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Author for correspondence:

Thomas Adcock

e-mail:

thomas.adcock@new.oxon.org

Non-linear dynamics of wave-groups in random seas: Unexpected walls of water in the open ocean

Thomas A.A. Adcock[1], Paul H. Taylor[1] and Scott Draper[2]

[1]Department of Engineering Science, University of Oxford, UK

[2]School of Civil, Environmental and Mining Engineering, University of Western Australia, WA 6009, Australia.

This paper investigates the size and structure of large waves on the open ocean. We investigate how non-linear physics modifies waves relative to those predicted by a linear model. We run linear random simulations and extract extreme waves and the surrounding sea state. For each extreme event we propagate the waves back in time under linear evolution before propagating the wave-field forward using a non-linear model. The differences between large linear and non-linear wave-groups are then examined. The general trends are that under non-linear evolution, relative to linear evolution, there is, on average, little or no extra amplitude in the non-linear simulations; that there is an increase in the width of the crest of the wave-group and a contraction of large wave-groups in the mean wave direction; that large waves tend to move to the front of a wave-packet meaning that the locally largest wave is relatively bigger than the wave preceding it; and that non-linearity can increase the duration of extreme wave events. In all these trends there is considerable scatter, although the effects observed are clear. Our simulations show that non-linearity does play an important part in the formation of extreme waves on deep water.

1. Introduction

Extreme ocean waves, sometimes called freak or rogue waves, are of interest to scientists and engineers. In many locations around the world's oceans they produce the dominant environmental loads on ships and offshore structures. There is an ongoing debate as to whether physics beyond linear theory (corrected for bound waves) is needed to describe the short term evolution of large waves on the open ocean up to the point of breaking (see review papers [1–3]). Ocean waves have a finite amplitude and therefore their evolution will, to some degree, be non-linear. Perhaps of most interest is whether this non-linearity gives rise to waves of greater amplitude than would be predicted by linear theory for realistic ocean sea-states. Also of significant interest is whether non-linearity changes the shape or duration of the extreme wave event as has been suggested by some authors [4,5].

In the literature, there are two main approaches to studying this problem. One approach is to study the properties of a random wave field in either a numerical or physical wave tank (e.g [6–8]). This is demanding in terms of both computational and tank time because, for much of the space and time, one is simulating waves which are not at the extremes and close to linear. An alternative approach is to consider the non-linear dynamics of an isolated wave-group whose initial profile is set by the expected shape of an extreme event under linear evolution. Examples of this approach are [4,9–12]. This approach is efficient at investigating the physics of extreme events but does not account for the randomness of extreme events, or the interactions of extreme events with the surrounding waves.

The main conclusions drawn from studies of the non-linear dynamics of isolated wave-groups is that in directionally spread sea-states there is little or no extra elevation from the non-linear physics. However, there is a significant change in the shape of large wave-groups due to non-linearity. This is demonstrated in the extreme wave-groups shown in Figure 1. It can be seen that there is an expansion of the group in the lateral direction (at 90° to the mean wave propagation direction) and a contraction of the group in the mean wave direction.

In this work, instead of starting with an isolated and dispersed wave-group that gives rise to the expected shape of a large wave under linear evolution, we start with a number of examples of large waves produced by chance from linear random wave simulations. Our procedure after this is similar to the studies of isolated wave-groups: we run the wave-field back in time under linear evolution before running it forward again using a non-linear model. We examine the differences in amplitude, group shape and directional spreading, wave amplitude relative to the preceding wave, duration of the extreme event and direction of the extreme waves comparing the linear and the non-linear cases.

2. Methods

The approach taken in this paper is conceptually similar to that taken by a number of groups to investigate isolated wave-packets (e.g. [4,9–12]). In previous work researchers have started with a wave-group which matches the average shape of an extreme wave in a linear sea – the so called 'NewWave'. This was run backwards in time for a given number of periods under linear evolution before being run forward using a non-linear model (or in a physical wave-tank). In this paper, instead of starting with an isolated NewWave, we start with a number of examples of large waves from random linear simulations. These are then run backwards in time under linear evolution before being run forward again using the Modified Non-Linear Schrödinger equation (i.e. with a weakly non-linear model).

The key assumptions we have made in this paper are:

(i) That the formation of an extreme event will be initiated by sufficient random wave components coming into phase and that the circumstances which give rise to large events can be captured by a linear model.

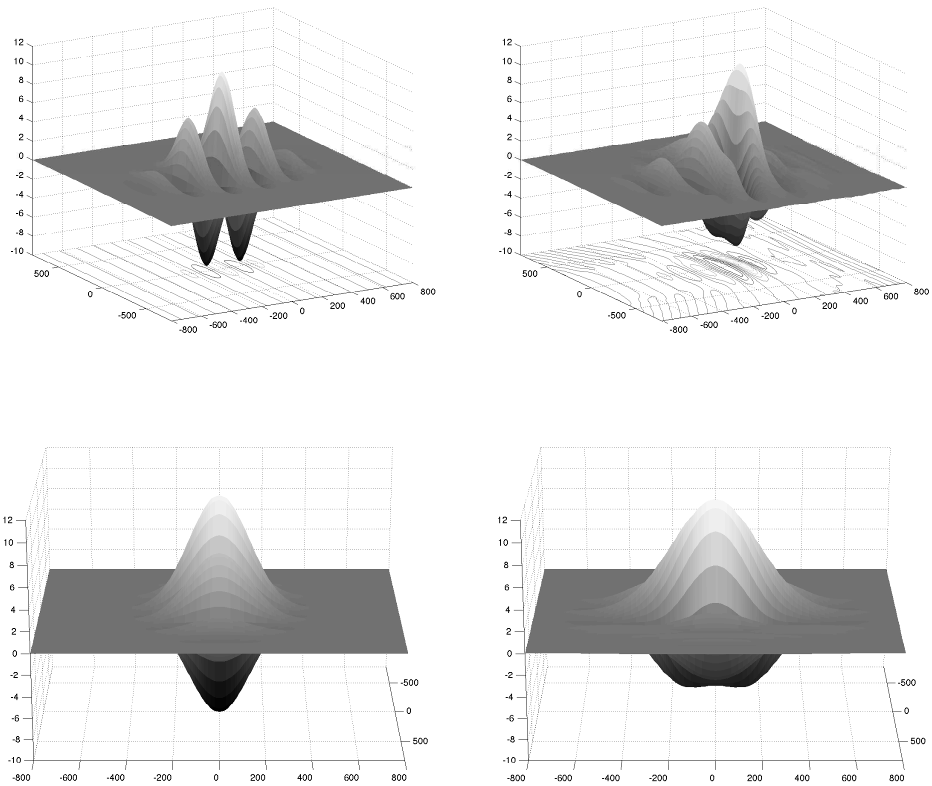

Figure 1. Comparison of 'NewWave' type wave-groups at the maximum point in their evolution under linear evolution (left) and evolution using Modified Non-linear Schrödinger equation (right). Top – view from behind wave-group with wave propagating from left to right; bottom – view from in front of wave-group with waves propagating towards observer. Dimensions in m.

(ii) That we can investigate non-linear changes to extreme waves by using as initial conditions a wave-field which, under linear evolution, would produce a large wave ten periods later.

In practice we generate linear simulations of random waves from which we extract wave-fields containing extreme wave events. For each of these wave-fields we propagate the waves backwards in time linearly and then forwards again using a non-linear model of wave evolution.

The approach we have adopted in this paper has a number of advantages over the isolated wave-group approach described in the introduction:

(i) We capture much of the randomness of real ocean waves.
(ii) We can capture the interaction of large wave-groups with the groups around them. For instance, if we use isolated wave-groups we might expect to wrongly estimate the lateral expansion of the group as there are no wave-groups adjacent which might help or hinder the expansion.

Our approach also has advantages over long duration non-linear random wave simulation:

(i) Increased computational speed.
(ii) Rather than comparing general statistics between linear and non-linear cases we can compare the differences in the evolution of essentially the same wave-packets in each model. Thus we hope to get a greater insight into the changes the non-linear physics makes to extreme waves.

Set against these advantages, our approach makes it difficult to generate robust statistics for the properties of extreme waves in a given sea-state. However, we can draw strong and robust qualitative conclusions as to how non-linear dynamics influence the shape and structure of extreme waves.

The first part of each run is to simulate a linear random sea. For these simulations we use the same general approach as Tucker [13,14] using a method adapted slightly from that appearing in Goda [15]. The numerical code produces results consistent with expectations [16]. We simulate 5km $\times$ 5km of ocean for a duration of 1 hour. We do this for 744 different instances. From this we search for the largest wave whose crest occurs in the central 3 km square of ocean. A 2 km square of ocean around the giant crest is then extracted and converted from a free surface to a complex envelope as described by [17]. The complex wave envelope was then propagated backwards for ten periods under 'linear evolution' before being propagated forwards under 'non-linear evolution' (see below for model details). A time of ten periods was chosen as a compromise between capturing all the non-linear changes and maintaining the identity of particular wave-groups. For comparison Gibbs & Taylor [4] used 20 periods. To analyse the output we compare the initial linear wave-group (i.e. when the wave-group was at its largest under linear evolution) with the point of maximum amplitude in the non-linear simulation – because of the non-linearity this is generally at a slightly different point in space and time. We analyse the largest event in the non-linear run within 4 periods and 1 km of the original linear maximum. In a few instances this leads to different wave-groups being analysed in linear and non-linear case (although the precise identification of a wave-group can be ambiguous). This means there are a small number of outliers in these results which are the result of the analysis method rather than interesting non-linear physics.

In this paper we use the Modified Non-Linear Schrödinger equation (MNLSE) derived in a series of papers by Dysthe and Trulsen [18,19]. This is given by

$$\begin{aligned}\frac{\partial U}{\partial t} + \frac{\omega}{2k}\frac{\partial U}{\partial x} + i\frac{\omega}{8k^2}\frac{\partial^2 U}{\partial x^2} - i\frac{\omega}{4k^2}\frac{\partial^2 U}{\partial y^2} - \frac{\omega}{16k^3}\frac{\partial^3 U}{\partial x^3} + \frac{3\omega}{8k^3}\frac{\partial^2 U}{\partial y^2 \partial x} - i\frac{5\omega}{128k^4}\frac{\partial^4 U}{\partial x^4} \\ - i\frac{3\omega}{32k^4}\frac{\partial^4 U}{\partial y^4} + i\frac{15\omega}{32k^4}\frac{\partial^4 U}{\partial y^2 \partial x^2} + \frac{7\omega}{256k^5}\frac{\partial^5 U}{\partial x^5} - \frac{35\omega}{64k^5}\frac{\partial^5 U}{\partial y^2 \partial x^3} + \frac{21\omega}{64k^5}\frac{\partial^5 U}{\partial y^4 \partial x} \\ = -\frac{i\omega k^2}{2} U\left|U\right|^2 - \frac{3}{2}\omega k U^2 \frac{\partial U}{\partial x} - \frac{1}{4}\omega k U^2 \frac{\partial U^*}{\partial x} - ikU \left.\frac{\partial \phi}{\partial x}\right|_{z=0},\end{aligned} \tag{2.1}$$

where ω and k are the frequency and wavenumber of the carrier wave and related by the linear dispersion equation. U is the complex wave envelope of the close to linear free waves (as opposed to the bound waves). The waves are mostly travelling in the x direction. The left hand side of this equation is a high order approximation to linear evolution (linear dispersion of components in a directionally spread sea). The right hand side contains non-linear terms. For linear evolution we simply set the right hand side of equation 2.1 to zero. The final term in equation 2.1 simulates the interaction with the induced local surface current driven by spatial variations in the wave envelope. At a given time this return induced current term, ϕ, can be found from

$$\left.\frac{\partial \phi}{\partial z}\right|_{z=0} = \frac{\omega}{2}\frac{\partial \left|U\right|^2}{\partial x}, \tag{2.2}$$

and within the fluid

$$\nabla^2 \phi = 0, \tag{2.3}$$

with ϕ tending to zero as z goes to infinity (i.e. at infinte water depth).

Equation 2.1 is solved using a pseudo-spectral scheme to evaluate the left hand side with the non-linear terms being evaluated in the spatial domain. The induced current is solved analytically at every timestep. A spatial discretisation of 10 m is used in both x and y directions. A timestep of 0.5 s coupled with a 4th order Runge-Kutta solver are used to time-march the equations. The maximum error in energy between the beginning and end of a run was 0.7% but in most cases the energy conservation was more than an order of magnitude better than this. The code is reversible with a maximum error of 1 part in 10^4 when a wave-group with steepness $ak = 0.3$ was run through non-linear focus and back again.

The MNLSE is a narrowbanded approximation to the full water wave evolution equations. For this reason we confine our study to narrowbanded spectra. However, during extreme wave events there is a localised increase in the bandwidth that the MNLSE may struggle to simulate [20]. It is expected that, relative to the full water wave equations, the MNLSE will fractionally over-estimate the amplitude of the wave but under-estimate the local change in group shape, though capturing the group-scale changes to the shape of the wave-group accurately [20]. This is an important limitation on this study. On the other hand, the 'leakage' of energy to high wavenumbers, which is a well known facet of the non-linear Schrödinger type equations [21], is not a significant issue due to the relatively small number of wave periods for which the simulation is run.

In this paper we concentrate on analysis of the envelope of the free waves. The absolute phase of the wave-field is arbitrary but the relative phase within the group is not. In some places we do consider the properties of the (linear) free surface which we do by assuming that the maximum of the wave envelope corresponds with the position of a wave crest. The free surface, and its bound wave components, can be found from the equations in [20].

In this study we simulate linear waves from a Gaussian wavenumber spectrum with a bandwidth of $s_x = 0.0046\text{m}^{-1}$ and a fundamental wavenumber of $k = 0.0279\text{m}^{-1}$ corresponding to a period of 12 s. A wrapped normal directional spreading is used with a directional spreading of 15°. The wrapped normal directional spreading (as used in [4]) is given by

$$D(\theta) = \frac{1}{\sqrt{2\pi}\mu} \exp\left(-\left(\frac{\theta^2}{2\mu^2}\right)\right), \tag{2.4}$$

where θ is the angle from the mean wave direction and μ is the directional spread. The spectrum is a reasonable approximation to extreme sea-states in the North Sea although for the largest value of steepness considered herein we would expect a rather broader spectrum (a broader spectrum implies a reduction in the non-linearity). Note that extreme wave-groups approximately scale as the solutions to the nonlinear Schrodinger equation do, so that, roughly, a doubling the amplitude and bandwidth (as measured in both the inline and transverse directions) keeps the non-linearity the same [5,22].

The amplitude of the linear waves, before being propagated back in time, varies considerably. The distribution of extreme wave-heights studied (one per spatial realisation) is shown in Figure 2. The amplitudes in this study are larger than those predicted by a Rayleigh distribution as we are considering the maximum elevation over a finite area – instead the distribution of amplitudes closely follows the expected Gumbel distribution [23]. We group our analysis by the amplitude of the background sea state. Thus there is a considerable variation in the amplitudes of the extreme events which we group together for analysis purposes, but these variations reflect the variation in the amplitudes of extreme waves in a given background sea-state. We examine cases with different wave-heights and background sea-states but the same underlying spectral shape. We consider different background significant wave-heights of 6 m, 8 m , 10 m, and 12 m and compare these with linear evolution (effectively $H_s = 0$ m). We present these sea-states in terms of significant steepness which we define as $S = H_s k_p/2$. This gives steepness of 0.084, 0.111, 0.140 and 0.167. The different wave-fields are simply generated by multiplying the linear initial conditions by the appropriate factor – i.e. the same waves were used for each amplitude with their amplitudes simply scaled. The largest sea-state ($S = 0.167$) is probably unrealistic for a bandwidth as narrow as that studied here and is sufficiently steep to push the assumptions of the MNLSE. In reality wave breaking would also be significant for waves of this steepness, something we do not

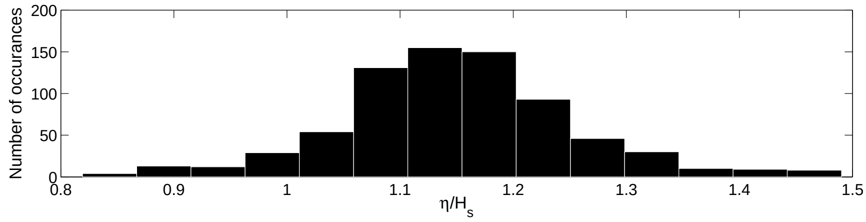

Figure 2. Distribution of linear crest amplitudes.

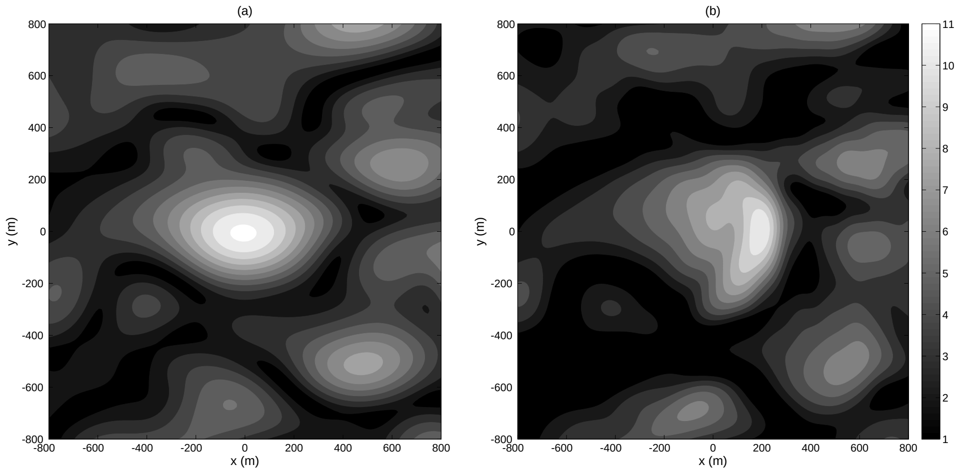

Figure 3. Wave envelopes for linear (a) and non-linear (b) for a randomly chosen example. Waves are moving from negative to positive x (from left to right). x and y axes are centered on the linear maximum in the wave-field. Envelope elevation is in m.

try to capture in our model. As such the results for this case should be treated with caution but are included to help demonstrate general trends.

To demonstrate the general method and results Figure 3 shows the first case in our simulation. The non-linear case we present is for $S = 0.140$ and the linear wave-field has a maximum of 11.2 m. Relative to the linear case, the non-linear result shows very similar maximum envelope amplitude (in this case a small decrease), but a significant change in the main group shape in space: with a steep front to the group, a contraction of the group in the mean wave (x) direction and an expansion in the lateral (y) direction. For this case the results are strikingly similar to those of Gibbs & Taylor [4] for isolated wavegroups (see also Figure 1). The rest of this paper looks at how general these conclusions are given random variation in the linear waves.

3. Elevation

Firstly we consider the changes in the maximum amplitude of the envelope between the linear and non-linear cases. We refer to the maximum amplitude of a non-linear simulation, divided by the maximum amplitude of the equivalent linear simulation as the 'normalised amplitude'. Firstly, we look at the $S = 0.140$ case. Figure 4 shows a scatter plot of the normalised amplitude against linear amplitude. There is obviously a significant scatter with some wave-groups reaching

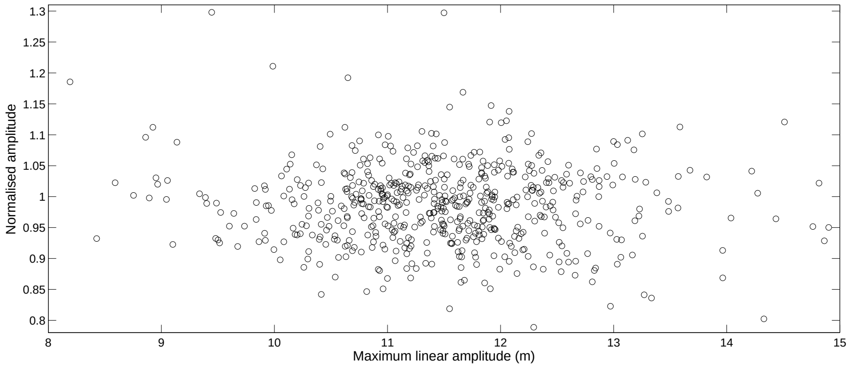

Figure 4. Plot of normalised amplitude (maximum amplitude in non-linear run to maximum amplitude under linear evolution) against the maximum amplitude under linear evolution for $S = 0.140$.

Steepness	1st percentile	Mean	99th percentile
0.084	0.90	0.97	1.10
0.111	0.86	0.97	1.11
0.140	0.84	0.99	1.19
0.167	0.84	1.00	1.19

Table 1. Statistics of maximum non-linear amplitude divided by maximum linear amplitude.

larger amplitudes under linear evolution and some larger under non-linear evolution. The mean value of the normalised non-linear wave-height is 0.99.

Similar results were found for the other background sea-states – increasing the size of the sea-state increases the variability in the amplitude of the non-linear results relative to the linear. Table 3 presents the mean, 1st percentile, and 99th percentile of the non-linear amplitudes divided by the amplitude under linear evolution.

Although the mean amplification remains close to one, in some cases there is significant amplification. As an example of this we present the contours of the case where there is the largest amplification in Figure 5. The key question is whether this implies weakly non-linear dynamics (i.e. excluding wave-breaking) increase the probability of exceptionally large waves. To examine this question we plot the sorted crests from our simulations in Figure 6. These do not correspond to probabilities of exceedances as there may be wave wave-groups which were smaller under linear evolution but would have been amplified under non-linear evolution. The plot does suggest that at the extremes there may be extra elevation even if the circumstances for this to occur are rare.

It is of interest to know whether there is a correlation between the normalised amplitudes for different non-linearities, i.e. if a particular normalised amplitude is larger than unity for $S = 0.111$ will the same initial wave-field produce a normalised amplitude greater than unity in the $S = 0.140$ case. Figure 7 presents the normalised amplitudes for $S = 0.111$ plotted against the normalised amplitude for $S = 0.140$m. There is obviously a high degree of correlation between the two suggesting that whether the normalised amplitude is greater or less than unity is strongly dependent of the structure of initial linear wave-field.

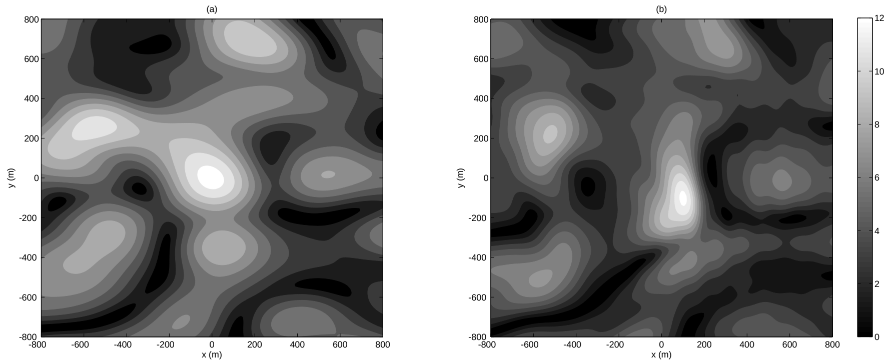

Figure 5. Wave envelopes for linear (a) and non-linear (b) for the case with the biggest gain in elevation. Waves are moving from negative to positive x (from left to right). x and y axes are centered on the linear maximum in the wave-field. Envelope elevation is in m.

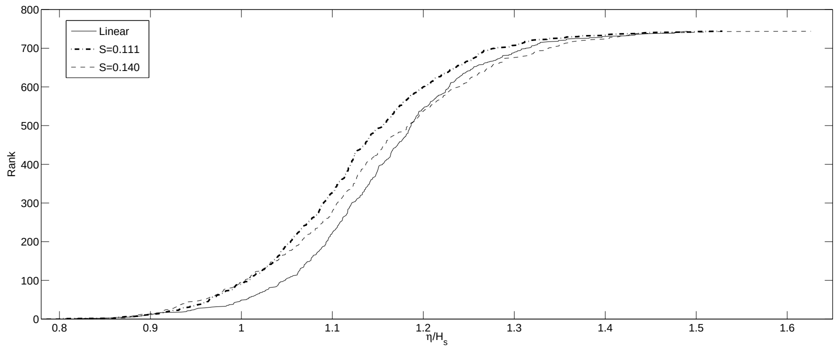

Figure 6. Ranked maximum amplitudes from simulations normalised by significant wave-height.

This raises the question as to whether an increase or decrease in amplitude can be predicted with any certainty from the shape of the initial linear wave-group. We introduce some bandwidth-type simple parameters to define the shape of a wave-group using the assumption that the wave-group can be fitted with a Gaussian given by

$$A_{max}\exp(-\frac{1}{2}s_x^2x^2). \tag{3.1}$$

for the x or mean-wave direction (with an equivalent expression for the y lateral direction). We consider both 'local bandwidths', sx_l and sy_l, which are defined by the shape around the maximum point in the wave-group, and 'group bandwidths', sx_g and sy_g, which are based on the larger scale width of the wave-group. More precisely we define the local bandwidth following Gibbs & Taylor [4] as

$$sx_l = \sqrt{\frac{\pi}{2}}\frac{d|\hat{U}|}{dx}\frac{1}{\sqrt{|U|^2+|\hat{U}|^2}}, \tag{3.2}$$

evaluated at the maximum of the 'envelope of the envelope' given by $\sqrt{|U|^2+|\hat{U}|^2}$ where $|\hat{U}|$ is the Hilbert transform of $|U|$. An analogous expression is used in the y direction. The 'group bandwidth' we evaluate from the width of the wave-group at $2/3$ its maximum height, and relate this to an equivalent Gaussian.

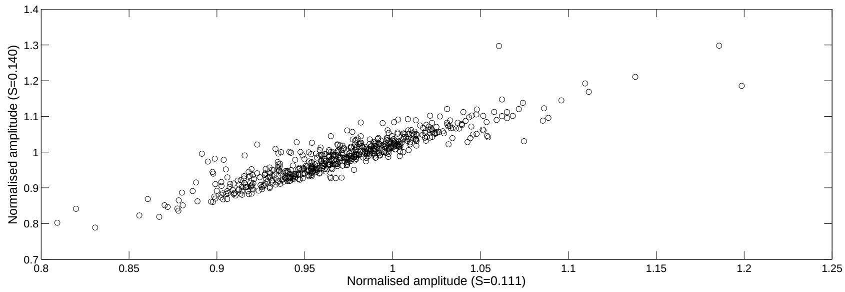

Figure 7. Normalised amplitudes (maximum non-linear amplitude/maximum linear amplitude) for individual run from different background sea states.

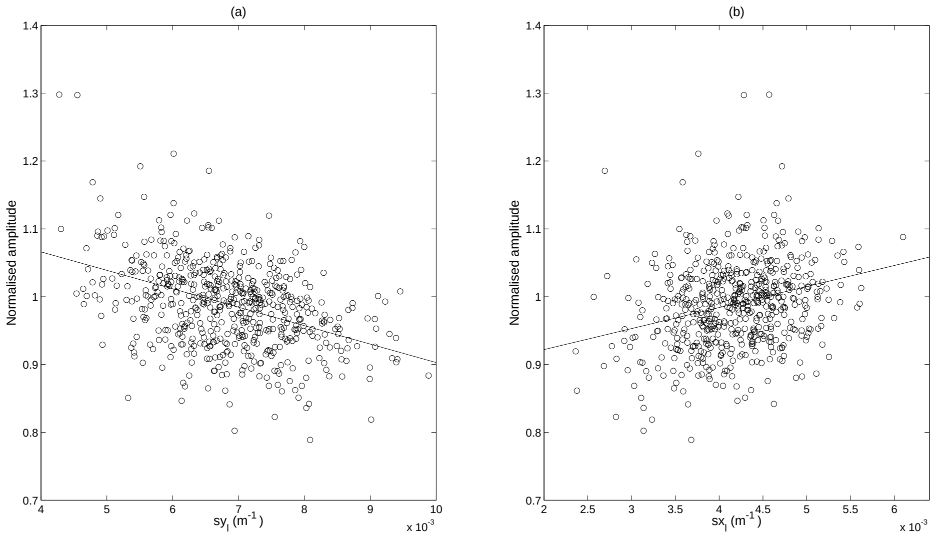

Figure 8. Relationship between local bandwidth of linear group and change in maximum amplitude under non-linear evolution for $S = 0.140$. (a) lateral bandwidth; (b) bandwidth in mean wave direction. Linear fit shown with solid line.

The first observation is that none of these parameters, or combination of parameters, is a good predictor of whether non-linearity will lead to extra elevation. The strongest trend we have observed is that the more long-crested a wave-group (based either on local or global y bandwidth) the more likely is extra elevation as shown in Figure 8a. This is in line with expectation – the closer to uni-directional a group is the stronger the non-linearity. However, although a rather weak trend, our data suggests a positive correlation between sx_l and extra amplitude (Figure 8b). Physically this implies that more compact wave-groups (also implying a broader local spectrum) will be more likely to increase in amplitude. The parameter we use here to describe the spectral width, sx_l, is closely related to the bandwidth of the frequency spectrum which is used by Janssen in his classic paper [24] to derive the Benjamin-Feir index (although strictly this is derived for uni-directional waves). Thus, the apparent correlation between extra wave-amplitude and sx_l is unexpected as the classic Benjamin-Feir type non-linearity would imply the reverse.

It is impossible to fully explain the physics which is causing the trends presented here but it seems appropriate to speculate on this. The dominant trend is, probably, that non-linear physics is changing, subtly, the phases of the wave components bringing the waves either slightly more into phase, or slightly more out of phase, at the wave crest. It is clear that there is no strong non-linear mechanism causing an increase in the amplitude of wave-groups in random seas. There does appear to be a very weak mechanism which is causing small increases in the size of the waves which is correlated to amplitude and negatively correlated with directional spreading. Despite the contrary evidence of this not being correlated to the width of the group in the mean wave direction we think this is likely to be related to the Benjamin-Feir instability, only very much weakened due to directional spreading.

4. Changes in group shape

As noted in the introduction, analysis of isolated wave-packets suggests that the dominant change to wave-groups under non-linear evolution is a dramatic expansion in the lateral direction coupled with a contraction in the mean wave direction. As shown in [20], the MNLSE does not accurately reproduce the changes to local bandwidths (defined in the previous section) – we therefore concentrate on group bandwidths in this section although noting that local bandwidths are perhaps of more importance for some calculations such as wave kinematics under crests.

We start by considering the average shape of the wave-group. Under linear evolution the average shape of the wave-group is closely approximated by the auto-correlation function [25,26] – the so called 'NewWave' [27]. The average shape of the linear simulations (normalised by the maximum amplitude with the maximum crest at $x = 0$, $y = 0$) is shown in Figure 9a. As expected this is very close to the NewWave shape and is the starting point for this investigation.

Figure 9b presents the average shape of the extreme waves in the $S = 0.140$ simulations (normalised by the maximum amplitude with the maximum crest at $x = 0$, $y = 0$). To calculate the free surface we assume that the phase of the carrier wave is in phase with the maximum of the envelope. There are a number of striking differences to the linear simulations:

- The maximum crest has moved towards the front of the wave-group.
- The group has expanded in the lateral direction (i.e. normal to the mean propagation direction).
- The group is more compact in the mean wave direction.
- The front of the wave-group is slightly curved when viewed from above.

These are identical features to those reported for isolated wave-groups by [4] and also those observed for isolated wave-groups in Figure 1. The simulations in the present paper demonstrate qualitatively that these changes are robust features of large waves in random sea-states.

It is interesting to examine whether simulations of isolated wave-groups produce the same average change to the group shape as occurs in random waves. We investigate this by running the average shape of the linear waves (Figure 9a) back in time for ten periods under linear evolution, before running them forward again under non-linear evolution. We consider three different amplitudes for these runs so that under linear evolution they would have amplitudes of 12 m (Figure 9c), 11 m (Figure 9e) and 10 m (Figure 9g). We also plot the difference between the average shape of the non-linear runs, and the average shape of each of the different isolated wave-groups.

Each of the non-linear changes to the wave-groups itemized above has a different sensitivity to non-linearity. The movement of the largest wave to the front of the group and the contraction of the group in the mean wave direction are both sensitive to the non-linearity. Isolated wave-groups with amplitudes between 11 m and 12 m show approximately the same changes as those for the random waves (whose average amplitude is 11.15 m) suggesting that these non-linear changes are relatively insensitive to the surrounding waves. By contrast, the lateral expansion of the wave-group is very similar for all the isolated wave cases (the relatively insensitivity to amplitude for this range of amplitudes is consistent with [20] and the random simulation presented below). In all cases the lateral expansion of the isolated wave-groups is slightly larger than the average

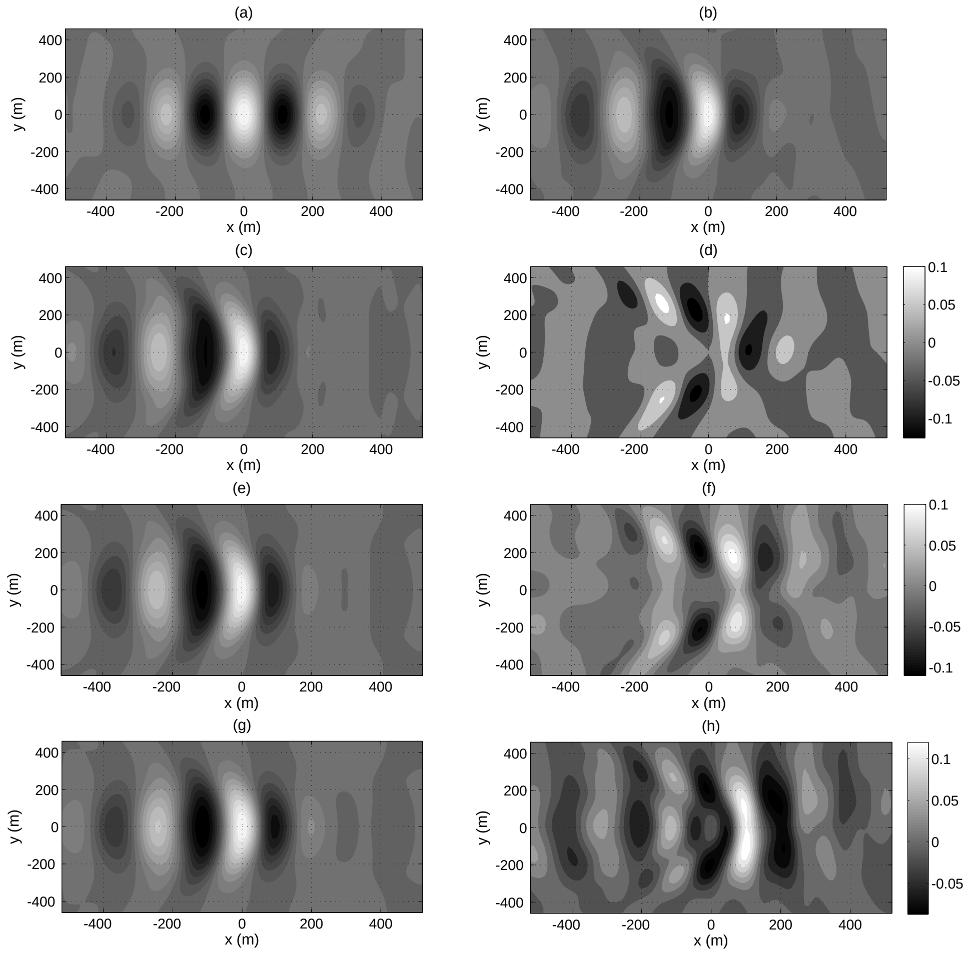

Figure 9. (a) Average shape of extreme linear waves (derived from simulation). (b) Average shape of non-linear waves ($S = 0.140$) from simulations. (c), (e) and (g) Average shape of isolated wave-group after non-linear evolution for groups which under linear evolution would have amplitudes 12 m, 11 m, and 10 m respectively. (d), (f) and (h) plot the difference between (b) and (c), (e), and (g) respectively. Plots (a), (b), (c), (e) and (g) are normalised so the extreme crest is at unity amplitude and contours are at 0.1 intervals. Waves moving from left to right.

expansion of the random waves, suggesting the presence of waves around the large wave-group will slightly inhibit this expansion.

We introduce the parameter σ as the ratio of the group bandwidth under non-linear evolution to the group bandwidth under linear evolution. Thus $\sigma_x = sx_{g,nlin}/sx_{g,lin}$ and $\sigma_y = sy_{g,nlin}/sy_{g,lin}$ where $nlin$ and lin in the subscripts indicate the bandwidth is measured at the maximum amplitude under non-linear and linear evolution respectively.

Figure 10 shows the changes in the group bandwidths for the different extreme waves with $S = 0.140$. There is a clear trend that shows wave-groups contracting in the mean wave direction. Figure 10 also shows that isolated wave-groups also show the same trend. However, the analysis of isolated wave-groups generally over-estimates the changes in the group shape relative to that

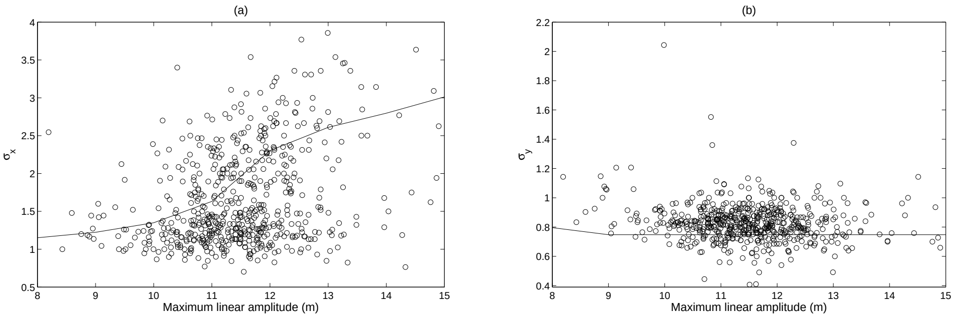

Figure 10. Change in group bandwidth in (a) mean wave direction; (b) lateral direction for background $S = 0.140$. Black lines shows change predicted from NewWave.

Steepness	σ_x	σ_y
0.084	1.06	0.87
0.111	1.18	0.83
0.135	1.45	0.83
0.167	2.41	0.86

Table 2. Median change in the x and y bandwidths for different background significant waveheights.

observed in random waves. In less than 10% of cases the lateral expansions and mean-wave contraction is reversed. Visual checking of these cases suggests that this is due to nearby large wave-groups either joining or leaving the main wave-group.

Table 2 presents the median change in the group shape for different background sea-states. In the mean wave direction the average contraction of the group goes up rapidly for increasingly severe sea-states. By contrast, the lateral expansion predicted by our simulations is largest for $S = 0.111$ and $S = 0.140$ sea states. The lateral expansion is however significant even for $S = 0.084$ where an average increase in the width of a crest is predicted to be 15%. This also implies a small reduction in the directional spreading under large waves even in modest sea states relative to that given by linear theory. These are consistent with the trends observed for isolated wave-groups.

There does not appear to be any significant correlation between the lateral expansion and mean wave direction contraction, i.e. a larger lateral expansion than average does not appear to imply an above average, or below average, contraction in the mean wave-direction. In contrast, as with the changes in maximum amplitude discussed in the previous section, there is strong correlation between the changes is group shape for differing non-linearity, i.e. if there is an above average change in a group with background $S = 0.111$ then there will also be for $S = 0.140$.

Given this we seek to correlate the change in group shape under non-linear evolution with the shape of the linear wave-group. Unfortunately we only observe extremely weak trends. The contraction in the x direction is correlated with the linear wave-group having a longer crest and greater amplitude but these correlations are rather weak. In addition to searching for correlations in the data, we also consider the approximate analytical results of Adcock *et al.* [5]. These results are derived assuming the focussing of isolated wave-groups over infinite time, but we might expect some degree of agreement with the data in this study. Adcock *et al.* predict the change in x bandwidth ($\sigma_{x,p}$) as

$$\sigma_{x,p}^2 = \frac{1}{4}\left(2 + \left(\frac{2a_{lin}k^2}{sx_{g,lin}}\right)^2 - 4\left(\frac{sy_{g,lin}}{sx_{g,lin}}\right)^2 + \sqrt{2 + \left(\frac{2a_{lin}k^2}{sx_{g,lin}}\right)^2 - 4\left(\frac{sy_{g,lin}}{sx_{g,lin}}\right)^2 + 32\left(\frac{sy_{g,lin}}{sx_{g,lin}}\right)^2}\right). \tag{4.1}$$

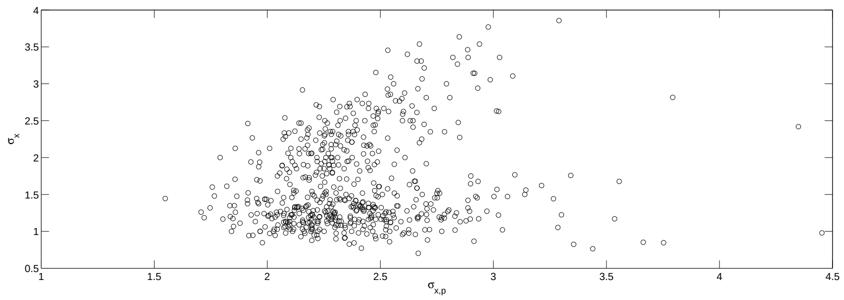

Figure 11. Mean-wave direction group bandwidth change predicted by [5] compared with simulations for case with $S = 0.140$.

Figure 11 presents the predicted change in mean wave direction group bandwidth against the result of the simulations. There is only a very weak correlation, with the prediction generally overestimating the change in group shape. Curiously, visual analysis of the figure does appear to suggest that there are two populations: one which roughly follows the expected trend and one where there are only small changes and σ_x is close to one. Despite careful analysis we have not identify what might lead there to be two populations.

The lack of any strong correlations between the initial linear group shape and the non-linear changes to it are somewhat surprising. A probable reason for this is that trying to describe the complicated shapes of random wave-packets with just two parameters is overly simplistic.

5. Waves preceding extreme wave

The term 'unexpected wave' was termed by Gemmrich & Garrett [28] to describe waves which are much larger than the waves preceding them. Gemmrich & Garrett confined their study to linear waves – in this paper we consider whether non-linear dynamics modifies the probability of extreme waves arriving without large waves preceding them in deep water. We choose to study the magnitude of the wave envelope one period prior the maximum of the envelope at the spatial location where the maximum of the envelope occurs one period later. We introduce the parameter ζ defined as

$$\zeta = \frac{|U_{max}|}{|U_p|} \tag{5.1}$$

where U_{max} is the the complex wave envelope at the maximum spatio-temporal extreme and U_p is the complex envelope of the wave preceding the maximum.

Figure 12 plots ζ under linear evolution (ζ_l) against the ratio ζ_{nl}/ζ_l where ζ_{nl} is the value under non-linear evolution. The Figure considers background sea-states of 6 m and 10 m significant wave-height. For the extreme waves in the $S = 0.084$ sea there is, on average, a small increase in the value of ζ. However, for steeper waves in the $S = 0.140$ sea there is a very dramatic change in the value of ζ. The non-linear dynamics shift the extreme wave to the front of the wave-group (see Figures 1 and 3) significantly increasing the chance of a large wave being immediately preceded by a small one.

A prediction from the above analysis is that, in deep water, the wave preceding a large event would be expected to be smaller than the one following it. This contrasts with linear theory, with the work of Lindgren [25] and Boccotti [26] suggesting that, on average, the waves preceding and following a giant wave will be the same size. Of course, most high quality wave measurements are made from fixed platforms and are installed in finite water depth. For instance, Santo *et al.* [29] looked at the Hurriance Camille dataset and found minimal asymmetry for waves in 100 m of water depth (although the small asymmetry observed was in the direction predicted here). An early study in 170 m water depth at Shell's Term platform [30] does not show any

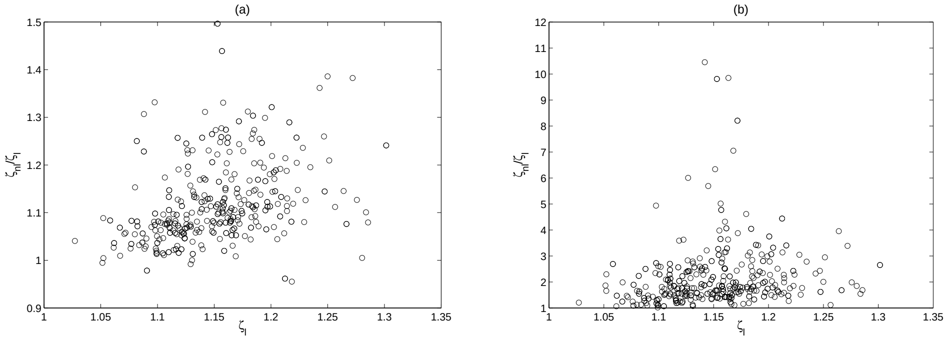

Figure 12. Amplitude of largest wave divided by the wave preceding it – x-axis shows this for linear evolution; y-axis gives non-linear evolution divided by linear evolution. (a) $S = 0.084$; (b) $S = 0.140$. Note the different vertical scales between (a) and (b).

statistically significant difference between waves preceding and following a large event. It would be interesting to analyse horizontal asymmetry in data sets of storm waves on greater water depths or steep but unusually short period storms than these to see if the asymmetry predicted by these simulations is observed.

6. Duration of extreme events

Adcock & Taylor [31] analysed uni-directional wave-groups analytically and numerically and predicted that there was a characteristic non-linear timescale associated with large waves which caused extreme wave-groups to persist for longer under non-linear evolution than under linear. Caution must be applied when applying uni-directional results to real waves on a spread sea but somewhat similar, but weaker, results were found by Gibbs [32] for isolated directionally spread groups.

In random sea-states it is impossible to define uniquely a wave-group as these merge and separate through dispersive and non-linear mechanisms. We choose to study simply the duration of time for which the maximum of the envelope of the extreme wave packet is greater than 95% of the maximum value which it reached. This simple analysis does throw up a few outliers due to second and even third large wave-packets coming along and maintaining large crests for long periods of time. Visual analysis of these cases suggests no particularly significant non-linear physics drives these. To reduce the impact of such cases on our analysis we choose to analyse the median time of extreme events rather than the mean.

Table 3 shows the duration of extreme events for different background significant wave-heights. This suggests that non-linearity can lead to an increased persistence of wave-groups. However, this effect appears to be greatest around $S = 0.111$ and for the most extreme wave-groups in this study the duration of the extreme event ($S = 0.167$) is in fact shorter than that predicted by a linear model. The differences observed here are not huge but might have implications for wave statistics, either at spatial points or over finite areas.

7. Direction of extreme waves

Finally we briefly consider the direction in which the extreme crests are travelling. There is anecdotal evidence (see [33]) that extreme waves may sometimes move in a direction different from that of the average wave field (anecdotal evidence must, of course, be treated with caution). As shown qualitatively in Figure 9, and discussed above, non-linearity can cause the crest-line of an extreme wave to become slightly curved. However, even for extremely non-linear cases, this effect is rather small around the crest itself.

Steepness	Time $U > U_{max} \times 0.95$ (periods)
0 (linear)	5.02
0.084	5.38
0.111	6.29
0.140	5.85
0.167	4.60

Table 3. Median duration of extreme wave events.

To investigate the direction of extreme waves we examine the direction of the crest-line though the extreme crest. We determine crestlines by finding where the instantaneous phase (the argument of the analytic signal of the free surface along the mean wave direction) is zero. For the linear random simulations there is some variation in the direction of the largest crest relative to the mean crest direction. This variation is approximately normally distributed with a standard deviation of 3.2°. The main result here is that this does not appear to change significantly in the non-linear simulations – all the non-linearities considered have standard deviations on the variability of the crest direction of between 3.16° and 3.37° with no obvious dependence on non-linearity.

8. Conclusion

In this paper we have investigated the changes that non-linear dynamics make to the formation of extreme waves in random seas with realistic sea-state parameters in deep water.

Our results suggest that non-linear dynamics will have a relatively small influence on wave-height and wave-crest statistics. This is consistent with the vast majority of studies on this topic where, for directionally spread seas, the statistics of extreme waves are close to those predicted by linear theory (with second-order correction). There is some evidence in this paper that non-linearity can give small amounts of extra elevation above linear theory although from the approach taken here we cannot quantify this. We have, of course, neglected wave breaking which in practice will limit the magnitude of very large waves.

Non-linear dynamics do produce a change in the shape and structure of extreme waves. A process which occurs even for relatively moderate non-linearity (extreme waves in sea-states with $H_s = 6$ m and significant steepness $S = 0.084$) is the elongation or extension of the length of the crests relative to that expected by linear theory. This would tend to reduce the directional spreading of large waves which would tend to increase loads on offshore structures. A mechanism which becomes very significant in the more non-linear seas is the contraction of the wave-group in the mean wave direction and the movement of the largest wave to the front of the wave packet. This latter is significant in the study of the interaction of floating bodies with extreme waves. Beyond the purely hydrodynamic response it may mean that the largest waves tend to be unexpected, at least if expectation is based on the wave which precedes it. Non-linearity can also increase the duration of extreme wave events, although this process is relatively weak.

There is considerable scatter in the data analysed herein due to the random nature of ocean waves. We have found it difficult to pin-point the localised structures which will promote non-linearity (apart from wave amplitude) although localised reductions in the directional spreading appears to be the most significant factor in triggering the non-linear changes to a wave-group.

Ethics statement This research poses no ethical considerations.

Data accessibility statement This work does not have any experimental data.

Competing interests statement We have no competing interests.

Authors' contribution TAAA and SD developed the computational methods used in this paper. All authors helped analyse the data and contributed to writing the paper.

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