# Problems in Random Walks in Random Environments

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**Abstract**

Recent years have seen progress in the analysis of the heat kernel for certain reversible random walks in random environments. In particular the work of Barlow (2004) showed that the heat kernel for the random walk on the infinite component of supercritical bond percolation behaves in a Gaussian fashion. This heat kernel control was then used to prove a quenched functional central limit theorem. Following this work several examples have been analyzed with anomalous heat kernel behaviour and, in some cases, anomalous scaling limits.

We begin by generalizing the first result - looking for sufficient conditions on the geometry of the environment that ensure standard heat kernel upper bounds hold. We prove that these conditions are satisfied with probability one in the case of the random walk on continuum percolation and use the heat kernel bounds to prove an invariance principle.

The random walk on dynamic environment is then considered. It is proven that if the environment evolves ergodically and is, in a certain sense, geometrically $d$-dimensional then standard on diagonal heat kernel bounds hold. Anomalous lower bounds on the heat kernel are also proven - in particular the random conductance model is shown to be "more anomalous" in the dynamic case than the static.

Finally, the reflected random walk amongst random conductances is considered. It is shown in one dimension that under the usual scaling, this walk converges to reflected Brownian motion.

# Contents

# 1. Introduction

As the title suggests, this thesis considers several questions concerning random walks in random environments - in particular questions concerning decay of transition probabilities and the proof of functional central limit theorems. There are three distinct models that we consider. The models and the tools used to analyze the various models are mathematically similar but to avoid confusion we will introduce the specific notation, precise definitions and results in their respective chapters and give only an outline here.

This introduction begins by discussing some of the questions one can ask about random walks in random environments - particularly reversible random walks - briefly discussing models that are well studied in the literature before summarizing the problems that will be considered and the contributions made by each chapter.

Let's start at the beginning and consider the simple random walk on $\mathbb{Z}^d$. This is the Markov chain on $\mathbb{Z}^d$ with transition probabilities

$$P\left(X_n = y \mid X_{n-1} = x\right) = \begin{cases} \frac{1}{2d} & \text{if } x = y \pm e_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases},$$

where $\{e_i\}_{i=1,\ldots,d}$ are the unit coordinate vectors. Various combinatorial or analytic arguments can be used to prove that there exist $c_i = c_i(d) > 0$ such that

$$c_1 n^{-d/2} e^{-c_2|x-y|^2/n} \le P\left(X_n = y \mid X_0 = x\right) \le c_3 n^{-d/2} e^{-c_4|x-y|^2/n}, \tag{1.1}$$

for all $n \ge |x-y|$. In particular this proves that the walk is recurrent in dimension $d = 1, 2$ and transient in all higher dimensions.

Donsker's Theorem gives the scaling limit for the simple random walk: the rescaled, linearly interpolated process

$$B_n(t) := \frac{1}{n}\left(X_{\lfloor n^2 t\rfloor} + \left(n^2 t - \lfloor n^2 t\rfloor\right)\left(X_{\lfloor n^2 t\rfloor+1} - X_{\lfloor n^2 t\rfloor}\right)\right) \tag{1.2}$$

converges weakly to isotropic Brownian motion with diffusion constant $\sigma^2 = d^{-1}$.

To generalize this example, give each edge a non-negative weight and let the walk's transition probabilities be proportional to the weights: taking $\mathbb{E}^d$ to be the edge set of the square lattice, consider the weighted graph $\mathcal{G} = \left(\mathbb{Z}^d, \mathbb{E}^d, (\omega_e)_{e\in\mathbb{E}^d}\right)$ and define the random walk on $\mathcal{G}$ to be the Markov chain with transition probabilities

$$P^\omega\left(X_n = y \mid X_{n-1} = x\right) = \begin{cases} \frac{\omega_{xy}}{\pi(x)} & \text{if } x = y \pm e_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases}, \tag{1.3}$$

for $\pi(x) = \pi_\omega(x) := \sum_z \omega_{xz}$. The simple random walk on $\mathbb{Z}^d$ corresponds to the case $\omega \equiv 1$. What happens if we perturb the square lattice and consider a less regular graph? What happens to the transition probabilities? Is Brownian motion still the correct scaling limit? We will consider these questions in the case where the graph - or environment - is itself randomly chosen.

A well studied example is that of supercritical percolation: for each undirected edge $e \in \mathbb{E}^d$ take $\omega_e = 1$ with probability $p$ and $\omega_e = 0$ otherwise, independently of all other edges. It is well known (see, for example, [32]) that for $d \geq 2$ there exist $p_c(d) \in (0,1)$ such that if $p > p_c(d)$ then with probability one the graph $\mathcal{G}$ contains a unique infinite component connected by bonds of unit weight, and for $p < p_c(d)$ with probability one no such infinite component exists. We use $\mathcal{C}_\infty$ to denote this infinite component. The random walks on the finite connected components of $\mathcal{G}$ are not so interesting as they are just finite irreducible Markov chains. The walk on $\mathcal{C}_\infty$, however, provides a number of interesting questions. Is the walk transient if $d \geq 3$? Do bounds of the form (1.1) hold? Does the random walk converge to Brownian motion under the scaling described in (1.2)?

There are in fact two settings for these questions: annealed and quenched. We describe these settings for the percolation example. Take $\Omega = \{0,1\}^{\mathbb{E}^d}$, $\mathcal{F}$ to be the $\sigma$-field generated by the finite dimensional subsets of $\Omega$ and $\mathbb{P} = \mu_e^{\mathbb{E}^d}$ to be product measure for Bernoulli $\{0,1\}$ random variables. Then one can either consider for a fixed environment, $\omega \in \Omega$, the Markov chain with transition probabilities $P^\omega$ as stated above or one can consider the annealed measure - the result of averaging over environments:

$$\mathcal{P}(\cdot) := \int P^\omega(\cdot)\,\mathbb{P}(d\omega).$$

At times the annealed walk can be an easier object to handle. For example the annealed invariance principle requires primarily only the reversibility of the random walk [27]. The proof of the quenched invariance principle on the other hand requires strong quenched heat kernel control analogous to (1.1) (see [53] and [10] for the invariance principle and [4] and [40] for the quenched heat kernel bounds).

We have alluded to quenched heat kernel bounds for the random walk on supercritical percolation. Note that these bounds cannot be uniform over space and time as detailed in (1.1). For example for any $m \in \mathbb{N}$ there will exist two points $x$ and $y$ in the infinite cluster that are connected only via a one dimensional path of length $m$ - that is a path where each vertex has exactly two edges emanating from it. If $m$ is taken large then $P^\omega(X_n = y \mid X_0 = x)$ will resemble the one-dimensional transition probabilities for all small $n$ and hence equation (1.1) cannot hold for small times. The work [4] shows that in fact the effect of these local irregularities in the graph become negligible as $n$ becomes large. The walk will travel through sufficiently regular regions of the graph to average out this initial irregular behaviour. For concreteness we state the main theorem of [4]: consider the continuous time Markov process $Y_t = X_{M_t}$ for $M_t$ an independent Poisson process of unit intensity, then the following bounds on the transition probabilities of $Y_t$ hold:

**Theorem 1 (Barlow, 2004).** *Let $d \geq 2$ and $p > p_c(d)$. There exists $\Omega_1 \subseteq \Omega$ with $\mathbb{P}_p(\Omega_1) = 1$ and random variables $\left(S_x, x \in \mathbb{Z}^d\right)$, such that $S_x(\omega) < \infty$ for all $\omega \in \Omega_1$, $x \in \mathcal{C}_\infty(\omega)$. There also exist constants $c_i = c_i(d,p)$ such that for $x, y \in \mathcal{C}_\infty(\omega)$ and $|x-y|_1 \vee S_x(\omega) \vee 1 \leq t$, the transition density of $Y$ satisfies*

$$c_1 t^{-d/2} e^{-c_2 |x-y|_1^2/t} \leq p_t^\omega(x,y) \leq c_3 t^{-d/2} e^{-c_4 |x-y|_1^2/t}. \tag{1.4}$$

The random variables $S_x(\omega)$ describe how long the walker must travel before the initial irregularity of the graph around $x$ is averaged out. An adaptation of this theorem to the discrete time walk is contained in [8] and the on diagonal upper bound is also proven in [40].

Leaving the percolation model for the moment and returning to general random walks of the form (1.3), what can be said? The model becomes more tractable if (as in the percolation example) a symmetry condition on the edge weights is assumed: $\omega_{xy} = \omega_{yx}$. Under this condition the detailed balance equations are satisfied:

$$\pi(x) P^\omega(X_1 = y | X_0 = x) = \pi(y) P^\omega(X_1 = x | X_0 = y)$$

and in particular the random walk is reversible with respect to the measure $\pi$. We define the heat kernel by

$$q_n^\omega(x,y) = \frac{P^\omega(X_t = y | X_0 = x)}{\pi(y)}.$$

Under the assumption of reversibility there is a well known link between the geometry of the graph and heat kernel estimates: the isoperimetric profile ([58], [50], [43]). When proving estimates on the heat kernel of the random walk on infinite percolation clusters, both [4] and [40] show that the isoperimetry of the infinite cluster is comparable in some sense to that of the $d$-dimensional square lattice. The isoperimetric profile will also form a key part of our approach.

A natural model to follow the random walk on supercritical bond percolation is the random walk on supercritical continuum percolation. Continuum percolation is the graph with vertex set given by a realization of a $d$-dimensional Poisson point process of intensity $\lambda$ and edge set formed by connecting two vertices by an edge if and only if they lie within unit distance of each other. A simulation is shown in Figure 1.1.

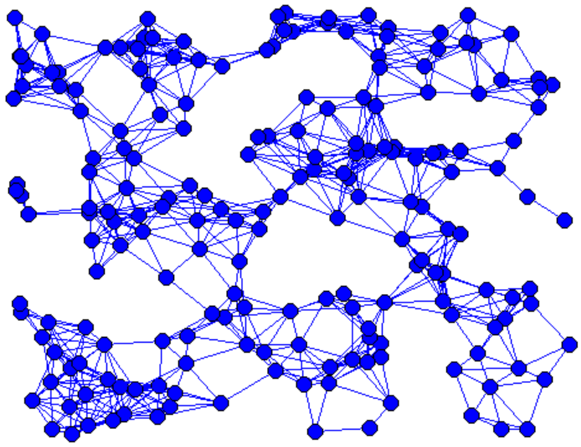

Figure 1.1: The graph induced by continuum percolation

It is well known [46] that if $d \geq 2$ and $\lambda$ is sufficiently large then with probability one the graph contains a unique infinite connected component, again written as $\mathcal{C}_\infty$. We will consider the discrete time simple random walk on this infinite cluster - the walk with transition probabilities

$$P^\omega \left( X_n = y | \, X_{n-1} = x \right) = \begin{cases} \frac{1}{\pi(x)}, & \text{if } x \sim y \in \mathcal{C}_\infty \\ 0 & \text{otherwise} \end{cases},$$

for $\pi(x) = |\{y : x \sim y\}|$, where we write $x \sim y$ if the vertices are connected by an edge. Although this graph at first viewing looks rather different to edge percolation on $\mathbb{Z}^d$ we will show that it shares many of the same geometric properties and in fact similar combinatorial approaches are employed to analyze both models.

This model was the starting point for my research - asking whether or not heat kernel bounds exist in this setting. Over time we have generalized this question to look at more general graphs and ask what geometric properties does a graph have to satisfy for heat kernel bounds to hold? In particular, Chapter 2 looks at reversible random walks on graphs with weights bounded below and provides sufficient conditions - volume and isoperimetric in nature - for the on diagonal upper bounds of Theorem 1 to hold.

One of the main motivations for this question is the fact that there are natural examples for which Theorem 1 does not hold. The work [11] shows that for the case of the random walk on the random conductance model - the random walk on $\mathcal{G} = \left(\mathbb{Z}^d, \mathbb{E}^d, (\omega_e)_{e \in \mathbb{E}^d}\right)$, where $\omega_e \in [0,1]$ and the weights are symmetric and iid - the best general on diagonal upper bound on the heat-kernel in dimensions $d \geq 5$ is of order $n^{-2}$, with the best upper bound for $d = 4$ being $n^{-2} \log n$ [12] and standard upper bounds for $d = 2, 3$. Further, for the case $\omega_e \in [1, \infty)$ it is shown in [7] that if $\mathbb{E}\omega_e = \infty$ then Brownian motion is not the correct scaling limit for the random walk. In fact, in [6] and [21] convergence to a fractional kinetics process is proved in the infinite mean case, under additional assumptions on the tail of the weights, and hence the process is subdiffusive.

These examples all share the property that their one step transition probabilities are not bounded below - we say that they are not uniformly elliptic. When uniform ellipticity fails many of the standard techniques that link a graph's geometry to its associated heat kernel also fail. This is the main technical challenge to obtaining general conditions for standard heat kernel bounds tackled in Chapter 2.

Chapter 3 provides an exploration of the geometry of continuum percolation; proving that the geometric conditions detailed in Chapter 2 hold and hence obtaining standard upper bounds on the heat kernel for the random walk. As well as showing that the conditions for standard heat kernel upper bounds are satisfied, we will also show that the scaling limit for the random walk on continuum percolation is Brownian motion. This is also presented in Chapter 3. We follow standard methods and show that for almost every environment $\omega \in \Omega$ it is possible to construct a corrector $\chi$ such that

$$X_n = M_n + \chi\left(X_n, \omega\right),$$

where $M_n$ is a martingale. A standard martingale convergence theorem is used to show that $M_n$ scales to Brownian motion. It is then proved that the corrector is sublinear, proving that $X_n$ also scales to Brownian motion. The methods used are very similar to those presented in [10] and [13].

If spatial inhomogeneity can lead to both standard and anomalous heat kernel behaviour, what happens if time inhomogeneity is also permitted? This is the question considered in Chapter 4 where we consider the random walk in a dynamic environment - an environment that evolves over time. Take a space-time environment of the form $\left(\omega_e\left(n\right)\right)_{e \in \mathbb{E}^d, n \in \mathbb{R}}$ and consider the random walk with transition probabilities

$$P^{\omega}\left(X_n = y \middle| X_{n-1} = x\right) = \begin{cases} \frac{\omega_{xy}(n-1)}{\pi(x,n-1)} & \text{if } x = y \pm e_i \text{ for some } i \\ 0 & \text{otherwise} \end{cases},$$

for $\pi\left(x, n\right) = \sum_z \omega_{xz}\left(n\right)$. Several authors have developed various approaches to proving central limit theorems in this setting, with the environment taken to be Markov in time but where there is no symmetry assumption $\omega_{xy}\left(n\right) = \omega_{yx}\left(n\right)$ (for example [14], [3] and [49]). These papers generally show that if the walk is uniformly elliptic and the environment is well mixing in time then the rescaled process converges to non-degenerate Brownian motion. Is it also possible to obtain heat kernel bounds?

To make the question more tractable we assume the symmetry condition $\omega_{xy}(n) = \omega_{yx}(n)$ for all edges and times and consider the variable speed walk on $\omega$. Assume that $\pi(x,n) \leq 1$ for all $x \in \mathbb{Z}^d, n \in \mathbb{N}$ then the variable speed walk is the Markov chain with transition probabilities

$$P^{\omega}\left(X_n = y \middle| X_{n-1} = x\right) = \begin{cases} \omega_{xy}(n-1) & \text{if } x = y \pm e_i \text{ for some } i \\ 1 - \pi(x,n) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

The flat measure $\pi \equiv 1$ is invariant for the variable speed walk $X_n$ but the walk is not reversible. We will in fact mostly consider the continuous time version of this walk (note that as the environment evolves over time, the continuous time variable speed walk is not simply a Poisson time change of $X_n$).

We will consider the heat kernel for $X_n$ with two main results. The first consists of an upper bound on the on diagonal heat kernel. We show that if over the time interval $[0,T]$ the spatial environment "looks" $d$-dimensional for at least a linear amount of time then the heat kernel can be bounded above by $O\left(n^{-d/2}\right)$. We will show that if the environment is ergodic in time then this condition is natural.

The second result of this chapter concerns anomalous behaviour of the heat kernel for the variable speed random walk on dynamic environment. In the static environment case where the edge weights are iid the paper [11] shows that if the weights are bounded above but not below then the best general upper bound on the heat kernel is of order $n^{-2}$ for $d \geq 5$. There are two steps to prove this kind of result: one must first prove that an upper bound of order $n^{-2}$ holds and then, as this is a weaker upper bound than is standard, give examples of environments where the heat kernel is bounded below arbitrarily close to $n^{-2}$. We will show that in the dynamic setup where edge weights are spatially independent and evolve in a Markov manner, such that $\omega_e(n) \in [0,1]$ for all edges and times, there are environments where the on diagonal heat kernel is bounded below by functions close to $O(n^{-1})$.

This result is somewhat unintuitive - one would perhaps expect that the environment evolving over time would reduce the time that the walk spends in locally anomalous regions due to the time dynamic removing these regions before the random walk can spend a large quantity of time in them. Intriguingly it is this dynamic - the disappearance of anomalous regions - that leads to the change in the heat kernel. Without going into detail, anomalous heat kernel decay in the static case is due to the walk becoming "trapped" close to the origin so that when the walk escapes the trap it is much closer to the origin than would normally be expected. However, the random walk must pay a price to enter and exit the trap - a price of $O(n^{-1})$ for both entrance and exit leading to the $O(n^{-2})$ return probabilities stated above. In the dynamic case there is a tradeoff - for such traps to be effective they must persist so that the walk remains in the traps for a large length of time. The key idea that we present is that it is possible to choose a dynamic environment where the traps persist for long enough to trap the walk for a good length of time - so that the walk is much closer to the origin than would be expected - but then the trap disappears leaving the walk unimpeded to return to the origin. As the walk only has to pay to enter the trap and not to exit we show that lower heat kernel bounds close to $O(n^{-1})$ can be achieved.

Note that this implies that the dynamic heat kernel has at least three extrema: when the environment is strongly mixing the walk resembles the walk on the annealed graph and hence has heat kernel bounds of order $n^{-d/2}$; when the environment is highly persistent the walk is close to the walk on the static graph and hence the heat kernel is bounded above by order $n^{-2}$; in between these two cases sit the environments we have outlined with heat kernel lower bounds of order $n^{-1}$.

We also present heuristics for why we believe $O(n^{-1})$ is the correct order for general upper bounds in the dynamic case. At present we have no rigorous justification of this claim.

In Chapter 5 we return to the static environment setting of earlier chapters and consider the reflected random walk on a weighted, reversible graph. We take $\mathcal{G}$ to be a realization of the random conductor model described above and consider the random walk on $\mathcal{G}$ restricted to the box $[-n,n]^d$, with reflection at the boundary. When the weights are bounded above it is known that the rescaled random walk on the full graph $\mathcal{G}$ converges to Brownian motion. It is therefore natural to anticipate that the reflected random walk on $[-n,n]^d$ rescaled in space by $n^{-1}$ and in time by $n^2$ will converge weakly to reflected Brownian motion on $[-1,1]^d$. This problem has proved to be more challenging than it at first appears. We present a proof of the claimed convergence only for the case $d=1$ and $\omega \in [a,b]$ for $0 < a \leq b < \infty$. Our approach is to reflect the restricted graph to produce a graph on the entirety of $\mathbb{Z}$ that is periodic of period $4n$. The random walk on this reflected graph is closely linked to the random walk that we are interested in. We show via the standard martingale/corrector decomposition that the walk on the reflected graph converges to Brownian motion and use this to prove that the reflecting random walk converges to reflecting Brownian motion. Full details are contained in the chapter along with an explanation of the failure of the method in higher dimensions.

We finish this introduction with some comments about notation. The questions that we consider are almost exclusively in the quenched setting outlined above. We will use $\mathbb{P}$ for the law on environments and for a given environment $\omega$, use $P^\omega$ for the law of the random walk on $\omega$ (or the infinite component of $\omega$ if appropriate). We will write $c_i$ for positive constants whose precise values are unimportant. These constants will be recycled through various sections and subsections. Where a constant from a previous section or subsection is required we will write $c_{i.j.k}$ for constant $c_k$ in Section $j.k$. Within proofs we write $c, c'$ etc for positive constants that change from line to line. Chapters 2 and 3 are linked as the latter proves that conditions detailed in the former hold. Thus they share common notation. All other chapters are independent and although most of the notation is the same, some will inevitably change. This is particularly important for the random walk as, for example, $X_n$ refers to a different random walk in each chapter.

# 2. Heat kernel bounds for the random walk on non-uniformly elliptic graphs embedded in $\mathbb{R}^d$

## 2.1. The model

In this chapter we consider the simple random walk on weighted graphs; investigating geometric conditions that lead to standard on diagonal $d$-dimensional upper bounds for the heat kernel associated with the walk. We begin by introducing the necessary notation.

Take $\mathcal{G} = (\mathcal{V}, \mathcal{E}, (\omega_e : e \in \mathcal{E}))$ to be a countably infinite, locally finite, connected, weighted graph embedded in $\mathbb{R}^d$ for some $d \in \mathbb{N}$, where $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the edge set and the weights on the edges, $\omega_e$, are bounded below: $1 \leq \omega_e < \infty$.

We consider the discrete time simple random walk on $\mathcal{G}$, that is, the Markov chain $(X_n : n \in \mathbb{N}_0)$ with transition probabilities

$$P^\omega(x, y) := \begin{cases} \frac{\omega_{xy}}{\pi(x)} & \text{if } (xy) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}, \quad \pi(x) := \sum_{y:(xy)\in\mathcal{E}} \omega_{xy}, \ x \in \mathcal{V},$$

for $x, y \in \mathcal{V}$. It is straightforward to check that $\pi$ is the invariant measure for the random walk on $\mathcal{G}$ and that the random walk is reversible with respect to $\pi$. Note that as we do not assume that $\omega_e$ is bounded above (or that the number of edges emanating from a single point is bounded), the transition probabilities do not necessarily satisfy a uniform ellipticity condition.

The heat kernel of the random walk is defined by

$$q_n^\omega(x, y) = \frac{P_x^\omega(X_n = y)}{\pi(y)}.$$

Bounds on $q_n^\omega$ were discussed in the Introduction for the specific case where the environment $\omega$ is a realization of bond percolation in $\mathbb{Z}^d$: Theorem 1 shows that for almost every $\omega \in \Omega$, $q_n^\omega(x, y)$ decays in a Gaussian fashion for all $n \geq S_\omega(x)$, dependent on the local irregularity of the graph. The proof of this theorem in [4] consists of providing a set of geometric conditions on an infinite subset of the square lattice that, if satisfied, lead to Gaussian bounds on the heat-kernel of the random walk on the infinite subset. It is then proven that almost every environment satisfies these conditions and hence Theorem 1 follows. We will take a similar approach here - in this chapter we will detail geometric conditions for graphs of the above form that, if satisfied, lead to on diagonal upper bounds on the heat-kernel for the associated random walk. In Chapter 3 we will introduce continuum percolation and show that it is an environment that satisfies these conditions.

Many of the standard techniques to obtain heat kernel bounds from the geometry of the graph fail to hold without the assumption of uniform ellipticity - in particular, many of the ideas of [4] fail to pass over into the non-elliptic setting. To bypass this problem we follow the ideas of [11] and "thin" the graph by removing vertices (and their associated edges) in a way such that for some $k = k(\mathcal{G}) > 0$ we have a unique infinite graph, $\mathcal{C}_\infty^k = (\mathcal{V}_k^\infty, \mathcal{E}_k^\infty, (\omega_e))$, with $\mathcal{V}_k^\infty \subseteq \mathcal{V}$, $\mathcal{E}_k^\infty \subseteq \mathcal{E}$ and $\pi(x) \leq k$ for all $x \in \mathcal{V}_k^\infty$. Call the components of $\mathcal{G} \backslash \mathcal{C}_\infty^k$ traps. We consider our random walk time changed so that it only walks on the thinned graph (see Definition 3). This walk does satisfy a uniform ellipticity condition. If the thinned graph satisfies certain isoperimetric and volume conditions then Gaussian upper bounds for the time changed random walk can be obtained. If the time the walk spends in traps can also be controlled then the heat kernel for the original random walk on $\mathcal{G}$ also has standard on diagonal upper bounds. From these on diagonal bounds it can often be straightforward to pass to full Gaussian type bounds. Conditions on the geometry of the traps will be presented that imply that the walk spends large amounts of time in traps with sufficiently small probability. These ideas are formalized in Section 2.2 and proved in Section 2.5.

Note that although we follow ideas analogous to those of [11], that paper considers the case $\omega_e \in [0,1]$ and examples of anomalous behaviour are given. Our results are restricted to the bounded below case. In fact the bounded above case has recently received further attention with [18] showing that if the weights satisfy

$$\mathbb{P}(\omega_e \leq a) \sim a^\gamma, a \downarrow 0$$

then provided $\gamma > \frac{d}{2}$

$$\lim_{n\to\infty} \frac{\log P_0^\omega(X_{2n} = 0)}{\log n} = -\frac{d}{2}, \ \mathbb{P}\text{-almost surely.}$$

Our methods will not allow as sharp a bound for the case $\omega_e \in [1,\infty)$

The chapter is structured as follows. The main results of the chapter are detailed in Section 2.2: we give geometric conditions on the graph that lead to standard on diagonal upper bounds on the heat kernel. Their proof waits until Section 2.5. Sections 2.3 and 2.4 are self contained sections. In Section 2.3 we show that control over the on diagonal heat kernel, long range heat kernel, control over the volume growth of the graph and a linear relationship between graph distance and Euclidean distance, together imply full Gaussian type upper bounds. This follows closely the methods of [4] and also utilizes ideas from [13]. In Section 2.4, we give generic bounds on the probability of spending large amounts of time in a trap of a particular type. We make clear what we mean by types of trap in the section. The results of Sections 2.3 and 2.4 are used in Section 2.5 to extend the ideas of [11], so that if the time the walk spends in traps is controlled, we get standard on diagonal heat kernel behaviour. In many situations the results of Section 2.3 can then be applied again to extend the result to full off-diagonal upper bounds.

## 2.2. Statement of results

In this section we introduce the main results of the current chapter: presenting conditions on a weighted, symmetric graph that, if satisfied, ensure standard on diagonal upper bounds on the heat kernel. The conditions are given in terms of a set of constants. Unless otherwise stated, the only assumption on these constants is that they lie in $(0, \infty)$.

As noted in the introduction, the lack of uniform ellipticity is a technical challenge. Our approach follows the methods of [11], thinning the graph in such a way that the transition probabilities for the random walk on this thinned graph are uniformly elliptic. The conditions we place on the original graph must therefore ensure the existence of such a thinning and control the geometry of both the thinned and unthinned graph. The first of these conditions is the existence of a thinned graph. We then wish to show that the walk, time changed so that it only walks on the thinned graph, displays standard heat kernel behaviour. We use standard results to gain this heat kernel control - assuming isoperimetric and volume control of the thinned graph. These asuumptions form Conditions 4 and 5. Finally, for the heat kernel control to pass over to the initial random walk we must show comparability between the time scales for the initial random walk and the time changed random walk. This is equivalent to showing that the time the walk spends in traps grows linearly. We are able to prove this under Conditions 6 and 8, that show volume control for the full graph and a fairly strong control over the location and size of traps.

**Condition 2 (Existence of a thinned graph).** *There exists $k \in \mathbb{N}$ and a thinned graph, $\mathcal{G}_k = (\mathcal{V}_k, \mathcal{E}_k) \subseteq (\mathcal{V}, \mathcal{E})$, such that $\mathcal{G}_k$ contains exactly one infinite component, call this $\mathcal{C}_\infty^k$, and for every $x \in \mathcal{G}_k$ we have $\pi(x) \leq k$. We further require that all components of $\mathcal{G} - \mathcal{C}_\infty^k$ are finite and that any Euclidean box of unit side contains at most $K$ points of $\mathcal{C}_\infty^k$ for some constant $K$ and any point in $\mathcal{G} - \mathcal{C}_\infty^k$ has at most $K$ neighbours in $\mathcal{C}_\infty^k$.*

We refer to the components of $\mathcal{G}-\mathcal{C}_\infty^k$ as traps. For $x \in \mathcal{C}_\infty^k$, define $T_x$ to be the trap located at $x$, that is, the component of $\left(\mathcal{G}-\mathcal{C}_\infty^k\right) \cup \{x\}$ that contains $x$, where the union includes all edges eminating from $x$. Define $\pi\left(T_x\right) := \sum_{y \in T_x} \pi\left(y\right)$. Note that if $x \in \mathcal{C}_\infty^k$ then it is possible for $x$ to be connected to at most $K-1$ traps and hence $T_x$ may be the union of several traps with the vertex $x$.

**Definition 3.** *We define three random walks.*

1. $\left(X_n\right)_{n\geq 0} = \left(X_n\left(\mathcal{G}\right)\right)_{n \geq 0}$, *the simple random walk on* $\mathcal{G}$.
2. $\left(\tilde{X}_n\right)_{n\geq 0} = \left(\tilde{X}_n\left(\mathcal{C}_\infty^k\right)\right)_{n\geq 0}$, *the time changed random walk on* $\mathcal{C}_\infty^k$. *Let* $S_0 = 0$ *and* $S_i = \inf\left(t > 0 : X_{S_{i-1}+t}\left(\omega\right) \in \mathcal{C}_\infty^k\right)$ *for* $i > 0$. *Define* $\tilde{X}_n = X_{S_1 + \ldots + S_n}$.
3. $\left(Y_n\right)_{n\geq 0} = \left(Y_n\left(\mathcal{C}_\infty^k\right)\right)_{n\geq 0}$, *the simple random walk on the infinite component of the thinned graph* $\mathcal{C}_\infty^k\left(\mathcal{G}\right)$.

Note that the time changed walk, $\tilde{X}$, can jump traps and thus potentially travel a large Euclidean distance in a small number of steps. $\tilde{X}$ thus induces a distance function on $\mathcal{C}_\infty^k$ that is significantly different to both the natural graph distance and Euclidean distance. We use $\tilde{d}$ to denote the induced distance, reserving $d$ for graph distance with respect to the original graph, $\mathcal{G}$.

Following the definition in [13], we introduce

$$C_{vol}\left(y, a\right) := \sup_{0<r\leq a} r^d \sum_{z \in \mathcal{V}} \pi\left(z\right) e^{-rd(y,z)}.$$

We will use $\tilde{C}_{vol}$ to denote this quantity with respect to the distance $\tilde{d}$.

For $x \in \mathbb{R}^d$ and $n > 0$, write $B_x\left[n\right] := x + \left[-n, n\right]^d$ for the box of side $2n$ centred at $x$. We now introduce the remaining conditions. For $A \subseteq \mathcal{V}$ let

$$\pi\left(\partial A\right) := \sum_{x \in A, y \in A^c} \omega_{xy}.$$

**Condition 4 (Isoperimetry of the thinned graph).** *There exist positive, finite constants* $C_i, \psi, \phi > 0$ *and* $\{R_0\left(x\right) : x \in \mathcal{V}_k\}$ *such that* $\psi < \frac{1-\phi}{2}$ *and for every* $x \in \mathcal{V}_k$, $r \geq R_0\left(x\right)$ *and each connected* $\Lambda$ *such that*

$$\Lambda \subseteq \mathcal{C}_\infty^k \cap B_x\left[r\right] \text{ and } \pi\left(\Lambda\right) \geq C_1 r^\psi$$

*we have*

$$\pi\left(\partial\Lambda\right) \geq C_2 \pi\left(\Lambda\right)^{\frac{d-1}{d}}.$$

*Further, the diameter of the largest trap with at least one vertex in* $B_x\left[r\right]$ *is bounded above by* $r^\phi$ *for all* $r \geq R_0\left(x\right)$.

**Condition 5 (Thinned volume and graph distance).** *There exist constants* $C_3, C_4$ *and* $\{R_1(x) : x \in \mathcal{V}_k\}$ *such that* $\tilde{d}(x,y) \geq C_3 |x-y|$ *for all* $x, y \in \mathcal{V}_k$ *such that* $|x-y| \geq R_1(x)$*. We also have for all* $a \leq R_1(x)^{-1}$

$$\tilde{C}_{vol}(x,a) \leq C_4.$$

**Condition 6 (Volume of the full graph).** *Set* $A_n(x) := B_n[x] - B_{n-1}[x]$*. There exist constants* $\{R_2(x) : x \in \mathcal{V}\}$*,* $C_6$ *such that*

$$\sum_{y \in A_n(x) \cap \mathcal{C}_\infty^k} \pi(T_y) \leq C_6 n^{d-1}, \forall n \geq R_2(x) \tag{2.1}$$

$$\sum_{y \in B_x[R_2(x)] \cap \mathcal{C}_k^\infty} \pi(T_y) \leq C_6 R_2(x)^d. \tag{2.2}$$

Note that (2.1) and (2.2) combine to say that

$$\sum_{y \in B_x[n] \cap \mathcal{C}_k^\infty} \pi(T_y) \leq C_6 r^d, \forall n \geq R_2(x).$$

As in [11], standard behaviour in dimensions $d = 2, 3$ follows from the above conditions. However, in higher dimensions anomalous behaviour can occur due to trapping effects. Note that it is not only large traps that can have strong trapping effects as clusters of traps can also be difficult for the walk to escape from. Any configuration of traps that encourages the walk to travel a short distance in a large time can lead to anomalous heat kernel behaviour. For this reason we insist on control of both the size, number and spread of traps.

Define the external boundary of $T_x$, by

$$\partial_{ext} T_x := \left\{ y \in \mathcal{C}_\infty^k : (yz) \in \mathcal{E} \text{ for some } z \in T_x - \{x\} \right\}$$

and define the worst case constant

$$R_0(T_x) := \sup_{y \in \partial_{ext} T_x} R_0(y).$$

We now categorize the traps.

**Definition 7.** *We say a trap* $T_x$ *is of type* $(m,r)$ *for* $m, r \in \mathbb{N}$ *if* $\lceil \pi(T_x) \rceil = m$ *and* $r - 1 < R_0(T_x) \leq r$*.*

We will say that a trap $T_x$ of type $(m,r)$ is worse than a trap $T_y$ of type $(m', r')$ if $m \geq m'$ and $r \geq r'$. Call $\gamma = (\gamma_0, \ldots, \gamma_n)$ a nearest neighbour path with respect to $\tilde{d}$ if $\gamma_i \in \mathcal{C}_\infty^k$ for $i \in \{0, \ldots, n\}$ and $\tilde{d}(\gamma_i, \gamma_{i+1}) = 1$ for all $i \in \{0, \ldots, n-1\}$.

**Condition 8 (Spread of traps).** *There exist constants* $\alpha < \frac{1}{5}, \theta, \vartheta > 0, \gamma \geq 5, C_7$ *and* $\{R_3(x) : x \in \mathcal{V}\}$ *such that for* $d \geq 4$*:*

*1. All traps,* $T$ *of type* $(m, r)$*, with* $T \cap B_x[n] \neq \phi$ *have* $m \leq n^{\alpha}$ *and* $r \leq (\log n)^{\theta}$ *for* $n \geq R_3(x)$*.*

*2. For any* $(m, r)$*,* $n \geq R_3(x)$ *and* $\sigma$ *a nearest neighbour path with respect to* $\tilde{d}$*, started at* $x$ *and of length* $n$

$$\sum_{i=0}^{n} 1_{\left\{T_{\sigma_i}=(m,r)\right\}} \leq C_7 \left[\exp\left(-r^{\vartheta}\right) \wedge m^{-\gamma}\right] n.$$

Define

$$N_R(x) := \max_{y \in B_x[R]} \{R_i(y) : i \in \{0, 1, 2, 3\}\}.$$

**Definition 9.** *Suppose* $\mathcal{G}$ *is a graph for which Condition 2 holds. We call a box* $B_x[R] \subseteq \mathcal{G}$ *good with respect to the set of constants* $C_i, \alpha, \gamma, \theta, \vartheta$ *and* $\beta < \frac{2}{d+2}$ *if Conditions 4, 5, 6 and 8 hold and* $N_R(x) \leq R^{\beta}$*.*

**Theorem 10.** *Suppose that* $B_{x_1}[R]$ *is a good box. Then for* $x \in B_{x_1}[R/10]$ *the simple random walk started at* $x$ *satisfies the standard heat kernel upper bound:*

$$q_n(x, y) \leq C_8 n^{-d/2}$$

*for all* $C_{10} N_R^{q(d)}(x_1) \leq n \leq C_{11} \frac{R^2}{\log R}$ *and* $y \in \mathcal{G}$*. The constants* $C_8, C_9, C_{10}$ *and* $C_{11}$ *are dependent on the constants* $C_i, \alpha, \gamma, \theta, \vartheta$ *from Definition 9 along with the dimension* $d$ *and*

$$q(d) := \begin{cases} (2+d) \vee \frac{4}{1-\phi} & d = 2, 3 \\ 2+d & d \geq 4 \end{cases}.$$

When applying the theorem to random environments we look to show that for all $x \in \mathcal{V}$, the boxes $B_x[R]$ are good for all sufficiently large $R$, with the constant $\beta < \frac{1}{d+2}$. Theorem 10 then provides standard on diagonal type upper bounds on the heat kernel for all $x, y \in \mathcal{V}$ and all sufficiently large $n$, where the local irregularity of the graph around the point $x$ determines how sufficiently large $n$ must be.

In fact, provided that long range off-diagonal upper bounds can be shown to hold, the upper bound can be extended to full Gaussian upper bounds via Theorem 12 that will be proven in the next Subsection.

The proof of Theorem 10 can be found in Section 2.5. Before we begin the proof we present two self-contained sections that form important planks of the proof.

## 2.3. From on diagonal to off

In this section we work in the general setting described in Section 2.1, extending the results given in [4] and [13]: demonstrating that uniform control over the on diagonal heat kernel for all points in a box along with long range heat kernel bounds and control over volume growth in the box combine to give full Gaussian type upper bounds. The methods follow [4] closely, with [13] providing the modifications required for irregular volume and [8] the modifications for discrete time.

All constants in this subsection are independent of Section 2.2. We begin by defining a fine ball and stating the section's main result.

**Definition 11.** *We call the ball $B_{x_1}[R]$, $(C_A, C_B, C_C, N_R)$-fine if for every $x, y \in B_{x_1}[R]$ we have*

$$\begin{aligned} q_n(x,y) &\leq C_A n^{-d/2} \text{ for all } n \geq \frac{N_R}{2}, && (2.3)\\ q_n(x,y) &\leq C_B \exp\left(-\frac{C_C d(x,y)^2}{n}\right) \text{ for all } n \geq d(x,y) && (2.4)\end{aligned}$$

*and*

$$C_{vol}(y,a) := \sup_{0<r\leq a} r^d \sum_{z\in\mathcal{V}} \pi(z) e^{-rd(y,z)} \leq 1 \qquad (2.5)$$

*for all $a \leq N_R^{-1/2}$. In order for our results to be useful, we assume that there is some $\beta < 2$ such that $N_R \leq R^\beta$.*

The choice of one as the upper bound on $C_{vol}(y,a)$ in equation 2.5 is purely for notational convenience. Any other positive constant could take its place.

Note also that the long range bounds on transition probabilities stated in (2.4) are standard and in many situations can be proven using known results such as [25]. In particular, for the time changed random walk, [25] implies that the long range bounds hold as the conductances $\omega_e$ are uniformly bounded above. Note that for simplicity the long range bounds are stated as being uniform in time and space. This could be relaxed such that (2.4) only holds for large times. We choose not to do this here as the uniform bounds will hold in all the examples we explore.

**Theorem 12.** *Suppose $B_{x_1}[R]$ is $(C_A, C_B, C_C, N_R)$-fine and $x \in B_{x_1}\left[\frac{1}{2}R\right]$. Then for $y \in \mathcal{V}$ and for $\alpha > 0$ there exist constants $c_i = c_i(d, C_A, C_B, C_C, \alpha)$ such that for*

$$N_R^{1+\alpha} \leq n \leq c_1 \frac{R^2}{\log R} \qquad (2.6)$$

*we have*

$$q_n(x,y) \leq c_2 n^{-d/2} \exp\left(-\frac{c_3 d(x,y)^2}{n}\right). \qquad (2.7)$$

For the remainder of the section we suppress the constants $(C_A, C_B, C_C, N_R)$ and just refer to fine balls.

As noted in, amongst others, [8], the continuous time heat kernel is smoother than the discrete time heat kernel and at times this enables an easier analysis. We thus introduce the continuous time walk: the Markov process, $(Z_t : t \geq 0)$ with distribution

$$\bar{P}_x (Z_t = y) = P_x (X_{M_t} = y),$$

for $M_t$ an independent Poisson process of unit rate. The continuous time heat kernel is

$$\bar{q}_t (x, y) := \frac{\bar{P}_x (Z_t = y)}{\pi (y)}.$$

We first show that control for the discrete time walk gives control over the continuous time walk.

**Lemma 13.** *Suppose $B_{x_1}[R]$ is fine. There exist $c_i$ such that for $x, y \in B_{x_1}[R]$ and $t \geq N_R$ we have*

$$\bar{q}_t (x, y) \leq c_4 t^{-d/2}.$$

*Further, for $x \in B_{x_1}\left[\frac{8}{9}R\right]$, $y \in \mathcal{V} - B_{x_1}[R]$ and $t \leq \frac{c_5 R^2}{\log R}$*

$$\bar{q}_t (x, y) \leq c_4 t^{-d/2}.$$

**Proof.** Let $M_t$ be a Poisson process, independent of the random walks, of unit rate, then

$$\bar{q}_t (x, y) = \mathbb{E}\left[q_{M_t} (x, y)\right].$$

Now, $\mathbb{P}\left(M_t \leq \frac{t}{2} \text{ or } M_t \geq 2t\right)$ is exponentially small in $t$. Further, $q_n (x, y)$ is uniformly bounded on the range $\left[\frac{t}{2}, \ldots, 2t\right]$ for $t \geq N_R$ by (2.3) and hence the first result follows.

The second result is identical to [4], Corollary 3.2. ■

We now follow the methods of [4], which employ the methods of Bass and Nash, to show that in a fine box we can control the expected distance that the walk travels in time $t$ and thus the time it takes to exit a ball. The methods are very similar, but as we do not assume standard upper bounds on volume, the volume must be controlled through the proof. This is a problem dealt with in [13] and we employ the same methods.

We begin by introducing the entropy and expected distance:

$$\begin{aligned} Q(x,t) \;&:= -\sum_y \bar{q}_t (x, y) \log \bar{q}_t (x, y) \pi (y), \\ M(x,t) \;&:= E_x (d (x, Z_t)). \end{aligned}$$

$Q(x, 0) := \log \pi (x) > 0$ for all $x$. The following results are standard.

**Lemma 14.** *1. There exist $c_i$ such that if $B_{x_1}[R]$ is fine then for $N_R \le t \le \frac{c_5 R^2}{\log R}$ and $x \in B_{x_1}\left[\frac{8}{9}R\right]$*

$$Q(x,t) \ge -c_4 + \frac{1}{2} d \log t.$$

*2. For $x \in \mathcal{V}$ and any $t \ge 0$ we have*

$$M(x,t) \ge c_7(d) \exp\left(\frac{Q(x,t)}{d} - \frac{C_{vol}\left(x, M(x,t)^{-1}\right)}{d}\right).$$

*3. For $t > 0$ and $x \in \mathcal{V}$*

$$Q'(x,t) \ge c_6 M'(x,t)^2.$$

**Proof.** 1. is trivial from Lemma 13 and the definition of $Q$.

For 2, see [13], Lemma 6.3.

3. is standard - found in [4] - and makes use of the long range bounds given in (2.4). ■

**Proposition 15.** *There exist $c_i$ such that if $B_{x_1}[R]$ is fine then*

$$M(x,t) \le c_8 t^{1/2},$$

*for $x \in B_{x_1}\left[\frac{8}{9}R\right]$ and $N_R \log N_R \le t \le \frac{c_5 R^2}{\log R}$.*

**Proof.** Fix $x \in B_{x_1}\left[\frac{8}{9}R\right]$ and write $M(t) := M(x,t)$, $Q(t) := Q(x,t)$. Suppose that $M(t) \ge t^{1/2}$ as otherwise the proposition holds with $c_8 = 1$ and there is nothing to prove.

Set $R(t) := d^{-1}\left(Q(t) + c_4 - \frac{1}{2} d \log t\right)$, then by part (1) of Lemma 14, $R(t) \ge 0$ for $N_R \le t \le \frac{c_5 R^2}{\log R}$. Now, set $t_1 := \frac{c_5 R^2}{\log R}$ and define

$$t_0 = \begin{cases} 1, & \text{if } R(t) \ge 0 \text{ on } [1, N_R] \\ \sup\{t \le t_1 : R(t) < 0\} & \text{otherwise.} \end{cases}$$

If $t_0 > 1$ then $t_0 \le N_R$ and

$$\begin{aligned} M(t_0) &= \int_0^{t_0} M'(s)\, ds \\ &\le c_6^{1/2} \int_0^{t_0} Q'(s)^{1/2} ds \\ &\le c_6^{1/2} \left(\int_0^{t_0} Q'(s)\, ds\right)^{1/2} t_0^{1/2} \text{ (by Cauchy-Schwarz)} \\ &\le c t_0^{1/2} Q(t_0)^{1/2} \\ &\le c t_0^{1/2} \left(c_3 + \frac{1}{2} d \log t_0\right)^{1/2} \\ &\le c (N_R \log N_R)^{1/2}, \end{aligned}$$

where $Q(0) > 0$ was required on the fourth line.

If $t_0 = 1$ then $M(t_0) \leq c$ from the definition of the continuous walk.

Now let $t_0 < t < t_1$. Then, by Lemma 14 part (3), we have since $t > t_0$

$$\begin{aligned} M(t) &\leq M(t_0) + c_6^{1/2} \int_{t_0}^{t} Q'(s)^{1/2} \, ds \\ &\leq M(t_0) + (c_6 d)^{1/2} \int_{t_0}^{t} \left( R'(s) + \frac{1}{2s} \right)^{1/2} ds \\ &\leq M(t_0) + ct^{1/2} + c \int_{t_0}^{t} s^{1/2} R'(s) \, ds \end{aligned}$$

by the inequality $(a+b)^{1/2} \leq b^{1/2} + a/(2b)^{1/2}$. We now integrate the final term by parts. As $R(s) \geq 0$ for $s > t_0$, the integral is bounded by $cR(t)\, t^{1/2}$. Hence

$$M(t) \leq c(1 + R(t))\, t^{1/2} + c(N_R \log N_R)^{1/2}. \tag{2.8}$$

Lemma 14 part (2), gives

$$M(t) \geq c_7 \exp\left( \frac{Q(t)}{d} - \frac{C_{vol}\left(x, M(x,t)^{-1}\right)}{d} \right).$$

Hence, if $t \geq N_R$ then under the assumption that $M(t) \geq t^{1/2}$ and the volume condition (2.5) we obtain

$$M(t) \geq c_7 \exp\left( R(t) - \frac{c_4}{d} + \frac{1}{2} \log t - d^{-1} \right). \tag{2.9}$$

Combining (2.8) and (2.9) we see that when $N_R \log N_R < t < t_1$

$$c' t^{1/2} \exp\left( R(t) - \frac{c_3}{d} - d^{-1} \right) \leq c(1 + R(t))\, t^{1/2}.$$

Dividing through by $t^{1/2}$ we see that $R(t)$ is bounded independently of $t$. Hence, by (2.8) there exists $c'' > 0$ such that for $N_R \log N_R < t < t_1$

$$M(t) \leq c'' t^{1/2}.$$

Take $c_8 := c'' \wedge 1$ and we are done. ■

The previous result enables us to give bounds on the probability of exiting a ball quickly. Let $\bar{\tau}(x,r) := \inf\{t \geq 0 : Z_t \notin B_x[r]\}$. As in [4]:

**Proposition 16.** *There exist $c_i$ such that if $B_{x_1}[R]$ is fine then for $x \in B_{x_1}\left[\frac{7}{9}R\right]$, $c_9 (N_R \log N_R)^{1/2} \leq r \leq R$ and $0 \leq t \leq \frac{1}{2} c_5 \frac{R^2}{\log R}$ we have*

$$P_x(\bar{\tau}(x,r) < t) \leq \frac{1}{2} + \frac{c_{10} t}{r^2}. \tag{2.10}$$

**Proof.** Write $\bar{\tau} = \bar{\tau}(x,r)$. Let

$$A := B_x[r] \cup \partial_{\text{ext}} B_x[r].$$

Firstly assume that $r < \frac{R}{9}$. For $N_R \log N_R \leq t \leq \frac{c_5 R^2}{2\log R}$, by Proposition 15:

$$\begin{aligned}
c_8 t^{1/2} &\geq E_x\left(d\left(x, Z_{2t}\right)\right) \\
&\geq E_x\left(d\left(x, Z_{t\wedge\bar{\tau}}\right) - d\left(Z_{t\wedge\bar{\tau}}, Z_{2t}\right)\right) \\
&\geq E_x 1_{\{\bar{\tau}<t\}} d\left(x, Z_{\bar{\tau}}\right) - E_x\left[E_{Z_{t\wedge\bar{\tau}}} d\left(Z_{t\wedge\bar{\tau}}, Z_{2t-t\wedge\bar{\tau}}\right)\right] \\
&\geq P_x\left(\bar{\tau} < t\right) r - \sup_{z\in A, s\leq t} E_z d\left(z, Z_{2t-s}\right) \\
&\geq P_x\left(\bar{\tau} < t\right) r - c_8 t^{1/2}.
\end{aligned}$$

We conclude, by rearranging, that

$$P_x\left(\bar{\tau} < t\right) \leq 2c_8 t^{1/2} r^{-1}. \tag{2.11}$$

Since $\lambda \leq \frac{1}{2}\left(1+\lambda^2\right)$ we obtain (2.10).

Now, if $t < N_R \log N_R$ then by (2.11)

$$\begin{aligned}
P_x\left(\bar{\tau} < t\right) &\leq P_x\left(\bar{\tau} < N_R \log N_R\right) \\
&\leq 2c_8\left(N_R \log N_R\right)^{1/2} r^{-1} \\
&\leq \frac{1}{2}
\end{aligned}$$

for $r > 4c_8\left(N_R \log N_R\right)^{1/2}$.

Finally, if $\frac{R}{9} \leq r \leq R$ then $\bar{\tau}(x,r) \geq \bar{\tau}\left(x, \frac{R}{9}\right)$. Thus, adjusting the earlier constants, the result holds. ■

As noted in [8], we can prove an analogous result for discrete time. Let $\tau$ refer to the exit time for the discrete walk.

**Proposition 17.** *There exist $c_i$ such that if $B_{x_1}[R]$ is fine then for $x \in B_{x_1}\left[\frac{7}{9}R\right]$, $c_9\left(N_R \log N_R\right)^{1/2} \leq r \leq R$ and $c_{12} \leq t \leq \frac{1}{2}c_5 \frac{R^2}{\log R}$ we have*

$$P_x\left(\tau(x,r) < t\right) \leq \frac{2}{3} + \frac{c_{11} t}{r^2}.$$

**Proof.** $Z_t = X_{M_t}$ for $M_t$ a unit rate Poisson process independent of $X$. Hence

$$\begin{aligned}
P_x\left(\tau(x,r) < t\right) P_x\left(M_{2t} > t\right) &= P_x\left(\tau(x,r) < t, M_{2t} > t\right) \\
&\leq P_x\left(\bar{\tau}(x,r) < 2t\right) \\
&\leq \frac{1}{2} + \frac{c_{10} t}{r^2}.
\end{aligned}$$

Now, there exists $c_{12}$ such that $P_x\left(M_{2t} > t\right) \geq \frac{3}{4}$ for all $t \geq c_{12}$. The result now follows. ■

**Proposition 18.** *There exist $c_i$ such that if $B_{x_1}[R]$ is fine, $x \in B_{x_1}\left[\frac{2}{3}R\right]$ and $t, \rho > 0$ satisfy*

$$\rho < R \text{ and } c_{13}\left(N_R \log N_R\right)^{1/2} \rho \le t \le \frac{R^2}{\log R},$$

*then*

$$P_x\left(\tau\left(x,\rho\right) < t\right) \le c_{14} \exp\left(-\frac{c_{15}\rho^2}{t}\right).$$

**Proof.** Identical to Proposition 3.7 of [4]. ■

We can now prove Theorem 12.

**Proof of Theorem 12.** For ease of notation set $D := d(x,y)$.

If $n \log n \le 2C_C d^{-1} D^2$ then by (2.4)

$$q_n\left(x,y\right) \le C_B \exp\left(-\frac{C_C d\left(x,y\right)^2}{n}\right) \le C_B n^{-d/2}$$

and hence (2.7) holds.

Now, suppose that

$$n \log n > 2C_C d^{-1} D^2. \tag{2.12}$$

If $c_1$ is chosen small enough in (2.6) then (2.12) forces $y \in B_x\left[\frac{2}{3}R\right]$.

Set

$$A_x := \left\{z : d\left(x,z\right) \le d\left(y,z\right)\right\}$$

and $A_y := G - A_x$. Set $s = \lfloor n/2 \rfloor$ and $\rho = D/2$. The idea is the following: we can write

$$\pi\left(x\right) P_x\left(X_n = y\right) = \pi\left(x\right) P_x\left(X_n = y, X_s \in A_y\right) + \pi\left(x\right) P_x\left(X_n = y, X_s \in A_x\right). \tag{2.13}$$

We can bound the first term on the right-hand side:

$$\begin{aligned}
P_x\left(X_n = y, X_s \in A_y\right) &= P_x\left(\tau\left(x,\rho\right) < s, X_s \in A_y, X_n = y\right) \\
&\le E_x\left(1_{\{\tau(x,\rho)<s\}} P^{X_\tau}\left(X_{n-\tau} = y\right)\right) \\
&\le P_x\left(\tau\left(x,\rho\right) < s\right) \sup_{\substack{z \in \partial B(x,\rho) \\ s \le n/2}} q_{n-s}\left(z,y\right) \pi\left(y\right)
\end{aligned}$$

For $n \ge 2N_R$ the second term is bounded by $C_A n^{-d/2}$ by Definition 11. To bound the first part of the right hand side we look to use Proposition 18 and so must check that the conditions are satisfied. Now, $\rho < D < R$ and $s < \frac{c_5}{2}\frac{R^2}{\log R}$ for $c_1$ small. By (2.12) and the assumption $n \ge N_R^{1+\alpha}$ we see

$$\begin{aligned}
c_{13}\left(N_R \log N_R\right)^{1/2} \rho &\le c\left(N_R \log N_R\right)^{1/2} \left(n \log n\right)^{1/2} \\
&\le \frac{1}{3} n \le s.
\end{aligned}$$

Thus, applying Proposition 18 we see that

$$P_x\left(X_n = y, X_s \in A_y\right) \le cn^{-d/2} \exp\left(-c' \frac{D^2}{s}\right) \pi\left(y\right).$$

By reversibility we can bound the second term of (2.13) in the same way. Divide through by $\pi(x)\pi(y)$ and the proof is complete. ■

## 2.4. Controlling the time in traps

As discussed in the introduction, we consider two graphs in this chapter: the original graph $\mathcal{G}$ and a thinned graph $\mathcal{C}_k^\infty \subseteq \mathcal{G}$. This structure naturally induces what we call traps - the connected components of $\mathcal{G} - \mathcal{C}_k^\infty$.

As will become clear later, it is crucial that we can control the time that the walk spends in these traps and we explore this here.

In Section 2.4.1 we consider the time spent in a generic trap, $T$, when $\pi(T) = M$. We use finite Markov chain arguments to bound the probability that the walk spends a large amount of time in $T$ in one visit in terms of $M$.

In dimension $d \ge 3$ we will show in Section 2.5 that, under the conditions of Definition 9, the random walk is transient. Therefore the total time spent by the walk in $T$ is finite and we can ask how this behaves. We consider this in Section 2.4.2, giving bounds on the probability that a large amount of time is spent in the trap in terms of $M$ and the probability that the walk returns to the trap.

### 2.4.1. Time spent in a single trap of measure $M$ in one visit

We look at a trap, $T_1$, of measure $\pi(T_1) = M$. We start the walk at one of the neighbours of $T_1$, this is by definition a vertex in $\mathcal{C}_\infty^k$. Call this vertex $x_0$. We use the work of Aldous and Brown [1] to obtain exponential decay for the distribution of the exit time from $T_1$.

Now, as we are interested in the exit time from $T_1$ we assume that the walk steps directly into $T_1$ from $x_0$. We therefore consider the augmented trap $T$ with vertex set $V(T) = V(T_1) \cup \{x_0\}$ and edge set induced from $\mathcal{G}$. Note that we may have deleted other edges that connected $T$ to $\mathcal{C}_k^\infty$, however their deletion can only increase the exit time from $T$.

The quantity we wish to investigate is thus

$$S_{x_0} = S_{x_0}(T) := \inf\{n > 0 : X_n = x_0\}$$

where $X_n$ is the simple random walk on $T_1$ started at the point $x_0$. For the continuous time version of the walk, $(Z_t)_{t\ge 0}$, we similarly define

$$\bar{S}_{x_0} = \bar{S}_{x_0}(T) := \inf\{t > 0 : Z_t = x_0\}.$$

Instead of analyzing $S$ directly we attack the smoother $\bar{S}$. Write $Z_t = X_{M_t}$ for $M_t$ an independent Poisson point process of unit rate and let $\bar{M}_n := \inf\{t > 0 : M_t = n\}$ be its inverse. We deduce results for $S$ from the following:

$$\begin{aligned}
\mathbb{P}(S_{x_0} \geq n) &= \mathbb{P}\left(S_{x_0} \geq n, \bar{M}_n > c_1 n\right) + \mathbb{P}\left(S_{x_0} \geq n, \bar{M}_n \leq c_1 n\right) \\
&\leq \mathbb{P}\left(\bar{S}_{x_0} \geq c_1 n\right) + \mathbb{P}\left(\bar{M}_n \leq c_1 n\right) \\
&= \mathbb{P}\left(\bar{S}_{x_0} \geq c_1 n\right) + \mathbb{P}\left(M_{c_1 n} > n\right). \qquad (2.14)
\end{aligned}$$

When $c_1 < 1$ the second term will decay exponentially quickly in $n$ and hence upper bounds on $\bar{S}$ will transfer to upper bounds on $S$.

We quote the following from [1]:

$$\mathbb{P}_\pi\left(\bar{S}_{x_0} > t\right) \leq (1 - \pi(x_0)) \exp\left(-\frac{t}{\mathbb{E}_\alpha\left(\bar{S}_{x_0}\right)}\right), \qquad (2.15)$$

where $\pi$ is the stationary probability distribution for the chain and $\alpha$ is the eigenvector corresponding to the smallest eignenvalue of $-Q_{x_0}$, the rate matrix of the simple random walk on $T$ restricted to $T - x_0$. The subscripts $\pi$ and $\alpha$ refer to the initial distribution being $\pi$ and $\alpha$ respectively. The bottom term inside the exponential is difficult to calculate explicitly. We can use another result from [1], namely:

$$\mathbb{E}_\alpha\left[\bar{S}_{x_0}\right] \leq \frac{\tau}{\pi(x_0)}, \qquad (2.16)$$

where $\tau$ is the inverse of the spectral gap - the spectral gap being the smallest non-zero eigenvalue of the rate matrix $-Q$.

It is important to emphasize that the $\pi$ quoted is a probability distribution. When we come to apply these results to finite traps in our model, we will take

$$\pi(x) = \frac{\sum_{y \sim x} \omega_{xy}}{\sum_{y \in T} \sum_{z \sim y} \omega_{zy}},$$

this is the normalized $\pi$ used in the rest of this chapter.

We should say something about why these results hold. It is easy to see where the first result comes from. If we restrict the transition matrix of our finite Markov chain to $T - x_0$, then we obtain a strictly substochastic matrix and hence when we raise this to the power $n$, rows will sum to strictly less than one. If we look at the row corresponding to the point that is connected to $x_0$ and sum the entries, the difference between this sum and one is the probability that we have hit $x_0$ before time $n$.

To bound the spectral gap we use the isoperimetric constant. We therefore introduce the isoperimetric profile for a graph.

**Definition 19.** *For a Markov chain on a vertex set $V$, with transition probabilities $P$ and invariant measure $\pi$, let $Q(x,y) := \pi(x) P(x,y)$. For sets $A_1, A_2 \subseteq V$, write*

$$Q(A_1, A_2) := \sum_{x \in A_1} \sum_{y \in A_2} Q(x,y).$$

*For $A \subseteq V$ with $0 < \pi(A) < \infty$ set*

$$\Phi_A := \frac{Q(A, A^c)}{\pi(A)},$$

*and define the isoperimetric profile*

$$\Phi(r) := \inf\{\Phi_A : \pi(A) \leq r\}. \tag{2.17}$$

The isoperimetric profile will be used extensively in Section 2.5. We consider only the isoperimetric constant here:

$$\Phi = \Phi(\pi(V)/2).$$

From the isoperimetric constant we can use Cheeger's inequality to bound the spectral gap (see, for example, Saloff-Coste's St Flour notes [50]).

Considering the simple random walk we obtain

$$\begin{aligned} Q(A, A^c) &= \sum_{x \in A} \sum_{y \in A^c} \pi(x) P(x,y) \\ &= \pi(\partial A), \end{aligned}$$

where $\partial A := \{e = (xy) : x \in A, y \notin A\}$. Thus,

$$\Phi = \min\left\{\frac{\pi(\partial A)}{\pi(A)} : A \subseteq V(G), \pi(A) \leq \frac{\pi(V)}{2}\right\}.$$

We need to bound the isoperimetric constant from below. Since $\pi(V) \leq M + k$ and $\omega_e \geq 1$ for any edge $e$, we have

$$\Phi \geq \frac{2}{M+k}$$

and in fact $M^{-1}$ is the correct order as can be seen by considering the graph comprising two complete graphs with edges of weight 1 on $\sqrt{\frac{M}{2}}$vertices connected by exactly one edge.

Cheeger's inequality (see eg [50]) gives a bound on the spectral gap in terms of the isoperimetric profile:

$$\frac{\Phi^2}{8} \leq \lambda \leq \Phi.$$

Hence we obtain the lower bound

$$\lambda \geq \frac{1}{2(M+k)^2}.$$

Now, combining this with (2.15) and (2.16) we obtain:

$$\mathbb{P}_{\pi}\left(\bar{S}_{x_0} > t\right) \leq \left(1 - \pi\left(x_0\right)\right) \exp\left(-\frac{t\pi\left(x_0\right)}{2\left(M+k\right)^2}\right).$$

Splitting the left hand side into constituent parts:

$$\mathbb{P}_{x_0}\left(\bar{S}_{x_0} > t\right) \leq \frac{1-\pi\left(x_0\right)}{\pi\left(x_0\right)} \exp\left(-\frac{t\pi\left(x_0\right)}{2\left(M+k\right)^2}\right).$$

By (2.14) we can recover a bound on the return time for the discrete walk:

$$\mathbb{P}_{x_0}\left(S_{x_0} > n\right) \leq c_2 M \exp\left(-\frac{c_3 n}{M^2}\right), \tag{2.18}$$

for constants $c_i$ independent of $M$.

We thus have exponential decay for the tail of $S_{x_0}$ and that $\mathbb{E}\left[e^{\theta S_{x_0}}\right]$ exists for suitably chosen values of $\theta$; this in turn allows us to consider the amount of time spent in $T$ over several visits to the trap.

Using equation (2.18) we see that

$$\begin{aligned}
\mathbb{E}\left[\exp\left(\theta S_{x_0}\right)\right] &= \int_0^\infty \mathbb{P}\left[\exp\left(\theta S_{x_0}\right) \geq x\right] dx \\
&= 1 + \int_1^\infty \mathbb{P}\left[\exp\left(\theta S_{x_0}\right) \geq x\right] dx \\
&= 1 + \int_0^\infty \theta \exp\left(\theta x\right) \mathbb{P}\left[S_{x_0} > x\right] dx \\
&\leq 1 + \int_0^\infty \theta \exp\left(\theta x\right) c_2 M \exp\left(-\frac{c_3 x}{M^2}\right) dx \\
&= 1 + c_2 \frac{\theta M}{c_3 M^{-2} - \theta},
\end{aligned} \tag{2.19}$$

for $\theta < c_3 M^{-2}$. We will choose $\theta$ later.

### 2.4.2. Total time spent in a trap for transient walks

In Section 2.5 we will investigate transient random walks. This leads to the natural question concerning the total time spent in a trap. This will depend not only on the size and shape of the trap, but also on how many times the walk returns to the trap. This in turn will be determined by the trap's return probability.

For a trap $T$, we define the worst case return probability:

$$p_T := \max\left\{P_x\left(X_n \in T \text{ for some } n > 0\right) : x \in \partial_{ext} T\right\}.$$

For the purpose of this section we assume that $p_T < 1$.

Now, let $T$ be a trap of total measure $M$ with worst case return probability $p_T$. Write

$$\tau := |\{n : X_n \in \partial_{ext} T, X_{n+1} \in T\}|$$

for the number of visits to the trap $T$. Then, if we let $Z_i$ be random variables corresponding to time spent in $T$ on successive visits, the total amount of time spent in $T$, call this $Y$, is given by

$$Y := \sum_{i=1}^{\tau} Z_i.$$

Note that

$$\mathbb{E}\left[e^{\theta Z_i}\right] \leq \sup_{x_0 \in \partial T} \mathbb{E}\left[e^{\theta S_{x_0}}\right]$$

and the righthand side can in turn be bounded by equation (2.19). Define $z := \sup_{x_0 \in \partial T} \mathbb{E}\left[e^{\theta S_{x_0}}\right]$.

We look to calculate bounds on the moment generating function of $Y$ when the walk is started at any boundary point of the trap. We thus look to bound:

$$w := \sup_{x \in \partial T} \mathbb{E}_x\left[e^{\theta Y}\right].$$

Define $\sigma_1 := \inf\{n \geq 0 : X_n \in \partial T, X_{n+1} \in T\}$ and $\sigma_2 := \inf\{n > \sigma_1 : X_n \in \partial T\}$. Then

$$\begin{aligned}
w &= \sup_{x \in \partial T}\left[\sum_{y \in \partial T} \mathbb{E}_x\left[e^{\theta Y} | X_{\sigma_1 = y}\right] \mathbb{P}_x\left[X_{\sigma_1} = y\right] + \mathbb{P}_x\left[\sigma_1 = \infty\right]\right] \\
&\leq \sup_{x \in \partial T}\left[\sum_{y \in \partial T} \mathbb{E}_y\left[e^{\theta Z_1}\right] \mathbb{E}_y\left[e^{\theta(Z_2 + \ldots + Z_\tau)} | X_{\sigma_1} = y\right] \mathbb{P}_x\left[X_{\sigma_1} = y\right] + 1\right] \\
&\leq \sup_{x \in \partial T}\left[z \mathbb{E}_x\left[e^{\theta(Z_2 + \ldots + Z_\tau)} | \sigma_1 < \infty\right] \mathbb{P}_x\left[\sigma_1 < \infty\right]\right] + 1 \\
&\leq z p_T w + 1.
\end{aligned}$$

Hence provided $z p_T < 1$, then

$$w \leq (1 - z p_T)^{-1}.$$

Now, to satisfy $z < \frac{1}{p_T} < \infty$, we use equation (2.19):

$$z \leq 1 + c_2 \frac{\theta M}{c_3 M^{-2} - \theta}.$$

Thus the condition is satisfied if

$$\theta < \frac{c_3 (1 - p_T)}{c_2 p_T M^3 + M^2 (1 - p_T)}.$$

Take $c_4$ suitably small and choose

$$\theta := \frac{c_4 \left(1 - p_T\right)}{M^3},$$

then after some work we obtain the bound

$$\begin{aligned} w &\leq \frac{1}{1 - p_T} + \frac{p_T M}{\left(1 - p_T\right)^2 \left(M - 1\right)} \\ &\leq \frac{1}{1 - p_T} + c_5 \frac{p_T}{\left(1 - p_T\right)^2}. \end{aligned}$$

Thus for $Y_1, Y_2, \ldots$ iid random variables, equal in distribution to $Y$ we have for $l, n \in \mathbb{N}$

$$\begin{aligned} P_T\left(Y_1 + \ldots + Y_l \geq n\right) &\leq \exp\left(-\left(\frac{c_4 \left(1 - p_T\right)}{M^3}\right) n\right) w^l \\ &\leq \exp\left(-\left(\frac{c_4 \left(1 - p_T\right)}{M^3}\right) n\right) \left(\frac{1}{1 - p_T} + c_5 \frac{p_T}{\left(1 - p_T\right)^2}\right)^l \end{aligned}$$

Now, we can group our traps in the following way.

**Definition 20.** *We say a trap $T$ is of type $[m, r]$ for $m, r \in \mathbb{N}$ if $\lceil \pi\left(T\right) \rceil = m$ and*

$$1 - \frac{1}{c_6 \left(r - 1\right)} \leq p_T \leq 1 - \frac{1}{c_6 r},$$

*where the constant $c_6 := c_{2.5.1}$, which in turn is defined in Theorem 24.*

It will be shown in Section 2.5 that a trap of type $(m, r)$ as defined in Definition 7 is a trap of type $[m, r']$ in Definition 20 for some $r' \leq r$.

We can now summarize this section in the following proposition.

**Proposition 21.** *Suppose $T_i$ are traps of type $[m, r]$ and $Y_i$ is the total time spent in trap $T_i$. Then there exist constants $c_i$ dependent only on $c_{2.5.1}$ such that for $l, n \in \mathbb{N}$*

$$\mathbb{P}\left(Y_1 + \ldots + Y_l \geq n\right) \leq \exp\left(-\left(\frac{c_7}{m^3 r}\right) n\right) \left(c_8 r + c_9 r^2\right)^l \tag{2.20}$$

## 2.5. Heat kernel upper bounds

In this section we will use the results of Sections 2.3, 2.4 and the ideas of [4] and [11] to prove Theorem 10.

Recall the three random walks introduced in Definition 3: the simple random walk on $\mathcal{G}$, $X_n$; the time changed random walk, $\tilde{X}_n$; and the simple random walk on $\mathcal{C}^k_\infty$, $Y_n$.

Our method to obtain upper bounds for the on diagonal heat kernel behaviour for $X_n$ is the following: we show that the isoperimetric profiles of the time changed random walk and the simple random walk on the thinned graph are comparable, then, since we have good control of the isoperimetric profile of the thinned graph through Condition 4, we obtain standard heat kernel behaviour for the time changed random walk. This together with Condition 5 and Section 2.3 give full Gaussian upper bounds for the time changed walk. This is contained in Section 2.5.1 and follows closely [11].

We then show that we can transfer these bounds to the random walk on the full graph provided we have sufficient control over the time spent in traps - this boils down to controlling the size and frequency of traps. Section 2.5.2 proves that Condition 8 is sufficient to control the time spent in traps. These results are then brought together to prove Theorem 10 in Section 2.5.3.

For $X_n$ and $\tilde{X}_n$ we write respectively $P^n(x,y) := P\left(X_n = y \,|X_0 = x\right)$, $q_n(x,y) = \frac{P^n(x,y)}{\pi(y)}$, $\tilde{P}^n(x,y) := P\left(\tilde{X}_n = y \,\middle|\tilde{X}_0 = x\right)$ and $\tilde{q}_n(x,y) = \frac{\tilde{P}^n(x,y)}{\pi(y)}$.

### 2.5.1. The time changed random walk

Recall the definition of the isoperimetric profile, Definition 19. There are several results providing a link from isoperimetric inequalities to heat kernel bounds; we choose to invoke the following, taken from [43].

**Theorem 22 (Morris and Peres 2005).** *Consider a graph $G = (V, E)$ and let $P$ be the transition matrix of a random walk on $G$ with invariant measure $\pi$. Suppose that $\gamma \in (0,1]$ is such that $P(x,x) \geq \gamma$ for all $x \in V$. If*

$$\eta \geq 1 + \frac{(1-\gamma)^2}{\gamma^2} \int_{4(\pi(x)\wedge\pi(y))}^{4/\varepsilon} \frac{4}{u\Phi(u)^2} du$$

*then $\left|\frac{P^\eta(x,y)}{\pi(y)}\right| \leq \varepsilon$.*

The above theorem requires $P(x,x) \geq \gamma$ for some $\gamma > 0$. Hence we consider the transition densities given by $\tilde{P}^2$ and $P_k^2$, corresponding to two step transitions of the time changed random walk and the random walk on the thinned graph respectively, since for all $x \in \mathcal{C}_\infty^k$ we have $\tilde{P}^2(x,x), P_k^2(x,x) \geq k^{-2}$.

We write $\tilde{\Phi}$ for the isoperimetric profile corresponding to $\tilde{P}^2$ and $\Phi^k$ for the isoperimetric profile associated to $P_k^2$.

The following proposition, adapted from [11], shows that we can bound the isoperimetric profile of the time-changed random walk by the isoperimetric profile of the simple random walk on the thinned graph.

**Proposition 23.** *For any finite set $\Lambda \subseteq \mathcal{C}_\infty^k$,*

$$\tilde{\Phi}_\Lambda \geq k^{-3}\Phi_\Lambda^k,$$

*where $\tilde{\Phi}_\Lambda$ and $\Phi^k_\Lambda$ are with respect to the time changed walk and the simple walk on the thinned graph respectively.*

**Proof.** For $x \in \mathcal{C}^k_\infty$, define

$$\nu(x) := \sum_{y \in C^k_\infty} \omega_{xy},$$

the invariant measure for the simple random walk on the thinned graph. Note that $\nu \leq \pi$. Note also that $\pi$ is the invariant measure for the time changed random walk $\tilde{X}$.

For $x \in \mathcal{C}^k_\infty$ we have

$$\pi(x) \geq \nu(x) \geq 1 \geq \frac{1}{k}\pi(x). \tag{2.21}$$

Set

$$B(x,y) := \left\{z \in \mathcal{C}^k_\infty : z \sim x, z \sim y\right\}.$$

Then, on restricting $\tilde{P}^2(x,y)$ to transitions that fail to visit a trap and using (2.21) we see:

$$\begin{aligned}\tilde{P}^2(x,y) &\geq \sum_{z \in B(x,y)} \frac{\omega_{xz}}{\pi(x)} \frac{\omega_{zy}}{\pi(z)} \\ &\geq \frac{1}{k^2} \sum_{z \in B(x,y)} \frac{\omega_{xz}}{\nu(x)} \frac{\omega_{zy}}{\nu(z)} = k^{-2} P^2_k(x,y). \end{aligned} \tag{2.22}$$

The claim follows from (2.21), (2.22) and Definition 19. ■

By this proposition and Condition 4 we have control of the isoperimetric profile of the time changed walk. We therefore look to control its heat kernel.

**Theorem 24.** *Suppose Condition 4 holds with set of constants $\{R_0(x) : x \in \mathcal{V}\}$. Then there exist constants $c_i(d, k, \psi, \phi, C_1) < \infty$ such that for $x_1, x \in \mathcal{V}$,*

$$\tilde{q}_{2n}(x_1, x) \leq \frac{c_1}{n^{d/2}}$$

*for all $n \geq c_2 \vee R_0(x_1)$.*

**Proof.** For simplicity we assume that $x_1 = 0 \in \mathcal{V}$. For general $x_1$ we simply translate the graph - mapping $x_1$ to the origin.

By Condition 4, the largest distance the time changed random walk can travel in $n$ steps is bounded by $L = n^{1/(1-\phi)}$, provided $n \geq R_0$. We thus only consider sets $\Lambda \subseteq \mathcal{C}^k_\infty$ entirely contained in $[-L, L]^d$ when calculating $\tilde{\Phi}(r)$. By Proposition 23 and the above comment we restrict our attention to sets $\Lambda \subseteq \mathcal{C}^k_\infty \cap [-L, L]^d$ that are connected in the graph structure of $\mathcal{C}^k_\infty$. We consider the isoperimetric profile of this truncated, finite graph. Take $\theta \in \left(\psi, \frac{1-\phi}{2}\right)$.

If $n \geq R_0$ then $L \geq R_0^{1/(1-\phi)} \geq R_0$. Thus if $\pi(\Lambda) \geq L^\theta$, then Condition 4 and equation (2.21) combine to say that there is a constant $c > 0$ such that:

$$\Phi_\Lambda^k \geq c\pi(\Lambda)^{-1/d}.$$

If $\pi(\Lambda) < L^\theta$, then we trivially have

$$\Phi_\Lambda^k \geq \pi(\Lambda)^{-1} \geq L^{-\theta}.$$

From Proposition 23 and the above two equations we see that

$$\tilde{\Phi}_\omega(r) \geq c\left(r^{-1/d} \wedge L^{-\theta}\right).$$

We wish to use Theorem 22 and intend to take $\varepsilon \approx n^{-d/2}$. The switch between the two regimes occurs when $r = L^{d\theta}$, which is less than $4/\varepsilon$ since $\theta < 1/2$. The integral we need to calculate can thus be bounded by

$$\begin{aligned} & 1 + k^2 \int_{4(\pi(x)\wedge\pi(y))}^{4/\varepsilon} \frac{4}{u\tilde{\Phi}(u)^2} du \\ \leq \; & 1 + k^2 \int_{4(\pi(x)\wedge\pi(y))}^{L^{d\theta}} \frac{4}{ucL^{-2\theta}} du + k^2 \int_{L^{d\theta}}^{4/\varepsilon} \frac{4}{cu^{1-2/d}} du \\ \leq \; & 1 + cL^{2\theta} \log L + c'\varepsilon^{-2/d} \end{aligned}$$

Take $\varepsilon$ such that $2c'\varepsilon^{-2/d} = n$. Now, take $N_1$ such that for all $n > N_1$ we have

$$cL^{2\theta} \log L < n/4,$$

this is possible due to our choice of $\theta$. Take $N_2$ such that for $n > N_2$ we have

$$L^{d\theta} < \frac{4}{\varepsilon}$$

and hence the splitting of the integral above is valid. Note that $N_1$ and $N_2$ are both deterministic, dependent only on $d, \theta, \psi, \phi$ and $C_1$. In particular they do not depend on $R_0$.

Hence if $n > N_1 \vee N_2 \vee R_0$ we have

$$1 + k^2 \int_{4(\pi(x)\wedge\pi(y))}^{4/\varepsilon} \frac{4}{u\tilde{\Phi}(u)^2} du < n$$

and appealing to Theorem 22 with $\gamma = k^{-2}$, we obtain the claimed result. ■

To ease notation, from now on assume that $R_0(x) \geq c_2$ for all $x$.

**Corollary 25.** *If $B_{x_1}[R]$ is good then for any $x \in B_{x_1}[4R/5]$ and $y \in \mathcal{V}$ we have*

$$\tilde{q}_n(x, y) \leq \frac{c_1}{n^{d/2}}$$

*for $n \geq R_0(x)$.*

**Proof.** For even $n$ this follows directly from Theorem 24 and the definition of a good ball. For $n = 2l + 1$ we see:

$$\begin{aligned}
\tilde{q}_n(x,y) &= \frac{1}{\pi(y)} \sum_{z \sim y} P_x(X_{2l} = z) P_z(X_1 = y) \\
&= \frac{1}{\pi(y)} \sum_{z \sim y} \tilde{q}_{2l}(x,z) \pi(z) P_z(X_1 = y) \\
&\leq \frac{1}{\pi(y)} c_1 l^{-d/2} \sum_{z \sim y} \omega_{zy} \\
&\leq c_1 n^{-d/2},
\end{aligned}$$

where the value of $c_1$ is altered in the final line. ■

From Section 2.3 and the condition on volume growth and graph distance for the thinned graph, Condition 5, we obtain:

**Corollary 26.** *Define* $\xi := \left(\frac{d+2}{d+1}\right)^{1/2} - 1$. *Suppose* $B_{x_1}[R]$ *is good. Then there exist* $c_i$ *such that for* $x \in B_{x_1}[2R/5]$ *and* $y \in \mathcal{V}$ *we have:*

$$\tilde{q}_n(x,y) \leq c_3 n^{-d/2} \exp\left(-\frac{c_4 |x-y|^2}{n}\right)$$

*for* $N_R^{1+\xi} \leq n \leq c_5 \frac{R^2}{\log R}$.

**Proof.** This is an application of Theorem 12, with $\alpha := \xi > 0$. We see that the conditions of the theorem are satisfied by Corollary 25, [25] and Condition 5 for good balls. ■

We keep the choice of $\xi$ constant for the remainder of the section.

Note in particular, that for $d \geq 3$ the walk is transient. For $x \in \mathcal{C}_\infty^k(\omega)$, let

$$p(x,\omega) := P_x^\omega(X_n = x \text{ for some } n > 0).$$

Since the random walk $X_n$ returns to $x$ iff the time changed random walk $\tilde{X}_n$ returns to $x$, by a simple application of the Markov property we have:

$$E_x^\omega\left[\sum_{n=0}^\infty 1_{\{\tilde{X}_n = x\}}\right] = \frac{1}{1 - p(x,\omega)}.$$

Now, by Corollary 25, if Condition 4 holds then for all $x \in \mathcal{C}_\infty^k(\omega)$:

$$\begin{aligned}
E_x^\omega\left[\sum_{n=0}^\infty 1_{\{\tilde{X}_x = 0\}}\right] &= \sum_{n=0}^\infty P_x^\omega\left(\tilde{X}_n = x\right) \\
&\leq R_0(x) + \sum_{n=R_0(x)+1}^\infty c_1 n^{-d/2} \\
&\leq c_1 R_0(x),
\end{aligned}$$

where $c_1$ has changed value. Thus

$$p(x,\omega) \leq 1 - \frac{1}{c_1 R_0(x)}, \tag{2.23}$$

validating the link between Definitions 7 and 20 claimed after the latter definition.

The following proposition would normally be proved by the ergodicity of the environment from the point of view of the particle. However, we require control over the speed of the convergence of the following sum and hence use the above Gaussian bounds.

**Proposition 27.** *If $B_{x_1}[R]$ is good then there exists $c_6$ such that for any $x \in B_{x_1}[2R/5]$ we have*

$$E_x\left[\frac{1}{n}\sum_{j=0}^{n} \pi\left(T_{\tilde{X}_j}\right)\right] \leq c_6,$$

*for all $N_R^{(1+\xi)(d+1)} \leq n \leq c_5 \frac{R^2}{\log R}$.*

**Proof.** Set $N := N_R^{1+\xi}$.

Now, by Corollary 26,

$$\begin{aligned}
& E_x\left(\frac{1}{n}\sum_{j=0}^{n} \pi\left(T_{\tilde{X}_j}\right)\right) \\
= \; & \frac{1}{n}\sum_{j=0}^{n}\sum_{y\in\mathcal{C}_\infty^k} \pi(T_y)\tilde{P}^j(x,y) \\
\leq \; & \frac{1}{n}\sum_{j=0}^{N}\sum_{y\in B_x[N]} \pi(T_y)\tilde{P}^j(x,y) + \frac{1}{n}\sum_{j=N}^{n}\sum_{y\in B_x[N]} \pi(T_y)\tilde{P}^j(x,y) \\
& + \frac{1}{n}\sum_{j=N}^{n}\sum_{y\in\mathcal{C}_\infty^k - B_x[N]} c_3\pi(y)\pi(T_y)j^{-d/2}\exp\left(-c_4\frac{|x-y|^2}{j}\right).
\end{aligned} \tag{2.24}$$

Recall Condition 6. For the first term we have by (2.1) and (2.2) standard volume control as $N > N_R \geq R_2(x)$ :

$$\begin{aligned}
\sum_{j=0}^{N}\sum_{y\in B_x[N]} \pi(T_y)\tilde{P}^j(x,y) &= \sum_{y\in B_x[N]}\sum_{j=0}^{N} \pi(T_y)\tilde{P}^j(x,y) \\
&\leq N\sum_{y\in B_x[N]\cap\mathcal{C}_\infty^k} \pi(T_y) \\
&\leq cN^{d+1}.
\end{aligned}$$

For the second term: by (2.1), (2.2) and Corollary 25,

$$\begin{aligned}
\sum_{j=N}^{n}\sum_{y\in B[N]} \pi(y)\pi(T_y)\tilde{q}_j(x,y) &\leq \sum_{y\in B_x[N]\cap\mathcal{C}_\infty^k} \pi(y)\pi(T_y)\sum_{j=N}^{n} cj^{-d/2} \\
&\leq c'\sum_{y\in B_x[N]\cap\mathcal{C}_\infty^k} k\pi(T_y) \\
&\leq c''N^d,
\end{aligned}$$

for $d \geq 3$. Similarly, for $d = 2$:

$$\sum_{j=N}^{n} \sum_{y \in B[N]} \pi(y)\, \pi(T_y)\, \tilde{q}_j(x, y) \leq c'' N^d \log n.$$

For the third term, when $d \geq 3$, by comparison with the corresponding integral and using the substitution $z = \frac{c_4 |x-y|^2}{j}$:

$$\begin{aligned} &\sum_{j=N}^{n} \sum_{y \in \mathcal{C}_\infty^k - B_x[N]} \pi(y)\, \pi(T_y)\, j^{-d/2} \exp\left(-c_4 \frac{|x-y|^2}{j}\right) \\ \leq\ & \sum_{y \in \mathcal{C}_\infty^k - B_x[N]} k \pi(T_y)\, c(d)\, |x-y|^{2-d} \exp\left(-c' \frac{|x-y|^2}{n}\right). \end{aligned}$$

Now, splitting $\mathcal{C}_\infty^k - B_x[N]$ into annuli and appealing to (2.1):

$$\begin{aligned} &\sum_{y \in \mathcal{C}_\infty^k - B_x[N]} \pi(T_y)\, |x-y|^{2-d} \exp\left(-c' \frac{|x-y|^2}{n}\right) \\ \leq\ & \sum_{i=N}^{\infty} \sum_{y \in A_i} c \pi(T_x)\, i^{2-d} \exp\left(-c' \frac{i^2}{n}\right) \\ \leq\ & \sum_{i=N}^{\infty} c i^{d-1} i^{2-d} \exp\left(-c' \frac{i^2}{n}\right). \end{aligned}$$

Evaluate this to conclude that

$$\begin{aligned} &\sum_{j=N}^{n} \sum_{y \in \mathcal{C}_\infty^k - B_x[N]} \pi(y)\, \pi(T_y)\, j^{-d/2} \exp\left(-c_4 \frac{|x-y|^2}{j}\right) \\ \leq\ & c''' n \exp\left(-c' \frac{N^2}{n}\right) \leq c''' n. \end{aligned}$$

For the $d = 2$ case, again using (2.1):

$$\begin{aligned} &\sum_{j=N}^{n} \sum_{y \in \mathcal{C}_\infty^k - B_x[N]} \pi(y)\, \pi(T_y)\, j^{-1} \exp\left(-c_4 \frac{|x-y|^2}{j}\right) \\ \leq\ & \sum_{j=N}^{n} j^{-1} \sum_{i=N}^{\infty} \sum_{y \in A_i} k \pi(T_y) \exp\left(-c_4 \frac{i^2}{j}\right) \\ \leq\ & \sum_{j=N}^{n} j^{-1} \sum_{i=N}^{\infty} c i \exp\left(-c_4 \frac{i^2}{j}\right) \\ \leq\ & \sum_{j=N}^{n} c \leq c''' n. \end{aligned}$$

Now, when we combine the three terms in equation (2.24), we see

$$\begin{aligned} E_x\left(\frac{1}{n} \sum_{j=0}^{n} \pi\left(T_{X_j}\right)\right) &\leq \frac{1}{n}\left(c N^{d+1} + c'' N^d \log n + c''' n\right) \\ &\leq c_6 \end{aligned}$$

for $n \geq N^{d+1}$. ■

Recall the definition of $S_i$, the time the walk takes to make the $i$th step on the thinned graph. By the methods of [11] we have:

**Proposition 28.** *There exists $c_7 = c_7(k, K)$ such that for $x \in \mathcal{C}_\infty^k$ we have*

$$E_x(S_1) \leq c_7 \pi(T_x).$$

**Proof.** We add a vertex $\Delta$ to the finite graph $T_x$ and connect it by an edge to every $y \in T_x$ such that the graph $\mathcal{G}$ contains an edge from $y$ to $\mathcal{C}_\infty^k$, giving the edge $(y\Delta)$ weight

$$\omega_{y\Delta} := \sum_{z \in \mathcal{C}_\infty^k} \omega_{yz}.$$

The resulting finite graph we call $T_x'$. Now, the simple random walk started at $x$ on $\mathcal{G}$ and the simple random walk started at $x$ on $T_x'$ have the same law up until they first hit $\Delta$. In particular $S_1$ for the walk on $\mathcal{C}_\infty$ started at $x$ is stochastically dominated by $S_x'$, the first time that the walk started at $x$ on $T_x'$ returns to $x$.

Now, $\pi$ is an invariant measure for the walk on $T_x'$ provided we set

$$\pi(\Delta) = \sum_{y \in T_x} \omega_{y\Delta}.$$

By standard Markov chain theory $z \mapsto (E_z S_z')^{-1}$, where $E_z$ is expectation with respect to the walk on $T_x'$ started at $z$, is the invariant distribution and

$$E_x S_x' = \frac{\pi(T_x')}{\pi(x)}.$$

Taking $K$ to be the maximum number of neighbours in $\mathcal{C}_\infty^k$ that a point in $\mathcal{G} - \mathcal{C}_\infty^k$ can have, we see that $\pi(\Delta) \leq Kk\pi(T_x)$. Hence $\pi(T_x') \leq \pi(T_x) + Kk\pi(T_x)$. Hence we obtain the required result:

$$E_x(S_1) \leq E_x(S_x') \leq c_7 \pi(T_x).$$

■

The following also follows from the methods of [11].

**Lemma 29.** *Suppose $B_{x_1}[R]$ is good. Then there exists a constant $c_8 < \infty$ such that for all $x \in B_{x_1}[R/2] \cap \mathcal{C}_\infty^k$, $n \geq 1$ and $2N_R^{(1+\xi)(d+1)} \leq l \leq c_5 \frac{R^2}{\log R}$ we have*

$$P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_l \geq n\right) \leq c_8 \pi(x) \frac{l^{1-d/2}}{n}.$$

**Proof.** By reversibility of $\tilde{X}$, for $i < l$

$$P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_i \geq n/2\right) = P_x\left(\tilde{X}_l = x, S_l + \ldots + S_{l-i+1} \geq n/2\right).$$

Applying this with $i = l/2$ then using Markov's inequality

$$\begin{aligned} P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_l \geq n\right) &\leq 2P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_{l/2} \geq n/2\right) \\ &\leq \frac{4}{n} E_x\left[1_{\{\tilde{X}_l = x\}}\left(S_1 + \ldots + S_{l/2}\right)\right]. \end{aligned}$$

If we now condition on the path of $\tilde{X}$ we see

$$\begin{aligned} & P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_l \geq n\right) \\ \leq & \sum_{j=1}^{l/2}\sum_{y,z}\frac{4}{n}P_x\left(\tilde{X}_{j-1} = y\right)E_y\left(S_1 1_{\{\tilde{X}_1 = z\}}\right)P_z\left(\tilde{X}_{l-j} = x\right). \end{aligned} \tag{2.25}$$

By Corollary 25, for $l - j \geq N_R \vee l/2$

$$P_z\left(\tilde{X}_{l-j} = x\right) = \frac{\pi(x)}{\pi(z)}P_x\left(\tilde{X}_{l-j} = z\right) \leq c_1\pi(x)\left(\frac{l}{2}\right)^{-d/2},$$

since $\tilde{X}_i \in \mathcal{C}_\infty^k$. As $j \leq l/2$, if $l > 2N_R$ then $l - j \geq N_R$.

Now, $\sum_z E_y\left(S_1 1_{\{\tilde{X}_1 = z\}}\right) = E_y(S_1) \leq c_7\pi(T_y)$ by Proposition 28. By Proposition 27 we have

$$\sum_{j=1}^{m}\sum_{y}P_x\left(\tilde{X}_{j-1} = y\right)\pi(T_y) = E_x\left(\sum_{j=0}^{m-1}\pi\left(T_{\tilde{X}_j}\right)\right) \leq c_6 m$$

for $N_R^{(1+\xi)(d+1)} \leq m \leq c_5\frac{R^2}{\log R}$.

Combining these results with (2.25) we obtain the claimed result. ■

### 2.5.2. Time spent in traps

Recall from Definition 20 the definition of traps of type $(m, r)$.

If $B_{x_1}[R]$ is a good ball then Condition 8 and equation (2.23) tells us that the traps are well spread. We will combine this condition with the results from Section 2.4 to deduce bounds for the probability that the walk spends a large proportion of its time in traps.

Recall the definition of $S_i$ from Definition 3.

**Proposition 30.** *Suppose $B_{x_1}[R]$ is good. For $x \in B_{x_1}[R/2]$ we have the following:*

*For $d = 2, 3$: Then there are constants $c_9, c_{10}$ such that*

$$\sum_{l=1}^{N}P_x\left(S_1 + \ldots + S_l \geq n\right) \leq c_9\exp\left(-c_{10}n^{1-\frac{3}{4-\phi}}\right) \tag{2.26}$$

*for $n \geq N^{1+\frac{3}{1-\phi}}$ and $N \geq N_R$.*

*For $d \geq 4$: There exists $c_{11} \in (0,1)$ and a function $f : \mathbb{N} \to \mathbb{R}$ with $f(n) = o\left(n^{-d/2+1}\right)$, such that for $c < c_{11}$ and $n \geq N_R$,*

$$\sum_{l=1}^{cn} P_x\left(S_1 + \ldots + S_l \geq n\right) \leq f(n). \tag{2.27}$$

**Proof.** We start with the $d = 2, 3$ case. By Condition 4, as $N \geq N_R$ the largest trap we encounter in $N$ steps of the time changed walk, started at $x$, is bounded above by $N^{1/(1-\phi)}$. We therefore see that

$$\begin{aligned}\sum_{l=1}^{N} P_x\left(S_1 + \ldots + S_l \geq n\right) &\leq NP_x\left(S_1 + \ldots + S_N \geq n\right) \\ &\leq NP_x\left(Z_1 + \ldots + Z_N \geq n\right)\end{aligned}$$

where $Z_i$ are exit times from a trap of size $N^{1/(1-\phi)}$. By (2.19) we have

$$E\left(\exp\left(\theta Z_1\right)\right) \leq 1 + \frac{c_{2.4.2}\theta N^{1/(1-\phi)}}{c_{2.4.3}N^{-2/(1-\phi)} - \theta}$$

for $\theta < \frac{c_{2.4.3}}{N^{2/(1-\phi)}}$. Hence, taking $\theta = \frac{c_{2.4.3}}{2N^{2/(1-\phi)}}$, by Markov's inequality we see

$$\begin{aligned}P_x\left(Z_1 + \ldots + Z_N \geq n\right) &\leq \exp\left(-\frac{c_{2.4.3}}{2N^{2/(1-\phi)}}n\right)\left(cN^{1/(1-\phi)}\right)^N \\ &\leq \exp\left[-\left(\frac{c_{2.4.3}}{2N^{2/(1-\phi)}}n - N\log c - \frac{N}{1-\phi}\log N\right)\right].\end{aligned}$$

Equation (2.26) now follows by taking $n \geq N^{1+\frac{3}{1-\phi}}$.

We now move to the $d \geq 4$ case.

The idea is the following: to spend a large amount of time in traps, the walker must spend a large amount of time in a particular type of trap. However, since all traps of a particular type are spread out, the walker cannot move quickly between traps and so can only spend large amounts of time in traps of a particular type by remaining in traps for large periods of time. This has a small probability of occurring. We now make this rigorous.

Let $Y_i^{m,r}$ be iid random variables corresponding to exit times from a trap of type $(m, r)$. From Proposition 21

$$\mathbb{P}\left[Y_1^{m,r} + \ldots + Y_l^{m,r} \geq n\right] \leq \exp\left(-\frac{c_{2.4.7}}{m^3 r}n\right)\left(c_{2.4.8}r + c_{2.4.9}r^2\right)^l.$$

In particular, for $\delta = \delta(m, r)$ and $\varepsilon = \varepsilon(m, r)$

$$\begin{aligned}&\mathbb{P}\left[Y_1^{m,r} + \ldots + Y_{\delta n}^{m,r} \geq \varepsilon n\right] \\ \leq\;& \exp\left(-\frac{c_{2.4.7}\varepsilon n}{m^3 r}\right)\left(c_{2.4.8}r + c_{2.4.9}r^2\right)^{\delta n} \\ =\;& \exp\left[-n\left(\frac{c_{2.4.7}\varepsilon}{rm^3} - \delta\log\left(c_{2.4.8}r\left(1 + c'r\right)\right)\right)\right].\end{aligned} \tag{2.28}$$

Now, note that if we take values of $\varepsilon(m,r)$ such that

$$\sum_{m,r} \varepsilon(m,r) \le 1,$$

then

$$\begin{aligned}\sum_{l=1}^{cn} P_x\left[S_1 + \ldots + S_l \ge n\right] &\le cnP_x\left[S_1 + \ldots + S_{cn} \ge n\right] \\ &\le cn \sum_{m,r} P_x\left[S_1^{m,r} + \ldots + S_{cn}^{m,r} \ge \varepsilon(m,r)\, n\right]\end{aligned}$$

where $S_l^{m,r} = 1_{\left\{T\left(X_{S_l}\right)=(m,r)\right\}} S_l$.

Hence if we take

$$\varepsilon(m,r) = \frac{36}{\pi^4 r^2 m^2}$$

and define $Y_i^{m,r}$ to be iid random variables corresponding to exit times from a trap of type $(m,r)$ as above, then:

$$\begin{aligned}\sum_{l=1}^{cn} P_x\left[S_1 + \ldots + S_l \ge n\right] &\le cnP_x\left[S_1 + \ldots + S_{cn} \ge n\right] \\ &\le cn \sum_{m=1}^{n^\alpha} \sum_{r=2}^{(\log n)^\theta} P_x\left[S_1^{m,r} + \ldots + S_{cn}^{m,r} \ge \varepsilon(m,r)\, n\right].\end{aligned}$$

By using equation (2.28) and Condition 8, for $n \ge N_R$ we have:

$$\begin{aligned}&\sum_{l=1}^{cn} P_x\left[S_1 + \ldots + S_l \ge n\right] \\ \le\quad & cn \sum_{m=1}^{n^\alpha} \sum_{r=2}^{(\log n)^\theta} P_0\left[Y_1^{m,r} + \ldots + Y^{m,r}_{cC_7 n\left[\exp\left(-r^\vartheta\right)\wedge m^{-\gamma}\right]} \ge \varepsilon(m,r)\, n\right]\end{aligned}$$

and

$$\begin{aligned}& cn \sum_{m=1}^{n^\alpha} \sum_{r=2}^{(\log n)^\theta} P_0\left[Y_1^{m,r} + \ldots + Y^{m,r}_{cC_7 n\left[\exp\left(-r^\vartheta\right)\wedge m^{-\gamma}\right]} \ge \varepsilon(m,r)\, n\right] \\ \le\ & cn \sum_{m=1}^{n^\alpha} \sum_{r=2}^{(\log n)^\theta} \exp\left[-n\left(\frac{6c_{2.4.7}}{\pi^2 r^3 m^5} - cC_7\left[\exp\left(-r^\vartheta\right) \wedge m^{-\gamma}\right] \log\left(c_{2.4.8} r\left(1 + c'r\right)\right)\right)\right] \\ \le\ & cn \sum_{m=1}^{n^\alpha} \sum_{r=2}^{(\log n)^\theta} \exp\left[-n\left(\frac{c'}{r^3 m^5}\right)\right] \\ \le\ & cn^{1+\alpha} (\log n)^\theta \exp\left(-\frac{c'n}{(\log n)^\theta n^{5\alpha}}\right),\end{aligned}$$

where the penultimate line requires $c$ to be sufficiently small and $\gamma > 5$.

Thus, as $\alpha < \frac{1}{5}$ and $\gamma > 5$ we have the required decay. This proves equation (2.27). ■

### 2.5.3. Proof of Theorem 10

We are now able to prove the standard upper bound for the on diagonal heat kernel decay for the random walk on graphs that satisfy the conditions of Definition 9. The control on the time spent in traps enables the extension of the ideas of [11] to:

**Theorem 31.** *Suppose that $B_{x_1}[R]$ is good. Then there exists $c_{12}, c_{13} < \infty$ such that for $x \in B_{x_1}[2R/5] \cap \mathcal{C}_\infty^k$ and*

$$q_{2n}(x,x) \leq c_{12} n^{-d/2} \tag{2.29}$$

*for all $c_{13} N_R^{\upsilon(d)} \leq n \leq c_5 \frac{R^2}{2\log R}$ where*

$$\nu(d) := \begin{cases} (d+1)(1+\xi) \vee \frac{4}{1-\phi} & \text{for } d = 2,3 \\ (d+1)(1+\xi) & \text{for } d \geq 4 \end{cases}.$$

**Proof.** Define

$$R_m := \sup\{l \geq 0 : S_1 + \ldots + S_l \leq m\}.$$

Then for fixed $l$:

$$\sum_{m \geq n} P_x[X_m = x, R_m = l] = P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_l \geq n\right).$$

Now,

$$\begin{aligned} \sum_{n \leq m < 2n} P^m(x,x) &= \sum_{n \leq m < 2n} \sum_{l=1}^{2n} P_x(X_m = x, R_m = l) \\ &= \sum_{l=1}^{2n} \sum_{n \leq m < 2n} P_x(X_m = x, R_m = l) \\ &\leq \sum_{l=1}^{2n} P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_l \geq n\right). \end{aligned} \tag{2.30}$$

For $d = 2,3$, by Lemma 29 and (2.26), defining $N := 2N_R^{(d+1)(1+\xi)} \vee N_R^{\frac{4}{1-\phi}}$, we see that for $N < 2n \leq c_5 \frac{R^2}{\log R}$, there exists $c$ such that

$$\begin{aligned} \sum_{n \leq m < 2n} P^m(x,x) &\leq \sum_{l=1}^{N} P_x(S_1 + \ldots + S_l \geq n) \\ &\quad + \sum_{l=N}^{2n} P_x\left(\tilde{X}_l = 0, S_1 + \ldots + S_l \geq n\right) \\ &\leq c_9 \exp\left(-c_{10} n^{1-\frac{3}{4-\phi}}\right) + \sum_{l=N}^{2n} \frac{c_8 l^{1-d/2}}{n} \pi(x) \\ &\leq c\pi(x) n^{1-d/2}. \end{aligned}$$

In the case $d \geq 4$ we split the sum in equation (2.30) slightly differently: take $c \leq c_{11}$ then

$$\begin{aligned} & \sum_{n \leq m < 2n} P^m(x,x) \\ \leq & \sum_{l=1}^{cn} P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_l \geq n\right) \\ & + \sum_{l=cn}^{2n} P_x\left(\tilde{X}_l = x, S_1 + \ldots + S_l \geq n\right). \end{aligned} \tag{2.31}$$

The first term of (2.31) can be bounded using equation (2.27) and the second by using Lemma 29. Hence if $N \leq cn$ and $2n \leq c_5 \frac{R^2}{\log R}$ then

$$\sum_{n \leq m < 2n} P^m(x,x) \leq f(n) + \pi(x) \sum_{l=cn}^{2n} \frac{c_8 l^{1-d/2}}{n}.$$

Now, there exists $c' = c'(d,c) < \infty$ such that

$$\sum_{l=cn}^{2n} l^{1-d/2} \leq c' n^{2-d/2}.$$

Hence, as $f(n) = o\left(n^{-d/2+1}\right)$, for $d \geq 2$ and $n \geq N$

$$\sum_{n \leq m < 2n} q_m(x,x) \leq c n^{1-d/2}.$$

Since $p_{2m}(x,x)$ is decreasing in $m$, the sum on the left is bounded below by $\frac{1}{2} n q_{2n}(x,x)$. Dividing through by $n$ then gives the claimed result. ■

**Proof of Theorem 10.** By the Cauchy-Schwarz inequality

$$q_{2n}(x,y)^2 \leq q_{2n}(x,x)\, q_{2n}(y,y),$$

and hence

$$q_{2n}(x,y) \leq c_{12} n^{-d/2}$$

for $x, y \in B_{x_1}[2R/5] \cap \mathcal{C}_\infty^k$ and $c_{13} N_R^{\nu(d)} \leq n \leq c_5 \frac{R^2}{2 \log R}$.

For $n = 2l+1$ we see

$$\begin{aligned} q_n(x,y) & \leq \sum_z \frac{\pi(z)}{\pi(y)} q_{2l}(x,z)\, p_1(z,y) \\ & \leq \sum_z \frac{\omega_{zy}}{\pi(y)} c_{12} l^{-d/2} \\ & \leq c_{12} n^{-d/2}, \end{aligned}$$

provided $x, y \in B[R/5]$ and $c_{13} N_R^{\nu(d)} \leq l \leq c_5 \frac{R^2}{2 \log R}$.

We can now extend the on diagonal bound (2.29) to vertices in traps. Take $x \in T$ for some trap $T$. By Condition 4, we have $\pi(T) \leq R_0(x)^{\phi}$.

Fix $m \in \mathbb{N}$. Define

$$S_T := \inf\{n > 0 : X_n \notin T\}.$$

On the event $\{S_T < m\}$ define

$$\tilde{S}_T := \sup\{n \leq m : X_n \notin T\}.$$

$S_T$ is the first exit time from $T$ and $\tilde{S}_T$ will be the last entrance time if the walk returns to $x$. Conditioning on $S_T$ and $\tilde{S}_T$:

$$\begin{aligned}
P_x(X_m = x) \leq & \sum_{n,l \leq cm} \sum_{y,z \in \partial_{ext} T} P\left(X_m = x \middle| S_T = n, X_{S_T} = y, m - \tilde{S}_T = l, X_{\tilde{S}_T} = z\right) \\
& P_x\left(S_T = n, X_{S_T} = y, m - \tilde{S}_T = l, X_{\tilde{S}_T} = z\right) \\
& + P_x\left(\{S_T > cm\} \cup \left\{m - \tilde{S}_T > cm\right\}\right) \\
\leq & \sup_{n,l \leq cm} \sup_{y,z \in \partial_{ext} T} P_y(X_{m-n-l} = z) \\
& + P_x(S_T > cm) + P_x\left(m - \tilde{S}_T > cm\right) \\
\leq & K(m(1-2c))^{-d/2} + P_x(S_T > cm) + P_x\left(m - \tilde{S}_T > cm\right).
\end{aligned}$$

From Section 2.4 (and reversibility) we know that the final two terms decay exponentially quickly, dependent on $\pi(T)$. More precisely, equation (2.18) says that for $m \geq \pi(T)^3$ we have

$$P_x(S_T > cm) \leq c_1 m^{1/3} \exp\left(-c_2 m^{1/3}\right).$$

Since $\pi(T) \leq R_0(x)^{\phi}$, the choice of $\nu$ implies that $m \geq \pi(T)^3$ whenever $m \geq 2N_R^{\nu}$. Thus, dividing through by $\pi(x)$ and altering the constant $c$ we have

$$q_m(x,x) \leq cm^{-d/2}$$

for all $c_{13} N_R^{\nu(d)} \leq m \leq c_5 \frac{R^2}{2\log R}$.

Applying the Cauchy-Schwarz and parity arguments outlined earlier we have:

$$q_n(x,y) \leq cn^{-d/2}$$

for all $x, y \in \mathcal{G}$ and $c_{13} N_R^{\nu(d)} \leq n \leq c_5 \frac{R^2}{2\log R}$.

This completes the proof. However, in cases where it is known that the long range bounds stated in (2.4) hold, one can obtain full off-diagonal upper bounds by appealing to Theorem 12, with Condition 6 providing the necessary volume bounds. More, precisely, apply Theorem 12 with

$$\alpha := \begin{cases} \xi \wedge \frac{1}{3} & d = 2,3 \\ \xi & \text{for } d \geq 4 \end{cases}.$$

■

# 3. Continuum percolation

## 3.1. The model

In this chapter we consider the simple random walk on the infinite component of supercritical continuum percolation. We will prove that the heat kernel associated with this walk is bounded above by the standard Gaussian bounds and that non-degenerate Brownian motion is the walk's scaling limit. We begin by formally introducing continuum percolation.

For $d \geq 2$, take $\mathcal{B}^d$ to be the Borel sets in $\mathbb{R}^d$ and let $\Omega$ be the set of counting measures on $\mathcal{B}^d$ which assign finite measure to bounded Borel sets and for which the measure of a point is at most one. We take the natural $\sigma$-algebra $\mathcal{N}$, generated by sets of the form

$$\{n \in \Omega : n(A) = k\}, \quad k \in \mathbb{N} \cup \{0\}, \ A \in \mathcal{B}^d.$$

Let $\mathbb{P} = \mathbb{P}_\lambda$ be the measure on $(\Omega, \mathcal{N})$ corresponding to a Poisson point process (PPP) of intensity $\lambda$ on $\mathbb{R}^d$ (see, for example, [36]). We write $\mathcal{H}_\lambda(\omega)$ for a realization of the PPP and take $\mathcal{H}_\lambda(\omega)$ to be the vertex set of our graph. We define the edge set to be

$$\mathcal{E}_\lambda(\omega, r) := \left\{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : x, y \in \mathcal{H}_\lambda(\omega), \ 0 < |x - y| \leq r\right\},$$

and define the graph $\mathcal{G}_{\lambda,r}(\omega) = (\mathcal{H}_\lambda(\omega); r) := (\mathcal{H}_\lambda(\omega), \mathcal{E}_\lambda(\omega, r))$. We work exclusively with $r = 1$ and write $\mathcal{G}_\lambda$, suppressing the dependence on $r$.

In the language of continuum percolation, this is the discrete analogue of the Boolean model with fixed radii: taking the centres of the spheres as vertex set and joining vertices if their respective spheres overlap. Continuum percolation has been studied for many years (see [42] for a survey of continuum percolation and [46] for a review from the graphical perspective) and it has been shown to share many of the geometric properties of bond percolation on $\mathbb{Z}^d$, the most basic of which is that there exists a critical intensity, $\lambda_c = \lambda_c(d)$, above which there almost surely exists a unique infinite connected component of the graph $\mathcal{G}_\lambda$ and below which there almost surely does not. We work in the supercritical case, taking $\Omega_1 \subset \Omega$ such that $\mathbb{P}(\Omega_1) = 1$ and for every $\omega \in \Omega_1$ there exists a unique infinite component of $\mathcal{G}_\lambda(\omega)$, written $\mathcal{C}_\infty(\omega)$.

For $\omega \in \Omega_1$, define the simple random walk on $\mathcal{C}_\infty(\omega)$ to be the Markov chain $(X_n)_{n\geq 0} = (X_n(\omega))_{n\geq 0}$ with transition probabilities

$$P^\omega\left(X_n = y \middle| X_{n-1} = x\right) = \begin{cases} \frac{1}{\pi(x)} & \text{if } (x,y) \in \mathcal{E}_\lambda \\ 0 & \text{otherwise} \end{cases}, \tag{3.1}$$

where for $x \in \mathcal{C}_\infty(\omega)$

$$\pi(x) = \pi_\omega(x) := \left|\{y \in \mathcal{C}_\infty(\omega) : |x-y| \leq 1\}\right|.$$

As previously, define the heat kernel to be

$$q_n^\omega(x,y) = \frac{P^\omega\left(X_n = y \middle| X_0 = x\right)}{\pi(y)}.$$

In this chapter we will prove the following upper bound on the long range decay of the heat kernel.

**Theorem 32.** *There exist $C_i = C_i(d,\lambda)$ such that for almost every $\omega \in \Omega$, there exist constants $\{N_\omega(x) : x \in \mathcal{C}_\infty(\omega)\}$ such that for every $x, y \in \mathcal{C}_\infty(\omega)$*

$$q_n^\omega(x,y) \leq C_1 n^{-d/2} \exp\left(-\frac{C_2|x-y|^2}{n}\right) \tag{3.2}$$

*for all $n \geq N_\omega(x)$.*

The proof of Theorem 32 will employ Theorems 10 and 12 of Chapter 2. As the edge weights are exactly equal to one in this example, the results of [25] imply the long range bounds of equation (2.4). Hence it will be sufficient to prove that the conditions of Definition 9 are satisfied for the box $B_x[R]$ for all $x \in \mathbb{R}^d$ and sufficiently large $R$. These conditions will be proven in Section 3.2. Note that, unlike [4], we give no bounds on the tail behaviour of $N_\omega(x)$. This is due to the difficulty controlling the tail behaviour for the random variable controlling the spread of the traps.

The second result of this chapter concerns the quenched scaling limit of $X_n$. In order for the walk to have a distinct starting vertex we modify the environment by adding the origin to the vertex set and connect the origin to any edges within unit distance. It is straightforward to show that the infinite component of this augmented graph will contain the origin with positive probability. We will prove that for almost every environment where the origin is contained in the infinite component, the random walk started at the origin weakly converges to Brownian motion.

To be more precise, the vertex set of the augmented graph is $\mathcal{H}_\lambda(\omega) \cup \{0\}$. This is the superposition of a Poisson point process with the point process $\delta_0$, where $\delta_0$ is the counting measure with $\delta_0(A) = 1$ if $0 \in A$ and zero otherwise. Define

$$\mathbb{P}_0 := \mathbb{P} * \delta_0,$$

where $*$ denoting superposition of point processes. Then $\mathbb{P}_0$ is the distribution for the vertex set of the augmented graph.

We shall see in Section 3.3 that $\mathbb{P}_0$ is equivalent to the so called Palm distribution for $\mathbb{P}$ - the probability measure 'conditioned' on the origin being contained in $\mathcal{H}_\lambda$. This alternative 'conditioning' viewpoint will be important when proving ergodic properties for the environment viewed from the particle.

Introduce the measure $\mathbb{P}'(.) := \mathbb{P}_0(.|\, 0 \in \mathcal{C}_\infty)$. Writing $\Omega_0 = \Omega_1 \cap \{0 \in \mathcal{C}_\infty\}$ and modifying $\mathcal{N}$ to $\mathcal{N}_0$ in the obvious way, we consider the triple $(\Omega_0, \mathcal{N}_0, \mathbb{P}')$, a measure on graphs whose infinite component contains the origin. Let $(X_n^0)_{n\geq 0}$ be the Markov chain with $X_0^0 = 0$ and transition probabilities given by (3.1). Then we prove the following invariance principle.

Let $(C[0,T], \mathcal{W}_T)$ be the space of continuous functions $f : [0,T] \to \mathbb{R}^d$ with Borel $\sigma$-algebra defined with respect to the supremum topology.

**Theorem 33.** *Take $d \geq 2$, $\lambda > \lambda_c(d)$. Define the scaled, interpolated random walk*

$$B_n(t) := \frac{1}{\sqrt{n}}\left(X^0_{\lfloor tn\rfloor} + (tn - \lfloor tn\rfloor)\left(X^0_{\lfloor tn\rfloor+1} - X^0_{\lfloor tn\rfloor}\right)\right),\ t \geq 0.$$

*Then for every $T > 0$ and $\mathbb{P}'$-almost every $\omega \in \Omega_0$, the law of $(B_n(t) : 0 \leq t \leq T)$ converges weakly on $(C[0,T], \mathcal{W}_T)$ as $n \to \infty$ to an isotropic Brownian motion whose diffusion constant $D(\lambda, d) > 0$, is strictly positive and depends only on $\lambda$ and $d$.*

**Remark 34.** *The annealed version of Theorem 33 will come for free in the course of proving Theorem 33 due to the work of [27].*

**Remark 35.** *Note that Theorem 33 gives an on diagonal lower bound for (3.2) of order $n^{-d/2}$. This is detailed in Remark 2.2 of [13].*

To obtain the invariance principle we follow the corrector method, used, for example, in the proof of the invariance principle for the simple random walk on supercritical bond percolation in [10]. We define a corrector, $\chi : \mathcal{H}_\lambda \times \Omega_0 \to \mathbb{R}^d$ such that $X_n + \chi(X_n, \omega)$ is a martingale. We then check criterion for weak convergence of this martingale to a non-degenerate Brownian motion. Finally we show that $\chi$ is sublinear and hence when we subtract the effect of $\chi$ from the martingale our conclusion is not altered.

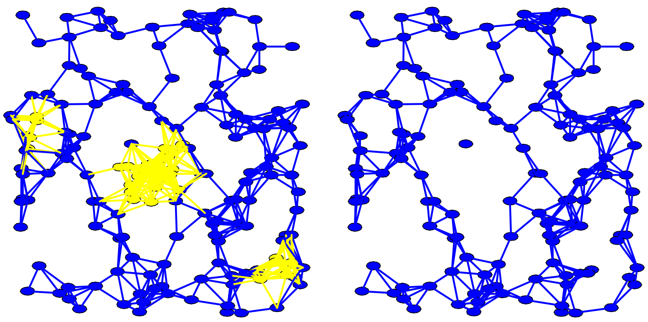

Figure 3.1: The unthinned graph (with points of high degree highlighted) and its thinned counterpart

This Chapter is structured as follows. Section 3.2 contains the combinatorial arguments required to analyze the environment, proving Theorem 32. The necessary ergodicity properties of the environment as viewed from the position of the random walk are proven in Section 3.3. These proofs are somewhat more involved than in the case of bond percolation [10] or the random conductor model [13] due to the lack of a square lattice. We present the necessary modifications. The chapter concludes with the proof of Theorem 33 in Section 3.4, where the existence and sublinearity of the corrector function described above is proven.

## 3.2. Geometry of the environment

In this section we present the combinatorial arguments necessary to prove that the conditions laid out in Section 2.2 hold for continuum percolation. We will begin by defining the thinned graph and proving its existence and uniqueness - the methods employed are standard [32]. We then prove that the thinned graph satisfies the necessary isoperimetric profile by adapting the methods of [11] to the continuum percolation setting. The size and spread of traps is then considered - we show that traps can be bounded using the same methods as would normally be employed to bound the second largest component of percolation. The volume bounds are then obtained via standard properties of a Poisson point process. Finally we adapt the methods of [2] to compare thinned graph distance with Euclidean distance.

Before we begin the section proper, we state a result that we will use repeatedly: let $\left(X_z : z \in \mathbb{Z}^d\right)$ be a family of Bernoulli random variables. For $l \in \mathbb{N}$, we say that $\left(X_z : z \in \mathbb{Z}^d\right)$ is $l$-dependent if $X_z$ is independent of $X_y$ whenever $|y - z| \geq l$ (we call this an $l$-dependent random field).

Given two families $\left(X_z : z \in \mathbb{Z}^d\right)$, $\left(Y_z : z \in \mathbb{Z}^d\right)$ of Bernoulli random variables we say that $X$ stochastically dominates $Y$ (written $X \geq_{st} Y$) if

$$\mathbb{E}\left[f\left(X\right)\right] \geq \mathbb{E}\left[f\left(Y\right)\right]$$

for all bounded, increasing, measurable functions $f : \{0,1\}^{\mathbb{Z}^d} \to \mathbb{R}$.

The following theorem is a weak version of the main theorem of [37].

**Theorem 36 (Liggett, Schonmann and Stacey).** *Fix $d$, $l \geq 1$ and take $p \in (0,1)$. Suppose $\left(X_z : z \in \mathbb{Z}^d\right)$ is an $l$-dependent family of Bernoulli random variables with $\mathbb{P}\left[X_z = 1\right] \geq p$ for all $z \in \mathbb{Z}^d$. Then there exists a non-decreasing function $\pi : [0,1] \to [0,1]$ with $\pi\left(\delta\right) \to 1$ as $\delta \to 1$, such that*

$$\left(X_z : z \in \mathbb{Z}^d\right) \geq_{st} \left(Y_z^{\pi(p)} : z \in \mathbb{Z}^d\right),$$

*where $\left(Y_z^{\pi(p)} : z \in \mathbb{Z}^d\right)$ is a family of independent Bernoulli $\pi\left(p\right)$ random variables.*

Applications of this result normally go as follows: $X_z$ are taken to be $\{0,1\}$ events dependent on the graph in a bounded region contained in $B_{2Rz}\left[R\right]$ for some $R > 0$, with the events $X_{z_1}$ and $X_{z_2}$ being independent provided $|z_1 - z_2| > l$. One then shows that $\mathbb{P}\left(X_z = 1\right)$ gets close to one as $R$ gets large. By taking $R$ sufficiently large Theorem 36 then says that $X_z$ stochastically dominates supercritical percolation and we appeal to some known property of supercritical percolation to prove a property for the original graph. These renormalization techniques will be key to our progress.

### 3.2.1. Existence and uniqueness of the infinite component for the thinned graph

Our first result shows that if $\lambda > \lambda_c\left(d\right)$ then there exists $k_0 = k_0\left(d, \lambda\right) < \infty$ such that for any $k \geq k_0$, if all points of degree greater than $k$ are removed then with probability one the remaining graph contains a unique infinite component.

Define the thinned graph $\mathcal{G}_{\lambda,k}\left(\omega\right)$ to be the subgraph of $\mathcal{G}_\lambda\left(\omega\right)$ with vertex set

$$\mathcal{V}_k = \{x \in \mathcal{H}_\lambda : \pi\left(x\right) < k\}$$

and edge set

$$\mathcal{E}_k = \{e = (x,y) \in \mathcal{E}_\lambda : x, y \in \mathcal{V}_k\}.$$

We begin by introducing some notation. Write

$$B_x\left[s\right] := x + \left[-s, s\right]^d,$$

for the box centred at $x \in \mathbb{R}^d$ of sidelength $2s$. For a cube $Q = B_x\left[s\right]$ define

$$\tilde{Q} = \tilde{B}_x\left[s\right] := x + \left[-3s, 3s\right]^d.$$

We define a disjoint tiling of $\mathbb{R}^d$ by cubes in the following manner: for $x \in \mathbb{Z}^d$ and $s > 0$, let

$$T_s(x) := B_{2sx}\left[s\right].$$

Take $Q$ to be a cube in $\mathbb{R}^d$. For a connected component $\mathcal{C}$ contained in $Q$ we say that $\mathcal{C}$ is crossing for $Q$ if each face of $Q$ is within unit distance of at least one vertex of $\mathcal{C}$. For a subcube $Q' \subset Q$ we say that $\mathcal{C}$ is crossing for $Q'$ if there exists a connected component of $\mathcal{C} \cap Q'$ that is crossing for $Q'$ in the sense outlined previously.

Now, for a cube $Q$ of side $s$, let $R_0^k(Q)$ be the event:

$$R_0^k(Q) := \begin{array}{l} \{\text{there exists a unique crossing cluster } \mathcal{C} \text{ in } \tilde{Q} \text{ for } \tilde{Q}, \text{ all open paths} \\ \text{contained in } \tilde{Q} \text{ of diameter greater than } \frac{s}{8} \text{ are connected to } \mathcal{C} \text{ in } \tilde{Q}, \\ \mathcal{C} \text{ is crossing for each cube } Q' \subset Q \text{ such that } s(Q') \geq \frac{s}{8}, \text{ and for} \\ \text{every } y \in \tilde{Q} \text{ we have } \deg y \leq k\ \}, \end{array}$$

where $s\left(Q'\right)$ is the sidelength of $Q'$ and diameter is with respect to Euclidean distance. Let $\mathcal{C}^{\vee}\left(Q\right)$ be the cluster contained in $\mathcal{G}_\lambda^k\left(\omega\right) \cap Q$ with the largest number of vertices and define the event

$$R^k(Q) = R_0^k(Q) \cap \left\{\mathcal{C}^{\vee}\left(Q\right) \text{ is crossing for } Q\right\} \cap \left\{\mathcal{C}^{\vee}(\tilde{Q}) \text{ is crossing for } \tilde{Q}\right\}.$$

We now set $\phi_{k,s}(x) := 1_{R^k(T_s(x))}$ and show that this process dominates supercritical Bernoulli site percolation when $k$ and $s$ are sufficiently large.

**Proposition 37.** *For $\lambda > \lambda_c$ there exists $k_0\left(\lambda, d\right)$ such that for $k > k_0$, the graph $\mathcal{G}_{\lambda,k}$ contains an infinite component $\mathbb{P}$-almost surely.*

**Proof.** From [46], Chapter 10, we know that the probability that there is either no unique crossing component for $Q$ or that there exists some other component of metric diameter greater than $\frac{s}{8}$ within $Q$ (we could equally replace $\frac{s}{8}$ with any other increasing function $f(s) \leq s$ such that $\frac{f(s)}{\log s} \to \infty$) decays exponentially in the sidelength of $Q$. So for $s(Q) > 16$ the probability that $Q$ and $\tilde{Q}$ are not both crossed by the same cluster decays exponentially in $s$. Thus, if we consider the set $\{A_1, \ldots, A_j\}$ ($j$ is of order $s^{d+1}$) of all subcubes of $Q$ with integer corners and sidelengths greater than or equal to $\frac{s}{8}$, we see that the probability that there fails to be a unique crossing of every $A_i$ and $\tilde{A}_i$ decays exponentially in $s$. Considering the way two neighbouring cubes overlap we see that this extends to all subcubes of side greater than or equal to $\frac{s}{8}$. Further, note that for fixed $Q$ we can make $\mathbb{P}\left\{\exists y \in Q : \deg y > k\right\}$ arbitrarily small by taking $k$ large. Hence we can take a sequence $k\left(s\right)$ such that

$$\mathbb{P}(\phi_{k(s),s}(x) = 1) \to 1 \text{ as } s \to \infty. \tag{3.3}$$

Now, by Theorem 36 and the fact that the random variables $\phi(x), \phi(y)$ are independent if $|x-y| \geq 4$, we can stochastically dominate Bernoulli site percolation of parameter $\pi(s)$ by $\left(\phi_{k(s),s}(x); x \in \mathbb{Z}^d\right)$, where $\pi(s) \to 1$ as $\mathbb{P}(\phi_{k(s),s}(x) = 1) \to 1$. Thus, if we take $s$ large then we dominate a supercritical site percolation and hence there exists a unique infinite cluster in $\left(\phi_{k(s),s}(x); x \in \mathbb{Z}^d\right)$ with probability one.

If $x \sim y$ in $\mathbb{Z}^d$ and $\phi_{k,s}(x) = \phi_{k,s}(y) = 1$, then the cube $T_s(x)$ has a crossing component, which must be of length at least $s$. However, $T_s(x) \subset \widetilde{T_s(y)}$ and $T_s(x) \subset \widetilde{T_s(x)}$ and so by the definition of $R_0^k$ we must have that the crossing component of $T_s(x)$ is connected to crossing components of both $\widetilde{T_s(y)}$ and $\widetilde{T_s(x)}$. So on the event $R^k$, any neighbours in $\mathbb{Z}^d$ with $\phi_{k,s} = 1$ have the property that their largest components are crossing and are connected to each other. Hence, taking $s$ large so that $\pi(s) > p_c(d)$, $\left(\phi_{k(s),s}(x)\right)_{x \in \mathbb{Z}^d}$ stochastically dominates supercritical site percolation and hence an infinite connected cluster exists in $\mathcal{G}_{\lambda,k}$. ■

This is a standard method to show the existence of an infinite component; it is used, for example, in [4], [46] and [42].

In the work that follows we assume several times that $k$ is suitably large, for example we at times need to know that the probability that a box is crossed by the thinned graph is suitably large. However, we do not always mention explicitly that we require $k$ to be large; it is assumed that $k$ has already been taken sufficiently large.

We now show that the thinned infinite component is unique. We do this following the methods introduced in [20], that are described, for example, in [42] and [32]. Since we know that the unthinned graph will almost surely have a unique infinite component, $\mathcal{C}_\infty$, and the thinned graph will almost surely have an infinite component, pick one and call it $\mathcal{C}_\infty^k$, defining traps to be components of $\mathcal{C}_\infty \backslash \mathcal{C}_\infty^k$, then proving that all traps are almost surely finite will suffice.

We do this in two steps: the first step shows that the number of traps of infinite cardinality is almost surely either zero or infinite and the second step rules out the possibility that it is infinite.

**Theorem 38.** *For $\lambda > \lambda_c(d)$ there exists $k_0(\lambda, d)$ such that for $k > k_0$ the graph $\mathcal{G}_{\lambda,k}$ contains a unique infinite component $\mathbb{P}$-almost surely. Further, all traps are finite almost surely.*

**Proof.** Let $N$ be the number of infinite traps. By ergodicity it follows that $N$ is almost surely constant, say $N = b$ with probability one. We first assume that $1 \leq b < \infty$ and look for a contradiction so to reduce the possibilities to $b \in \{0, \infty\}$.

Now, we have already shown that there almost surely exists an infinite component with bounded degree, $\mathcal{C}_\infty^k$. Hence, choose $n$ sufficiently large such that the box $B_0[n]$ intersects both an infinite trap, $T$, and $\mathcal{C}_\infty^k$ with strictly positive probability. Assume $k_0$ is taken sufficiently large that with probability one the points of degree greater than $k_0$ do not percolate. Let $\varepsilon > 0$. We can choose $n$ large such that the following event holds with strictly positive probability:

$$F = \left\{ \begin{array}{c} B_0[n] \text{ intersects both } \mathcal{C}_\infty^k \text{ and } T, \text{ there exists } y \in T - B_0[n], \\ x \in \mathcal{C}_\infty^k - B_0[n] \text{ with } \deg x, \deg y < k - c \text{ and } |x - \partial B_0[n]|, |y - \partial B_0[n]| < \varepsilon \end{array} \right\},$$

where $c = c(d) < k$ is some constant we will choose shortly. Note that on the event $\{N = b\}$, $F$ is dependent only on $B_0[n]^c$.

We now introduce an event on the interior of the box $B_0[n]$. Let $a < \frac{1}{2\sqrt{d}}$ and split $B_0[n]$ into disjoint subcubes of sidelength $a$ (making the obvious corrections at the edges), call these $B_1, \ldots, B_l$ where $l = O(n^d)$. Define the event

$$E = \{|B_i \cap \mathcal{H}_\lambda| = 1 \text{ for all } i\}.$$

Then $\mathbb{P}[E] > 0$. Further, on the event $E$, the interior of $B_0[n]$ is exactly one connected component, as if two points lie in neighbouring subcubes then they are within unit distance of each other and are thus connected. It is also clear that the degree of any point within $B_0[n]$ is bounded by some constant, take this to be $c - 1$.

Since the event $E$ depends on the configuration within the box $B_0[n]$ and the event $F$ depends on the configuration outside the box, the events are independent and hence

$$\mathbb{P}[E \cap F] = \mathbb{P}[E]\,\mathbb{P}[F] > 0.$$

If $\varepsilon$ is sufficiently small, with both $E$ and $F$ holding then we have a path of points of degree less than $k$ that connects $T$ to $C_\infty^k$. This contradicts the definition of a trap. Hence $N \in \{0, \infty\}$ almost surely.

It suffices now to rule out the infinite case. We suppose $N = \infty$ a.s. and look for a contradiction via the methods of [20]. We consider only the infinite traps and ignore $C_\infty^k$ for this part of the proof. If we take $n$ large enough, there is positive probability that $B_0[n]^c$ contains at least three distinct unbounded traps that hit $\partial B_0[n]$. By considering a suitable set of configurations on the interior of $B_0[n]$ we find that the following event has probability $\nu > 0$, say, of occurring:

$$A(n) := \left\{ \begin{array}{c} \exists \text{ unbounded trap } T' \text{ such that } T' \cap B_0[n]^c \text{ contains at} \\ \text{least three unbounded traps} \end{array} \right\}.$$

Call the three unbounded traps in $T' \cap B_0[n]^c$ branches. Take $K$ large and choose $M$ such that the following event has probability at least $\frac{\nu}{2}$:

$$\begin{aligned} &A(n, M) := \\ &\quad A(n) \cap \{\text{all branches of } B_0[n] \text{ contain at least } K \text{ points in } B_0[Mn] \backslash B_0[n]\}. \end{aligned}$$

For $z \in \mathbb{Z}^d$ define the events $A^z(n)$, $A^z(n,M)$ by translating $A(n)$, $A(n,M)$ over the vector $z$. For large $L$ (chosen later), let $R$ be the set:

$$R = R(L) := \left\{ z \in \mathbb{Z}^d : B_{nz}[Mn] \subset B_0[Ln], A^{nz}(n,M) \text{ occurs} \right\}.$$

Since $\mathbb{P}[A^z(n,M)]$ is constant for all $z \in \mathbb{Z}^d$, we have:

$$\mathbb{E}|R| = \mathbb{P}[A(n,M)] \left| \left\{ z \in \mathbb{Z}^d : B_{nz}[Mn] \subset B_0[Ln] \right\} \right| \geq \eta L^d \tag{3.4}$$

for some $\eta > 0$ and large enough $L$.

For $z \in \mathbb{Z}^d$, take $C_z^1, C_z^2, C_z^3$ to be the points in three of the branches of $z$ contained in $B_{nz}[Mn] - B_{nz}[n]$. Then $C_z^i \cap C_z^j = \phi$ for $i \neq j$, and $|C_z^i| \geq K$ for all $i$. For $z \in R$ we identify $z$ with a point in $B_{nz}[n] \cap T'$ (which exists by the definition of $A^{nz}(n,M)$). We now apply Proposition 39, stated at the end of this proof and taken from Lemma 3.2 of [42]. We need only check condition (b). Take distinct $z, z' \in R$. If $C_z$ and $C_{z'}$ are in different components of $B_0[Ln]$ then (i) holds and (ii) holds otherwise. Thus, writing $|B_0[Ln]|$ for the number of points contained in the cube $B_0[Ln]$, the Proposition and (3.4) give:

$$\mathbb{E}|B[Ln]| \geq K\left(\eta L^d + 2\right).$$

Note that trivially $\mathbb{E}|B[Ln]| = \lambda(LN)^d$, implying that for large enough $L$

$$K\left(\eta L^d + 2\right) \leq \lambda(Ln)^d,$$

this gives a contradiction when $K$ is large. ■

**Proposition 39.** *Let $S$ be a set and $R$ a nonempty finite subset of $S$. Suppose:*

*(a) $\forall r \in R$, we have a family $(C_r^1, C_r^2, C_r^3)$ of disjoint nonempty subsets of $S$, not containing $r$, and $|C_r^i| \geq K$ for all $r$ and $i$.*

*(b) $\forall r, r' \in R$, one of the following events occur, with $C_r = \cup_{i=1}^3 C_r^i$ :*

*(i) $(\{r\} \cup C_r) \cap (\{r'\} \cup C_{r'}) = \phi$*

*(ii) $\exists i, j$ such that $C_r^i \supset \{r'\} \cup C_{r'} \backslash C_{r'}^j$ and $C_{r'}^j \supset \{r\} \cup C_r \backslash C_r^i$.*

*Then $|S| \geq K(|R| + 2) + |R|$.*

### 3.2.2. Isoperimetric profile

We demonstrate that, using an adaptation of the methods of Berger et al in [11], we can compute bounds on the isoperimetric profile of the thinned graph. Recall the definition of the isoperimetric profile from Definition 19 and the definition of $\tilde{B}_x[R] := B_x[3R]$.

We say that the event $G_M(x)$ holds if the following are satisfied

1. For each neighbour $y$ of $x$ (with respect to the square lattice), the side of $B_{2My}[M]$ adjacent to $B_{2Mx}[M]$ is connected to the opposite side of $B_{2My}[M]$ by a path.

2. Any two paths connecting $B_{2Mx}[M]$ to $\partial\tilde{B}_{2Mx}[M]$ are connected to each other by a path contained entirely in $\tilde{B}_{2Mx}[M]$.

From the proof of Proposition 37, we know that the probability of $G_M(x)$ holding can be taken as close to one as we require by considering large $M$ and $k$. We use Theorem 36 to stochastically dominate supercritical site percolation by $\left\{G_M(x) : x \in \mathbb{Z}^d\right\}$.

For finite discrete sets $\Lambda \subseteq \mathbb{R}^d$, let

$$\Lambda^M := \left\{x \in \mathbb{Z}^d : \Lambda \cap B_{2Mx}[M] \neq \phi\right\},$$

and take $\bar{\Lambda}^M$ to be the complement of the unique infinite component of $\mathbb{Z}^d - \Lambda^M$. For a realization, $\mathcal{C}_\infty^k(\omega)$, of the thinned graph and a subset $A \subseteq \mathcal{C}_\infty^k(\omega)$ we define the $\omega$-edge boundary of $A$ by

$$\partial^\omega(A) := \left\{e = xy : y \in \mathcal{C}_\infty^k(\omega) - A,\ x \in A \text{ with } |x-y| \leq 1\right\}.$$

Finally, for sets $\triangle \subseteq \mathbb{Z}^d$, define their internal vertex boundary by

$$\partial_i(\triangle) := \left\{x \in \triangle : \exists y \in \mathbb{Z}^d - \triangle \text{ with } |x-y| = 1\right\}.$$

Note that, since edges in continuum percolation can go from within $B_0[1]$ to $B_{\mathbf{1}}[1]$ (where $\mathbf{1} = (1, \ldots, 1)$), if $\Lambda$ is a connected subset of the graph it does not follow that $\bar{\Lambda}^M$ is connected in $\mathbb{Z}^d$ under the standard relation $x \sim y$ iff $|x-y| = 1$. We do, however, have that $\bar{\Lambda}^M$ is $*$-connected in $\mathbb{Z}^d$ ($x, y$ being $*$-connected iff $|x-y|_\infty = 1$). See [32] and [46] for a discussion of $*$-connectedness.

Now we are in a position to attack the isoperimetric profile of the thinned graph. Set $\triangle := \bar{\Lambda}^M$. The proof will show that off an event of small probability, for suitably large $\Lambda$, we have $|\partial^\omega\Lambda| > c|\partial_i\triangle|$. Also, since we are working on the thinned graph there can be at most $K$ points of the thinned graph in a square of unit side. Hence $|\triangle| \geq K|\Lambda|$. Putting these two comments together, since $\triangle$ must satisfy the standard isoperimetry in $\mathbb{Z}^d$, off a set of small probability we get the required profile for our thinned graph. We show we have good control over this probability which then gives us the isoperimetric profile.

**Lemma 40.** *For $\omega \in \Omega$, let $\Lambda \subseteq \mathcal{C}_\infty^k$ be connected and finite with* $\operatorname{diam}\Lambda \geq 3M+1$. *Then*

$$|\partial^\omega\Lambda| < \frac{1}{2 \times 3^d}|\partial_i\triangle| \tag{3.5}$$

*implies that*

$$|\{x \in \partial_i\triangle : G_M(x)^c\}| > \frac{1}{2}|\partial_i\triangle|.$$

**Proof.** First note that $x \in \partial_i \triangle$ implies that $x \in \Lambda^M$ and hence $\Lambda \cap B_{2Mx}(M) \neq \phi$. The result will follow from the following claim for $x \in \partial_i \triangle$:

$$G_M(x) \subseteq \left\{ \tilde{B}_{2Mx}(M) \text{ contains an edge of } \partial^\omega \Lambda \right\}. \tag{3.6}$$

Suppose that $G_M(x)$ occurs. Since $\operatorname{diam} \Lambda \geq 3M + 1$ and $\Lambda$ is $\omega$-connected, there must be a path, $\gamma_1$, in $\mathcal{G}^k_\lambda(\omega) \cap \Lambda$ connecting $B_{2Mx}(M)$ to $\partial \tilde{B}_{2Mx}(M)$ (as above, we use 'connecting' to mean a path that slightly overshoots). Take $y \notin \triangle$, neighbouring $x$. Then by condition (1) on $G_M(x)$ there is a path, $\gamma_2$, in $\mathcal{G}^k_\lambda(\omega)$ that crosses $B_{2My}(M)$. This is a path from $B_{2Mx}(M)$ to $\partial \tilde{B}_{2Mx}(M)$ and hence by condition 2 must be connected to $\gamma_1$ by a path contained in $\tilde{B}_{2Mx}(M)$. Since $y \notin \triangle$, we see that $\gamma_2 \cap \Lambda = \phi$. So $\gamma_1 \subseteq \Lambda$, $\gamma_2 \cap \Lambda = \phi$ and hence the path that links the two must contain an edge of $\partial^\omega \Lambda$ and we know this path lies in $\tilde{B}_{2Mx}(M)$. The claim is proven.

Now, an edge can lie in at most $3^d$ cubes of type $\tilde{B}_{2Mx}(M)$ and so by (3.6) the number of sites in $\partial_i \triangle$ at which $G_M$ can occur is bounded by $3^d |\partial^\omega \Lambda|$. Hence

$$|\partial_i \triangle| - |\{x \in \partial_i \triangle : G^c_M \text{ occurs}\}| \leq 3^d |\partial^\omega \Lambda|$$

from which the result follows from assumption (3.5). ∎

**Proposition 41.** *For $d \geq 2$, $\lambda > \lambda_c$ there exist constants $c_i$, $\zeta \in (0, \infty)$ such that for all $t > 0$*

$$\mathbb{P}\left( \exists \Lambda,\ |\Lambda - 0| \leq 1, \omega\text{-connected},\ |\Lambda| \geq t^{\frac{d}{d-1}},\ |\partial^\omega \Lambda| < c_1 |\Lambda|^{\frac{d-1}{d}} \right) \leq c_2 e^{-\zeta t}$$

*(where $|\Lambda - x| \leq 1$ means that there exists $y \in \Lambda$ with $|x - y| \leq 1$).*

**Proof.** Take $K = K(k, d)$ to be the maximum number of points of degree less than or equal to $k$ that can fit into a unit box in $d$ dimensions. Define $c = (2 \times 3^d)^{-1}$. Fix $\triangle \subseteq \mathbb{Z}^d$, $*$-connected with connected complement. Suppose that $\Lambda$ is $\omega$-connected with $\bar{\Lambda}^M = \triangle$. Then $|\triangle| \geq (KM)^{-d} |\Lambda|$ and by the standard isoperimetry on $\mathbb{Z}^d$ we see, for some $c' = c'(d) > 0$:

$$|\partial_i \triangle| \geq c' |\triangle|^{\frac{d-1}{d}} \geq c' (KM)^{1-d} |\Lambda|^{\frac{d-1}{d}}.$$

Setting $c_1 = cc' (MK)^{1-d}$ we have

$$\left\{ |\partial^\omega \Lambda| < c_1 |\Lambda|^{\frac{d-1}{d}} \right\} \subseteq \{ |\partial^\omega \Lambda| < c |\partial_i \triangle| \}. \tag{3.7}$$

Further, we see that $|\partial_i \triangle| \geq c' (KM)^{1-d} t$ whenever $|\Lambda| \geq t^{\frac{d}{d-1}}$. To apply Lemma 40 we need $\Lambda$ to have large enough diameter and hence we suppose that

$t^{\frac{d}{d-1}} \geq (3KM)^d$. Using (3.7), Lemma 40 and the stochastic domination of product measure of parameter $1 - \varepsilon_M$ we see

$$\begin{aligned}
& \mathbb{P}\left(\exists \Lambda,\ |\Lambda - 0| \leq 1,\ \omega\text{-connected},\ |\Lambda| \geq t^{\frac{d-1}{d}},\ \bar{\Lambda}^M = \triangle,\ |\partial^\omega \Lambda| < c_1 |\Lambda|^{\frac{d-1}{d}}\right) \\
\leq\ & \mathbb{P}\left(\exists \Lambda,\ |\Lambda - 0| \leq 1,\ \omega\text{-connected},\ |\Lambda| \geq t^{\frac{d-1}{d}},\ \bar{\Lambda}^M = \triangle,\ |\partial^\omega \Lambda| < c\,|\partial_i \triangle|\right) \\
& \leq \mathbb{P}\left(\sum_{x \in \partial_i \triangle} 1_{G_M(x)} \leq \frac{1}{2} |\partial_i \triangle|\right) \qquad (3.8) \\
& \leq 2^{|\partial_i \triangle|} (\varepsilon_M)^{\frac{1}{2}|\partial_i \triangle|}. \qquad (3.9)
\end{aligned}$$

We now need to sum over all possible $\triangle$'s.

Note that if $M \geq 2$ then $|\Lambda - 0| \leq 1$ implies $0 \in \triangle$. Let $\alpha = \alpha(d)$ be the number such that $\alpha^n$ bounds the number of $*$-connected sets $\triangle \subseteq \mathbb{Z}^d$ with connected complement containing the origin and having $|\partial_i \triangle| = n$ (the exponential bound is proved in [46]). Since $\varepsilon_M \to 0$ as $M \to \infty$, we can choose $M$ such that $2\alpha\sqrt{\varepsilon_M} < \frac{1}{2}$. Hence, summing (3.8) over all $\triangle$ with connected complement,

$$\begin{aligned}
& \mathbb{P}\left(\exists \Lambda,\ |\Lambda - 0| \leq 1,\ \omega\text{-connected},\ |\Lambda| \geq t^{\frac{d-1}{d}},\ |\partial^\omega \Lambda| < c_1 |\Lambda|^{\frac{d-1}{d}}\right) \\
\leq\ & \sum_{n \geq c'(NM)^{1-d} t} 2^n (\varepsilon_M)^{n/2} \alpha^n \leq \sum_{n \geq c'(NM)^{1-d} t} 2^{-n} \leq 2^{1 - \lfloor c'(KM)^{1-d} t \rfloor},
\end{aligned}$$

giving the claimed result. ■

**Theorem 42.** *Take $\psi \in (0,1)$. For all $d \geq 2$ and $\lambda > \lambda_c(d)$, there are positive, finite constants $c_i(d, \lambda, k, \psi)$ and an almost surely finite random variable $R_0' = R_0'(\omega)$ such that for $r \geq R_0'$ and each $\omega$-connected $\Lambda$ with*

$$\Lambda \subseteq \mathcal{C}_\infty^k \cap [-r, r]^d \text{ and } |\Lambda| \geq r^\psi$$

*we have*

$$|\partial^\omega \Lambda| \geq c_1 |\Lambda|^{\frac{d-1}{d}}.$$

*Further,*

$$\mathbb{P}(R_0' \geq r) \leq c_2 e^{-c_3 r^\psi}.$$

**Proof.** By Proposition 41 and translation invariance, the probability that there exists a set $\Lambda \subseteq \mathbb{Z}^d \cap [-r, r]^d$ that is $\omega$-connected with $|\Lambda| \geq t^{\frac{d}{d-1}}$ and $|\partial^\omega \Lambda| < c_1 |\Lambda|^{\frac{d-1}{d}}$ is bounded by a constant times $r^d e^{-\zeta t}$. Take $t = r^\psi$, then this probability is summable in $r$ and hence by the Borel-Cantelli lemma this event occurs for only finitely many $r$. In fact,

$$\mathbb{P}(R_0' \geq r) \leq \sum_{R=r}^{\infty} R^d e^{-\zeta R^\psi} \leq c_2 e^{-c_3 r^\psi}.$$

■

By translation invariance, the above holds for any $x \in \mathbb{Z}^d$ and hence we have good control over $R'_0$. This, together with Corollary 44 below proves that Condition 4 is satisfied.

### 3.2.3. Size and spread of traps

We now look to bound the size of traps seen in a box of side $r$, centred at $x$ as $r$ gets large. Our methods follow those used to gain control over the second largest component in percolation. In the case of the second largest component, in order for a set to be large but not connected to the infinite component its boundary must be very badly connected. This is unlikely in the supercritical case and hence it is unlikely for the second largest component to be large. Analogously, for a trap to be large the boundary has to be a combination of very badly connected regions and regions full of high density points. We show that this is also unlikely. The combinatorial methods presented are standard and can be found in, for example, [32] and [46]. We modify them in the necessary ways.

Recall that for $x \in \mathcal{C}_\infty^k$ we define the trap at $x$ to be $T_x$, the connected component of $\left(\mathcal{C}_\infty - \mathcal{C}_\infty^k\right) \cup \{x\}$ containing $x$.

**Proposition 43.** *Let $\lambda > \lambda_c$. There exists a set $N$ of $\mathbb{P}$-measure zero such that for all $\omega \in \Omega - N$ there exists $c_1(\omega) < \infty$ such that the number of points of the largest trap in $\mathcal{G}_\lambda(\omega) \cap [-s,s]^d$ is bounded above by $c_1(\omega)(\log s)^{d/(d-1)}$.*

**Proof.** Let $W_s$ be the number of points in the box $B[s]$ that are part of a trap of size greater than $c(\log s)^{d/(d-1)}$. We want to show quick decay for $\mathbb{P}[W_s \geq 1]$. For $x \in \mathbb{R}^d$, write $T_x(\omega)$ for the trap at $x$ in the graph $G(\mathcal{H}_\lambda(\omega) \cup \{x\};1)$ if it exists, and take $T_x(\omega) = \phi$ otherwise. By Markov's inequality, $\mathbb{P}[W_s \geq 1] \leq \mathbb{E}[W_s]$, and by Theorem 56, to be introduced in Section 3.3.1,

$$\begin{aligned}\mathbb{E}[W_s] &= \lambda \int_{B(s)} \mathbb{P}_x\left[|T_x| \geq c(\log s)^{d/(d-1)}\right] dx \\ &= \lambda s^d \mathbb{P}_0\left[|T_0| \geq c(\log s)^{d/(d-1)}\right]. \end{aligned} \tag{3.10}$$

Here, $\mathbb{P}_x = \mathbb{P} * \delta_x$ is the superposition of a Poisson point process with the point at $x$.

We introduce the random field $\left(X_z; z \in \mathbb{Z}^d\right)$ in a similar fashion to the proof of Proposition 37 but with slightly larger boxes. Let $B_z := B_{2Mz}[M]$, $B_z^+ := B_{2Mz}[5M]$. For $z \in \mathbb{Z}^d$, we set $X_z = 1$ iff

1. there exists a path in $G(\mathcal{H}_\lambda \cap B_z; 1)$ that is crossing for $B_z$,

2. for every $z' \in \mathbb{Z}^d$ with $\|z'-z\|_\infty \leq 2$, there is exactly one component of $G\left(\mathcal{H}_\lambda \cap B^+_{z'};1\right)$ of metric diameter at least $\frac{M}{3}$,

3. $B_z$ fails to contain any point of degree higher than $k$.

For $l \geq 12$, $\left(X_z; z \in \mathbb{Z}^d\right)$ is an $l$-dependent random field. Fix $\delta > 0$. By Theorem 10.9 of [46], we can choose $M_\delta$ s.t. $\forall M \geq M_\delta$, $\mathbb{P}\left[(1) \text{ and } (2) \text{ hold}\right] > 1 - \frac{\delta}{2}$ and we can take $k(\delta, M)$ sufficiently large so that condition (3) fails with probability less than $\frac{\delta}{2}$. Choosing such $M$ and $k$ we have $\mathbb{P}\left[X_z = 1\right] > 1-\delta$. Hence, by Theorem 36, by taking $M$ sufficiently large we can take $p$ as close as we wish to 1 such that

$$\left(X_z; z \in \mathbb{Z}^d\right) \geq_{sd} \left(Z^p_z; z \in \mathbb{Z}^d\right),$$

where the second family is standard Bernoulli site percolation.

We let $\mathcal{T}_x$ be the set of $y \in \mathbb{Z}^d$ such that the cube $B_y$ contains at least one vertex of $T_x$. Then $\mathcal{T}_x$ is $*$-connected, where $x \overset{*}{\sim} y$ iff $|x-y|_\infty \leq 1$ for $x, y \in \mathbb{Z}^d$. We define the external vertex boundary of $\mathcal{T}_x$ by

$$\partial_{ext}\mathcal{T}_x = \left\{y \in \mathbb{Z}^d - \mathcal{T}_x : |x-y| = 1\right\}.$$

Due to translation invariance of the underlying Poisson point process, we only need consider $\mathcal{T}_0$. We first show that if $\mathcal{T}_0$ is sufficiently large then all the points of $\partial_{ext}\mathcal{T}_0$ must take value 0 in our random field.

Suppose $|\mathcal{T}_0| > 3^d$ and that there is some $z \in \partial_{ext}\mathcal{T}_0$ with $X_z = 1$. Then there would be a crossing component for $B_z$ and also a vertex $w \in \mathcal{T}_0$ with $\|w-z\|_\infty \leq 1$. Then there would be a crossing component for $B_z$ and a part of $T_0$ contained in $B^+_z$, both of metric diameter greater than $\frac{M}{3}$. The second condition would then imply that these components would be connected, implying that $T_0 \cap B^+_z \neq \phi$, a contradiction.

Hence clusters in $\left\{z \in \mathbb{Z}^d : X_z = 1\right\}$ are either contained in $\mathcal{T}_0$ or disjoint from $\mathcal{T}_0$. For a finite set $A \subseteq \mathbb{Z}^d$, write $\bar{A}$ for the complement of the unique infinite cluster of $\mathbb{Z}^d - A$. The external boundary $\partial_{ext}\bar{\mathcal{T}}_0$ is $*$-connected due to Lemma 96 of [46], which in turn uses a result of Kesten. Applying the isoperimetric inequality for $*$-connected subsets of $\mathbb{Z}^d$ (see [46] or [32]) we have:

$$|\partial_{ext}\mathcal{T}_0| \geq \beta\, |\mathcal{T}_0|^{(d-1)/d}, \tag{3.11}$$

for $\beta = (2d)^{-1}\left(1 - \left(\frac{2}{3}\right)^{1/d}\right) > 0$.

Let $\mathcal{A}_{m,s}$ denote the collection of $*$-connected subsets of $B_0[s] \cap \mathbb{Z}^d$ of cardinality $m$. If $|\mathcal{T}_0| \geq (\log s)^{d/(d-1)}$ then by (3.11) there exists a set $A \in \mathcal{A}_{m,s}$ such that $X_z = 0$ for all $z \in A$, for some $m \geq \beta \log s$. We thus have

$$\mathbb{P}\left[(\log s)^{\frac{d}{d-1}} \leq |\mathcal{T}_0|\right] \leq \mathbb{P}\left[\bigcup_{m \geq \beta \log s} \bigcup_{\sigma \in \mathcal{A}_{m,s}} \left\{X_z = 0, \forall z \in \sigma\right\}\right]. \tag{3.12}$$

The cardinality $|\mathcal{A}_{m,s}|$ is bounded by $s^d \gamma^m$, with $\gamma := 2^{3^d}$. Hence, by equation (3.12)

$$\begin{aligned}
\mathbb{P}\left[|\mathcal{T}_0| \geq (\log s)^{d/(d-1)}\right] &\leq s^d \sum_{m \geq \beta \log s} \gamma^m (1-p)^m \\
&\leq s^d (\gamma (1-p))^{\beta \log s} \\
&= s^d s^{\beta \log(\gamma(1-p))} \leq s^{-2-d} \qquad (3.13)
\end{aligned}$$

where the last two lines hold if we choose our $p = p(d)$ such that

$$\beta \log \left(2^{3^d} (1-p)\right) < -(2d+2).$$

To complete the proof it suffices to show that the probability of the event

$$H := \left\{|T_0| \geq c (\log s)^{d/(d-1)}\right\} \cap \left\{|\mathcal{T}_0| \leq (\log s)^{d/(d-1)}\right\}$$

is sufficiently small for large enough $c$.

First note that we have at most $\rho^{(\log s)^{d/(d-1)}}$ subsets of $\mathbb{Z}^d$ that are $*$-connected, of cardinality $(\log s)^{d/(d-1)}$ and containing the origin (see [46], Chapter 10 for a proof). If $H$ occurs then one of these subsets must contain more than $c(\log s)^{d/(d-1)}$ vertices. Writing $Y_\gamma$ for a Poisson random variable of parameter $\gamma$, for sufficiently large $c$ we have:

$$\begin{aligned}
&\mathbb{P}\left[\left\{|T_0| \geq c (\log s)^{d/(d-1)}\right\} \cap \left\{|\mathcal{T}_0| \leq (\log s)^{d/(d-1)}\right\}\right] \\
\leq\ & \rho^{(\log s)^{d/(d-1)}} \mathbb{P}\left[Y_{nM_0^d \lambda} \geq c (\log s)^{d/(d-1)}\right] \qquad (3.14) \\
\leq\ & \rho^{(\log s)^{d/(d-1)}} \exp\left(-\frac{c (\log s)^{d/(d-1)}}{2} \log\left(\frac{c}{M_0^d \lambda}\right)\right), \qquad (3.15)
\end{aligned}$$

where the last line comes from the observation that if $a \geq e^2 \lambda$ then

$$\mathbb{P}[Y_\lambda \geq a] \leq \exp\left(-\frac{a}{2} \log \frac{a}{\lambda}\right).$$

Hence by (3.10), (3.13) and (3.15) we see that

$$\begin{aligned}
&\mathbb{P}[W_s \geq 1] \\
\leq\ & \lambda \theta s^d \left(2 M_0^{-d} s^{-2-d} + \rho^{(\log s)^{d/(d-1)}} \exp\left(-\frac{c(\log s)^{d/(d-1)}}{2} \log\left(\frac{c}{M_0^d \lambda}\right)\right)\right).
\end{aligned}$$

This is summable over $s \in \mathbb{N}$, hence the Borel-Cantelli lemma tells us that the event that the largest trap in $B[s]$ is bigger than $c(\log s)^{d/(d-1)}$ holds for only finitely many $s \in \mathbb{N}$ with probability one, and this is enough to complete the proof.
■

In particular, the following weaker bounds hold:

**Corollary 44.** *Take $\phi > 0$. If $\lambda > \lambda_c$ and $k > k_0$, then with probability one, there exists $\left\{R_0''(x) < \infty : x \in \mathbb{Z}^d\right\}$ such that for all $r > R_0''(x)$ and $y \in B_x[r]$*

$$\pi(T_y) \leq r^{\phi}.$$

*Further, there exist constants $c_i = c_i(d, \lambda, k, \phi)$ such that*

$$\mathbb{P}\left[R_0'' > r\right] \leq c_2 \exp\left(-r^{c_3}\right).$$

Recall the definition of $R_0'$ in Theorem 42. Define $R_0 := R_0' \vee R_0''$ then Condition 4 holds with respect to $R_0$. For notational simplicity fix the constants $\psi$ and $\phi$ in Condition 4 to be suitably small and equal from now on. One can do this due to the exponential decay observed above and in the proof of the isoperimetric profile.

Note that the macroscopic arguments of the proof of Proposition 43 are all local arguments and hence provide a way to localize the probability of a trap: employing the above method, the event that there is a trap at $x \in \mathcal{C}_\infty^k$ of size $m$ can be bounded using macroscopic balls that depend only on the environment within $B_x[M(m+12)]$, where $M$ is chosen sufficiently large in the proof. Abusing notation by changing the value of $M$ we have the following:

**Corollary 45.** *There exists $M$ such that if $(\omega_e)_{e \in B_x[Mm]^c}$ is any environment off the set $B_x[Mm]$, then*

$$\mathbb{P}\left(|T_y| = m \text{ for some } y \in B_x[1] \,\middle|\, (\omega_e)_{e \in B_x[Mm]^c}\right) \leq c_2 \exp\left(-m^{c_3}\right).$$

As the arguments of Section 3.2.2 also follow local macroscopic arguments we also have:

**Corollary 46.** *Take $\phi \in (0,1)$. There exists $M$ such that if $(\omega_e)_{e \in B_x[Mm]^c}$ is any environment off the set $B_x[Mm]$, then*

$$\begin{aligned} &\mathbb{P}\left(\exists \Lambda \subseteq \mathcal{C}_\infty^k, B_x[1] \cap \Lambda \neq \phi \text{ with } |\Lambda| = r^{\phi} \text{ and } |\partial^{\omega}\Lambda| < c_1 |\Lambda|^{(d-1)/d} \,\middle|\, (\omega_e)_{e \in B_x[Mr]^c}\right) \\ \leq\ & c_4 \exp\left(-r^{c_5}\right). \end{aligned}$$

Recall the definition of a trap of type $(m, r)$ from Definition 7. Note in particular that the $r$ is dependent on $R_0(x)$ which depends on the full graph and not just the graph locally around $x$. This presents a problem as for $x, y \in \mathbb{R}^d$ the variables $R_0(x)$ and $R_0(y)$ will be correlated. In turn the type of traps $T_x$ and $T_y$ will also be correlated.

It is much easier to work with local events. With this in mind we will introduce local traps and use Corollaries 45 and 46 to bound correlations.

Say that there is a local trap $T^{loc}(x)$ at $x \in \mathcal{C}_\infty^k$ of type $(m, r)$ if $\pi(T_x) = m$ and $r$ is the largest integer such that there exists connected $\Lambda \subseteq \mathcal{C}_\infty^k$ such that $x \in \Lambda$, $|\Lambda| = r^{\phi}$ and $|\partial_\omega \Lambda| \leq c_1 |\Lambda|^{(d-1)/d}$. From Corollaries 45 and 46 we have:

**Proposition 47.** *There exist $c_i$ and $M$ such that*

$$\mathbb{P}_0\left(T_0^{loc} \text{ is a local trap of type } (m,r) \middle| (\omega_e)_{e\in B_x[M(r\vee m)]^c}\right) \le c_6 \exp\left(-(m\vee r)^{c_7}\right).$$

Suppose that there is a local trap $T^{loc}$ of type $(m,r)$ at $x \in \mathbb{R}^d$. Then the diameter of $T^{loc}$ is bounded above by $m \vee r$. Hence, if $y \in \mathbb{R}^d$ is such that $|x-y| \ge (m\vee r)^{1/\phi} + m \vee r$, then the trap $T^{loc}$ has no effect on $R_0(y)$ since $\left|y - T^{loc}\right| \ge (m \vee r)^{1/\phi}$ and a trap of type $(m,r)$ is admissible at this distance (see Condition 4). With this in mind we introduce a deterministic algorithm that produces a configuration of global traps from a configuration of local ones.

For a configuration of local traps $\left(T^{loc}(x)\right)_{x\in\mathcal{C}_\infty^k}$, define an associated configuration of global traps $\bar{T}$ in the following way: for all $y \in \mathcal{C}_\infty^k$, set

$$\bar{T}(y) := \left(\pi(T_y), (m\vee r)^{1/\phi}\right),$$

where $m, r \in \mathbb{N}$ are the largest numbers such that there exists $x_1, x_2 \in \mathbb{R}^d$ with $T^{loc}(x_1) = (m, r'), T^{loc}(x_2) = (m', r)$ and $|x_i - y| \le 2(m\vee r)^{1/\phi}$ for some $m', r'$.

It is perhaps easier to see this process the other way round - a local trap's influence is spread to all points within distance $(m \vee r)^{1/\phi}$ of the local trap.

**Definition 48.** *Let $(T_1(x))_{x\in\mathcal{C}_\infty^k}, (T_2(x))_{x\in\mathcal{C}_\infty^k}$ be two trap configurations. Say that $T_1$ dominates $T_2$ if whenever $T_2(x)$ is of type $(m,r)$, $T_1(x)$ is of type $(m',r')$ for some $m' \ge m, r' \ge r$.*

**Proposition 49.** *For any graph with local trap configuration $T^{loc}$ and global configuration $T$, $\bar{T}$ dominates $T$.*

**Proof.** Suppose $T_x = (m,r)$ is a global trap. If $R_0(x)$ is decided by the local trap at $x$, then $\bar{T}(x) = (m,r)$.

Suppose instead that $R_0(x)$ is not decided by the local trap at $x$. Then there exists at most two points $y_1, y_2 \in \mathcal{C}_\infty^k$ where the local traps at $y_i$ determine $R_0(x)$. Suppose the local trap at $y_i$ is of type $(m_i, r_i)$. Then by the above reasoning $|x - y_i| \le 2(m_i \vee r_i)^{\phi}$ and hence the local traps $T^{loc}(y_i)$ implies that

$$\bar{T}(x) = \left(\pi(T_x), (m_1 \vee r_1 \vee m_2 \vee r_2)^{1/\phi}\right).$$

By the definition of $\bar{T}$ we have $\bar{T}(x) > T(x)$. ■

From a series of combinatorial identities contained in Appendix A, we obtain the following bound on how many traps of each type the walk sees.

**Proposition 50.** *Take $x \in \mathbb{R}^d$. Take $\beta < (d+1)^{-1}$. There exist constants $c_i, \alpha, \theta'$ and an almost surely finite random variable, $R_3(x)$, such that for $d > 3$:*

*1. For $n \geq R_3(x)$, if the trap $T$ is of type $(m, r)$ and $T \cap B_x[n] \neq \phi$ then $m, r \leq (c_8 \log n)^{\theta'}$.*

*2. For any $(m, r)$ and $n \geq R_3(x)$ the maximum number of traps in a non-self-intersecting path of length $n$ started at $x$ is bounded above by*

$$[c_9 \exp(-(r \vee m)^{c_{10}})]\, n.$$

*Further, $\sup_{x \in B_0[R]} R_3(x) \leq R^\beta$ for sufficiently large $R$.*

**Proof.** We begin by proving the upper bound on the worst type of trap encountered in a box, $B_0[n]$.

Take $\theta' := \frac{1}{c_3}$. Define

$$M_x(n) := \sup\{m : \exists y \in B_x[n] \text{ with } T_y = (m, r) \text{ for some } r\}.$$

Then by Corollary 45

$$\begin{aligned} \mathbb{P}\left(M_x(n) \geq ((d+2)(\log n))^{\theta'}\right) &\leq n^d \sum_{m \geq [(d+2)(\log n)]^{\theta'}} c_2 \exp(-m^{c_3}) \\ &\leq n^d c' \exp - [(d+2)(\log n)]^{\theta' c_3} \\ &= n^d c' n^{-(d+2)} \\ &= c' n^{-2}. \end{aligned}$$

Now, setting $\beta < \frac{1}{d+1}$, $\gamma > \beta^{-2}$, $\kappa := d + \gamma + 1$ and

$$R_3^1(x) := \inf\left\{n \in \mathbb{N} : M_x(m) \leq (\kappa \log m)^{\theta'} \text{ for all } m \geq n\right\}.$$

Then by the above work we have

$$\mathbb{P}\left(R_3^1(x) \geq R\right) \leq c R^{-\gamma}.$$

Hence

$$\mathbb{P}\left(\sup_{x \in B_0[R]} R_3^1(x) \geq R^\beta\right) \leq R^d R^{-\gamma\beta} \leq R^{d - \beta^{-1}},$$

which is summable over $R \in \mathbb{N}$ and hence by the Borel-Cantelli Lemma, there exists an almost surely finite random variable $L_1$ such that for $R \geq L_1$, $\sup_{x \in B_0[R]} R_3^1(x) < R^\beta$.

The same argument applies when replacing $m$ with $r$ with $L_2$ replacing $L_1$.

We now prove the second claim, showing that we have good control over the spread of the traps. We bound the total number of local traps of each type seen in a non self intersecting path of length $n$ by looking to stochastically dominate by a subcritical site percolation model. The claim will then follow from Proposition 49.

Write $\mathbb{Z}^d$ as the disjoint union of $2^d$ subsets in the following way: for each choice of $z \in \left\{(z_1, \ldots, z_d) \in \mathbb{Z}^d : z_i \in \{0,1\}\right\}$, we define the set

$$\mathcal{Z}_z := \left\{x \in \mathbb{Z}^d : x = z + 2y \text{ for some } y \in \mathbb{Z}^d\right\}.$$

There are $2^d$ such sets and we write them as $\mathcal{Z}_1, \ldots, \mathcal{Z}_{2^d}$. It is easy to check that they are disjoint and that their union is $\mathbb{Z}^d$. Further, for $i \in \left\{1, \ldots, 2^d\right\}$ and distinct $x, y \in \mathcal{Z}_i$ we see that $|x-y|_\infty \geq 2$.

For $x \in \mathbb{Z}^d$ associate the box $\mathcal{B}(x) := B_{6M(m\vee r)x}\left[3M(m \vee r)\right] \subseteq \mathbb{R}^d$. Then we split $\mathbb{R}^d$ into $2^d$ disjoint sublattices by defining, for $i \in \left\{1, \ldots, 2^d\right\}$,

$$\mathcal{B}_i := \{\mathcal{B}(x) : x \in \mathcal{Z}_i\}.$$

For distinct $x \in \mathcal{Z}_i$, the probability that $\mathcal{B}(x)$ contains a local trap of type $(m, r)$ given any information about $\mathcal{B}(y_j)$ for a set of $y_j \in \mathcal{Z}_i$ is bounded above by Proposition 47. Hence, defining the process, $Z^i_{m,r}$ on $\mathcal{Z}_i$, by $Z^i_{m,r}(x) = 1$ if $\mathcal{B}(x)$ contains a local trap of type $(m, r)$ and $Z^i_{m,r}(x) = 0$ otherwise, then for any $i \in \left\{1, \ldots, 2^d\right\}$, the process $Z^i_{m,r}$ is stochastically dominated by site percolation, $S^i_{m,r}$ on $\mathcal{Z}_i$, with probability

$$\begin{aligned} p(m,r) : &= \mathbb{P}\left(B_0\left[3M(m \vee r)\right] \text{ contains a local trap of type } (m,r)\right) \\ &\leq 6^d M^d (m \vee r)^d c_6 \exp\left(-(m \vee r)^{c_7}\right). \end{aligned}$$

Let $y \in \mathcal{C}^k_\infty$ and define, for every triple $(m, r, n)$,

$$\mathcal{N}_y(m,r,n) := \sup_\gamma \left\{\sum_{i=1}^n 1_{\left\{T^{loc}_{\gamma_i} = (m,r)\right\}}\right\},$$

where $\gamma$ is a non-self-intersecting path in $\mathcal{C}^k_\infty$ of length $n$ with $\gamma_1 = y$. Also define

$$N^i_y(m,r,n) := \sup_\sigma \left\{\sum_{j=1}^n S^i_{m,r}(\sigma_j)\right\}$$

for $\sigma$ a non-self-intersecting path in $\mathbb{Z}^d$ with $\sigma_1 \in \mathbb{Z}^d$ such that the distance

$$d_\nu(y) := \inf\left\{|x-y| : x \in \mathcal{B}_i(\nu)\right\}$$

is minimized by $\sigma_1$.

In a box of side $6M(m \vee r)$ we can find at most $K\left(6M(m \vee r)\right)^d$ points of $\mathcal{C}^k_\infty$ connected to local traps of type $(m, r)$. Hence

$$\mathcal{N}_y(m,r,n) \leq K 6^d M^d (m \vee r)^d \sum_{i=1}^{2^d} N^i_y(m,r,n).$$

It thus follows that for any $m, r$

$$\begin{aligned} & \mathbb{P}\left(\text{for all } l>0,\ n\geq l \text{ and } y\in B_0\left[l^{\theta}\right],\ \mathcal{N}_y\left(m,r,n\right)\leq n\mu_{m,r}\right) \\ \geq\ & \mathbb{P}\left(\text{for all } i,l>0,\ n\geq l \text{ and } y\in B_0\left[l^{\theta}\right],\ N_y^i\left(m,r,n\right)\leq n\mu'_{m,r}\right), \end{aligned} \quad (3.16)$$

where for $\theta > d+1$

$$\begin{aligned} \mu'_{m,r} &:= 2^6 p\left(m,r\right)^{1/4\theta d} \\ \mu_{m,r} &:= 2^d K\left(6M\left(m\vee r\right)\right)^d \mu'_{m,r}. \end{aligned}$$

By Proposition 110 and (3.16) we have for $\theta > d+1$ and any $m, r$

$$\begin{aligned} & \mathbb{P}\left(\exists\, l>0, n\geq l \text{ and } y\in B_0\left[l^{\theta}\right] \text{ such that } \mathcal{N}_y\left(m,r,n\right)>n\mu_{m,r}\right) \\ \leq\ & 2^d 3\sqrt{p\left(m,r\right)} \\ \leq\ & 2^{4d}M^{d/2}\left(m\vee r\right)^{d/2}\sqrt{c_6}\exp\left(-\left(m\vee r\right)^{c_7/2}\right) \\ \leq\ & 2^{4d}M^{d/2}\left(m+r\right)^{d/2}\sqrt{c_6}\exp\left(-\left(m\vee r\right)^{c_7/2}\right). \end{aligned}$$

Now,

$$\sum_{m=1}^{\infty}\sum_{r=2}^{\infty}\left(m+r\right)^{d/2}\exp\left(-\left(m+r\right)^{c_7/2}\right)<\infty.$$

Hence we conclude that

$$\sum_{m=k}^{\infty}\sum_{r=2}^{\infty}\mathbb{P}\left(\exists l>0, n\geq l \text{ and } y\in B_0\left[l^{\theta}\right] \text{ such that } \mathcal{N}_y\left(m,r,n\right)>n\mu_{m,r}\right)<\infty$$

and hence by Borel-Cantelli $\mathcal{N}_y\left(m,r,n\right)\leq \mu_{m,r}n$ for all $n>\left|y\right|^{1/\theta}$ and $m, r$ sufficiently large, say $m, r\geq R_3^2\left(\omega\right)$.

For any fixed $m, r\leq R_3^2\left(\omega\right)$, the proof of Proposition 110 proves that there exists an almost surely finite random variable $J\left(m,r,\omega\right)$ such that $\mathcal{N}_y\left(m,r,n\right)\leq \mu_{m,r}n$ for all $n>\left|y\right|^{1/\theta}\vee J\left(m,r,\omega\right)$. Setting

$$R_3^3\left(\omega\right):=R_3^2\left(\omega\right)\vee\sup_{m,r\leq R_3^2\left(\omega\right)}J\left(m,r,\omega\right),$$

then for any $y\in\mathcal{C}_\infty^k$, $n\geq R_3^3\left(\omega\right)\vee\left|y\right|^{1/\theta}$ and $m, r\in\mathbb{N}$

$$\mathcal{N}_y\left(m,r,n\right)\leq\left[c_9\exp\left(-\left(r\vee m\right)^{c_{10}}\right)\right]n.$$

Hence the second statement holds and taking $\theta=\beta^{-1}$ we have that for sufficiently large $R$, $\sup_{x\in B_0[R]}R_3\left(x\right)\leq R^{\beta}$. ■

### 3.2.4. Volume control

We now prove that Condition 5 holds. Recall the definition of annuli $A_n = A_n(x) := B_n[x] - B_{n-1}[x]$.

**Proposition 51.** *There exists constants $c_i$ and $\rho = \rho(d,k,\lambda)$ such that for almost every $\omega \in \Omega$ there exist $\left\{R_1(x) < \infty : x \in \mathcal{C}_\infty^k\right\}$ such that for all $y \in \mathcal{C}_\infty^k$ with $|x-y| \geq R_1(x)$ we have*

$$\tilde{d}(x,y) \geq \rho\,|x-y|\,.$$

*Further, $\mathbb{P}(R_1(x) \geq r) \leq c_1 e^{-c_2 r}$.*

*We also have that for almost every $\omega \in \Omega$ and every $x \in \mathbb{Z}^d$:*

$$\tilde{C}_{vol}(x,a) \leq c_3$$

*for all $a \leq R_1(x)^{-1}$.*

**Proof.** The proof of the comparison between Euclidean and thinned graph distance can be found in [13], Lemma 3.1, as can the bounds on the probability of $R_1$ being large.

Note that since $\left|\mathcal{C}_\infty^k \cap B_x[1]\right| \leq K$ for any $x \in \mathbb{R}^d$, we have the standard bound $\left|\mathcal{C}_\infty^k \cap A_n\right| \leq cn^{d-1}$ for all $n$. Hence for $N \geq R_1(x)$

$$\begin{aligned}\sum_y \pi(y)\,e^{-ad(x,y)} &\leq \sum_{y\in B_x[N]} \pi(y)\,e^{-ad(x,y)} + \sum_{y\notin B_x[N]} \pi(y)\,e^{-a\rho|x-y|} \\ &\leq cN^d + ca^{-d},\end{aligned}$$

where the second sum is approximated via annuli. Hence there exists $c_3$ such that if $a \leq R_1(x)^{-1}$ then $\tilde{C}_{vol}(x,a) \leq c_3$. ■

We now approach Condition 6.

**Proposition 52.** *There exists $c_i$, $\left\{R_2(x) < \infty : x \in \mathcal{C}_\infty^k\right\}$ such that*

$$\sum_{y\in A_n\cap\mathcal{C}_\infty^k} \pi(T_y) \leq c_4 n^{d-1},\ \forall n \geq R_2(x) \tag{3.17}$$

$$\sum_{y\in B_x[R_2(x)]\cap\mathcal{C}_k^\infty} \pi(T_y) \leq c_4 R_2(x)^d\,. \tag{3.18}$$

*Further $\mathbb{P}(R_2(x) \geq r) \leq c_5 e^{-r^{c_6}}$*

**Proof.** This proof uses macroscopic arguments as in the proof of Proposition 50.

Recall Corollary 45 that states that there exists $M$ such that

$$\mathbb{P}\left(|T_x| = m \,\middle|\, (\omega_e)_{e\in B_x[Mm]^c}\right) \leq c_{3.2.3.2}\exp\left(-m^{c_{3.2.3.3}}\right). \tag{3.19}$$

Following the proof and notation of Proposition 50, break $\mathbb{Z}^d$ into $2^d$ sublattices $\mathcal{Z}_1, \ldots \mathcal{Z}_{2^d}$. Set $\mathcal{K}_i = \mathcal{Z}_i \cap A_n$, then $|\mathcal{K}_i| = O\left(n^{d-1}\right)$. For $m \in \mathbb{N}$ set

$$\mathcal{B}(x) := B_{Mmx}[Mm] \subseteq \mathbb{R}^d.$$

For $x \in \mathcal{K}_i$ define $Z_m^i(x) = 1$ if $\mathcal{B}(x)$ contains a trap of size $m$ and $Z_m^i(x) = 0$ otherwise. By (3.19),

$$\mathbb{P}\left(Z_m^i(x) = 1 \middle| \left\{Z_m^i(y) = z(y)\right\}_{y \in \mathcal{K}_i - x}\right) \leq c_{3.2.3.2} \exp\left(-m^{c_{3.2.3.3}}\right)$$

for any environment $\{z(y)\}_{y \in \mathcal{K}_i - x}$ off $x$.

Now, on the event $Z_m^i(x) = 1$

$$\sum_{\substack{y \in \mathcal{B}(x) \cap \mathcal{C}_\infty^k \\ \pi(T_y) = m}} \pi(T_y) \leq K M^d m^{d+1}.$$

Hence

$$\sum_{y \in A_n \cap \mathcal{C}_\infty^k} \pi(T_y) \leq K M^d \sum_{i=1}^{2^d} \sum_{x \in \mathcal{K}_i} \sum_{m=k}^{\infty} m^{d+1} Z_m^i(x).$$

As $|\mathcal{K}_i| \leq c n^{d-1}$, for fixed $i$ and $m$, $\sum_{x \in \mathcal{K}_i} Z_m^i(x)$ can be stochastically dominated by a Binomial distribution $X_m :=\mathrm{Bi}\left(c n^{d-1}, c_{3.2.3.2} \exp\left(-m^{c_{3.2.3.3}}\right)\right)$. Hence

$$\mathbb{P}\left(\sum_{y \in A_n \cap \mathcal{C}_\infty^k} \pi(T_y) \geq C n^{d-1}\right) \leq 2^d \sum_{m=k}^{\infty} \mathbb{P}\left(X_m \geq \frac{C n^{d-1} c_m}{2^d K M^d m^{d+1}}\right)$$

where $\sum_m c_m \leq 1$. Taking $c_m = \frac{m^{-2}}{2}$,

$$\mathbb{P}\left(\sum_{y \in A_n \cap \mathcal{C}_\infty^k} \pi(T_y) \geq C n^{d-1}\right) \leq 2^d \sum_{m=k}^{\infty} \mathbb{P}\left(X_m \geq \frac{C n^{d-1}}{2^{d-1} K M^d m^{d+3}}\right).$$

These probabilities can then be controlled by standard large deviations for the Binomial distribution: from [46], we quote Lemma 1.1: if $l \geq e^2 np$ then

$$P(\mathrm{Bi}(n,p) \geq l) \leq \exp\left(-\left(\frac{l}{2}\right) \log\left(\frac{l}{np}\right)\right).$$

Hence,

$$\mathbb{P}\left(X_m \geq \frac{C n^{d-1}}{2^{d-1} K M^d m^{d+3}}\right) \leq \exp\left(-\frac{C n^{d-1}}{2^{d+1} K M^d m^{d+3}} \log \frac{C \exp\left(m^{c_{3.2.3.3}}\right)}{c c_{3.2.3.2} 2^{d-1} K M^d m^{d+3}}\right).$$

There exists $c'$ such that $\log \frac{C \exp(m^{c_{3.2.3.3}})}{c c_{3.2.3.2} 2^{d-1} K M m^{d+3}} \geq c'$ for all $m \geq k$. Further, by Corollary 44 the worst trap in $B_x[n]$ is bounded above by $n^\phi$ for all $n \geq R_0'(x)$. Hence,

$$\begin{aligned}
\mathbb{P}\left(\sum_{y \in A_n \cap \mathcal{C}_\infty^k} \pi(T_y) \geq C n^{d-1}\right) &\leq 2^d \sum_{m=k}^{n^\phi} \exp\left(-\frac{c'' n^{d-1}}{m}\right) + \mathbb{P}\left(R_0'(x) \geq n\right) \\
&\leq 2^d n^\phi \exp\left(-c'' n^{d-1-\phi}\right) + \mathbb{P}\left(R_0'(x) \geq n\right).
\end{aligned}$$

As both terms decay exponentially we are done.

Equation (3.18) follows from similar arguments. ■

The following is required to ensure that the conditions of Theorem 12 are satisfied and hence that full upper bounds on the heat kernel are achieved.

**Proposition 53.** *There exists a deterministic constant $c_7 = c_7(d, \lambda)$ such that for almost every $\omega \in \Omega$ we have*

$$C_{vol}(x, a) \leq c_7$$

*for all $a \leq R_2(x)^{-1}$.*

**Proof.** As we work on the unthinned graph we trivially have

$$d(x, y) \geq |x - y|.$$

Now, for any $N \geq R_2(x)$ we have by Proposition 52

$$\begin{aligned}
\sum_y \pi(y) e^{-ad(x,y)} &\leq \sum_y \pi(y) e^{-a|x-y|} \\
&\leq \pi(B_N) + \sum_{n=N}^{\infty} \sum_{y \in A_n} \pi(y) e^{-a|x-y|} \\
&\leq c_4 N^d + \sum_{n=N}^{\infty} \sum_{y \in A_n} \pi(y) e^{-a(n-1)} \\
&\leq c_4 N^d + \sum_{n=0}^{\infty} c_4 n^{d-1} e^{-an} \\
&\leq c_4 N^d + c_4 a^{-d}.
\end{aligned}$$

Hence, if $a \leq N^{-1}$ then $C_{vol}(x, a) \leq c_7$ for some suitable constant. ■

### 3.2.5. Relating graph distance to Euclidean distance

We now prove a large deviations estimate for the graph distance between two points in the infinite cluster.

For two points $x, y \in \mathcal{C}_\infty$ let the graph distance $D(x, y)$ to be the length of the shortest path connecting $x$ to $y$ within $\mathcal{C}_\infty$. For $x, y \in \mathbb{R}^d$, if there exists $x', y' \in \mathcal{C}_\infty$ such that $x' \in B_x\left[(2d)^{-1/2}\right]$, $y' \in B_y\left[(2d)^{-1/2}\right]$ write $B_x\left[(2d)^{-1/2}\right] \sim B_y\left[(2d)^{-1/2}\right]$ and set $d(x, y) := \inf\{D(x', y') : x'y' \in \mathcal{C}_\infty,\, x' \in B_x[1],\, y' \in B_y[1]\}$. The choice of $(2d)^{-1/2}$ ensures that any two points within $B_x\left[(2d)^{-1/2}\right]$ must be joined by an edge. Thus the difference between $d(x, y)$ and $D(x', y')$ is at most one for points $x' \in B_x[1]$, $y' \in B_y[1]$.

Take $\mathbb{P}'$ to be $\mathbb{P}$ 'conditioned' on the origin belonging to the infinite component - this will be formally introduced in the next Subsection.

**Theorem 54.** *Let $\lambda > \lambda_c(d)$. Then there exists a constant $\rho = \rho(\lambda, d) \in [1, \infty)$ such that*

$$\limsup_{|y|\to\infty} \frac{1}{|y|} \log \mathbb{P}' \left[ B_0 \left[ (2d)^{-1/2} \right] \sim B_y \left[ (2d)^{-1/2} \right],\ d(0,y) > \rho |y| \right] < 0.$$

**Sketch Proof.** The proof runs analogously to Theorem 1.1 of [2], the corresponding result for bond percolation on the square lattice. For completeness we sketch the main ideas of the proof.

Reintroduce the boxes

$$B_x[s] := x + [-s, s]^d,$$

and, for a cube $Q = B_x[s]$, we set

$$\tilde{Q} = \tilde{B}_x[s] := x + [-3s, 3s]^d.$$

For a cube $Q$ of side $s$, define the events:

$$\begin{aligned} R_0(Q) \quad : \quad = \quad & \Big\{ \text{there exists a unique crossing cluster } \mathcal{C} \text{ in } \tilde{Q} \text{ for } \tilde{Q}, \text{ all open paths} \\ & \text{contained in } \tilde{Q} \text{ of diameter greater than } \tfrac{s}{8} \text{ are connected to } \mathcal{C} \text{ in } \tilde{Q}, \\ & \mathcal{C} \text{ is crossing for each cube } Q' \subset Q \text{ such that } s(Q') \geq \tfrac{s}{8} \Big\}, \\ R(Q) \quad = \quad & R_0(Q) \cap \{\mathcal{C}^{\vee}(Q) \text{ is crossing for } Q\} \cap \left\{ \mathcal{C}^{\vee}(\tilde{Q}) \text{ is crossing for } \tilde{Q} \right\}, \end{aligned}$$

where $\mathcal{C}^{\vee}(Q)$ is the largest connected component contained in $Q$.

Define the macroscopic process $\phi(x) := 1_{R(B_{nx}[n])}$ for $x \in \mathbb{Z}^d$. We call $x \in \mathbb{Z}^d$ white if $\phi(x) = 1$ and black otherwise. As before, taking $n$ sufficiently large, we can apply Theorem 36 and stochastically dominate site percolation of parameter $p > p_c(d)$.

From the definition of the event $R$ we see that if $x, y \in \mathbb{Z}^d$ are macroscopic $*$-neighbours ($|x - y|_\infty = 1$) and are both white then the microscopic crossing clusters of their respective cubes must be joined. We can think of the black points as obstacles and look to find a path that avoids them.

Take a black component $A \subseteq Z^d$. Since we are dominating supercritical percolation we know that with probability one $|A| < \infty$. For $A$ finite, we have that

$$\mathbb{Z}^d - A = \Lambda_1 \cup \Lambda_2 \cup \ldots \cup \Lambda_k$$

for disjoint connected (connected, not $*$-connected) components $\Lambda_i$, exactly one of which, $\Lambda_1$ say, is infinite. We define

$$\hat{A} := A \cup \Lambda_2 \cup \ldots \cup \Lambda_k.$$

$\hat{A}$ is $A$ with its holes filled in and set $\bar{A} := \hat{A} \cup \partial_{ext}\hat{A}$.

Now, suppose that $B_0[1]$ is connected to $B_y[1]$ by some path $\gamma$ (not necessarily the shortest path) in $\mathcal{C}_\infty$. In the macroscopic process take $\tilde{y} \in \mathbb{Z}^d$ to correspond to the cube containing $y$ and write $\tilde{0}$ to distinguish the origin in the macroscopic process. Let $(x_0 = \tilde{0}, x_1, \ldots, x_m = \tilde{y})$ be some deterministic path in $\mathbb{Z}^d$ from $\tilde{0}$ to $\tilde{y}$ of minimal length. If $x_i$ is black, set $\bar{A}_{x_i}$ to be the black component containing $x_i$ with its holes filled and unioned with its external neighbours; if $x_i$ is white set $\bar{A}_{x_i} := \{x_i\}$.

We now define a new microscopic path from 0 to $y$. Suppose $\gamma \nsubseteq \bar{A}_{x_0}$. If $x_0$ is black we follow $\gamma$ until it hits the boundary $\partial_{int}\bar{A}_{x_0}$, giving us a path from 0 to a point in $\mathcal{C}_\infty$ contained in a white macroscopic cube, $z_1$, say. If $x_m$ is black then we follow $\gamma$ towards 0, until it hits $\partial_{int}\bar{A}_{x_m}$; again giving us a microscopic path from $y$ to a point in $\mathcal{C}_\infty$ contained in a white cluster, $z_2$, say. Since we have filled in all the holes, we can find a macroscopic $*$-connected white path from $z_1$ to $z_2$ that is contained in

$$W := \partial_{int}\left(\bigcup_{i=0}^{m} \bar{A}_{x_i}\right).$$

By the definition of the events $R$, this allows us to join our two ends of the microscopic path together while remaining in $W$. Clearly if $\gamma \subseteq \bar{A}_{x_0}$ then the path $\gamma$ lies in $W$.

Hence, if the graph distance between $x$ and $y$ is large then $W$ must also be large. This requires large black components, we can control this probability since we are dominating supercritical site percolation. The details of how we obtain this control are presented in [2] and completes the proof. ∎

### 3.2.6. Proof of Theorem 32

We apply Theorem 10, showing that for any $x \in \mathbb{R}^d$, the box $B_x[r]$ is good for all $r \geq R(\omega)$. As commented earlier, we will not give tail estimates for the random variable $R$ due to a lack of control over $R_3$.

Condition 2, referring to the existence of a unique thinned infinite component, is satisfied by Theorem 37.

Take $\beta < (d+1)^{-1}$. In the language of Section 2.2, the random variable $R_0(x)$ is controlled by Theorem 42 and Corollary 44. $R_1(x)$ is controlled by Proposition 51. $R_2(x)$ is controlled by Propositions 52 and 53 . All have exponentially decaying tails. Define

$$L_r(x) := \max_{y \in B_x[r]} \{R_0(y), R_1(y), R_2(y), R_3(y)\}.$$

By the exponential decay mentioned and Proposition 50 (to control $R_3$) there exists a random variable $R(\omega)$ that is almost surely finite and such that $L_R(x) \leq R^\beta$ for $R \geq R(\omega)$.

We can now apply Theorem 10 to obtain the claimed result, with Euclidean distance replacing graph distance due to Theorem 54.

## 3.3. The environment from the point of view of the particle

In this section we prove ergodicity for the Markov chain "on environments". Previously we have considered a fixed environment with a particle moving around within it. Equivalently we could keep the particle fixed and shift the environment around the particle. Thus the random walk $(X_n)_{n\geq 0}$ induces a Markov chain on $\Omega_0$ corresponding to the environment viewed from the perspective of the particle.

For $x \in \mathbb{R}^d$, $\omega \in \Omega$, define $\tau_x \omega(A) = \omega(A+x)$ for $A \in \mathcal{B}^d$ (where $A + x := \{a + x : a \in A\}$). Recall that we write $\mathbb{P}_0$ for the Palm distribution with a point at the origin and $\mathbb{P}'(.) := \mathbb{P}_0(.|\, 0 \in \mathcal{C}_\infty)$. Introduce the weighted measure $\mu$

$$\mu(d\omega) = \frac{1}{\mathbb{E}'[\deg 0]} (\deg_\omega 0)\, \mathbb{P}'[d\omega], \tag{3.20}$$

where $\mathbb{E}'$ is expectation with respect to $\mathbb{P}'$.

The main theorem of this section is:

**Theorem 55.** *Let $f \in L^1(\Omega_0, \mathcal{N}_0, \mathbb{P}')$. Then for $\mathbb{P}'$-almost all $\omega \in \Omega_0$*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f \circ \tau_{X_k}(\omega) = \mathbb{E}_\mu(f), \quad P_{0,\omega}\text{-almost surely.}$$

To prove this requires subtle manipulations with the repeated use of the soon to be introduced Campbell-Mecke Theorem linking the Palm distribution to the initial distribution $\mathbb{P}$.

The section is divided into three parts: section 3.3.1 introduces the Palm distribution formally and details the Campbell-Mecke Theorem. In section 3.3.2 we show the reversibility of the operator $Q : L^2(\Omega_0, \mathcal{B}, \mu) \to L^2(\Omega_0, \mathcal{B}, \mu)$ defined by

$$Qf(\omega) := \frac{1}{\deg_\omega 0} \sum_{x \overset{\omega}{\sim} 0} f(\tau_x \omega),$$

with respect to $\mu$. The final part of the section proves Theorem 55.

### 3.3.1. Point processes and the Palm distribution

For $d \geq 2$, take $\mathcal{B}^d$ to be the Borel sets of $\mathbb{R}^d$ and let $\Omega$ the set of counting measures on $\mathcal{B}^d$ which assign finite measure to bounded Borel sets and for which the measure of a point is at most one. We take the natural $\sigma$-algebra $\mathcal{N}$, generated by sets of the form

$$\{n \in \Omega : n(A) = k\}, \quad k \in \mathbb{N},\, A \in \mathcal{B}^d.$$

A simple point process corresponds to a measure on $(\Omega, \mathcal{N})$ (the word simple refers to the condition that the measure of a point is at most one. We drop the use of this word from now on).

Let $\mathcal{P}$ be a point process on $\mathbb{R}^d$. We wish to condition on there being a point at a fixed location, $x \in \mathbb{R}^d$, of the point process. However, since $x$ being contained in the point process is a null event, we need to define how to do this conditioning.

What properties should one expect from the conditioned process? Firstly, if we are working with a stationary point process we should expect

$$\mathcal{P}\left(Y \,|x \in Y\right) = \mathcal{P}\left(\tau_x Y \,|0 \in Y\right).$$

Further, if we take a measurable function $h : \mathbb{R}^d \times \mathcal{N} \to \mathbb{R}$, and partition $\mathbb{R}^d$ into domains $D_1, D_2, \ldots$ of positive volume, then

$$\mathbb{E}_{\mathcal{P}}\left[\sum_{x\in\omega} h\left(x,\omega\right)\right] = \sum_k \mathbb{E}_{\mathcal{P}}\left[\left.\sum_{x\in\omega\cap D_k} h\left(x,\omega\right)\right| \omega\left(D_k\right) > 0\right] \mathcal{P}\left[\omega\left(D_k\right) > 0\right]. \quad (3.21)$$

Define the intensity measure $\Lambda$ by $\Lambda\left(A\right) = \mathbb{E}_{\mathcal{P}}\left[\omega\left(A\right)\right]$ for $A \in \mathcal{B}^d$. Suppose we can allow the volumes of the $D_k$ to tend down to infinitesimal volume elements $dx$. Then $\mathcal{P}\left[\omega\left(D_k\right) > 0\right]$ should converge to $\Lambda\left(dx\right)$ and the conditional expectation should tend to what we hope to define: $\mathbb{E}_{\mathcal{P}}\left[h\left(x,\omega\right)\right| x \in \omega]$. Thus (3.21) suggests a relation of the form:

$$\mathbb{E}_{\mathcal{P}}\left[\sum_{x\in\omega} h\left(x,\omega\right)\right] = \int_{\mathbb{R}^d} \mathbb{E}_{\mathcal{P}}\left[h\left(x,\omega\right)\right| x \in \omega] \Lambda\left(dx\right).$$

In the homogeneous Poisson point process case, stationarity and the fact that $\Lambda = \lambda v_d$ for $v_d$ Lebesgue measure, the above equation becomes

$$\mathbb{E}\left[\sum_{x\in\omega} h\left(x,\omega\right)\right] = \lambda \int_{\mathbb{R}^d} \mathbb{E}\left[h\left(0,\tau_x\omega\right)\right| 0 \in \omega] \, dx.$$

We will obtain a relation like this in Theorem 56.

We now look to construct the conditioned probability $\mathcal{P}_x\left(.\right) := \mathcal{P}\left(.|x \in \omega\right)$. We begin by defining the Campbell measure $\mathfrak{C}$ on the product space $(W, \mathcal{W}) := \left(\mathbb{R}^d \times \Omega, \mathcal{B}^d \times \mathcal{N}\right)$: for $A \in \mathcal{B}^d$, $U \in \mathcal{N}$ define

$$\mathfrak{C}_{\mathcal{P}}\left(A \times U\right) := \mathbb{E}_{\mathcal{P}}\left[\omega\left(A\right) 1_U\left(\omega\right)\right],$$

that is we weight the indicator function of $U$ by the number of points in the Borel set $A$. It is straight forward to see that this defines a $\sigma$-finite measure on the product space.

By the standard route from indicator functions to simple functions and limits of simple functions we obtain, for $\mathcal{W}$-measurable functions $h$, the integrable representation

$$\begin{aligned}\int_W h(x,\omega)\,\mathfrak{C}_{\mathcal{P}}(dx\times d\omega) &= \mathbb{E}_{\mathcal{P}}\left[\sum_{x\in\omega} h(x,\omega)\right]\\ &= \int_\Omega \sum_{x\in\omega} h(x,\omega)\,\mathcal{P}(d\omega)\,. \qquad (3.22)\end{aligned}$$

Now, for fixed $Y\in\mathcal{N}$, the measure

$$B\mapsto \mathfrak{C}_{\mathcal{P}}(B\times Y)$$

is absolutely continuous with respect to the intensity measure $\Lambda$. Hence by the Radon–Nikodym theorem there exists a $\mathcal{B}^d$-measurable function $\mathcal{P}_x(U)$ such that for every $A\in\mathcal{B}^d$

$$\int_A \mathcal{P}_x(U)\,\Lambda(dx) = \mathfrak{C}_{\mathcal{P}}(A\times U)\,. \qquad (3.23)$$

This defines $\mathcal{P}_x$ uniquely up to values on sets of $\Lambda$-measure zero. In fact it is possible (see [24] for the details) to choose the family $\{\mathcal{P}_x(U)\}$ such that

- for each fixed $U\in\mathcal{B}^d$, $\mathcal{P}_x(U)$ is a measurable function of $x$, $\Lambda$-measurable on bounded subsets of $\mathbb{R}^d$,
- for each fixed $x\in\mathbb{R}^d$, $\mathcal{P}_x(.)$ is a probability measure on $(\Omega,\mathcal{N})$.

We call the family $\{\mathcal{P}_x(U)\}$ the Palm distribution for $\mathcal{P}$.

Combining (3.22) and (3.23) we see

$$\begin{aligned}\mathbb{E}_{\mathcal{P}}\left[\sum_{x\in\omega} h(x,\omega)\right] &= \int_W h(x,\omega)\,\mathfrak{C}_{\mathcal{P}}(dx\times d\omega)\\ &= \int_{\mathbb{R}^d}\int_W h(x,\omega)\,\mathcal{P}_x(d\omega)\,\Lambda(dx)\,. \qquad (3.24)\end{aligned}$$

It is well known (see for example [24] or [54]) that if $\mathcal{P}=\mathbb{P}$, a Poisson point process, then the Palm distributions are given by $\mathcal{P}_x=\mathbb{P}_x=\mathbb{P}*\delta_x$, where $\delta_x$ is the point process with exactly one point at $x$ and $*$ is superposition of point processes. Inserting this into (3.24) we obtain the following Theorem:

**Theorem 56 (Campbell-Mecke).** *For any measurable function $h : \mathbb{R}^d \times \Omega \to \mathbb{R}^+$ we have*

$$\mathbb{E}\left[\sum_{x\in\omega} h(x,\omega)\right] = \lambda \int_{\mathbb{R}^d} \mathbb{E}_x\left[h(x,\omega_x)\right] dx, \tag{3.25}$$

*where we are taking $\omega_x := \omega \cup \{0\}$ and $\mathbb{E}_x$ is expectation with respect to $\mathbb{P}_x$.*

**Remark 57.** *This section borrows heavily from the books [54] and [24]. It is noted in [54] that for stationary point processes we could equivalently define the Palm distribution at zero by*

$$\mathcal{P}_0(Y) = \frac{1}{\lambda v_d(B)} \int_\Omega \sum_{x\in B\cap\omega} 1_Y(\tau_x\omega)\, \mathcal{P}(d\omega)$$

*for arbitrary Borel $B$ of positive volume and $v_d$ being Lebesgue measure. Stationarity then extends this definition. We do not use this approach as Theorem 56 is obtained more cleanly via the Radon-Nikodym method.*

### 3.3.2. Reversibility

We prove that $Q$ is reversible with respect to the measure $\mu$. This is not as trivial as in other ergodic environments since we are using the Palm distribution and not classical conditioning, as the event $\{0 \in \mathcal{C}_\infty\}$ is null with respect to $\mathbb{P}$.

The technical issue that this presents is that translations under the Palm measure $\mathbb{P}_0$ are not measure preserving due to the additional point at the origin. One therefore has to use the Campbell-Mecke Theorem to switch from $\mathbb{P}_0$ to $\mathbb{P}$, use the fact that translations are measure preserving under $\mathbb{P}$, before switching back to $\mathbb{P}_0$ by Campbell-Mecke again to obtain results about shifts under $\mathbb{P}_0$.

The reversibility of $Q$ will follow from the following proposition, showing that shifting the origin to a (well defined) second point of $\omega$ is measure preserving. Recall that $\Omega_0 = \{0 \in \mathcal{C}_\infty\}$.

**Proposition 58.** *Suppose $A \subseteq \Omega_0$ is such that, for some $\varepsilon > 0$ and $y \in \mathbb{R}^d \setminus \{0\}$ with $|y| \leq 1$, we have*

$$\begin{aligned} A &\subseteq \{\omega \in \Omega_0 : \omega(B_0[2\varepsilon]) = 1, \exists x \in B_y[\varepsilon] \text{ with } \omega(\{x\}) = \omega(B_x[2\varepsilon]) = 1\} \\ &: = \Omega_0^{\varepsilon,y}. \end{aligned} \tag{3.26}$$

*For $\omega \in A$, define $\tau_{e_y}\omega := \tau_x\omega$ for $x \in B_y[\varepsilon]$ such that $\omega\{x\} = 1$, the shift to the unique point of the infinite cluster lying in $B_y[\varepsilon]$. Then*

$$\mathbb{P}'(A) = \mathbb{P}'\left(\tau_{e_y} A\right).$$

**Proof.** Our plan is to use the set $A \subseteq \Omega_0$ to generate a set in $B \subseteq \Omega$, apply the stationarity of $\mathbb{P}$ to shift by the $y$ given in the statement of the proposition and then use the Campbell-Mecke theorem to show that this shifted set can be generated by $\tau_{e_y} A$. We then bring this together to show that $\mathbb{P}_0(A) = \mathbb{P}_0\left(\tau_{e_y} A\right)$, which implies the result.

We start by generating our set $B \subseteq \Omega$. Define

$$B := \{\omega \in \Omega : \tau_x \omega \in A \text{ for some } |x| \leq \varepsilon,\ \omega(B_y[\varepsilon]) = 1\}.$$

We set $h(x,\omega) = 1_{\{\tau_x\omega \in A\} \cap \{|x| \leq \varepsilon\} \cap \{\omega(B_y[\varepsilon]) = 1\}}$ in Theorem 56. The left hand side of (3.25) becomes $\mathbb{P}(B)$ and the right hand side is

$$\lambda \int_{B_0[\varepsilon]} \mathbb{P}_0\left[A \cap \{\omega(B_{y-x}[\varepsilon]) = 1\}\right] dx. \tag{3.27}$$

since $\mathbb{E}_x(h(x,\omega_x)) = \mathbb{E}_0(h(x,\omega_0) \circ \tau_x) = \mathbb{P}_0\left[A \cap \{\omega(B_{y-x}[\varepsilon]) = 1\}\right]$ for $|x| \leq \varepsilon$.

Now, by stationarity with respect to $\mathbb{P}$ we have

$$\mathbb{P}(\tau_y B) = \mathbb{P}(B).$$

Set $A' := \tau_y B \cap \Omega_0$, then we claim that $A' = \tau_{e_y} A$ and $\mathbb{P}_0(A') = \mathbb{P}_0(A)$. The first claim follows trivially from (3.26), since for each $\omega \in A$ there is exactly one $x \in B_0[\varepsilon]$ such that $(\tau_x\omega)\{y\} = 1$. The second claim is what we set out to prove. Now, apply Campbell Mecke again to see

$$\mathbb{P}(\tau_y B) = \lambda \int_{B_0[\varepsilon]} \mathbb{P}_0\left[A' \cap \{\omega(B_{-y+x}[\varepsilon]) = 1\}\right] dx. \tag{3.28}$$

By translation invariance, $\mathbb{P}(B) = \mathbb{P}(\tau_y B)$ and so (3.27) and (3.28) are equal.

Now suppose that $\mathbb{P}_0[A'] \neq \mathbb{P}_0[A]$; without loss of generality assume $\mathbb{P}_0[A'] > \mathbb{P}_0[A]$. Then since $\mathbb{P}_0$ is non-atomic, there exists $c > 0$, $0 < \delta < \varepsilon$ such that

$$\mathbb{P}_0\left[A' \cap \{\omega(B_{-y+x'}[\varepsilon]) = 1\}\right] - \mathbb{P}_0\left[A \cap \{\omega(B_{y-x'}[\varepsilon]) = 1\}\right] > c \tag{3.29}$$

for all $x' \in B_0[\delta]$. In particular, the integrals of the terms on the left hand side of (3.29) over any subcube $B_x[\delta'] \subseteq B_0[\delta]$ are not equal. To obtain a contradiction we will show that these integrals should agree.

We split our cubes around 0 and $y$ into disjoint subcubes. Take $n \in \mathbb{N}$. Split $B_0[\varepsilon]$ into $3^{nd}$ disjoint subcubes of side $\varepsilon 3^{-n}$ and write the centres of these cubes as $X_n = \{x_1, \ldots, x_{3^{nd}}\}$ where we assume that $x_1 = 0$. We similarly split $B_y[\varepsilon]$ into subcubes with centres $Y_n = \{y_1, \ldots, y_{3^{nd}}\}$ with $y_1 = y$ and $y_i = y + x_i$. Write $C_{x_i} = B_{x_i}[\varepsilon 3^{-n}]$ for the subcube with centre $x_i \in X_n$ and similarly for $y_i$. Now, define

$$B_{x_i,y_j} := \left\{\omega \in B : \omega(C_{x_i}) = \omega\left(C_{y_j}\right) = 1\right\} \subseteq \Omega,$$

that is, the elements of $\omega$ that have points in the subcubes corresponding to $x_i$ and $y_j$. Note that since we defined $B$ as being generated by $A$ and $\mathbb{P}$ is translation invariant, we have, that if $x_i, x_k \in X_n,\ y_j \in Y_n$ are such that $x_i + x_k \in X_n$ and $y_j + x_k \in Y_n$, then

$$\mathbb{P}\left(B_{x_i,y_j}\right) = \mathbb{P}\left(B_{x_i+x_k,y_j+x_k}\right).$$

In particular,

$$\mathbb{P}\left(B_{x_1,y_j}\right) = \mathbb{P}\left(B_{x_1-x_j,y_1}\right) = \mathbb{P}\left(\tau_y\left(B_{-x_j,y_1}\right)\right) = \mathbb{P}\left(B_{x_1,-y_j}\right). \tag{3.30}$$

By the work above we have

$$\begin{aligned}
\mathbb{P}\left(B_{x_i,y_j}\right) &= \lambda \int_{B_{x_i}} \mathbb{P}_0\left(A \cap \left\{\omega\left(B_{y_j-x}\left[\varepsilon 3^{-n}\right]\right) = 1\right\}\right) dx \\
&= \lambda \int_{B_{x_j}} \mathbb{P}_0\left(A' \cap \left\{\omega\left(B_{-y_i+x}\left[\varepsilon 3^{-n}\right]\right) = 1\right\}\right) dx.
\end{aligned} \tag{3.31}$$

Hence, combining (3.30) and (3.31) we see

$$\begin{aligned}
&\lambda \int_{B_0[\varepsilon 3^{-n}]} \mathbb{P}_0\left(A \cap \{\omega\left(B_{y-x}[\varepsilon]\right) = 1\}\right) dx \\
&= \lambda \sum_{y_j \in Y_n} \int_{B_{x_1}} \mathbb{P}_0\left(A \cap \left\{\omega\left(B_{y_j-x}\left[\varepsilon 3^{-n}\right]\right) = 1\right\}\right) dx \\
&= \sum_{y_j \in Y_n} \mathbb{P}_0\left(B_{x_1,y_j}\right) = \sum_{y_j \in Y_n} \mathbb{P}_0\left(B_{x_1,-y_j}\right) \\
&= \lambda \sum_{y_j \in Y_n} \int_{B_{x_1}} \mathbb{P}_0\left(A' \cap \left\{\omega\left(B_{-y_j+x}\left[\varepsilon 3^{-n}\right]\right) = 1\right\}\right) dx \\
&= \lambda \int_{B_0[\varepsilon 3^{-n}]} \mathbb{P}_0\left(A' \cap \{\omega\left(B_{-y+x}[\varepsilon]\right) = 1\}\right) dx
\end{aligned}$$

Now, by taking $n$ large enough, we have $B_0\left[\varepsilon 3^{-n}\right] \subseteq B_0[\delta]$ and we obtain our contradiction, since the two integrals above cannot be equal by assumption (3.29).
■

Recall the definition of $\mu$ in equation (3.20).

**Proposition 59.** *The measure $\mu$ is reversible for the Markov kernel $Q$, ie for $f, g \in L^2$*

$$\mathbb{E}_\mu\left(f\left(Qg\right)\right) = \mathbb{E}_\mu\left(g\left(Qf\right)\right). \tag{3.32}$$

**Proof.** Take $A, B \in \mathcal{N}$. Now,

$$\mathbb{E}_\mu \left(1_A \left(Q 1_B\right)\right) = \int\limits_{\Omega_0} \sum_{x \overset{\omega}{\sim} 0} 1_A(\omega) 1_B(\tau_x \omega) \, d\mathbb{P}'(\omega) \, \mathbb{E}' [\deg 0]^{-1}$$
$$= \int\limits_{A} \sum_{x \overset{\omega}{\sim} 0} 1_B(\tau_x \omega) \, d\mathbb{P}'(\omega) \, \mathbb{E}' [\deg 0]^{-1} \tag{3.33}$$

Write $D_0[1] := \left\{ x \in \mathbb{R}^d : |x| \le 1 \right\}$. Define, for $\omega \in A$,

$$N(AB)(\omega) \ := \sup \{ i : \exists y_1, \ldots, y_i \in D_0[1] \ , \ y_j = y_k \text{ iff } j = k, \\ \omega\{y_j\} = 1, \ \tau_{y_j} \omega \in B \text{ for } j = 1, \ldots, i, \}$$

that is the number of neighbours of 0 in $\omega$ that we can shift the origin to and lie in $B$. Then, for $i \in \mathbb{N}$, we define the set $(AB)_i := \{\omega \in A : N(AB)(\omega) = i\}$. (3.33) then becomes

$$\mathbb{E}_\mu \left(1_A \left(Q 1_B\right)\right) = \sum_{i=1}^{\infty} i \mathbb{P}' \left[(AB)_i\right] \mathbb{E}' [\deg 0]^{-1}. \tag{3.34}$$

Now, take $i, j$ such that $1 \le j \le i$. We can write

$$(AB)_i = \coprod_{\substack{\varepsilon \in \mathbb{Q} \\ y \in \mathbb{Q}^d}} (AB)^{i,j}_{\varepsilon,y},$$

a disjoint union of sets of the form (3.26) with the added condition that for $\omega \in (AB)^{i,j}_{\varepsilon,y}$, the point of $\omega$ lying in $B_y[\varepsilon]$ is the $j$th neighbour of 0 in $\omega$ such that $\tau_\cdot \omega \in B$, with respect to some deterministic numbering. As the left hand side is independent of the choice of $j$, we have

$$\mathbb{E}_\mu \left(1_A \left(Q 1_B\right)\right) = \sum_{i=1}^{\infty} \sum_{j=1}^{i} \sum_{\varepsilon,y} \mathbb{P}' \left[(AB)^{i,j}_{\varepsilon,y}\right] \mathbb{E}' [\deg 0]^{-1}. \tag{3.35}$$

Note that for $\omega \in (AB)^{i,j}_{\varepsilon,y}$ there is exactly one point in $B_y[\varepsilon]$ and hence for fixed $\varepsilon, y$ the sets $\left\{(AB)^{i,j}_{\varepsilon,y}\right\}_{1 \le j \le i}$ are disjoint. Hence, writing

$$(AB)_{\varepsilon,y} = \coprod_{1 \le j \le i} (AB)^{i,j}_{\varepsilon,y},$$

(3.35) becomes:

$$\mathbb{E}_\mu \left(1_A \left(Q 1_B\right)\right) = \sum_{\varepsilon,y} \mathbb{P}' (AB)_{\varepsilon,y} \, \mathbb{E}' [\deg 0]^{-1},$$

where the sum is still being taken over $\mathbb{Q} \times \mathbb{Q}^d$. Each of the sets $(AB)_{\varepsilon,y}$ satisfies (3.26), applying Proposition 58:

$$\mathbb{E}_\mu \left(1_A \left(Q 1_B\right)\right) = \sum_{\varepsilon,y} \mathbb{P}' \left[(BA)_{\varepsilon,y}\right] \mathbb{E}' \left[\deg 0\right]^{-1}, \tag{3.36}$$

where $(BA)_{\varepsilon,y} := \left\{\tau_{e_y}\omega : \omega \in (AB)_{\varepsilon,y}\right\}$.

We now follow the same steps in reverse: for $1 \leq j \leq i$ define

$$(BA)^{i,j}_{\varepsilon,y} := (BA)_{\varepsilon,y} \cap \{N(BA) = i\} \cap \{\phi(\varepsilon, y) = j\}$$

where $\phi(\varepsilon, y)(\omega)$ gives the position of the point of $\omega$ in $B_y[\varepsilon]$, in the ordering of neighbours of 0 in $\omega$ such that $\tau_. \omega \in A$. By (3.36)

$$\mathbb{E}_\mu \left(1_A \left(Q 1_B\right)\right) = \sum_{\varepsilon,y} \sum_{1 \leq j \leq i} \mathbb{P}' (BA)^{i,j}_{\varepsilon,y} \mathbb{E}' \left[\deg 0\right]^{-1}. \tag{3.37}$$

For fixed $1 \leq i \leq j$, the sets $\left\{(BA)^{i,j}_{\varepsilon,y}\right\}_{(\varepsilon,y) \in \mathbb{Q} \times \mathbb{Q}^d}$ are disjoint, hence

$$\mathbb{E}_\mu \left(1_A \left(Q 1_B\right)\right) = \sum_{1 \leq j \leq i} \mathbb{P}' (BA)^{i,j} \mathbb{E}' \left[\deg 0\right]^{-1}$$

for $(BA)^{i,j} := \coprod_{(\varepsilon,y) \in \mathbb{Q} \times \mathbb{Q}^d} (BA)^{i,j}_{\varepsilon,y}$.

Now, for fixed $i \in \mathbb{N}$ and $j, j' \leq i$, working through the definitions, it is straight forward to see that $(BA)^{i,j} = (BA)^{i,j'} = \{\omega \in B : N(BA)(\omega) = i\} =: (BA)^i$. Thus

$$\begin{aligned}
\mathbb{E}_\mu \left(1_A \left(Q 1_B\right)\right) &= \sum_{i=1}^{\infty} i \mathbb{P}' (BA)^i \mathbb{E}' \left[\deg 0\right]^{-1} \\
&= \int_B \sum_{x \overset{\omega}{\sim} 0} 1_A (\tau_x \omega)\, d\mathbb{P}'(\omega) \mathbb{E}' \left[\deg 0\right]^{-1} \\
&= \int_{\Omega_0} 1_B(\omega) 1_A(\tau_x \omega)\, d\mathbb{P}'(\omega) \mathbb{E}' \left[\deg 0\right]^{-1} \\
&= \mathbb{E}_\mu \left(1_B \left(Q 1_A\right)\right).
\end{aligned} \tag{3.38}$$

Now use the monotone convergence theorem twice to complete the proof. ■

### 3.3.3. Ergodicity

We continue towards a proof of Theorem 55. Recall that for $\omega \in \Omega_0$, $P_{\omega,0}$ refers to the law of the simple random walk started at the origin on the infinite graph induced by $\omega$.

**Proposition 60.** *Let $A \subseteq \Omega_0$ be measurable and such that for $\mathbb{P}'$-almost every $\omega \in A$*

$$P_{\omega,0}\left(\tau_{X_1}\omega \in A\right) = 1, \tag{3.39}$$

*and $\mathbb{P}'(A) > 0$ then $\mathbb{P}'(A) = 1$.*

**Proof.** Note that (3.39) implies that for $\mathbb{P}'$-almost every $\omega \in A$, we have $\tau_x\omega \in A$ for every $x \in \mathcal{C}_\infty(\omega)$.

Write $\Omega^0 := \{|0 - \mathcal{C}_\infty| \leq 1\}$, the set of environments for which the origin would be in the infinite component if we augment the graph. Define the set $B := \{\omega \in \Omega : \omega = \tau_x a \text{ for some } a \in A,\ |x| \leq 1\} \subseteq \Omega^0$. Then $\mathbb{P}(B) > 0$ by the Campbell-Mecke Theorem. From the definition of $B$, (3.39) and Campbell-Mecke we see that there is a null set $N \subset B$ such that for $b \in B - N$, $\tau_x b \in B$ for all $x \in \mathcal{C}_\infty(b)$. We claim that if we have $y \in \mathbb{R}^d,\ b \in B - N$ such that $\tau_y b \in \Omega^0$ then $\tau_y b \in B$.

Suppose we have such a $y$ and $b$. Since $B \subseteq \Omega^0$, by the definition of $B$ we can find $x \in B_0[1]$ such that $x \in \mathcal{C}_\infty(b)$ and $\tau_x b \in A$ (where for Borel set $C$, $\tau_x b(C) = b(C + x)$ and $C + x = \{c + x : c \in C\}$). Take $x' \in B_0[1]$ such that $x' \in \mathcal{C}_\infty(\tau_y b)$. Then $y + x' \in \mathcal{C}_\infty(b)$, hence $y + x' - x \in \mathcal{C}_\infty(\tau_x b)$ and thus by (3.39)

$$\tau_{y+x'-x}(\tau_x b) = \tau_{y+x'}b \in A.$$

But $x' \in B_0[1]$, so by definition of $B$ we see $\tau_{-x'}\tau_{y+x'}b = \tau_y b \in B$. So $B$ is invariant under shifts from $\Omega^0$ to itself.

Fix $e \in \mathbb{R}^d - \{0\}$. Now, $n_e(\omega) := \min\{k \in \mathbb{N} : \tau_{ke}(\omega) \in \Omega^0\}$ is finite almost surely and hence we can define $\sigma_e : \Omega^0 \to \Omega^0$ by

$$\sigma_e(\omega) := \tau_{n_e(\omega)e}\omega.$$

Since $\tau_e$ is measure preserving and ergodic with respect to $\mathbb{P}$, the methods of Berger and Biskup's paper [10] immediately give us that $\sigma_e$ is measure preserving and ergodic with respect to $\mathbb{P}(\,.\,|\Omega^0)$. We have shown that $B$ is invariant under this shift and hence $\mathbb{P}(B) = \mathbb{P}(\Omega^0)$.

We now show that this implies that $\mathbb{P}'(A) = 1$. Define the event $F(x,\omega) = \{B_0[|x|] \cap \mathcal{C}_\infty(\omega) = \{x\}, |x| \leq 1\}$. By Campbell-Mecke, with $\omega_x := (\mathcal{H}_\lambda(\omega) \cup \{x\}; 1)$,

$$\begin{aligned}
\mathbb{P}(\Omega^0) &= \int_\Omega \sum_{x\in\omega} 1_{F(x,\omega)} d\omega \\
&= \lambda \int_\Omega \int_{\mathbb{R}^d} 1_{F(x,\omega_x)} dx d\omega \\
&= \lambda \int_\Omega \int_{B_0[1]} 1_{F(x,\omega_x)} dx d\omega \\
&= \lambda \int_{B_0[1]} \mathbb{P}_x \{\mathcal{C}_\infty \cap B_0[|x|] = \{x\}\} dx. \qquad (3.40)
\end{aligned}$$

Similarly, define the event $D(x,\omega) = \{\omega \in B, |x| \leq 1, B_0[|x|] \cap \mathcal{C}_\infty(\omega) = \{x\}\}$ then

$$\begin{aligned}
\mathbb{P}(B) &= \int_\Omega \sum_{x\in\omega} 1_{D(x,\omega)} d\omega \\
&= \lambda \int_{B_0[1]} \mathbb{P}_x [\{\mathcal{C}_\infty \cap B_0[|x|] = \{x\}\} \cap B] dx. \qquad (3.41)
\end{aligned}$$

Comparing (3.40) and (3.41), noting that the values of the integral are the same and the term inside (3.40) dominates that inside (3.41) we see that they must be equal almost everywhere, and by continuity we have that they must be identical. In particular

$$\mathbb{P}'[\{0 \in \mathcal{C}_\infty\} \cap B] = \mathbb{P}'[\{0 \in \mathcal{C}_\infty\}].$$

To complete the proof we show that $\{0 \in \mathcal{C}_\infty\} \cap B = A$, $\mathbb{P}'$-a.s.

Suppose $b \in \{0 \in \mathcal{C}_\infty\} \cap B - N$. Then since $b \in B$, $b = \tau_x a$ for some $x$ with $|x| \leq 1$ and some $a \in A$. But then $x \in \mathcal{C}_\infty(a)$ and, since $b \in B - N$, $\tau_x a \in A$. Thus $b \in A$. The other inclusion is trivial and we are done. Note that since the set $N$, null with respect to $\mathbb{P}$, is induced by the $\mathbb{P}'$-null set referred to in (3.39), $\{0 \in \mathcal{C}_\infty\} \cap N$ is $\mathbb{P}'$-null. ■

We now show that the Markov chain on environments is ergodic with respect to $\mu$.

Let $\mathcal{X} = \Omega_0^{\mathbb{Z}}$ and $\zeta = \mathcal{N}_0^{\otimes\mathbb{Z}}$ the product $\sigma$-algebra on $\mathcal{X}$. The space $\mathcal{X}$ is that of two-sided sequences $(\ldots, \omega_{-1}, \omega_0, \omega_1, \ldots)$, the trajectories of the Markov chain on environments. We take $\pi$ to be the measure on $(\mathcal{X}, \zeta)$ such that for any $B \in \mathcal{N}^{2n+1}$

$$\pi((\omega_{-n}, \ldots, \omega_n) \in B) = \int_B \mu(d\omega_{-n}) Q(\omega_{-n}, d\omega_{-n+1}) \ldots Q(\omega_{n-1}, d\omega_n),$$

where

$$Q(a,A)=\frac{1}{\deg_a 0}\sum_{x\overset{a}{\sim}0}1_{\{\tau_x a\in A\}}.$$

Define the shift $T:\mathcal{X}\to\mathcal{X}$ by $(T\omega)_n:=\omega_{n+1}$. Then following the methods of [10] we obtain:

**Proposition 61.** *$T$ is measure-preserving and ergodic with respect to $\pi$.*

**Proof.** Take $A\subseteq\mathcal{X}$, measurable and $T$ invariant. Let $f:\Omega_0\to R$ be defined as $f(\omega_0)=E_\pi(1_A|\omega_0)$. We show that $f=1_A$ almost surely. Since $A$ is $T$-invariant, by approximation by finite dimensional events, there exist $A_+\in\sigma(\omega_k:k>0)$ and $A_-\in\sigma(\omega_k:k<0)$ such that $A$, $A_+$ and $A_-$ differ only by null sets from each other. Conditional on $\omega_0$, $A_+$ is independent of $\sigma(\omega_k:k<0)$, thus

$$\begin{aligned}E_\pi(1_A|\omega_0) &= E_\pi\left(1_{A_+}|\omega_0\right)=E_\pi\left(1_{A_+}|\omega_0,\omega_{-1},\ldots,\omega_{-n}\right)\\ &= E_\pi\left(1_{A_-}|\omega_0,\ldots,\omega_{-n}\right)\to 1_{A_-}=1_A,\end{aligned}$$

$\pi$-almost surely.

Now, let $B\subset\Omega_0$ be defined by $B=\{\omega_0:f(\omega_0)=1\}$. Then $B$ is $\mathcal{N}$-measurable and since the $\omega_0$-marginal of $\pi$ is $\mu$,

$$\pi(A)=E_\pi(f)=\mu(B).$$

Since $A$ is $T$-invariant, up to null sets, if $\omega_0\in B$ then $\omega_1\in B$, satisfying the condition of Proposition 60, hence $\mathbb{P}'(B)\in\{0,1\}$, equivalently $\mu(B)\in\{0,1\}$. ■

We can now prove the convergence of the ergodic average.

**Proof of Theorem 55.** Since $(\tau_{X_k}(\omega))_{k\geq 0}$ has the same law under $\mathbb{E}_\mu(P_{0,\omega}(.))$ as $(\omega_0,\omega_1,\ldots)$ has under $\pi$, if $g(\ldots,\omega_{-1},\omega_0,\omega_1,\ldots)=f(\omega_0)$ then

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{\infty}f\circ\tau_{X_k}\overset{D}{=}\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{\infty}g\circ T^k.$$

The second limit exists by Birkhoff's Ergodic theorem and Proposition 61 and equals $E_\pi(g)=\mathbb{E}_\mu(f)$ almost surely. ■

**Remark 62.** *In particular, our environment process satisfies the condition of Theorem 2.1 of the work of Ferrari et al [27] and thus we have the following annealed invariance principle:*

**Theorem 63.** *Define*

$$\mathbf{P}(.):=\int_{\Omega_0}P_{\omega,0}(.)\,\mu(d\omega).$$

*Then the interpolated random walk with respect to the annealed measure* $\mathbf{P}$*, converges weakly to a nondegenerate Brownian motion.*

## 3.4. Invariance principle

We now prove the quenched invariance principle for the simple random walk, Theorem 33. This section draws strongly on the quenched invariance principles for random walk on discrete percolation [10], for random walks amongst bounded random conductances [13] and the annealed invariance principle for both proved in [27].

The underlying principle is the following: we want to define a corrector $\chi : \Omega \times \mathcal{C}_\infty \to \mathbb{R}^d$ such that for $\mathbb{P}'$-almost every $\omega$

$$M_n(\omega) := \chi(\omega, X_n) + X_n$$

is a martingale. Standard results about weak convergence of martingales prove that $M_n$ converges to Brownian motion. We thus have to show that the corrector is sublinear. The proof of this fact takes up the majority of this section.

In Section 3.4.1 the corrector is defined following the methods of [10] and using the results of Section 3.3. Sublinearity of the corrector is then proved via Sections 3.4.2 and 3.4.3, again following the methods of [10], with the ideas brought together in Section 3.4.4 to prove Theorem 33.

### 3.4.1. The corrector

We want to define a corrector, $\chi : \Omega \times \mathcal{C}_\infty \to \mathbb{R}^d$, but have the problem that for $x \in \mathbb{R}^d - \{0\}$, $\mathbb{P}'[x \in \mathcal{C}_\infty] = 0$. To get round this we make use of the fact that the vertex set (and in fact the edge set) of $\mathcal{C}_\infty$ is almost surely countable. We begin by splitting $\mathbb{R}^d$ into disjoint cubes: for $z \in \mathbb{Z}^d$ we associate the cube $B_{2z}[1]$ to $z$, and take $\Phi : \Omega \times \mathbb{Z}^d \times \mathbb{N} \to \mathbb{R}^d$ to be a deterministic map such that for almost every $\omega \in \Omega$ and every $z \in \mathbb{Z}^d$

$$\{\Phi(\omega, z, n)\}_{n \in \mathbb{N}} \tag{3.42}$$

is a complete list of the points in $\mathcal{H}(\lambda) \cap B_{2z}[1]$. We assume that there are no repetitions and once all points of $\mathcal{H}(\omega) \cap B_{2z}[1]$ are exhausted, $\Phi$ returns a null value; $\delta$, say. This map gives rise to a deterministic inverse: for $\mathbb{P}_0$-almost every $\omega$ we have

$$\Phi^{-1}(\omega, .) : \mathcal{C}_\infty(\omega) \to \mathbb{Z}^d \times \mathbb{N}$$

such that $\Phi(\omega, \Phi^{-1}(\omega, x)) = x$ for all $x \in \mathcal{C}_\infty(\omega)$. We insist that the deterministic map lists first the points of degree less than or equal to $k$, so that the numbering remains consistent when we consider the thinned graph later. We also insist that the numbering within each box stays the same when we shift by $2z$ for some $z \in \mathbb{Z}^d$, so the numbering of the points in the box about the origin in $\omega$ is consistent with the numbering of the box about $-2z$ in $\tau_{2z}\omega$. Other than these conditions, the numbering used is unimportant, we could for example order by distance from the centre of the box.

We abuse notation by writing, for $z \in \mathbb{Z}^d$, $n \in \mathbb{N}$

$$\tau_{(z,n)}(\omega) = \tau_{\Phi(\omega,z,n)},$$

note that our numberings will change when we apply this shift.

Recall the operator $Q : L^2(\Omega_0, \mathcal{B}, \mu) \to L^2(\Omega_0, \mathcal{B}, \mu)$ defined by

$$Qf(\omega) = \frac{1}{\deg_\omega 0} \sum_{x \overset{\omega}{\sim} 0} f(\tau_x \omega).$$

**Proposition 64.** *$Q$ is a self adjoint contraction on $L^2(\mu) := L^2(\Omega_0, \mathcal{N}_0, \mu)$.*

**Proof.** For $n, m \in \mathbb{N}$ define

$$A_{n,m} := \left\{\omega \in \Omega : \Phi\left(\tau_{(0,n)}\omega, 0, m\right) = -\Phi(\omega, 0, n)\right\},$$

that is the environments such that the $n$th neighbour of the origin considers the origin to be its $m$th neighbour.

Then

$$\begin{aligned}
(Qf, f)_\mu &= \mathbb{E}_P\left[f(\omega) \sum_{n,m} 1_{\{0 \in \mathcal{C}_\infty\}} 1_{A_{n,m}}(\omega)\, 1_{\left\{0 \overset{\omega}{\sim} \Phi(\omega,0,n)\right\}} f\left(\tau_{(0,n)}\omega\right)\right] \\
&= \sum_{n,m} \mathbb{E}_P\left[f(\omega)\, 1_{\{0 \in \mathcal{C}_\infty\}} 1_{A_{n,m}}(\omega)\, 1_{\left\{0 \overset{\omega}{\sim} \Phi(\omega,0,n)\right\}} 1_{\left\{0 \overset{\tau_{(0,n)}\omega}{\sim} \Phi\left(\tau_{(0,n)}\omega,0,m\right)\right\}} f\left(\tau_{(0,n)}\omega\right)\right] \\
&\leq \sum_{n,m} \mathbb{E}_P\left[f(\omega)^2\, 1_{\{0 \in \mathcal{C}_\infty\}} 1_{A_{n,m}}(\omega)\, 1_{\left\{0 \overset{\omega}{\sim} \Phi(\omega,0,n)\right\}}\right]^{1/2} \\
&\qquad \mathbb{E}_P\left[1_{\{0 \in \mathcal{C}_\infty\}} 1_{A_{n,m}}(\omega)\, 1_{\left\{0 \overset{\tau_{(0,n)}\omega}{\sim} \Phi\left(\tau_{(0,n)}\omega,0,m\right)\right\}} f\left(\tau_{(0,n)}\omega\right)^2\right]^{1/2},
\end{aligned}$$

where the last line follows from Cauchy-Schwarz. Hence,

$$\begin{aligned}
(Qf, f)_\mu &\leq \sum_{n,m} \mathbb{E}_P\left[f(\omega)^2\, 1_{\{0 \in \mathcal{C}_\infty\}} 1_{A_{n,m}}(\omega)\, 1_{\left\{0 \overset{\omega}{\sim} \Phi(\omega,0,n)\right\}}\right]^{1/2} \\
&\qquad \mathbb{E}_P\left[1_{\{0 \in \mathcal{C}_\infty\}} 1_{A_{m,n}}(\omega)\, 1_{\left\{0 \overset{\omega}{\sim} \Phi(\omega,0,m)\right\}} f(\omega)^2\right]^{1/2} \\
&\leq \sum_{n,m} \mathbb{E}_P\left[f(\omega)^2\, 1_{\{0 \in \mathcal{C}_\infty\}} 1_{A_{n,m}}(\omega)\, 1_{\left\{0 \overset{\omega}{\sim} \Phi(\omega,0,n)\right\}}\right] \\
&= \mathbb{E}_P\left[1_{\{0 \in C_\infty\}} \deg_\omega 0 f(\omega)^2\right] = (f, f)_\mu .
\end{aligned}$$

■

As $Q$ is a contraction, for $\varepsilon > 0$, we can define $\psi_\varepsilon : \Omega_0 \to \mathbb{R}^d$, a function in $L^2(\mu)$, to be the solution to

$$(1 + \varepsilon - Q)\,\psi_\varepsilon = V,$$

where

$$V(\omega) = \sum_{x \overset{\omega}{\sim} 0} \frac{x}{\deg_\omega 0} = E^\omega(X_1).$$

We now look to define the corrector, following the methods of [10] which in turn follow [35].

**Theorem 65.** *There exists a function $\chi : \mathbb{Z}^d \times \mathbb{N} \times \Omega_0 \to \mathbb{R}^d$ such that for every $(z,n) \in \mathbb{Z}^d \times \mathbb{N}$,*

$$\lim_{\varepsilon \downarrow 0} 1_{\{\Phi(\omega,z,n)\in C_\infty\}} \left(\psi_\varepsilon \circ \tau_{(z,n)} - \psi_\varepsilon\right) = \chi((z,n),.)\,, \text{ in } L^2(\mu)\,.$$

*We also have the following properties:*

1. *(Shift invariance) For $\mathbb{P}'$-almost every $\omega \in \Omega_0$*

$$\chi(x,\omega) - \chi(y,\omega) = \chi(x - y, \tau_y \omega)$$

   *holds for all $x, y \in \mathcal{C}_\infty(\omega)$, where we have abused notation and $\chi(x,\omega)$ in fact means $\chi(\Phi^{-1}(\omega,x),\omega)$and similarly for the other terms.*

2. *(Harmonicity) For $\mathbb{P}'$-almost every $\omega \in \Omega_0$, the function*

$$x \mapsto x + \chi(x,\omega)$$

   *is harmonic with respect to the transition probabilities $P_{0,\omega}$.*

3. *(Uniform square integrability on the thinned graph) There exists a constant $C < \infty$ such that*

$$\left\| [\chi((z,n),.) - \chi((z',n'),.)]\, 1_{\left\{\Phi(\omega,z,n),\Phi(\omega,z',n')\in \mathcal{C}^k_\infty\right\}} 1_{\{(z',n')\sim(z,n)\}} \right\|_2 < C$$

   *holds for $z, z' \in \mathbb{Z}^d$, $n, n' \in \mathbb{N}$, where $\|.\|_2 = \left(\mathbb{E}_\mu |.|^2\right)^{1/2}$.*

We recall properties of $\psi_\varepsilon$, from [10], required to prove the above theorem. Let $\mu_V$ be the spectral measure of $Q : L^2(\Omega_0, \mathcal{N}_0, \mu) \to L^2(\Omega_0, \mathcal{N}_0, \mu)$, ie for every bounded, continuous $\Phi : [-1,1] \to \mathbb{R}$ we have

$$\mathbb{E}_\mu(V\Phi(Q)V) = (V, \Phi(Q)V) = \int_{-1}^{1} \Phi(\lambda)\,\mu_V(d\lambda)\,.$$

Then by the results of Section 3.3 and the work of Ferrari et al (Lemma 2.5 of [27]) we have $\int_{-1}^{1} \frac{1}{1-\lambda}\mu_V(d\lambda) < \infty$ and $\lim_{\varepsilon\downarrow 0} \varepsilon \|\psi_\varepsilon\|_2^2 = 0$.

**Lemma 66.** *Let $n \in \mathbb{N}$. Define*

$$G_n^{\varepsilon} = 1_{\{0 \in C_\infty\}} 1_{\{|\Phi(\cdot,\mathbf{0},n)| \le 1\}} \left(\psi_\varepsilon \circ \tau_{\Phi(\cdot,\mathbf{0},n)} - \psi_\varepsilon\right).$$

*Then for every $n \in \mathbb{N}$*

$$\lim_{\varepsilon_1,\varepsilon_2 \downarrow 0} \|G_n^{\varepsilon_1} - G_n^{\varepsilon_2}\|_2 = 0.$$

**Proof.** Firstly, suppose that instead of picking the $n$th neighbour in our deterministic sequence we picked a neighbouring vertex of the origin uniformly at random, call this selection $e$. Then, with $\mathbb{E} = \mathbb{E}_\mu \times \mathbb{E}_U$, where $U$ is the uniform choice of edge $e$, we see

$$\begin{aligned}
& \mathbb{E}\left[1_{\{0\in\mathcal{C}_\infty\}}\left[\psi_{\varepsilon_1}\circ\tau_e - \psi_{\varepsilon_1} - \psi_{\varepsilon_2}\circ\tau_e + \psi_{\varepsilon_2}\right]^2\right] \\
= \;& \mathbb{E}_\mu \times \mathbb{E}_U\left[1_{\{0\in\mathcal{C}_\infty\}}\left[\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)\circ\tau_e - \left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)\right]^2\right] \\
= \;& \mathbb{E}_\mu\left[1_{\{0\in\mathcal{C}_\infty\}}\sum_{x\sim 0}\frac{1}{\deg 0}\left[\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)\circ\tau_x - \left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)\right]^2\right] \\
= \;& \mathbb{E}_\mu\left[Q\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)^2 - 2\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)Q\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right) + \left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)^2\right] \\
\le \;& 2\mathbb{E}_\mu\left[\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)^2\right] - 2\mathbb{E}_\mu\left[\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)Q\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)\right] \\
= \;& 2\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}, (1-Q)\left(\psi_{\varepsilon_1} - \psi_{\varepsilon_2}\right)\right) \\
= \;& 2\int_{-1}^{1}\frac{\left(\varepsilon_1 - \varepsilon_2\right)^2(1-\lambda)}{\left(1+\varepsilon_1-\lambda\right)^2\left(1+\varepsilon_2-\lambda\right)^2}\mu_V(d\lambda),
\end{aligned}$$

where the fact that $Q$ is a contraction is used for the fifth line. This integrand is bounded by $\frac{1}{1-\lambda}$ for all $\varepsilon_1, \varepsilon_2$ and by the work of Ferrari et al tends to zero as $\varepsilon_1, \varepsilon_2 \to 0$.

By Fubini's theorem

$$\begin{aligned}
\mathbb{E}\left|G_e^{\varepsilon_1} - G_e^{\varepsilon_2}\right|^2 &= \int_U \sum_{k=1}^{\infty}\mathbb{E}_\mu\left[\left|G_e^{\varepsilon_1} - G_e^{\varepsilon_2}\right|^2 \middle| \deg 0 = k\right]\mathbb{P}_\mu\left[\deg 0 = k\right]du \\
&\ge \sum_{k=1}^{\infty}\frac{1}{k}\sum_{i=1}^{k}\mathbb{E}_\mu\left|G_i^{\varepsilon_1} - G_i^{\varepsilon_2}\right|^2\mathbb{P}_\mu\left[\deg 0 = k\right] \\
&\ge c(i)\,\mathbb{E}_\mu\left|G_i^{\varepsilon_1} - G_i^{\varepsilon_2}\right|^2
\end{aligned}$$

for arbitrary $i$ and constants $c(i) > 0$. Since the term on the left hand side tends to zero as $\varepsilon_1, \varepsilon_2 \to 0$ we have the claimed result. ■

Now, for $z \in \mathbb{Z}^d$, $m \in \mathbb{N}$, since $\Phi(., z, m)$ may equal $\delta$ or may fail to lie in $\mathcal{C}_\infty$, we have

$$\left\|G_n^{\varepsilon_1}\circ\tau_{(z,m)} - G_n^{\varepsilon_2}\circ\tau_{(z,m)}\right\|_2 \le \left\|G_n^{\varepsilon_1} - G_n^{\varepsilon_2}\right\|_2$$

and hence the limit is zero as $\varepsilon_1, \varepsilon_2 \to 0$. Hence for all $z \in \mathbb{Z}^d$, $m \in \mathbb{N}$, $G_n^{\varepsilon}\circ\tau_{(z,m)}$ converges in $L^2$ as $\varepsilon \downarrow 0$. Write $G_{(z,m)}^{n} = \lim_{\varepsilon\downarrow 0} G_n^{\varepsilon}\circ\tau_{(z,m)}$. Take $z, z' \in \mathbb{Z}^d$,

$m, m', n, n' \in \mathbb{N}$ then on the event that $(z', m')$ is the $n$th neighbour of $(z, m)$ and $(z, m)$ is the $n'$th neighbour of $(z', m')$:

$$G^{\varepsilon}_{(z,m)} \circ \tau_n + G^{\varepsilon}_{(z',m')} \circ \tau_{n'} = 0$$

and hence the limit is zero off a null set. Similarly, for any closed loop $(x_0, \ldots, x_i)$ on $\mathcal{C}_\infty$, writing $G_{x,y}(\omega) = G^n_{(z,m)}(\omega)$ for $x = \Phi(\omega, z, m)$ and $y = x + \Phi\left(\tau_{(z,m)}\omega, \mathbf{0}, n\right)$, we have

$$\sum_{k=0}^{i} G_{x_k, x_{k+1}}(\omega) = 0$$

for almost every $\omega \in \Omega_0$. Thus, we can define

$$\chi(x, \omega) := \sum_{k=0}^{i-1} G_{x_k, x_{k+1}}(\omega),$$

where $(x_0, \ldots, x_i)$ is a nearest neighbour path on $\mathcal{C}_\infty(\omega)$ connecting $x_0 = 0$ to $x_i = x$.

**Proof of Theorem 65.** Shift invariance follows trivially from the definition.

Assuming that $\Phi(., \mathbf{0}, 1) = \mathbf{0}$, to prove $x \longmapsto x + \chi(x, \omega)$ is harmonic we require:

$$\frac{1}{\deg 0} \sum_{n=2}^{\deg 0} \left[\chi(\mathbf{0}, 1, .) - \chi(\mathbf{0}, n, .)\right] = V.$$

Now, since $\chi(\mathbf{0}, 1, .) - \chi(\mathbf{0}, n, .) = -G^n_{(\mathbf{0},1)}$, the left hand side is the limit of

$$\frac{1}{\deg 0} \sum_{n=2}^{\deg 0} \left[\psi_\varepsilon - \psi_\varepsilon \circ \tau_n\right] = (1 - Q)\psi_\varepsilon.$$

By the definition of $\psi_\varepsilon$ we have $(1 - Q)\psi_\varepsilon = -\varepsilon\psi_\varepsilon + V$. Since $\varepsilon\psi_\varepsilon$ tends to zero in $L^2$ we have the desired result.

Finally,

$$\begin{aligned} & \left\| \left[\chi((z, n), .) - \chi((z', n'), .)\right] 1_{\Phi(\cdot,z,n),\Phi(\cdot,z',n') \in \mathcal{C}^k_\infty} 1_{\Phi(\cdot,z',n') \sim \Phi(\cdot,z,n)} \right\|_2 \\ \leq \; & \sup_{1 \leq n \leq k} \|\chi(\mathbf{0}, n, .)\|_2 \\ \leq \; & C \end{aligned}$$

for some $C < \infty$. ∎

### 3.4.2. Sublinearity along coordinate axis

Take $e \in \mathbb{Z}^d$ with $|e| = 1$, and let $\rho_e : \mathbb{R}^d \to \mathbb{R}$ be the projection on to the coordinate direction $e$. Let $S_e$ be the strip emanating from the origin in the direction $e$:

$$S_e = \left\{ B_0[1] + ne : n \geq \frac{1}{2} \right\}.$$

We firstly look to define the first element of $\mathcal{C}_\infty^k$ in the strip $S_e$. Define

$$\eta_e^1(\omega) = \inf\left\{n > 0 : \exists x \in \mathcal{C}_\infty^k(\omega) \cap S_e \text{ with } \rho_e(x) = n\right\}$$

and let $v_e^1(\omega) \in \mathcal{C}_\infty^k(\omega) \cap S_e$ be the vector with $\rho_e(v_e^1(\omega)) = \eta_e^1(\omega)$ ($v_e^1$ is well defined off a set of measure zero).

For $l \in \mathbb{N}$, $l > 1$, we define the $l$th member of $\mathcal{C}_\infty^k$ in the strip:

$$\eta_e^l(\omega) = \inf\left\{n > \eta_e^{l-1}(\omega) : \exists x \in \mathcal{C}_\infty^k(\omega) \cap S_e \text{ with } \rho_e(x) = n\right\}$$

and take $v_e^l(\omega) \in \mathcal{C}_\infty^k \cap S_e$ to be the point that achieves the infimum.

In this section we look to prove the following.

**Theorem 67.** *For $\mathbb{P}'$-almost every $\omega \in \Omega_0^0$,*

$$\lim_{l\to\infty} \frac{\chi\left(v_e^l(\omega), \omega\right)}{l} = 0. \tag{3.43}$$

To prove this we follow [10], making use of the ergodic properties of the original environment under $\mathbb{P}$. Note that we work with $\mathbb{P}$ and not $\mathbb{P}_0$ or $\mathbb{P}'$ as these latter measures are not preserved under translation due to the point at the origin.

For $e \in \mathbb{R}^d$, $|e| = 1$, let $\tau_e : \Omega \to \Omega$ be the unit shift in the $e$ direction:

$$\tau_e\omega(A) := \omega\{a + e : a \in A\}$$

for Borel $A \in \mathbb{R}^d$. Then we know that $\tau_e$ is an ergodic transformation with respect to $\mathbb{P}$.

Now, working on the thinned graph, the number of points in a box of unit side is bounded by $N = N(d,k)$. Take $N$ to be minimal. The deterministic numbering of the points of bounded degree in a box thus runs through at most $n = 1, \ldots, N$. Take $n \in \{1, \ldots, N\}$ and define

$$\Omega_n = \left\{\omega \in \Omega : \Phi^k(\omega, \mathbf{0}, n) \in \mathcal{C}_\infty^k\right\} \subset \Omega.$$

Note that $\mathbb{P}(\Omega_n) > 0$. Define for $\omega \in \Omega$,

$$\hat{\eta}_1^e(\omega) = \inf\{m \in \mathbb{N} : \tau_{me}\omega \in \Omega_n\},$$

giving how far in direction $e$ we have to travel to find a box containing at least $n$ thinned points. For $j > 1$ we define the $j$th such box

$$\hat{\eta}_j^e(\omega) = \hat{\eta}_{j-1}^e(\omega) + \hat{\eta}_1^e\left(\tau_{\hat{\eta}_{j-1}^e e}\omega\right).$$

Write $w_{e,n}^j(\omega) = w_e^j(\omega) := \Phi\left(\omega, \hat{\eta}_j^e(\omega)e, n\right)$ for the corresponding point. Note that $\hat{\eta}_j^e$ is almost surely finite whether we start with $\omega$ in $\Omega_n, \Omega$ or $\Omega_0$.

Now, $\sigma_e^n : \Omega_n \to \Omega_n$ defined by

$$\sigma_e^n(\omega) = \tau_e^{\hat{\eta}_1^e(\omega)}(\omega)$$

is well defined up to a null set. Further, following the arguments of Berger and Biskup [10] we have:

**Proposition 68.** *$\sigma_e^n$ is measure preserving and ergodic with respect to $\mathbb{P}(.|\,\Omega_n)$. It is also almost surely invertible with respect to the same measure.*

Our plan is the following: we define $f : \Omega_n \to \mathbb{R}^d$ by

$$f(\omega) = \chi\left(w_e^1, \tau_{(0,n)}\omega\right), \tag{3.44}$$

and show that $f \in L^1(\mathbb{P}_n)$ for $\mathbb{P}_n(.) := \mathbb{P}(.|\Omega_n)$. We also show that $f$ has zero mean. Then, by Proposition 68, we can apply Birkhoff's ergodic theorem to $f$, giving us a formula similar to (3.43). Then, since we can apply this to $n = 1, \ldots, N$, we can put everything together and prove Theorem 67.

We abused the notation a little in the definition of $f : \Omega_n \to \mathbb{R}^d$. When we translate $\omega$ by $\tau_{(0,n)}$, $w_e^1(\omega)$ ceases to be a vertex of the graph and of course all the numberings will change. The $w_e^1$ should in fact read:

$$\Phi^{-1}_{\tau(0,n)\omega}\left(w_e^1(\omega) - \Phi((0,n),\omega)\right),$$

that is, the point to which $w_e^1$ is moved under the translation. In words, the function $f$ is the corrector between the $n$th point of the box $B_0[1]$ and the $n$th point of the first box in the direction $e$ that contains $n$ points in $\mathcal{C}_\infty^k$.

The fact that $f$ is well defined up to a set of measure zero with respect to $\mathbb{P}_n$ is due to the fact that $\chi$ is defined up to a set of null measure with respect to $\mathbb{P}'$ and hence the Campbell-Mecke Theorem gives that this set will remain null under $\mathbb{P}_n$.

The next proposition shows that we have good control over both the graph and Euclidean distance of $w_e^1$ from the origin.

**Proposition 69.** *For $\lambda > \lambda_c(d)$ there exists a constant $a = a(\lambda) > 0$ such that for all $e$ with $|e| = 1$,*

$$\mathbb{P}'\left(\left|w_e^1\right| > l\right) \leq e^{-al}. \tag{3.45}$$

*Further, let $L = L(\omega)$ be the length of the shortest path from $0$ to $w_e^1$. Then there exists a constant $C < \infty$ such that for all $n \geq 1$*

$$\mathbb{P}'(L > l) < Ce^{-al}. \tag{3.46}$$

**Proof.** We know that (3.46) holds for site percolation and will look to use this fact by dominating supercritical site percolation by a certain random field and then demonstrating that a long path in the original continuum graph forces a long path in the dominated site percolation, an event that has exponentially decaying probability.

Take the strip in the direction $e$ out from the origin:

$$S = S_e := \{ne + B_0[1] : n \in \mathbb{R}\}.$$

We slightly modify the event we have been using before: as we look up from the origin we need a point of the infinite cluster to be in the strip $S$, thus we require our cubes to have a point of the crossing cluster within the strip $S$. For a cube $Q = x + B_0[s]$ for some $x \in \mathbb{R}^d$ and $s > 0$ define

$$\mathcal{S}_0^k(Q) := \begin{array}{l} \{\text{there exists a unique crossing cluster } \mathcal{C} \text{ in } \tilde{Q} \text{ for } \tilde{Q}, \text{ all open paths} \\ \text{contained in } \tilde{Q} \text{ of diameter greater than } \frac{s}{8} \text{ are connected to } \mathcal{C} \text{ in } \tilde{Q}, \\ \mathcal{C} \text{ is crossing for each cube } Q' \subset Q \text{ such that } s(Q') \geq \frac{s}{8}, \text{ for} \\ \text{every } y \in \tilde{Q} \text{ we have } \deg(y) \leq k \text{ and } (\tau_x S) \cap \mathcal{C} \neq \phi\ \}, \end{array}$$

$$\mathcal{S}^k(Q) := \mathcal{S}_0^k(Q) \cap \{\mathcal{C}^{\vee}(Q) \text{ is crossing for } Q\} \cap \left\{\mathcal{C}^{\vee}(\tilde{Q}) \text{ is crossing for } \tilde{Q}\right\}.$$

Define the macroscopic process $\phi(x) := 1_{\mathcal{S}^k(B_{nx}[n])}$ for $x \in \mathbb{Z}^d$. As before, taking $s$ sufficiently large, we can make the probability of the above event arbitrarily close to one and thus we can apply Theorem 36 and stochastically dominate site percolation of parameter $p > p_c(d)$.

Write $\tilde{w}_e$ for the first macroscopic point of the macroscopic infinite cluster of the form $\tilde{w}_e = ne$ for some $n \in \mathbb{N}$. Then, by comparison with the first point of the dominated supercritical site percolation model, a trivial modification of the arguments in [10] give that

$$\mathbb{P}(|\tilde{w}_e| > l) \leq e^{-al}. \tag{3.47}$$

Abusing notation a little and writing $w_e^1$ to be the first member of $\mathcal{C}_\infty^k$ in the strip $\mathcal{S}$ irrespective of whether we are in $\Omega_0$ or $\Omega$, we see that there exists a $c > 0$ such that

$$\begin{aligned} \mathbb{P}'\left(\left|w_e^1\right| > l\right) &\leq c\mathbb{P}\left(\left|w_e^1\right| > l\right) \\ &\leq c\mathbb{P}\left(|\tilde{w}_e| > \frac{l-1}{s}\right) \\ &\leq ce^{-a'l}. \end{aligned}$$

The first line is a triviality with the constant $c$ a result of conditioning. The second line comes from the domination of supercritical site percolation and the final line uses the bounds on $\tilde{w}_e$ in (3.47). This proves (3.45).

The proof of (3.46) is now just a combination of (3.45) with Theorem 54, relating the graph distance between two connected points to the Euclidean distance between them. This argument is straight forward and identical to that given in [10]. ■

**Proof of Theorem 67.** Fix $1 \leq n \leq N$ and recall that $w_e^1(\omega)$ is the first point in the strip in direction $e$ that lies in a cube with at least $n$ points contained in it. We show first that $\omega \mapsto \chi(w_e^1(\omega), .\omega)$ lies in $L^1(\mathbb{P}')$ with zero mean.

Now, on the event $x := (z, n) \in \mathcal{C}_\infty^k$, $\chi(x, \cdot)$ is an $L^2(\mu)$ limit of functions $\chi_\varepsilon(x, \cdot) = \psi_\varepsilon \circ \tau_x - \psi_\varepsilon$, as $\varepsilon \downarrow 0$. Since $\mathbb{P}'(\cdot) \leq \mathbb{E}'(\omega\{0\})\mu(\cdot)$, the limit is in $L^2(\mathbb{P}')$ too. Now,

$$\left|\chi\left(w_e^1, \tau_{(0,n)}\omega\right)\right| \leq \sum_{|m|_\infty \leq L} \sum_{i=1}^N \left|G_i^\varepsilon \circ \tau_m(\omega)\right|,$$

where $L$ is the length of the path from $\tau_{(0,n)}\omega$ to $w_e^1$.

We know from Theorem 65 that $\|G_i^\varepsilon \circ \tau_m\|_2 \leq \|G_i^\varepsilon\|_2 < C$ for all $i, m$ and that the number of terms in the sum is bounded by $N(2L(\omega) + 1)^d$ which has all moments by (3.46). We then apply Lemma 4.5 from [10] to see that $\sup_{\varepsilon > 0} \|\chi_\varepsilon(w_e, .)\|_r < \infty$ for all $r \in [1, 2)$.

Now, note first that a uniform bound on the $L^r$-norm of $\chi(w_e^1, .)$ for some $r > 1$ implies that the family $\{\chi_\varepsilon(w_e^1, .)\}_{\varepsilon > 0}$ is uniformly integrable. By conditioning on the events $\{\hat{\eta}_1^e = k\}$ as $k$ runs through the naturals, we can show $\chi_\varepsilon(w_e^1, .) \to \chi(w_e^1, .)$ in probability as $\varepsilon \to 0$. Combining with the uniform integrability we see that the convergence is in $L^1$ also. The convergence in probability is a result of the fact that $\mathbb{P}(\hat{\eta}_1^e > k) \to 0$ as $k \to \infty$ and the fact that for all $k$ and $n$, $\chi_\varepsilon((ke, n), .) \to \chi((ke, n), .)$ in $L^2$. Let $\delta > 0$ then for $n \in \mathbb{N}$,

$$\begin{aligned}
& \mathbb{P}\left\{\left|\chi_\varepsilon\left(w_e^1, .\right) - \chi\left(w_e^1, .\right)\right| \geq \delta\right\} \\
\leq \ & \sum_{i=1}^k \mathbb{P}\left\{\omega : \left|\chi_\varepsilon((ie, n), \omega) - \chi((ie, n), \omega)\right| \geq \delta\right\} + \mathbb{P}\left\{\hat{\eta}_1^e > k\right\} \\
\leq \ & 2\sum_{k > n} \mathbb{P}\left\{\hat{\eta}_1^e > k\right\},
\end{aligned}$$

for small enough $\varepsilon$. Letting $k$ tend to infinity we have convergence in probability.

Note that if we take $\omega \in \Omega_n$ then $w_e^1(\omega)$ is not necessarily equal to $w_e^1\left(\tau_{(0,m)}\omega\right)$. However, since our viewpoint is shifting by less than a unit distance all the above arguments will hold with trivial modifications. By Campbell-Mecke and the above we instantly see that $f \in L^1(\mathbb{P}_n)$. Further, as before we have that $f$ is the $L^1$

limit of functions $\chi_\varepsilon\left(w_e^1, \tau_{(0,n)}\cdot\right) = \psi_\varepsilon \circ \sigma_e^n - \psi_\varepsilon$, but since $\sigma_e^n$ is $\mathbb{P}_n$-preserving and ergodic this has mean zero. Thus $\mathbb{E}_n(f) = 0$.

Now,

$$\chi\left(w_e^k(\omega), \tau_{(\mathbf{0},n)}\omega\right) = \sum_{i=0}^{k-1} f \circ (\sigma_e^n)^i.$$

Applying Birkhoff's ergodic theorem to $f$, by ergodicity of $\sigma_e^n$ we obtain

$$\lim_{k\to\infty} \frac{\chi\left(w_e^k(\omega), \tau_{(\mathbf{0},n)}\omega\right)}{k} = 0$$

for almost every $\omega \in \Omega_n$. Further, for all $1 \leq n \leq N$, $\chi\left(w_{e,n}^1(\omega), \tau_{(\mathbf{0},1)}\omega\right)$ is $\mathbb{P}_1$-almost surely finite and hence for every $1 \leq n \leq N$

$$\begin{aligned} \lim_{k\to\infty} \frac{\chi\left(w_{e,n}^k(\omega), \tau_{(\mathbf{0},1)}\omega\right)}{k} &= \lim_{k\to\infty} \left[\frac{\chi\left(w_{e,n}^k(\omega), \tau_{(\mathbf{0},1)}\omega\right)}{k} + \frac{\chi\left(w_{e,n}^k(\omega), \tau_{(\mathbf{0},n)}\omega\right)}{k}\right] \\ &= 0 \end{aligned} \tag{3.48}$$

$\mathbb{P}_1$-almost surely. Using Campbell-Mecke we obtain the same results $\mathbb{P}'$-almost surely. Now note that

$$\left\{w_{e,n}^k : 1 \leq n \leq N,\ k \in \mathbb{N}\right\} = \left\{v_e^k : k \in \mathbb{N}\right\}$$

and if $w_{e,n}^k = v_e^{k'}$ then $k' \geq k$. Hence, if (3.43) fails to hold then (3.48) fails for some $1 \leq n \leq N$. Thus (3.43) holds off a $\mathbb{P}'$-null set and we are done. ■

### 3.4.3. Sublinearity on average

We now detail how these results can be extended to the full space. We present the result first for $d = 2$ and then a somewhat weaker result for $d \geq 3$. The techniques are exactly those used in [10] to move from sublinearity in coordinate directions to sublinearity on average. We are therefore quite short with some of the explanations. For a fuller exposition see [10].

$\mathbf{d = 2}$

Fix a complete set of coordinate vectors $\{e_1, \ldots, e_d\}$ with $|e_i| = 1$ for all $i$.

**Definition 70.** *Given $K > 0$, $\varepsilon > 0$ and $\omega \in \Omega$ we say that $x \in \mathcal{C}_\infty(\omega)$ is $K$, $\varepsilon$-good if*

$$|\chi(y,\omega) - \chi(x,\omega)| < K + \varepsilon|x - y|$$

*holds for every $y \in \mathcal{C}_\infty(\omega)$ such that $y \in B_x[1] + ne_i$ for $n \in \mathbb{R}$ and coordinate direction $e_i$. Write $\mathcal{G}_{K,\varepsilon} = \mathcal{G}_{K,\varepsilon}(\omega)$ for the set of $K, \varepsilon$-good sites in $\omega$.*

By Theorem 67 and the fact that the average number of points per unit volume in a strip of finite length converges to $\lambda$ as the length of the strip tends to infinity, we see that for each $\varepsilon > 0$ there is a $K < \infty$ such that $\mathbb{P}_0 (0 \in \mathcal{G}_{K,\varepsilon}) > 0$.

Fix a unit vector $e = e_i$. Define the sequence $\{y_i\}_{i\geq 0} = \{y_i(\omega)\}_{i\geq 0}$ to be the complete list of points in $\mathcal{G}_{K,\varepsilon}(\omega) \cap \{B_0[1] + ne : n \geq 1/2\}$, ordered so that $\rho_e(y_i) < \rho_e(y_{i+1})$ for all $i$, where $\rho_e$ is the projection onto the $e$th coordinate direction. Let

$$\Delta_n(\omega) = \max_{j=1,\ldots,n} (y_j - y_{j-1}),$$

then

$$\lim_{n\to\infty} \frac{\Delta_n}{n} = 0, \text{ } \mathbb{P}\text{-almost surely.}$$

For the proof, let $A$ be the event that $B_0[1]$ contains a point of $\mathcal{G}_{K,\varepsilon}(\omega)$. Then by the ergodicity of $\tau_e$ with respect to $\mathbb{P}$ we have

$$\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} 1_{\{A\}} \circ \tau_e^k = \mathbb{P}(A) > 0$$

$\mathbb{P}$-almost surely. Now, if $\frac{\Delta_n}{n}$ does not tend to zero, then there exists $\varepsilon > 0$ and a sequence $\{n_k\}_{k\in\mathbb{N}}$ with $n_k \to \infty$ such that $\Delta_{n_k} > \varepsilon n_k$ for all $k \in \mathbb{N}$. In particular one can choose the sequence such that $y_{n_k} - y_{n_k-1} > \varepsilon n_k$ for all $k$. With this choice one has:

$$\frac{1}{n_k(1+\varepsilon)} \sum_{j=0}^{n_k(1+\varepsilon)} 1_{\{A\}} \circ \tau_e^j = \frac{1}{1+\varepsilon} \frac{1}{n_k} \sum_{j=0}^{n_k} 1_{\{A\}} \circ \tau_e^j$$

for all $k \in \mathbb{N}$. Thus $\lim_{n\to\infty} \frac{1}{n+1} \sum_{k=0}^{n} 1_{\{A\}} \circ \tau_e^k$ cannot exist and we have a contradiction to the assumption that $\frac{\Delta_n}{n}$ fails to converge to zero.

**Theorem 71.** *$d = 2$. For $\mathbb{P}'$-almost every $\omega \in \Omega_0$*

$$\lim_{n\to\infty} \max_{\substack{x\in\mathcal{C}_\infty(\omega) \\ |x|_\infty \leq n}} \frac{\chi(x,\omega)}{n} = 0.$$

**Proof.** Fix $\varepsilon > 0$ and let $K_0$ be such that $\mathbb{P}(0 \in \mathcal{G}_{K,\varepsilon}) > 0$ for all $K \geq K_0$. Take $\omega \in \Omega_0$. Write $S_i^x = \{B_x[1] + ne_i : n \in \mathbb{R}\} \subseteq \mathbb{R}^2$ for a two sided strip about $x$ in the coordinate direction $e_i$. Let $(p_1(x_k))_{k\in\mathbb{Z}}$ be the projection onto $e_1$ of an exhaustive two–sided sequence of points in the strip $S_1^0 = \{B_0[1] + ne_1 : n \in \mathbb{R}\}$ that lie in $\mathcal{G}_{K,\varepsilon}(\omega)$, ordered by their value under the projection $\rho_1$. If $\Delta_n$ is the maximal gap between consecutive $x_j$'s that lie in $[-n, n]$, then define

$$\eta_1(\omega) := \inf\{N \in \mathbb{N} : \Delta_n \leq \varepsilon n \text{ for all } n \geq N\}.$$

We do the same in the direction $e_2$ with sequence $(p_2(y_k))_{k\in\mathbb{Z}}$ and corresponding $\eta_2(\omega)$. Set $\eta_0 = \max\{\eta_1, \eta_2\}$. We claim that for $n \geq \eta_0(\omega)$

$$\max_{\substack{x\in\mathcal{C}_\infty^k(\omega)\\|x|\leq n}} |\chi(x,\omega)| \leq 2K + 6\varepsilon n. \tag{3.49}$$

Define the grid of good points

$$\begin{aligned}\mathbb{G} &= \mathbb{G}(\omega)\\ &= \mathcal{C}_\infty^k(\omega) \cap \left[\{S_2^{x_i} : i\in\mathbb{Z}\} \cup \{S_1^{y_i} : i \in \mathbb{Z}\}\right].\end{aligned}$$

Now, let $x \in \mathcal{C}_\infty^k \backslash \mathbb{G}$. Then every path connecting $x$ to infinity must include at least one grid point. Applying the maximum principle for harmonic functions and using our control on $\Delta_n$ we obtain, for large enough $n$:

$$\max_{\substack{x\in\mathcal{C}_\infty^k\backslash\mathbb{G}\\|x|_\infty\leq n}} |\chi(x,\omega)| \leq 2\varepsilon n + \max_{\substack{x\in\mathbb{G}\\|x|_\infty\leq 2n}} |\chi(x,\omega)|.$$

Let $x \in \mathbb{G}\cap[-2n, 2n]^2$. Then $x$ lies in some strip, say $S_{x_k}^2$. But then

$$|\chi(x,\omega) - \chi(x_k,\omega)| \leq K + 2\varepsilon n$$

and

$$|\chi(x_k,\omega) - \chi(0,\omega)| \leq K + 2\varepsilon n.$$

Thus the triangle inequality gives

$$|\chi(x,\omega)| \leq 2K + 4\varepsilon n,$$

proving (3.49). As this holds for any $\varepsilon > 0$ we are done. ■

$\mathbf{d \geq 3}$

In higher dimensions we are only able to prove the following, weaker, result.

**Theorem 72.** *For all $\varepsilon > 0$, all $d \geq 3$ and $\mathbb{P}'$-almost every $\omega \in \Omega_0$*

$$\lim_{n\to\infty} \frac{1}{n^d} \sum_{\substack{x\in\mathcal{C}_\infty^k\\|x|\leq n}} 1_{\{|\chi(x,\omega)|\geq\varepsilon n\}} = 0.$$

We prove this by an induction argument over $\upsilon$-dimensional sections of the $d$-dimensional box $\left\{x \in \mathbb{R}^d : |x| \leq n\right\}$. For $\upsilon = 1, \ldots, d$ let

$$\Lambda_n^\upsilon = \left[-\frac{n}{2}, \frac{n}{2}\right]^\upsilon \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-\upsilon}.$$

Let $\upsilon \leq d$ and define the upper density

$$\varrho_\upsilon(\omega) = \lim_{\varepsilon\downarrow 0} \limsup_{n\to\infty} \inf_{y\in\mathcal{C}_\infty^k(\omega)\cap\Lambda_n^1} \frac{1}{|\Lambda_n^\upsilon|} \sum_{x\in\mathcal{C}_\infty^k(\omega)\cap\Lambda_n^\upsilon} 1_{\{|\chi(x,\omega)-\chi(y,\omega)|\geq\varepsilon n\}}. \tag{3.50}$$

**Lemma 73.** *Let $1 \le v < d$. If $\varrho_v = 0$ $\mathbb{P}$-almost surely, then $\varrho_{v+1} = 0$ $\mathbb{P}$-almost surely.*

**Proof.** In this proof we make use of the fact that the thinned graph has bounded degree and hence there can be no more than $N = N(d, k)$ points in a $d$-dimensional box of unit side. Note, however, that for any $\varepsilon > 0$ a ball of radius $\varepsilon > 0$ can contain $N$ points of the thinned graph. Hence there is no way to bound the number of thinned points in a set $A \subset \mathbb{R}^d$ with respect to the Lebesgue measure of $A$.

To take this into account we slightly modify the definition of a good ball:

Given $K > 0$, $\varepsilon > 0$ and $\omega \in \Omega$ we say that $x \in \mathcal{C}_\infty(\omega)$ is $K,\ \varepsilon$-good if

$$|\chi(y, \omega) - \chi(x, \omega)| < K + \varepsilon |x - y|$$

holds for every $y \in \mathcal{C}_\infty(\omega)$ such that $y \in B_x[2] + ne_i$ for $n \in \mathbb{R}$ and coordinate direction $e_i$. Write $\mathcal{G}_{K,\varepsilon} = \mathcal{G}_{K,\varepsilon}(\omega)$ for the set of $K, \varepsilon$-good sites in $\omega$.

Define $\Lambda^v_{n,j} = \tau_{je_{v+1}}(\Lambda^v_n)$ for $j \in \mathbb{N}$ and take a stack of translates of $\Lambda^v_n$, $\{\Lambda^v_{n,j}\}_{j=0,1,\ldots,L-1}$ for some $L > 0$ that we shall choose shortly. For $z \in \mathbb{Z}^d \cap \Lambda^v_n$, we say that $z$ is good if there exists $x \in \Lambda^v_{n,j} \cap \mathcal{C}^k_\infty$ with $x \in B_z[1] + je_{v+1}$ for some $j \in \{0, \ldots, L-1\}$. For a given $\delta > 0$ we pick $L = L(\delta, \lambda, d)$ deterministically such that $\Delta_0 := \{z \in \mathbb{Z}^d \cap \Lambda^v_n : z \text{ good}\}$ satisfies $|\Delta_0| \ge (1 - \delta)\left|\Lambda^v_n \cap \mathbb{Z}^d\right|$ for large enough $n$ (dependent on $\omega$).

Suppose $\varrho_v = 0$ for some $v < d$, $\mathbb{P}$-almost surely. Fix $0 < \delta < \frac{1}{2}P^2_\infty := \frac{1}{2}\mathbb{P}\left(\left|0 - \mathcal{C}^k_\infty\right| \le 1\right)^2$. Take $\varepsilon > 0$ such that

$$L\varepsilon + \delta < \frac{1}{2}P^2_\infty. \tag{3.51}$$

For a fixed, large $K$ and $\mathbb{P}$-almost every $\omega$ and $n$ sufficiently large (dependent on $\omega$), for each $j = 1, \ldots, L$ we can find $\Delta_j \subseteq \Lambda^v_{n,j} \cap \mathcal{C}^k_\infty(\omega)$ satisfying

1. $\left|\left(\Lambda^v_{n,j} \cap \mathcal{C}^k_\infty\right) \setminus \Delta_j\right| \le \varepsilon \left|\Lambda^v_{n,j}\right|$,
2. $|\chi(x, \omega) - \chi(y, \omega)| \le \varepsilon n$, $x, y \in \Delta_j$,
3. $\Delta_j \subset \mathcal{G}_{K,\varepsilon}$,
4. $(\cup_j \Delta_j) \cap \Lambda^1_n \neq \phi$,

(note that the left hand side of 1. refers to the number of points and the right hand side refers to Lebesgue measure). We can do this since $\varrho_v = 0$, the pointwise ergodic theorem and the fact that $\mathbb{P}_0(0 \in \mathcal{G}_{K,\varepsilon})$ converges to $\mathbb{P}_0\left(0 \in \mathcal{C}^k_\infty\right)$ as $K$ grows to infinity.

Set $\hat{\Delta}_j \subseteq \Lambda^v_{n,j} \cap \mathbb{Z}^d$ to be

$$\hat{\Delta}_j := \left\{z \in \Lambda^v_{n,j} \cap \mathbb{Z}^d : x \in B_z[1] \text{ for some } x \in \Delta_j\right\}.$$

Then by construction, the projection of $\hat{\Delta}_j$ onto $\Lambda_n^v \cap \mathbb{Z}^d$ covers all of $\Delta_0$ save at most $L\varepsilon |\Lambda_n^v|$ points. Hence the number of points of $\Lambda_n^v \cap \mathbb{Z}^d$ that cannot be realized as the projection of some point in $\cup\hat{\Delta}_j$ is at most $(\delta + L\varepsilon)\left|\Lambda_n^v \cap \mathbb{Z}^d\right|$.

We write $\Lambda$ to be the set of sites in $\Lambda_n^{v+1} \cap \mathcal{C}_\infty^k$ that when projected onto $\Lambda_n^v$ belong to boxes corresponding to the projections of $\Delta_1 \cup \ldots \cup \Delta_L$.

Take $1 \le i < j \le L$, then for sufficiently large $K$, sites $x \in \mathcal{C}_\infty^k(\omega)$ such that there are points $z_i \in B_x[1] + ie_{v+1}$, $z_j \in B_x[1] + je_{v+1}$ make up at least $\frac{1}{2}P_\infty^2$ fraction of the points of $\mathcal{C}_\infty^k(\omega)$ that lie in the strip $\mathbb{R}^v \times \left[-\frac{1}{2}, \frac{1}{2}\right]^{d-v}$. Since we have assumed (3.51) we know that for large enough $n$ for each pair $1 \le i < j \le L$ we can find such $z_i \in \Delta_i, z_j \in \Delta_j$. Since the $\Delta_i, \Delta_j$ satisfy condition 2 above and $z_i, z_j$ are $K, \varepsilon$-good, for any $x \in \Delta_i$, $y \in \Delta_j$ we have

$$\begin{aligned}
|\chi(x,\omega) - \chi(y,\omega)| &\le |\chi(x,\omega) - \chi(z_i,\omega)| + |\chi(z_i,\omega) - \chi(z_j,\omega)| + |\chi(y,\omega) - \chi(z_j,\omega)| \\
&\le \varepsilon n + K + \varepsilon L + \varepsilon n = K + \varepsilon L + 2\varepsilon n.
\end{aligned}$$

Hence, for every $r, s \in \Lambda$ we have for $x, y$ such that $r, s$ lie in the strips emanating from $x$ and $y$ respectively,

$$\begin{aligned}
|\chi(r,\omega) - \chi(s,\omega)| &\le |\chi(r,\omega) - \chi(x,\omega)| + |\chi(x,\omega) - \chi(y,\omega)| + |\chi(s,\omega) - \chi(y,\omega)| \\
&\le K + \varepsilon n + K + \varepsilon n + L\varepsilon + 2\varepsilon n + K + \varepsilon n \\
&= 3K + L\varepsilon + 4\varepsilon n < 5\varepsilon n
\end{aligned}$$

for large enough $n$.

If $\varrho_{v,\varepsilon}$ denotes the right hand side of (3.50) before taking $\varepsilon \downarrow 0$, we see that

$$\varrho_{v+1,5\varepsilon}(\omega) \le \delta + L\varepsilon$$

for $\mathbb{P}$-almost every $\omega$. The left hand side increases as $\varepsilon \downarrow 0$ but the right hand side decreases. Thus taking $\varepsilon \downarrow 0$, $\delta \downarrow 0$ proves that $\varrho_{v+1} = 0$ $\mathbb{P}$-almost surely. ■

**Proof of Theorem 72.** Firstly, by Theorem 67, we see that $\varrho_1(\omega) = 0$ for $\mathbb{P}'$-almost every $\omega$ and by Campbell-Mecke we extend this to $\mathbb{P}$-almost surely. Using induction on dimension we then see that $\varrho_d(\omega) = 0$ for $\mathbb{P}$-almost every $\omega$, applying Campbell-Mecke in reverse we see that this holds for $\mathbb{P}'$ almost every $\omega$.

Let $\omega \in \Omega_0$. By Theorem 67 for each $\varepsilon > 0$ there is $n_0 = n_0(\omega)$ such that $\mathbb{P}(n_0 < \infty) = 1$ and for all $n \ge n_0$ we have $|\chi(x,\omega)| < \varepsilon n$ for all $x \in \Lambda_n^1 \cap \mathcal{C}_\infty^k(\omega)$. Hence for $n \ge n_0$

$$\sum_{\substack{x \in \mathcal{C}_\infty^k \\ |x| \le n}} 1_{\{|\chi(x,\omega)| \ge \varepsilon n\}} \le \inf_{y \in \mathcal{C}_\infty^k \cap \Lambda_n^1} \sum_{\substack{x \in \mathcal{C}_\infty^k \\ |x| \le n}} 1_{\{|\chi(x,\omega) - \chi(y,\varepsilon)| \ge 2\varepsilon n\}}.$$

Thus by the definition of $\varrho_d$, the fact that it is zero proves the desired result. ■

### 3.4.4. Proof of the invariance principle

We now follow the scheme of Biskup and Prescott in [13] to show sublinearity of the corrector on the thinned graph, which we can easily extend to the full graph by harmonicity. We will then be able to draw our results together and prove the invariance principle.

**Theorem 74.** *For $\mathbb{P}'$-almost every $\omega$*

$$\lim_{n\to\infty} \max_{\substack{x\in\mathcal{C}_\infty^k \\ \cap B_0[n]}} \frac{|\chi(x,\omega)|}{n} = 0.$$

This will follow from several results taken from [13].

**Lemma 75.** *For every $\theta > d$, $\mathbb{P}'$-almost surely*

$$\lim_{n\to\infty} \max_{\substack{x\in\mathcal{C}_\infty^k \\ \cap B_0[n]}} \frac{|\chi(x,\omega)|}{n^\theta} = 0$$

**Proof.** Set

$$R_n = \max_{\substack{x\in\mathcal{C}_\infty^k \\ |x|\le n}} |\chi(x,\omega)|.$$

By the large deviations result comparing Euclidean and graph distances, Theorem 54,

$$\lambda(\omega) := \sup_{x\in\mathcal{C}_\infty^k} \frac{d_\omega^k(0,x)}{|x|} < \infty$$

$\mathbb{P}'$-almost surely. Thus we need only show that $R_n/n^\theta \to 0$ on $\{\lambda(\omega) \le \lambda\}$ for every $\lambda < \infty$.

Now, on the event $\{\lambda(\omega) \le \lambda\}$, every $x \in \mathcal{C}_\infty^k$ with $|x| \le n$ can be reached by a path on $\mathcal{C}_\infty^k$ within $[-\lambda n, \lambda n]^d$. Hence on $\{\lambda(\omega) \le \lambda\}$

$$R_n \le \sum_{\substack{x\in\mathcal{C}_\infty^k \\ x\le\lambda n}} \sum_{\substack{y\sim x \\ y\in\mathcal{C}_\infty^k}} |\chi(x,\omega) - \chi(y,\omega)|.$$

Invoking the uniform bound on the change of $\chi$ along an edge in the thinned cluster we see

$$\left\| R_n 1_{\{\lambda(\omega)\le\lambda\}} \right\|_2 \le Cn^d$$

for some constant $C = C(\lambda, k, d) < \infty$. Take $\varepsilon > 0$, then applying Chebychev's inequality we see

$$\mathbb{P}'\left(\frac{R_n}{n^\theta} > \varepsilon\right) \le \frac{1}{\varepsilon^2 n^{2\theta}} Cn^{2d} = Cn^{-(2\theta-2d)}.$$

Setting $n = 2^m$ and summing over $m$, Borel-Cantelli yields

$$\mathbb{P}'\left(\frac{R_n}{n^\theta} > 2\varepsilon \text{ infinitely often}\right) = 0.$$

As $\varepsilon > 0$ is arbitrary we are done. ■

Let $(Z_t : t \geq 0)$ be the continuous version of the time changed random walk on $\mathcal{C}_\infty^k$. That is, $Z_t := \tilde{X}_{N_t}$, where $N_t$ is an independent Poisson process of rate $t$. The proof of the following bounds comes directly from Corollary 26, Theorem 42 and Proposition 51.

**Lemma 76.** *For a deterministic sequence $b_n = o(n^2)$ and $\mathbb{P}'$-almost every $\omega$,*

$$\sup_{n\geq 1} \max_{\substack{x\in\mathcal{C}_\infty^k \\ |x|\leq n}} \sup_{t\geq b_n} \frac{E_{\omega,x}|Z_t - x|}{\sqrt{t}} < \infty$$

*and*

$$\sup_{n\geq 1} \max_{\substack{x\in\mathcal{C}_\infty^k \\ |x|\leq n}} \sup_{t\geq b_n} t^{d/2} P_{\omega,x}[Z_t = x] < \infty.$$

Note that as we do not currently have good control over the tail of the random variable corresponding to the spread of traps, this result is only proven to hold for the time changed random walk.

Sublinearity of the corrector on the thinned graph will be proved directly from the following lemma, which uses the diffusive bounds.

**Lemma 77.** *Let*

$$R_n := \max_{\substack{x\in\mathcal{C}_\infty^k \\ |x|\leq n}} |\chi(x,\omega)|.$$

*Then for each $\varepsilon > 0$ and $\delta > 0$ there exists an almost surely finite random variable $n_0 = n_0(\omega, \varepsilon, \delta)$ such that*

$$R_n \leq \varepsilon n + \delta R_{3n},\ n \geq n_0.$$

**Sketch of Proof.** The proof is identical to that in [13]. We give a summary.

Let $z \in \mathcal{C}_\infty^k \cap B_0[n]$ be the point at which the maximum of $R_n$ is achieved and let $Z_t$ be the continuous time, time changed random walk started at $z$. By harmonicity and the optional stopping theorem

$$R_n \leq E_{\omega,z}|\chi(Z_{t\wedge S_n}) + Z_{t\wedge S_n} - z|. \tag{3.52}$$

Take $S_n$ to be the stopping time

$$S_n := \inf\{t > 0 : |Z_t - z| \geq 2n\},$$

then for large enough $n$, $|\chi(Z_{S_n},\omega)| \leq R_{3n}$.

We take $t := \xi n^2$ for suitably small $\xi > 0$, and break it down into two cases. Either we remain in $B_0[3n]$ up to time $t$, ie $S_n \geq t$. In this case by sublinearity on average (Section 3.4.3) and Lemma 76, for large enough $n$ the probability of the particle being in the set

$$\mathcal{O}_n := \left\{ x \in \mathcal{C}_\infty^k : |x| \leq n, |\chi(x,\omega)| \geq \frac{1}{2}\varepsilon n \right\}$$

at time $t = \xi n^2$ tends to zero as $n \to \infty$. If $S_n \geq t$ and $Z_t \notin \mathcal{O}_n$ then the expectation in (3.52) is small.

Otherwise $t > S_n$. On this event, the right hand side of (3.52) is bounded by $R_{3n} + 3n$. Further, Lemma 76 shows that the probability that $t > S_n$ is small if the constant $\xi > 0$ is chosen suitably.

Combining these arguments gives the result. ■

We can now prove the sublinearity of the corrector on the thinned graph.

**Proof of Theorem 74.** Take $R_n$ as above and suppose $R_n/n \not\to 0$. Pick $c$ with

$$0 < c < \limsup_{n\to\infty} R_n/n.$$

Let $\theta$ be as in Lemma 75 and set $\varepsilon := c/2$ and $\delta := 3^{-(\theta+1)}$. Note that for $c' \geq c$, we trivially have $c' - \varepsilon \geq 3^\theta \delta c'$. If $R_n \geq cn$, which happens infinitely often, and $n \geq n_0$ then Lemma 77 implies

$$R_{3n} \geq \frac{c-\varepsilon}{\delta} n \geq 3^\theta cn.$$

Inductively we have $R_{3^k n} \geq 3^{k\theta} cn$. But this contradicts Lemma 75, since we have

$$\frac{R_{3^k n}}{3^{k\theta}} \geq cn$$

and so fails to tend to zero as $k$ tends to infinity. ■

We are now ready to prove the functional CLT.

**Proof of Theorem 33.** We know that the order of the largest trap in $B_0[n]$ is bounded above by $C(\omega)(\log n)^{d/(d-1)}$. Take a trap, $T$, contained in $B_0[n]$, then by harmonicity and the maximum principle

$$\max_{x\in T} |x + \chi(x,\omega)| \leq \max_{x\in\partial_{ext}T} |x + \chi(x,\omega)|,$$

and hence

$$\max_{x\in T} |\chi(x,\omega)| \leq \max_{x\in\mathcal{C}_\infty^k\cap B_0[n]} |\chi(x,\omega)| + C(\omega)(\log n)^{d/(d-1)}.$$

Thus, by Theorem 74, the corrector is sublinear on the unthinned graph $\mathcal{C}_\infty(\omega)$.

Set $\varphi_\omega(x) = x + \chi(x,\omega)$ for $x \in \mathcal{C}_\infty(\omega)$ and define $M_n := \varphi_\omega(X_n)$, a martingale by property 2. of Theorem 65. Fix $v \in \mathbb{R}^d$ and define

$$f_K(\omega) := E_{\omega,0}\left((v \cdot M_1)^2 1_{\{|v \cdot M_1| \geq K\}}\right).$$

By square integrability of the corrector, $f_K \in L^1(\Omega, \mathcal{F}, \mu)$ for all $K$. By the ergodicity of the Markov chain from the view of the particle

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} f_K \circ \tau_{X_k}(\omega) = \mathbb{E}_\mu f_K,$$

for $\mathbb{P}'$-almost every $\omega$ and $P_{\omega,0}$-almost every path $X = (X_k)$ of the random walk. Setting $K = 0$ and $K = \varepsilon\sqrt{n}$ along with the monotonicity of $K \longmapsto f_K$ verifies the Lindeberg-Feller Martingale Functional CLT, see [30] page 411. Hence

$$t \mapsto \frac{1}{\sqrt{n}}\left(v \cdot M_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\, v \cdot \left(M_{\lfloor nt \rfloor + 1} - M_{\lfloor nt \rfloor}\right)\right)$$

converges weakly to Brownian motion with mean zero and covariance $\mathbb{E}_\mu f_0$. We then invoke the Cramer-Wold device to see that the linear interpolation of

$$t \mapsto \frac{M_{\lfloor nt \rfloor}}{\sqrt{n}}$$

scales to $d$-dimensional Brownian motion with covariance matrix $D$ given by

$$D_{ij} = \mathbb{E}_\mu E_{\omega,0}\left((e_i \cdot M_1)(e_j \cdot M_1)\right).$$

The sublinearity of the corrector implies that

$$X_n - M_n = \chi(\omega, X_n) = o(|X_n|) = o(|M_n|) = o\left(\sqrt{n}\right),$$

and hence $X_n$ converges weakly to Brownian motion with covariance matrix $D$.

Now, note that the reflection and rotation symmetry of $\mu$ force $D$ to be of the form $D = \frac{1}{d}\sigma^2 1$ for

$$\sigma^2 = \mathbb{E}_\mu E_{\omega,0} |M_1|^2.$$

The only way this can be zero is if $\mu$-almost everywhere we have $\chi(.,x) = -x$ for all $x \in \mathcal{C}_\infty$, but this cannot be true since the corrector is sublinear. Thus we have convergence to non-degenerate Brownian motion. ■

# 4. Random walk in dynamic environment

In this chapter we consider the space-time random conductance model. We take environments of the form $\omega = \left(\omega_{xy}(t)\right)_{\substack{x,y\in\mathbb{Z}^d \\ t\in K}}$ for $K$ either $\mathbb{R}$ or $\mathbb{N}$, and consider the random walk on $\omega$. We assume the symmetry condition $\omega_{xy}(t) = \omega_{yx}(t)$ for all edges and times - this will prove to be key in our analysis. There are several different walks that one can consider - walks in both continuous or discrete time moving at either constant speed or variable speed. We will begin by introducing the various possibilities.

Previously we have considered the constant speed walk in discrete time. In the dynamic setting this corresponds to the discrete time Markov process on $\mathbb{N}\times\mathbb{Z}^d$ with transition probabilities

$$\begin{aligned} &P\left(Y_n = (m,y)\,|Y_{n-1} = (m-1,x)\right) \\ &= \begin{cases} \frac{\omega_{xy}(m-1)}{\sum_z \omega_{xz}(m-1)} & \text{if } x\sim y \text{ and } \sum_z \omega_{xz}(m-1)\neq 0 \\ 1 & \text{if } x=y \text{ and } \sum_z \omega_{xz}(m-1) = 0 \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

This walk presents several technical challenges: if one looks for an invariant measure for the walk considered on the space-time graph then the measure does not generally have a simple form and the walk is never reversible. Further, this invariant space-time measure is not spatially consistent - the projection of the invariant measure at time $t$ onto $\mathbb{Z}^d$ will be different from the projection at time $s\neq t$. This restricts the tools available with which to analyze the walk.

With this in mind we introduce the variable speed random walk in continuous time. This is the Markov process $X_t$ with generator at time $t$

$$\mathcal{L}_t f(x) = \sum_{y\sim x}\omega_{xy}(t)\left(f(y)-f(x)\right). \tag{4.1}$$

We will restrict our environments to those where every edge changes only finitely often in a bounded interval. Thus the edge weights are step functions:

$$\omega_e(t) = \sum_{i=1}^{\infty}\alpha_i 1_{A_i}(t),$$

for $\alpha_i \geq 0$, $A_i = [a_i, b_i]$ and $|\{i : t\in A_i\}| < \infty$ for all $t\in\mathbb{R}$.

The flat measure $\pi(x) = 1$ for all $x \in \mathbb{Z}^d$, is the invariant measure for the variable speed walk with

$$\langle \mathcal{L}_t f, g\rangle_\pi = \langle f, \mathcal{L}_t g\rangle_\pi \tag{4.2}$$

for all $t \in \mathbb{R}$. Although this does not imply $P(X_t = y \,|X_s = x) = P(X_t = x \,|X_s = y)$ the fact that $\mathcal{L}_t$ is self-adjoint will be important in our analysis. It is this variable speed, continuous time walk that provides most of our results.

One can also consider the discrete time, variable speed walk: suppose that $\sum_z \omega_{xz}(n) \in [0,1]$ for all $x \in \mathbb{Z}^d$ and $n \in \mathbb{N}$. Define $Z_n$ to be the random walk with transitions

$$P(Z_n = y \,|Z_{n-1} = x) \begin{cases} \omega_{xy}(n-1) & \text{if } x \sim y \\ 1 - \sum_z \omega_{xz}(n-1) & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}.$$

We will see that this walk can be very different from $X_t$. For example note that there is the possibility for $Z_n$ to have deterministic steps - an impossibility for $X_t$.

We will primarily consider heat kernel behaviour for the variable speed, continuous time random walk in various cases - placing conditions on the geometry of $\omega$ that lead to upper bounds on the heat kernel. We then give examples of random environments that satisfy these conditions with probability one.

We begin with the case $\mathbb{P}(\omega_e \in [a,b]) = 1$ for $0 < a \leq b < \infty$. In the static time model, a standard Nash inequality holds and full off-diagonal upper and lower heat kernel bounds hold - see for example [5]. This is also true in the dynamic setting as the Nash inequality holds uniformly in time and full off-diagonal heat kernel bounds can be achieved through the same methods as in the static case. This was proved in [31] and we recap the proof of the on diagonal upper bounds as the techniques are those that we will modify to attack more general cases.

As discussed in previous sections, if the lower bound is missing: $\mathbb{P}(\omega \in (0,1]) = 1$, one can see more interesting behaviour. The work [11] in the static (constant speed) case shows that anomalous heat kernel behaviour can occur. By simple modifications to this argument one can show that this behaviour extends to the variable speed continuous time walk in the static case. In particular it is proven that the best upper bound for the heat kernel in $d \geq 5$ are $O(t^{-2})$. We present two main results for this case. We show that in the dynamic time setting with dimension $d \geq 3$ lower bounds on the heat kernel close to $O(t^{-1})$ can be achieved - ensuring a more anomalous heat kernel behaviour than seen in the static environment. Anomalous heat kernel behaviour in this setting occurs when the random walk remains closer to the origin than one would normally expect. A trap that facilitates this behaviour involves a vertex surrounded by weak bonds. If the weak bonds are of order $O(n^{-1})$ then one would expect the random walk to remain in the trap for time $O(n)$ provided that the trap is not removed by the time dynamic. To enter or exit the trap the walk has to pay $O(n^{-1})$. However, if the time dynamic ensures that traps of this type last for time $O(n)$ then the walk can wait until the trap disappears and escape from the trap for free. This strategy leads to stronger anomalous behaviour than the static case but relies on dynamic environments existing whose traps disappear at the appropriate times. We will give examples of such environments.

In particular this ensures at least three extrema for the heat kernel for the dynamic random walk. As the mixing of the random environment increases the walk resembles the random walk on the annealed graph - this is just the simple random walk on $\mathbb{Z}^d$ and hence has heat kernel bounded above and below by $O\left(t^{-d/2}\right)$. Conversely, as the mixing of the environment decreases, the walk resembles the random walk on the static graph and as described this can have heat kernel behaviour $O(t^{-2})$. Our example shows that in between these cases are environments with heat kernel decay close to $O(t^{-1})$. We do not present corresponding upper bounds but do give heuristics for why $O(t^{-1})$ is the most anomalous the heat kernel can be.

We also, as in Chapter 2, look for conditions that ensure standard heat kernel behaviour in the case $\omega \in (0,1]$. We approach this problem via isoperimetric techniques - looking for the spatial graphs at a fixed time to satisfy $d$-dimensional isoperimetric inequalities. When isoperimetric inequalities hold we obtain Nash inequalities and they in turn enable upper bounds on the heat kernel. In particular we show that if the environment evolves in time in an ergodic way and geometrically the spatial environment looks $d$-dimensional at every time $t$, then the space-time walk will also look $d$-dimensional in its heat kernel. We will make these ideas precise later.

Next, we consider the case $\mathbb{P}(\omega \in [1,\infty)) = 1$. In this case standard on diagonal upper heat kernel bounds hold due to a uniform Nash inequality being satisfied as $\omega$ is bounded below. The question is thus can we control the off-diagonal behaviour? In the static case this question is attacked in [7], where it is shown that by introducing an auxiliary distance function, the methods of [25] provide long range bounds. The methods of Bass-Nash then enable full off-diagonal heat-kernel upper bounds to be proven. In the dynamic case we have struggled to find an appropriate space-time distance to use in the methods of [25]. We outline properties that a suitable space-time distance should satisfy and describe the problems faced in finding such a distance. We then give an example where incomplete upper bounds can be proven - bounds that we believe not to be sharp.

Finally, we will discuss functional central limit theorems in this setting. These models have been considered previously in the literature, often without the assumption that $\omega_{xy} = \omega_{yx}$. Various techniques have been employed including heavy, analytic cluster expansions, probabilistic arguments via regeneration times, and the Kipnis-Varadhan methods discussed earlier in this thesis. The results obtained thus far normally require assumptions on the ellipticity of the walk and mixing of the environment. We discuss these methods in more detail and why currently our heat kernel techniques cannot be used in a fashion similar to Chapter 3 to prove a functional CLT.

In terms of heat kernel estimates, there is not a huge amount in the literature. The Appendix of [31] proves full off-diagonal results in the case where weights are bounded away from zero and infinity. The paper [22] proves that if

$$m(x) := \sum_{y} \omega_{xy}(k)$$

is independent of $k$ and the environment satisfies both a uniform ellipticity condition and uniform Sobolev inequality then standard off-diagonal upper bounds and on diagonal lower bounds hold. There are also results in the case where space is taken to be finite, these can be found in [51] and [52].

We finish the introduction with some notation. As we are walking on a space-time graph, starting points should be in space-time and hence we write

$$P^{\omega}_{(s,x)}(X_t = y)$$

for the random walk on the space-time graph started at $(s, x) \in \mathbb{R} \times \mathbb{Z}^d$, and abuse notation by implicitly assuming that the time coordinate of $X_t$ is equal to $t$ and hence the walk has been run for time $t - s$. When $P^{\omega}_{x}$ is written it should be assumed that we are taking $s = 0$.

## 4.1. Variable Speed, Bounded Conductances

We begin with the simplest case: we consider the variable speed continuous time random walk on an environment with conductances bounded above and below: there exist $0 < a \leq b < \infty$ such that $\mathbb{P}(\omega_e(s) \in [a, b]) = 1$. This is the only assumption that we make on the environment. This case has been considered previously in [31], with Proposition 78 proven. We present the details of the on diagonal upper bounds both for completeness and to show how Nash inequalities lead to heat kernel control as this idea will be crucial in the unbounded cases. The bounded conductance example is particularly simple as uniform Nash inequalities hold and hence standard heat kernel technology can be used to prove standard upper bounds.

Let $(X_t)_{t\in\mathbb{R}}$ be the continuous time, variable speed walk on $\omega = (\omega_e(s))_{\substack{e\in\mathbb{E}^d \\ s\in\mathbb{R}}}$ as introduced in the introduction. As flat space measure, $\pi(x) \equiv 1$, is the unique (up to multiplicative constant) invariant measure for each time marginal transition kernel, $\pi$ is also the unique invariant measure for $P^{\omega}$. It thus makes sense to refer to $P^{\omega}(\cdot,\cdot)$ as the heat kernel for the space-time walk.

**Proposition 78.** *Take $0<a\leq b<\infty$. Suppose $\omega$ is a space-time environment satisfying $\omega_e(s)\in[a,b]$ for all $e\in\mathbb{E}^d$ and $s\in\mathbb{R}$. Then there exists $C=C(a,b,d)$ such that*

$$P^{\omega}\left(X_{s+t}=y\,|X_s=x\right)\leq Ct^{-d/2}. \tag{4.3}$$

*for all $s\in\mathbb{R},t>0$.*

**Proof.** This Proposition is Proposition B.2 of [31]. For simplicity take $s=0$, the general case follows by the same arguments.

Due to the ellipticity condition and the fact that $\mathcal{L}_s$ is reversible with respect to $\pi$ for all $s$, there exists a constant $c=c(a,b,d)$, independent of $s$ such that the following Nash inequality holds uniformly in $s$: for all $f\in l_0\left(\mathbb{Z}^d\right)$ we have

$$\mathcal{E}_{\mathcal{L}_s}(f,f)\geq c\left\|f\right\|_{L^2(\pi)}^{2+\frac{4}{d}}\left\|f\right\|_{L^1(\pi)}^{-\frac{4}{d}}, \tag{4.4}$$

for the Dirichlet form

$$\mathcal{E}_{\mathcal{L}_s}(f,g):=\sum_{x,y}\omega_{xy}(s)\left(f(y)-f(x)\right)\left(g(y)-g(x)\right).$$

Obtaining the Nash inequality is standard under the ellipticity condition since $\mathcal{L}_s$ is reversible with respect to $\pi$ (see, for example, [55]).

Take $f\in L^1(\pi)$ to be non-negative with $\|f\|_1=1$ and define $u(t):=\|P_{0,t}f\|_{L^2(\pi)}^2$. Then by (4.4)

$$\begin{aligned}\frac{d}{dt}u(t) &= -2\mathcal{E}_{\mathcal{L}_t}\left(P_{0,t}f,P_{0,t}f\right)\\ &\leq -c\left\|P_{0,t}f\right\|_2^{2+\frac{4}{d}}\end{aligned}$$

which implies that

$$u(t)\leq c't^{-d/2}$$

and hence

$$\left\|P_{0,t}\right\|_{1\to2}\leq c't^{-d/4}.$$

We will show in Proposition 84 that $P_{0,t}^{*}$ is the transition semi-group for the random walk on the "reversed graph" for more general environments. Leaving the details until then, it is enough to know that walk on the reversed graph must also satisfy the uniform Nash inequalities and thus

$$\left\|P_{0,t}^{*}\right\|_{1\to2}\leq c't^{-d/4}.$$

Hence, by Cauchy-Schwarz

$$\begin{aligned}
\|P_{0,t}\|_{1\to\infty} &= \left\|P_{t/2,t}\circ P_{0,t/2}\right\|_{1\to\infty} \\
&\leq \left\|P_{t/2,t}\right\|_{2\to\infty}\left\|P^*_{0,t/2}\right\|_{2\to\infty} \\
&= \left\|P^*_{t/2,t}\right\|_{1\to 2}\left\|P_{0,t/2}\right\|_{1\to 2} \\
&\leq ct^{-d/2}.
\end{aligned}$$

■

As all our constants are uniform across starting points, standard methods lead to full off-diagonal bounds. We do not give the details here but refer the reader to [31], where near diagonal lower estimates are also proven.

## 4.2. Continuous time, variable speed, $\omega \in (0,1]$.

In this section we will consider the case of conductances that are bounded above but not below, presenting examples of dynamic environments where standard heat kernel behaviour is observed. Our main result being that if the environment at time $t$ looks geometrically $d$-dimensional and the environment is ergodic over time then the on diagonal heat kernel behaviour will look like $t^{-d/2}$.

As conductances are bounded above, the methods of [25] pass into the dynamic setting, bounding the probability that the walk travels a large distance in a short period of time and giving Carne-Varopoulos type bounds - upper bounds on the probability that the walk travels a large distance in a short time. In particular we obtain bounds on the time it takes the walk to exit a box - see Proposition 79 and Corollary 80 below.

With control of the probability that the walk exits a finite box, we can pick a box sufficiently large that the random walk in unlikely to be affected by the boundary of the box. Analysis of the original random walk is then reduced to the analysis of the random walk restricted to the finite box. Isoperimetric inequalities on the restricted graph imply Nash inequalities with respect to the walk on the finite box which in turn garners control on the heat kernel for the walk on the finite box. This idea is used in [40] to obtain standard on diagonal upper bounds for the heat kernel for the random walk on supercritical percolation. We will extend this to the dynamic time case, although we will not allow $\omega_e = 0$.

The section is thus broken into three distinct parts. Section 4.2.1 provides the adaptations to [25] that give long range bounds on the heat kernel and control the exit time for boxes. Section 4.2.2 analyzes the walk restricted to the box and gives conditions on the geometry of the space-time environment that, if satisfied, ensure standard on diagonal heat kernel decay. Finally, Section 4.2.3 provide examples of environments that satisfy these geometrical conditions.

#### 4.2.1. Methods of Davies

We begin by stating the main results obtained through the application of the methods of [25] to the time dynamic case. For $x, y \in \mathbb{Z}^d$ let $d(x,y)$ be the graph distance between $x$ and $y$.

**Proposition 79.** *Suppose $\omega_e(s) \in [0,1]$ for all $e \in \mathbb{E}^d$ and all $s \in \mathbb{R}$. There exists constant $C_{dav}$ independent of $\omega$ such that for $x, y \in \mathbb{Z}^d$: if $t \geq d(x,y)$ then*

$$P^{\omega}_{(s,x)}(X_{s+t} = y) \leq \exp\left(-C_{dav}\frac{d(x,y)^2}{t}\right). \tag{4.5}$$

*If $t \geq 0$ then*

$$P^{\omega}_{(s,x)}(X_{s+t} = y) \leq \exp\left(-d(x,y)\log\left(\frac{d(x,y)}{edt}\right)\right). \tag{4.6}$$

Equation (4.6) is (up to constants) in essence the best general bound one can obtain for all $t \geq 0$ as the large deviations when $t$ is much smaller than $d(x,y)$ are due to the Poisson distribution and not the geometry of the graph. See [45] for the corresponding lower bound.

Introduce the exit time from the box $[-n,n]^d$ for the random walk started at the origin

$$\tau_n := \inf\left\{t : X_t^{(0,0)} \notin [-n,n]^d\right\}.$$

The bounds of Proposition 79 lead directly to probabilistic control on $\tau_n$.

**Corollary 80.** *Suppose $\omega_e(s) \in [0,1]$ for all $e \in \mathbb{E}^d$ and $s \in \mathbb{R}$. There exist constants independent of $\omega$ such that for $t \geq n$*

$$P^{\omega}(\tau_n \leq t) \leq C_1 n^{-d} e^{-C_2 n} + C_3 n^{d-1} t \exp\left(-\frac{C_4 n^2}{t}\right).$$

**Proof.** Let $N_\lambda$ be an independent Poisson process of rate $\lambda = 1$ and note that

$$\begin{aligned}
P_{(0,x)}(X_t = y) &\geq P_{(0,x)}(X_u = y \text{ for all } u \in [s,t] \text{ for some } s \in [t-1,t]) \\
&\geq P_{(0,x)}\left(X_s = y \text{ and } N_{(t-s)2d} = 0 \text{ for some } s \in [t-1,t]\right) \\
&\geq P_{(0,x)}(X_s = y \text{ for some } s \in [t-1,t])\, P(N_{2d} = 0) \\
&= C(d)\, P_{(0,x)}(X_s = y \text{ for some } s \in [t-1,t])
\end{aligned}$$

and hence the bounds (4.5) and (4.6) apply (up to multiplicative constant) to $P_{(0,x)}(X_u = y \text{ for some } u \in [t-1,t])$.

We now apply these bounds

$$\begin{aligned}
P^{\omega}(\tau_n \le t) &\le \sum_{y\in\partial[-n,n]^d}\sum_{i=1}^{t} P_{(0,x)}(X_u = y \text{ for some } u\in[i-1,i]) \\
&= \sum_{y\in\partial[-n,n]^d}\sum_{i=1}^{n} P_{(0,x)}(X_u = y \text{ for some } u\in[i-1,i]) \\
&\quad + \sum_{y\in\partial[-n,n]^d}\sum_{i=n+1}^{t} P_{(0,x)}(X_u = y \text{ for some } u\in[i-1,i]) \\
&\le \sum_{y\in\partial[-n,n]^d}\sum_{i=1}^{n} C(d)^{-1} e^{-c_1 n} \\
&\quad + \sum_{y\in\partial[-n,n]^d}\sum_{i=n+1}^{t} C(d)^{-1}\exp\left(-c_2\frac{d(x,y)^2}{i}\right) \\
&\le c_3 n^d e^{-c_1 n} + c_4 n^{d-1} t \exp\left(-\frac{c_2 n^2}{t}\right).
\end{aligned}$$

■

Note that the choice of a box centred at the space-time origin in Corollary 80 is purely for notational convenience.

**Proof of Proposition 79.** Let $(P_{s,t})_{s\le t}$ be the semigroup associated with the variable speed walk on $\omega$. Let $f$ and $g$ be functions of finite support. Then as the environment changes locally a finite number of times in a finite time interval, for almost all $t$ the following derivative is well defined:

$$\begin{aligned}
\frac{d}{dt}\langle g, P_{s,t} f\rangle_{\pi} &= \langle g, \mathcal{L}_t P_{s,t} f\rangle_{\pi} \\
&= -\mathcal{E}_{\mathcal{L}_t}(g, P_{s,t} f) \\
&= \frac{1}{2}\sum_{x\in\mathbb{Z}^d}\sum_{y\sim x}\omega_{xy}(t)(g(x)-g(y))(P_{s,t}f(x)-P_{s,t}f(y)).
\end{aligned}$$

We are interested in the transition kernel

$$P_{(s,x)}(X_{s+t} = y) = P_{s,t}\delta_y(x).$$

We follow [25] and set $\mathcal{L}_{t,\phi} := \phi\mathcal{L}_t\phi^{-1}$ for positive $\phi, \phi^{-1}\in l^{\infty}$. Let $\left(P^{\phi}_{s,t}\right)_{s\le t}$ be the semigroup driven by $(\mathcal{L}_{t,\phi})$. We look for integral estimates for $P^{\phi}$. Note that the integral kernel satisfies $P^{\phi}((s,x),(t,y)) = \phi(x)P((s,x),(t,y))\phi(y)^{-1}$. For simplicity set $s=0$. For $f$ of bounded support on $\mathbb{Z}^d$, let

$$f_t := P^{\phi}_{0,t} f.$$

We see

$$\frac{d}{dt}\|f_t\|_2^2 = 2\left\langle \phi\mathcal{L}_t\phi^{-1} f_t, f_t\right\rangle_{\pi}.$$

We look for the best constant $c_t(\phi)$ such that

$$\left\langle \phi \mathcal{L}_t \phi^{-1} f, f \right\rangle \leq c_t(\phi) \|f\|_2^2, \tag{4.7}$$

for all $f \in \mathrm{Dom}\left(\phi \mathcal{L}_t \phi^{-1}\right)$. Now, by Section 3 of [25] for all $t \in \mathbb{R}$

$$c_t(\phi) \leq \sup_{x \in \mathbb{Z}^d} b_t(\phi, x)$$

where

$$\begin{aligned} b_t(\phi, x) \quad : &= \frac{1}{2} \sum_{y \sim x} \omega_{x,y}(t) \left\{ \frac{\phi(y)}{\phi(x)} - \frac{\phi(x)}{\phi(y)} - 2 \right\} \\ &\leq \frac{1}{2} \sum_{y \sim x} \left\{ \frac{\phi(y)}{\phi(x)} - \frac{\phi(x)}{\phi(y)} - 2 \right\}. \end{aligned}$$

As $b_t(\phi, x)$ is bounded independently of $t$, there exists $c(\phi)$ such that $c_t(\phi) \leq c(\phi)$. Hence, for all $t$

$$\partial_t \|f_t\|_2^2 \leq 2c(\phi) \|f_t\|_2^2.$$

Solving the differential equation yields

$$\|f_t\|_2^2 \leq e^{2c(\phi)t} \|f\|_2^2. \tag{4.8}$$

Setting $f = \delta_y$ we obtain

$$f_t = P^\phi((0, x), (t, y)),$$

where

$$P^\phi((s, x), (t, y)) = \phi(x) P((s, x), (t, y)) \phi(y)^{-1}$$

is the kernel of the semigroup $P_{0,t}^\phi$. By (4.8) we have

$$P^\phi((0, x), (t, y)) \leq e^{c(\phi)t}.$$

We conclude that

$$P_{(0,x)}(X_t = y) \leq \inf_\phi \left\{ \phi(x)^{-1} \phi(y) e^{c(\phi)t} \right\}.$$

We continue to follow [25] and set $\phi(u) = e^{-\lambda(d(x,u) \wedge d(x,y))}$ and then by carefully choosing $\lambda$ we obtain

$$P_{(0,x)}(X_t = y) \leq \exp\left( -\frac{d(x,y)^2}{4dt} \left( 1 - \frac{d(x,y)^2}{40d^2t^2} \right) \right).$$

In particular, for $t \geq d(x, y)$ we have the bound

$$P_{(0,x)}(X_t = y) \leq \exp\left( -C \frac{d(x,y)^2}{t} \right). \tag{4.9}$$

For $t$ that only satisfy $t \geq 0$ we obtain

$$P_{(0,x)}(X_t = y) \leq \exp\left(-d(x,y)\log\left(\frac{d(x,y)}{edt}\right)\right).$$

■

**Remark 81.** *Note that this argument fails if $\omega$ is unbounded above. In this case the uniform upper bound on the function $b_t(\phi, x)$ does not hold. This was addressed in the static variable speed case by [7], where a more subtle, environment dependent distance was used in the construction of the function $\phi$. We will come back to this question when addressing the case $\omega \in [1, \infty)$.*

**Remark 82.** *It is natural to ask whether or not Carne-Varopoulos type bounds hold in the discrete time case. In the most general case, where we take $\omega_e \in [0,1]$ with the only condition being*

$$\sum_{y \sim x} \omega_{xy}(n) \leq 1,$$

*the Carne-Varopoulos bounds cannot hold as it is straightforward to construct environments where the random walk becomes entirely deterministic. However, if we were to instead consider the case $\omega_e \in \left[0, \frac{1}{2d}\right]$, then one would expect the bounds to hold. However, a proof does appear to be more difficult as the discrete time walk is some how "further away" from being reversible than its continuous time counterpart.*

*For example, for the continuous time walk one can introduce the following forward and backward martingales:*

$$M_u \; : \; = d(x, X_u) - d(x, X_0) - \int_0^u \mathcal{L}_s(d(x, X_s))\, ds \tag{4.10}$$

$$M_u^{*t} \; : \; = d\left(x, X_u^{*t}\right) - d\left(x, X_0^{*t}\right) - \int_0^u \mathcal{L}_s^{*t}\left(d\left(x, X_s^{*t}\right)\right) ds, \tag{4.11}$$

*where $X^{*t}$ is the Markov process with semigroup $P_{0,t}^*$ (see Proposition 84 below for a more detailed analysis of $X^{*t}$). It is then possible to show that*

$$\mathbb{E}_{x \otimes y}\left[M_t - M_t^{*t} \,\middle|\, X_t^{*t} = x, X_t = y\right] = -2d(x,y), \tag{4.12}$$

*where $\mathbb{P}_{x \otimes y}$ is the law for independent $X$ and $X^{*t}$ started at $x$ and $y$ respectively. This gives us another method to attack long range bounds as (4.12) says that if $M_t - M_t^*$ is large then both $M$ and $M^*$ must be simultaneously large, yet $M$ and $M^*$ are independent. Thus the probability that either is large can be controlled. See [47] for full details.*

*If one were to attempt to set up these martingales in the discrete time case then the integrals in (4.10) and (4.11) would be replaced by a sum of the form*

$$\sum_{s=1}^{u-1}\left[\mathbb{E}_{X_s}\left[d\left(x,X_1\right)\right]-d\left(x,X_s\right)\right].$$

*Although this does give us a forward and a backward martingale we do not obtain control over their difference as one can construct natural examples where we do not have the equality*

$$\begin{aligned}&\mathbb{E}\left[\sum_{s=1}^{t-1}\left[\mathbb{E}_{X_s}\left[d\left(x,X_1\right)\right]-d\left(x,X_s\right)\right]\middle| X_0=x,X_t=y\right]\\ =\;&\mathbb{E}\left[\sum_{s=1}^{t-1}\left[\mathbb{E}_{X_s^{*t-1}}\left[d\left(x,X_1^{*t-1}\right)\right]-d\left(x,X_s^{*t-1}\right)\right]\middle| X_0^{*t-1}=y,X_t^{*t-1}=x\right].\end{aligned}$$

*Thus the method outlined above to control the probability that the walk travels a large distance will not apply.*

### 4.2.2. Analysis of the random walk restricted to a finite box

Let $(X_t^n)_{t\geq 0}$ be the random walk restricted to the box $[-n,n]^d$, that is set $\omega_{xy}(t)=0$ if either $x$ or $y\notin[-n,n]^d$. In the static time case considered in [40], control over $X^n$ was obtained by showing that $G|_{[-n,n]^d}$ satisfied a close to standard isoperimetric profile, in turn proving a close to standard Nash inequality. The methods described in [50] then transfer the partial Nash inequality to control over the distance of the heat kernel for $X^n$ from uniform measure on the cube.

We will take a similar approach, with the methods of [50] altered for the time dynamic setting. Proposition 83 will demonstrate the control we require over the heat kernel of $X^n$. Proposition 85 and Theorem 87 will then modify the methods of [50] to prove that such control does exist provided the graph satisfies certain geometric conditions.

**Proposition 83.** *Take $d\geq 1$ and set*

$$f(n,t):=\sup_{x,y\in[-n,n]^d}\left|\frac{1}{(2n+1)^d}-P^{\omega}_{(0,x)}\left[X_t^n=(t,y)\right]\right|. \tag{4.13}$$

*Take $c_1<C_4/2\wedge 2C_{dav}/d$ for constants $C_{dav}$ and $C_4$ defined in Proposition 79 and Corollary 80 respectively. Suppose that there exists $C_1>0$ such that*

$$f\left(\left(t\log t/c_1\right)^{1/2},t\right)<C_1t^{-d/2} \tag{4.14}$$

*for all $t\geq 1$. Then there exists a constant $c_2$ such that for all $t\geq 0$*

$$\sup_{x\in\mathbb{Z}^d}P^{\omega}\left((0,0),(x,t)\right)\leq c_2t^{-d/2}. \tag{4.15}$$

**Proof.** Assume that $t \geq 2$ as otherwise the case $t \in [0,2]$ holds by suitable choice of $c_2$.

Suppose $|x|^2 \geq \frac{dt \log t}{2C_{dav}}$. The bound (4.15) follows trivially from equation (4.5) of Proposition 79.

Now take $|x|^2 < \frac{dt \log t}{2C_{dav}}$ and set $c_1 n^2 = t \log t$ for $c_1 < C_4/2 \wedge 2C_{dav}/d$.

Note that

$$P^{\omega}\left((0,0),(x,t)\right) \leq P^{\omega}_{(0,0)}\left(X^n_t = (x,t)\right) + P^{\omega}_{(0,0)}\left(\tau_n \leq t\right).$$

We can bound the first term by (4.13) and the second by Corollary 80:

$$\begin{aligned} P^{\omega}\left((0,0),(x,t)\right) &\leq (2n+1)^{-d} + f\left(\left(t \log t / c_1\right)^{1/2}, t\right) \\ &\quad + C_1 n^{-d} e^{-C_2 n} + C_3 n^{d-1} t \exp\left(-\frac{C_4 n^2}{t}\right). \end{aligned}$$

The right hand side can be bounded term by term.

The first term can be trivially bounded: $(2n+1)^{-d} = \left(\frac{t \log t}{c_1}\right)^{-d/2} \leq ct^{-d/2}$ for all $t \geq 2$.

The second term is bounded by (4.14).

Invoking Corollary 80 and the relationship between $n$ and $t$ allows us to bound the third and fourth terms:

$$\begin{aligned} & C_1 n^{-d} e^{-C_2 n} + C_3 n^{d-1} t \exp\left(-\frac{C_4 n^2}{t}\right) \leq ct^{-d/2} \\ \iff\; & C_3 n^{d-1} t \exp\left(-\frac{C_4 \log t}{c_1}\right) \leq ct^{-d/2} \\ \iff\; & C_3 \left(\frac{t \log t}{c_1}\right)^{\frac{d-1}{2}} t \cdot t^{-C_4/c} \leq ct^{-d/2} \\ \iff\; & c_2 t^{(d+1)/2 - C_4/c_1} (\log t)^{(d-1)/2} \leq ct^{-d/2} \end{aligned}$$

this will be true for $c_1 < C_4/2$ and all $t$ provided that $c$ is suitably chosen. ■

Gaining on diagonal control is thus reduced to proving that (4.14) holds - that the random walk on the finite box mixes at a quick enough rate. We will use the methods described in [50] to prove this mixing via Nash inequalities.

Introduce the adjoint of $P_{s,t}$, call this $P^*_{s,t}$. $P^*_{s,t}$ can be most easily described as the semigroup associated to the random walk on the space-time environment reversed around $t$. Define the graph reflected in time about $t$

$$G^{*t} = \left(V, E, \omega^{*t}\right)$$

to have identical vertex and edge sets as the original time-space graph $G$ and edge weights reflected about time $t$:

$$\omega^{*t}_r(x,y) := \omega_{t-r}(x,y) \text{ for } r \in \mathbb{R}.$$

Let $Q_{s,t}$ be the semigroup for the random walk on $G^{*t}$ started at zero time and running to time $t-s$.

**Proposition 84.** $P^*_{s,t} = Q_{s,t}$.

**Proof.** This is clear in the discrete time case as we are simply transposing matrices: recall the symmetry assumption $\omega_{xy}(t) = \omega_{yx}(t)$, then $Q_i = Q_i^*$ for all $i$ and hence

$$P^*_{s,t} = \left(\prod_{u=s}^{t-1} Q_u\right)^* = \prod_{u=s}^{t-1} Q^*_{t+s-u} = \prod_{u=s}^{t-1} Q_{t+s-u} = Q_{s,t}.$$

Note that the reflection in the discrete time case is at $t-1$ and not $t$.

For the continuous time case there are several ways to approach this. We simply note that $P^*_{s,s} = I = Q_{s,s}$ and the generators for both semigroups at time $s \leq u \leq t$ is $\mathcal{L}^*_u$. ■

This result is useful as it ensures that the graphs controlling the random walk and the reversed random walk are simply reversed in time. In particular the spatial graph for the forward walk at time $u$ is the spatial graph for the reversed walk at time $t-u$. As we will obtain heat kernel bounds via assumptions on "enough" spatial graphs satisfying a Nash inequality, the fact that the set of forward and backward spatial graphs are identical will prove to be crucial.

Write $P^n_{s,t}$ and $P^{*n}_{s,t}$ for the random walk and reversed random walk restricted to $[-n,n]^d$. Let $\pi_n$ be the uniform distribution on $[-n,n]^d$, $\pi_n(x) = \frac{1}{(2n+1)^d}$ if and only if $x \in [-n,n]^d$. Write $\|\cdot\|_p$ for the $L^p(\pi_n)$ norm. Define $u_n(s,t) := \left\|P^n_{s,t}(f - \pi_n(f))\right\|_2^2 = Var\left(P^n_{s,t} f\right)$ and $v_n(s,t) := \left\|P^{*n}_{s,t}(f - \pi_n(f))\right\|_2^2$ for some function $f$ such that $\|f\|_1 = 1$. Control over $u_n(s,t)$ and $v_n(s,t)$ leads to a bound on (4.13).

**Proposition 85.** *Take $t > 0$. Suppose that for all $f$ with $\|f\|_1 = 1$*

$$\begin{aligned} u_n(0,t/2) &\leq \left(\frac{C}{t}\right)^{d/2}, \\ v_n(t/2,t) &\leq \left(\frac{C}{t}\right)^{d/2}, \end{aligned}$$

*for some constant $C$. Then*

$$\sup_{x,y \in [-n,n]^d} \left| \frac{1}{(2n+1)^d} - P^\omega_{(0,x)}\left[X^n_t = (t,y)\right] \right| \leq \left(\frac{2C}{t}\right)^{d/2}. \tag{4.16}$$

**Proof.** From the assumptions and definition of $u$ we have

$$\left\|P^n_{0,t/2} - \pi_n\right\|_{1\to 2}, \left\|P^{*n}_{t/2,t} - \pi_n\right\|_{1\to 2} \leq \left(\frac{C}{t}\right)^{d/4},$$

where

$$\left\|P^n_{0,t/2}-\pi_n\right\|_{1\to 2}:=\sup\left\{\left\|P^n_{0,t/2}\left(f-\pi_n\left(f\right)\right)\right\|_2:\left\|f\right\|_1=1\right\}.$$

By standard results (see, for example, Section 1.2.4 of [50])

$$\begin{aligned}\left\|P^{*n}_{0,t/2}-\pi_n\right\|_{2\to\infty} &= \left\|P^n_{0,t/2}-\pi_n\right\|_{1\to 2}\leq\left(\frac{C}{t}\right)^{d/4}, &(4.17)\\ \left\|P^n_{t/2,t}-\pi_n\right\|_{2\to\infty} &= \left\|P^{*n}_{t/2,t}-\pi_n\right\|_{1\to 2}\leq\left(\frac{C}{t}\right)^{d/4}. &(4.18)\end{aligned}$$

Whence, by Cauchy-Schwarz,

$$\begin{aligned}&\sup_{x,y}\left\langle\left(P_{0,t}-\pi_n\right)\delta_x,\delta_y\right\rangle\\ =\;&\sup_{x,y}\left\langle\left(P_{0,t/2}-\pi_n\right)\left(P_{t/2,t}-\pi_n\right)\delta_x,\delta_y\right\rangle\\ \leq\;&\sup_{x,y}\left\langle\left(P_{t/2,t}-\pi_n\right)\delta_x,\left(P^*_{0,t/2}-\pi_n\right)\delta_y\right\rangle\\ \leq\;&\pi_n\left(x\right)^{1/2}\pi_n\left(y\right)^{1/2}\left\|\left(P_{t/2,t}-\pi_n\right)\frac{\delta_x}{\pi_n\left(x\right)}\right\|_2\left\|\left(P^*_{0,t/2}-\pi_n\right)\frac{\delta_y}{\pi_n\left(y\right)}\right\|_2\\ \leq\;&\left(\frac{C}{t}\right)^{d/2}\left(2n+1\right)^{-d}\end{aligned}$$

by (4.17) and (4.18), completing the proof. ■

The problem is now reduced to controlling the growth of $u_n$ and $v_n$. This can be achieved by considering spatial Nash inequalities through time. We elucidate this idea below.

Recall Theorem 3.3.11 from [50], linking the isoperimetric profile of a graph to a Nash inequality for the random walk on the graph.

For $t\in\mathbb{R}$, $n\in\mathbb{N}$ and $A\subseteq[-n,n]^d$ define

$$Q_t\left(\partial A\right)=\sum_{\substack{x\in A\\ y\in[-n,n]^d-A}}\omega_{xy}\left(t\right)\pi_n\left(x\right).$$

**Theorem 86 (Saloff-Coste).** *Assume that for some constant $S>0$, $\left(Q_t,\pi_n\right)$ satisfies*

$$\pi_n\left(A\right)^{(d-1)/d}\leq SQ_t\left(\partial A\right)$$

*for all $A\subseteq\mathbb{Z}^d$ such that $\pi_n\left(A\right)\leq\frac{1}{2}$. Then*

$$\forall g\in L^2\left(\pi\right),\; Var_{\pi_n}\left(g\right)^{1+2/d}\leq 8S^2\mathcal{E}_t\left(g,g\right)\left\|f\right\|_1^{4/d}.$$

Assume for the moment that for every $t\in\mathbb{R}$ the space graph $G_t$ has some asymptotic spatial dimension $d=d\left(t\right)$ that can be seen through the isoperimetric profile. More formally, we choose the isoperimetric bound of [40], and say that the spatial $d$-dimensional isoperimetric profile holds for $t \in \mathbb{R}$ if

$$\begin{aligned} I^t_{\varepsilon(n)} \quad &:= \inf_{A\subseteq[-n,n]^d,\pi(A)\leq\frac{|C^n|}{2}} \frac{Q_t\left(\partial_{C^n}A\right)}{\pi_n\left(A\right)^{(\varepsilon(n)-1)/\varepsilon(n)}} \\ &\geq \frac{\beta}{n^{1-\frac{2d}{\varepsilon(n)}}}, \end{aligned} \tag{4.19}$$

for all $n \geq N\left(\omega, t\right)$, where $\varepsilon\left(n\right) := d + 2d\frac{\log\log n}{\log n}$ and $C^n = [-n,n]^d$. Note that this is a different approach to that of Chapter 2 where the standard isoperimetry was shown to hold for all large enough sets. This different approach is simply due to the fact that a different method is being employed to move from isoperimetric bounds to heat kernel bounds.

For $u, t \in \mathbb{R}$ define

$$D_{u,t} := \left\{ s \in [u,t] : I^s_{\varepsilon(n)} \geq \beta n^{-1+2d/\varepsilon(n)} \text{ for } n = \left(\frac{(t-u)\log(t-u)}{c}\right)^{1/2} \right\},$$

where $c < C_4/2 \wedge 2C_{dav}/d$ as in Proposition 83.

**Theorem 87.** *Suppose there exists a constant $c_2$ such that*

$$\int_0^{t/2} 1_{D_{0,t/2}}\left(s\right)ds, \int_{t/2}^{t} 1_{D_{t/2,t}}\left(s\right)ds \geq c_2 t \tag{4.20}$$

*for all $t \geq T$, then there exists $C > 0$ such that*

$$P^\omega\left(\left(0,0\right),\left(x,t\right)\right) \leq Ct^{-d/2} \text{ for } t \geq T.$$

**Proof.** We will show that (4.20) can be used to prove the conditions of Proposition 85 for suitable $C$ and $d$. We will then show that this choice of $C$ and $d$ ensures that the right hand side of (4.16) can be bounded by $O\left(t^{-d/2}\right)$. Proposition 83 will then complete the proof.

Theorem 86 implies that for $n, s$ for which (4.19) holds the following Nash inequality holds:

$$\forall g \in L^2\left(\pi_n\right), Var_{\pi_n}\left(g\right)^{1+2/\varepsilon(n)} \leq \frac{8n^{2-4d/\varepsilon(n)}}{\beta^2}\mathcal{E}_{\mathcal{L}_s}\left(g,g\right)\left\|f\right\|_1^{4/d}. \tag{4.21}$$

Fix $n = \left(\frac{t\log t}{c}\right)^{1/2}$ and abuse notation by setting $u\left(t\right) = u_n\left(0,t\right) = \left\|\left(P^n_{0,t} - \pi_n\right)f\right\|_2^2$ for $f$ such that $\left\|f\right\|_1 = 1$. By a simple calculation we see that for almost all $t$ we have the differential equality $u'\left(t\right) = -2\mathcal{E}_t\left(P_{0,t}f, P_{0,t}f\right)$. Then (4.21) implies

$$u\left(s\right)^{1+2/\varepsilon(n)} \leq -\frac{4n^{2-4d/\varepsilon(n)}}{\beta^2}u'\left(s\right)$$

for $s,n$ where (4.19) holds.

Set $w(t) := \frac{2\varepsilon(n)n^{2-4d/\varepsilon(n)}}{\beta^2} u(t)^{-2/\varepsilon(n)}$, then combining the above we have

$$\frac{\partial}{\partial t} w(t) \geq 1 \tag{4.22}$$

for $s,n$ satisfying (4.19).

Note that since the definition of $D_{0,t}$ is dependent only on the finite spatial box $\left[-\frac{t\log t}{c}, \frac{t\log t}{c}\right]^d$ and a finite spatial box can have at most a finite number of edge changes in the time interval $[0,t]$, $D_{0,t}$ must be of the form

$$D_{0,t} = \bigcup_{i=1}^{m} [s_i, t_i).$$

On an interval $[s_i, t_i) \subset D_{0,t}$, inequality (4.22) gives

$$w(t) \geq w(s_i) + t - s_i$$

for $t \in [s_i, t_i]$. By the definition of $w$ this translates to:

$$\frac{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}{\beta^2} u(t)^{-2/\varepsilon(n)} \geq \frac{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}{\beta^2} u(s_i)^{-2/\varepsilon(n)} + t - s_i$$

so

$$u(t) \leq \left(u(s_i)^{-2/\varepsilon(n)} + \frac{\beta^2}{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}(t - s_i)\right)^{-\varepsilon(n)/2}. \tag{4.23}$$

Note that $u(t)$ is decreasing in $t$ since $u'(t) = -2\mathcal{E}_t(P_{0,t}f, P_{o,t}f)$.

Trivially $u(s_1) \leq 1$. Hence, by (4.23)

$$\begin{aligned} u(t) &\leq \left(1 + \frac{\beta^2}{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}(t - s_1)\right)^{-\varepsilon(n)/2} \\ &\leq \left(\frac{\beta^2}{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}(t - s_1)\right)^{-\varepsilon(n)/2} \end{aligned}$$

for $t \in [s_1, t_1]$. As $u(t)$ is decreasing $u(s_{i+1}) \leq u(t_i)$. Thus, by a simple induction

$$u(t) \leq \left(\frac{\beta^2}{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}\left(\sum_{j=1}^{i-1}(t_j - s_j) + t - s_i\right)\right)^{-\varepsilon(n)/2}$$

for $t \in [s_i, t_i]$. In particular, if $\int_0^t D_{0,t}(s)\,ds \geq c_1 t$ then

$$u(t) \leq \left(\frac{c_1\beta^2 t}{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}\right)^{-\varepsilon(n)/2}.$$

Applying this method to the forward and reversed graph, noting that the set of spatial environments for the forward and reversed walks are identical, we obtain

$$\begin{aligned} u_n(0, t/2), u_n(t/2, t) &\leq \left(\frac{c_1\beta^2 t}{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}\right)^{-\varepsilon(n)/2} \\ v_n(0, t/2), v_n(t/2, t) &\leq \left(\frac{c_1\beta^2 t}{2\varepsilon(n) n^{2-4d/\varepsilon(n)}}\right)^{-\varepsilon(n)/2}. \end{aligned}$$

Appealing to Proposition 85 we have

$$\sup_{x,y\in[-n,n]^d}\left|\frac{1}{(2n+1)^d}-P^{\omega}_{(0,x)}\left[X^n_t=(t,y)\right]\right|\leq\left(\frac{c_1\beta^2 t}{\varepsilon(n)\,n^{2-2d/\varepsilon(n)}}\right)^{-\varepsilon(n)/2}. \tag{4.24}$$

Insert the choice $n=\left(\frac{t\log t}{c}\right)^{1/2}$ into (4.24) and check if the conditions of Proposition 83 are satisfied. The conditions will be satisfied if

$$\left(\frac{c_1\beta^2 t}{\varepsilon(n)\,n^{2-2d/\varepsilon(n)}}\right)^{-\varepsilon(n)/2}\leq Ct^{-d/2}.$$

By the definition of $\varepsilon(n)$ above,

$$\begin{aligned}
&\left(\frac{c_1\beta^2 t}{\varepsilon(n)\,n^{2-2d/\varepsilon(n)}}\right)^{-\varepsilon(n)/2}\leq Ct^{-d/2}\\
\iff\ &\beta^{-\varepsilon(n)}t^{-\varepsilon(n)/2}\varepsilon(n)^{\varepsilon(n)/2}n^{\varepsilon(n)-d}\leq Ct^{-d/2}\\
\iff\ &\log c+d\frac{\log\log n}{\log n}\log\left(\frac{\log t}{c}\right)\leq\log C
\end{aligned}$$

and this holds for all sufficiently large $n$ and hence we are done. ■

The conditions of equation (4.20) look rather complicated. However, all they ask for is that the amount of time for which the spatial graph looks geometrically $d$-dimensional grows linearly with time. In particular, if all the spatial graphs $G_t$ are spatially $d$-dimensional at large distances from the origin and the environment's time dynamic is ergodic then the conditions will be satisfied:

**Corollary 88.** *Suppose $\omega$ is an environment such that for all $t\in\mathbb{R}$, $G_t$ satisfies (4.19) for some $d$ and all sufficiently large $n$. Suppose further that the time dynamic of the environment is ergodic, then there exist constants $C(\omega),T(\omega)$ such that*

$$P^{\omega}((0,0),(x,t))\leq C(\omega)\,t^{-d/2}$$

*for $t>T$.*

**Proof.** Let $N(\omega,t)$ be the random variable such that (4.19) holds for all $n\geq N(\omega,t)$. As $(\omega_t)_{t\geq 0}$ is ergodic

$$\frac{1}{t}\int_0^t 1_{\{N(\omega,s)\leq k\}}ds\to\mathbb{P}(N(\omega,0)\leq k).$$

We have

$$D_{0,t/2}\supseteq\left\{0\leq s\leq t:N(\omega,s)\leq\frac{t\log t}{c}\right\}$$

and hence

$$\begin{aligned}
\int_0^{t/2}1_{D_{0,t/2}}(s)\,ds\ &\geq\ \int_0^{t/2}1_{\left\{N(\omega,s)\leq\frac{t\log t}{c}\right\}}ds\\
&\geq\ \frac{t}{2}
\end{aligned}$$

for all sufficiently large $t$. Similarly $\int_{t/2}^t 1_{D_{t/2,t}}(s)\,ds\geq\frac{t}{2}$ for all sufficiently large $t$. Now apply Theorem 87. ■

#### 4.2.3. Examples

Here we give examples of environments for which the conditions of Theorem 87 hold.

We begin with a dynamic random conductance model. For $e \in \mathbb{E}^d$, let $\{\omega_e(t)\}_{t\in\mathbb{R}}$ be an ergodic Markov process with distribution $P$ and invariant measure $\mu$ supported on $(0,1]$, with $\omega_e(0) =^d \mu$. The distribution of the full space-time environment $\omega = (\omega_e(t))_{\substack{e\in\mathbb{E}^d\\ t\in\mathbb{R}}}$ is then given by product measure: $\mathbb{P} := P^{\mathbb{E}^d}$. We will show that if $\mu$ decays slowly at the origin then standard upper bounds on the heat kernel hold for $\mathbb{P}$-almost every $\omega \in \Omega$.

In the case of static environment, it has been shown in [18] that one can obtain standard heat kernel decay where the isoperimetric inequality (4.19) fails to hold. Thus one cannot expect to fully classify those environments for which standard heat kernel decay holds by their isoperimetric profile. Instead, we simply give sufficient conditions on $\mu$ that ensure standard on diagonal upper bounds. This will be achieved by a rather crude argument that only depends on the decay of the weakest edge in the box $[-n,n]^d$ as $n$ gets large.

Define

$$g(n) := \inf\{\omega_e(0) : e \in B_0[n]\}$$

and recall the definition

$$\varepsilon = \varepsilon(n) := d + \frac{2d\log\log n}{\log n}.$$

**Proposition 89.** *Take $d \geq 2$. Suppose $\mu$ is such that there exists $\delta > 0$ such that for all $\omega \in \Omega$ there exists $N_1(\omega) < \infty$ such that*

$$g(n) \geq \frac{\left((\log n)^{d/(d-1)+\delta}\right)^{(\varepsilon-1)/\varepsilon}}{n^{1-d/\varepsilon}}, n \geq N_1(\omega). \tag{4.25}$$

*Then there exist $C_1$ such that for almost every $\omega \in \Omega$ there exists $N_2(\omega)$ such that*

$$P^\omega(X_n = 0 \,|X_0 = 0) \leq C_1 n^{-d/2}$$

*for all $n \geq N_2$.*

**Proof.** By Corollary 88 we only need prove the requisite isoperimetric decay for a static environment. For this reason we will consider the static graph $(\omega_e(0))_{e\in\mathbb{E}^d}$ and for convenience shall write $\omega_e = \omega_e(0)$.

We need to prove that there exists $\beta > 0$ such that for all sufficiently large $n$

$$\inf_{A\subseteq[-n,n]^d} \frac{\sum_{e\in\partial A}\omega_e}{\pi(A)^{(\varepsilon-1)/\varepsilon}} \geq \frac{\beta}{n^{1-d/\varepsilon}}. \tag{4.26}$$

Take $\alpha \in (0,1)$ and let $\mathcal{G}_\alpha$ be the thinned graph formed by removing all edges with weight $\omega_e \leq \alpha$. Take $\alpha$ sufficiently small to guarantee an infinite connected component in $\mathcal{G}_\alpha$ and call this component $\mathcal{C}_\infty^\alpha$. As in the arguments of Section 3.2, if $\alpha$ is chosen to be sufficiently small then the largest components of $\mathbb{Z}^d - \mathcal{C}_\infty^\alpha$ are of order $(\log n)^{d/(d-1)}$. Hence there exists $N_3(\omega,\delta)$ such that for all $n \geq N_3(\omega,\delta)$ and $A \subseteq [-n,n]^d$ with $\pi(A) \geq (\log n)^{d/(d-1)+\delta}$ we have

$$A \cap \mathcal{C}_\infty^\alpha \neq \phi$$

and hence

$$\sum_{e \in \partial A} \omega_e \geq \alpha.$$

In particular, for any $A \subseteq [-n,n]^d$ with $\pi(A) \in \left[(\log n)^{d/(d-1)+\delta}, n^{(\varepsilon-d)/\varepsilon-1}\right]$, equation (4.26) holds with $\beta = \alpha$. Note that $n^{(\varepsilon-d)/(\varepsilon-1)} > (\log n)^{3d/2(d-1)}$ for all large $n$ and hence the interval is non-empty.

For $A$ with $\pi(A) < (\log n)^{d/(d-1)+\delta}$, the choice of $g(n)$ in (4.25) ensures that (4.26) holds for sufficiently large $n$.

Finally, for $A$ with $\pi(A) > n^{(\varepsilon-d)/\varepsilon-1}$ we use the results of page 13 onwards of [40]. Taking their results into our setting, they show that there exists a constant $c > 0$ and $N_4(\omega)$ such that if $n \geq N_4$ and $\pi(A) > n^{(\varepsilon-d)/\varepsilon-1}$ then

$$\sum_{e \in \partial A} 1_{\{\omega_e \geq \alpha\}} \geq c\,|\partial A|\,.$$

This, together with the standard $d$-dimensional isoperimetric inequality is sufficient to prove (4.26) in this case. ∎

There are other models to which one could apply Theorem 87. For example, one could construct a dynamic environment that changes dimension over time. Theorem 87 would then give upper bounds of order $n^{-d/2}$ where $d$ is the highest dimension that the environment spends a linear amount of time in.

As many of the ideas for this section have come from [40], we would like to be able to apply our methods to the case of dynamic supercritical percolation - where the Markov chain on the edge weights is supported on $\{0,1\}$. At the time being this model is outside the range of our methods. This is due to the fact that for a given time $t$, the isoperimetric inequalities claimed will only hold on subsets of the infinite cluster at time $t$ and not on all subsets of $\mathbb{Z}^d$.

This is natural as if the walk happens to be in one of the finite subclusters at time $t$ then its movement is fundamentally restricted, whereas if the walk is on the infinite cluster it will be able to mix well. Over time, by the ergodicity of the environment from the point of the particle, we know that the walk will spend a linear amount of time on the infinite cluster and hence will mix well. It is therefore our belief that standard Gaussian upper bounds should hold but this question remains open.

## 4.3. Anomalous behaviour for variable speed, continuous time when $\omega \in (0, 1]$

We give an example of anomalous heat kernel behaviour in this setting. In particular our example will demonstrate a stronger trapping effect than in the static case with the heat kernel shown to be both more anomalous and anomalous in more dimensions.

Theorem 90 states the main result of this section - showing that there exist environments where the on diagonal heat kernel is at least close to order $n^{-1}$ infinitely often. The example environment described borrows much from the proof of Theorem 2.2 of [11] where behaviour close to $O(n^{-2})$ is observed for static environments. Both examples require the walk to become trapped close to the origin and remain in the trap for a large time before escaping and returning to the origin. In the static case there is a price to be paid both to enter the trap and to exit the trap - a price of order $n^{-1}$ each time and thus paying $O(n^{-2})$ in total. We will give an example in the dynamic setting where the walk only has to pay to enter the trap and escapes the trap for free - hence only paying $O(n^{-1})$. This is possible since the time dynamic enables traps to disappear and when this happens the walk escapes the trap without having to move.

One could also ask whether it is possible to enter a trap for free - this corresponds to the walk being at the trap site at the time when the trap forms. This would lead to an even more anomalous heat kernel. We do not expect this to be the case. It will become clear that there is a trade-off between persistence of traps and their frequency of occurrence. As we wish the traps to be persistent so that the walk remains trapped for long time periods the traps cannot occur frequently and thus the walk is highly unlikely to be at the trap site when the trap forms.

To make this belief rigorous requires showing that corresponding upper bounds hold of order $n^{-1}$. In higher dimensions ($d \geq 5$) we have strong heuristics for why we believe this to be the case. In lower dimensions we are less sure. In neither case do we have a proof. We will discuss this problem in Section 4.3.1.

We begin by stating the main theorem of the section. We will then introduce the example environment to analyze, providing a heuristic argument for the lower bounds, before giving a rigorous proof.

**Theorem 90.** *Take $\kappa > \frac{1}{d}$ and $d \geq 3$. There exists a law on environments $\mathbb{P}$ with iid edge marginals that evolve in an ergodic Markov fashion with edge weights* *bounded by one such that for almost all $\omega \in \Omega$ there exists a constant $C(\omega) > 0$ and a sequence $(n_i(\omega))$ with $\lim_{i\to\infty} n_i = \infty$ such that*

$$P^{\omega}_{(0,0)}\left(X_{n_i} = 0\right) \geq \frac{C(\omega)\, e^{-(\log n_i)^{\kappa}}}{n_i}$$

*for all $i$, where $X_n$ is the variable speed continuous time random walk on $\omega$.*

Before beginning the formal proof we introduce an example that exhibits this lower bound and heuristically explain the trapping phenomena. We take a space-time environment that only switches at discrete time points. This choice of discrete environment does lead to a somewhat peculiar hybrid pair as the walk is in continuous time. The choice does, however, make the combinatorics easier. We will discuss extensions of this model later. Take weights $\omega_e$ to be supported on $\{2^{-n} : n \geq 0\}$ and let the transition probabilities for the Markov chain $(\omega_e(n))_{n\in\mathbb{Z}}$ be:

$$\begin{aligned} K\left(1, 2^{-n}\right) &= s_n \\ K\left(2^{-n}, 1\right) &= p_n \\ K\left(2^{-n}, 2^{-n}\right) &= 1 - p_n. \end{aligned}$$

Then $\omega_e(n)$ has an invariant distribution if and only if the Markov chain is positive recurrent. Hence, the invariant distribution, $\mu$, exists if and only if

$$s_0 + \sum_{n>0} \frac{s_n}{p_n} < \infty.$$

Assume this to be the case and set

$$\mu\left(2^{-n}\right) = \mu(1)\frac{s_n}{p_n}.$$

We define our space-time environment to be $\omega = (\omega_e(n))_{\substack{e\in\mathbb{E}^d \\ n\in\mathbb{N}^{\geq 0}}}$ with $\omega$ being iid in space with $\mathbb{P}\left(\omega_e(0) = 2^{-k}\right) = \mu\left(2^{-k}\right)$. Take $\mu(1) > p_c(d)$ to ensure that for every $t \in \mathbb{R}$, the bonds of unit conductance percolate in $G_t$.

The traps that we consider are of the form shown in Figure 4.1. They consist of the following. At time zero there is a strong spatial path (made of bonds of unit conductance) connecting the origin to a vertex $x$. $x$ is connected to $y$ by a weak bond of strength $2^{-n}$ and all other bonds are of lesser conductance. The trap, without necessarily the path to the origin, remains in place until time $T_n$ at which point $\omega_{xy}$ returns to unit conductance. At time $T_n$ there again exists a strong spatial path to the origin.

If such a trap exists, we obtain a lower bound on $P^{\omega}_{(0,0)}\left(X_{T_n+1} = 0\right)$ by conditioning on the walk moving directly to $x$ within one unit of time, jumping from $x$ to

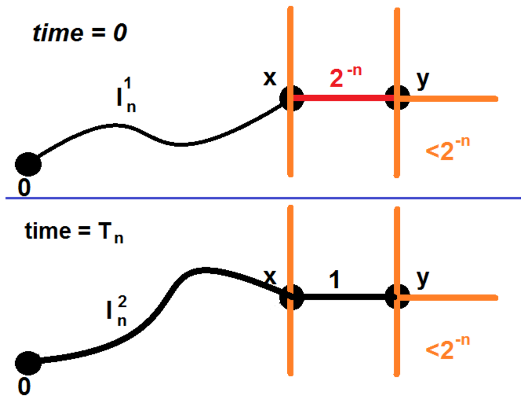


Figure 4.1: A space time trap

$y$ in one unit of time, then not jumping from $y$ until time $T_n$ and then proceeding directly back to the origin within one unit of time:

$$\begin{aligned} & P^{\omega}_{(0,0)}\left(X_{T_n+1}=0\right) \\ \geq \; & P^{\omega}_{(0,0)}\left(X_1=x\right) P^{\omega}_{(1,x)}\left(X_i=y \text{ for all } i\in[2,T_n]\right) P^{\omega}_{(T_n,x)}\left(X_{T_n+1}=0\right) \\ \geq \; & ce^{-c_1\left(l_n^1\log l_n^1+l_n^2\log l_n^2\right)}2^{-n}c_2, \end{aligned} \tag{4.27}$$

where $l_n^1$ and $l_n^2$ are the graph distances between the origin and $x$ at times 0 and $T_n$ respectively. In the final line the $2^{-n}$ is the cost to the walk of crossing the bond $\omega_{xy}(1)$, the exponential terms come from the lower bounds on the probability of the walk moving a large distance in a short time proven in [45], and we have assumed that $T_n \leq \frac{1}{2d}2^n$ and hence the walk does not jump from $y$ between time 2 and $T_n$ with probability bounded independently of $n$. If we can take $l_n^i = O(\log n)$ and $T_n = O(2^n)$ then we would see

$$P^{\omega}_{(0,0)}\left(X_{2^n}=(2^n,0)\right) \geq p(n)\,2^{-n}$$

for some function $p$ that decays much slower than $2^{-n}$.

In order to prove the theorem we require some combinatorial estimates on the strong paths that will connect the traps to the origin. For $t\in\mathbb{R}$ write $x\leftrightarrow y$ at time $t$ if there exists a nearest neighbour path from $x$ to $y$ consisting purely of bonds of unit conductance in the graph $G_t$. As $\mu(1)>p_c(d)$ there exists a unique infinite cluster at each $t\in\mathbb{R}$. Call this $\mathcal{C}_\infty(t)$. The following proposition encapsulates the results required.

**Proposition 91.** *Suppose $\mu(1)>p_c(d)$. For any $x\in\mathbb{Z}^d$ and $k\geq 1$ define the events*

$$\begin{aligned} C_0(k,x) &= \{x\leftrightarrow \partial B_x[k] \;\; \textit{at time } 0\} \\ \textit{and } C_m(k,x) &= \{x\leftrightarrow \partial B_x[k] \;\; \textit{at time } m\}. \end{aligned}$$

*There exists constant $c_1 > 0$ such that*

$$\mathbb{P}\left(C_0\left(k,x\right)\cap C_m\left(k,x\right)\right)\geq c_1$$

*for all $k$ and all large enough $m$.*

*Further, for $t\in\mathbb{R}$ define $D_t\left(n\right)$ to be the event that every connected component contained in $B_0\left[n\right]\cap G_t$ of size at least $c_2\left(\log n\right)^2$ is connected to $\mathcal{C}_\infty\left(t\right)$. Then there exists $c_2>0$ such that for almost every $\omega\in\Omega$ there exists $N\left(\omega\right)<\infty$ such that $D_0\left(n\right)\cap D_n\left(n\right)$ occurs for all $n\geq N\left(\omega\right)$.*

**Proof.** For the first claim we use the mixing properties of the Markov chain on edge weights. We have to be careful as if $B_x\left[k\right]$ contains edges with very light conductance then these edges can take a long time to mix. To avoid this problem we delete all light edges and percolate on what remains of the box.

Let

$$\varepsilon:=\frac{\mu\left(1\right)-p_c\left(d\right)}{4}$$

and take $M$ such that

$$\sum_{n=M}^{\infty}\mu\left(2^{-n}\right)<\varepsilon.$$

Define $E_M:=\left\{e\in B_x\left[k\right]:\omega_e\left(0\right)\leq 2^{-M}\right\}$, the set of all light edges - we will throw these edges away as we cannot control their mixing properties.

Take $M_1$ sufficiently large so that for all $m\geq M_1$

$$\mathbb{P}\left(\omega_e\left(m\right)=1\,\middle|\,\omega_e\left(0\right)=2^{-n}\right)>p_c\left(d\right)+2\varepsilon$$

for all $n<M$. $M_1$ exists since the Markov chain on edge weights is irreducible, aperiodic, positive recurrent and $n$ is bounded.

Take $\left(\tilde{\omega}_e\right)_{e\in\mathbb{Z}^d}$ to be an independent bond percolation realization with

$$\mathbb{P}\left(\tilde{\omega}_e=1\right)=p_c\left(d\right)+2\varepsilon.$$

Define $\left(\bar{\omega}_e\left(m\right)\right)_{e\in\mathbb{Z}^d}$ to be the environment defined by

$$\bar{\omega}_e\left(m\right)=\begin{cases}0 & \text{if } e\in E_M\\ \tilde{\omega}_e & \text{otherwise}\end{cases},$$

then $\bar{\omega}$ is stochastically dominated by $G_m$ for $m\geq M_1$. Hence the event $\bar{C}_m\left(k,x\right):=\left\{x\leftrightarrow\partial B_x\left[k\right]\text{ in }\bar{\omega}_e\left(m\right)\right\}$ is dominated by $C_m\left(k,x\right)$. Conditioned on $E_M$, the events $C_0$ and $\bar{C}_m$ are independent and hence

$$\begin{aligned}
\mathbb{P}\left(C_0\left(k,x\right)\cap C_m\left(k,x\right)\right) &= \sum_e \mathbb{P}\left(C_0\left(k,x\right)\cap C_m\left(k,x\right)|E_M=e\right)\mathbb{P}\left(E_M=e\right)\\
&\geq \sum_e \mathbb{P}\left(C_0\left(k,x\right)\cap \bar{C}_m\left(k,x\right)|E_M=e\right)\mathbb{P}\left(E_M=e\right)\\
&= \sum_e \mathbb{P}\left(C_0\left(k,x\right)|E_M=e\right)\mathbb{P}\left(\bar{C}_m\left(k,x\right)|E_M=e\right)\mathbb{P}\left(E_M=e\right)\\
&\geq \sum_e \mathbb{P}\left(\bar{C}_m\left(k,x\right)|E_M=e\right)^2\mathbb{P}\left(E_M=e\right)\\
&\geq C^2,
\end{aligned}$$

where the final line follows from the definition of $E_M$ and standard percolation estimates.

The second claim is straight forward to verify via standard percolation arguments. Consider first the event $D_i\left(n\right)$ for $i\in\{0,n\}$. In either case this corresponds to static percolation and hence Theorem 8.65 of [32] gives upper bounds on $\mathbb{P}\left(D_i\left(n\right)^c\right)$ that are independent of $i$ and summable over $n$. Now,

$$\begin{aligned}
\mathbb{P}\left(D_0\left(n\right)\cap D_n\left(n\right)\right)^c &= \mathbb{P}\left(D_0\left(n\right)^c\cup D_n\left(n\right)^c\right)\\
&\leq \mathbb{P}\left(D_0\left(n\right)^c\right)+\mathbb{P}\left(D_n\left(n\right)^c\right)
\end{aligned}$$

and hence is also summable over $n$. Thus Borel-Cantelli ensures that the event $D_0\left(n\right)\cap D_n\left(n\right)$ happens only finitely often with probability one. ∎

**Proof of Theorem 90.** Take $\varepsilon>0$. For $n>0$ set $l_n:=n^{(1+(2d+1)\varepsilon)/d}$ and $T_n=2^n$. For $n\in\mathbb{N}$ choose $s_n=c2^{-n}n^{-(1+\varepsilon)}$, $p_n=2^{-n}$ so that $\mu\left(2^{-n}\right)=cn^{-(1+\varepsilon)}$ and $c$ chosen sufficiently small so that $\mu\left(1\right)>p_c\left(d\right)$.

For a fixed point $x\in\mathbb{Z}^d$, let $y=x+e_1$ and $A_n\left(x\right)$ be the event that:

- In the spatial environment at time zero, $G_0$, $x$ is connected to the boundary of the spatial box of side $c_2\left(\log l_n\right)^2$ centred at $x$ by a path of unit conductors.
- $\omega_{xy}\left(i\right)=2^{-n}$ for $i\in[0,T_n-1]$, $\omega_{xy}\left(T_n\right)=1$,
- $\omega_{yz}\left(i\right)\leq 2^{-n}$ for $i\in[0,T_n]$, $z\neq x$,
- $x$ is connected to the boundary of the spatial box of side $c_2\left(\log l_n\right)^2$ centred at $x$ by a path of unit conductors in the spatial environment $G_{T_n}$.

Proposition 91 holds with the events $C_i$ modified to ensure that the paths connecting $x$ to $\partial B_x\left[k\right]$ avoid using the edge $(xy)$. Call these modified events $\tilde{C}_i$.

Then

$$\begin{aligned}
\mathbb{P}\left(A_n\left(x\right)\right) &= \mathbb{P}\left(\tilde{C}_0\left(c_2\left(\log l_n\right)^2, x\right) \cap \tilde{C}_{T_n}\left(c_2\left(\log l_n\right)^2, x\right)\right) \\
&\quad \times \mathbb{P}\left(\omega_{xy}\left(i\right) = 2^{-n} \text{ for } i \in \left[0, T_n - 1\right), \omega_{xy}\left(T_n\right) = 1\right) \\
&\quad \times \mathbb{P}\left(\omega_e\left(i\right) \leq 2^{-n} \text{ for } i \in \left[0, T_n\right]\right)^{2d-1} \\
&\geq c_1 \mu\left(2^{-n}\right) c_3 \left(\sum_{i \geq n} \mu\left(2^{-i}\right)\right)^{2d-1} \\
&\geq c_4 n^{-1-2d\varepsilon}.
\end{aligned}$$

Taking $\mathbb{G}_n$ to be a grid of sites in $\left[-l_n, l_n\right]^d \cap \mathbb{Z}^d$ that are spaced by distance $2(\log l_n)^2$. The events $\{A_n(x) : x \in \mathbb{G}_n\}$ are independent, so using $1 - x \leq e^{-x}$ for $x \in [0, 1]$,

$$\mathbb{P}\left(\bigcap_{x \in \mathbb{G}_n} A_n\left(x\right)^c\right) \leq \exp\left\{-c_5\left(\frac{l_n}{\left(\log l_n\right)^2}\right)^d n^{-[1+2d\varepsilon]}\right\} \leq e^{-cn^{\varepsilon}},$$

hence by Borel-Cantelli, the intersection occurs for only finitely many $n$.

By Proposition 91, every connected component of diameter at least $\left(\log l_n\right)^2$ in $\left[-l_n, l_n\right]^d \cap G_i$ will be connected to the largest component of $\left[-2l_n, 2l_n\right]^d \cap G_i$ for $i \in \{0, T_n\}$ and all large enough $n$. Now, the origin at time zero will not necessarily be connected to this largest component. Take $z$ to be the closest space-time point to the origin that lies on the infinite component. On the event $A_n\left(x\right)$ for $n$ large, take $l_n^1$ to be the shortest path from the origin to $x$ in $G_0$ that goes from 0 to $z$ and then follows a strong path to $x$.

Take $n_i$ to be a subsequence such that there is a strong path from $x$ to 0 at time $T_{n_i}$ of length $l_n^2$ with $l_n^2$ bounded by $c_9 l_n$. Such a subsequence (and constant $c_9$) exist by [2]. The results of [2] also imply that $l_n^1 \leq c_9 l_n$ for all large enough $n$. We take a subsequence as we then avoid the complication of the origin being surrounded by many weak bonds as such a situation would make it difficult for the walk to return to the origin.

It is shown in [45] that for a one dimensional walk the following lower bound holds: there exist constants $c_i$ such that for any $x, y \in \mathbb{Z}^d$ and $d\left(x, y\right) \geq t \geq 1$

$$P_x\left(X_t = y\right) \geq c_6 \exp\left(-c_7 d\left(x, y\right)\left(1 + \log d\left(x, y\right)/t\right)\right). \tag{4.28}$$

We wish to bound the probability that the walk travels fully along a strong path of length $l_n^i$ in a unit of time. As we can bound the probability that the walk deviates from this one dimensional path from below by $e^{-c_8 l_n^1}$ we can condition so that the walk only sees the one dimensional path and hence

$$P^{\omega}_{(x,T_n)}\left(X_{T_n+1} = 0\right) \geq c_6 \exp\left(-c_{10} l_n^2\left(1 + \log d l_n^2/t\right)\right) \exp\left(-c_8 l_n^2\right).$$

Similarly for the initial strong path from $z$ to $x$, with a constant dependent on the local environment around the origin replacing $c_6$.

Plugging this into (4.27) we obtain

$$\begin{aligned} P^{\omega}_{(0,0)}\left(X_{2^{n_i}+1}=0\right) &\geq C\left(\omega\right)2^{-n_i}\exp\left(-cn_i^{(1+(2d+1)\varepsilon)/d}\log n\right) \\ &\geq C\left(\omega\right)2^{-n_i}\exp\left(-c'n_i^{(1+(3d+1)\varepsilon)/d}\right), \end{aligned}$$

taking $\varepsilon$ small concludes the proof. ■

With this example of $\omega$ supported on $\{2^{-n}: n\in\mathbb{N}\}$ in mind we can demonstrate three distinct behaviours for the heat kernel for the dynamic random walk. Let $G_t\left(\omega\right)$ be the environment at time $t$ for $\omega\in\Omega$. For $m>0$ define the dynamic graph $\mathcal{G}^m_t := G_{tm}$, that is the graph speeded up so that edges flip $m$ times per unit of time. Let $X^m_t$ be the space-time random walk on $\mathcal{G}^m_t$ started at $(0,0)$. We have already proved that if $m=1$ we have a lower bound close to $O\left(t^{-1}\right)$. As we let $m$ tend to zero and infinity then we have two distinct behaviours.

**Proposition 92.** *For almost all $\omega\in\Omega$ there exist a constant $C\left(\omega\right)$ such that for all $t>0$ we have the limits:*

$$\begin{aligned} \frac{C\left(\omega\right)}{t^2} &\geq \lim_{m\to 0} P\left(X^m_t=0\,|X^m_0=0\right)\geq C\left(\omega\right)\frac{e^{-(\log t)^{\kappa}}}{t^2} \text{ for } d\geq 5 \\ C_1 t^{-d/2} &\geq \lim_{m\to\infty} P\left(X^m_t=0\,|X^m_0=0\right)\geq C_2 t^{-d/2} \text{ for } d\geq 1. \end{aligned}$$

**Proof.** The first line is due to [11], as $m\to 0$ corresponds to the static case.

As $m$ gets large the probability that the walk crosses an edge in time $t$ tends towards the probability that the random walk crosses an edge of conductance $\mathbb{E}\left(\omega_e\right)$ in time $t$. Hence the limit as $m\to\infty$ corresponds to the annealed random walk. This is the simple random walk on $\mathbb{Z}^d$ with speed $2d\mathbb{E}\left(\omega_e\right)$ and hence exhibits standard on diagonal heat kernel behaviour. ■

We have been slightly disingenuous when suggesting that this shows behaviour more anomalous than presented in [11] as we are considering a different model - the variable speed walk as opposed to the constant speed walk investigated in [11]. In the constant speed case the above trap is not a trap at all as the transition rates are normalized by $\sum_{y\sim x}\omega_{xy}$ so that the walk always moves at unit speed. There are thus two obvious questions: what is the behaviour of the variable speed walk in the static case and what is the behaviour of the constant speed walk in the dynamic case?

We begin with the first question. The trapping demonstrated above will lead to lower bounds close to $O\left(n^{-2}\right)$ in the static case (the $n^{-2}$ is due to the walk now having to pay a price of $O\left(n^{-1}\right)$ to exit the trap as well as to enter). The upper

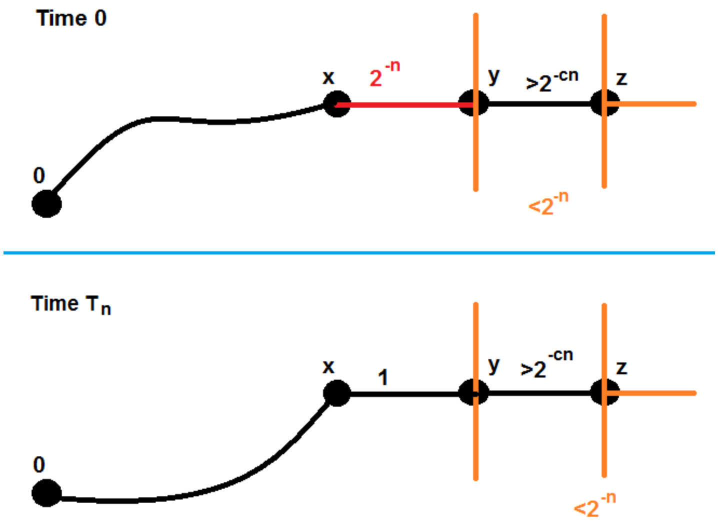


Figure 4.2: A space time trap for the constant speed walk

bounds follow from slight modifications to the arguments of [11], with summations replaced by integrals.

The answer to the second question is presented in Proposition 93 below where lower bounds close to $O\left(n^{-1}\right)$ are again proven. The example that displays these lower bounds is a very similar example of space-time trapping in the constant speed case - again the environment changes at discrete time points with the walk being a continuous time Markov process.

**Proposition 93.** *Take $d \geq 3$. For any $\alpha > 0$ and $\kappa > \frac{1}{d}$, there exist non-static random space-time environments of the above form such that for almost every $\omega \in \Omega$ there exists $C(\omega) > 0$ and an increasing sequence $(n_i)_{i \geq 0}$ such that $\lim_{i\to\infty} n_i = \infty$ and for all $i$*

$$P_\omega^{n_i}(0,0) \geq C(\omega)\frac{e^{-(\log n_i)^\kappa}}{n_i^{1+\alpha}}.$$

**Proof.** The proof is very similar to above. We will simply outline the type of traps that lead to this behaviour.

Figure 4.2 demonstrates the types of trap we consider around a point $x$. Take $y = x + e_1$ and $z = y + 2e_1$. We initially take the bond between $y$ and $z$ to be of weight 1, with the bond between $x$ and $y$ being of weight $2^{-n}$ and all other bonds emanating from $y$ and $z$ being of weight $\omega_e \leq 2^{-n}$. As time evolves all the weak bonds remain at their initial value. The strong bond will weaken but will never be weaker than $2^{-cn}$ for some constant $c$ that we can take to be as small as we like. At time $T_n$ the bond between $x$ and $y$ switches to unit weight. We condition on there existing strong paths between 0 and $x$ in the spatial environments $G_0$ and $G_{T_n}$.

Take $r_n$ to be the length of the space-time path from 0 to $x$ at time zero and $r_n'$ to be the length from $x$ to 0 at time $T_n$. As in equation (4.27) above, if $T_n = O\left(2^{cn}\right)$ we have

$$\begin{aligned}
P^{\omega}_{(0,0)}\left(X_1 = x\right) &\geq c_1 \exp\left(-c_2 r_n \left(1+\log r_n\right)\right)\\
P^{1}_{(1,x)}\left(X_1 = y\right) &\geq c_1 2^{-n}\\
P_{(2,y)}\left(\text{stay on } yz \text{ for } T_n\right) &\geq C\\
P_{r_n'}\left(x,0\right) &\geq c_1 \exp\left(-c_2 r_n' \left(1+\log r_n'\right)\right)
\end{aligned}$$

and hence

$$P^{\omega}_{(0,0)}\left(X_{T_n+1} = 0\right) \geq c_2 e^{-c_3\left(r_n(1+\log r_n)+r_n'(1+\log r_n')\right)} 2^{-n}.$$

The details are similar to the proof of Theorem 90. ■

These results will also go through into discrete time provided that some further combinatorics can be proven. In discrete time lower bounds of the form (4.28) do not hold. The walk must take at least time $l$ to traverse a path of length $l$. We thus introduce the notion of a strong space-time path:

Call a space-time path $\gamma = \left(\gamma_i\right)_{i=0,\ldots,n}$ a strong space (discrete) time path from $(m,x)$ to $(m+n,y)$ if $\gamma_0 = (m,x)\,, \gamma_n = (m+n,y)$ and writing $\gamma_i = (t_i, z_i)$ we have $t_i = m+i$, $z_i \sim z_{i+1}$ and $\omega_{z_i,z_{i+1}} = 1$ for all $i$. Write $(n,x) \leftrightarrow (n+m,y)$ if there exists a strong path between the two points. Suppose that we have the following

- There exists a constants $c_i$ such that for any $x \in \mathbb{Z}^d$,

$$\mathbb{P}\left((0,x) \leftrightarrow \left\{c_1 (\log n)^2\right\} \times \partial B_x\left[(\log n)^2\right]\right) \geq c_2.$$

- For almost all $\omega \in \Omega$ there exist $N(\omega)$ such that all connected components of size at least $(\log n)^2$ in $B_0[n]$ are connected to the infinite cluster for $n \geq N(\omega)$.

- The events

$$\begin{aligned}
&\left\{(0,x) \leftrightarrow \left\{c_1 (\log n)^2\right\} \times \partial B_x\left[(\log n)^2\right]\right\}\\
\text{and}\qquad &\left\{(m,x) \leftrightarrow \left\{m + c_1 (\log n)^2\right\} \times \partial B_x\left[(\log n)^2\right]\right\}
\end{aligned}$$

  are asymptotically independent.

Then the probability that the walk takes the direct space-time path from 0 to $x$ will be bounded by an $e^{-cl_n}$ where $l_n$ is the length of the path and by the same calculations as before we would obtain anomalous heat kernel as stated in Theorem 90. Whether or not these conditions are realistic is not clear. We are working in the setting of oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}$ without independence in the $\mathbb{Z}$ direction. Proving these conditions in this setting would appear to be a challenge.

### 4.3.1. Upper Bounds?

To prove that the lower bounds presented are of the correct order requires corresponding upper bounds. We will discuss a couple of heuristics that may lead to the upper bound. The first is a modification of the argument given in [11] to prove upper bounds in the static environment case. Where the arguments fail in the dynamic case are discussed with hurdles presented, that if overcome would give an upper bound of $Cn^{-1}$ for $d \geq 5$.

The second heuristic looks for a link between the heat kernel on the static environment and the heat kernel on the dynamic graph. We conjecture that for large enough time the dynamic on diagonal heat kernel $P^t_{\text{dyn}}(0,0)$ is bounded above by the off-diagonal heat-kernel of the static graph. More precisely take $(\omega(t))_{t\in\mathbb{R}}$ to be a time-space environment, iid in space, then for almost all $\omega \in \Omega$ and all sufficiently large $t$

$$P^t_\omega(0,0) \leq \max_{y\in\mathbb{Z}^d} P^t_{\omega(0)}(0,y), \tag{4.29}$$

where the left hand side refers to the kernel of the random walk on the dynamic graph $\omega$ started at $(0,0)$ and the right hand side refers to the random walk on the static environment $(\omega_e(0))_{e\in\mathbb{Z}^d}$. We discuss why we believe this to be the case and the obstacles to obtaining an upper bound.

In [11] the random walk is analyzed via a time changed walk: all bonds of conductance $\omega_e \leq \alpha$ are removed. If $\alpha$ is chosen to be small then the remaining graph contains a unique infinite component. The time changed walk is the walk only observed when it is on the infinite component. The geometry of this component can be shown to satisfy standard isoperimetric inequalities and this leads to standard heat kernel behaviour for the time changed random walk. By controlling the time in traps the following conclusion can be reached ((3.51) of [11]):

$$\sum_{n\leq m<2n} P^m_\omega(0,0) \leq C(\omega) \begin{cases} n^{1-d/2} & d=2,3 \\ n^{-1}\log n & d=4 \\ n^{-1} & d\geq 5 \end{cases}. \tag{4.30}$$

It is then argued that since

$$\begin{aligned} P_\omega^{2m}(0,0) &= \left\langle \delta_0, P_\omega^{2m}\delta_0 \right\rangle_\omega \\ &= \langle P_\omega^m \delta_0, P_\omega^m \delta_0 \rangle_\omega \\ &= \left\| P^1 P^{m-1}\delta_0 \right\|_2^2 \\ &\leq \left\| P^{m-1}\delta_0 \right\|_2^2 \\ &= P_\omega^{2(m-1)}(0,0), \end{aligned} \tag{4.31}$$

$P^{2m}$ is decreasing and so the left hand side of (4.30) can be bounded below by $\frac{1}{2}nP^{2n}(0,0)$. Thus dividing through by $n$ gives sharp upper bounds.

How can this argument be used in the dynamic case? Well, the time changed walk is not Markov and so obtaining heat kernel bounds is certainly more tricky. However, as the time changed walk only walks on geometrically $d$-dimensional graphs it does not appear unreasonable to suppose that its heat kernel will satisfy standard on diagonal decay. If this were true, then together with results from Section 4.5 on the ergodicity of the environment from the point of view of the particle, equation (4.30) could be obtained (in integral form as we are dealing with a continuous time process). However, in the dynamic case the monotonicity argument at (4.31) fails to hold as $P^m \neq P^{*m}$. Instead we use the trivial bound

$$P_\omega^n(0,0) \leq \sum_{n \leq m < 2n} P_\omega^m(0,0).$$

For $d \geq 5$ this gives the upper bound that we desire. So this idea will work provided that on diagonal control of the time changed walk can be achieved.

We now discuss the second idea. Our examples for the lower bound are good examples of the non-monotonicity of the on diagonal kernel. While the trap is in situ paths that go directly to the trap, stay in the trap for a long time (but not long enough for the trap to disappear) before returning to the origin contribute $O(n^{-2}\exp(p(n)))$ to the on diagonal bound. However, as soon as the trap disappears these paths contribute $O(n^{-1}\exp(p(n)))$. In the static model the traps never disappear so paths that contribute to the on diagonal probabilities must exit any trap that they enter. However, if we consider

$$\max_{y \in \mathbb{Z}^d} P_{\omega(0)}^t(0,y) \tag{4.32}$$

then paths that do not have to exit traps and thus do not have the additional probabilistic penalty to do so are included. Hence equation (4.29) looks reasonable. Further, if this were true then by the Cauchy Schwarz inequality

$$\begin{aligned} P_{\omega(0)}^t(0,y) &\leq P_{\omega(0)}^t(0,0)^{1/2} P_{\omega(0)}^t(y,y)^{1/2} \\ &\leq C(\omega)\, n^{-1} \text{ for } d \geq 5. \end{aligned}$$

So again we would obtain the bound we seek in high dimensions. The major issue is whether (4.29) is true and if so proving it. For example, how do you discount the event that the walk gets into a trap for free by being at the trap site as the trap forms? As noted in the introduction, if traps are persistent that they must begin infrequently so this probability should be negligible for all large enough $t$, but how do we prove it? There are plenty of other possible trapping strategies that can be cooked up and need to be dismissed. We have no method currently to prove (4.29).

## 4.4. Continuous time, variable speed $\omega \in [1, \infty)$

When conductances are bounded below the existence of standard on diagonal heat kernel upper bounds are assured and the question moves to controlling off-diagonal behaviour. This is due to the fact that with $\omega$ bounded below at each time $t$ the graph $G_t$ satisfies a standard isoperimetric inequality and thus a uniform Nash inequality holds at all times. Thus, similarly to the on diagonal bounds of Proposition 78 we have:

**Proposition 94.** *There exists a constant $C = C(d)$, independent of $\omega$, such that for all $x, y \in \mathbb{Z}^d$ and $s \in \mathbb{R}$, $t > 0$ :*

$$P^{\omega}(X_{s+t} = y \,|X_s = x) \leq Ct^{-d/2}.$$

**Proof.** As $\omega \geq 1$ and the invariant measure is flat: $\pi \equiv 1$, by comparison with the simple random walk on $\mathbb{Z}^d$, for each $t \in \mathbb{R}$ the spatial graph $G_t$ satisfies the standard $d$-dimensional isoperimetric inequality: there exists $c = c(d)$ such that

$$\forall A \subseteq \mathbb{Z}^d \text{ finite, } Q(A, A^c) \geq c(d)\, \pi(A)^{(d-1)/d}.$$

By well known results (for example, Proposition 14.1 of [58]) this implies the uniform $d$-dimensional Nash inequality: there exists $c' = c'(d)$ such that

$$\forall f \in l_0\left(\mathbb{Z}^d\right), \ \ \|f\|_2^{2+4/d} \|f\|_1^{-4/d} \leq \mathcal{E}_t(f, f).$$

The result then follows by the standard methods outlined previously. ■

As mentioned when looking for Carne-Varopoulos type bounds in Section 4.2.1, obtaining long range bounds is more complicated when $\omega$ is not bounded above. For example, if there exists a finite subset $A \subseteq \mathbb{Z}^d$ with $\omega_e(t) = l$ for all $e \in A$ and $t \in \mathbb{R}$ and $\omega_e = 1$ for $e \in \partial A$, then the random walk will move at speed $2dl$ on $A$ and cross the boundary at unit speed. In particular for any $x, y \in A$, if $t$ is small and $l$ is large then

$$P^{\omega}(X_t = y \,|X_0 = y) \approx |A|^{-1}.$$

Hence for $x, y \in A$ with $d(x,y) = o(t^2)$, the usual long range bounds cannot hold. There can be no general result of the flavour of Proposition 79 with respect to Euclidean distance.

This problem was tackled in the static time case in [7]. There a second distance function, $\tilde{d}$, obtained through a first passage percolation procedure replaces Euclidean distance in the statements and proof of Proposition 79. This distance takes account of paths between points $x, y \in \mathbb{Z}^d$ that consist of a series of high conductance edges. If two points can be joined by such a path then their $\tilde{d}$ distance is shorter than their corresponding graph distance. The methods of [25] transfer to this setting giving bounds as in Proposition 79 with respect to $\tilde{d}$ as opposed to $d$. It is then shown that for large enough distances, $\tilde{d}$ is comparable to $d$ and hence standard long range bounds hold. Together with the on diagonal bounds and the methods of Bass-Nash, full off diagonal upper bounds on the heat kernel can be proven.

We would like to prove a statement similar - we would like a space-time distance $\tilde{d}^\omega((s,x),(s+t,y))$ such that

$$P^\omega(X_{s+t} = y \,|X_s = x) \leq c_1 \exp\left(-\frac{c_2 \tilde{d}^\omega((s,x),(s+t,y))^2}{t}\right) \tag{4.33}$$

for $t \geq c_3 \tilde{d}^\omega((s,x),(s+t,y))$ and Poisson type bounds for small $t$. We fail to provide such a distance. To illustrate the difficulties faced, we will give an incomplete upper bound in Proposition 95 and then comment on the difficulties adapting the proof to standard long range bounds - showing the limitations of Davies' perturbation method in the time dynamic setting with unbounded conductances.

**Proposition 95.** *Define $f(x,r,t) := \sup_{e \in B_x[r], 0 \leq s \leq t} \omega_e(s)$ then*

$$P^\omega_{0,t}(x,y) \leq \exp\left[|x-y| \, F(\gamma)\right]$$

*for $\gamma = \frac{|x-y|}{2dtf(|x-y|,t)}$ and*

$$F(\gamma) = \gamma^{-1}\left(\sqrt{1+\gamma^2} - 1\right) - \log\left\{\gamma + \sqrt{1+\gamma^2}\right\}.$$

**Proof.** We use the techniques outlined in the proof of Proposition 79.

Take $\phi$ such that $\phi, \phi^{-1} \in L^\infty(\mathbb{Z}^d, \pi)$. Set

$$P^\phi_{s,t} = \exp\left(\phi \int_s^t \mathcal{L}_u du \phi^{-1}\right)$$

and function $f$ of bounded support set

$$f_t := P^\phi_{0,t} f.$$

As $\phi$ is independent of $t$ we obtain the differential inequality

$$\partial_t \|f_t\|_2^2 \leq c_t(\phi) \|f_t\|_2^2$$

and hence

$$\|f_t\|_2 \leq \exp\left\{\int_0^t c_s(\phi)\,ds\right\} \|f\|_2.$$

For $\lambda > 0$ take

$$\phi(x) = \exp\left(-\lambda\left(|x - x_0| \wedge |y_0 - x_0|\right)\right).$$

By [25] we have

$$c_s(\phi) \leq \sup_x b_s(\phi, x)$$

for

$$b_s(\phi, x) := \frac{1}{2}\sum_{y\sim x} \omega_{xy}(s)\left(e^{\psi(x)-\psi(y)} + e^{\psi(y)-\psi(x)} - 2\right),$$

for $\psi(x) := -\lambda\left(|x - x_0| \wedge |y_0 - x_0|\right)$. Thus

$$c_s(\phi) \leq d \sup_{e\in B_{x_0}[|x_0-y_0|]} \omega_e(s)\left(e^{-\lambda} + e^{\lambda} - 2\right). \tag{4.34}$$

Putting these arguments together we have

$$\|f_t\|_2 \leq \exp\left(d\int_0^t \sup_{e\in B_{x_0}[|x_0-y_0|]} \omega_e(s)\left(e^{-\lambda} + e^{\lambda} - 2\right)ds\right)\|f\|_2. \tag{4.35}$$

The kernels of $P_{s,t}$ and $P_{s,t}^{\phi}$ are related via:

$$P_{s,t}^{\phi}(x, y) = \phi(x)\,P_{s,t}(x, y)\,\phi(y)^{-1}.$$

Hence, taking $f = \delta_{y_0}$ in the definition of $f_t$ we have by (4.35)

$$\begin{aligned} P_{0,t}(x_0, y_0) &\leq \phi(x_0)^{-1}\phi(y_0)\|f_t\|_2 \\ &\leq \exp\left(-\lambda|y_0 - x_0| + d\int_0^t \sup_{e\in B_{x_0}[|x_0-y_0|]} \omega_e(s)\left(e^{-\lambda} + e^{\lambda} - 2\right)ds\right) \\ &= \exp\left(-\lambda|y_0 - x_0| + d\left(e^{-\lambda} + e^{\lambda} - 2\right) w_{B_{x_0}[|y_0-x_0|]}(t)\right), \end{aligned} \tag{4.36}$$

where

$$w_B(t) = \int_0^t \sup_{e\in B} \omega_e(s)\,ds.$$

For $f(x_0, |x_0 - y_0|, t) = \sup_{e\in B_{x_0}[|x_0-y_0|], 0\leq s\leq t} \omega_e(s)$:

$$w_{B_{x_0}[|x_0-y_0|]}(t) \leq tf(|x_0 - y_0|, t).$$

Hence, we wish to minimize

$$\exp\left(-|x_0 - y_0|\lambda + d\left(e^{-\lambda} + e^{\lambda} - 2\right)tf(|x_0 - y_0|, t)\right).$$

By the arguments detailed in page 70 of [25] we obtain

$$p\left(0, x_0; t, y_0\right) \leq \exp\left[\left|x_0 - y_0\right| F\left(\gamma\right)\right],$$

for $\gamma = \frac{|x_0 - y_0|}{2dtf(|x_0-y_0|,t)}$ and

$$F\left(\gamma\right) = \min_{\lambda > 0}\left\{-\lambda + \frac{e^{\lambda} + e^{-\lambda} - 2}{2\gamma}\right\}.$$

The minimum is achieved at $\lambda = \log\left\{\gamma + \sqrt{1+\gamma^2}\right\}$. ■

For large enough $t$ this result reduces to a close to standard bound provided that the function $f$ is slowly growing.

**Corollary 96.** *There exists constant $C$ such that for $t > |x - y|$*

$$P^{\omega}\left(X_t = y \,|X_0 = x\right) \leq \exp\left(-\frac{|x-y|^2}{tf\left(|x-y|, t\right)}\right).$$

**Proof.** As noted in [25], $F\left(\gamma\right) \leq -\gamma/2 + \gamma^3/20$, hence by Proposition 95

$$P^{\omega}\left(X_t = y\,|X_0 = x\right) \leq \exp\left(-\frac{|x-y|^2}{2dtf\left(|x-y|, t\right)}\left(1 - \frac{|x-y|^2}{10d^2t^2f\left(|x-y|, t\right)}\right)\right)$$

and the Corollary follows. ■

The proof of Proposition 95 is unsatisfactory - it is a blunt adaptation of the static methods with the only change being requiring control over the strongest edge in a space-time box. The long range heat kernel behaviour is not controlled by the strongest edge that the walk can see and so the Proposition cannot tell the full story. We have tried other adaptations to Davies' perturbation methods and we outline why these fail to work.

First we describe the methods of [7] where the variable speed random walk with unbounded conductances is dealt with in the static environment case.

Take $t\left(e\right) := \omega_e^{-1/2}$ and define

$$\tilde{d}\left(x, y\right) := \inf_{\gamma}\left\{\sum_{i=1}^{n} t\left(e_i\right)\right\}$$

where the infimum is over paths $\gamma = (e_1, \ldots, e_n)$ from $x$ to $y$. This distance satisfies

$$\omega_{y,y'}\left|\tilde{d}\left(x, y\right) - \tilde{d}\left(x, y'\right)\right|^2 \leq 1$$

for all $x \in \mathbb{Z}^d$ and $y \sim y'$. This is important as if we take $\psi\left(x\right) := \lambda\left(\tilde{d}\left(x_0, y_0\right) \wedge \tilde{d}\left(x_0, x\right)\right)$ and $\phi\left(x\right) = e^{-\psi(x)}$ then equation (4.34) of the above proof can be bounded independently of $\omega$ :

$$\begin{aligned} c\left(\phi\right) &\leq \sup_{x} \frac{1}{2}\sum_{y\sim x}\omega_{xy}\left(e^{-\psi(x)+\psi(y)} + e^{\psi(x)-\psi(y)} - 2\right) \\ &\leq d\left(e^{-\lambda} + e^{\lambda} - 2\right). \end{aligned}$$

Hence the equivalent statement to Proposition 95 would involve the distance $\tilde{d}$ but otherwise be independent of the environment.

With these methods in mind, perhaps the most natural adaptation would be to consider the semigroup:

$$P_{s,t}^{\phi} := \exp\left(\int_s^t \phi_u \mathcal{L}_u \phi_u^{-1} du\right),$$

for functions $(\phi_u)_{u\in\mathbb{R}}$ defined in terms of the first passage distance at each time $u$ :

$$\phi_u = \exp\left(-\lambda \tilde{d}_u (x_0, y_0) \wedge \tilde{d}_u (x_0, x)\right).$$

We then set

$$f_t = \exp\left(\int_s^t \phi_u \mathcal{L}_u \phi_u^{-1} du\right) f$$

and obtain the differential equation

$$\frac{d}{dt} \|f_t\|_2^2 = \left\langle 2\phi_t \mathcal{L}_t \phi_t^{-1} f_t, f_t \right\rangle.$$

As before we wish to find $c_t(\phi_t)$ such that

$$\left\langle 2\phi_t \mathcal{L}_t \phi_t^{-1} f_t, f_t \right\rangle \leq 2c_t(\phi_t) \|f_t\|_2^2.$$

By the choice of $\phi_u$ we can bound $c_t$ independently of the environment

$$c_t(\phi_t) \leq d\left(e^{-\lambda} + e^{\lambda} - 2\right).$$

Hence

$$\begin{aligned} \|f_t\|_2 &\leq \|f\|_2 \exp\left(\int_s^t c_u(\phi_u)\, du\right) \\ &\leq \|f\|_2 \exp\left((t-s)\, d\left(e^{-\lambda} + e^{\lambda} - 2\right)\right). \end{aligned}$$

As before taking $f = \delta_x$

$$P_{s,t}^{\phi}(x,y) \leq \exp\left((t-s)\, d\left(e^{-\lambda} + e^{\lambda} - 2\right)\right).$$

This looks promising. However, we wish to bound the kernel $P_{s,t}(\cdot,\cdot)$ and so need a relationship between $P_{s,t}^{\phi}$ and $P_{s,t}$. This is much more complicated than in the static time case. We illustrate this with an example.

Suppose that $\phi_u \mathcal{L}_u \phi_u^{-1}$ is a step function:

$$\phi_u \mathcal{L}_u \phi_u^{-1} = \sum_{i=1}^{n} 1_{[u_i, u_{i+1}]}(u)\, \phi_{u_i} \mathcal{L}_{u_i} \phi_{u_i}^{-1},$$

then restricting to the interval $[s,t]$

$$\begin{aligned} P^{\phi}_{s,t} &= \exp\left(\sum_{\substack{i=0 \\ u_0=s,u_{n+1}=t}}^{n} (u_{i+1}-u_i)\,\phi_{u_i}\mathcal{L}_{u_i}\phi^{-1}_{u_i}\right) \\ &= \prod_{i=0}^{n} \exp\left((u_{i+1}-u_i)\,\phi_{u_i}\mathcal{L}_{u_i}\phi^{-1}_{u_i}\right) \end{aligned}$$

and hence

$$\begin{aligned} P^{\phi}_{s,t}(x,y) &= \sum_{y_i\in\mathbb{Z}^d} \phi_s(x)\,P_{s,u_1}(x,y_1)\,\phi_s(y_1)^{-1}\,\phi_{u_1}(y_1)\,P_{u_1,u_2}(y_1,y_2)\,\phi_1(y_2) \\ &\quad \ldots\phi_{u_n}(y_n)\,P_{u_n,t}(y_n,y)\,\phi^{-1}_t(y)\,. \end{aligned} \tag{4.37}$$

From (4.37) it is apparent that there is not an easy way to transfer bounds on $P^{\phi}$ to bounds on $P$.

With this method not leading to the conclusion we would hope for, one could instead start by requiring a relationship between $P$ and $P^{\phi}$ :

$$P^{\phi}_{s,t}(x,y) = \phi_t(x)\,P_{s,t}(x,y)\,\phi_t(y)^{-1}\,,$$

where we assume $\phi_t$ is differentiable in $t$. Here we run into a different problem. Set

$$f_t = P^{\phi}_{0,t}f$$

for $f$ of finite support. Then,

$$\begin{aligned} \frac{d}{dt}\|f_t\|_2^2 &= 2\left\langle \frac{d}{dt}f_t, f_t\right\rangle \\ &= 2\left\langle \phi'_t\phi^{-1}_t f_t, f_t\right\rangle + 2\left\langle \phi_t\mathcal{L}_tP_{s,t}\phi^{-1}_t f, f_t\right\rangle - 2\left\langle \phi_tP_{s,t}\phi'_t\phi^{-1}_t f, f_t\right\rangle. \end{aligned}$$

Although the middle term can be controlled as before it would appear difficult to choose functions $\phi_t$ that give the requisite control over the remaining terms.

A sensible space-time distance that satisfies equation (4.33) has thus far proved elusive.

## 4.5. Central limit theorems

In this section we discuss various central limit theorems obtained previously in the dynamic setting. More general environments where we allow $\omega_{xy}(t) \neq \omega_{yx}(t)$ are considered and jumps are not necessarily nearest neighbour or in fact bounded. For the case when the environment is iid in space and time quenched invariance principles hold under very relaxed conditions. When the environment evolves in a Markovian way it is harder to analyze. Known results in this setting tend to assume an ellipticity condition on the random walk and a mixing condition on the random environment in order to prove both annealed and quenched central limit theorems. We will discuss several approaches to this problem. Unless otherwise stated the following results refer to a discrete time walk on a dynamic environment that evolves at integer points.

The early work on this problem can be found in [14], [15] and [16], where highly functional analytic arguments are used to prove functional CLTs under fairly strong assumptions.

In [3] a probabilistic regeneration time argument is used instead. Here a direct tradeoff between the ellipticity of the walk - an assumption that the transition probabilities for the walk are not far away from a deterministic set of transitions - and the mixing of the environment is assumed. When this tradeoff holds an annealed functional CLT is shown to hold with a quenched CLT also holding for large enough dimension. The proof comes from establishing moment bounds on regeneration times. These are times $t$ at which the current weights of the edges emanating from points that the walk has visited:

$$\{\omega_{xy}(t) : x \in X_n \text{ for } n \in [0,t]\}$$

are independent of the values of those edges at the point at which the walk visited them:

$$\{\omega_{X_n y}(n) : n \in [0, t-1]\}.$$

In essence, all the information that the walk has picked up between time 0 and $t-1$ is irrelevant to the current state of the environment. As the environment mixes rapidly and the chain can step deterministically, it is shown that these regeneration times exist and moment bounds hold. If $T$ is such a regeneration time then $\{X_n : n \geq T\}$ is shown to be independent of $\{X_n : n < T\}$. In particular the walk can be considered as a series of iid, finite walks between regeneration times. The annealed central limit theorem then follows from standard results regarding summing iid random variables, with the moment bounds on $T$ allowing these bounds to be interpolated between regeneration times. The quenched result is somewhat more involved, requiring stronger moment bounds and higher dimensions in order for the proof to hold.

Methods analogous to the Kipnis-Varadhan methods discussed in Chapter 3 are introduced in [49], considering the walk from the point of view of the particle. The environment is taken to be iid in space and time and the walk is viewed as taking place in $d+1$ dimensions with deterministic moves in the time dimension. As the environment is iid in space and time the invariant measure for the environment process is straight forward to write down: for $n \geq 0$ write

$$f_n(\omega) = \sum_{x: x \cdot e_1 = -n} P_x^{\omega}(X_n = 0),$$

where $e_1$ is the unit vector in the positive time direction. Then $f_n$ is a martingale with respect to the filtration

$$\left\{ \mathcal{G}_{-n} = \sigma\left( (\omega_{xy})_y, x \cdot e_1 \geq -n \right) \right\}_{n \geq 0},$$

that is the filtration generated by the edges connected to points whose time coordinate is at least $-n$. We can therefore define $\mathbb{P}_\infty$ by

$$\frac{d\mathbb{P}_\infty|_{\mathcal{G}_{-n}}}{d\mathbb{P}|_{\mathcal{G}_{-n}}} = f_n, \tag{4.38}$$

and an induction on $n$ shows that $\mathbb{P}_\infty$ is invariant for the environment from the point of view of the particle. This measure is shown to be ergodic and due to the iid nature of the environment the results of [41] are sufficient to prove a functional central limit theorem. This derivation of $\mathbb{P}_\infty$ is intuitively nice but its scope is limited by the fact that $f_n$ is only a martingale in the iid case or under an assumption that $\omega_e(n)$ is a martingale which in turn violates the existence of a stationary measure for $\omega_e$ in the bounded conductance case.

The Kipnis-Varadhan inspired methods are again employed in [29] to prove a central (but not functional) limit theorem. Their setting is more general, showing that if the environment is mixing in time and space and satisfies an ellipticity condition then a central limit theorem holds. Environments that satisfy these conditions are presented and it is proven that the conditions are weaker than those assumed in [3]. There is no nice formulation for the invariant measure for the environment process analogous to (4.38) and analytic arguments are required to prove that a unique invariant measure exists and that it is ergodic.

It is interesting to compare invariant distributions for the environment process in the static and dynamic cases. Consider the 1-dimensional environment with conductances $\omega_e \in \{1, 2\}$ and the constant speed walk. Then in the static case with iid conductances the invariant measure is simply product measure multiplied by the degree at the origin: $(\omega_{0,1} + \omega_{-1,0}) \mathbb{P}(d\omega)$. The dynamic case turns out to be more complicated. Suppose that $\omega_e(n) = \omega_e(n-1)$ with probability $1-p$ and with probability $p$ changes to the other possible weight. For this dynamic environment one can explicitly show that the invariant distribution for the particle process has the form:

$$\pi(\cdot) = \pi_{-1,1}(\cdot) \prod_{k \in \mathbb{Z} - \{0,1\}} \pi_k(\cdot),$$

where $\pi_{-1,1}(\cdot)$ depends only on the edges $(0,1)$ and $(-1,0)$ and $\pi_k$ depends only on the edge $(k-1,k)$. The measures $\pi_k$ satisfy for $\omega$ such that $\omega_{k-1,k}(0) = 1$

$$\pi_k(\omega) = \frac{1}{2} + c_2 f(p)^k,$$

with $c_2 = 0$ if and only if $p = \frac{1}{2}$. Note that this is a one dimensional invariant distribution - we are not considering the time-space graph as at (4.38).

The obvious question is can we extend these results to our setting? In Section 3.4 adaptations to the papers [10] and [13] were made that use heat kernel control to gain sufficient control over the corrector to imply convergence to Brownian motion of the scaled random walk. As we have heat kernel results can we do the same here?

Potentially the variable speed walk looks promising as we will show that the walk on the space-time graph is invariant and ergodic under $\mathbb{P}$.

Take $\mathbb{P}$ to be the measure on the time space environments: product measure in space with each edge marginal being a discrete time Markov process with transition function $K$ and stationary measure $\mu$. Define the process from the point of the environment to be the discrete Markov process on $\mathbb{Z} \times \mathbb{Z}^d$ with transitions

$$Q(\omega, A) = P_0^\omega\left(\tau_{(1,X_1)}\omega \in A\right).$$

**Proposition 97.** *$\mathbb{P}$ is invariant and ergodic for $Q$.*

**Proof.** Let $B$ be a finite set of edges in $\mathbb{Z}\times\mathbb{Z}^d$, $C$ be a fixed configuration on $B$ and set $A = \{\omega \in \Omega : \omega|_B = C\}$. Define $f(\omega) = 1_A(\omega)$. For $x \in \mathbb{Z}^d$ let $A_x = \tau_{(1,x)}(A)$, then since in the variable speed case $\sum_x P_x^\omega(X_1 = 0) = 1$, we have

$$\begin{aligned}
\mathbb{E}(Qf) &= \sum_x \mathbb{E}_{A_x}\left[P_0^\omega(X_1 = -x)\right] \\
&= \sum_x \mathbb{E}_A\left[P_x^\omega(X_1 = 0)\right] \\
&= \mathbb{E}_A\left[\sum_x P_x^\omega(X_1 = 0)\right] \\
&= \mathbb{E}(f).
\end{aligned}$$

Invariance then follows.

Ergodicity is standard. Take $D \subseteq \Omega$ such that for all $\omega \in \Omega$ we have

$$P_0^\omega\left(\tau_{(1,X_1)} \in D\right) = 1.$$

Thus as $P_0^\omega(X_1 = x) > 0$ for all $|x| = 1$, $\tau_{(1,x)}D \subseteq D$. Hence as $D$ is $\tau_{(1,x)}$-invariant, by spatial translation invariance $\mathbb{P}(D) \in \{0,1\}$. The methods outlined in [27] or [10] then complete the proof. ■

Although ergodic, the environment process is not reversible. In particular one cannot appeal to the results of [27] in order to easily prove the existence of a corrector (as we previously did in Section 3.3). For non-reversible walks the methods of [41] are used instead. Introduce $g := E_0^\omega (X_1) - e_1$ to be the local spatial drift at the origin ($-e_1$ centres the time dimension as time increments at unit rate: $X_1 \cdot e_1 = 1$). One then requires the following bound:

$$\sum_{n=1}^{\infty} n^{-3/2} \left\| \sum_{k=0}^{n-1} Q^k g \right\|_2 = \sum_{n=1}^{\infty} n^{-3/2} \sqrt{\mathbb{E}\left[ \left| E_0^\omega (X_n) - ne_1 \right|^2 \right]} < \infty \tag{4.39}$$

in order to obtain the martingale/remainder decomposition.

Note that even in the bounded conductance example the heat kernel bounds will only explicitly give

$$\sqrt{\mathbb{E}\left[ \left| E_0^\omega (X_n) - ne_1 \right|^2 \right]} \leq O\left(n^{1/2}\right), \tag{4.40}$$

and hence are not enough to prove (4.39).

In the iid space-time case (4.39) follows from the independence structure of the annealed environment - the drift is independent (under the annealed measure) of the path that the walk has taken and hence the right hand side of (4.40) can be replaced by $O\left(n^{1/4}\right)$ - see [49] for details. In the Markovian evolution of [29], equation (4.39) follows from strong mixing assumptions on the environment as seen from the point of view of the particle.

The question of whether these or other methods along with heat kernel bounds are sufficient to prove a central limit theorem for the variable speed walk remains open. It is worth noting that the methods described above all require that the environment is well mixing. For the environment with bounded conductances no such assumption is required to prove uniform heat kernel bounds - should it be a necessary condition for a functional central limit theorem?

# 5. Convergence of reflected random walk to reflected Brownian motion

We have previously discussed the random walk amongst random conductances, both in dynamic and static settings. In this section we again look at the static environment model, but restrict the random walk to a finite subset of $\mathbb{Z}^d$, reflecting the walk at the boundary and ask if this reflected walk converges to reflected Brownian motion under the usual scaling.

More precisely, if $\mathcal{G} = \left(\mathbb{Z}^d, \mathbb{E}^d, \omega\right)$ is a weighted graph then define the graph restricted to $[-n,n]^d$ by $\mathcal{G}^n = (V^n, E^n, \omega^n)$ for

$$\begin{aligned} V^n &= [-n,n]^d, E^n = \left\{(x,y) \in \mathbb{E}^d : x, y \in [-n,n]^d\right\}, \\ \omega_e^n &= \begin{cases} \omega_e & e \in E^n \\ 0 & \text{otherwise} \end{cases}. \end{aligned}$$

We let $(X_i^n)_{i\in\mathbb{N}^0}$ be the constant speed, discrete time random walk on $\mathcal{G}^n$; the Markov chain with transition probabilities

$$P^{\mathcal{G}^n}\left(X_i^n = y \,\middle|\, X_{i-1}^n = x\right) = \frac{\omega_{xy}^n}{\sum_z \omega_{xz}^n},$$

as before we define the heat kernel

$$q_t^{\mathcal{G}^n}(x,y) = \frac{P^{\mathcal{G}^n}\left(X_t^n = y \,|\, X_0^n = x\right)}{\pi(y)}. \tag{5.1}$$

We will take the environment $\mathcal{G} = \mathcal{G}(\omega)$ to be a realization of the random conductance model with bounded conductors. We are interested in the limiting behaviour of the scaled, linearly interpolated random walk: let $X_0^n = 0$ and define

$$B_n(t) := \frac{1}{n}\left(X^n_{\lfloor n^2t\rfloor} + \left(n^2t - \lfloor n^2t\rfloor\right)\left(X^n_{\lfloor n^2t\rfloor+1} - X^n_{\lfloor n^2t\rfloor}\right)\right), t \geq 0. \tag{5.2}$$

Given that the random walk on the unrestricted graph converges weakly to Brownian motion (see [53]), it is natural to expect $B_n(t)$ to converge to reflected Brownian motion.

This question is motivated by the work [23] where similar families of random walks on finite graphs are considered and results concerning the convergence of mixing times are given. They show that if the triple consisting of the graph, invariant measure and heat kernel converge in some suitable way to a limiting

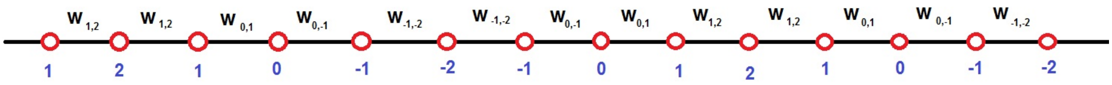


Figure 5.1: The reflected graph

triple, then convergence of the mixing time follows. The simple random walk on $[-n, n]^d$ is an example for which their results apply and one would anticipate that the walk on $\mathcal{G}^n$ should also satisfy these conditions - at least if the conductances are bounded.

In fact, it transpires that even proving convergence of $B_n$ to reflected Brownian motion is difficult. We present a proof only for the case of bounded conductances in dimension one. The one dimensional case is easier to attack as one can explicitly define a corrector for the random walk on the unrestricted graph $\mathcal{G}$ for any $\mathcal{G}$. In particular the existence of the corrector does not depend on the environment being ergodic. We turn the question around - reflecting the graph instead of the walk, so that for example the edges emanating from the vertex at $n+1$ are the same weight as those emanating from $n-1$ but in the opposite positions (see figure 5.1). We run a simple random walk on the reflected graph and show that if this converges to Brownian motion then the random walk on the $\mathcal{G}^n$ converges to reflected Brownian motion.

The random walk on the reflected graph is an object that we have good control over: by [26] we have upper and lower bounds on the heat kernel and as mentioned above it is possible to define a corrector for the random walk. We thus look to break the random walk up as a sum of a martingale and corrector then show that the martingale converges to Brownian motion and that the corrector is sublinear.

Once a functional central limit theorem is shown to hold, the techniques of [23] allow us to prove that the mixing time for the reflected random walk, scaled by $n^{-2}$, converges to the mixing time for reflecting Brownian motion.

Our main results are stated in the following theorems.

**Theorem 98.** *Let $\mathcal{G} = \mathcal{G}(\omega)$ be a realization of the random conductance model on $\mathbb{Z}$. Take $T > 0$, then for almost every $\omega \in \Omega$ the linearly interpolated reflected random walk $(B_n(t))_{t\in[0,T]}$ converges weakly in $(C[0,T], \mathcal{W}_T)$ to reflected Brownian motion on $[-1,1]$, started at the origin, with diffusion constant $\sigma^2 > 0$. Further, $\sigma^2$ is the diffusion constant as in the limit for the non-reflected random walk.*

**Theorem 99.** *Fix $p \in [1, \infty]$. If $t^p_{mix}(\mathcal{G}^n)$ is the $L^p$-mixing time of the reflected random walk on $\omega$ restricted to $[-n, n]$, then*

$$n^{-2} t^p_{mix}(\mathcal{G}^n) \to t^p_{mix}([-1,1]),$$

*where $t^p_{mix}([-1,1])$ is the $L^p$-mixing time of the Brownian motion on $[-1,1]$ reflected at the boundary.*

Formally, the $L^p$-mixing time is defined as

$$t^p_{mix}(\mathcal{G}) := \inf\left\{m > 0 : \sup_{x\in\mathcal{V}(\mathcal{G})} D^{\mathcal{G}}_p(x,m) \leq \frac{1}{4}\right\}$$

for

$$D^{\mathcal{G}}_p := \left\| \frac{q^{\mathcal{G}}_m(x,\cdot) + q^{\mathcal{G}}_{m+1}(x,\cdot)}{2} - 1 \right\|_{L^p(\pi^{\mathcal{G}})},$$

where $\pi^{\mathcal{G}}$ is taken both here and in (5.1) to be the invariant probability measure for the random walk. As noted in Remark 1.3 of [23], Theorem 99 can also be stated for mixing times with respect to total variation distance.

It is in fact true that Theorem 98 holds for an arbitrary starting point: for $x \in [-1,1]$ take $g_n(x)$ to be the point of $[-n,n]\cap\mathbb{Z}$ closest to $xn$ and set $X^n_0 := g_n(x)$. Then $B_n(t)$ converges weakly to reflected Brownian motion on $[-1,1]$ started at $x$. As the proof of Theorem 98 is notationally heavy, the arguments are presented for the case $x = 0$, however they also hold for general $x \in [-1,1]$. This fact will be important in the proof of Theorem 99.

The proof of Theorems 98 and 99 can be found in Section 5.1.

In Section 5.2 we present the results that we have in higher dimensions. Motivated by [19] we consider random walks started uniformly on subsets of $\mathbb{R}^d$. The initial aim of this study was to prove the higher dimensional analogue of Theorem 98 by proving convergence of the excursions of the reflected random walk to excursions of reflected Brownian motion. This has not yet proved to be possible. We present the results that we currently have.

## 5.1. The one dimensional case

From now on assume $d = 1$ and $\omega_e \in [a,b]$ for $0 < a \leq b < \infty$. We will discuss relaxing these assumptions later.

We begin by formally introducing the reflected graph. Let $G^n$ be the graph with vertex set, edge set and weights

$$V^n = \mathbb{Z}, E^n = \mathbb{E},$$
$$\left\{\begin{array}{rcl} \bar{\omega}^n_{4nk+i,4nk+i+1} &=& \omega_{i,i+1} \\ \bar{\omega}^n_{(4k+2)n-i-1,(4k+2)n-i} &=& \omega_{i,i+1} \\ \bar{\omega}^n_{(4k+2)n+i,(4k+2)n+i+1} &=& \omega_{-i,-i-1} \\ \bar{\omega}^n_{4nk-i,4nk-i-1} &=& \omega_{-i,-i-1} \end{array}\right. , i \in [0,n-1], k \in \mathbb{Z},$$

where the weights $\bar{\omega}^n$ are symmetric and the weights $\omega$ correspond to a realization of the random conductance model on $(\mathbb{Z},\mathbb{E})$. Figure 5.1 illustrates this definition.

We will write $\left(\bar{X}_i^n\right)_{i\geq 0}$ for the random walk on $G^n$ and write $\bar{B}_n(t)$ for the scaled, linearly interpolated walk analogous to the definition (5.2).

Note that

$$\left(X_i^n\right)_{i\geq 0} =^d \left(f\left(\bar{X}_i^n\right)\right)_{i\geq 0},$$

where

$$f(x) = 4n\left|\frac{x}{4n} + \frac{1}{4} - \left\lfloor \frac{x}{4n} + \frac{3}{4} \right\rfloor\right| - n.$$

The function $f$ folds the graph $G^n$ onto $[-n, n]$.

**Proposition 100.** *Let $T > 0$. If $\left(\bar{B}_n(t)\right)_{t\in[0,T]}$ weakly converges to Brownian motion started at the origin with diffusion constant $\sigma^2$ then $\left(B_n(t)\right)_{t\in[0,T]}$ weakly converges to reflected Brownian motion on $[-1,1]$ started at the origin with diffusion constant $\sigma^2$.*

**Proof.** As $f$ is continuous, this follows from the continuous mapping theorem. ■

Hence Theorem 98 will follow from the following proposition:

**Proposition 101.** *Take $T > 0$. For almost every $\omega \in \Omega$, $\left(\bar{B}_n(t)\right)_{t\in[0,T]}$ converges weakly in $\left(C[0,T], \mathcal{W}_T\right)$ to Brownian motion, started at the origin, with diffusion constant $\sigma^2 > 0$.*

As discussed in the introduction, we follow standard methods to prove Proposition 101, decomposing the walk into a martingale and a corrector. As we are in one dimension, the corrector can be explicitly defined:

$$\begin{aligned} G_{i,i+1} &:= \frac{1}{\mathbb{E}\left(1/\bar{\omega}_{0,1}\right)\bar{\omega}_{i,i+1}} - 1 \\ \chi(x, \bar{\omega}) &:= \sum_{i=0}^{x-1} G_{i,i+1}, \end{aligned} \tag{5.3}$$

and set $M_i^n = \bar{X}_i^n + \chi\left(\bar{X}_i^n, \bar{\omega}\right)$. Let $\mathcal{F}_i = \sigma\left(\bar{X}_0^n, \ldots, \bar{X}_i^n\right)$.

**Proposition 102.** *For every $\omega \in \Omega$, the process $\left(M_i^n\right)_{i\geq 0}$ is a martingale with respect to $\left(\left(\mathcal{F}_i\right)_{i\geq 0}, P^{\bar{\omega}}\right)$.*

**Proof.** By direct computation

$$\begin{aligned} E^{\bar{\omega}}\left[M_{i+1}^n \middle| \mathcal{F}_i\right] &= \frac{\bar{\omega}_{i,i+1}}{\bar{\omega}_{i-1,i} + \bar{\omega}_{i,i+1}}\left[\bar{X}_i^n + 1 + \chi\left(\bar{X}_i^n + 1, \bar{\omega}\right)\right] \\ &\quad + \frac{\bar{\omega}_{i-1,i}}{\bar{\omega}_{i-1,i} + \bar{\omega}_{i,i+1}}\left[\bar{X}_i^n - 1 + \chi\left(\bar{X}_i^n - 1, \bar{\omega}\right)\right] \\ &= \bar{X}_i^n + \chi\left(\bar{X}_i^n, \bar{\omega}\right) + \frac{\bar{\omega}_{i,i+1}\left(1 + G_{i,i+1}\right)}{\bar{\omega}_{i-1,i} + \bar{\omega}_{i,i+1}} - \frac{\bar{\omega}_{i-1,i}\left(1 + G_{i-1,i}\right)}{\bar{\omega}_{i-1,i} + \bar{\omega}_{i,i+1}} \\ &= \bar{X}_i^n + \chi\left(\bar{X}_i^n, \bar{\omega}\right) = M_i^n. \end{aligned}$$

■

The strategy employed is to show that $M^n$, scaled and interpolated, converges to Brownian motion via a standard martingale convergence theorem. Showing that for almost every $\omega \in \Omega$ for all $t > 0$

$$\frac{1}{n}\chi\left(\bar{X}^n_{[tn^2]}, \bar{\omega}\right) \overset{n\to\infty}{\to} 0 \text{ in } P_{0,\bar{\omega}}\text{-probability}$$

will complete the proof of Proposition 101 and hence Theorem 98.

In fact the above statement would only imply convergence of the finite dimensional distributions, however since the reflected graph $(\mathbb{Z}, \mathbb{E}, (\bar{\omega}^n_e))$ has uniformly bounded weights there is uniform control of the heat kernel and hence the laws of $\bar{B}_n$ are tight under $P_{0,\bar{\omega}}$. We recall the uniform heat kernel bounds from [26]:

**Theorem 103.** *There exist constants $c_i = c_i(a,b) > 0$ such that for all $\omega \in \Omega, x, y \in \mathbb{Z}$ and $t \geq |x-y|$ :*

$$c_1 t^{-d/2} \exp\left(-\frac{c_2 |x-y|^2}{t}\right) \leq q^{\bar{\omega}}_t(x,y) \leq c_3 t^{-d/2} \exp\left(-\frac{c_4 |x-y|^2}{t}\right). \tag{5.4}$$

We will require control over how long the walk $X^n$ spends close to the boundary of $[-n,n]$. This corresponds to how long the walk $\bar{X}^n$ spends close to the points $\{n + 2kn : k \in \mathbb{Z}\}$. This control is obtained via Theorem 103.

**Proposition 104.** *Let $T > 0$. For $\varepsilon > 0$, define the random variable*

$$Y^n_\varepsilon := \left|\left\{0 \leq i \leq Tn^2 : \bar{X}^n_i \in ((k-\varepsilon)n, (k+\varepsilon)n) \text{ for some } k\right\}\right|,$$

*the time the walk spends within distance $\varepsilon n$ of the boundary. Then for all $\eta, \delta > 0$ there exists $\varepsilon > 0$ and $N_1(\eta, \delta, T, a, b)$ such that for all $n \geq N_1$*

$$P^\omega\left(Y^n_\varepsilon > \eta n^2\right) < \delta \text{ for all } \omega \in \Omega.$$

**Proof.** The proof is an application of Markov's inequality.

Note that by the uniform heat kernel upper bounds in (5.4), for fixed $k \in \mathbb{N}$

$$\begin{aligned} & E^\omega\left[\left|\left\{1 \leq i \leq Tn^2 : \bar{X}^n_i \in ((k-\varepsilon)n, (k+\varepsilon)n)\right\}\right|\right] \\ = \quad & \sum_{x\in((k-\varepsilon)n,(k+\varepsilon)n)} \sum_{i=1}^{Tn^2} \bar{P}^n(0,x) \\ \leq \quad & \sum_{x\in((k-\varepsilon)n,(k+\varepsilon)n)} \sum_{i=1}^{Tn^2} 2bc_3 i^{-1/2} \exp\left(-\frac{c_4|x|^2}{i}\right) \\ \leq \quad & \sum_{i=1}^{Tn^2} (2\varepsilon n \vee 1)\, 2bc_3 i^{-1/2} \exp\left(-\frac{c'|kn|^2}{i}\right), \end{aligned}$$

where $\varepsilon$ is assumed to be small, say $\varepsilon < 1$, to ensure that $c'$ can be chosen independently of $\varepsilon$. This can now be approximated via the corresponding integral and integration by parts to yield:

$$E^{\omega}\left[\left|\left\{1 \leq i \leq Tn^2 : \bar{X}_i^n \in ((k-\varepsilon)\, n, (k+\varepsilon)\, n)\right\}\right|\right] \leq (2\varepsilon n \vee 1)\, cn \exp\left(-c'k^2\right),$$

for $c, c'$ dependent on $T, a, b$.

Hence, for $n > 1/2\varepsilon$

$$E^{\omega}\left[\left|\left\{1 \leq i \leq Tn^2 : \bar{X}_i^n \in ((k-\varepsilon)\, n, (k+\varepsilon)\, n) \text{ for some } k\right\}\right|\right] \leq \varepsilon cn^2.$$

Applying Markov's inequality

$$P^{\omega}\left(Y_{\varepsilon}^n > \eta n^2\right) \leq \frac{\varepsilon cn^2}{\eta n^2}$$

and this can be made less than $\delta$ by taking $\varepsilon$ suitably small.

With such an $\varepsilon > 0$ chosen, define $N_1 := \inf\{n : \varepsilon n > 1\}$. ■

To prove that $M^n$ converges weakly to Brownian motion we appeal to the Lindeberg-Feller CLT (see [30]). Convergence to Brownian motion will follow from the following proposition. Introduce the measure weighted by the conductance at the origin:

$$\mathbb{P}_0 := \frac{(\omega_{-1,0} + \omega_{1,0})}{\mathbb{E}\left(\omega_{-1,0} + \omega_{1,0}\right)}\mathbb{P}\left(d\omega\right).$$

Recall that $\mathbb{P}_0$ is the invariant measure for the random walk from the point of view of the particle. Define $\Delta M_i^n = M_i^n - M_{i-1}^n$ for $i > 0$. For the random walk on the full graph $\mathcal{G}$, write $M_i$ for the associated martingale defined as in (5.3) and $\Delta M_i$ for martingale differences.

**Proposition 105.** *For $T > 0$ and almost all $\omega \in \Omega$,*

$$\frac{1}{Tn^2}\sum_{i=1}^{Tn^2}\left(\Delta M_i^n\right)^2 \to \mathbb{E}_0\left(\Delta M_1\right)^2 \text{ in } P^{\omega}\text{-probability as } n \to \infty.$$

**Proof.** We require to prove that for almost every $\omega \in \Omega$ and all $\varepsilon > 0, \delta > 0$ there exists $N(\omega, \delta, \varepsilon)$ such that

$$P^{\omega}\left[\left|\frac{1}{Tn^2}\sum_{i=1}^{Tn^2}\left(\Delta M_i^n\right)^2 - \mathbb{E}_0\left(\Delta M_1\right)^2\right| > \varepsilon\right] < \delta \tag{5.5}$$

for all $n > N$. We will prove this statement by breaking the walk up into the time it spends well away from the "boundary points" $\{(2k-1)\, n : k \in \mathbb{Z}\}$, and the time the walk spends close to these points. We will invoke Proposition 104 to give control over how long the walk spends near these points and use the ergodicity of the original graph $\mathcal{G}$ to show that we have good control over the martingale when the walk is away from the boundary.

By the ergodicity from the point of view of the particle for the random walk on $\mathcal{G}$, we know that for any starting point $x \in \mathbb{Z}$ and almost every $\omega \in \Omega$

$$\frac{1}{n}\sum_{i=1}^{n}(\Delta M_i)^2 \overset{n\to\infty}{\to} \mathbb{E}_0(\Delta M_1)^2, \; P^{\omega}\text{-almost surely.}$$

In particular, for every $\varepsilon, \delta > 0$ and every $x \in \mathbb{Z}$ there exists $N_1(x, \varepsilon, \delta, \omega) < \infty$ such that

$$P^{\omega,x}\left[\left|\frac{1}{n}\sum_{i=1}^{n}(\Delta M_i)^2 - \mathbb{E}_0(\Delta M_1)^2\right| > \varepsilon\right] < \delta \tag{5.6}$$

for all $n > N_1$ (where the superscript $x$ indicates starting vertex).

Let $c_6, c_7, c_8 > 0$ be constants to be chosen later. Define the sets

$$\begin{aligned}
\bar{L}(n,\gamma) \; &: = \left\{x \in [-n,n] : N_1\left(x, \frac{\varepsilon}{3}, \frac{\delta c_6}{4T}, \omega\right) < \gamma\right\}, \\
L(n,\gamma) \; &: = \left\{x \in \mathbb{Z} : f(x) \in \bar{L}(n,\gamma)\right\}.
\end{aligned}$$

By ergodicity of the full environment, $\frac{|\bar{L}(n,\gamma)|}{2n} \to \mathbb{P}\left(N_1\left(x, \frac{\varepsilon}{3}, \frac{\delta c_6}{4T}, \omega\right) < \gamma\right)$ almost surely as $n \to \infty$.

Define

$$\begin{aligned}
H_1 \; &: = 0, \\
H_i \; &: = \inf\left\{j \geq H_{i-1} + c_6 n^2 : \bar{X}_j^n \in L\left(n, c_6 n^2\right), \left|f\left(\bar{X}_j^n\right)\right| < c_8 n\right\}, \\
K \; &: = \sup\left\{i : H_i + c_6 n^2 \leq T n^2\right\}, J = H_K + c_6 n^2.
\end{aligned}$$

The idea is to break the walk into finite subwalks started at points in $L(n, c_6 n^2)$ where we have good control over the local ergodicity of the walk. We need the extra assumption that $\left|f\left(\bar{X}_j^n\right)\right| < c_8 n$, for $c_8$ close to one, so that if the walk is run for a short time (dependent on $c_8$) it is unlikely to hit the boundary points $\{(2k-1)n : k \in \mathbb{Z}\}$. If the walk fails to hit these boundary points then the martingale increments for $M^n$ are identical to those for $M$, which (5.6) enables us to control.

With this notation we write

$$\sum_{i=1}^{Tn^2}(\Delta M_i^n)^2 = \sum_{j=1}^{K}\left[\sum_{i=H_j}^{H_j+c_6n^2}(\Delta M_i^n)^2 + \sum_{i=H_j+c_6n^2+1}^{H_{j+1}}(\Delta M_i^n)^2\right] + \sum_{i=J}^{Tn^2}(\Delta M_i^n)^2.$$

Thus

$$
\begin{aligned}
&P^{\omega}\left[\left|\frac{1}{Tn^2}\sum_{i=1}^{Tn^2}(\Delta M_i^n)^2-\mathbb{E}_0\left(\Delta M_1\right)^2\right|>\varepsilon\right]\\
\leq\ &\sum_{j=1}^{T/c_6}P^{\omega}\left[\left|\frac{c_6n^2}{Tn^2}\frac{1}{c_6n^2}\sum_{i=H_j}^{H_j+c_6n^2}(\Delta M_i^n)^2-\mathbb{E}_0\left(\Delta M_1\right)^2\right|>\frac{\varepsilon c_6}{3T}\right]\\
&+P^{\omega}\left[\frac{1}{Tn^2}C_1\sum_{j=1}^{K}H_j-\left(H_{j-1}+c_6n^2\right)>\frac{\varepsilon}{3}\right]\\
&+P^{\omega}\left[\frac{1}{Tn^2}C_1\times\left(Tn^2-J\right)>\frac{\varepsilon}{3}\right], \qquad (5.7)
\end{aligned}
$$

where $C_1$ is

$$
\begin{aligned}
C_1:=\ &\max_{\omega}\left|(\Delta M_1)^2-\mathbb{E}_0\left(\Delta M_1^2\right)\right|\\
\leq\ &2\max_{\omega}\left(G_{0,1}+1\right)^2\\
\leq\ &2\frac{1}{a\mathbb{E}\left(1/\omega_{0,1}\right)}^2.
\end{aligned}
$$

The first term of (5.7) corresponds to the finite subwalks mentioned above, with the second and third terms bounding the time at which the walk is not in one of these good subwalks.

We begin by treating the second term of (5.7). We must control the time it takes to find a point in $L\left(n,c_6n^2\right)$. Note that for any $c_6>0$ and almost every $\omega\in\Omega$,

$$
\frac{\left|\bar{L}\left(n,c_6n^2\right)\right|}{2n}\to 1 \text{ as } n\to\infty.
$$

In particular, for any $c_7>0$, and almost every $\omega\in\Omega$, there exists $N_2\left(\omega,c_7,c_6,\delta,\varepsilon,T\right)<\infty$ such that for all $n\geq N_2$, the largest distance from any point $x\in[-n,n]$ to a point in $L\left(n,c_6n^2\right)$ is bounded by $c_7n$.

We must control

$$
P^{\omega}\left[\frac{1}{Tn^2}C_1\sum_{j=1}^{K}H_j-\left(H_{j-1}+c_6n^2\right)>\frac{\varepsilon}{3}\right].
$$

For $c_7>0$ we break this into two cases: either $\left|f\left(\bar{X}^n_{H_j+c_6n^2}\right)\right|\leq\left(c_8-c_7\right)n$ or $\left|f\left(\bar{X}^n_{H_j+c_6n^2}\right)\right|>\left(c_8-c_7\right)n$. We choose $c_8-c_7$ to guarantee that in the first case the walk is close to a point that satisfies both $x\in L\left(n,c_6n^2\right)$ and $\left|f\left(x\right)\right|<c_8n$. Figure 5.2 shows how the choice of $c_7$ and $c_8$ decide the boundary region of $[-n,n]$. For the first case note that the walk must hit a point of $L\left(n,c_6n^2\right)$ before exiting

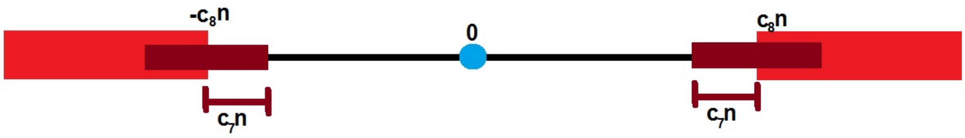


Figure 5.2: The boundary region

the ball of radius $c_7 n$ centred at $\bar{X}_{H^{j-1}+c_6 n^2}$. Hence, for $n \geq N_2$

$$\begin{aligned} & P^{\omega}\left(H^j - \left(H^{j-1} + c_6 n^2\right) > \frac{\varepsilon c_6}{6C_1} n^2 \left| \left| f\left(\bar{X}^n_{H_j + c_6 n^2}\right)\right| \leq (c_8 - c_7)\, n\right.\right) \\ \leq \;& P^{\omega, \bar{X}_{H^{j-1}+c_6 n^2}}\left(\tau\left(\bar{X}_{H^{j-1}+c_6 n^2}, c_7 n\right) > \frac{\varepsilon c_6}{6C_1} n^2\right). \end{aligned}$$

By the upper bound of Theorem 103, there exists $\zeta > 0$ such that for all $c_7 > 0$ the right hand side is bounded uniformly above by $T\delta/6c_6$. Hence, taking $0 < c_7 < \zeta$, there exists $N_2\left(\omega, c_7, c_8, c_6, \varepsilon, \delta\right)$ such that

$$P^{\omega}\left(H^j - \left(H^{j-1} + c_6 n^2\right) > \frac{\varepsilon c_6}{6C_1} n^2 \left| \left| f\left(\bar{X}^n_{H_j + c_6 n^2}\right)\right| \leq (c_8 - c_7)\, n\right.\right) < \frac{T\delta}{6c_6}$$

for all $n \geq N_2$.

For the second case, we use Proposition 104. Let $C_1$ be the maximum value of $\left|\Delta M_1^2 - \mathbb{E}_0\left(M_1^2\right)\right|$, then by Proposition 104 there exists $c_9 > 0$, $N_3\left(\omega, \varepsilon, \delta, c_9, T\right) < \infty$ such that

$$P^{\omega}\left(Y^n_{\alpha} > \frac{T\varepsilon}{6C_1} n^2\right) < \frac{\delta}{6},\; n \geq N_3, \alpha > c_9.$$

Set

$$c_8 := \frac{1 + c_9}{2},$$

noting that $c_8$ is independent of $c_6$ and $c_7$. Now, if the walk is in $\{x \in [-n, n] : |x| > (c_8 - c_7)\, n\}$ then it must hit a point $y \in L\left(n, c_6 n^2\right)$ such that $|f(y)| < c_8 n$ by the time the walk exits $\{x \in [-n, n] : |x| > (c_8 - 2c_7)\, n\}$. Hence

$$\begin{aligned} & P^{\omega}\left(\sum_{j : \left|f\left(\bar{X}^n_{H_j + c_6 n^2}\right)\right| > (c_8 - 2c_7) n} H^j - \left(H^{j-1} + c_6 n^2\right) > \frac{T\varepsilon}{6C_1} n^2\right) \\ \leq \;& P^{\omega}\left(Y^n_{c_8 - 2c_7} > \frac{T\varepsilon}{6C_1} n^2\right), \end{aligned}$$

and if $c_7 > 0$ is chosen small such that $1 - (c_8 - 2c_7) < c_9$ then

$$P^{\omega}\left(\sum_{j : \left|f\left(\bar{X}^n_{H_j + c_6 n^2}\right)\right| > (c_8 - 2c_7) n} H^j - \left(H^{j-1} + c_6 n^2\right) > \frac{T\varepsilon}{6C_1} n^2\right) < \frac{\delta}{6}, n \geq N_3.$$

Combining these results, for all $c_6 > 0$ there exists $\beta > 0$ such that if $0 < c_7 < \beta \wedge (-1 + c_9 + c_8)/2 \wedge \zeta$ then there exists $N_2(\omega, c_6, c_8, c_7, \delta, \varepsilon)$ such that

$$\begin{aligned}
& P^{\omega}\left[\frac{1}{Tn^2}C_1\left[\sum_{j=1}^{K} H_j - \left(H_{j-1} + c_6 n^2\right)\right] > \frac{\varepsilon}{3}\right] \\
\leq \; & \sum_{j=1}^{K} P^{\omega}\left(H^j - \left(H^{j-1} + c_6 n^2\right) > \frac{\varepsilon c_6}{6C_1} n^2 \left|\left|f\left(\bar{X}^n_{H_j + c_6 n^2}\right)\right| \leq (c_8 - 2c_7)\, n\right.\right) \\
& + P^{\omega}\left(Y^n (c_8 - 2c_7) > \frac{\varepsilon T}{6C_1} n^2\right) \\
\leq \; & \delta/6 + \delta/6 = \delta/3, \;\; n \geq N_2 \vee N_3. \qquad (5.8)
\end{aligned}$$

We have picked $c_8$ independently of $c_6$. We will choose $c_6$ dependent on $c_8$ and then $c_7$ dependent on $c_8$ and $c_6$.

Similarly, as the remainder term $Tn^2 - J$ can be bounded by $H_{K+1} - H_K$, there exists $N_4 = N_4(\omega, c_6, c_8, c_7, \varepsilon, \delta)$ such that

$$P^{\omega}\left[\frac{1}{Tn^2}C_1 \times \left(Tn^2 - J\right) > \frac{\varepsilon}{3}\right] < \frac{\delta}{3}, \;\; n \geq N_4. \qquad (5.9)$$

We now move to the first term of (5.7). Note that the walks $(X_i)_{i\in[0,c_6 n^2]}$ and $\left(f\left(\bar{X}^n_i\right)\right)_{i\in[0,c_6 n^2]}$ started at the same point in $[-n, n]$, have the same distribution until they hit $\{\pm n\}$. Note further that if $f(x) = f(y)$ then $E^x\left(\Delta M^n_1\right)^2 = E^y\left(\Delta M^n_1\right)^2$. Thus conditioning on the walk hitting or failing to hit $\{\pm n\}$:

$$\begin{aligned}
& P^{\omega, X_{H_i}}\left[\left|\frac{1}{c_6 n^2}\sum_{i=0}^{c_6 n^2}\left(\Delta M^n_i\right)^2 - \mathbb{E}_0\left(\Delta M_1\right)^2\right| > \frac{\varepsilon}{3}\right] \\
\leq \; & P^{\omega, X_{H_i}}\left[\max_{i\leq c_6 n^2}\left|\bar{X}_i\right| \geq n\right] + P^{\omega, X_{H_i}}\left[\left|\frac{1}{c_6 n^2}\sum_{i=0}^{c_6 n^2}\left(\Delta M_i\right)^2 - \mathbb{E}_0\left(\Delta M_1\right)^2\right| > \frac{\varepsilon}{3}\right] \\
< \; & P^{\omega, X_{H_i}}\left[\max_{i\leq c_6 n^2}\left|\bar{X}_i\right| \geq n\right] + \frac{\delta c_6}{4T}. \qquad (5.10)
\end{aligned}$$

By Theorem 103, for fixed $c_8$, the first term on the right hand side can be made small by taking $c_6$ small since $|X_{H_i}| < c_8 n$.

By the definition of $H_i$ and (5.10), we can control the terms in the sum:

$$\begin{aligned}
& \sum_{j=1}^{T/c_6} P^{\omega}\left[\left|\frac{c_6 n^2}{Tn^2}\frac{1}{c_6 n^2}\sum_{i=H_j}^{H_j + c_6 n^2}\left(\Delta M^n_i\right)^2 - \mathbb{E}\left(\Delta M^2_1\right)\right| > \frac{c_6 \varepsilon}{3T}\right] \\
\leq \; & \frac{T}{c_6}\left(\sup_{|x|<c_8 n} P^{\omega,x}\left[\max_{i\leq c_6 n^2}|X_i| \geq n\right] + \frac{\delta c_6}{4T}\right).
\end{aligned}$$

Now by Theorem 103,

$$\lim_{c_6\downarrow 0}\frac{T}{c_6}\sup_{|x|<c_8 n} P^{\omega, X_{H_i}}\left[\max_{i\leq c_6 n^2}|X_i| \geq n\right] = 0$$

Hence by choosing small $c_6$ (dependent on $c_8$)

$$\sum_{j=1}^{T/c_6} P^{\omega}\left[\left|\frac{c_6 n^2}{Tn^2}\frac{1}{c_6 n^2}\sum_{i=H_j}^{H_j+c_6n^2}\left(\Delta M_i^n\right)^2-\mathbb{E}\left(\Delta M_1\right)^2\right|>\frac{\varepsilon c_6}{3T}\right]\leq\frac{\delta}{3}. \tag{5.11}$$

Combining (5.11), (5.8) and (5.9) we are done, taking $N(\omega)=N_2\vee N_3\vee N_4$ in (5.5). ■

**Proof of Proposition 101.** Proposition 105 and the Lindeberg-Feller functional central limit theorem imply that for almost every $\omega\in\Omega$,

$$\frac{1}{n}\left(M^n_{\lfloor n^2t\rfloor}+\left(n^2t-\lfloor n^2t\rfloor\right)\left(M^n_{\lfloor n^2t\rfloor+1}-M^n_{\lfloor n^2t\rfloor}\right)\right)$$

converges in law to a Brownian motion with diffusion constant $\sigma^2$, for

$$\sigma^2=2\mathbb{E}_0\left[\frac{\omega_{0,1}}{\omega_{0,1}+\omega_{-1,0}}\left(1+G_{0,1}\right)^2\right],$$

where $G_{0,1}$ is defined in (5.3). Note that $\sigma^2$ is identical to the variance for the scaling limit of the non-reflecting random walk.

Now, as noted in [53], the upper bounds of Theorem 103 imply that the laws of $\bar{B}^n(\cdot)$ are tight under $P_0^{\bar{\omega}}$. The proposition will thus follow once we show that for almost every $\omega\in\Omega$ and $t>0$

$$\frac{1}{n}\chi\left(X^n_{n^2t},\bar{\omega}\right)\to 0 \text{ in } P_0^{\bar{\omega}}\text{-probability.}$$

Again appealing to the uniform upper bounds on the heat kernel, it suffices to show that for any $M>0$, for almost every $\omega\in\Omega$

$$\lim_n n^{-1}\sum_{|x|<Mn}\frac{|\chi(x,\bar{\omega})|}{n}=0. \tag{5.12}$$

This will be proved using the fact that (5.12) holds for the non-reflecting corrector $\chi(x,\omega)$ and the periodicity of $\chi(x,\bar{\omega})$.

As

$$\chi(x,\bar{\omega}):=\sum_{i=0}^{x}G^{\bar{\omega}}(i,i+1),$$

by the periodicity of the environment $\bar{\omega}$ we have that for $x=(4k+l)n+r$, $0\leq r<n$, $k\in\mathbb{N}_0$ and $0\leq l<3$:

$$\chi(x,\bar{\omega})=\begin{cases}2k\left(\chi(n,\bar{\omega})+\chi(-n,\bar{\omega})\right)+\chi(r,\bar{\omega}) & \text{if } l=0\\ (2k+2)\chi(n,\bar{\omega})-\chi(n-r,\bar{\omega})+2k\chi(-n,\bar{\omega}) & \text{if } l=1\\ (2k+2)\chi(n,\bar{\omega})+\chi(-r,\bar{\omega})+2k\chi(-n,\bar{\omega}) & \text{if } l=2\\ (2k+2)\left(\chi(n,\bar{\omega})+\chi(-n,\bar{\omega})\right)-\chi(-n+r,\bar{\omega}) & \text{if } l=3\end{cases},$$

and for $x = -(4k+l)\,n - r, 0 \le r < n, k \in \mathbb{N}_0$ and $0 \le l \le 3$ :

$$\chi(x,\bar{\omega}) = \begin{cases} 2k\left(\chi(n,\bar{\omega}) + \chi(-n,\bar{\omega})\right) + \chi(-r,\bar{\omega}) & \text{if } l = 0 \\ (2k+2)\,\chi(-n,\bar{\omega}) - \chi(-n+r,\bar{\omega}) + 2k\chi(n,\bar{\omega}) & \text{if } l = 1 \\ (2k+2)\,\chi(-n,\bar{\omega}) + \chi(r,\bar{\omega}) + 2k\chi(n,\bar{\omega}) & \text{if } l = 2 \\ (2k+2)\left(\chi(n,\bar{\omega}) + \chi(-n,\bar{\omega})\right) - \chi(n-r,\bar{\omega}) & \text{if } l = 3 \end{cases}.$$

By the triangle inequality

$$\begin{aligned} & \lim_n n^{-1} \sum_{|x|<Mn} \frac{|\chi(x,\bar{\omega})|}{n} \\ \le\ & \lim_n n^{-1} \sum_{k=0}^{M} \sum_{l=0}^{3} \sum_{r=0}^{n-1} \frac{|\chi((4k+l)\,n + r,\bar{\omega})| + |\chi(-(4k+l)\,n - r,\bar{\omega})|}{n} \\ \le\ & \lim_n n^{-2} \sum_{k=0}^{M} \sum_{r=0}^{n-1} [4(k+1)\left(|\chi(n,\bar{\omega})| + |\chi(-n,\bar{\omega})|\right) \\ & +2\left(|\chi(r,\bar{\omega})| + |\chi(-r,\bar{\omega})| + |\chi(n-r,\bar{\omega})| + |\chi(r-n,\bar{\omega})|\right)] \\ \le\ & 2M(M+2) \lim_n \frac{|\chi(n,\bar{\omega})| + |\chi(-n,\bar{\omega})|}{n} \\ & +4(M+1) \lim_n \sum_{r=0}^{n} \frac{|\chi(r,\bar{\omega})| + |\chi(-r,\bar{\omega})|}{n^2}. \end{aligned}$$

For almost every $\omega \in \Omega$ the first limit is zero since $\chi(n,\bar{\omega})$ is the sum of iid mean zero random variables. For the second limit note that we can replace $\chi(\pm r,\bar{\omega})$ by $\chi(\pm r,\omega)$ as $|r| \le n$. Hence the second limit is zero for almost every $\omega \in \Omega$ by line (1.22) of [53], where the corrector for the walk on the unreflected graph is controlled. ■

**Proof of Theorem 98.** This follows from Propositions 100 and 101. ■

Note that, as we are in one dimension, sublinearity of $\chi$ requires only ergodicity and that $\mathbb{E}\left(\frac{1}{\omega_{xy}}\right) < \infty$. Hence the proof extends to all such ergodic environment where Proposition 104 holds.

Theorem 98 allows us to prove the convergence of mixing times.

**Proof of Theorem 99.** The result will follow directly from Theorem 1.4 of [23] once we prove that the conditions of that theorem hold. As reflecting Brownian motion satisfies the regularity conditions necessary, Theorem 1.4 will apply once we show that the triple

$$\left(\left(V^n, n^{-1}\left|\cdot\right|_1\right), \pi_n, \left(q^n_{n^2 t}(x,y)\right)_{x,y\in V^n, t\in I}\right)$$

converges in a spectral Gromov-Hausdorff sense to the triple: $[-1,1]$ with the Euclidean metric, one dimensional Lebesgue measure on $[-1,1]$ and the transition density of Brownian motion on $[-1,1]$ reflected at the boundary. Here, $\pi_n$ is the normalized stationary distribution for the reflecting random walk on $[-n,n]$.

We will not discuss what this convergence means here, but instead note that this claim follows from Proposition 2.4 of the same paper. The rest of this proof consists of showing that the conditions of Proposition 2.4 hold.

It is straight forward to show that $(V^n, n^{-1}|\cdot|_1)$ converges to $([-1,1], |\cdot|_1)$ with respect to Hausdorff distance. By ergodicity $\pi_n$ converges to Lebesgue measure on $[-1,1]$ with respect to Prohorov distance (both of these convergences assume scaling of the space $[-n,n]$ by $n^{-1}$).

As we are working in one dimension with conductances bounded below we have the following upper bound on the resistance metric:

$$R_{\mathcal{G}^n}(x,y) \leq c_1 |x-y|,$$

for some $c_1 = c_1(a) > 0$. By Lemma 2.5 of [23], this in turn proves the tightness condition required.

The final condition we must prove is the following: there exists a dense subset $F^*$ of $[-1,1]$ such that for any compact interval $I \subset (0,\infty)$, $x \in F^*$, $y \in [-1,1]$ and $r > 0$:

$$\lim_{n\to\infty} P^{\mathcal{G}_n}_{g_n(x)}\left(\frac{1}{n}X^n_{\lfloor n^2 t\rfloor} \in B_y[r]\right) = P_x(W_t \in B_y[r]) \tag{5.13}$$

uniformly in $t \in I$, where $g_n(x)$ is the vertex in $[-n,n]$ that is closest to $nx$ and $W_t$ is reflecting Brownian motion on $[-1,1]$ with diffusion constant as in Theorem 98.

As noted after the statement of Theorems 98 and 99, Theorem 98 holds irrespective of the starting point $x \in [-1,1]$. In fact, the only modifications to the arguments presented are a slight change in the transition density used to prove Proposition 104 and $H_1$ becoming a random variable in the proof of Proposition 105.

Equation (5.13) will follow from the functional central limit theorem, the control of exit times and the regularity of reflecting Brownian motion.

Fix $\varepsilon > 0, x \in \mathbb{Q}\cap[-1,1]$, $r > 0$ and compact $I \subset (0,\infty)$. Choose $\gamma > 0$ and $N_1$ such that

$$\begin{aligned}
\sup_{s\in I}|P_x(W_s\in B_y[r]) - P_x(W_s \in B_y[r+\gamma])| &< \frac{\varepsilon}{6},\\
\sup_{s\in I}\left|P^{\mathcal{G}_n}_{g_n(x)}\left(\frac{1}{n}X^n_{\lfloor n^2 s\rfloor}\in B_y[r+\gamma]\right) - P^{\mathcal{G}_n}_{g_n(x)}\left(\frac{1}{n}X^n_{\lfloor n^2 s\rfloor}\in B_y[r-\gamma]\right)\right| &< \frac{\varepsilon}{13}, n \geq N_1.
\end{aligned}$$

This is possible since $I$ is compact and $0 \notin I$. Justification for the second line also requires the heat kernel bounds of Theorem 103 which show that for sufficiently large $n$

$$P^{\mathcal{G}_n}_{g_n(x)}\left(\frac{1}{n}X^n_{\lfloor n^2 s\rfloor} \in [y+r-\gamma, y+r+\gamma]\cup[y-r-\gamma, y-r+\gamma]\right) \leq c\gamma$$

holds and hence a suitable choice of $\gamma$ is possible.

Now take $\delta > 0$ such that for all $n > 0$

$$\sup_{z\in[-n,n]} P_z^{\mathcal{G}_n}\left(\tau_{B_x[\gamma n]} < \delta n^2\right) < \frac{\varepsilon}{13}.$$

and

$$\sup_{s\in I}\left|P_x\left(W_s \in B_y[r]\right) - P_x\left(W_{s+\delta} \in B_y[r]\right)\right| < \frac{\varepsilon}{3} \tag{5.14}$$

The first line is possible as the uniform bounds on conductances imply uniform bounds on exit times (this can be seen, for example, via the arguments in the proof of Proposition 18). The second line is possible due to $I$ being compact and $0 \notin I$.

Choose a finite set $\{t_1, \ldots, t_i\} \subseteq I$ such that for all $j \in \{1, \ldots, i\}$ we have $\sup_l |t_j - t_l| < \delta$. Take $t \in I$ and let $i$ be such that $|t - t_i|$ is minimized, then:

$$\begin{aligned}
&\left|P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t\rfloor}^n \in B_y[r]\right) - P_x\left(W_t \in B_y[r]\right)\right| \\
\leq\ &\left|P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t\rfloor}^n \in B_y[r]\right) - P_{g_n(x)}^{G_n}\left(X_{\lfloor n^2 t_i\rfloor}^n \in B_y[r]\right)\right| \\
&+\left|P_{g_n(x)}^{\mathcal{G}_n}\left(X_{\lfloor n^2 t_i\rfloor}^n \in B_y[r]\right) - P_x\left(W_{t_i} \in B_y[r]\right)\right| \\
&+\left|P_x\left(W_t \in B_y[r]\right) - P_x\left(W_{t_i} \in B_y[r]\right)\right|.
\end{aligned}$$

The third term is less than $\frac{\varepsilon}{3}$ by choice of $\delta$ in (5.14). Due to the functional central limit theorem we have convergence of finite dimensional distributions and hence there exists $N_2$ such that the second term is less than $\frac{\varepsilon}{3}$ for all $n \geq N_2$.

It remains to control the first term. Suppose that $t > t_i$ then

$$\begin{aligned}
&P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t\rfloor}^n \in B_y[r]\right) \\
\geq\ &P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t_i\rfloor}^n \in B_y[r-\gamma]\right)\sup_{z\in B_y[r-\gamma]} P_{zn}^{\mathcal{G}_n}\left(\tau_{B_{zn}[\gamma n]} \geq \delta n^2\right) \\
\geq\ &P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t_i\rfloor}^n \in B_y[r-\gamma]\right)\left(1 - \frac{\varepsilon}{13}\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t\rfloor}^n \in B_y[r]\right) &\leq P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t_i\rfloor}^n \in B_y[r+\gamma]\right)\left(1 - \frac{\varepsilon}{13}\right)^{-1} \\
&\leq \left(P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t_i\rfloor}^n \in B_y[r-\gamma]\right) + \frac{\varepsilon}{13}\right)\left(1 - \frac{\varepsilon}{13}\right)^{-1}
\end{aligned}$$

In particular

$$\left|P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t\rfloor}^n \in B_y[r]\right) - P_{g_n(x)}^{\mathcal{G}_n}\left(\frac{1}{n}X_{\lfloor n^2 t_i\rfloor}^n \in B_y[r-\gamma]\right)\right| \leq \frac{\varepsilon}{6}. \tag{5.15}$$

Similar arguments show that (5.15) also holds for $t < t_i$.

To complete the proof, note that by the functional central limit theorem there exists $N_3$ such that

$$\left| P^{\mathcal{G}_n}_{g_n(x)} \left( \frac{1}{n} X^n_{\lfloor n^2 t_i \rfloor} \in B_y [r-\gamma] \right) - P_x \left( W_{t_i} \in B_y [r-\gamma] \right) \right| \leq \frac{\varepsilon}{6}, n \geq N_3.$$

Hence:

$$\begin{aligned} & \left| P^{\mathcal{G}_n}_{g_n(x)} \left( \frac{1}{n} X^n_{\lfloor n^2 t \rfloor} \in B_y [r] \right) - P^{\mathcal{G}_n}_{g_n(x)} \left( X^n_{\lfloor n^2 t_i \rfloor} \in B_y [r] \right) \right| \\ \leq & \left| P^{\mathcal{G}_n}_{g_n(x)} \left( \frac{1}{n} X^n_{\lfloor n^2 t \rfloor} \in B_y [r] \right) - P^{\mathcal{G}_n}_{g_n(x)} \left( \frac{1}{n} X^n_{\lfloor n^2 t_i \rfloor} \in B_y [r-\gamma] \right) \right| \\ & + \left| P^{\mathcal{G}_n}_{g_n(x)} \left( \frac{1}{n} X^n_{\lfloor n^2 t_i \rfloor} \in B_y [r-\gamma] \right) - P_x \left( W_{t_i} \in B_y [r-\gamma] \right) \right| \\ & + \left| P_x \left( W_{t_i} \in B_y [r-\gamma] \right) - P_x \left( W_{t_i} \in B_y [r] \right) \right| \\ \leq & \frac{\varepsilon}{6} + \frac{\varepsilon}{6} + \frac{\varepsilon}{6} = \frac{\varepsilon}{3}. \end{aligned}$$

Hence for all $t \in I$ and $n \geq N_1 \vee N_2 \vee N_3$

$$\left| P^{\mathcal{G}_n}_{g_n(x)} \left( \frac{1}{n} X^n_{\lfloor n^2 t \rfloor} \in B_y [r] \right) - P_x \left( W_t \in B_y [r] \right) \right| < \varepsilon.$$

Hence (5.13) converges uniformly over $t \in I$. Thus Proposition 2.4 and Theorem 1.4 of [23] apply. ■

In fact the proof of Theorem 99 can be shortened and simplified considerably. The following method was explained to me by my examiner, David Croydon.

**Alternative Proof of Theorem 99.** Let $B^n_t = \frac{1}{n} X^n_{\lfloor n^2 t \rfloor}$. As $C(I, \mathbb{R})$ is separable for any compact $I$, one can apply Skorokhod's representation theorem to define $\tilde{B}^n$ and $\tilde{W}$ on the same probability space with the same laws as $B^n$ started from $g_n(x)$ and $W$ started from $x$, respectively, and $\tilde{B}^n$ converging almost surely to $\tilde{W}$.

Now, for any ball $A$ and $\varepsilon > 0$ let $A_{\pm\varepsilon}$ be the $\varepsilon$ expansion/contraction of $A$. Then

$$\begin{aligned} \sup_{t \in I} \left| P^{\mathcal{G}_n}_{g_n(x)} \left( B^n_t \in A \right) - P_x \left( W_t \in A \right) \right| & \leq \sup_{t \in I} \left( \tilde{B}^n_t \in A, \tilde{W}_t \notin A \right) + \sup_{t \in I} \left( \tilde{B}^n_t \notin A, \tilde{W}_t \in A \right) \\ & \leq 2P \left( \sup_{t \in I} \left| \tilde{B}^n_t - \tilde{W}_t \right| > \varepsilon \right) + \sup_{t \in I} P \left( \tilde{W}_t \in A_{\varepsilon} \backslash A_{-\varepsilon} \right). \end{aligned}$$

Letting $n \to \infty$, the first term tends to 0 by the weak convergence established in Theorem 98. The second term can be made arbitrarily small by taking $\varepsilon$ small. ■

## 5.2. Higher dimensions

The above proof does not transfer to higher dimensions as there is no consistent martingale/corrector decomposition for the random walk on the reflected graph for $d > 1$. In particular there is no way to define a periodic corrector due to the higher dimensionality of the boundary and this leads the above scheme to fail.

There are various other potential avenues to explore. For example [19] considers approximations to reflecting Brownian motion on bounded domains through discrete time simple random walks. There, due to the uniformity of the heat kernel and symmetry of the simple random walk, it is possible to show uniform convergence for the excursions of the two processes and convergence of the random walk to reflecting Brownian motion follows. Perhaps we can follow a similar method?

This led us to consider the non-reflecting random walk started away from the origin at a fixed point $x \in \mathbb{Z}^d$. More precisely, take $\mathcal{G} = \mathcal{G}(\omega)$ to be a realization of the random conductance model in $d$ dimensions with $\omega \in (0, 1]$ to ensure that a functional CLT holds for the random walk started at the origin. Then for $n \in \mathbb{N}$ let $\left(X_i^{2^n x}\right)_{i \geq 0}$ be the non-reflecting random walk on $\mathbb{Z}^d$ with transition probabilities dependent on $\mathcal{G}$ as defined previously and starting point

$$X_0^{2^n x} = 2^n x.$$

It is natural to believe that the rescaled process

$$2^{-n} \left(X_{[2^{2n} t]}^{2^n x}\right)_{t \in [0,T]}$$

weakly converges under the quenched law to Brownian motion starting at $x$.

The proof of this claim is not as straight forward as we initially thought. One can use the corrector function to give a martingale/corrector decomposition:

$$X_i^{2^n x} = M_i^{2^n x} + \chi\left(X_i^{2^n x}, \omega\right), i \in \left[0, 2^{2n} T\right].$$

Standard facts about the corrector show that it is still sublinear in this case. However, it is not clear how to prove that the martingale converges to Brownian motion. In the $x = 0$ case the ergodicity of the environment from the point of view of the particle is used to prove that the conditions of the Lindeberg-Feller Theorem hold. We do not have access to this machinery here as the starting point of the walk changes with $n$.

We have not found a route around this complication and so the results we present are much weaker.

For $C \subseteq \mathbb{R}^d$, define $C^n := \left\{nx \in \mathbb{Z}^d : x \in C\right\}$ to be the lattice points within $C$ at stage $n$. The uniform distribution on $C^n$, rescaled by $n^{-1}$, converges to the uniform distribution on $C$ as $n \to \infty$. We show that for suitable $C$ the random walk started uniformly over $C^n$, rescaled and linearly interpolated converges to Brownian motion started uniformly on $C$.

Write $\left(X_i^{U(C^n)}\right)_{i \in \mathbb{N}}$ for the random walk started uniformly on $C^n \subseteq \mathbb{Z}^d$.

**Proposition 106.** *Suppose $C \subseteq \mathbb{R}^d$ is of the form*

$$C = \bigcup_{i=1}^{\infty} B_{x_i}[r_i], r_i > 0, \sum_i r_i < \infty, x_i \in \mathbb{R}^d,$$

*then*

$$B_n^C := \frac{1}{n}\left(X_{\lfloor n^2 t\rfloor}^{U(C^n)} + \left(n^2 t - \lfloor n^2 t\rfloor\right)\left(X_{\lfloor n^2 t\rfloor+1}^{U(C^n)} - X_{\lfloor n^2 t\rfloor}^{U(C^n)}\right)\right)$$

*converges weakly to Brownian motion started uniformly on $C$.*

We first show that the statement holds for the particular choice $C = [-1,1]^d$ and then show how this proof can be adapted to prove Proposition 106.

**Proposition 107.** *Let $C = [-1,1]^d$, then the rescaled process $B_n^C$ converges weakly to Brownian motion started uniformly on $[-1,1]^d$.*

**Proof.** As $\omega \in (0,1]$, [13] gives that for all bounded continuous $g : C[0,T] \to \mathbb{R}$ and $\varepsilon > 0$ there exists $N = N(\omega, \varepsilon)$ such that

$$\left|E^{\omega}\left(g\left(\frac{1}{n}X_{[n^2 t]}^{0}\right)\right) - \mathbb{E}^0 g(\mathcal{W})\right| < \varepsilon, \text{ for } n \geq N(\omega,\varepsilon), \tag{5.16}$$

where $\mathbb{E}^x g(\mathcal{W})$ is expectation with respect to driftless Brownian motion started at $x \in \mathbb{R}^d$ with diffusion constant $\sigma^2 > 0$.

Fix a bounded, continuous function $f : C[0,T] \to \mathbb{R}$.

By (5.16), for all $\omega \in \Omega$, $\varepsilon > 0$ and $x \in \mathbb{R}^d$ there exists minimal $N_1(x,\omega,\varepsilon) < \infty$ such that

$$\left|E^{\omega}\left[f\left(\frac{1}{n}X_{n^2 t}^{0} + x\right)\right] - \mathbb{E}^x\left[f(\mathcal{W})\right]\right| < \varepsilon$$

for all $n > N_1(x,\omega,\varepsilon)$. Note that $N_1$ is dependent on $x$ as the entire walk is shifted by $x$.

We will end up considering scaled walks on an environment $\omega$, started at some point $x \in \mathbb{Z}^d$. Equivalently one can consider the walk started at 0 on the shifted environment $\tau_x\omega$ and then shift the whole path

$$E^{\tau_x\omega}\left[f\left(\frac{1}{n}X_{n^2 t}^{0} + \frac{x}{n}\right)\right] = E^{\omega}\left[f\left(\frac{1}{n}X_{n^2 t}^{x}\right)\right].$$

It therefore becomes important to control $N_1$ over $x \in [-1,1]^d$.

With this in mind set

$$h(x) := N_1(x,\omega,\varepsilon),$$

then as $f$ is continuous, so too is $h$ with $h(x) < \infty$ for all $x \in \mathbb{R}^d$. In particular

$$N_2(\omega,\varepsilon) := \max_{x\in[-1,1]^d} h(x)$$

exists and is finite.

Set

$$N_3(x,\omega,\varepsilon) := N_2(\tau_x\omega,\varepsilon),$$

then for all $\omega \in \Omega, \varepsilon > 0$, $x \in \mathbb{Z}^d$ and $n \geq N_3(x,\omega,\varepsilon) \vee |x|$

$$\left| E^\omega \left[ f\left(\frac{1}{n} X^x_{n^2t}\right)\right] - \mathbb{E}^{x/n}\left[f(\mathcal{W})\right]\right| < \varepsilon.$$

Further, by the continuity of $f$ there exists $\delta = \delta(\varepsilon) > 0$ such that

$$\left| E^y\left[f(\mathcal{W})\right] - \mathbb{E}^z\left[f(\mathcal{W})\right]\right| < \varepsilon$$

for all $y, z \in [-1,1]^d$ with $|z-y| < \delta$. Under this assumption define

$$N_4(x,\omega,\varepsilon) = N_4 := N_3(x,\omega,\varepsilon) \vee (2\delta)^{-1/d} \vee |x|.$$

Then

$$\left| E^\omega \left[ f\left(\frac{1}{n} X^x_{n^2t}\right)\right] - \mathbb{E}^{U\left(B_{x/n}\left[n^{-d}/2\right]\right)}\left[f(\mathcal{W})\right]\right| < 2\varepsilon \tag{5.17}$$

for all $n \geq N_4$, where $U(B_x[r])$ is the uniform distribution of starting point over $B_x[r]$.

Now, since for any $n$, the function $1_{\{N_3(\omega,x,\varepsilon)>n\}}$ is measurable and integrable, the multidimensional ergodic theorem implies that for almost every $\omega \in \Omega$ :

$$\begin{aligned}\tfrac{1}{(2R)^d}\left|\left\{x \in [-R,R]^d : N_3(x,\omega,\varepsilon) > n\right\}\right| &= \frac{1}{(2R)^d}\sum_{x\in B_0[R]} 1_{\{N_3(x,\omega,\varepsilon)>n\}} \\ &\overset{R\to\infty}{\longrightarrow} \mathbb{P}\left[N_3(0,\omega,\varepsilon) > n\right].\end{aligned} \tag{5.18}$$

Note that the right hand side tends to zero as $n \to \infty$.

Hence, splitting $[-n,n]^d \cap \mathbb{Z}^d$ into subsets with $N_4(x,\omega,\varepsilon)$ large and subsets with $N_4(x,\omega,\varepsilon)$ small (equivalently $N_3$ large and small) and applying (5.17):

$$\begin{aligned}
&\left| E^{U[-n,n]^d}_\omega \left[ f\left(\frac{1}{n} X^\cdot_{n^2t}\right)\right] - \mathbb{E}^{U[-1,1]^d} f(\mathcal{W})\right| \\
\leq\ & \frac{1}{(2n)^d}\sum_{x\in[-n,n]^d}\left| E^x_\omega \left[ f\left(\frac{1}{n} X^x_{n^2t}\right)\right] - \mathbb{E}^{U\left(B_{x/n}\left[n^{-d}/2\right]\right)} f(\mathcal{W})\right| \\
\leq\ & \frac{2\varepsilon}{(2n)^d}\left|\left\{x \in [-n,n]^d : N_4(\omega,x,\varepsilon) < n^2\right\}\right| \\
&+\frac{1}{(2n)^d}\left|\left\{x \in [-n,n]^d : N_4(\omega,x,\varepsilon) > n^2\right\}\right| \|f\|_\infty \\
\leq\ & 2\varepsilon + \frac{1}{(2n)^d}\left|\left\{x \in [-n,n]^d : N_3(\omega,x,\varepsilon) > n^2\right\}\right| \|f\|_\infty \\
\to\ & 2\varepsilon \text{ as } n\to\infty,
\end{aligned}$$

where the penultimate line holds for $n \geq (2\delta)^{-1/d}$.

As this holds for all continuous and bounded $f$ and all $\varepsilon > 0$, the random walk started uniformly on $[-1,1]^d$ converges weakly to Brownian motion started uniformly on $[-1,1]^d$. ■

**Proof of Proposition 106.** By the spatial ergodic theorem, provided $nC^n \subseteq (n+1)\, C^{n+1}$ for all $n$, we can replace (5.18) with

$$\frac{1}{|C^n|}\left|\{x \in C^n : N_3(\omega, nx, \varepsilon) > a\}\right| \overset{n\to\infty}{\to} \mathbb{P}\left[N_3(\omega, 0, \varepsilon) > a\right].$$

The above arguments then yield

$$E_\omega^{U(C^n)}\left[f\left(\frac{1}{n}X_{n^2t}^{\cdot}\right)\right] \overset{n\to\infty}{\to} \mathbb{E}^{U(C)} f(\mathcal{W}).$$

We can extend to the more general class of subsets by noting that if $C \subseteq D \subseteq \mathbb{R}^d$ both satisfy the above condition then

$$\begin{aligned}
& E_\omega^{U(D-C)^n}\left[f\left(\frac{1}{n}X_{n^2t}^{\cdot}\right)\right] \\
= & \frac{|D^n|}{|(D-C)^n|}E^{U(D^n)}\left[f\left(\frac{1}{n}X_{n^2t}^{\cdot}\right)\right] - \frac{|C^n|}{|(D-C)^n|}E^{U(C^n)}\left[f\left(\frac{1}{n}X_{n^2t}^{\cdot}\right)\right] \\
& \overset{n\to\infty}{\to} \left(\frac{\mathcal{L}(D)}{\mathcal{L}(D-C)}\mathbb{E}^{U(D)} - \frac{\mathcal{L}(C)}{\mathcal{L}(D-C)}\mathbb{E}^{U(C)}\right) f(\mathcal{W}) \\
= & \mathbb{E}^{U(D-C)} f(\mathcal{W}).
\end{aligned} \tag{5.19}$$

We wish to prove that we can take $C = B_x[R]$ for any $x \in \mathbb{R}^d$ and $R > 0$. If $0 \in B_x[R]$ then we are done. Suppose that this is not the case. Write $x = (x_1, \ldots x_d)$ and let

$$\begin{aligned}
A &:= [(x_1 - R)\wedge 0, (x_1+R)\vee 0] \times \ldots \times [(x_d - R)\wedge 0, (x_d + R)\vee 0] \\
B &:= A - B_x[R].
\end{aligned}$$

Both $A$ and $B$ satisfy the condition $nA^n \subseteq (n+1)\, A^{n+1}$ and hence the result holds for $A$ and $B$. By (5.19) the result holds for their difference, $B_x[R]$.

The extension to starting uniformly on sets that are unions of boxes follows trivially. ■

# A. Last passage percolation estimates

Here we present the estimates required in Section 3.2.3.

We consider subcritical site percolation, $S$, with parameter $p \in (0, p_c(d))$ on the square lattice $\mathbb{Z}^d$: for $x \in \mathbb{Z}^d$ set $S(x) = 1$ with probability $p$ and $S(x) = 0$ otherwise, independent of all other sites. Take $(\Omega, \mathcal{F}, \mathbb{P})$ to be the appropriate probability triple.

For $\omega \in \Omega$, $n \in \mathbb{N}$, $x \in \mathbb{Z}^d$ define

$$N_x(n, \omega) := \sup_{\gamma} \left\{ \sum_{i=1}^{n} S(\gamma_i) \right\},$$

where the supremum is taken over all non self intersecting paths of length $n$, $\gamma = (\gamma_1 = x, \ldots, \gamma_n)$ started at the point $x$.

In this section we look to bound the growth of $N_x(n, \omega)$ in terms of $n$ and $p$. We suspect that these results are already contained in the literature but do not have the appropriate references.

Our first result is the following.

**Theorem 108.** *There exists a constant $C_1$, independent of $p$ and $d$ such that for almost all $\omega \in \Omega$ and every $x \in \mathbb{Z}^d$ there exists $A_x(\omega)$ such that*

$$N_x(n, \omega) \leq C_1 n p^{1/d} \tag{A.1}$$

*for all $n \geq A_x(\omega)$. Further*

$$\mathbb{P}\left(\exists n \geq N, x \in B_0[n] : N_x(n, \omega) \geq n C_1 p^{1/d}\right) < c_1 p^{-1/d} \exp\left(-c_2 \left(N p^{1/d}\right)^{c_3}\right) \tag{A.2}$$

*where $c_3 = \frac{\log 4}{\log 2}$, $c_1, c_2$ are independent of $p$ and $d$ and $C_1$ is as in (A.1).*

The proof of the Theorem 108 will follow from the following proposition.

**Proposition 109.** *Suppose $p < 32^{-d}$. There exists a map $\tau : \Omega \times \mathbb{Z}^d \to \mathbb{N} \cup \{\delta\}$ and $j := \frac{1}{32} p^{-1/d}$ such that:*

1. *$\tau_\omega(x) = \delta$ iff $S_\omega(x) = 0$.*

2. *If $\tau_\omega(x) = k \in \mathbb{N}$ then $S_\omega(x) = 1$ and there exist points $x_1 = x, x_2, \ldots, x_{2^k}$ such that $\tau_\omega(x_i) = k$, $|x_i - x_l| \leq 2^{2k} j$ for all $i, l$ and if $y \notin \{x_1, \ldots, x_{2^k}\}$ is such that $\tau_\omega(y) = k$ then $|y - x_i| > 2^{2k} j$.*

*Further,*

$$\mathbb{P}(\tau_\omega(x) \geq k) \leq 2^{-2^{k-1}}. \tag{A.3}$$

**Proof.** We prove the proposition by constructing $\tau_\omega$ via a deterministic algorithm. We begin by defining the algorithm on a finite box, $B_0[L]$, and then proceed to extend this to the infinite case. Take an ordering on $\mathbb{Z}^d$ such that if $|x| < |y|$ then $x$ comes before $y$ in the ordering. This induces an ordering on $B_0[L]$.

Consider $\omega \in \Omega$ restricted to $\{0,1\}^{B_0[L]}$. We define $\tau_\omega$ inductively. As the algorithm progresses, we define a function $\tilde{\tau}_\omega(x)$ and sets $F_k(x)$ for $x \in B_0[L]$ and $k \geq 0$. These will aid our definition of $\tau_\omega$.

To begin, for $x \in B_0[L]$, such that $\omega(x) = 0$ we define $\tau_\omega(x) := \delta$ and $F_k(x) := \phi$ for all $k \geq 0$. For $x \in \mathbb{Z}^d$ such that $\omega(x) = 1$, we set $\tilde{\tau}_\omega(x) = 1$ and set $F_0(x) = \{x\}$. This is the step 0.

Now, at the $(k+1)$th step, let $E_k := \{x \in B_0[L] : \tilde{\tau}(x) = k\}$. Let $z$ be the first point of $E_k$ in the ordering. Now, if there exists a point $x \in E_k \backslash F_k(z)$ and $z_i \in F_k(z)$ such that $|x - z_i|_\infty < 4^k j$ then we set $\tilde{\tau}_\omega(y) = k+1$ and $F_{k+1}(y) = F_k(x) \cup F_k(z)$ for all $y \in F_k(x) \cup F_k(z)$. There may be several such $x$, take the first in the ordering. If there fails to exist such an $x$ then we set $\tau_\omega(y) = k$ for all $y \in F_k(z)$ and set $F_i(y) = F_k(y)$ for all $i \geq k$ and $y \in F_k(z)$.

We now remove the points of $F_{k+1}(x)$ from $E_k$ and repeat this step with the new first element of the set $E_k \backslash F_{k+1}(x)$. We continue until $E_k = \phi$. The $(k+1)$th step is now complete.

The algorithm concludes when $\tau_\omega$ has been defined for every point of $B_0[L]$.

As we are working in a finite box, the algorithm must conclude after at most $k$ steps for $k$ such that $(2L)^d \leq 2^k$.

If we write $r_k$ for the maximum possible radius of the set $F_k(x)$, then $r_k$ must satisfy the relation:

$$r_k \leq \left(2r_{k-1} + 4^{k-1} j\right), \; r_0 = 1.$$

Thus

$$\begin{aligned} r_k &\leq \left(2^k + 2^{2k-1}\right) j \\ &\leq 2^{2k} j. \end{aligned}$$

Hence the event $\{\tau_\omega(0) < k\}$ is determined by the value of $\omega$ on the vertices of the box $B_0\left[r_{k-1} + 4^k j + r_{k-1}\right] \subset B_0\left[2^{2k+1}\right]$.

Note that since the origin is the first point of the ordering we have for any $x \in B_0[L]$:

$$\tau_\omega(x) \leq \tau_{\sigma_x(\omega)}(0), \tag{A.4}$$

where $\sigma_x$ is the spatial shift that takes 0 to $x$. If we write $\omega|_{B_0[L]}$ for $\omega$ considered on the finite box $B_0[L]$, then by (A.4) we see if $\mathbb{P}\left\{\lim_{L\to\infty}\tau_{\omega|_{B_0[L]}}(0)<\infty\right\}=1$ then for almost every $\omega\in\Omega$ the limit

$$\tau_\omega(x) := \lim_{L\to\infty}\tau_{\omega|_{B_0[L]}}(x)$$

exists for every $x\in\mathbb{Z}^d$. Take this as the definition of $\tau_\omega$ on an infinite graph.

It follows easily from the construction that $\tau_\omega$ satisfies properties 1 and 2. We now prove equation (A.3).

Set $p_k := \mathbb{P}(\tau_\omega(0)\geq k)$. By our construction, for $\tau_\omega(0)\geq k$ there must be a point $y\in B_0\left[2^{2k}j+4^kj\right]-F_{k-1}(0)$ such that $\tau_\omega(y)\geq k-1$ as well as $\tau_\omega(0)\geq k-1$. By (A.4) we obtain the recursive formula

$$p_k \leq p_{k-1}^2\left(2^{3k}j\right)^d, p_1 = p$$

which is satisfied by

$$p_k \leq 2^{a_kd}j^{b_kd}p^{2^{k-1}},$$

where

$$a_k = 3\left(2^k-k-1\right),\quad b_k = 2^{k-1}-1.$$

In particular, as by assumption $j\geq 1$

$$p_k \leq \left(16^dj^dp\right)^{2^{k-1}}.$$

Note that if we take $j=j(p,d)$ sufficiently small then $p_k\to 0$ as $k\to\infty$, and thus with probability one

$$\tau_\omega(x) := \lim_{L\to\infty}\tau_{\omega|_{B_0[L]}}(x)<\infty$$

for all $x\in\mathbb{Z}^d$.

In particular, with

$$j := \frac{1}{32}p^{-1/d},$$

we obtain

$$p_k \leq 2^{-2^{k-1}}$$

as claimed. ■

We can now prove Theorem 108.

**Proof of Theorem 108.** If $p\geq 32^{-d}$, take $C_1\geq 32$ and (A.1) follows trivially.

Now suppose $p<32^{-d}$. From (A.3) we obtain:

$$\sum_{k=1}^{\infty}\mathbb{P}\left\{\exists x\in\left[-2\times 4^kj, 2\times 4^kj\right]^d : \tau(x)\geq k\right\} \leq \sum_{k=1}^{\infty} j^d 4^{(k+1)d}2^{-2^{k-1}} < \infty, \quad \text{(A.5)}$$

thus by Borel-Cantelli, the event happens infinitely often with probability zero.

Now, by the definition of the labelling $\tau$, if $\tau(x) = k$ and

$$y \in \bigcup_{z \in F_k(x)} B_z\left[4^k j\right]$$

with $\tau(y) = k$, then we must have $y \in F_k(x)$. Hence a 'trap' of size $2^k$ has a 'buffer' of radius $4^k j$ surrounding it. If $n = 4^l j$ for some $l \in \mathbb{N}$, then a non self-intersecting path of length $n$ contained in the box $[-2n, 2n]^d$ can contain at most $2^k 4^{(l-k)\vee 0}$ points with $\tau(x) = k$. Now, if the largest value of $\tau$ we see in $[-2n, 2n]^d$ is $\tau(x) = l - 1$, then for any self-avoiding path of length $n$ contained in $[-2n, 2n]^d$ the maximum number of occupied sites seen is bounded above by

$$\sum_{i=0}^{l} 2^i 4^{l-i} = 4^l \sum_{i=0}^{l} 2^{-i} \leq 2.4^l. \tag{A.6}$$

Bringing together (A.5) and (A.6), we get an upper bound proving (A.1):

$$\frac{N_x(n,\omega)}{n} \leq \frac{2.4^l}{4^l j} = \frac{2}{j} = 64p^{1/d}.$$

Note that this will in particular hold for paths of length $n$ started at some point $x \in [-n, n]^d$. Set $C_1 = 64$.

We further see that for $M > 0$, defining $m := \left\lfloor (\log 4)^{-1} \log\left(\frac{M}{j}\right)\right\rfloor$

$$\begin{aligned}
& \mathbb{P}\left(\exists n \geq M : N_0(n,\omega) \geq nC_1 p^{1/d}\right) \\
\leq\ & \mathbb{P}\left(\exists k \geq \left\lfloor (\log 4)^{-1} \log\left(\frac{M}{j}\right)\right\rfloor \text{ and } x \in B_0\left[4^k\right] : \tau(x) \geq k\right) \\
\leq\ & \sum_{k=m}^{\infty} j^d 4^{(k+1)d} 2^{-2^{k-1}} \\
\leq\ & c_1 j^d \sum_{k=m}^{\infty} 2^{-2^{k-2}} \\
\leq\ & c_1 j^d \exp\left(-c_2 2^m\right) \\
=\ & c_1 p^{-1/d} \exp\left(-c_2\left(Mp^{1/d}\right)^{c_3}\right)
\end{aligned}$$

where $c_3 := \frac{\log 2}{\log 4}$. ■

## A.1. Speed of convergence

We in fact require slightly different results - taking decay in equation (A.2) to be in $p$. The following proposition gives our results in this direction.

**Proposition 110.** *Let $\theta > d+1$. Take $j = 2^{-6}p^{-1/4\theta d}$. Defining*

$$\tilde{N}_\theta(\omega) := \inf\left\{n : \text{for all } l \geq n,\ N_x(m,\omega) \leq \frac{8m}{j} \text{ for all } m \geq l \text{ and } x \in B_0\left[l^\theta\right]\right\},$$

*then*

$$\mathbb{P}\left(\tilde{N}_\theta(\omega) > 0\right) \leq 3p^{1/2}.$$

**Proof.** Note first that if $j \leq 1$ then the result is certainly true. Suppose $j > 1$.

Take $\theta > d+1$. Take $y \in \mathbb{R}^d$. Note that if, for $4^{k-1}j \leq l \leq 4^k j$ and $\tau(x) < k$ for all $x \in B_y[l]$, then

$$\begin{aligned} N_y(l,\omega) &\leq \sum_{i=0}^{k-1} 2^i 4^{k-i} \leq 2\cdot 4^k \\ &\leq \frac{8l}{j}. \end{aligned}$$

Hence, if $4^{k-1}j \leq l \leq 4^k j$, $y \in B_0\left[l^\theta\right]$ and for all $x \in B_0\left[\left(2\times 4^k j\right)^\theta\right]$ we have $\tau(x) < k$ then $N_y(l,\omega) \leq \frac{8l}{j}$.

From this observation we see

$$\begin{aligned} \mathbb{P}\left(\tilde{N}_\theta(\omega) > 0\right) &\leq \sum_{k=1}^{\infty} \mathbb{P}\left(\exists x \in B_0\left[\left(2\times 4^k j\right)^\theta\right] : \tau(x) \geq k\right) \\ &\leq \sum_{k=1}^{\infty} 4^{(k+1)d\theta} j^{d\theta} p_k(j,p) \\ &\leq \sum_{k=1}^{\infty} 4^{(k+1)d\theta} j^{d\theta}\left(16^d j^d p\right)^{2^{k-1}} \\ &= \sum_{k=1}^{\infty} 2^{d\left(2(k+1)\theta + 4\cdot 2^{k-1}\right)} j^{d\left(\theta + 2^{k-1}\right)} p^{2^{k-1}}. \end{aligned}$$

Inserting our choice of $j \geq 1$ we have

$$\begin{aligned} \mathbb{P}\left(\tilde{N}_\theta(\omega) > 0\right) &\leq \sum_{k=1}^{\infty} 2^{d\left(2(k+1)\theta + 4\cdot 2^{k-1}\right)}\left(2^{-6}p^{-1/4\theta d}\right)^{d\left(\theta+2^{k-1}\right)} p^{2^{k-1}} \\ &\leq \sum_{k=1}^{\infty} 2^{6\theta 2^k d}\left(2^{-6}p^{-1/4\theta d}\right)^{d\theta 2^k} p^{2^{k-1}} \\ &\leq \sum_{k=1}^{\infty} p^{-2^{k-2}} p^{2^{k-1}} \\ &\leq \sum_{k=1}^{\infty} p^{2^{k-2}} \\ &\leq 3p^{1/2}, \end{aligned}$$

where the final line follows since $p < p_c(d) \leq 1/2$. ∎

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