 = v_2' \cup \xi^\perp$ as maps from M to S^2. If p is a common regular value of v_1 and v_2, then

$$f_\xi(\mathfrak{s}_1') - f_\xi(\mathfrak{s}_2') = [v_1^{-1}(p) - v_2^{-1}(p)] \in H_1(M').$$

But v_1 and v_2 agree outside M'; moreover,

$$v_1^{-1}(p) - v_2^{-1}(p) = (v_1')^{-1}(p) - (v_2')^{-1}(p).$$

The right-hand side, thought of as a 1-cycle in M, represents $e_*(\mathfrak{s}_1' - \mathfrak{s}_2')$, which proves the lemma. □

Proposition 9.11. *Let* (M',γ') *be a sutured submanifold of* (M,γ), *and let* ξ *be a contact structure on* $M \setminus \mathit{Int}(M')$ *with convex boundary, and dividing set* γ *on* ∂M *and* γ' *on* $\partial M'$. *Pick a* Spin^c *structure* $\mathfrak{s}' \in \mathit{Spin}^c(M',\gamma')$, *and choose an element* $x' \in \mathit{SFH}(-M',-\gamma',\mathfrak{s}')$. *Then*

$$\Phi_\xi(x') \in \mathit{SFH}(-M,-\gamma,f_\xi(\mathfrak{s}')).$$

Proof. Fix an arbitrary contact structure ξ' on (M',γ') such that $\partial M'$ is a convex surface with dividing set γ'. Choose a partial open book decomposition defining (M',γ',ξ'), and let $(-\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}')$ be the corresponding balanced diagram of $(-M',-\gamma')$. Extend $(-\Sigma',\boldsymbol{\alpha}',\boldsymbol{\beta}')$ to a diagram $(-\Sigma,\boldsymbol{\alpha},\boldsymbol{\beta})$ defining $(-M,-\gamma)$, as in the definition of the map Φ_ξ. More precisely, we have $\boldsymbol{\alpha} = \boldsymbol{\alpha}' \cup \boldsymbol{\alpha}''$ and $\boldsymbol{\beta} = \boldsymbol{\beta}' \cup \boldsymbol{\beta}''$, and there is a distinguished intersection point $\mathbf{x}'' \in \mathbb{T}_{\alpha''} \cap \mathbb{T}_{\beta''}$ such that $\phi_\xi(\mathbf{y}) = (\mathbf{y},\mathbf{x}'')$ for every $\mathbf{y} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$. Let $\mathbf{y}_0 \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ be the distinguished intersection point representing the contact class $EH(M',\gamma',\xi')$.

Again, let $e\colon M' \hookrightarrow M$ be the embedding. Then it follows from Lemma 9.10 and the above description of ϕ_ξ that for any $\mathbf{y} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ we have

$$\begin{aligned} &f_\xi(\mathfrak{s}(\mathbf{y})) - f_\xi(\mathfrak{s}(\mathbf{y}_0)) = e_*(\mathfrak{s}(\mathbf{y}) - \mathfrak{s}(\mathbf{y}_0)) = \\ &\mathfrak{s}(\mathbf{y},\mathbf{x}'') - \mathfrak{s}(\mathbf{y}_0,\mathbf{x}'') = \mathfrak{s}(\phi_\xi(\mathbf{y})) - \mathfrak{s}(\phi_\xi(\mathbf{y}_0)) \in H_1(M). \end{aligned}$$

Note that $\phi_\xi(\mathbf{y}_0) = (\mathbf{y}_0,\mathbf{x}'')$ is the distinguished intersection point representing $EH(M,\gamma,\xi)$, see Theorem 9.6. So the proof of Proposition 9.4 implies that $\mathfrak{s}(\mathbf{y}_0) = \mathfrak{s}_{\xi'}$ and $\mathfrak{s}(\phi_\xi(\mathbf{y}_0)) = \mathfrak{s}_{\xi\cup\xi'}$. Of course, $f_\xi(\mathfrak{s}_{\xi'}) = \mathfrak{s}_{\xi'\cup\xi}$. We conclude that

$$f_\xi(\mathfrak{s}(\mathbf{y})) = \mathfrak{s}(\phi_\xi(\mathbf{y}))$$

for every $\mathbf{y} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$. Since $\mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$ freely generates $CF(-\Sigma,\boldsymbol{\alpha}',\boldsymbol{\beta}')$, this implies the claim of the proposition. □

Remark 9.12. It is not true in general that for every $x' \in \mathit{SFH}(-M',-\gamma')$ there exists a contact structure ξ' on (M',γ') such that $x' = EH(M',\gamma',\xi')$.

Corollary 9.13. *For any* $\mathfrak{s}' \in \mathit{Spin}^c(M',\gamma')$, *let*

$$\Phi_{\xi,\mathfrak{s}'} = \Phi_\xi|_{\mathit{SFH}(-M',-\gamma',\mathfrak{s}')}.$$

Then

$$\Phi_{\xi,\mathfrak{s}'}\colon \mathit{SFH}(-M',-\gamma',\mathfrak{s}') \to \mathit{SFH}(-M,-\gamma,f_\xi(\mathfrak{s}')).$$

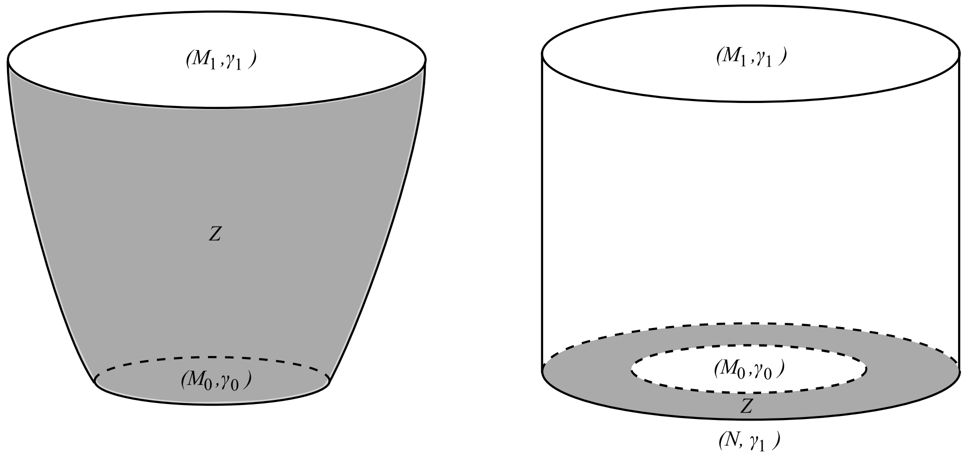

FIGURE 11. A balanced cobordism $\mathcal{W}$ having no isolated components on the left, and the corresponding special cobordism $\mathcal{W}_1$ on the right.

Remark 9.14. In the remark on page 11 of [14], a construction is outlined for a map from $SFH(-M', -\gamma')$ to $SFH(-M, -\gamma)$ that depends on a Spin^c structure $\mathfrak{s}' \in \text{Spin}^c(M', \gamma')$. Specifically, one would use a contact structure ξ' on (M', γ') that represents $\mathfrak{s}'$ to construct the balanced diagram for $(-M', -\gamma')$ via a partial open book decomposition. This would replace the diagrams for $(-M', -\gamma')$ that are contact compatible near the boundary. It is natural to conjecture that the restriction of this map to $SFH(-M', -\gamma', \mathfrak{s}')$ is precisely $\Phi_{\xi,\mathfrak{s}'}$, which might simplify concrete computations of $\Phi_{\xi,\mathfrak{s}'}$.

10. CONSTRUCTION OF THE COBORDISM MAP $F_{\mathcal{W}}$

We can now describe the construction of the map $F_{\mathcal{W}}$ for an arbitrary balanced cobordism $\mathcal{W} = (W, Z, [\xi])$. It is a composition of the gluing map $\Phi_{-\xi}$ induced by the contact structure $-\xi$ and the cobordism map induced by a special cobordism.

Definition 10.1. Let $\mathcal{W} = (W, Z, [\xi])$ be a balanced cobordism from (M_0, γ_0) to (M_1, γ_1). Following [14], we say that a component Z_0 of Z is *isolated* if $Z_0 \cap M_1 = \emptyset$. First, we define $F_{\mathcal{W}}$ if Z has no isolated components, see Figure 11.

Observing the orientation conventions in Remark 2.5, note that $(-M_0, -\gamma_0)$ is a sutured submanifold of $(-N, -\gamma_1) = (-M_0 \cup Z, -\gamma_1)$. Furthermore, $-\xi$ is a positive contact structure on Z with dividing set $-\gamma_0$ on ∂M_0 and $-\gamma_1$ on ∂M_1. Thus Theorem 9.6 provides us with a map

$$\Phi_{-\xi} \colon SFH(M_0, \gamma_0) \to SFH(N, \gamma_1).$$

The sutured manifolds (N, γ_1) and (M_1, γ_1) have the same boundary and same sutures, and $\partial W = -N \cup M_1$. We can think of this as a special cobordism from (N, γ_1) to (M_1, γ_1), call it $\mathcal{W}_1$. For each such cobordism we constructed a map

$$F_{\mathcal{W}_1} \colon SFH(N, \gamma_1) \to SFH(M_1, \gamma_1)$$

in Section 5. Finally, we set

$$F_{\mathcal{W}} = F_{\mathcal{W}_1} \circ \Phi_{-\xi}.$$

In the general case, for each isolated component Z_0 of Z, choose a small standard contact ball $B_0 \subset \text{Int}(Z_0)$ with convex boundary and connected dividing set δ_0. Let B be the union of all such balls, and δ the dividing set on ∂B. Then we replace the cobordism $\mathcal{W}$ with $\mathcal{W}' = (W, Z', [\xi'])$, where $Z' = Z \setminus \text{Int}(B)$ and $\xi' = \xi|_{Z'}$. The morphism $\mathcal{W}'$ has source (M_0, γ_0) and range $(M_1', \gamma_1') = (M_1, \gamma_1) \sqcup (B, \delta)$. Since $SFH(M_1', \gamma_1') \cong SFH(M_1, \gamma_1)$, and now Z' has no isolated components, we can define $F_{\mathcal{W}} = F_{\mathcal{W}'}$. It follows from Proposition 10.3 that this is independent of the choice of B.

The map $F_{\mathcal{W}}$ has a refinement along relative Spin^c structure on $\mathcal{W}$. We describe this next.

Definition 10.2. Let $\mathcal{W} = (W, Z, [\xi])$ be a balanced cobordism from (M_0, γ_0) to (M_1, γ_1). Choose a relative Spin^c structure $\mathfrak{s} \in \text{Spin}^c(\mathcal{W})$, and for $i \in \{0, 1\}$ set

$$\mathfrak{t}_i = \mathfrak{s}|_{M_i} \in \text{Spin}^c(M_i, \gamma_i).$$

First, suppose that Z has no isolated components. Then there is a natural restriction map from $\text{Spin}^c(\mathcal{W})$ to $\text{Spin}^c(\mathcal{W}_1)$, we denote the image of $\mathfrak{s}$ by $\mathfrak{s}_1 = \mathfrak{s}|_{\mathcal{W}_1}$. If we view $\mathfrak{t}_0$ as an element of $\text{Spin}^c(-M_0, -\gamma_0)$, then

$$\mathfrak{s}_1|_{(-N, -\gamma_1)} = f_{-\xi}(\mathfrak{t}_0) \in \text{Spin}^c(-N, -\gamma_1).$$

Hence the special cobordism $\mathcal{W}_1$, endowed with the Spin^c structure $\mathfrak{s}_1$, induces a map

$$F_{\mathcal{W}_1, \mathfrak{s}_1} \colon SFH(N, \gamma_1, f_{-\xi}(\mathfrak{t}_0)) \to SFH(M_1, \gamma_1, \mathfrak{t}_1).$$

By Corollary 9.13, the sutured submanifold $(-M_0, -\gamma_0)$ of $(-N, -\gamma_1)$, together with the Spin^c structure $\mathfrak{t}_0$, give rise to a map

$$\Phi_{\xi, \mathfrak{t}_0} \colon SFH(M_0, \gamma_0, \mathfrak{t}_0) \to SFH(N, \gamma_1, f_{-\xi}(\mathfrak{t}_0)).$$

So we can define

$$F_{\mathcal{W}, \mathfrak{s}} \colon SFH(M_0, \gamma_0, \mathfrak{t}_0) \to SFH(M_1, \gamma_1, \mathfrak{t}_1)$$

by the formula $F_{\mathcal{W}, \mathfrak{s}} = F_{\mathcal{W}_1, \mathfrak{s}_1} \circ \Phi_{-\xi, \mathfrak{t}_0}$.

When Z does have isolated components, then as before, we take the cobordism $\mathcal{W}'$ from (M_0, γ_0) to $(M_1', \gamma_1') = (M_1, \gamma_1) \sqcup (B, \delta)$. Then set $\mathfrak{s}' = \mathfrak{s}|_{\mathcal{W}'}$, and notice that $\mathfrak{t}_1' = \mathfrak{s}'|_{(M_1', \gamma_1')}$ agrees with $\mathfrak{t}_1$ on (M_1, γ_1), and is the vertical Spin^c structure $\mathfrak{t}_B$ on the product (B, δ). Hence

$$SFH(M_1', \gamma_1', \mathfrak{t}_1') = SFH(M_1, \gamma_1, \mathfrak{t}_1) \otimes SFH(B, \delta, \mathfrak{t}_B) \cong SFH(M_1, \gamma_1, \mathfrak{t}_1) \otimes \mathbb{Z}_2,$$

and we can set $F_{\mathcal{W}, \mathfrak{s}} = F_{\mathcal{W}', \mathfrak{s}'}$.

Proposition 10.3. *Let $\mathcal{W} = (W, Z, [\xi])$ be a balanced cobordism and $\mathfrak{s} \in$ Spin$^c(\mathcal{W})$. Then the map $F_{\mathcal{W}, \mathfrak{s}}$ defined above is independent of the choice of balls in the isolated components of Z.*

Proof. Let $\mathcal{W}'$ and $\mathcal{W}''$ be cobordisms obtained from $\mathcal{W}$ by removing from Z the standard contact balls B' and B'' with dividing sets δ' and δ'', respectively, and adding them to (M_1, γ_1). In particular, $\mathcal{W}'$ is a cobordism from (M_0, γ_0) to

$$(M_1', \gamma_1') = (M_1, \gamma_1) \sqcup (B', \delta'),$$

and $\mathcal{W}''$ is a cobordism from (M_0, γ_0) to

$$(M_1'', \gamma_1'') = (M_1, \gamma_1) \sqcup (B'', \delta'').$$

By the homogeneity of Z, there is a diffeomorphism $d\colon \mathcal{W}' \to \mathcal{W}''$ that is the identity on $M_0 \cup M_1 \subset \partial W$ and is isotopic to Id_W. Furthermore, we can choose d such that $d(\xi') = d(\xi|_{Z'})$ and $\xi'' = \xi|_{Z''}$ are equivalent. Indeed, choose a tight contact ball $B \subset Z$ containing both B' and B'' with convex boundary and connected dividing set δ. Let $C' = B \setminus \mathrm{Int}(B')$ and $C'' = B \setminus \mathrm{Int}(B'')$. Choose properly embedded arcs $a' \subset C'$ and $a'' \subset C''$ connecting δ with δ' and δ with δ'', respectively. Then let $A' \subset C'$ and $A'' \subset C''$ be properly embedded annuli parallel to a' and a'', respectively, in convex position. The contact structures $\xi'|_{C'}$ and $\xi''|_{C''}$ are both tight as they are restrictions of the tight contact structure $\xi|_B$, and they are determined up to isotopy by their dividing sets on A' and $\partial C'$, and on A'' and $\partial C''$, respectively. By convex surface theory, the dividing sets ν' and ν'' on A' and A'' consist of two parallel arcs connecting the two boundary components. Hence, there is a diffeomorphism $d\colon Z \to Z$ that is the identity outside B, maps (B', δ') to (B'', δ''), and (A', ν') to (A'', ν''). The restriction of d to Z' gives the desired map. Indeed, decomposing C' along A' gives two standard contact balls, and each has a unique contact structure up to isotopy fixing the dividing sets on their boundaries. These are mapped by d to the contact balls obtained by decomposing C'' along A''.

Then we can apply Theorem 11.1 for cobordisms without isolated components to obtain the commutative diagram

$$\begin{array}{ccc} SFH(M_0, \gamma_0, \mathfrak{t}_0) & \xrightarrow{F_{\mathcal{W}', \mathfrak{s}'}} & SFH(M_1', \gamma_1', \mathfrak{t}_1') \\ \big\downarrow {\scriptstyle (d|_{M_0})_*} & & \big\downarrow {\scriptstyle (d|_{M_1'})_*} \\ SFH(M_0, \gamma_0, \mathfrak{t}_0) & \xrightarrow{F_{\mathcal{W}'', \mathfrak{s}''}} & SFH(M_1'', \gamma_1'', \mathfrak{t}_1''), \end{array}$$

where $\mathfrak{s}' = \mathfrak{s}|_{\mathcal{W}'}$ and $\mathfrak{s}'' = \mathfrak{s}|_{\mathcal{W}''}$. By construction, $(d|_{M_0})_*$ is the identity of the group $SFH(M_0, \gamma_0, \mathfrak{t}_0)$. Furthermore, and $d|_{M_1'}$ is the identity on M_1 and sends B' to B''. Hence, after canonically identifying $SFH(B', \delta', \mathfrak{t}_{B'})$ and $SFH(B'', \delta'', \mathfrak{t}_{B''})$ with $\mathbb{Z}_2$,

$$(d|_{M_1'})_*\colon SFH(M_1, \gamma_1, \mathfrak{t}_1) \otimes \mathbb{Z}_2 \to SFH(M_1, \gamma_1, \mathfrak{t}_1) \otimes \mathbb{Z}_2$$

is the identity as well. It follows that $F_{\mathcal{W}', \mathfrak{s}'} = F_{\mathcal{W}'', \mathfrak{s}''}$. □

The above construction motivates the following definition.

Definition 10.4. A cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0, γ_0) to (N, γ_1) is called a *boundary cobordism* if $\mathcal{W}$ is balanced, N is parallel to $M_0 \cup (-Z)$, and we are also given a retraction $r\colon W \to M_0 \cup (-Z)$ such that $r|_N$ is an orientation preserving diffeomorphism from N to $M_0 \cup (-Z)$.

Given a boundary cobordism, we can view $(-M_0, -\gamma_0)$ as a sutured submanifold of $(-N, -\gamma_1)$, and $-\xi$ is a contact structure such that $\partial M_0 \cup \partial N$ is a convex surface with dividing set $-\gamma_0 \cup -\gamma_1$. Hence a boundary cobordism gives rise to a map

$$\Phi_{-\xi}\colon SFH(M_0, \gamma_0) \to SFH(N, \gamma_1).$$

Definition 10.5. Let $\mathcal{W} = (W, Z, [\xi])$ and $\mathcal{W}' = (W', Z', [\xi'])$ be boundary cobordisms from (M_0, γ_0) to (N, γ_1), together with retractions r and r', respectively. Then we say that $\mathcal{W}$ and $\mathcal{W}'$ are *equivalent* if there is an equivalence $d\colon W \to W'$ in the sense of Definition 2.7 that also respects the retractions; i.e., $d \circ r = r' \circ d$. Such a d is called an *equivalence*.

If $\mathcal{W}$ is a boundary cobordism from (M_0, γ_0) to (N, γ_1), and $\mathcal{W}'$ is a boundary cobordism from (M_0', γ_0') to (N', γ_1'), then $\mathcal{W}$ and $\mathcal{W}'$ are said to be *diffeomorphic* if there is a diffeomorphism $d\colon W \to W'$ in the sense of Definition 2.7 that also satisfies $d \circ r = r' \circ d$. We call such a d a *diffeomorphism.*

The naturality of $\Phi_{-\xi}$ implies the following statement.

Proposition 10.6. *Suppose that d is a diffeomorphism from the boundary cobordism $\mathcal{W} = (W, Z, [\xi])$ from (M_0, γ_0) to (N, γ_1) to $\mathcal{W}' = (W', Z', [\xi'])$ from (M_0', γ_0') to (N', γ_1'). Furthermore, let $\mathfrak{t} \in \text{Spin}^c(M_0, \gamma_0)$ and $\mathfrak{t}' = d_*(\mathfrak{t})$. Then there is a commutative diagram*

$$\begin{array}{ccc} SFH(M_0, \gamma_0, \mathfrak{t}) & \xrightarrow{\Phi_{-\xi,\mathfrak{t}}} & SFH(N, \gamma_1, f_{-\xi}(\mathfrak{t})) \\ \Big\downarrow (d|_{M_0})_* & & \Big\downarrow (d|_N)_* \\ SFH(M_0', \gamma_0', \mathfrak{t}') & \xrightarrow{\Phi_{-\xi',\mathfrak{t}'}} & SFH(N', \gamma_1', f_{-\xi'}(\mathfrak{t}')), \end{array}$$

and there is an analogous diagram for Φ_ξ and $\Phi_{\xi'}$ that does not refer to Spin^c structures.

Lemma 10.7. *Suppose that $\mathcal{W} = (W, Z, [\xi])$ is a balanced cobordism from (M_0, γ_0) to (M_1, γ_1) such that Z has no isolated components. Then $\mathcal{W}$ can be written as a composition $\mathcal{W}_1 \circ \mathcal{W}_0$, where $\mathcal{W}_0$ is a boundary cobordism from (M_0, γ_0) to some sutured manifold (N, γ_1), and $\mathcal{W}_1$ is a special cobordism from (N, γ_1) to (M_1, γ_1).*

This decomposition is unique in the following sense. Suppose that d is a diffeomorphism from $\mathcal{W}$ to another cobordism $\mathcal{W}'$ from (M_0', γ_0') to (M_1', γ_1'), and we have a splitting $\mathcal{W}' = \mathcal{W}_1' \circ \mathcal{W}_0'$ along (N', γ_1'). Then there is a diffeomorphism d' such that $d'(N, \gamma_1) = (N', \gamma_1')$ and $d(x) = d'(x)$ for every $x \in M_0 \cup M_1$. Furthermore, the map $d'|_{W_0}$ is a diffeomorphism from the boundary cobordism $\mathcal{W}_0$ to $\mathcal{W}_0'$, and $d'|_{W_1}$ is a diffeomorphism from the special cobordism $\mathcal{W}_1$ to $\mathcal{W}_1'$. Finally, $d_*(\mathfrak{s}) = d_*'(\mathfrak{s})$ for every $\mathfrak{s} \in \text{Spin}^c(\mathcal{W})$.

Proof. First, choose a collar neighborhood $Z_1 = \partial M_1 \times I$ of ∂M_1 in Z such that $\partial M_1 = \partial M_1 \times \{0\}$, and for every $t \in I$ the surface $\partial M_1 \times \{t\}$ is convex with dividing set $\gamma_1 \times \{t\}$ in $\xi_1 = \xi|_{Z_1}$. Set Z_0 to be the closure of $Z \setminus Z_1$. Then choose a properly embedded 3-manifold $N \subset W$ such that $\partial N = \partial M_0 \times \{1\}$, and N is parallel to $M_0 \cup (-Z_0)$. Cutting $\mathcal{W}$ along N gives the required decomposition.

The uniqueness follows from the uniqueness of the "product" collar neighborhood (Z_1, ξ_1), see [16, Theorem 2.5.23]. So one can isotope d relative to $M_0 \cup M_1$ thorough diffeomorphisms of cobordisms until we obtain a diffeomorphism d' such that $d'(N, \gamma_1) = (N', \gamma_1')$. Then a further isotopy in $\text{Int}(W')$ ensures that $d' \circ r = r' \circ d'$, where r and r' are the retractions for the boundary cobordisms $\mathcal{W}_0$ and $\mathcal{W}_0'$, respectively. Hence $d'|_{W_0}$ is an equivalence of boundary cobordisms, and $d'|_{W_1}$ is an equivalence of special cobordisms. The last statement follows from the fact that we got d' from d by isotoping it through diffeomorphisms of cobordisms. □

Let $\mathcal{W} = (W, Z, [\xi])$ be a balanced cobordism such that Z has no isolated components. An alternative way of thinking about the map $F_{\mathcal{W}}$ is to write $\mathcal{W} = \mathcal{W}_1 \circ \mathcal{W}_0$ as in Lemma 10.7; i.e., $\mathcal{W}_0 = (W_0, Z_0, [\xi_0])$ is a boundary cobordism from (M_0, γ_0) to (N, γ_1), and $\mathcal{W}_1$ is a special cobordism from (N, γ_1) to (M_1, γ_1). Then set

$$F_{\mathcal{W}} = F_{\mathcal{W}_1} \circ F_{\mathcal{W}_0},$$

where $F_{\mathcal{W}_1}$ is the map defined in Section 5, and $F_{\mathcal{W}_0} = \Phi_{-\xi_0}$.

Proposition 10.8. *Let $\mathcal{W}$ be a cobordism from (M_0, γ_0) to (M_1, γ_1). If $\mathcal{W}$ is a special cobordism, then the map $F_{\mathcal{W}}$ defined here agrees with the special cobordism map $F_{\mathcal{W}}$ defined in Section 5. If $\mathcal{W} = (W, Z, [\xi])$ is a boundary cobordism, then we have $F_{\mathcal{W}} = \Phi_{-\xi}$.*

Proof. If $\mathcal{W}$ is a special cobordism, then $\mathcal{W}_0 = (W_0, Z_0, [\xi_0])$ is the trivial cobordism from (M_0, γ_0) to itself, so by Theorem 9.7, the map $\Phi_{-\xi_0}$ is the identity of $SFH(M_0, \gamma_0)$, and $F_{\mathcal{W}} = F_{\mathcal{W}_1}$. Furthermore, $\mathcal{W} = \mathcal{W}_1$.

On the other hand, if $\mathcal{W}$ is a boundary cobordism, then $\mathcal{W}_1$ is the trivial cobordism from (M_1, γ_1) to itself, so $\mathcal{W} = \mathcal{W}_0$. By Proposition 8.13, the map $F_{\mathcal{W}_1}$ is the identity, hence $F_{\mathcal{W}} = \Phi_{-\xi_0}$. □

Consequently, the maps $F_{\mathcal{W}}$ generalize both special cobordism maps and gluing maps.

Definition 10.9. Suppose that $\mathcal{W} = (W, Z, [\xi])$ is a balanced cobordism such that Z has no isolated components. Write $\mathcal{W}$ as $\mathcal{W}_1 \circ \mathcal{W}_0$, where $\mathcal{W}_0$ is a boundary cobordism and $\mathcal{W}_1$ is a special cobordism. We say that the Spin^c structures $\mathfrak{s}$, $\mathfrak{s}' \in \mathrm{Spin}^c(\mathcal{W})$ are *equivalent*, in short $\mathfrak{s} \sim \mathfrak{s}'$, if $\mathfrak{s}|_{\mathcal{W}_i} = \mathfrak{s}'|_{\mathcal{W}_i}$ for $i \in \{0, 1\}$. We denote the set of equivalence classes by $\mathrm{Spin}^c(\mathcal{W})/_\sim$.

If $\mathcal{W}$ does have isolated components, then we let $\mathrm{Spin}^c(\mathcal{W})/_\sim = \mathrm{Spin}^c(\mathcal{W}')/_\sim$, where $\mathcal{W}'$ is the cobordism from (M_0, γ_0) to $(M_1, \gamma_1) \sqcup (B, \delta)$ introduced in Definition 10.1.

By construction, if $\mathfrak{s} \sim \mathfrak{s}'$, then $F_{\mathcal{W},\mathfrak{s}} = F_{\mathcal{W},\mathfrak{s}'}$. For $s \in \mathrm{Spin}^c(\mathcal{W})/_\sim$, let

$$F_{\mathcal{W},s} = F_{\mathcal{W},\mathfrak{s}},$$

where $\mathfrak{s}$ is an arbitrary representative of s.

Remark 10.10. Suppose that $\mathcal{W}_0$ is a boundary cobordism from (M_0, γ_0) to (N, γ_1), and $\mathcal{W}_1$ is a special cobordism from (N, γ_1) to (M_1, γ_1). Let $\mathcal{W} = \mathcal{W}_1 \circ \mathcal{W}_0$. Using a relative Mayer-Vietoris sequence, we see that $\mathrm{Spin}^c(\mathcal{W})/_\sim$ corresponds to the set of $\delta H^1(N, \partial N)$ orbits in $\mathrm{Spin}^c(\mathcal{W})$. Hence $\mathrm{Spin}^c(\mathcal{W}) = \mathrm{Spin}^c(\mathcal{W})/_\sim$ if $H_2(N) = 0$.

Also note that $\mathrm{Spin}^c(\mathcal{W}_0) \cong \mathrm{Spin}^c(M_0, \gamma_0)$. For $\mathfrak{t}_0 \in \mathrm{Spin}^c(M_0, \gamma_0)$, let $\mathfrak{s}_0$ be the corresponding element of $\mathrm{Spin}^c(\mathcal{W}_0)$. Then $\mathfrak{s}_0|_N = f_{-\xi}(\mathfrak{t}_0)$, and in general the map $f_{-\xi}$ is neither injective, nor surjective, cf. Lemma 9.10. Furthermore, $\mathfrak{s}$, $\mathfrak{s}' \in \mathrm{Spin}^c(\mathcal{W})$ are equivalent if and only if $\mathfrak{s}|_{M_0} = \mathfrak{s}'|_{M_0}$ and $\mathfrak{s}|_{\mathcal{W}_1} = \mathfrak{s}'|_{\mathcal{W}_1}$.

Proposition 10.11. *Given a balanced cobordism $\mathcal{W} = (W, Z, [\xi])$, we have*

$$F_{\mathcal{W}} = \bigoplus_{s \in Spin^c(\mathcal{W})/_\sim} F_{\mathcal{W},s}.$$

Proof. This follows from Proposition 8.12 and Corollary 9.13. □

11. Properties of the cobordism map $F_{\mathcal{W}}$

11.1. **Naturality and functoriality.** We start by proving a naturality result for the cobordism map $F_{\mathcal{W}}$.

Theorem 11.1. *Suppose that $\mathcal{W} = (W, Z, [\xi])$ is a balanced cobordism from the sutured manifold (M_0, γ_0) to (M_1, γ_1), and $\mathcal{W}' = (W', Z', [\xi'])$ is a balanced cobordism* *from (M_0',γ_0') to (M_1',γ_1'). Pick an $\mathfrak{s} \in Spin^c(\mathcal{W})$, and let $\mathfrak{t}_i = \mathfrak{s}|_{M_i}$ for $i \in \{0,1\}$. If d is a diffeomorphism from $\mathcal{W}$ to $\mathcal{W}'$, then we have a commutative diagram*

$$\begin{array}{ccc} SFH(M_0,\gamma_0,\mathfrak{t}_0) & \xrightarrow{F_{\mathcal{W},\mathfrak{s}}} & SFH(M_1,\gamma_1,\mathfrak{t}_1) \\ \Big\downarrow{\scriptstyle (d|_{M_0})_*} & & \Big\downarrow{\scriptstyle (d|_{M_1})_*} \\ SFH(M_0',\gamma_0',\mathfrak{t}_0') & \xrightarrow{F_{\mathcal{W}',\mathfrak{s}'}} & SFH(M_1',\gamma_1',\mathfrak{t}_1'), \end{array}$$

where $\mathfrak{s}' = d_*(\mathfrak{s})$ and $\mathfrak{t}_i' = \mathfrak{s}'|_{M_i'}$ for $i \in \{0,1\}$. An analogous statement holds for $F_{\mathcal{W}}$.

Proof. This follows from the corresponding naturality result for special cobordisms, Theorem 8.2, and the naturality of the gluing map $\Phi_{-\xi}$. More precisely, first assume that Z and Z' have no isolated components. Set $N = M_0 \cup (-Z)$ and $N' = M_0' \cup (-Z')$. Then (M_0,γ_0) is a sutured submanifold of (N,γ_1), and (M_0',γ_0') is a sutured submanifold of (N',γ_1'). Furthermore, $d|_N$ is an orientation preserving diffeomorphism from N to N' such that $d(Z) = Z'$, $d(\gamma_0) = \gamma_0'$, $d(\gamma_1) = \gamma_1'$, and $d_*(\xi) = \xi'$. Hence the naturality of the gluing map gives the commutative diagram

$$\begin{array}{ccc} SFH(M_0,\gamma_0,\mathfrak{t}_0) & \xrightarrow{\Phi_{-\xi,\mathfrak{t}_0}} & SFH(N,\gamma_1,f_{-\xi}(\mathfrak{t}_0)) \\ \Big\downarrow{\scriptstyle (d|_{M_0})_*} & & \Big\downarrow{\scriptstyle (d|_{N})_*} \\ SFH(M_0',\gamma_0',\mathfrak{t}_0') & \xrightarrow{\Phi_{-\xi',\mathfrak{t}_0'}} & SFH(N',\gamma_1',f_{-\xi'}(\mathfrak{t}_0')). \end{array}$$

On the other hand, d gives rise to a diffeomorphism between the special cobordism $\mathcal{W}_1$ from (N,γ_1) to (M_1,γ_1) and the special cobordism $\mathcal{W}_1'$ from (N',γ_1') to (M_1',γ_1'). So Theorem 8.2 gives the commutative diagram

$$\begin{array}{ccc} SFH(N,\gamma_1,f_{-\xi}(\mathfrak{t}_0)) & \xrightarrow{F_{\mathcal{W}_1,\mathfrak{s}_1}} & SFH(M_1,\gamma_1,\mathfrak{t}_1) \\ \Big\downarrow{\scriptstyle (d|_{N})_*} & & \Big\downarrow{\scriptstyle (d|_{M_1})_*} \\ SFH(N',\gamma_1',f_{-\xi'}(\mathfrak{t}_0')) & \xrightarrow{F_{\mathcal{W}_1',\mathfrak{s}_1'}} & SFH(M_1',\gamma_1',\mathfrak{t}_1'). \end{array}$$

Putting the above two commutative diagrams together gives the required diagram. The case when Z and Z' have isolated components follows from this by deleting standard contact balls (B,δ) from Z, and then deleting the corresponding balls $B' = d(B)$ from Z'. □

Corollary 11.2. *If the balanced cobordisms $\mathcal{W}$ and $\mathcal{W}'$ from (M_0,γ_0) to (M_1,γ_1) are equivalent, then $F_{\mathcal{W}} = F_{\mathcal{W}'}$. Furthermore, if d is an equivalence and $\mathfrak{s} \in Spin^c(\mathcal{W})$, then $F_{\mathcal{W},\mathfrak{s}} = F_{\mathcal{W}',d_*(\mathfrak{s})}$.*

Proof. This follows from Theorem 11.1 by observing that for a strong equivalence d and for $i \in \{0,1\}$ the map $d|_{M_i}$ is the identity of M_i, hence $(d|_{M_i})_*$ is the identity of $SFH(M_i,\gamma_i)$. □

So $F_{\mathcal{W}}$ can be defined on strong equivalence classes of cobordisms; i.e., on morphisms of the category of **BSut**. We now show that $F_{\mathcal{W}}$ is in fact functorial.

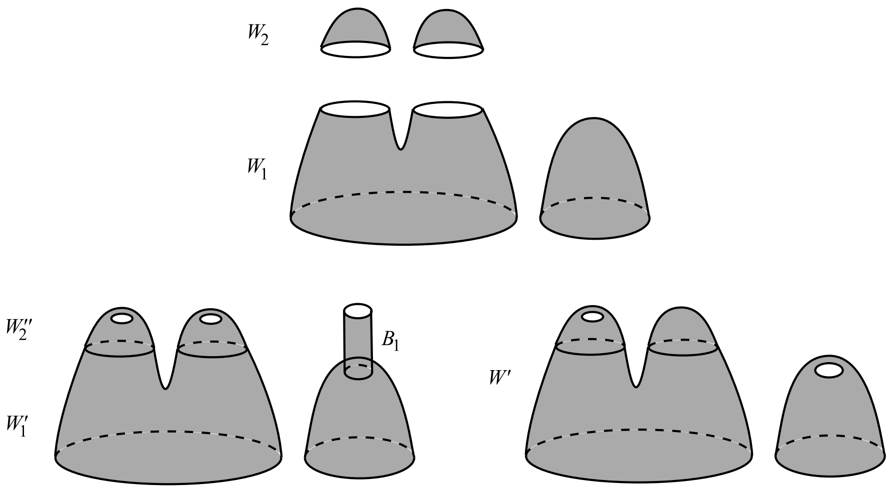

FIGURE 12. The cobordism $\mathcal{W}_1$ has one isolated component, while $\mathcal{W}_2$ has two. In this example, $\mathcal{W}_2'' \circ \mathcal{W}_1'$ and $\mathcal{W}'$ are different.

Theorem 11.3. *Let $\mathcal{W}_1$ be a balanced cobordism from (M_0, γ_0) to (M_1, γ_1), and let $\mathcal{W}_2$ be a balanced cobordism from (M_1, γ_1) to (M_2, γ_2). We write $\mathcal{W}$ for the composition $\mathcal{W}_2 \circ \mathcal{W}_1$. Then*

$$F_{\mathcal{W}_2} \circ F_{\mathcal{W}_1} = F_{\mathcal{W}}.$$

Furthermore, if $s_i \in Spin^c(\mathcal{W}_i)/_\sim$ for $i \in \{1,2\}$, then

$$F_{\mathcal{W}_2, s_2} \circ F_{\mathcal{W}_1, s_1} = \sum_{\{s \in Spin^c(\mathcal{W})/_\sim \,:\, s|_{\mathcal{W}_1} = s_1,\, s|_{\mathcal{W}_2} = s_2\}} F_{\mathcal{W}, s}.$$

Proof. We show how to reduce to the case when neither $\mathcal{W}_1$, nor $\mathcal{W}_2$ has isolated components. Recall that in the general case, for $i \in \{1,2\}$, the map $F_{\mathcal{W}_i}$ was defined to be $F_{\mathcal{W}_i'}$, where $\mathcal{W}_i'$ is a cobordism from (M_{i-1}, γ_{i-1}) to

$$(M_i', \gamma_i') = (M_i, \gamma_i) \sqcup (B_i, \delta_i).$$

The cobordisms $\mathcal{W}_1'$ and $\mathcal{W}_2'$ do not compose if $B_1 \neq \emptyset$, see Figure 12. To get around this problem, let $\mathcal{B}_1$ be the trivial cobordism from (B_1, δ_1) to itself, and take $\mathcal{W}_2''$ to be the disjoint union of $\mathcal{W}_2'$ and $\mathcal{B}_1$. Then, assuming the result for no isolated components, we have

$$F_{\mathcal{W}_2''} \circ F_{\mathcal{W}_1'} = F_{\mathcal{W}_2'' \circ \mathcal{W}_1'}.$$

Now $F_{\mathcal{W}_2''} = F_{\mathcal{W}_2'}$ by Proposition 8.13, so the left-hand side is just $F_{\mathcal{W}_2} \circ F_{\mathcal{W}_1}$.

To define the map $F_{\mathcal{W}}$, we use a cobordism $\mathcal{W}'$ with no isolated components. Notice that $\mathcal{W}_2'' \circ \mathcal{W}_1'$ almost agrees with $\mathcal{W}' = (W, Z', \xi')$, except that some standard contact balls might be removed from (Z', ξ') and added to (M_2', γ_2'). Such a situation is depicted on Figure 12. However, the following lemma will ensure that

$$F_{\mathcal{W}_2'' \circ \mathcal{W}_1'} = F_{\mathcal{W}'} = F_{\mathcal{W}}$$

still holds, and hence we still have $F_{\mathcal{W}_2} \circ F_{\mathcal{W}_1} = F_{\mathcal{W}}$ in the presence of isolated components.

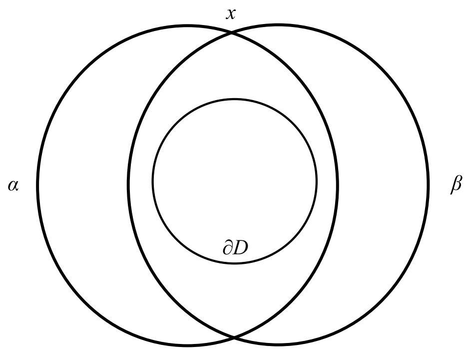

FIGURE 13

Lemma 11.4. *Suppose that* $\mathcal{W} = (W, Z, [\xi])$ *is a balanced cobordism from* (M_0, γ_0) *to* (M_1, γ_1)*. Let* $B \subset Int(Z)$ *be a standard contact ball in* (Z, ξ) *with convex boundary and connected dividing set* δ*. Furthermore, let* $\mathcal{V} = (W, Z_0, \xi_0)$ *be the cobordism from* (M_0, γ_0) *to* $(M_1, \gamma_1) \sqcup (B, \delta)$*, where* $Z_0 = Z \setminus Int(B)$ *and* $\xi_0 = \xi|_{Z_0}$*. Then* $F_{\mathcal{W}} = F_{\mathcal{W}_0}$*, after we identify* $SFH(M_1, \gamma_1)$ *with* $SFH((M_1, \gamma_1) \sqcup (B, \delta))$*.*

Proof. First, assume that Z has no isolated components, then the same holds for Z_0. By definition, $F_{\mathcal{W}} = F_{\mathcal{W}_1} \circ \Phi_{-\xi}$ and $F_{\mathcal{V}} = F_{\mathcal{V}_1} \circ \Phi_{-\xi_0}$, where $\mathcal{W}_1$ is a special cobordism from some sutured manifold (N, γ_1) to (M_1, γ_1). As explained on page 7 of [14], the map

$$\Phi_{-\xi_0} \colon SFH(M_0, \gamma_0) \to SFH(N, \gamma_1) \otimes V$$

agrees with $\Phi_{-\xi}$ composed with a map

$$SFH(N, \gamma_1) \to SFH(N, \gamma_1) \otimes V,$$

given by connected summing with the suture manifold $S^3(1)$. More precisely, take the same balanced diagram that gives $\Phi_{-\xi}$, remove an open ball D from the Heegaard surface, and add a circle α and a circle β that are small Hamiltonian translates of each other, and are parallel to ∂D, see Figure 13. We call this diagram $\mathcal{H}_0$. If $x \in \alpha \cap \beta$ is the intersection point with the smaller grading, and if $\phi_{-\xi}(\mathbf{y}) = (\mathbf{y}, \mathbf{x}'')$, then $\phi_{-\xi'}(\mathbf{y}) = (\mathbf{y}, \mathbf{x}'', x)$. Furthermore, $F_{\mathcal{V}_1}$ agrees with $F_{\mathcal{W}_1}$ composed with the 3-handle map corresponding to the connected sum sphere S, represented by the periodic domain bounded by α and β, see Definition 7.8. So, on the chain level, the map $F_{\mathcal{V}_1}$ is given by $f_{\mathcal{H}_0, S}(\mathbf{y}, \mathbf{x}'', x) = (\mathbf{y}, \mathbf{x}'')$. Hence

$$F_{\mathcal{V}} \colon SFH(M_0, \gamma_0) \to SFH(M_1, \gamma_1) \otimes SFH(B, \delta) \cong SFH(M_1, \gamma_1)$$

agrees with $F_{\mathcal{W}}$.

Now consider the general case. If B lies in an isolated component of Z, then the claim is obvious. So suppose that B lies in a non-isolated component of Z. Then one can apply the lemma to $\mathcal{W}'$ and $\mathcal{W}_0'$ to obtain that

$$F_{\mathcal{W}} = F_{\mathcal{W}'} = F_{\mathcal{W}_0'} = F_{\mathcal{W}_0},$$

which concludes the proof of the lemma. □

From now on, we can suppose that neither $\mathcal{W}_1$, nor $\mathcal{W}_2$ has isolated components. For $i \in \{1, 2\}$, we have a unique decomposition

$$\mathcal{W}_i = \mathcal{W}_{i1} \circ \mathcal{W}_{i0},$$

where $\mathcal{W}_{i0}$ is a boundary cobordism and $\mathcal{W}_{i1}$ is a special cobordism. By definition, $F_{\mathcal{W}_i} = F_{\mathcal{W}_{i1}} \circ F_{\mathcal{W}_{i0}}$. So

$$F_{\mathcal{W}_2} \circ F_{\mathcal{W}_1} = F_{\mathcal{W}_{21}} \circ F_{\mathcal{W}_{20}} \circ F_{\mathcal{W}_{11}} \circ F_{\mathcal{W}_{10}}.$$

We can uniquely write the cobordism $\mathcal{W}_{20} \circ \mathcal{W}_{11}$ in the form $\mathcal{V}_1 \circ \mathcal{V}_0$, where $\mathcal{V}_0$ is a boundary cobordism and $\mathcal{V}_1$ is a special cobordism. Then

$$\mathcal{W} = \mathcal{W}_{21} \circ \mathcal{V}_1 \circ \mathcal{V}_0 \circ \mathcal{W}_{10},$$

and $\mathcal{W}_{21} \circ \mathcal{V}_1$ is a special cobordism, whereas $\mathcal{V}_0 \circ \mathcal{W}_{10}$ is a boundary cobordism. So, by the uniqueness part of Lemma 10.7,

$$F_{\mathcal{W}} = F_{\mathcal{W}_{21} \circ \mathcal{V}_1} \circ F_{\mathcal{V}_0 \circ \mathcal{W}_{10}}.$$

Using Theorem 8.14, we get

$$F_{\mathcal{W}_{21} \circ \mathcal{V}_1} = F_{\mathcal{W}_{21}} \circ F_{\mathcal{V}_1},$$

and Proposition 9.8 implies that

$$F_{\mathcal{V}_0 \circ \mathcal{W}_{10}} = F_{\mathcal{V}_0} \circ F_{\mathcal{W}_{10}}.$$

So we are done as soon as we show that

$$F_{\mathcal{V}_1} \circ F_{\mathcal{V}_0} = F_{\mathcal{W}_{20}} \circ F_{\mathcal{W}_{11}}.$$

This follows from the next proposition.

Proposition 11.5. *Suppose that $\mathcal{W} = \mathcal{W}_1 \circ \mathcal{W}_0$, where $\mathcal{W}_1$ is a boundary cobordism and $\mathcal{W}_0$ is a special cobordism. Let $\mathcal{V}_1 \circ \mathcal{V}_0$ be the unique decomposition of $\mathcal{W}$ into a boundary cobordism $\mathcal{V}_0$ and a special cobordism $\mathcal{V}_1$. Then*

$$F_{\mathcal{W}_1} \circ F_{\mathcal{W}_0} = F_{\mathcal{V}_1} \circ F_{\mathcal{V}_0}.$$

Proof. Suppose that (M', γ') is a sutured submanifold of (M, γ), and assume ξ is a contact structure on $M \setminus \text{Int}(M')$ that has no isolated components. The result follows once we show that the gluing map $\Phi_{-\xi}$ commutes with 1-, 2-, and 3-handle maps, corresponding to handle attachments along $(-M', -\gamma')$. I.e., we can assume that $\mathcal{W}_0$ and $\mathcal{V}_1$ are both k-handle cobordisms for $k \in \{1, 2, 3\}$.

First, we consider the case $k = 2$. Let $\mathbb{L}' \subset (-M', -\gamma')$ be the framed link along which we attach the 2-handles, and we write $\mathbb{L}$ when we consider this link in $(-M, -\gamma)$. Then construct a triple diagram $(-\Sigma_0', \boldsymbol{\alpha}_0', \boldsymbol{\beta}_0', \boldsymbol{\delta}_0')$ subordinate to some bouquet for $\mathbb{L}'$. As explained after Theorem 9.6, we can extend $(-\Sigma_0', \boldsymbol{\alpha}_0', \boldsymbol{\beta}_0')$ to a diagram $(-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ of $(-M', -\gamma')$ that is contact compatible near $\partial M'$, which we then further extend to a diagram $(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ defining $(-M, -\gamma)$ in a way compatible with $-\xi$. Next, we extend $\boldsymbol{\delta}_0'$ to a set of curves $\boldsymbol{\delta}$ by taking small Hamiltonian translates of the curves in $\boldsymbol{\beta} \setminus \boldsymbol{\beta}_0'$, and denote by $\boldsymbol{\delta}'$ the translates of the curves in $\boldsymbol{\beta}' \setminus \boldsymbol{\beta}_0'$, together with $\boldsymbol{\delta}_0'$. Then the two-handle map $F_{\mathbb{L}'}$ is defined using the triple diagram $\mathcal{T}' = (-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}', \boldsymbol{\delta}')$, while $F_{\mathbb{L}}$ can be defined using $\mathcal{T} = (-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$.

Take intersection points $\mathbf{x} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ and $\mathbf{y} \in \mathbb{T}_{\alpha'} \cap \mathbb{T}_{\beta'}$. Furthermore, let $\Theta \in \mathbb{T}_\beta \cap \mathbb{T}_\delta$ and $\Theta' \in \mathbb{T}_{\beta'} \cap \mathbb{T}_{\delta'}$ be the distinguished intersection points. Then we have $\phi_\xi(\mathbf{y}) = (\mathbf{y}, \mathbf{x}'')$ and $E_{\mathcal{T}'}(\mathbf{y}) = F_{\mathcal{T}'}(\mathbf{y} \otimes \Theta')$, while $E_{\mathcal{T}}(\mathbf{x}) = F_{\mathcal{T}}(\mathbf{x} \otimes \Theta)$. Denote by $\mathbf{x}''' \in \mathbb{T}_{\alpha \setminus \alpha'} \cap \mathbb{T}_{\delta \setminus \delta'}$ the point lying closest to $\mathbf{x}''$, which is also $F_{\mathcal{T} \setminus \mathcal{T}'}(\mathbf{x}'' \otimes (\Theta \setminus \Theta'))$, see Figure 14. So what we need to check is

$$(F_{\mathcal{T}'}(\mathbf{y} \otimes \Theta'), \mathbf{x}''') = F_{\mathcal{T}}\left((\mathbf{y}, \mathbf{x}'') \otimes \Theta\right).$$

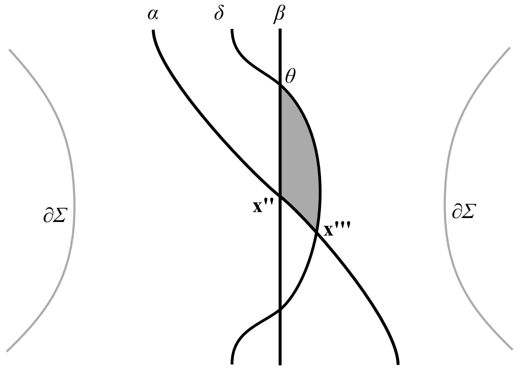

FIGURE 14

To see this, notice that the only domains in $\mathcal{T}$ that contain both Θ and $\mathbf{x}''$ as incoming corners consist of the small triangles next to $\mathbf{x}''$, plus a domain in $(-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$, as all the curves $\boldsymbol{\alpha} \setminus \boldsymbol{\alpha}'$, $\boldsymbol{\beta} \setminus \boldsymbol{\beta}'$, and $\boldsymbol{\delta} \setminus \boldsymbol{\delta}'$ are used up in the small triangles. These small triangles have corners $\mathbf{x}''$, $\Theta \setminus \Theta'$, and $\mathbf{x}'''$, and appear in the count for $F_{\mathcal{T}}((\mathbf{y}, \mathbf{x}'') \otimes \Theta)$. The above mentioned domains in $(-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ are exactly the ones counted in $F_{\mathcal{T}'}(\mathbf{y} \otimes \Theta')$. The argument is analogous to the proof of the invariance of the map $\Phi_{-\xi}$ under handleslides in $(-\Sigma_0', \boldsymbol{\alpha}_0', \boldsymbol{\beta}_0')$, found in [14].

The cases $k = 1$ and $k = 3$ are analogous to the invariance of $\Phi_{-\xi}$ under stabilization and destabilization in $(-\Sigma_0', \boldsymbol{\alpha}_0', \boldsymbol{\beta}_0')$. For example, if we attach a 1-handle to $(-M', -\gamma')$ along a framed pair of points $\mathbb{P}$, then we start out with an arbitrary diagram $(-\Sigma_0', \boldsymbol{\alpha}_0', \boldsymbol{\beta}_0')$ that defines $(-M', -\gamma')$ and is adapted to some bouquet $B(\mathbb{P})$ for $\mathbb{P}$, and extend it to $\mathcal{H}' = (-\Sigma', \boldsymbol{\alpha}', \boldsymbol{\beta}')$ that is contact compatible near $\partial M'$, and make it adapted to $\mathbb{P}$. Then we use the diagram

$$\mathcal{H}'_{\mathbb{P}} = (-(\Sigma')^0 \cup A, \boldsymbol{\alpha}' \cup \{\alpha\}, \boldsymbol{\beta}' \cup \{\beta\})$$

to define the 1-handle map by $f_{\mathcal{H}',\mathbb{P}}(\mathbf{y}) = \mathbf{y} \times \{\theta\}$, where $\theta \in \alpha \cap \beta$ is the intersection point with the larger relative grading. Note that we glue A to the subsurface $-\Sigma_0'$ of $-\Sigma'$. Then the result follows from

$$(\mathbf{y} \times \{\theta\}, \mathbf{x}'') = (\mathbf{y}, \mathbf{x}'') \times \{\theta\}.$$

The case $k = 3$ is very similar, we start with a diagram for $(-M', -\gamma')$ that represents the attaching sphere of the 3-handle, as in Definition 7.7, then "turn around" the argument for 1-handles. This proves the proposition. □

The second part, using the Spin^c structures, follows from the first part and Proposition 10.11. This concludes the proof of Theorem 11.3. □

Proposition 11.6. *If $\mathcal{W} = (W, Z, [\xi])$ is the trivial cobordism from (M, γ) to (M, γ), then $F_{\mathcal{W}}$ is the identity of $SFH(M, \gamma)$. Furthermore, the restriction map from $\mathrm{Spin}^c(\mathcal{W})$ to $\mathrm{Spin}^c(M, \gamma)$ is an isomorphism, and for every $\mathfrak{s} \in \mathrm{Spin}^c(\mathcal{W})$, the map $F_{\mathcal{W},\mathfrak{s}}$ is the identity of $SFH(M, \gamma, \mathfrak{s}|_M)$.*

Proof. This follows from Propositions 8.13 and 10.8. □

Remark 11.7. If (M', γ') is obtained from (M, γ) using a convex decomposition, then we can view $(-M', -\gamma')$ as a sutured submanifold of $(-M, -\gamma)$, and there is a natural contact structure ζ on $M \setminus \mathrm{Int}(M')$, with convex boundary and dividing set $-\gamma \cup -\gamma'$, see page 3 of [14]. Hence we can view this as a boundary cobordism $\mathcal{W}$ from (M', γ') to (M, γ). For nice sutured manifold decompositions, $F_{\mathcal{W}} = \Phi_\zeta$ is an embedding by [20].

11.2. **Duality and blow-up.** Let$(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta})$ be a diagram of the balanced sutured manifold (M, γ). Recall that there is a canonical affine isomorphism between $\mathrm{Spin}^c(M, \gamma)$ and $\mathrm{Spin}^c(-M, -\gamma)$. It follows from [7, Proposition 2.12] that there is a natural bilinear pairing

$$CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}) \otimes CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s}) \to \mathbb{Z}_2.$$

On the generators $\mathbf{x}$, $\mathbf{y} \in \mathbb{T}_\alpha \cap \mathbb{T}_\beta$ with $\mathfrak{s}(\mathbf{x}) = \mathfrak{s}(\mathbf{y}) = \mathfrak{s}$, it is given by $\langle \mathbf{x}, \mathbf{y} \rangle = 1$ if $\mathbf{x} = \mathbf{y}$, and $\langle \mathbf{x}, \mathbf{y} \rangle = 0$ otherwise. Just as in [38, Lemma 5.1], for every element $a \in CF(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s})$ and $b \in CF(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \mathfrak{s})$, we have

$$\langle a, \partial_{-\Sigma} b \rangle = \langle \partial_\Sigma a, b \rangle.$$

So there is an induced pairing $\langle\ ,\ \rangle$ on $SFH(M, \gamma, \mathfrak{s}) \otimes SFH(-M, -\gamma, \mathfrak{s})$. Then $F_{\mathcal{W}}$ satisfies the following duality result.

Theorem 11.8. *Let $\mathcal{W}$ be a special cobordism from (M_0, γ_0) to (M_1, γ_1). In Remark 2.13, we defined the cobordism $\overline{\mathcal{W}}$ from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$, obtained by "turning $\mathcal{W}$ upside down." Then the map $F_{\mathcal{W}}$ is dual to $F_{\overline{\mathcal{W}}}$. More precisely, for every $\mathfrak{s} \in \mathrm{Spin}^c(\mathcal{W})$ with $\mathfrak{s}|_{M_i} = \mathfrak{t}_i$ for $i \in \{1, 2\}$, and every $x \in SFH(M_0, \gamma_0, \mathfrak{t}_0)$ and $y \in SFH(-M_1, -\gamma_1, \mathfrak{t}_1)$, we have*

$$\langle F_{\mathcal{W},\mathfrak{s}}(x), y \rangle_1 = \langle x, F_{\overline{\mathcal{W}},\mathfrak{s}}(y) \rangle_0.$$

Here $\langle\ ,\ \rangle_i$ is the pairing on $SFH(M_i, \gamma_i, \mathfrak{t}_i) \otimes SFH(-M_i, -\gamma_i, \mathfrak{t}_i)$ for $i \in \{0, 1\}$.

Proof. This is completely analogous to [38, Theorem 3.5]. □

Question 11.9. Does Theorem 11.8 hold for arbitrary balanced cobordisms?

We will get back to this question shortly. Next, we generalize the hat version of the blow-up formula [38, Theorem 3.7] to our situation.

Theorem 11.10. *Let $\mathcal{W}_0$ be the trivial cobordism from the balanced sutured manifold (M, γ) to itself. Consider the blowup $\mathcal{W} = \mathcal{W}_0 \# \overline{\mathbb{C}P}^2$, where we take an internal connected sum, and write E for the exceptional divisor. Furthermore, fix a Spin^c structure $\mathfrak{s} \in \mathrm{Spin}^c(\mathcal{W})$, and let $\mathfrak{t} = \mathfrak{s}|_{M \times \{0\}}$. Then $\mathfrak{s}|_{M \times \{1\}}$ is also $\mathfrak{t}$, and the map*

$$F_{\mathcal{W},\mathfrak{s}} \colon SFH(M, \gamma, \mathfrak{t}) \to SFH(M, \gamma, \mathfrak{t})$$

is the identity if $\langle c_1(\mathfrak{s}), E \rangle = \pm 1$, and is zero otherwise.

Proof. This follows from the same local computation as [38, Theorem 3.7]. □

11.3. **Sutured Floer homology as a TQFT.** An axiomatic description of topological quantum field theories, in short TQFTs, was first given by Atiyah [2]. As explained by Blanchet [3], an $(n+1)$-dimensional TQFT over a field $\mathbb{F}$ is a symmetric monoidal functor from the cobordism category of n-manifolds to the category of finite dimensional vector spaces over $\mathbb{F}$.

Theorem 11.11. *The functor $SFH \colon \mathbf{BSut} \to \mathbf{Vect}_{\mathbb{Z}_2}$ is a $(3+1)$-dimensional TQFT over $\mathbb{Z}_2$ in the sense of Atiyah [2] and Blanchet [3].*

Proof. The axioms of Atiyah [2] and Blanchet [3] are equivalent. Since Blanchet takes the cobordism point of view, and we are working with cobordisms, we are going to check the axioms in [3].

In our case, we have a finitely generated $\mathbb{Z}_2$ vector space $SFH(M,\gamma)$ assigned to every balanced sutured manifold (M,γ), and a linear map

$$F_{\mathcal{W}}\colon SFH(M_0,\gamma_0) \to SFH(M_1,\gamma_1)$$

corresponding to every balanced cobordism $\mathcal{W}$ from (M_0,γ_0) to (M_1,γ_1).

Axiom (1) is naturality. We showed in [25, Theorem 1.9] that every orientation preserving diffeomorphism $\phi\colon (M_0,\gamma_0) \to (M_1,\gamma_1)$ induces an isomorphism

$$\phi_*\colon SFH(M_0,\gamma_0) \to SFH(M_1,\gamma_1),$$

cf. Section 5.2. Furthermore, this assignment is functorial; i.e., $(\psi \circ \phi)_* = \psi_* \circ \phi_*$. The naturality of the cobordism maps $F_{\mathcal{W}}$ was stated in Theorem 11.1.

Axiom (2) is functoriality; i.e.,

$$F_{\mathcal{W}_2 \circ \mathcal{W}_1} = F_{\mathcal{W}_2} \circ F_{\mathcal{W}_1}.$$

This was shown in Theorem 11.3.

Axiom (3), normalization, states that if $\mathcal{W}$ is the trivial cobordism from (M,γ) to (M,γ), then

$$F_{\mathcal{W}}\colon SFH(M,\gamma) \to SFH(M,\gamma)$$

is the identity. This is the content of Proposition 11.6.

Axiom (4) is multiplicativity. It states that there are functorial isomorphisms

$$SFH((M_1,\gamma_1) \sqcup (M_2,\gamma_2)) \cong SFH(M_1,\gamma_1) \otimes SFH(M_2,\gamma_2),$$

which easily follows from the definitions. Furthermore, we set $SFH(\emptyset) = \mathbb{Z}_2$, where $\emptyset$ is the empty balanced sutured manifold. It is straightforward to check that these isomorphisms fit into the commutative diagrams in [3] that describe associativity and the unit. Finally, using the above identifications, we also have

$$F_{\mathcal{W}_1 \sqcup \mathcal{W}_2} = F_{\mathcal{W}_1} \otimes F_{\mathcal{W}_2}.$$

Axiom (5), symmetry, states that the isomorphism

$$SFH((M_1,\gamma_1) \sqcup (M_2,\gamma_2)) \cong SFH((M_2,\gamma_2) \sqcup (M_1,\gamma_1))$$

induced by the obvious diffeomorphism $(x,i) \mapsto (x, 1-i)$ for $i \in \{1,2\}$ corresponds to the isomorphism of vector spaces

$$SFH(M_1,\gamma_1) \otimes SFH(M_2,\gamma_2) \cong SFH(M_2,\gamma_2) \otimes SFH(M_1,\gamma_1)$$

mapping $a \otimes b$ to $b \otimes a$. This is also straightforward. □

Remark 11.12. Similar axioms hold if we are dealing with balanced sutured manifolds endowed with Spin^c structures, and morphisms being balanced cobordisms endowed with relative Spin^c structures. These do not form a proper category, as the composition of $(\mathcal{W}_1, \mathfrak{s}_1)$ and $(\mathcal{W}_2, \mathfrak{s}_2)$ does not have a well defined relative Spin^c structure on it. But if we replace functoriality with the formula in Theorem 11.3, we do get a type of TQFT.

Recall that [2, Axiom 2] states that $SFH(M,\gamma)$ and $SFH(-M,-\gamma)$ are dual vector spaces. This was proved in [7, Proposition 2.14]. Furthermore, there has to be a functorial isomorphism between $SFH(-M,-\gamma)$ and $SFH(M,\gamma)^*$. This follows from the set of axioms in [3]. Indeed, if we view the trivial cobordism $\mathcal{W}$

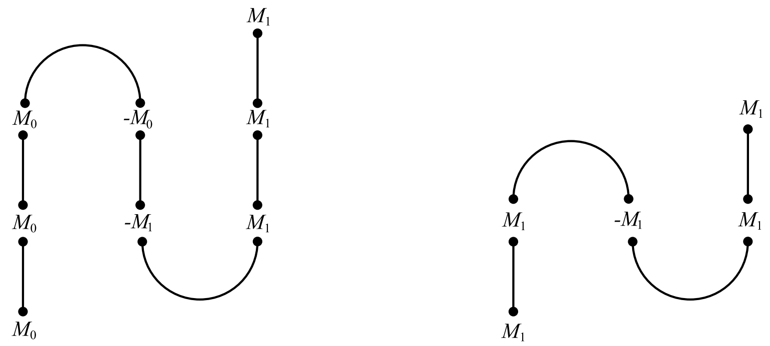

FIGURE 15. On the left, a cobordism is written as a product of its reverse plus other simpler pieces. The maps induced by these simpler pieces can be related to each other using the decomposition of the trivial cobordism on the right.

from (M,γ) to (M,γ) as a cobordism from $(M,\gamma) \sqcup (-M,-\gamma)$ to $\emptyset$, then we get a pairing

$$\langle\ ,\ \rangle' \colon SFH(M,\gamma) \otimes SFH(-M,-\gamma) \to \mathbb{Z}_2,$$

which is non-degenerate by [45, Theorem 2.1.1], see also the right hand-side of Figure 15.

Conjecture 11.13. *The pairing $\langle\ ,\ \rangle'$ agrees with the pairing $\langle\ ,\ \rangle$ appearing in Theorem 11.8.*

Corollary 11.14. *The above conjecture would give an affirmative answer to Question 11.9.*

Proof. Note that Theorem 11.8 is true for arbitrary balanced cobordisms if we replace $\langle\ ,\ \rangle$ with $\langle\ ,\ \rangle'$. To see this, note that a cobordism $\mathcal{W}$ from (M_0,γ_0) to (M_1,γ_1) can be decomposed as follows. Let $\overline{\mathcal{W}}$ be $\mathcal{W}$ viewed as a cobordism from $(-M_1,-\gamma_1)$ to $(-M_0,-\gamma_0)$, and for $i \in \{0,1\}$ let $\mathcal{I}_i$ be the trivial cobordism from (M_i,γ_i) to itself. Furthermore, let $\mathcal{I}_i'$ be $\mathcal{I}_i$ viewed as a cobordisms from $\emptyset$ to $(-M_i,-\gamma_i)\sqcup(M_i,\gamma_i)$ and $\mathcal{I}_i''$ is $\mathcal{I}_i$ viewed as a cobordism from $(M_i,\gamma_i)\sqcup(-M_i,-\gamma_i)$ to $\emptyset$. Then

$$\mathcal{W} = (\mathcal{I}_0'' \sqcup \mathcal{I}_1) \circ (\mathcal{I}_0 \sqcup \overline{\mathcal{W}} \sqcup \mathcal{I}_1) \circ (\mathcal{I}_0 \sqcup \mathcal{I}_1'),$$

see the left-hand side of Figure 15. Note that we can compute $F_{\mathcal{I}_1'}$ using the decomposition

$$\mathcal{I}_1 = (\mathcal{I}_1 \sqcup \mathcal{I}_1') \circ (\mathcal{I}_1'' \sqcup \mathcal{I}_1),$$

see the right-hand side of Figure 15. □

Using the same trick, one can rearrange the ingoing and outgoing ends of any balanced cobordism $\mathcal{W}$, and relate the induced map to $F_{\mathcal{W}}$ using $\langle\ ,\ \rangle'$.

If we were to follow the axioms of Atiyah [2], instead of maps induced by balanced cobordisms, we would need to talk about elements $EH(\mathcal{W}') \in SFH(M,\gamma)$ associated to balanced cobordisms $\mathcal{W}'$ from $\emptyset$ to (M,γ). It was explained in [2] how to translate between the two approaches. If we are given cobordism maps, to a balanced cobordism $\mathcal{W}'$ from $\emptyset$ to (M,γ), we can associate the element $EH(\mathcal{W}') = F_{\mathcal{W}'}(1)$, where $1 \in SFH(\emptyset) \cong \mathbb{Z}_2$.

In the other direction, if $\mathcal{W}$ is a balanced cobordism from (M_0,γ_0) to (M_1,γ_1), then we can also view $\mathcal{W}$ as a cobordism $\mathcal{W}'$ from $\emptyset$ to $(-M_0,-\gamma_0)\sqcup(M_1,\gamma_1)$. Hence

$$EH(\mathcal{W}') \in SFH((-M_0,-\gamma_0)\sqcup(M_1,\gamma_1)) \cong$$
$$SFH(M_0,\gamma_0)^* \otimes SFH(M_1,\gamma_1) \cong \mathrm{Hom}(SFH(M_0,\gamma_0), SFH(M_1,\gamma_1)).$$

So we do get a homomorphism $F_{\mathcal{W}}$ from $SFH(M_0,\gamma_0)$ to $SFH(M_1,\gamma_1)$ induced by the balanced cobordism $\mathcal{W}$. It is important to note that if we decide to follow Atiyah [2], we need to use the pairing $\langle\ ,\ \rangle'$ – and not $\langle\ ,\ \rangle$ – to identify $SFH(-M,-\gamma)$ with $SFH(M,\gamma)^*$.

Remark 11.15. Another consequence of Conjecture 11.13 would be a new definition of the maps $F_{\mathcal{W}}$, only using the EH class in SFH and special cobordism maps. Indeed, let $\mathcal{W}=(W,Z,[\xi])$ be a balanced cobordism from (M_0,γ_0) to (M_1,γ_1). Let $\gamma=-\gamma_0\cup\gamma_1$. If we view $\mathcal{W}$ as a cobordism $\mathcal{W}'$ from $\emptyset$ to $(-M_0,-\gamma_0)\sqcup(M_1,\gamma_1)$, then, by definition,

$$EH(\mathcal{W}') = F_{\mathcal{W}'}(1) = F_{\mathcal{W}_1'}(EH(Z,\gamma,\xi)),$$

where $\mathcal{W}_1'$ is the obvious special cobordism from (Z,γ) to $(-M_0,-\gamma_0)\sqcup(M_1,\gamma_1)$. But we saw above that, using $\langle\ ,\ \rangle'$, we can view $EH(\mathcal{W}')$ as a homomorphism from $SFH(M_0,\gamma_0)$ to $SFH(M_1,\gamma_1)$, which agrees with $F_{\mathcal{W}}$. A priori, to compute $\langle\ ,\ \rangle'$, one already needs to use some gluing map, but Conjecture 11.13 would eliminate this problem. In particular, if $\mathcal{W}$ is a boundary cobordism, then we would get a new definition for the gluing map $F_{\mathcal{W}}=\Phi_{-\xi}$.

However, if one starts out with this definition of $F_{\mathcal{W}}$, then Axioms 2 and 3 seem to be very difficult to prove without using the theory of gluing maps [14].

Remark 11.16. A cobordism $\mathcal{W}$ from the empty sutured manifold to itself is a smooth oriented 4-manifold W with contact boundary (Z,ξ). Here ξ is a positive contact structure when $Z=\partial W$ is given the boundary orientation. Then, for every relative Spin^c structure $\mathfrak{s}\in\mathrm{Spin}^c(\mathcal{W})$, we have a map $F_{\mathcal{W},\mathfrak{s}}\colon\mathbb{Z}_2\to\mathbb{Z}_2$. This can be computed using just the contact class and the cobordism map in Heegaard Floer homology. More precisely, let W_0 be the cobordism from Y to S^3 obtained by removing an open ball from the interior of W, and set $\mathfrak{s}_0=\mathfrak{s}|_W$. Then

$$F_{\mathcal{W},\mathfrak{s}}(1) = F_{W_0,\mathfrak{s}_0}(c(Y,\xi)) \in \widehat{HF}(S^3)\cong\mathbb{Z}_2,$$

where $c(Y,\xi)\in\widehat{HF}(Y,\mathfrak{s}(\xi))$ is the contact invariant defined by Ozsváth and Szabó [37], and agrees with $EH(Y,\xi)$ according to [17].

Recall that in Section 4, we defined a functor $\mathcal{W}\colon\mathbf{DLink}\to\mathbf{BSut}$, and for a decorated link $(Y,L,P)\in\mathbf{DLink}$, we have

$$SFH(\mathcal{W}(Y,L,P)) \cong \widehat{HFL}(Y,L)\otimes V^{\otimes d}.$$

Then Theorem 11.11 and Proposition 4.10 give the following.

Corollary 11.17. *The functor*

$$SFH\circ\mathcal{W}\colon \mathbf{DLink}_0\to\mathbf{Vect}_{\mathbb{Z}_2}$$

is also a TQFT in the sense of Atiyah [2] *and Blanchet* [3]*, making link Floer homology functorial.*

11.4. **Weinstein cobordisms.** We conclude with defining Weinstein cobordisms, and showing that they preserve the *EH* class. The following definition extends the corresponding notions of Bourgeois et. al [4] and Sidel [44] to sutured manifolds.

Definition 11.18. Suppose that (M_0, γ_0, ζ_0) and (M_1, γ_1, ζ_1) are contact manifolds. A *Liouville cobordism from* (M_0, γ_0, ζ_0) *to* (M_1, γ_1, ζ_1) is a pair $(\mathcal{W}, \theta)$, where $\mathcal{W} = (W, Z, [\xi])$ is a balanced cobordism from (M_0, γ_0) to (M_1, γ_1), and θ is a 1-form on W satisfying the following properties:

(1) $\omega = d\theta$ is symplectic,
(2) the *Liouville vector field* X defined by $\iota_X \omega = \theta$ is transverse to every face of ∂W, enters W through M_0, and exits W through $Z \cup M_1$,
(3) $\xi = \ker(\theta|_Z)$, $\zeta_0 = \ker(\theta|_{M_0})$, and $\zeta_1 = \ker(\theta|_{M_1})$.

The Liouville cobordisms $(\mathcal{W}, \theta)$ and $(\mathcal{W}', \theta')$ are equivalent (diffeomorphic) if there is an equivalence (diffeomorphism) d between $\mathcal{W}$ and $\mathcal{W}'$ such that $d^*(\theta') = \theta$.

Example 11.19. An instance of a Liouville cobordism is the *symplectization* of a contact manifold (M, γ, ζ). Let $\mathcal{W} = (W, Z, [\xi])$ be the trivial cobordism from (M, γ) to itself. Since ∂M is a convex surface, there is a contact vector field v on M which is transverse to ∂M, points out of M, and $v \in \zeta$ exactly along γ. Let α be a contact 1-form such that $\zeta = \ker(\alpha)$. Then $\mathcal{L}_v \alpha = \mu\alpha$ for some function $\mu \colon M \to \mathbb{R}$. By the Cartan formula, this is equivalent to $\iota_v d\alpha + d\iota_v \alpha = \mu\alpha$. The dividing set γ coincides with the zero set of $\alpha(v)$ along ∂M. If we multiply v by a sufficiently small positive scalar, then we can assume that $\mu < 1/2$. Take

$$\theta = e^t \alpha - d(e^t \alpha(v)).$$

Then

$$\omega = d\theta = e^t dt \wedge \alpha + e^t d\alpha$$

is symplectic since $\omega \wedge \omega = 2e^{2t} dt \wedge \alpha \wedge d\alpha$ is nowhere zero on W.

We claim that

$$X = (1 - \mu)\partial_t + v$$

is the Liouville vector field for θ. Indeed,

$$\iota_X \omega = (1 - \mu)\iota_{\partial_t} \omega + \iota_v \omega.$$

Since $\iota_{\partial_t} \omega = e^t \alpha$ and

$$\begin{gathered} \iota_v \omega = -e^t \alpha(v) dt + e^t \iota_v d\alpha = \\ -e^t \alpha(v) dt + e^t(\mu\alpha - d\iota_v \alpha) = e^t \mu\alpha - d(e^t \alpha(v)), \end{gathered}$$

we can conclude that $\iota_X \omega = \theta$. As $1 - \mu > 0$, the vector field X points into W along M_0 and points out of W along M_1. Furthermore, since v points out of M along ∂M, we see that X points out of W along $Z = \partial M \times I$.

For $i \in \{0, 1\}$, the contact structure

$$\ker(\theta|_{M_i}) = \ker(\alpha - d(\alpha(v)))$$

is isotopic to ζ through contact structures. To see this, note that, for every $0 \le \varepsilon \le 1$, the same argument as above shows that

$$\theta^\varepsilon = e^t \alpha - d(e^t \alpha(\varepsilon v))$$

induces a contact structure $\zeta^\varepsilon = \ker(\theta^\varepsilon|_{M_i})$ on M_i. For $\varepsilon = 0$, we have $\zeta^0 = \ker(\alpha) = \zeta$, while $\zeta^1 = \ker(\theta|_{M_i})$.

Finally, we show that $\ker(\theta|_Z) = \xi$. Since $\mathcal{L}_{\partial_t} d = d\mathcal{L}_{\partial_t}$, we have $\mathcal{L}_{\partial_t}\theta = \theta$. Hence $\partial_t|_Z$ is a contact vector field on $(Z, \theta|_Z)$. So, for every $0 \le t \le 1$, the surface $\partial M \times \{t\}$ is convex. The dividing set on this surface is given by the equation $\theta(\partial_t) = 0$. However,

$$\theta(\partial_t) = -d(e^t\alpha(v))(\partial_t) = -e^t\alpha(v),$$

and the function $\alpha(v)|_{\partial M}$ vanishes exactly along γ.

Definition 11.20. Let $(\mathcal{W}, \theta)$ from (M_0, γ_0, ζ_0) to (M_1, γ_1, ζ_1) be a Liouville cobordism, and let X be the corresponding Liouville vector field. A function $H\colon W \to \mathbb{R}$ is a *Lyapunov function for* X if H is a smooth Morse function, and there exists a constant $\delta \in \mathbb{R}_+$ and a Riemannian metric on W such that $dH(X) \ge \delta|X|^2$.

Example 11.21. If $(\mathcal{W}, \theta)$ is a symplectization, then $H(x, t) = t$ is a Lyapunov function for $X = (1 - \mu)\partial_t + v$.

Definition 11.22. We say that the Liouville cobordism $(\mathcal{W}, \theta)$ from (M_0, γ_0, ζ_0) to (M_1, γ_1, ζ_1) is *Weinstein* if there exists a Lyapunov function H for the Liouville vector field X such that

(1) a collar neighborhood $M_0 \times I$ of $M_0 = M_0 \times \{0\}$ is a symplectization of (M_0, γ_0, ζ_0), as in Example 11.19,
(2) $H(x, t) = t$ for $(x, t) \in M_0 \times I$,
(3) there is a collar neighborhood $\partial M_0 \times [0, 2]$ of ∂M_0 in Z that extends $\partial M_0 \times I$, where $H(x, t) = t$,
(4) $H \equiv 2$ on $(Z \cup M_1) \setminus (\partial M_0 \times [0, 2])$,
(5) H has no critical points on ∂W.

Remark 11.23. Since X points out of W along $Z \cup M_1$, the negative gradient flow lines of X can only exit W along M_0. As explained in [4], all critical points of the Lyapunov function H have Morse index at most two, and the stable manifolds intersected with regular levels $H^{-1}(c)$ are isotropic for the induced contact structure $\ker(\theta|_{H^{-1}(c)})$. Using the work of Weinstein [46], we see that one can build W, viewed as a special cobordism from M_0 to $Z \cup M_1$, from the symplectization of (M_0, γ_0, ζ_0) by attaching Weinstein 1- and 2-handles. A Weinstein 1-handle attachment changes the boundary by removing two standard contact balls, and gluing a standard contact $S^2 \times I$, or equivalently, by taking a connected sum with the standard contact $S^2 \times S^1$. Each Weinstein 2-handle is attached along some Legendrian knot K with framing $tb(K) - 1$.

Theorem 11.24. *Let $(\mathcal{W}, \theta)$ be a Weinstein cobordism from the contact manifold (M_0, γ_0, ζ_0) to (M_1, γ_1, ζ_1). Let $\mathfrak{s} \in Spin^c(\mathcal{W})$ be the $Spin^c$ structure associated with $\omega = d\theta$. As explained in Remark 2.13, we can view $\mathcal{W}$ as a balanced cobordism $\overline{\mathcal{W}}$ from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$. Then*

$$F_{\overline{\mathcal{W}}, \mathfrak{s}}(EH(M_1, \gamma_1, \zeta_1)) = EH(M_0, \gamma_0, \zeta_0).$$

Remark 11.25. Recall that, for $i \in \{0, 1\}$,

$$EH(M_i, \gamma_i, \zeta_i) \in SFH\left(-M_i, -\gamma_i, \mathfrak{s}_{\zeta_i}\right).$$

Furthermore, $\mathfrak{s}|_{M_i} = \mathfrak{s}_{\zeta_i}$, and

$$F_{\overline{\mathcal{W}}, \mathfrak{s}}\colon SFH\left(-M_1, -\gamma_1, \mathfrak{s}_{\zeta_1}\right) \to SFH\left(-M_0, -\gamma_0, \mathfrak{s}_{\zeta_0}\right).$$

Proof. Consider the cobordism $\overline{\mathcal{W}} = (W, Z, [-\xi])$ from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0)$. First, suppose that Z has no isolated components, and set $N = M_1 \cup Z$. Then $F_{\overline{\mathcal{W}}} = F_{\overline{\mathcal{W}}_1} \circ \Phi_\xi$, where $\overline{\mathcal{W}}_1$ is a special cobordism from $(-N, -\gamma_0)$ to $(-M_0, -\gamma_0)$. By Theorem 9.6, we have

$$\Phi_\xi(EH(M_1, \gamma_1, \zeta_1)) = EH(N, \gamma_0, \zeta_1 \cup \xi).$$

By turning it upside down, we can view $\overline{\mathcal{W}}_1$ as a special cobordism $\mathcal{W}_1$ from (M_0, γ_0) to (N, γ_0). As explained in Remark 11.23, we can build $\mathcal{W}_1$ from the symplectization of (M_0, γ_0, ζ_0) by first attaching Weinstein 1-handles, then attaching Weinstein 2-handles along a Legendrian link L with framing $tb(L) - 1$. Hence, we are done if we prove the result when $\mathcal{W}$ is a Weinstein 1- or 2-handle cobordism.

Suppose that $\mathcal{W}$ is a Weinstein 1-handle cobordism from (M_0, γ_0, ζ_0) to

$$(M_0 \# (S^1 \times S^2), \gamma_0, \zeta_1).$$

The contact structure ζ_1 agrees with ζ_0 minus a standard contact ball on $M_0 \setminus B^3$, and is the unique tight contact structure minus a standard contact ball ξ_{std} on $(S^1 \times S^2) \setminus B^3$. Fix a partial open book decomposition $(S_0, h_0 \colon P_0 \to S_0)$ for (M_0, γ_0, ζ_0), and let $(\Sigma_0, \boldsymbol{\alpha}_0, \boldsymbol{\beta}_0)$ be the associated balanced diagram. Furthermore, let $(S, h \colon P \to S)$ be the partial open book decomposition of

$$((S^1 \times S^2)(1), \xi_{std})$$

described in [34, Example 1]. It agrees with the partial open book of Example 9.3, except that the map h is the identity of P. We are going to denote by (Σ, α, β) the corresponding balanced diagram of $(S^1 \times S^2)(1)$. Then $\Sigma = T^2 \setminus B^2$, the curves α and β intersect in exactly two points, and α is a small Hamiltonian translate of β. Let $y \in \alpha \cap \beta$ be the intersection point with the smaller relative grading. If we take the boundary connected sum of S_0 and S along $\partial S_0 \setminus P_0$ and $\partial S \setminus P$, then we get a partial open book decomposition for $(M_0 \# (S^1 \times S^2), \gamma_0, \zeta_1)$, which induces the balanced diagram $(\Sigma_0 \natural \Sigma, \boldsymbol{\alpha}_0 \cup \{\alpha\}, \boldsymbol{\beta}_0 \cup \{\beta\})$. Let $\mathbf{x} \in \mathbb{T}_{\alpha_0} \cap \mathbb{T}_{\beta_0}$ be the distinguished generator representing $EH(M_0, \gamma_0, \zeta_0)$, and note that y represents the class $EH((S^1 \times S^2)(1), \xi_{std})$. Then $\mathbf{x} \times \{y\}$ represents $EH(M_0 \# (S^1 \times S^2), \gamma_0, \zeta_1)$. The curves α and β bound a periodic domain which represents a sphere $\{p\} \times S^2$ inside $S^1 \times S^2$. The cobordism $\overline{\mathcal{W}}$ corresponds to a 3-handle attached along $\{p\} \times S^2$, so by Definition 7.8, the map $F_{\overline{\mathcal{W}}}$ takes $\mathbf{x} \times \{y\}$ to $\mathbf{x}$, which proves the claim for Weinstein 1-handles.

Now assume that $\mathcal{W}$ is a Weinstein 2-handle cobordism corresponding to a Legendrian knot K in (M_0, γ_0, ζ_0). Then the proofs of [16, Proposition 4.4] and [38, Theorem 3.5] imply that the EH class is preserved by $F_{\overline{\mathcal{W}}}$. More concretely, in the proof of [16, Proposition 4.4], Honda, Kazez, and Matić construct a partial open book decomposition $(S, h \colon P \to S)$ for (M_0, γ_0, ζ_0) which contains the Legendrian knot K inside P. This gives rise to a triple diagram $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ which is subordinate to some bouquet for K. As described by Ozsváth and Szabó in the proof of [38, Theorem 3.5], if $(\Sigma, \boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\delta})$ corresponds to the 2-handle cobordism $\mathcal{W}$, then $(-\Sigma, \boldsymbol{\alpha}, \boldsymbol{\delta}, \boldsymbol{\beta})$ corresponds to $\overline{\mathcal{W}}$. We end up with the configuration depicted in Figure 14, where there is a distinguished triangle mapping the generator representing $EH(M_1, \gamma_1, \zeta_1)$ to the one representing $EH(M_0, \gamma_0, \zeta_0)$.

If Z does have isolated components, then $F_{\overline{\mathcal{W}}} = F_{\overline{\mathcal{W}}'}$, where $\overline{\mathcal{W}}'$ is the cobordism from $(-M_1, -\gamma_1)$ to $(-M_0, -\gamma_0) \sqcup (B, \delta)$ given by Definition 10.1. Note that $B \subset Z$, and $\ker(\theta|_B) = \xi|_B$ is a union of standard contact balls. Let $\mathcal{W}'$ be $\overline{\mathcal{W}}'$ viewed as a cobordism from $(M_0, \gamma_0) \sqcup (-B, -\delta)$ to (M_1, γ_1). Then $(\mathcal{W}', \theta)$ is not a Liouville cobordism, since the Liouville vector field X points out of W along B. To fix this, attach a Weinstein 1-handle to each component of B along one of its feet. Then θ plus the Liouville 1-forms on the 1-handles give a 1-form θ' such that the new Liouville vector field X' points in along the free feet of the 1-handles. The contact structure ξ on Z is left unchanged, since $I \times S^2$ has a unique tight contact structure. The cobordism $(\mathcal{W}', \theta')$ is also Weinstein, since a Lyapunov function H on $\mathcal{W}$ extends to the 1-handles with a unique index one critical point in each. Let $\xi_0 = \ker(\theta'|_{-B})$, then $(-B, \xi_0)$ is also a union of standard contact balls. If we apply the previous part to the Weinstein cobordism $(\mathcal{W}', \theta')$, then we get that

$$F_{\overline{\mathcal{W}}', \mathfrak{s}'}(EH(M_1, \gamma_1, \zeta_1)) = EH(M_0, \gamma_0, \zeta_0) \otimes EH(-B, -\delta, \xi_0),$$

where $\mathfrak{s}' = \mathfrak{s}|_{\mathcal{W}'}$. As $EH(-B, -\delta, \xi_0) = 1 \in SFH(B, \delta)$, the right-hand side maps to $EH(M_0, \gamma_0, \zeta_0)$ under the isomorphism

$$SFH(-M_0, -\gamma_0) \otimes SFH(B, \delta) \to SFH(-M_0, -\gamma_0),$$

which concludes the proof. □

Corollary 11.26. *Let (W, θ) from (M_0, γ_0, ζ_0) to (M_1, γ_1, ζ_1) be a Weinstein cobordism. If $EH(M_0, \gamma_0, \zeta_0) \neq 0$, then $EH(M_1, \gamma_1, \zeta_1) \neq 0$.*

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