Roadmap

The Pollica perspective on the (super)-conformal world

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Abstract

This manuscript samples a series of recent results in the quest for a systematic understanding of the space of conformal field theories, with a particular focus on theories with extended supersymmetry. The large majority of results reported here were presented during the second Pollica Summer Workshop which took place from June 3-21 2019 and focused on mathematical and geometric tools for superconformal field theories. This manuscript represents in many ways a partial summary of the workshop.

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Introduction

The analytic predictability of quantum field theory typically relies on calculations which require weak coupling. Unfortunately many observed phenomena are in the strongly coupled regime where most standard analytical techniques no longer apply. Recent years have witnessed a tremendous amount of progress in our systematic understanding of the physics in this regime by studying theories with special symmetries, like supersymmetry and conformal invariance. These theories are particularly suitable for these studies since new methods to perform exact computations at strong coupling relying on the enhanced symmetries, have become available.

Conformal field theories (CFTs) are signposts in the space of all quantum field theories (QFTs), appearing as fixed points of renormalization group trajectories. UV complete QFTs can thus be typically seen as RG flows between UV and IR scale invariant theories, which typically come with a larger conformal symmetry. By virtue of the large amount of symmetry these theories are very constrained, yet they remain physically relevant as their understanding is a first step towards the study of QFTs in general.

Most CFTs lack a lagrangian description. Progress in two dimensions is possible [1, 2], thanks to the fact that the conformal algebra becomes infinite dimensional, but the situation in higher dimensions is much less constrained. Through the AGT correspondence two-dimensional CFTs also teach us about four-dimensional $= 2$ SCFTs [3]. In the presence of supersymmetry, however, there is a vast set of tools which have been developed for studying three and higher dimensional theories. These include a systematic understanding of the geometry of moduli spaces [4, 5], their corresponding chiral rings [6] general superconformal representations [7], and analysis of extended operators [8, 9]. For holographic CFTs the bootstrap approach reveals hidden symmetries [10], and allows to compute string amplitudes [11], allowing for the study of string theory/supergravity in AdS. Finally, CFT techniques can also teach us about dS scattering processes [12]. Despite this progress, the most ambitious goal remains to have a complete catalog of and to solve all such theories , to know all observables: correlation functions of local and non-local operators.

Combining the insights obtained using all the aforementioned techniques and more, could be of tremendous help in advancing our systematic understanding and could bring us closer to a complete characterization of the space of CFTs. Arguably one of the main impediments is the breadth of the specialties involved. Our goal in this manuscript is to collect contributions from some of these research directions and explore how these pieces of information can be combined to chart out the space of consistent theories and to determine their basic properties. We believe that there is terrific gain in this approach.

The large majority of results reported here, were presented during the second *Pollica Summer Workshop* which took place from June 3-21 2019 and focused on *mathematical and geometric tools for superconformal field theories*. The workshop was held in the newly renovated Principi Capano Castle located in the medieval town of Pollica, in the Cilento region, southern Italy. Fed by the spectacular quality of the catered food offered by the Capital of the Mediterranean Diet (Pollica), and inspired by breathtaking views of the Mediterranean itself, leading experts in the aforementioned research communities came together for three weeks of stimulating discussions and vibrant interactions. This manuscript represents a partial summary of the workshop, sampling different current directions in the study of conformal field theories.

There are many important omissions among the topics and results treated here. The choice reflects the themes discussed during workshop rather than our own assessment of what research directions are important to pursue. We sincerely apologize in advance to those who are not represented in this volume. The rest of the introduction will be dedicated to quick descriptions of the broad themes developed in this publication.

Kinematics of (super)conformal field theories

The large amount of symmetry of the theories studied is very constraining. One must start by distilling what is dynamical information, , theory dependent, from what is kinematics, , completely fixed by symmetry alone. Local operators must lie in irreducible representations of the symmetry algebra, with their correlation functions constrained by symmetry. A preliminary exercise consists of identifying the representations of the symmetry algebra, and working out the constraints imposed on correlation functions. While unitarity representations of conformal and superconformal algebras have been studied in great detail, non-unitary representations have received considerably less attention in dimensions higher than two. In [7] the motivation for such a survey of non-unitary representations is discussed, together with a summary of the challenges of such an analysis.

Two dimensions

Two-dimensional conformal theories provide textbook examples of cases where symmetries allow for a complete solution of some models. The infinite dimensional conformal symmetry in two dimensions allows for the existence of *rational* theories, , theories with a finite number of representations of the extended symmetry algebra. Furthermore, the requirement that partition functions be modular invariant places strong constraints on the allowed spectrum of CFTs. This gives rise to the holomorphic bootstrap approach for a classification of rational theories where one organizes rational theories by the number of generalized characters present. Furthermore one has to also impose the conditions of modular invariance and integrality of the coefficients of the character in a series expansion, , integer degeneracies of states with a given dimension. The characters must transform as a vector valued modular function such that the full partition function is modular invariant. This condition is equivalent to the requirement that the characters are solutions of a linear modular differential equation. [1] gives a status report on this classification program.

Two-dimensional CFTs, in particular Liouville/Toda, also allow for computing observables in four-dimensional = 2 SCFTs [3]. Moving beyond conformal theories in two dimensions, in [2] the two-to-two scattering matrix of massive particles is constrained. The S-matrix of $O(N)$ massive models is bounded by using a convex optimization problem.

SCFT moduli spaces of vacua

The presence of supersymmetry allows for ground state configurations parametrized by a set of continuous variables which can in turn be interpreted as coordinates of a space called the *moduli space of vacua* . It is often the case that is a complex algebraic variety and many techniques from algebraic geometry can be readily used. Furthermore inherits an extensive extra geometric structure from the unbroken superymmety and the spontaneuously broken conformal invariance. In the case of SCFTs these spaces are therefore particularly suitable to be studied systematically with the hope of extracting information on the space of SCFTs itself.

Depending on the low-energy theory, is divided in different *branches*, whose specific name and characterization depends on the spacetime dimension and amount of supersymmetry. We report below a quick overview. These various branches of are described by the VEVs of operators with specific quantum numbers. By looking at the equation of motion, it is straightforward in lagrangian theories to establish whether each of these operators does or does not acquire a vev and consequently derive the moduli space structure. Answering a similar questions for generic SCFTs, and in absence of a lagrangian, it is instead an open question and it requires identifying a set of necessary and sufficient conditions purely formulated in terms of operator algebra data. In four dimension with $\mathcal{N} \geq 2$, it is largely believed that a necessary condition for operators to get a vev is that the operators should form a *chiral ring*. Studying the properties of chiral rings thus remains a very active field of research. Some recent progress is reported in [6].

Higgs Branches are hyperkähler varieties which are generically present for theories with eight or more supercharges. This structure implies in particular that the Higgs Branch (HB) is a *holomorphic symplectic variety* and its singularities have a canonical stratification in terms of symplectic leaves which can be analyzed with the useful tool of *Hasse diagrams* [5]. The HB structure is largely unaffected by dimensional reduction. Thus the techniques in [5] work for $\mathcal{N} = 2$ in four dimensions, as well as for $\mathcal{N} = 4$ SCFTs in three dimensions, $\mathcal{N} = 1$ SCFTs in five dimensions and $\mathcal{N} = (1, 0)$ in six dimensions.

Coulomb Branches are generically present in theories with eight or more supercharges, though it varies considerably upon dimensional reduction (it is often called the *tensor branch* in six dimensions). For four or fewer dimensions it is a complex variety and in three dimensions it generically exists even for a smaller amount of supersymmetry. Our focus here is $\mathcal{N} = 2$ in four dimensions where the Coulomb Branch (CB) is a singular *rigid special Kähler* variety. In the last few years a systematic study of these spaces which are also scale-invariant, has developed in a full-fledged classification program of four dimensional SCFTs. An update on the logic, the status and the direction of this program is given in [4].

Holography

Moving beyond the classification of SCFTs there has been much progress in recent years towards solving particular models. Through the AdS/CFT correspondence progress on the CFT side translates into results on scattering amplitudes in theories of gravity. In [11] scattering amplitudes of type IIB string theory on $AdS_5 \times S^5$ are studied through the corresponding dual theory, , = 4 super-Yang-Mills (SYM) with gauge group $SU(N)$. By studying the four-point function of stress tensors in = 4 in a perturbative expansion around large N and strong coupling, and using the analytic bootstrap, the authors review how one learns about graviton scattering amplitudes. It was also recently shown that such scattering amplitudes exhibit an unexpected hidden ten-dimensional and six-dimensional conformal symmetry in $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times K3$, respectively. These hidden symmetries, whose origin is still unclear, are summarized in [10], ending with a summary of currently open questions. While all this progress concerns AdS exchanges, [12] addresses the question of dS spaces, by studying CFT correlators.

Extended operators

While all the discussion so far focused only on local operators, these do not exhaust the observables of a given QFT. Non-local extended operators have to be included as well for a complete characterization of the theory. Since local operators form a closed subsector most of the contributions will focus on this restricted set of observables, which is quite rich on its own. As our understanding of local operators is increased, however, it is natural to start enlarging the set of observables, and this is the subject of the last two contributions.

Integrability of = 4 super-Yang-Mills has allowed for a large wealth of results for both the planar limit and beyond it. It is then appropriate to start enlarging the set of observables by adding extended operators. In [8] a status report is given for codimension one defects that are domain walls between two = 4 SYM theories with gauge groups $U(N)$ and $U(N-k)$. Results for one and two-point functions of bulk operators in the presence of the defect are given.

Along a different direction, [9] studies the algebra of half-BPS codimension two operators themselves. The results concern SCFTs with four supercharges, in particular the algebra of line operators in $3d = 2$ theories. Half-BPS line operators are in the simultaneous cohomology of two supercharges and the OPE of two parallel lines has a non-singular ring structure.The analytic predictability of quantum field theory typically relies on calculations which require weak coupling. Unfortunately many observed phenomena are in the strongly coupled regime where most standard analytical techniques no longer apply. Recent years have witnessed a tremendous amount of progress in our systematic understanding of the physics in this regime by studying theories with special symmetries, like supersymmetry and conformal invariance. These theories are particularly suitable for these studies since new methods to perform exact computations at strong coupling relying on the enhanced symmetries, have become available.

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References

[1] Sunil Mukhi. *Classification of RCFT from Holomorphic Modular Bootstrap: A Status Report.*

[2] Lucía Córdova et al. *Bounds on the S-matrix of 2d $O(N)$ symmetric massive models.*

[3] Elli Pomoni. *T_N , Toda and topological strings.*

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Classification of RCFT from Holomorphic Modular Bootstrap: A Status Report

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1. Background

Following the initial proposal in 1988, there has been much progress in classifying Rational Conformal Field Theories in 2 dimensions from the Holomorphic Bootstrap approach. This method starts by postulating a generic holomorphic Modular Linear Differential Equation of a given order and imposing the requirement of non-negative integrality of the coefficients in the series expansion of the solutions, which are then identified as admissible characters, from which a modular-invariant partition function is constructed. In this short note, the status of this project is summarised.

Conformal invariance in two dimensions has long been recognised as being a strong enough symmetry to partially classify, and solve, field theories having this invariance. An influential paradigm, initiated in [1, 2] has been to focus on the chiral symmetry algebra, identify its null vectors and search for the "minimal" models that arise on decoupling them. These lead to infinite families of conformal field theories with rational exponents. The Kac-Moody algebra is special because one can also take cosets of pairs of such models and obtain enormous families of new ones [3, 4]. Within each family (labelled by central charge or Kac-Moody level) the number of primaries under the chiral algebra increases rapidly down the list, and only the first few models are simple enough to be interesting.

The holomorphic bootstrap takes a different approach and gives rise to a classification based on the number of generalised characters. It is based on the fact that the characters $\chi_i(\tau),\ i-0,1,\cdots n-1$ of an RCFT are holomorphic functions of the modular parameter τ and transform as vector-valued modular functions of weight zero:

$$\chi_i(\gamma\tau)=V_{ij}(\tau)\chi_j(\tau),\qquad \gamma\tau\equiv\frac{a\tau+b}{c\tau+d},\ \ (\,a\,)\,b\qquad cd\in\text{PSL(2,Z)}\qquad(1)$$

which ensures that the partition function:

$$Z(\tau,\bar{\tau}) = \sum_{i=0}^{n-1} \bar{\chi}_i(\bar{\tau})\chi_i(\tau) \tag{2}$$

is modular invariant.

(1) holds if and only if $\chi_i(\tau)$ are the n independent solutions of a Modular Linear Differential Equation (MLDE) [5, 6, 7]. As proposed in these references, one therefore starts with a generic MLDE – which already guarantees that its solutions have the desired holomorphic and modular properties. The equation turns out to depend on finitely many real parameters. Now one solves the equation by inserting a series expansion in $q = e^{2\pi i\tau}$ and imposes the requirement that the coefficients in this expansion are non-negative integers. This is necessary in order to interpret them as degeneracies in a quantum field theory. The reader is referred to the original references for details on the procedure.

What will be important for us is that the MLDE is specified by two integers: n, the number of characters, and ℓ, the Wronskian index which is equal to the number of zeroes in moduli space of the Wronskian determinant of the solutions. For fixed (n,ℓ), the outcome of the bootstrap is to produce lists of "admissible" characters satisfying all the bootstrap conditions. A further problem, after such a list is generated, is to identify which candidates can be identified as the characters of actual conformal field theories. In the present note I will focus primarily on the classification of admissible characters, and refer to this additional question only in a few places.

2. State of the art

2.1. $n = 1$

For single-character CFT, candidate characters take the form [6]:

$$\chi(\tau) = j^{w_\rho}(j-1728)^{w_i} P_{w_\tau}(j) \tag{3}$$

where $j(\tau)$ is the Klein j-invariant, $w_\rho \in \{0, \frac{1}{3}, \frac{2}{3}\}$, $w_i \in \{0, \frac{1}{2}\}, w_\tau \in Z$ and $P_{w_\tau}(j)$ is a polynomial of degree w_τ in j. The polynomial must be chosen so that the final character is admissible, i.e. has non-negative coefficients. The Wronskian index for these characters is:

$$\ell = 6(w_\rho + w_i + w_\tau) \tag{4}$$

and the central charge of the corresponding theory, if any, is $c = 4\ell$. This set is finite for $\ell < 6$ ($c < 24$) and infinite beyond that. Corresponding CFT are completely classified for $\ell < 6$ while seminal work was done on the case of $\ell = 6$, equivalently $c = 24$ [8].

2.2. $n = 2, \ell < 6$

For $\ell = 0$, a list of ten pairs of admissible characters was found in the original work, [6]. Of these, one corresponds to a one-character theory re-discovered as a solution to a second-order MLDE, namely the WZW model $E_{8,1}$. Another corresponds to the Lee-Yang edge singularity, a non-unitary minimal CFT. Of the remaining cases, seven correspond to WZW models for Lie algebras in the "miraculous" Cvitanovic-Deligne series [9, 10]. One corresponds to the so-called $E_{7\frac{1}{2}}$ algebra [11] and has been identified, together with the Lee-Yang theory, as an "intermediate vertex operator algebra" in [12]. The central charges of this series satisfy $0 < c \leq 8$ in the "unitary presentation" (see [13]). A mathematically rigorous basis for this classification was provided in [14].

Subsequently it was shown in [15] that for $n = 2$, the Wronskian index ℓ is always even. The case of $(n, \ell) = (2, 2)$ was classified in the same reference. There are again 10 cases, much like the $\ell = 0$ set. Their central charges satisfy $16 \leq c < 24$. Many years later these were re-examined in [16]. In [17] it was shown that the $\ell = 0$ and $\ell = 2$ sets form dual coset pairs with their central charges adding up to 24. This novel coset procedure is quite general and will re-appear below.

For $(n, \ell) = (2, 4)$ the situation remains puzzling. After being missed in previous works, two of the admissible characters were found in [18] and a third was added in [13]. Their central charges lie between 32 and 35, but no CFT's corresponding to these characters have yet been identified. Besides these, there are tensor products of $\ell = 0$ characters with the $E_{8,1}$ character $j^{\frac{1}{3}}$.

2.3. $n = 2, \ell \geq 6$

The case of $\ell \geq 6$ was addressed by different methods. In [19] such characters were constructed as "Hecke images" of characters with lower ℓ. It became clear from this reference that there are infinitely many candidate characters in this case. In [13] it was shown that admissible characters can be constructed as semi-definite linear combinations of $\ell = 0$ "quasi-characters", which themselves have integer but possibly negative coefficients in their q-series. The latter reference classified all quasi-characters and showed that by suitable linear combinations, one generates the complete set of characters with $\ell \geq 6$.

The admissible characters so obtained are infinite in number and do not in general have any recognisable CFT interpretation. From experience with the one-character case, it is clear that once we cross the "$\ell = 6$ barrier" there should be far fewer CFT's than admissible characters. In [20], over a hundred genuine RCFT were constructed with $\ell = 6$ using the coset construction of [17] in conjunction with Kervaire lattice CFT. These have central charges satisfying $24 < c < 32$ and provide an existence proof for RCFT with $(n, \ell) = (2, 6)$.

2.4. $n = 3, \ell = 0$

For three characters the only well-studied case is $\ell = 0$. This was first addressed in [21] and discussed further via a different technique in [22]. It was noted that with three characters, even at $\ell = 0$ there are infinitely many known RCFT, and several examples as well as some bounds were obtained. In particular, the WZW model for $\mathrm{SO}(N)_1$,

$SU(4)_1$, $SU(5)_1$, $SU(2)_2$ as well as the Virasoro minimal models with $(p,p') = (3,4)$ (Ising) and $(2,7)$ (a non-unitary theory) all have three characters as well as $\ell = 0$.

Coset duals of many three-character $\ell = 0$ theories were obtained in [17]. These also turn out to have $\ell = 0$ and are listed in Table 2 of that reference. Also, all three-character theories with $\ell = 0$ and no Kac-Moody algebra were classified in [23, 24]. This turns out to be a finite set containing the Baby Monster CFT with $c = \frac{47}{2}$ [25] as well as exotic theories at $c = 8, 16$ related to the finite group $O^+_{10}(2)$. These CFT's were rediscovered in [26] in the context of the semi-definite programming approach to the conformal bootstrap.

Recently in [27], with the extra condition of "irreducible monodromy", a complete classification of three-character theories with $\ell = 0$ has been proposed. The results are the examples discussed in [21] along with a "U-series" (where U stands for "unknown") which includes the Baby Monster and a set of other theories with central charges in the range $\frac{27}{2}, \frac{29}{2}, \cdots \frac{47}{2}$. Many, but not all, of these U-series theories have previously appeared in Table 2 of [17], namely those with central charges:

$$c = \frac{31}{2}, \frac{35}{2}, \frac{37}{2}, \frac{39}{2}, \frac{41}{2}, \frac{43}{2}, \frac{45}{2} \tag{5}$$

There is a subtlety for the sub-series $SO(2N)_1$ for N=0 mod 4. Though these certainly exist as RCFT, they do not satisfy the irreducible monodromy condition of [27] (which is violated if two distinct primaries differ in dimension by an integer). This subtlety appears to be associated to congruence subgroups, which play an important role in the classification procedure of this work. The exotic $c = 8, 16$ models of [23, 24, 26] also do not satisfy the irreducible monodromy condition and therefore do not appear in this classification, though they appear to be sensible RCFT. Moreover, there are several other theories in Table 2 of [17], with central charges:

$$c = 14, 15, 17, 18, 19, 20, 21 \tag{6}$$

that do not appear in [27]. None of these has a pair of primaries differing in dimension by an integer, so naively it seems they should satisfy the criterion of irreducible monodromy. It will be interesting to establish why they do not appear in the classification of [27].

3. Outlook

It seems quite a realistic goal to complete the classification of admissible characters for $(n, \ell) = (3, 0)$, including cases with reducible monodromy, and even to associate most/all of them to actual RCFT. However little or nothing seems to be known about the case of $\ell > 0$.

3.1. $n = 4$

Very few results are available for this case. To my knowledge, the only ones (beyond the usual set of minimal and WZW models) are the $\ell = 0$ coset models obtained in Table 3 of [17]. There is a paper in the mathematical literature [28] where the 4th order MLDE is solved under very restrictive conditions on the degeneracies of the identity character. Four RCFT are obtained, all of which are either minimal models or simple-current extensions.

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Bounds on the S-matrix of 2d O(N) symmetric massive models

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1. Background

In this contribution we briefly discuss, following [1], the dual convex maximization problem that can be used to put bounds on the scattering matrix of two dimensional massive models. See also [1] for references to related work in the literature. In two dimensions, due to conservation of energy and momentum, if two particles scatter into two particles, we have that the momenta an energy of the final particles are the same as the initial ones. For an O(N) symmetric model we can write the S-matrix as where $S_{T,R,A}(s)$ are the S-matrices corresponding to transmission, reflection and annihilation and $s=(p_1+p_2)^2$ is a Mandelstam variable. Crossing implies $S_T(4m^2-s)=S_T(s)$ and $S_R(4m^2-s)=S_A(s)$. For a given representation of $O(N)$, the scattering matrix is

$$S_I = NS_A + S_T + S_R, \quad S_\pm = S_T \pm S_R \tag{2}$$

where the indices $I,+,-$ denote the isoscalar, anti-symmetric and traceless symmetric + representations. Crossing reads $S_a(4m^2-s)=C_{ab}S_b(s)$, $(a=I,-,+)$ for a certain constant matrix C_{ab} such that $C^2=1$. Unitarity implies $|S_{I,-,+}(s\geq 4m^2)|\leq 1$. Here we are going to describe a simple procedure to find bounds on linear combinations of the type

$$\mathcal{F} = \sum_{a=I,-,+} n_a S_a(s_0) \tag{3}$$

where n_a is a unit vector. The functions $S_a(s)$ are analytic with cuts on $(-\infty,0)\cup(4m^2,\infty)$.

2. State of the art

We can derive a dispersion relation by deforming the contour in the following integral

$$S_a(s_0) = \frac{1}{2\pi i}\oint_{\mathcal{C}} \frac{S_a(s)}{s-s_0}ds = \frac{1}{2\pi i}\left[\int_{4m^2}^{\infty} + \int_{-\infty}^{0}\right]\Delta S_a\, ds \tag{4}$$

where $\Delta S_a = 2i\text{Im}S_a(s^+)$ denotes the jump across the cuts (see figure). The simple observation is that we do not need to take $\frac{1}{s-s_0}$ as the function multiplying S_a. Any function having a simple pole at s_0 will suffice. However, to allow the same contour deformation it should not have any other singularity other than (possibly) cuts on $(-\infty,0)\cup(4m^2,\infty)$. Taking K_a to be analytic functions with a pole at $s_0=2m^2$ and residue n_a we obtain

$$\mathcal{F} = \frac{1}{2\pi i}\oint K_a(s)S_a(s) = \frac{1}{2\pi i}\left[\int_{4m^2}^{\infty} + \int_{-\infty}^{0}\right]\Delta(K_aS_a)ds \tag{5}$$

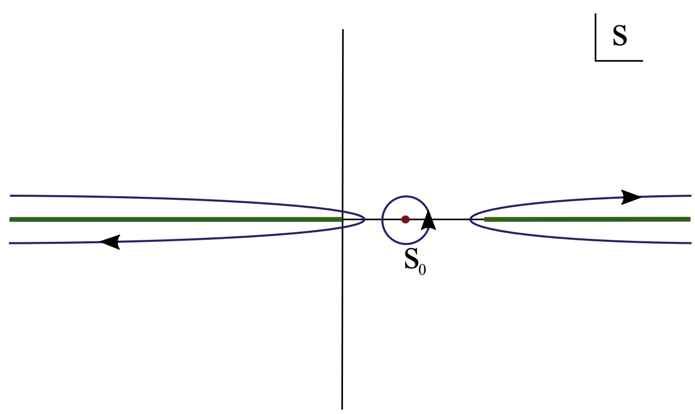

Figure 1: A contour around s_0 can be deformed to an integral around the cuts.

Crossing symmetry relates the two cuts: $S_a(s^+) = C_{ab}S_b(4m^2 - s^-)$. If we further impose the condition $K_a(4m^2 - s) = -C_{ba}K_b(s)$ and real analyticity $K_a(s^*) = (K_a(s))^*$ then we obtain that the jump is the same on both cuts $\Delta(K_aS_a)(s) = \Delta(K_aS_a)(4m^2 - s) = 2i\mathrm{Im}[K_aS_a]$ and therefore

$$\mathcal{F} = \frac{2}{\pi}\int_{4m^2}^{\infty} \mathrm{Im}[K_a(s^+)S_a(s^+)]ds \leq \frac{2}{\pi}\int_{4m^2}^{\infty} |K_a(s^+)S_a(s^+)|ds \leq \frac{2}{\pi}\int_{4m^2}^{\infty} |K_a(s^+)|ds \quad (6)$$

Since this is true for any set of functions K_a with the given properties, we find

$$\boxed{\max_{\{S_a\}}\left[\mathcal{F} = \sum_a n_a \,\mathrm{Re}\left[S_a\left(s_0 = 2m^2\right)\right]\right] \leq \min_{\{K_a\}}\left[\mathcal{F}_d \equiv \frac{2}{\pi}\int_{4m^2}^{+\infty}\sum_a |K_a(s)|\, ds\right]} \quad (7)$$

where the maximum is over all functions $S_a(\theta)$ analytic on the physical strip and obeying crossing and the unitarity constraint and the minimum is over all functions $K_a(s)$ analytic except for a pole at $s_0 = 2m^2$ with residue $\mathrm{Res}[K_a, s_0] = n_a$ cuts on $(-\infty, 0) \cup (4m^2, \infty)$ and obeying anticrossing with $C^\intercal$. As an example we can take

$$K_a(s) = \frac{n_a}{s-2}\frac{2}{\sqrt{4-s}\sqrt{s}} \quad (8)$$

that gives

$$\mathcal{F} \leq \min \int_{4m^2}^{\infty}\sum_a |K_a(s^+)|ds \leq \sum_a |n_a| \quad (9)$$

a simple but useful bound.

3. Outlook

More precise bounds can be obtained using a numerical procedure to minimize the dual functional $\mathcal{F}_D$ [1].

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T_N, Toda and topological strings

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1. Background

Famously, the class $\mathcal{S}$ of 4d CFTs is obtained via a twisted compactification of the 6d (2,0) theory on 2d Riemann surfaces [1, 2]. The decomposition of the Riemann surfaces into pants and tubes, corresponds to the building blocks of the 4d theories, matter and color factors. The T_N theories, *a.k.a.* trinions, correspond to the most general trifundamental "matter" in this construction. On the other hand, the 2d Toda CFT is a multi-field generalization of the Liouville CFT, the simplest non-rational 2d CFT which is the only example solvable using the bootstrap approach. While Liouville enjoys Virasoro symmetry and its primaries are in $sl(2)$ representations, Toda is governed by the higher spin W_N algebra and its primaries have $sl(N)$ quantum numbers.

2. State of the art

Our work is concerned with the following two problems open problems: on the one hand the computation, of the partition function of the T_N trinion theories on S^4 and, on the other hand, the three point functions of 2d Toda CFT with three generic primaries. The computation of the T_N partition function on S^4 has not been possible up to now as the T_N trinions do not have a Lagrangian description and thus Pestun's Localization is not applicable. The three point functions of 2d Toda CFT with three generic primaries were not attainable with 2d CFT techniques similar to the ones used for Liouville, as the Ward identities of the W_N algebra are not powerful enough.

Through the AGT map [3], partition functions of 4d CFTs in class $\mathcal{S}$ (with SU(2)/SU(N) symmetry) are related to 2d Liouville/Toda CFT correlation functions (with $sl(2)/sl(N)$ primaries), and the two problems described above are simply two different incarnations of the same problem. With out work we were able to overcome both problems, and propose a solution for both problems using topological strings, however, as we will see, it will come at a price. We have to first uplift the 4d T_N

theories to 5d (on $S^4 \times S^1$) and correspondingly the W_N Toda correlators to its q-deformed version (with $q = e^{-R_5}$ the radius of the 5th dimension).

For the 5d T_N theories we can use 5-brane junctions (webs) together with the topological vertex formalism to compute their partition functions on $S^4 \times S^1$ [4] and through AGT the q-Toda three point functions with three generic primaries [5, 6]. Our results have the correct symmetries, zeros and after appropriate specialisation simplify to known sub-cases (Fateev-Litvinov semidegeneration formula) whenever they are available; can be written in terms of known special functions. Nonetheless, they have a serious drawback, it is very hard to explicitly take the 4d limit ($q \to 1$) in the form they are written. This is because naturally from the topological vertex computation, the topological string partition functions of the T_N theories (which are the holomorphic half of the integrant of the 5d partition functions on $S^4 \times S^1$), contain a numbed of infinite sums which we do not know how to perform. What is worst, is that these sums do now look convergent if we naively try to take the 4d limit. To overcome this problem, we need turn to the matrix model representation (or free field representation) of the q-Toda three point functions with three generic primaries [7, 8]. In this form we can directly show that our formulas also have the correct poles and they reproduce the correct Dotsenko Fateev free field representation. What is more, we can explicitly take the 4d limit for this integral representation.

There are still many important questions left to understand. Most importantly, the topological string partition function of the T_N theories, has an integral representation (free field representation), the integrand of which is equal to the q-deformed A_{N-1} generalisation of Selberg integral (as it should), however, its contour prescription is very intricate and we still do not understand it for $N > 2$. Different choices of contours correspond to different versions of web diagrams [8] and changing the integration contours correspond to jumps when passing from one chamber in the parameter space to another (from one web diagram to another). These changes of the integration contours should be understood as changes in the bases for spaces of conformal blocks of the Toda CFT in order to have a complete understanding of the problem.

3. Outlook

The 4d formulas should also be possible to reproduce directly (without having to uplift to 5d) starting from the Seiberg-Witten curves of the T_N theories which we know [4] and applying Topological recursion or the techniques we developed in [9]. The Seiberg-Witten curves of the 4d T_N theories define integrable systems the quantization of which is encoded in isomonodromic tau-functions (free fermion representation of the 2d CFT blocks). The corresponding tau-functions should admit a series expansion of generalised theta series type from which we should be able to extract the topological string partition functions for each chamber. What is more, it seems worthwhile to explore weather it is possible to use an extension of the Fiber-Base duality [10, 11] and the symmetry enhancement the partition function of the T_3 theory enjoys, to try to completely fix this function. This way of thinking is also pointing to the need for defining new special functions which will have as one representation our sum formulas and on the other hand the A_{N-1} generalisation of Selberg integral.

Finally, the topological string partition functions of the T_N theories can also be though of as interwiners of three generic triples of representations of the Ding-Iohara-Miki algebra, a point of view which seems very promising and allows connections with yet new areas in mathematics.

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Construction and classification of CB geometries

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1. Background

We give a non-technical summary of the classification program, very dear to the heart of both authors, of four dimensional $\mathcal{N}=2$ superconformal field theories (SCFTs) based on the study of their Coulomb branch geometries. We outline the main ideas behind this program, review the most important results thus far obtained [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] and the prospects for future results. This note is organized as follows. In this first section, we provide a lighting review of the CB and outline in some details the logic of the classification program. In the second section, we report some of the most important results which our analysis has thus far produced. We conclude outlining the directions which we are currently pursuing and identify the main open questions of this program.

1.1. Logic of the program

The (complex) scalars in $\mathcal{N}=2$ gauge theories in four dimensions generically admit continuous vacuum expectation values which are not lifted by quantum corrections. This space of vacuum solution is called the moduli spaces of vacua $\mathcal{M}$. Depending on the properties of the low-energy physics, $\mathcal{M}$ is divided in *Coulomb*, *Higgs* and *Mixed* branches. We focus on the former and indicate the Coulomb Branch (CB) of a $\mathcal{N}=2$ theory by $\mathcal{C}$. The properties of both $\mathcal{M}$ and $\mathcal{C}$ can be formulated in a way that makes no reference to the quantum fields in the gauge theory. Therefore characterizing $\mathcal{N}=2$ field theories via the properties of their moduli space is immediately suitable to study theories with no known lagrangian formulation as it is the case for the majority of $\mathcal{N}=2$ superconformal field theories (SCFTs) which are the ultimate objective of our study.

1.2. Coulomb Branch generalities

The property defining $\mathcal{C}$ is that the low-energy theory on a generic point is extremely simple, just an $\mathcal{N}=2$ $U(1)^r$ theory with no massless charged states. r is called the *rank* of the theory and it coincides with the complex dimensionality of $\mathcal{C}$, $\dim_{\mathbb{C}}\mathcal{C}=r$. $\mathcal{C}$ is a singular space and its (complex co-dimension one) singular locus $\mathcal{V}$ is "generated" by charged states becoming massless. In other words, $\mathcal{V}$ represents precisely the locus where the low-energy physics is less boring and potentially interesting. But since $\mathcal{C}$ is singular there, it is not directly accessible.

The striking fact about CB geometry, is that the physics on $\mathcal{V}$ can be studied in a fairly detailed way by studying the theory in the non-singular region, $\mathcal{C}_{\rm reg}:=\mathcal{C}\setminus\mathcal{V}$ where the low-energy physics is as simple as it gets (just a bunch of non-interacting $\mathcal{N}=2$ vector multiplets!). This is due to the fact that no globally defined lagrangian description of the low energy $\mathcal{N}=2$ $U(1)^r$ is possible and non-trivial *monodromies* have to considered to describe the physics on $\mathcal{C}_{\rm reg}$. These are specific elements of the ${\rm sp}(2r,\mathbb{Z})$ *electromagnetic duality group* which depend on the physics at $\mathcal{V}$ and can be therefore used to characterize it. The object which transform non-trivially under the monodromy group is the vector of *special coordinates* σ, which provides a holomorphic section of an ${\rm sp}(2r,\mathbb{Z})$ bundle over $\mathcal{C}_{\rm reg}$. The special coordinates, also satisfy non trivial constraints which allow the definition of a Kähler metric on $\mathcal{C}_{\rm reg}$ and which can be extended in a non-trivial way to $\mathcal{V}$ (see stratification). All this together equips $\mathcal{C}_{\rm reg}$ with a *Rigid Special Kähler* (RSK) structure.

$\mathcal{N}=2$ SCFTs live at the origin of scale invariant CBs. Scale invariance makes the study of the geometry in the non-singular locus even simpler and strongly constraints the allowed monodromies. Our program aims at extracting the most information with the minimum (which for rank higher than one is not that low) effort, and characterize the space of $\mathcal{N}=2$ SCFTs by understanding the properties of the non-singular region of their CB geometries.

Other facts further motivate our approach. There is a belief that all interacting $\mathcal{N}=2$ SCFT have a CB (see the rank-0 section below) and thus can be captured by our classification method. This should be contrasted with the other branches of the moduli space, where an infinite number of interacting $\mathcal{N}=2$ SCFTs with trivial Higgs and/or Mixed branch are known. Also interestingly, the CB has the property that is only deformed and not lifted by $\mathcal{N}=2$ preserving relevant deformations and thus the RG-structure of $\mathcal{N}=2$ theories is immediately visible from the CB perspective.

There is a natural way to organize our classification program. First, theories with lower dimensional CBs are simpler, and in particular lower rank theories can be reached via RG-flows of higher ones but not viceversa (see Stratification below). Thus is reasonable to study CB geometries by their increasing dimensionality.

Secondly, if conformal invariance is unbroken, scale invariance of the corresponding geometry dramatically constraints the global structure. Quick progress can be made by classifying the *Scale invariant limit* of these geometries. In fact for $\mathcal{N}=2$ SCFTs, the $\mathbb{R}^+\times{\rm U}(1)$ action by the dilation and the $U(1)_r$ symmetry induces a $\mathbb{C}^*$ action on the full CB geometry. In the one dimensional case, requiring invariance under this $\mathbb{C}^*$ action immediately constraints the set of allowed geometries to only 7. We discuss the two dimensional case below.

Unfortunately it is a known fact, since the seminal papers on the topic [14, 15], that many SCFTs share the same scale invariant CB geometry, therefore this information is not enough to fully characterize the space of $\mathcal{N}=2$ SCFTs. Also in the conformal limit, $\mathcal{C}_{\rm reg}$ is "so simple" that not much information on the non-trivial physics of the $\mathcal{N}=2$ SCFT living at the origin of $\mathcal{C}$ can be "extracted". Here comes the third, and hardest, step of the story, understanding the *mass deformation* of the scale invariant geometries. As we said above, turning on ($\mathcal{N}=2$ SUSY preserving) relevant operators does not lift the CB and it instead deforms it in a precise manner so we can hope that it leaves an imprint on the geometry which can be captured by our analysis. In the rank-1 case, this information can be encoded via the *deformation pattern* of the scale invariant geometry. If refined with the deformation pattern, the initial scale invariant CB geometry data, almost uniquely characterizes a $\mathcal{N}=2$ SCFT‡.

‡ It is known that this data is not enough to distinguish also discretely gauged theories

In summary, our classification program then consists in picking an increasing CB complex dimension, determining the allowed scale invariant geometries and then understanding their mass deformation.

	CB:			HB:	ECB & flavor symm.:			Central charges:		
	SI sing.	$\Delta(u)$	deform.	d_{HB}	h	$\mathbf{2h}$	$\mathfrak{f}$	$k_{\mathfrak{f}}$	$24a$	$12c$
I_1 series	II^*	6	$\{{I_1}^{10}\}$	29	0	–	E_8	12	95	62
	III^*	4	$\{{I_1}^{9}\}$	17	0	–	E_7	8	59	38
	IV^*	3	$\{{I_1}^{8}\}$	11	0	–	E_6	6	41	26
	I_0^*	2	$\{{I_1}^{6}\}$	5	0	–	D_4	4	23	14
	IV	3/2	$\{{I_1}^{4}\}$	2	0	–	A_2	3	14	8
	III	4/3	$\{{I_1}^{3}\}$	1	0	–	A_1	8/3	11	6
	II	6/5	$\{{I_1}^{3}\}$	0	0	–	$\varnothing$	–	43/5	22/5
	I_1	1	–	0	0	–	U_1	$*$	6	3
I_4 series	II^*	6	$\{{I_1}^{6}, I_4\}$	16	5	$\mathbf{10}$	C_5	7	82	49
	III^*	4	$\{{I_1}^{5}, I_4\}$	8	3	$(\mathbf{6},\mathbf{1})$	C_3A_1	$(5,8)$	50	29
	IV^*	3	$\{{I_1}^{4}, I_4\}$	4	2	$\mathbf{4}_0$	C_2U_1	$(4,?)$	34	19
	I_0^*	2	$\{{I_1}^{2}, I_4\}$	0	1	$\mathbf{2}$	C_1	3	18	9
	I_4	1	–	0	0	–	U_1	$*$	6	3
I_1^* series	II^*	6	$\{{I_1}^{3}, I_1^*\}$	9	4	$\mathbf{4}\oplus\overline{\mathbf{4}}$	$A_3\rtimes\mathbb{Z}_2$	14	75	42
	III^*	4	$\{{I_1}^{2}, I_1^*\}$	?	2	$\mathbf{2}_+\oplus\mathbf{2}_-$	$A_1U_1\rtimes\mathbb{Z}_2$	$(10,?)$	45	24
	IV^*	3	$\{I_1, I_1^*\}$	0	1	$\mathbf{1}_+\oplus\mathbf{1}_-$	U_1	$*$	30	15
	I_1^*	2	–	0	0	–	$\varnothing$	–	17	8
$IV_{Q=1}^*$ ser.	II^*	6	$\{{I_1}^{2}, IV_{Q=1}^*\}$	?	3	$\mathbf{3}\oplus\overline{\mathbf{3}}$	$A_2\rtimes\mathbb{Z}_2$	14	71	38
	III^*	4	$\{I_1, IV_{Q=1}^*\}$	0	1	$\mathbf{1}_+\oplus\mathbf{1}_-$	$U_1\rtimes\mathbb{Z}_2$	$*$	42	21
	$IV_{Q=1}^*$	3	–	0	0	–	$\varnothing$	–	55/2	25/2
I_2 ser.	I_0^*	2	$\{{I_2}^{3}\}$	0	1	$\mathbf{2}$	C_1	3	18	9
	I_2	1	–	0	0	–	U_1	$*$	6	3

Table 1. Partial list of rank-1 $\mathcal{N}=2$ SCFTs. They are divided into 5 series; the CFTs within each series are connected by RG flows from top to bottom. The red rows give the characteristic IR-free theory each series flows to. Yellow rows are lagrangian CFTs, while blue and green singularities have enhanced $\mathcal{N}=4$ and $\mathcal{N}=3$ supersymmetry, respectively. The first 3 columns describe the CB geometry; the next column gives the HB dimension; the next 4 columns give properties of the ECB and the flavor symmetry; and the last five columns give the CFT central charges. The meaning of each column and the choice of the theories appearing in the rows are explained in the introduction.

1.3. Rank-0 theory

An $\mathcal{N}=2$ theory with no CB is a rank-0 theory and there is a belief that no interacting rank-0 $\mathcal{N}=2$ SCFT existed. This is largely based on the lack of counter-examples and could be a consequence of a *lamp post effect*; many techniques to study $\mathcal{N}=2$ SCFTs are based on the existence of the CB. There is also some further evidence from our rank-1 classification (more below). In carrying out this classification we explicitly assumes no interacting rank-0 SCFTs exists. If this is not the case, and a rank-0 with low enough central chargesΣ does instead exist, our results would be modified in a dramatic way. Instead of predicting the existence of a total of 28 theories$\|$, the number would be close to a hundred and those new rank-1 theories would have a CB and therefore would be detectable with a multitude of methods. The fact that our classification appears to be complete and there is no sign for the existence of these extra theories, therefore provides evidence supporting the *non existence of rank-0* theories belief, at least within a certain range of central charges. By extending our systematic classification to higher ranks, we can considerably strengthen this indirect evidence.

2. State of the art

2.1. Full classification of rank-1 geometries.

In a series of papers [1, 2, 3, 8] the program outlined above was carried out completely in the one complex dimensional CB case which led to a complete classification of rank-1 theories. The results of our analysis are summarized in table 1 of [8] which is reported here, see table 1. As discussed extensively in [4], it is possible to start from some of the theories in table 1 and gauge discrete subgroups without breaking $\mathcal{N}=2$ SUSY. This operation acts non-trivially on the CB but without lifting it, so it produces other rank-1 theories. This explains why table 1 in [4] differs from the table below. Recently it was shown that all the rank-1 theories can be obtained from 6 dimension. In particular, starting from specific 6d (1,0) theories, and compactifying on a T^2 twisting by non commuting (flavor) holonomies, it is possible to obtain the theories sitting at the top of each one of the series in table 1 [16]. The rest can be obtained turning on specific subsets of their mass deformations. [17] discusses instead the construction of all entries in table 1 in F-theory. To the best of our knowledge, all the known rank-1 $\mathcal{N}=2$ SCFTs are captured by our analysis.

2.2. Understanding the singularity structure of rank-2 geometries

The analysis of scale invariant CB geometries at complex dimension two is already considerably more challenging than the rank-1 case outlined above. In [6] we were able to show that the $\mathbb{C}^*$ action, along with some basic assumption which avoided pathological behavior of the CB (for instance that the $\mathcal{V}$ is not dense in $\mathcal{C}$)$\mathbb{P}$, constraints dramatically the topology of $\mathcal{V}_{\text{rank}-2}$. Specifically we showed that $\mathcal{V}_{\text{rank}-2}\cap S^3$, where the intersection with the three sphere S^3 is taken to get rid of the contractible direction corresponding to the scaling action, is in general a (p,q) n-link with unknots [6], see fig 1.
The set of allowed (p,q) is strongly restricted by the $\text{sp}(4,\mathbb{Z})$ monodromy structure and only a finite set of values is allowed. Yet the number of components of the link $n_{(p,q)}$ is not constrained in any obvious way by the single monodromies. In particular $n_{(p,q)}$ could be infinite therefore leading to an infinite set of topologically inequivalent CB geometries. A possible restriction could come from studying the whole monodromy group, and not just individual monodromy elements. The former provide a representation of the fundamental group of the smooth part of the CB $\mathcal{C}\setminus\mathcal{V}$ and thus is sensitive to global data. The fundamental group of the spaces of interest for rank-2 was computed recently [18].

2.3. Metric vs. Complex singularities

Σ For more details on this point see the discussion in σ 5 of [2].
$\|$ The exact number of rank-1 theories has been the source of some confusions particularly since each of the summary tables in [] seem to report contradicting results. 28 is the number of Coulomb branch geometries of non-discretely gauged theories which certainly exist.
$\mathbb{P}$ The reason why we have to make this assumption, is because we feel strongly that such geometries are unphysical though we have thus far been unable to prove it [6].

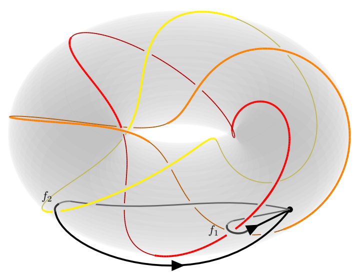

Figure 1. Depiction of an $L_{(1,2)}(0,3,0)$ link consisting of the red, orange, and yellow circles.

The singularities on $\mathcal{V}$ can occur in two types *metric* singularities, $\mathcal{V}_{\rm metric}$, and singularities in the *complex structure*, $\mathcal{V}_{\rm cplx}$. At $\mathcal{V}_{\rm metric}$, the CB as an algebraic (projective) variety is perfectly fine though the metric structure is ill-defined. $\mathcal{V}_{\rm cplx}$ is instead the set of points in which the CB is singular as an algebraic variety. The latter type were believed not to occur but counter-examples were pointed out in [19, 11].

The physical interpretation of $\mathcal{V}_{\rm metric}$ and $\mathcal{V}_{\rm cplx}$ is considerably different. $\mathcal{V}_{\rm metric}$ occur where charged states become massless. This can only happen when the BPS lower bound on their mass vanishes which restricts $\mathcal{V}_{\rm metric}$ to be complex co-dimension one in $\mathcal{C}$.

The locus of complex singularities $\mathcal{V}_{\rm cplx}$ is generically a proper subvariety of the locus of metric singularities. $\mathcal{V}_{\rm cplx}$ occur when the Coulomb branch chiral ring of the $\mathcal{N}=2$ SCFT is *not freely generated*. In this case new phenomena can take place like an apparent violation of the unitarity bound [7]. All the known cases of non-freely generated chiral rings arise by a non-trivial action of a discrete group on the CB [12, 19, 11].

2.4. Scaling dimensions of Coulomb branch coordinates

If the CB chiral ring is freely generated ($\mathcal{V}_{\rm cplx}=\varnothing$), the scaling dimension Δ_i of the CB coordinates are proportional to the $\mathrm{U}(1)_r$, r_i, charges of the CB operators of the $\mathcal{N}=2$ SCFT; $\Delta_i\propto r_i$. Superconformal representation theory does not set any constraints on these charges, so $\Delta_i\in\mathbb{R}$. A relatively elementary argument shows that these scaling dimensions are instead constrained by the low energy electromagnetic monodromy group and that they belong to a finite set of rational numbers, whose size depends on the rank r of the theory. Explicitly the allowed rational numbers for the Δ_i of a rank-r theory are [5, 9]:

$$\Delta\in\left\{\frac{n}{m}\;\middle|\; n,m\in\mathbb{N},\ 0<m\le n,\ \gcd(n,m)=1,\ \varphi(n)\le 2r\right\}$$

where $\varphi(n)$ is the Euler totient function and the maximal dimension allowed grows superlinearly with rank as $\Delta_{\max}\sim r\ln\ln r$.

If the CB chiral ring is not freely generated, the relation between scaling dimension of CB coordinates and $\mathrm{U}(1)_r$ charges of CB multiplet is less straightforward. Often (maybe always) the scaling dimensions are neither globally nor uniquely defined.

3. Outlook

3.1. Finite vs. infinite allowed geometries

One of the motivating reason behind carrying out the program of classifying $\mathcal{N}=2$ SCFTs in four dimensions, is the belief that at any given rank only a finite set of theories exist. If this belief is correct, it trivially follows that only a finite set of scale invariant CB geometries are *realized* as moduli spaces of consistent physical theories at any given rank. The surprising fact that all the CB geometries allowed in rank-1 are indeed realized, motivates instead the authors' belief that in fact only a finite number of scale invariant CB geometries are *allowed*. As mentioned above, our analysis of rank-2 scale invariant CB geometries, arrives close to show that this is the case for two complex dimensional CBs, but we fall short of showing that for any given value of (p,q) only a finite number of link components is allowed. If that was achieved it would be an important conceptual result.

It is important to also stress, that proving the existence of a finite set of scale invariant CB geometries at any given rank, is only a necessary condition for showing that the at that rank only a finite number of $\mathcal{N}=2$ SCFTs exist. In fact it is logically possible, that a given scale invariant geometry admits an infinite set of mass deformations and therefore corresponds to an infinite set of physically distinct $\mathcal{N}=2$ SCFTs.

3.2. Stratification of the Coulomb branch and importance of rank-1 theories

We have thus far said little about the structure of the singular locus $\mathcal{V}$. If we assume, again, that some pathological behaviors are avoided and in particular that $\mathcal{V}$ is a complex co-dimension one complex subvariety of $\mathcal{C}^{+}$, than it is possible to convincingly argue that $\mathcal{V}$ is itself a RSK variety of one comlpex dimension less, or more specifically the union of finite RSK varieties. This result might sound counter-intuitive. But the observation that there cannot be a transition among two inequivalent vacua at zero energy cost, implies that it has to be possible to induce a non-zero metric on $\mathcal{V}$ from the ambient space $\mathcal{C}$. This can be done by identifying an appropriate set of complex coordinates on $\mathcal{C}$, $(u^{\perp}, u^{\parallel})$, such that $\mathcal{V}$ is at $u^{\perp}=0$. The metric is induced considering the $\partial_{\parallel}\sigma$ and $\partial_{\parallel}\overline{\sigma}$ components, where σ labels the vector of special coordinates.

It is also possible to show that the entire RSK structure can be restricted to $\mathcal{V}$, thus providing a consistent lower dimensional RSK space. For more details on the argument see [6] and in particular [20] where the stratification of the singular locus is discussed in the context of $\mathcal{N}=3$ theories. This discussion, with some minor but important modifications, carries over to the $\mathcal{N}=2$ case [21, 22]. This powerful result, which parallels the structure of singularity of symplectic varieties, opens exciting perspectives of carrying out the classification of higher dimensional geometries using a sort of inductive argument from the completed rank-1 story.

Polarization and other $\mathcal{N}=4$ theory The RSK structure on $\mathcal{C}$ is richer than we have thus far discussed. In fact the $U(1)^r$ low-energy physics provides a natural integral skew-symmetric pairing on the special coordinates. This arises as follows. States in the low-energy theory are labeled by a set of $2r$ integers, their corresponding electromagnetic charges which we will collectively label Q^{*}. Denote by $\langle Q, Q'\rangle := Q^T\mathbb{D}Q'$, where $\mathbb{D}$ is an integer non-degenerate skew-symmetric $2r\times 2r$ matrix in canonical form, the pairing induced by the Dirac-Zwanziger-Schwinger quantization condition on the lattice of electromagnetic charges. Since the special coordinates are dual to the lattice of electromagnetic charges, $\mathbb{D}$ induces a pairing on them as well. $\mathbb{D}$ is called the *polarization* of the lattice of electromagnetic charges, and it has the very important property of determining the structure of the electromagnetic duality group, which is indeed $\mathrm{sp}_{\mathbb{D}}(2r,\mathbb{Z})$.

If the polarization can be brought to the canonical form $\mathbb{D}=\epsilon\otimes\mathbb{1}_r$, where ϵ is a 2×2 antisymmetric matrix, $\mathbb{D}$ is called *principal* and $\mathrm{sp}_{\mathbb{D}}(2r,\mathbb{Z})$ reduces to the more standard $\mathrm{sp}(2r,\mathbb{Z})$. Theories with non-principal polarization have not been studied in any real detail. There is a mild evidence, from the classification of rank-1 geometries, that such theories are indeed relative field theories [1, 13]. This result is very preliminary and this exciting subject certainly deserves more study.

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$^{+}$ $\mathcal{V}$ is identified by the zeros of the central charge Z_Q, where Q is the electromagnetic charge of a populated BPS state. If Z_Q were a function on $\mathcal{C}$ then it would be easy to prove that $\mathcal{V}$ is indeed a complex sub-variety. But Z_Q is indeed branched over $\mathcal{V}$ and thus the challenge in showing this result in generality.

* Recall that on $\mathcal{C}_{\rm reg}$ these states are massive, the theory there has no massless charged state.

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Hasse diagrams and Higgs branches

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1. Background

In this note, we investigate the structure of Higgs branches of supersymmetric field theories with 8 supercharges in 3, 4, 5 and 6 spacetime dimensions. Standard arguments show that the Higgs branch X of such a theory is hyperKähler, and in the absence of mass terms and Fayet-Iliopoulos terms, it is singular. Singular hyperKähler varieties are called symplectic singularities [1, 2]. They admit a partition into a finite number of so-called symplectic leaves [3, 4, 5, 6]

$$X = \mathcal{O}_1 \cup \cdots \cup \mathcal{O}_n \tag{1}$$

and these symplectic leaves are partially ordered using

$$\mathcal{O}_i \leq \mathcal{O}_j \qquad \Leftrightarrow \qquad \mathcal{O}_i \subset \overline{\mathcal{O}_j}\,. \tag{2}$$

As for any partial order, this ordering of the leaves can be represented in a graphical form using a so-called Hasse diagram, which is our main object of interest [7], see also [8, 9, 10, 11, 12].

The remainder of this note is organized as follows. In the first section, we explain what is the physical interpretation of the Hasse diagram, showing that it can be seen as partial Higgsing. In the second section, we demonstrate what methods can be used to compute these diagrams, and in the third we briefly describe some further applications.

1.1. Higgs and Hasse

For brevity and concreteness, we focus in this section on a particular example, namely the four-dimensional $\mathcal{N} = 2$ theory with gauge group SU(4) and matter content made of one second rank antisymmetric tensor and 12 fundamental hypermultiplets. The Hasse diagram for this theory is displayed in Figure 1. We now explain how this diagram is interpreted, and what is the physics underlying it.

Each black dot represents one leaf. Here we have five distinct leaves. Next to it, the integer number is the quaternionic dimension of the leaf – and throughout this note, all dimensions are quaternionic.

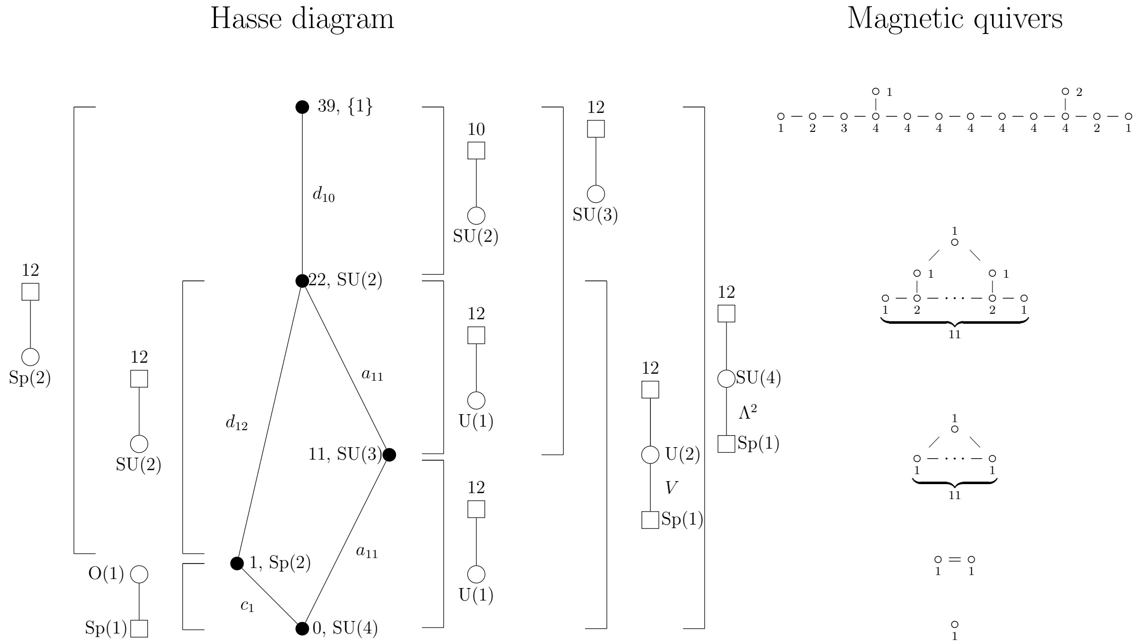


Figure 1: Hasse diagram for the SU(4) gauge theory with one second rank antisymmetric matter (denoted by Λ^2) and 12 fundamentals. For each leaf, denoted with a black dot, we give its quaternionic dimension next to it, and the associated magnetic quiver on the right. We also represent (using large brackets) electric quivers associated to every subdiagram between two points in the Hasse diagram. The V for the U(2) gauge group is a singlet of SU(2) and a vector of SO(2) ⊂ U(2).

The top leaf, of dimension 39, is dense in the Higgs branch, and consists (by definition) of the generic points on the Higgs branch. Physically, generic points are field configurations which break entirely the SU(4) gauge group. Going down along the diagram, we learn that a subset of dimension 22 of the Higgs branch is singular inside the closure of the top leaf. Physically, this corresponds to non-complete Higgsing: on that subspace, SU(4) is broken only to SU(2). Similarly, it turns out that each leaf corresponds to one subgroup of SU(4), which is the subgroup left unbroken by the hypermultiplet field configuration. This goes on until we reach the origin of the Higgs branch, which is the bottom leaf where the whole gauge group is unbroken. In the diagram, we indicate the unbroken gauge group next to the quaternionic dimension. Note that not all subgroups of SU(4) appear (for instance U(1) does not), this simply means that the matter content can only Higgs the gauge group to certain subgroups.

The Hasse diagram contains more information, located on the edges of the diagrams. Between to adjacent leaves in the partial order one can define a transverse slice. In many situations (though not all), these transverse slices belong to two families, first identified by Kraft and Procesi [5, 6]:

- Kleinian ADE singularities (of dimension 1), denoted A_n, D_n and E_n;
- Closures of minimal nilpotent orbits of simple Lie algebras, denoted similarly but using lowercase (their dimension is $h^\vee - 1$, where $h^\vee$ is the dual Coxeter number).

In our example, we learn from the diagram that the transverse slice from the top leaf to the next is the 17-dimensional closure of the minimal orbit of $\mathfrak{so}(20, \mathbb{C})$. Note that the SU(2) theory with 10 flavors has exactly this as a Higgs branch. We represent it on the diagram using quiver notation, next to the d_{10} slice. Similarly, to any pair of comparable leaves one can associate a (non necessarily unique) gauge theory which characterises the transverse slice. The Hasse diagrams for these theories are subdiagrams of the main Hasse diagram.

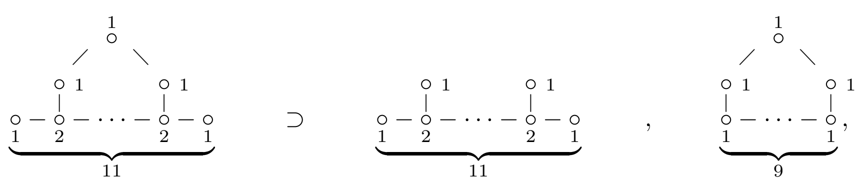
In each case, the non-abelian part of the global symmetry of a theory can be read from the lowest elementary slices. For our SU(4) for instance, the lowest elementary slices are c_1 and a_{11} giving a global symmetry including $\mathfrak{sp}(1) \oplus \mathfrak{su}(12)$.

2. State of the art

Now that we have explained what physical information can be extracted from a Hasse diagram, we turn to how it is computed. The most elementary method is the standard partial Higgsing. To find the lowest non-trivial leaves, one enumerates all the maximal subgroups of the gauge group, and check which ones can be left as residual after giving a VEV to some hypermultiplets.

2.1. Computing Hasse diagrams

The above methods only works for Lagrangian theories. More generally, one can in many cases see a symplectic singularity as the 3d Coulomb branch of a quiver with unitary gauge nodes and bifundamental matter. Such a quiver, called a *magnetic quiver* [13, 14, 15, 16, 17, 18, 19], can be seen as a combinatorial way to encode the geometry of the singularity. The algorithm of *quiver subtraction* [20, 21, 22] can then be used to derive mechanically the Hasse diagram, including the geometry of every elementary slice – elementary slices are identified as subquivers corresponding to affine Dynkin diagrams. In Figure 1 the magnetic quivers are represented, for each leaf, on the right. Let us look at an example: the magnetic quiver for the closure of the 22-dimensional leaf contains exactly two affine Dynkin diagrams



(3)

showing that two transitions d_{12} and a_{11} are possible. The quiver subtraction algorithm then produces the magnetic quivers shown in Figure 1.

Finally, when a brane realisation of the theory is available, it is possible to visualise geometrically the transitions from one leaf to another. This is the physical process underlying the quiver subtraction algorithm.

Using magnetic quivers and brane setups, one can go beyond perturbative Lagrangian theories. For instance, our SU(4) example has precisely the correct matter content to define, at strong coupling, a 6d $\mathcal{N} = (1, 0)$ SCFT. From the brane realisation, one can read that the Hasse diagram for its Higgs branch is identical to the one of Figure 1 with an additional e_8 (29 dimensional) slice on top. This is the generic effect of the small E_8 transition [23, 24] on a Higgs branch represented through its Hasse diagram : just add an e_8 transition on top.

3. Outlook

As we just saw, one of the virtues of representing Higgs branches through their Hasse diagrams is that it opens a window on non-perturbative regions of the moduli space, and gives insights about the geometry there, in a simple and precise way. In the case of the small E_8 transition, we saw that although the physics is non-trivial, the modification of the Hasse diagram is straightforward. Other situations are richer: for instance, it turns out certain generalisations of Argyres-Douglas theories have Higgs branches associated to magnetic quivers forming complete graphs [25, 26]. This generically gives rise to very intricate Hasse diagrams, unveiling a delicate structure of partial Higgsings.

The global shape of the diagram is also relevant. For some theories, the Higgs branch is a union of several cones, and in that case the Hasse diagram has several maximal leaves (a leaf is maximal if it is contained in the closure of no other leaf). This is the case for SU(N) SQCD in the appropriate range of parameters [7].

Hasse diagram constitute a new tool to study Higgs branches, and more generally symplectic singularities. Several features remain to be explored. From the mathematical perspective, although the Kraft-Procesi elementary slices seem to appear prominently, they by no means exhaust the possible elementary transitions in a Hasse diagrams [27], and a complete classification appears to be out of reach at the moment. Leaves can also admit multiplicities [28], a point that deserves further study. From the physical perspective, it would be interesting to extend our tools to theories that don't admit unitary magnetic quivers, but only ortho-symplectic magnetic quivers (equivalently, brane configurations with orientifold planes). This would expand the scope of (non-Lagrangian) theories attainable by Hasse diagrams techniques, allowing for instance to understand the relations between fixed-rank 4d $\mathcal{N} = 2$ SCFTs [29, 30, 31].

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Chiral ring stability

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1. Background

A general question in supersymmetric gauge theories is whether there are any constraints on the amount of accidental symmetry that may appear at a fixed point of the RG flow, or at special points on the moduli space of vacua of the theory.

Perhaps surprisingly, an answer to the question may come from studying the *chiral ring* of the theory, i.e. the ring of protected (BPS) gauge-invariant chiral operators, whose vev's parameterize the moduli space of vacua of the theory.

Several authors, both from the field theory perspective as well as from the holographic one, have proposed that the crucial criterion be that of checking the *stability* of this ring under certain *degenerations*.

We start with a lighting review of the chiral ring, and explain how the emergence of global symmetries in the IR of a gauge theory may be analyzed by studying this ring. We then present two methods (recently proposed in the literature) for carrying out such an analysis in practice and we discuss one example for each method.

1.1. Generalities about the chiral ring

1.2. The chiral ring

In any supersymmetric gauge theory, chiral operators $\mathcal{O}$ are those annihilated by the supercharges of one chirality, say $\overline{Q}$. A product of two chirals is still chiral. In fact we may consider equivalence classes, operators at large separations:

$$\frac{\partial}{\partial x_j} \langle \mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\cdots\mathcal{O}_{i_n}(x_n)\rangle = 0 \ \Rightarrow$$
$$\langle \mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\cdots\mathcal{O}_{i_n}(x_n)\rangle = \langle \mathcal{O}_{i_1}\rangle \langle \mathcal{O}_{i_2}\rangle \cdots \langle \mathcal{O}_{i_n}\rangle \ , \quad (1)$$

thanks to cluster decomposition. Moreover we have a simple OPE structure for the product of two such operators, $\mathcal{O}_i\mathcal{O}_j = \sum_k c_{ijk}\mathcal{O}_k$, with c_{ijk} some constants.

More generally, one can consider gauge-invariant polynomials (with complex coefficients) in the chiral operators. It is often possible to find a list of all nontrivial independent operators, which will be identified with the *generators* of the chiral ring. Here by independent we mean in different equivalence classes. However there are often relations satisfied by the generators (i.e. the ring is non-freely generated). E.g. in $4d$ $\mathcal{N} = 1$ theories, nontrivial relations follow from the tree-level superpotential, whose derivatives with respect to all chirals Φ must vanish (i.e. the EoM of the chirals must be satisfied):

$$\partial_\Phi \mathcal{W}(\Phi) = 0 \ . \quad (2)$$

Notice that these are *classical* relations, which may receive corrections in the quantum theory. That is, we expect every classical relation to correspond to a quantum one, which however may read different.

Therefore the quantum chiral ring $\mathcal{R}$ has the following conjectured structure:

$$\mathcal{R} = [\mathcal{O}_1, \ldots, \mathcal{O}_n]/I \ , \quad (3)$$

where $\mathcal{O}_1, \ldots, \mathcal{O}_n$ are the generators and I is the ideal of the relations satisfied by them.

A second crucial observation is that (polynomials in) the vev's of gauge-invariant chirals (the generators) are holomorphic functions (generators of the ring of holomorphic functions) on the moduli space $\mathcal{M}$ of vacua of the gauge theory (which may receive quantum corrections) [3, 4], which has the structure of an algebraic variety (possibly singular) [5].

The coordinate ring of the algebraic variety $\mathcal{M}$ may thus be identified with the chiral ring $\mathcal{R}$ of the theory. The vev's of the generators furnish coordinates on $\mathcal{M}$, which satisfy relations as dictated by I. Typically the relations cut out a (possibly singular) hypersurface inside ^{n}; e.g. if $I = (p)$, the ideal of a single polynomial in n variables, then $\mathcal{M} : p = 0 \subset^n$.

1.1.2. Emergence of symmetry and stability A crucial question one may want to ask is when a chiral ring is the chiral ring of a superconformal field theory (SCFT), namely whether flowing to an IR fixed point of the gauge theory modifies its chiral ring. This puts more constraints on the structure of the ring, as a (say) $4d$ $\mathcal{N} = 1$ SCFT has a distinguished $(1)_R$ symmetry, called superconformal R-symmetry, under which every chiral $\mathcal{O}$ satisfies

$$\Delta_\mathcal{O} = \frac{d-1}{2} R_\mathcal{O} = \frac{3}{2} R_\mathcal{O} \ , \quad (4)$$

with Δ its (conformal) dimension and R the charge under this $(1)_R$. Therefore the ring of an SCFT must be *graded*.

This distinguished $(1)_R$ may be identified with a specific linear combination of all global (1)'s visible in the UV via a-maximization [6]. Given the well-known relation between the a central charge and the R-symmetry 't Hooft anomalies [7], one practically maximizes a as a function of the trial R-charges R_Φ of all operators Φ appearing in the superpotential $\mathcal{W}(\Phi)$ of the theory. The R-charges are thus constrained in two ways: by the requirement that $R_\mathcal{W} = 2$ (which simply follows from supersymmetry) and that all NSVZ gauge couplings beta functions vanish at the fixed point.

Notice that the procedure implicitly assumes that *all* global (1)'s are already visible in the UV, and no other symmetry is expected to *emerge* at the IR fixed point. However there are situations in which the assumption is violated, such as when:

- an operator hits the *unitarity bound* (which requires $\Delta_{\mathcal{O}} \geq 1$ or $R_{\mathcal{O}} \geq \frac{2}{3}$ in $4d$); then it is argued to become free, and there is an "accidental" (1) symmetry acting only on it;
- some superpotential term is irrelevant at the IR fixed point, in which case some of the constraints imposed on the R-charges R_Φ from $R_{\mathcal{W}} = 2$ may not be valid.

In order to face such situations, the authors of [8] have proposed a practical criterion to tell when the chiral ring of a gauge theory is the same as the one at the SCFT point. The criterion is known as *K-stability*, and makes use of the fact that $\mathcal{M}$ is an algebraic variety.

1.1.3. Violations of unitarity bounds and flipping fields We would now like to explain how one can study the properties of the chiral ring in theories displaying unitarity bound violations. Usually when an operator $\mathcal{O}$ violates the unitarity bound, it is customary to compute the a, c central charges by subtracting the contribution of a chiral field with R-charge $R_{\mathcal{O}}$. Similarly, partition functions are computed by dividing by the contribution of a chiral field with R-charge $R_{\mathcal{O}}$.

With this recipe alone, it is not immediately clear what is the effect on the theory of removing "by hand" an operator: how are the chiral ring relations affected? Is it possible to obtain a lagrangian description of the interacting sector alone if the UV theory does have one? The way to answer these questions is the following [9]:
For each chiral ring operator $\mathcal{O}$ with $R_{\mathcal{O}} \leq R_{bound}$, add to the theory a gauge invariant chiral multiplet $\beta_{\mathcal{O}}$, and add to the superpotential a "flipping" term $\delta\mathcal{W} = \beta_{\mathcal{O}} \cdot \mathcal{O}$.
This formulation of the theory is complete, the F-terms of $\beta_{\mathcal{O}}$ imply that $\mathcal{O}$ is zero in the chiral ring, so there are no unitarity violations, moreover all standard computations can be performed.

As for the computations which are possible just stating that $\mathcal{O}$ is decoupled, our procedure is equivalent, as we now show. As we have said before, the $4d$ a and c central charges are known to be linear combinations of the cubic 't Hooft anomalies $\mathrm{tr}(R)$ and $\mathrm{tr}(R^3)$. Since $R_{\beta_{\mathcal{O}}} = 2 - R_{\mathcal{O}}$, adding to $\mathrm{tr}(R^3)$ (or to $\mathrm{tr}(R)$) the contribution of $\beta_{\mathcal{O}}$ is equivalent to subtracting the contribution of $\mathcal{O}$. So $\mathrm{tr}(R^3), \mathrm{tr}(R), c$ and a are the same. This is also true at the level of trial central charge, when performing a-maximization, so the prescription proposed in [10] to implement the decoupling gives the same result.

Similarly, the $4d$ superconformal index obtained with our prescription is the same: denoting by $\Gamma_{\mathrm{ell}}(r)$ the contribution of a chiral field with R-charge r, we have $\Gamma_{\mathrm{ell}}(r)\Gamma_{\mathrm{ell}}(2-r) = 1$. So adding the contribution of $\beta_{\mathcal{O}}$, i.e. multiplying the index by $\Gamma_{\mathrm{ell}}(R_{\beta_{\mathcal{O}}})$, is equivalent to removing the contribution of $\mathcal{O}$, i.e. dividing by $\Gamma_{\mathrm{ell}}(R_{\mathcal{O}})$. Analogous arguments hold for $3d$ superconformal indices or S^3 partition functions.

2. State of the art

Let us focus on $4d$ $\mathcal{N} = 1$ theories for concreteness.

2.1. K-stability

The idea behind *K-stability* is rather simple: instead of maximizing a with respect to just the "visible" (1)'s, we should maximize with respect to other (1)'s corresponding to certain *degenerations* of the chiral ring. Let us spell out what this all means.

The authors of [8] realize the gauge theory as (an appropriate decoupling limit of) the worldvolume theory of N D3-branes probing a non-compact threefold Gorenstein singularity X_6, i.e. the ten-dimensional spacetime of type IIB is $^{1,3} \times X_6$. It is well-known that for $N = 1$ the threefold X_6 coincides with the moduli space $\mathcal{M}$ of the $\mathcal{N} = 1$ gauge theory (whereas for $N > 1$ the moduli space should be given by the N-fold symmetric product $\mathrm{Sym}^N X_6$). K-stability of this space is a mathematical condition [11] insuring that the singularity actually admits a conical Ricci-flat metric; in the compact setup this would be called a conical Calabi–Yau (CY) space, $X_6 = C(L_5)$. Admitting a conical CY metric is in fact equivalent to the five-dimensional base L_5 of the cone admitting a Sasaki–Einstein metric: in the appropriate holographic limit the ten-dimensional spacetime becomes $\mathrm{AdS}_5 \times L_5$, and according to AdS/CFT this background is dual to an SCFT. Thus we expect the moduli space coordinate ring to be the chiral ring of a superconformal theory. The superconformal R-symmetry $(1)_R$ corresponds to the so-called Reeb vector on L_5, i.e. a vector generating a rescaling isometry.

In the simplest hypersurface case, that is when $\mathcal{M} : p(x, y, z, t) = 0 \subset^4$ for some polynomial p, the coordinates (x, y, z, t) of ^{4} will be provided by four generators of the chiral ring, and the hypersurface $p = 0$ encodes the relation satisfied by them (i.e. there is a single polynomial in the ideal I).

To check whether the space $\mathcal{M}$ is K-stable, i.e. admits a conical Ricci-flat metric (allowing us to apply the usual AdS/CFT logic), one has to consider special degenerations of the polynomial p known as *T-equivariant test configurations*, and compute the *Futaki invariant* of the degenerations. If this numerical invariant is strictly positive for all degenerations (which in favorable situations are finite in number, and can be explicitly constructed) then the space is said to be K-stable and admits a conical Ricci-flat metric [11]. The procedure is a “generalized volume minimization” (in a “generalized” space of Reeb vectors) which by AdS/CFT [12, 13] is related to a “generalized” a-maximization. In fact the degenerations of p are associated to (1) actions that are not symmetries (of the field theory) except in certain limits; thus K-stability gives a very concrete realization to the idea of emergent (1) symmetries. (For detailed definitions and calculations see [14].)

The claim is then that the chiral ring $\mathcal{R}$ of a theory can be that of an SCFT (i.e. is *stable*) if the associated algebraic variety $\mathcal{M}$ is K-stable. This physical interpretation of the K-stability criterion was first given in [8] and later clarified and expanded in [14, 15].

As an application to a well-known example, consider the class of singular Gorenstein threefolds defined by the following hypersurface equation:

$$\mathcal{M} : \; x^2 + y^2 + z^p + t^q = 0 \; \subset \; ^4 \; ; \quad p, q \geq 2 \; . \tag{5}$$

In this case all deformations can be classified, and the numerical Futaki invariants computed, showing that the space is K-stable iff $1/2 < p/q < 2$ [11]. For instance, when $p = 2$ we must either have $q = 2$ or $q = 3$ to insure K-stability of $\mathcal{M}$ (hence the stability of $\mathcal{R}$). The first case realizes the well-known conifold, whose base $L_5 = T^{1,1}$, and the superconformal model dual to this was famously studied in [16].‡

2.2. Chiral ring stability

Here we would like to discuss another notion of stability of the chiral ring proposed in [9], which is intimately related to the presence of emergent symmetries at the IR fixed point. The idea is to provide a criterion which allows us to understand when a superpotential term should be dropped, regardless of whether it is relevant or not.

First, let us state the criterion. Starting from a theory $\mathcal{T}$ with superpotential $\mathcal{W}_\mathcal{T} = \sum_i \mathcal{W}_i$ (where each term $\mathcal{W}_i$ is gauge invariant), one needs, for each i, to:

- consider the modified theory $\mathcal{T}_i$, where the term $\mathcal{W}_i$ is removed from $\mathcal{W}$;
- check if the operator $\mathcal{W}_i$ is in the chiral ring of $\mathcal{T}_i$.

If one of the terms $\mathcal{W}_i$ is not in the chiral ring of $\mathcal{T}_i$, it must be discarded from the full superpotential $\mathcal{W}_\mathcal{T}$. Notice that we are not requiring that $\mathcal{W}_i$ is a relevant deformation of $\mathcal{T}_i$, only that it is in the chiral ring of $\mathcal{T}_i$.

If a superpotential does not satisfy the above criterion, problematic terms should be dropped (which means that they belong to the Kähler, unprotected, part of the Lagrangian), since keeping them when computing protected quantities generally leads to wrong results.

For the purpose of illustration, let us discuss a simple example in $3d$: Consider $\mathcal{N} = 2$ SQED with one flavor and zero superpotential. As is well known [17], this theory is infrared dual to a Wess–Zumino

‡ For the $q = 3$ case see [14].

model (called XYZ model) with three chiral fields and a cubic superpotential $\mathcal{W} = XYZ$. The three chirals are dual to the meson and the monopole operators $M^{\pm}$ respectively in the gauge theory. From this dual description we immediately see that the monopole operators have R-charge 2/3 at the infrared fixed point and their product $M^{+}M^{-}$ vanishes in the chiral ring due to the F-terms in the XYZ model. On the gauge theory side the latter statement is due to non-perturbative effects.

Let us now modify the theory by adding to SQED a chiral multiplet α and turning on the superpotential term $\mathcal{W} = \alpha\, M^{+}M^{-}$. Naively one would argue as follows: without the superpotential term the multiplet α is free and therefore its R-charge is 1/2. Since the R-charge of the monopoles is 2/3, we conclude that the added superpotential term is relevant and therefore the deformation triggers a nontrivial RG flow to another fixed point. This conclusion is incorrect according to our chiral ring stability criterion: since the product of the monopole operators vanishes in the theory without a superpotential, the added superpotential term is unstable and should be dropped. We are therefore left with SQED and a decoupled free chiral multiplet. This conclusion is indeed confirmed in the dual XYZ description: the superpotential of the theory with the deformation included reads

$$\mathcal{W} = XYZ + \alpha YZ = (X + \alpha)YZ \ . \tag{6}$$

With the simple field redefinition

$$X' \equiv X + \alpha \ , \quad \alpha' \equiv X - \alpha \ , \tag{7}$$

we clearly see that the multiplet α' decouples from the theory and we are left with the XYZ model plus a free chiral, in agreement with our criterion.

It would be interesting to obtain a first principle derivation of the above criterion.

3. Outlook

We have highlighted the necessity of a better understanding of the structure of the chiral ring of a gauge theory, in order to infer properties of its infrared fixed point.

We have expounded two criteria (K-stability and chiral ring stability) which have been proposed in the literature to test whether the chiral ring of a gauge theory is *stable*, i.e. whether it describes the chiral ring of an SCFT in the IR of the gauge theory. Both methods provide a concrete way to test this stability, produce correct results whenever expected, and new examples as well. In particular the second method is readily applicable in $3d$, and K-stability can be rather easily extended to the $3d$ setup too. Therefore it would be interesting to apply this latter technique to $3d$ $\mathcal{N} = 2$ theories.

The intimate relation between the two methods is not presently clear however, and clearly begs for further investigations.

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Comments on Short Multiplets in Superconformal Algebras

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1. Background

The problem of classifying all short multiplets of superconformal algebras still seems to be an open question. A generic short multiplet is non-unitary, which nevertheless is of interest in various contexts. Even if one is interested in unitarity theories only, non-unitary short multiplets are of use in the analysis of (super)conformal blocks. The classification problem is mathematically formulated in terms of the representation theory of parabolic Verma modules, whose theory is known to be more challenging than that of more standard Verma modules associated with the Borel subalgebra. We comment on some recent developments of the representation theory, which could be of help in solving the classification problem

On the occasion of the Pollica Summer Workshop "Mathematical and Geometric Tools for Conformal Field Theories", the organizers have asked me to write up a very short note on an open question or a research direction. I have taken advantage of this opportunity to highlight the following problem, which seems to be little appreciated by the community. It is indeed surprising that the following problem is still unsolved:‡

Classification Problem

Classify all possible short multiplets (i.e. highest-weight irreducible representations) of (super)conformal algebras, in general ($D \geq 3$) spacetime dimensions.

Isn't This Well-Known Already?

The most likely reaction to the above-mentioned problem is that the answer to this problem should have been known since decades ago. I myself have long thought that this should be the case. However, almost all of the existing literature on the representation theory of (super)conformal algebras (e.g. [2–12]) have focused on *unitary* representations of the (super)conformal algebras. In general,

‡ The problem posed here seems to have been solved for the particular case of four-dimensional $\mathcal{N} = 1$ supersymmetry in a nice paper by Li and Stergiou [1]. The author would like to thank Andreas Stergiou for bring this paper to author's attention.

there are many more *non-unitary* irreducible representations, and a generic irreducible representation is non-unitary.§

I hope the importance of this question is obvious to readers. In the case of the two-dimensional conformal algebra (i.e. Virasoro algebra), one arrives at the celebrated minimal models by studying irreducible representations [14]. These models subsequently have been applied to universality classes for various systems in high energy theory, condensed matter theory and statistical mechanics. It is in my opinion rather natural to ask a similar question in higher dimensions.

While most of the short multiplets are non-unitary, non-unitary theories could still be applied to many problems in physics and mathematics, e.g. statistical mechanics. Even if one is interested only in unitary theories, non-unitary representations do appear in the analysis of (super)conformal blocks, which are crucial inputs for the modern bootstrap program [15]. Namely, we can study the superconformal blocks as an analytic complex function of the scaling dimension of the intermediate operator, and many of the poles of the block corresponds to non-unitary short multiplets (see the literature on the recursion relation for conformal blocks [16–19]).

2. State of the art

One can easily think of a straightforward approach for the problem. Again, we can learn from the study of the two-dimensional Virasoro algebra: we can study a Verma module, which is a universal module for a highest-weight state representation. We can then study when the module is irreducible (with the help of the Kac determinant formula), and we can decompose the Verma module into irreducible components.

Indeed, the study of the Verma module of a finite-dimensional semisimple Lie algebra is well-studied subject in mathematics (see e.g. the book [20] for a survey).

However, the situation is complicated by the fact that what is relevant for a physics of (super)conformal algebra is not the ordinary Verma module, but rather its variant, the parabolic Verma module (introduced first in [21] in mathematics literature). The latter is associated with a parabolic subalgebra, whereas the former to the Borel subalgebra: for this reason let us call an 'ordinary' Verma module a 'Borel' Verma module. This point concerning parabolic and Borel Verma modules seems to be known to some experts since long ago, except not emphasized enough in the literature; see [22] for a recent discussion.

One might think that the distinction between a Borel Verma module and a parabolic Verma module is minor. However, it is known in the mathematical literature there is a subtle difference between the representation theories between the two, and a complete understanding of this subtlety is still out of reach (see chapter 9 of [20] for a good summary and e.g. [21,23,24] for some early references).‖ For this reason the study of representation theory of parabolic Verma modules is still an active area of research.¶

§ For example, a free massless scalar field in flat space satisfies the Klein-Gordon Equation $\Box\phi = 0$. There is a generalization of this equation to $\Box^{k\geq 2}\phi = 0$ (see [13] for recent discussion), which describes a non-unitary theory.

‖ Here is a more technical explanation for mathematically-included readers, for illustration of the subtlety. Let us denote the Borel Verma module with highest weight λ by $M(\lambda)$, and its parabolic counterpart, associated with a parabolic subalgebra $\mathfrak{p}$, by $M_{\mathfrak{p}}(\lambda)$ (we here follow the notations of [22]). When $M(\lambda)$ is reducible at location μ there is a homomorphism $M(\mu) \to M(\lambda)$. This naturally induces a homomorphism $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ (the so-called *standard* homomorphism), and hence one is tempted to conclude that $M_{\mathfrak{p}}(\lambda)$ is also reducible. However, it is not guaranteed in general that this morphism is non-zero. It is also the case that not all the homomorphisms of $M_{\mathfrak{p}}(\mu) \to M_{\mathfrak{p}}(\lambda)$ have their counterparts $M(\mu) \to M(\lambda)$; such homomorphisms are called *non-standard*. In other words, irreducibility of $M(\lambda)$ is neither sufficient nor necessary for the irreducibility of the corresponding $M_{\mathfrak{p}}(\lambda)$

¶ The irreducible decomposition of the parabolic Verma module is already 'known' in the following sense. Suppose that one knows the decomposition of the Borel Verma module $M(\lambda)$ into irreducible

As in the case of the Virasoro algebra, irreducibility criterion of parabolic Verma modules is provided by the determinant formula (an analog of the Kac determinant formula for the Virasoro algebra). The precise expression for the determinant formula was derived long ago by a mathematician Jantzen in 1977 [25], however little attention has been paid to his work in the physics literature. For a superconformal algebra, the relevant determinant formula was conjectured in [22] and was later proven only recently in [26] by Oshima and the author (which generalizes the previous work by Gorelik and Kac [27]). The general irreducibility criterion of [26] was analyzed more concretely in [28].[+]

Of course, working out the irreducibility criterion is only part of the problem, and one still needs to first find the decompositions into irreducible representations, and find explicit characterization for each of the irreducible modules. Even for non-supersymmetric conformal algebras explicit and comprehensive description of the null states is recent [17, 19], and one can ask if we can complete the program for general superconformal algebras. Such a result will also be of great use in writing down a recursion relation for superconformal blocks along the lines of [17].

3. Outlook

In summary, despite various developments over the past decades, the classification problem of short multiplets of superconformal algebras is still an open problem, either mathematically or physically. Since this is a well-defined question with many potential applications, I would like to challenge (super)conformal-field-theory and/or representation-theory aficionados to settle this question once and for all. If anyone becomes interested in this problem and starts thinking about it after reading this article, then I could say that this article already served it purpose.

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modules $L(\mu)$: $M(\lambda) = \sum_{\mu \prec \lambda} c^{\mu}{}_{\lambda} L(\mu)$. On the other hand, one has the relation for the characters between Borel and parabolic Verma modules (see [25, Lemma 1] and [26, Lemma 2]): $\mathrm{ch}\, M_{\mathfrak{p}}(\lambda) = \sum_{w \in W_{\mathfrak{l}}} \det(w)\, \mathrm{ch}\, M(w.\lambda)$, where $W_{\mathfrak{l}}$ is a Weyl group of a reductive subalgebra $\mathfrak{l}$ defined in [26]. By combining the two equations, one obtains the decomposition of the parabolic Verma module $M_{\mathfrak{p}}$ into irreducible components. However, this approach involves the sum of the Weyl group $W_{\mathfrak{l}}$, and does not seem to be too efficient when the rank of the Weyl group is large. The author would like to thank Hisayoshi Matsumoto for discussion on this point.

[+] Note that irreducibility criterion of [26] for superconformal algebras is different from the the so-called "Kac criterion" [29]—the former applies to parabolic Verma modules, while the latter to Borel Verma modules. This point is worth emphasizing since several papers on representation theories on superconformal algebras, such as [5–8], refer to the Kac criterion, whose applicability to a physical (super)conformal algebra is not clear for the reason mentioned above. Note that the relevance of Jantzen's criterion is certainly known to some mathematicians—for example, the paper [30] who mathematically classified unitarity irreducible representations of the conformal algebra does refer to the Jantzen's criterion.

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Conformal Data for Defect CFTs

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1. Background

We summarize results on one- and two-point functions for certain defect CFTs based on $\mathcal{N}=4$ Super Yang-Mills Theory, the hope being that these might serve as input for the boundary conformal bootstrap program.

1.1. The dCFT set-up

The boundary conformal bootstrap program for defect CFT's based on $\mathcal{N}=4$ super Yang-Mills theory could profit from the underlying integrability of the bulk field theory and for certain special cases from the integrability of the defect theory as well. In the following we will report on correlation functions in a defect set-up where the defect can be represented by an integrable boundary state. In the field theory language the defect takes the form of a co-dimension one domain wall separating two regions of spacetime ($x_3 > 0$ and $x_3 \leq 0$) where the gauge group of the theory is respectively $U(N)$ and $U(N-k)$ and where the $U(N)$ invariant theory has non-trivial vevs for three out of the six scalar fields. More precisely,

$$\langle \phi_i \rangle_{\text{tree}} = \phi_i^{cl} = -\frac{1}{x_3}\, t_i^{(k)} \oplus 0_{(N-k)\times(N-k)}, \qquad x_3 > 0, \qquad i = 1,2,3, \tag{1}$$

where the $t_i^{(k)}$ are the generators of the k-dimensional irreducible representation of $\mathfrak{su}(2)$ [1, 2]. This set-up conserves half of the supersymmetries of $\mathcal{N}=4$ SYM and has symmetry group $\mathfrak{osp}(4|4)$. The extra bulk fields in the region $x_3 > 0$ turn into fundamental fields living on the defect as $x_3 \to 0_+$ [3, 4]. We note that this is different from the $k=0$ case studied in [5] where the fundamental fields on the defect are independent of the bulk fields. For $k \geq 2$, one-point functions of scalar operators are non-zero already at tree level and reveal an integrable structure at any loop order [6, 7, 8, 9]. The $k=1$ case is an interesting intermediate case with zero vevs but non-trivial boundary conditions on the fields [3, 10, 4]. This case is integrable as well [11, 12].

The defect CFT has a dual string theoretical description where a single probe D5-brane intersects a stack of N D3-branes with k of the D3-branes dissolving into the D5-brane as $x_3 \to 0_+$ [1, 2].

2. State of the Art

2.1. Integrable one-point functions

In the defect CFT described above conformal operators can have non-vanishing one-point functions which are bound to take the form

$$\langle \mathcal{O}_\Delta(x) \rangle = \frac{C}{x_3^\Delta}, \tag{2}$$

where Δ is the conformal dimension. In the following, we will restrict ourselves to considering the case $x_3 > 0$. Furthermore, for simplicity, let us consider an $SU(2)$ sub-sector of the theory consisting of conformal operators constructed from two complex scalars $Z = \phi_1 + i\phi_4$, $X = \phi_2 + i\phi_5$. Such an operator can be characterized as a Bethe eigenstate $|\mathbf{u}\rangle$ of the integrable Heisenberg spin chain and described in terms of a set of rapidities $\mathbf{u}$ [13]. The corresponding one-point function is conveniently described as the overlap of the Bethe eigenstate with a matrix product state [14, 15]

$$C_k = \frac{\langle \mathrm{MPS}_k|\mathbf{u}\rangle}{\sqrt{\langle \mathbf{u}|\mathbf{u}\rangle}}, \qquad |\mathrm{MPS}_k\rangle = \mathrm{tr} \prod_{n=1}^{L} \Big[t_1^{(k)} \otimes |\uparrow\rangle_n + t_2^{(k)} \otimes |\downarrow\rangle_n \Big], \tag{3}$$

where L is the length of the operator or spin chain state. Explicit calculations show that the one-point functions for $k \geq 2$, up to a trivial field theoretical pre-factor, can be expressed in a closed form as [14, 15]

$$C_k = T_k(0) \sqrt{\frac{Q(\frac{i}{2})Q(0)}{Q^2(\frac{ik}{2})}} \sqrt{\frac{\det G_+}{\det G_-}}, \tag{4}$$

where T_k is the transfer matrix of the Heisenberg spin chain in the k-dimensional representation, the Q's are Baxter polynomials and the matrices $G_\pm$ are related to the Gaudin norm of the Bethe state in question, $\langle \mathbf{u}|\mathbf{u}\rangle \propto \det G_+ \det G_-$. For details we refer to [14, 15]. The formula (4) can be generalized to the full scalar $SO(6)$ sector [16, 17] and likewise has a beautiful extension to one-loop order where the rapidities are replaced by loop corrected rapidities and the result (4) simply multiplied by the flux factor [6]

$$\mathbb{F}_k = 1 + g^2 \Big[\Psi(\tfrac{k+1}{2}) + \gamma_E - \log 2 \Big] \Delta^{(1)} + O(g^4)\,, \tag{5}$$

with $\Delta^{(1)}$ being the one-loop correction to the scaling dimension of the operator in question. We note that setting up the perturbative framework for the defect CFT is highly non-trivial and requires the diagonalization of a mass matrix which, due to the vevs, couples flavour as well as colour components of the various fields. Furthermore, the space-time dependence of the vevs implies that the relevant propagators are propagators in an auxiliary AdS_4 space which has the defect as its boundary. Finally, a careful choice of regularization procedure is necessary to preserve supersymmetry [18, 19].

Recently, it has been possible to extend the result for the one-point functions asymptotically to all loop orders and to all sectors of $\mathcal{N} = 4$ SYM by determining via bootstrap arguments an integrable boundary reflection matrix which represents the defect and which in combination with the bulk R-matrix fulfills the boundary Yang-Baxter equations [7, 8, 9]. In this approach the flux factor originates from a boundary dressing phase. It has furthermore been understood that the $k = 1$ result can be obtained by analytical continuation from the $k \geq 2$ result. As an illustration we give the result for the leading order contribution to the one-point function for $k = 1$ for the full $\mathcal{N} = 4$ Super Yang-Mills theory [9, 12]

$$C_1 = \sqrt{\frac{Q_2(0)Q_3(0)Q_4(0)Q_5(0)Q_6(0)}{Q_1(0)Q_1(\frac{i}{2})Q_3(\frac{i}{2})Q_4(\frac{i}{2})Q_5(\frac{i}{2})Q_7(0)Q_7(\frac{i}{2})}} \sqrt{\frac{\det G_+}{\det G_-}}. \tag{6}$$

Here, the indices on the Baxter polynomials refer to the numbering of the seven nodes in so-called Beauty Dynkin diagram of the super Lie algebra $\mathfrak{psu}(2,2|4)$.

2.2. Two-point functions

Defect CFTs allow for novel types of two-point functions compared to ordinary CFTs, namely two-point functions involving bulk operators with different conformal dimensions and two-point functions involving one bulk and one defect field. Having set up the perturbative framework in the bulk we have easy access to the former type of correlation functions. As an example, let us give the leading order contribution to the two-point function of two simple chiral primaries of different conformal weights [10]

$$\left\langle \mathrm{tr} Z^{J_1}(x)\, \mathrm{tr} \bar{Z}^{J_2}(y) \right\rangle_c = \frac{g_{\mathrm{YM}}^2}{16\pi^2} \frac{J_1}{x_3^{J_1}} \frac{J_2}{y_3^{J_2}} \sum_{\ell=0}^{\min\{J_1,J_2,k\}} \frac{\alpha_\ell^{J_1-1}\alpha_\ell^{J_2-1}}{\binom{2\ell+1}{\ell+1}} \frac{{}_2F_1(\ell,\ell+1;2\ell+2;-\xi^{-1})}{\xi^{\ell+1}}. \tag{7}$$

Here, the subscript c refers to the connected part, ξ is the conformal cross ratio,

$$\xi = \frac{|x-y|^2}{4x_3y_3}, \tag{8}$$

and the α's are some numerical constants which can be determined recursively [10]. Some additional examples of two-point functions where one operator has length two can be found in [20]. From the knowledge of the one- and two-point functions of the defect theory it is possible by means of the bulk OPE to extract information about the three-point functions of the theory without the defect. The two-point functions furthermore contain information about the bulk-to-boundary couplings via the boundary OPE. In [10] some examples of simple minded data mining along these lines were presented. Needless to say that our perturbative framework would allow us to calculate any other two-point function as well.

The string theory dual to the defect CFT becomes tractable in a supergravity approximation when the planar limit is supplemented by the the following double scaling limit [21]

$$\lambda \to \infty, \qquad k \to \infty, \qquad \frac{\lambda}{k^2} \quad \text{fixed}, \tag{9}$$

and our perturbative framework has made possible a comparison between string theory and gauge theory one-point functions to two leading orders in the double scaling parameter, λ/k^2 with a resulting complete match [18, 19]. In the same spirit it might be possible to match two-point functions between the two theories and we therefore list the leading k behaviour of some selected two-point functions

$$\left\langle \mathrm{tr}\, Z^{J_1}(x), \mathrm{tr}\, \bar{Z}^{J_2}(y) \right\rangle_c = \frac{\lambda}{16\pi^2} \frac{1}{N} \left(\frac{k}{2}\right)^{J_1+J_2-1} \frac{1}{x_3^{J_1} y_3^{J_2}} \frac{2\xi+1}{(\xi+1)\xi^2}, \qquad \text{as } k \to \infty, \tag{10}$$

$$\left\langle \mathrm{tr}\, Z^{J_1}(x)\ \mathrm{tr}\, Z^{J_2}(y) \right\rangle_c = \frac{\lambda}{16\pi^2} \frac{1}{N} \left(\frac{k}{2}\right)^{J_1+J_2-1} \frac{1}{x_3^{J_1} y_3^{J_2}} \frac{2\xi+1}{(\xi+1)^2\xi}, \qquad \text{as } k \to \infty, \tag{11}$$

$$\left\langle \mathrm{tr}\, Z^{J_1}(x)\ \mathrm{tr}\, X^{J_2}(y) \right\rangle_c = -\frac{\lambda}{16\pi^2} \frac{1}{N} \left(\frac{k}{2}\right)^{J_1+J_2-1} \frac{1}{x_3^{J_1} y_3^{J_2}} \frac{4}{(2\xi+1)^2}, \qquad \text{as } k \to \infty. \tag{12}$$

3. Outlook

It is our hope that the exact results for correlation functions in defect CFTs presented above could provide the seed for a cross-fertilization between the integrability and the boundary conformal bootstrap program. There exist two other rather similar but non-supersymmetric defect set-ups based on $\mathcal{N}=4$ SYM with non-trivial vevs. In one case five out of the six scalar fields have vevs whose commutators constitute an irreducible representation of $\mathfrak{so}(5)$ whereas in the other case all six scalar fields have vevs constituting two independent representations of $\mathfrak{su}(2)$ [22, 23, 24]. These set-ups correspond to two different D3-D7 probe brane configurations. The $SO(3)\times SO(3)$ symmetric case is not integrable at tree level [25], but the the SO(5) symmetric case is [17], and a closed formula for tree level one-point functions has been derived [26], providing a wealth of new data for this defect CFT. Whether or not the integrability extends to higher loop orders constitutes an important open problem.

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Half-BPS line operator algebras in 3d

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1. Background

In this short note, we give a non-technical introduction to a simple problem in supersymmetric theories with only 4 supercharges, which can be stated as follows: *Compute the algebra of half-BPS operators of codimension 2 in a supersymmetric QFT with four supercharges*—in particular, the algebra of line operators in 3d $\mathcal{N} = 2$ supersymmetric theories.

While some of the basic structure is well understood, a lot more remains to be studied.

1.1. The twisted chiral ring

Consider a supersymmetric theory with only four supercharges in $\mathbb{R}^d$. Such theories exist only for $d \leq 4$. Starting with minimal supersymmetry ($\mathcal{N} = 1$) in four dimensions, we can formally reduce on a circle to obtain 3d $\mathcal{N} = 2$ supersymmetry on $\mathbb{R}^3$, and further reduce on another circle to obtain 2d $\mathcal{N} = (2, 2)$ supersymmetry on $\mathbb{R}^2$. The 2d $\mathcal{N} = (2, 2)$ supersymmetric algebra takes the form:

$$\begin{aligned} \{Q_-, \bar{Q}_-\} &= H + P \ , \qquad & \{Q_+, \bar{Q}_+\} &= H - P \ , \\ \{Q_-, \bar{Q}_+\} &= Z \ , \qquad & \{Q_+, \bar{Q}_-\} &= \bar{Z} \ , \end{aligned} \tag{1}$$

with Z a complex central charge. For 3d $\mathcal{N} = 2$ theories on a circle, we have the same algebra but with $Z = Z_{\text{3d}} + iP_3$, with Z_{3d} the real central charge in 3d and P_3 the momentum along S^1. Similarly, we have $Z = P_4 + iP_3$ for a 4d $\mathcal{N} = 1$ theory on $\mathbb{R}^2 \times T^2$.

We are interested in the so-called *twisted chiral operators*, namely the half-BPS operators that commute with Q_- and $\bar{Q}_+$ [1]. In d dimensions, such operators are necessarily codimension-two operators, as the twisted chiral condition only preserves an $SO(2) \times SO(d-2)$ subset of the (Euclidean) Lorentz group $SO(d)$: they are local operators in 2d, line operators in 3d, and surface operators in 4d. We will focus on the 3d case, but the 4d case is certainly very interesting too (and rather less understood). Thus, from now on, let us consider half-BPS line operators, L, such that:

$$[Q_-, L] = 0 \ , \qquad [\bar{Q}_+, L] = 0 \ . \tag{2}$$

It follows from the supersymmetry algebra (1) that the lines must lie along the "third direction," namely the circle in $\mathbb{R}^2 \times S^1$. They are local operators from the point of view of $\mathbb{R}^2$. We then consider the associated twisted chiral ring, obtained by going to the simultaneous cohomology of Q_- and $\bar{Q}_+$.

Up to Q-exact terms, the OPE of two parallel lines (wrapped along S^1) is non-singular. We have a ring structure:

$$L_i L_j = \mathcal{N}_{ij}{}^k(y)\, L_k \ , \tag{3}$$

where the structure constants $\mathcal{N}_{ij}{}^k$ may themselves depend on various parameters, denoted by y, of the 3d supersymmetric QFT. This class of algebras includes, as a special case, the celebrated Verlinde algebra, seen as the algebra of topological (Wilson) lines in a pure Chern-Simons theory [2, 3, 4].

2. State of the art

There is a standard method to isolate the relevant twisted chiral ring physics from the full 3d QFT.

2.1. Computing the algebra of lines: the 3d A-model

We perform a topolological A-twist along $\mathbb{R}^2$ [5], which amounts to replacing the 2d spin S_0 by $S_0 + \frac{1}{2}R$, with R the vector-like R-charge, $U(1)_R$. Since $Q_\pm$ have spin $\mp\frac{1}{2}$ and R-charge -1, Q_- becomes a 2d scalar after the topological twist, and similarly for $\bar{Q}_+$. Passing to the cohomology of $Q_A \equiv Q_- + \bar{Q}_+$ (or, more generally, to the simultaneous cohomology of Q_- and $\bar{Q}_+$). ‡ We can then replace $\mathbb{R}^2$ with a closed Riemann surface of genus g, Σ_g. The A-model is the 2d topological QFT (TQFT) whose observables are cohomology classes of $Q_-, \bar{Q}_+$. In the present context, those operators are the lines L. The A-model observables are simply the correlation functions:

$$\big\langle L_1(x_1) L_2(x_2) \cdots \big\rangle_{\Sigma_g \times S^1} = \big\langle L_1 L_2 \cdots \big\rangle_{\Sigma_g \times S^1} \ . \tag{4}$$

They do not depend on the (distinct) positions along the Riemann surface, $x_1, x_2, \cdots \in \Sigma_g$, because of the topological invariance (in other words, 2d momenta are Q-exact). In particular, we define the genus-zero 2-point and 3-point functions:

$$g_{ij} = \big\langle L_i L_j \big\rangle_{\mathbb{P}^1 \times S^1} \ , \qquad \mathcal{N}_{ijk} = \big\langle L_i L_j L_k \big\rangle_{\mathbb{P}^1 \times S^1} \ , \tag{5}$$

given some complete basis $\{L_i\}$ of line operator. Note that they are fully symmetric in the indices $i, j, \cdots$. Here, g_{ij} is generally known as the topological metric. Its inverse is denoted by g^{ij}. The pairing $g(L_i, L_j)$ obviously satisfies the Frobenius property $g(L_i L_j, L_k) = g(L_i, L_j L_k)$, therefore the half-BPS line algebra with this pairing is also a (super-commutative) Frobenius algebra (it has a unit, the trivial line operator $L_0 = \mathbf{1}$). Given this structure, we can obtain the structure constants in (3) simply as $\mathcal{N}_{ij}{}^k = \mathcal{N}_{ijl} g^{lk}$. Furthermore, the data (5) also encode the observables at higher genus, because of the 2d TQFT structure. There exists an "handle-gluing operator" [6], canonically defined as:

$$\mathcal{H} \equiv L_i \, g^{ij} \, L_j \ . \tag{6}$$

This is the local operator obtained when shrinking an handle of Σ_g down to a point, going from Σ_g to Σ_{g-1} with $\mathcal{H}$ inserted at a point. One can thus write any A-model observable as a genus-zero correlator:

$$\big\langle L_1 L_2 \cdots \big\rangle_{\Sigma_g \times S^1} = \big\langle \mathcal{H}^g L_1 L_2 \cdots \big\rangle_{\mathbb{P}^1 \times S^1} \ . \tag{7}$$

The general question is to understand the half-BPS lines, and to compute the A-model observables (5) explicitly, for a given 3d $\mathcal{N} = 2$ field theory.

‡ Note that Q_- and $\bar{Q}_+$ are always nilpotent. In the standard description, we go to the cohomology of Q_A but Q_A is not nilpotent in general, since $Q_A^2 = Z$. One can still discuss the A-model at non-zero central charge in terms of Q_A, but in the language of equivariant cohomology.

2.2. 3d $\mathcal{N} = 2$ gauge theories

For 3d $\mathcal{N} = 2$ supersymmetric *gauge theories*, the answer is fully known. § Consider the 3d theory on $\mathbb{R}^2 \times S^1$, viewed as a 2d theory with an infinite number of fields, corresponding to the Fourier modes along the S^1. The zero-mode of the 3d vector multiplet is a 2d vector multiplet, which contain a complex scalar, $u = i\,(\beta\sigma_{\rm 3d} + ia_0)$, where a_0 is the holonomy of the 3d gauge field along the S^1, and β is the radius of the S^1, so that u is dimensionless. For a given gauge group G of rank r, we can go on the 2d Coulomb branch, $\langle u \rangle \neq 0$. The Coulomb branch coordinates (and low-energy excitations) are denoted by u_a, $a = 1, \cdots, r$—that is, using the gauge freedom, we take u to be valued in the Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g} = {\rm Lie}(G)$. Under large gauge transformations along the S^1, we have the identifications $u_a \sim u_a + 1$ (in some appropriate basis for $\mathfrak{h}$). It is often very convenient to use the variables $x_a = e^{2\pi i u_a}$, which are single-valued.

Let us assume that, at generic values of u, all other fields in the gauge theory are massive and can be integrated out. We are then left with a low-energy effective description on the 2d Coulomb branch, which is entirely determined by an effective twisted superpotential, $\mathcal{W}(u, \nu)$. Here, ν denotes various additional "flavor parameters" of the theory, including, for instance, FI terms; we also define $y = e^{2\pi i \nu}$ their exponentiated form. The vacua of the 3d A-model (namely, the 2d vacua in the low-energy Coulomb branch description) correspond to critical points of the twisted superpotential. More precisely, due to the gauge invariance along the S^1, the twisted superpotential is only defined modulo $\mathcal{W} \sim \mathcal{W} + n^a u_a + n_0$, with $n^a, n_0 \in \mathbb{Z}$. Therefore, the vacuum equations take the form $\partial_{u_a}\mathcal{W} = n^a \in \mathbb{Z}$, or equivalently:

$$\Pi_a(x, y) \equiv \exp\left(2\pi i \frac{\partial \mathcal{W}}{\partial u_a}\right) = 1 \;, \qquad a = 1, \cdots, r \;, \tag{8}$$

It is interesting to note that the objects Π_a are actually a rational function of the variables x_a (and y). Thus, we "only" need to solve certain definite polynomial equations of r variables. The solutions to these equations, modulo the residual Weyl group that acts on the Coulomb branch coordinates u_a, are called the *Bethe vacua*. The name comes from the Bethe/gauge correspondence of Nekrasov and Shatashvili [7, 8], which relates this kind of 2d gauge theories to integrable models. Let us denote by $\mathcal{S}_{\rm BE}$ the set of Bethe vacua.

The effective twisted superpotential can be computed exactly in the UV [7, 8], as it is one-loop exact—see also [9, 10] for a thorough discussion of $\mathcal{W}$, including important subtleties related to Chern-Simons terms and the parity anomaly. Any A-model line operator also takes a specific form as an operator on the Coulomb branch, namely any well-defined L corresponds to a rational function $L(x)$ of the variables x_a (and possibly also of the flavor parameters y). Then, one can easily derive an explicit formula for the observables:

$$\big\langle L_1 L_2 \cdots \big\rangle_{\Sigma_g \times S^1} = \sum_{\hat{x} \in \mathcal{S}_{\rm BE}} \mathcal{H}(\hat{x}, y)^{g-1}\, L_1(\hat{x})\; L_2(\hat{x}) \cdots \;, \tag{9}$$

where the sum is over all the Bethe vacua. This formula was first obtained in [6], and further studied in [11, 12, 13, 14], including in terms of supersymmetric localization of the 3d path integral. The formula (9) can be understood as generalization of a classic formula by Vafa [15] for 2d Landau-Ginzburg models.

2.3. Target space and K-theory

In two dimensions, $\mathcal{N} = (2, 2)$ gauge theories are also known as "gauged linear sigma models" (GLSM), whenever they can be interpreted as a UV completion of a non-linear sigma model with target space X, some Kähler manifold [16]. In that case, the local operators in the twisted chiral ring are in one-to-one correspondence with the Dolbeault cohomology classes of the target, $[\omega_{q,p}] \in H^{p,q}(X)$. The 2d twisted

§ At least for a simply-connected gauge group, or for a $U(N)$ gauge group, and products thereof. The completely general case has additional subtleties which deserve further study.

chiral ring then provides a deformation of the cohomology ring of X, known as the *quantum cohomology,* with "quantum parameter" $y = q$. Moreover, the A-model observables encode enumerative invariants of X, such as the Gromov-Witten (GW) invariants (at least at genus zero) [17].

Similarly, we may consider 3d gauge theories as GLSM into a target space X. Now, the set of possible line operators is rather broad. One can always consider a basis of Wilson lines, but that may not always be the most natural basis from the target-space point of view. The lines L can often be identified with K-theory classes on X—equivalence classes of bundles or, more generally, coherent sheaves. The A-model observables should then compute quantum K-theoretic enumerative invariants [18, 19]. There has been some interesting recent work on the subject on the Physics side–see *e.g.* [20, 21], but much remains to be done in order to clarify the precise map between 3d supersymmetric theories and quantum K-theory.

2.4. Hilbert space interpretation

One mathematical property of the quantum K-theory invariants, in contrast with the GW invariants, is that they are always naturally *integers.* In Physics, this property has a simple explanation. We can always consider the setup $\Sigma_g \times S^1$ as defining a quantum mechanics (QM) along the "time direction" S^1. Then the observables are Witten indices in that QM, and the coefficients of the expansion in powers of q (and, more generally, y) are integers because they simply compute the dimensions of Hilbert spaces of the 3d theory on Σ_g at fixed charges (and, generally, defect Hilbert spaces living along the half-BPS lines).

It is an ongoing effort to derive and study these Hilbert spaces systematically—see for instance [22]. In fancy language, they give us a "categorification" the $\Sigma_g \times S^1$ observables—in particular, of the so-called twisted indices.

3. Outlook

It will be very interesting to complete the program outlined above, for any 3d $\mathcal{N} = 2$ gauge theories. More generally, this problem can be carried out for more general 3d supersymmetric field theories, possibly without a Lagrangian description, such as the ones arising in the 3d/3d correspondence [23]. There is also a fascinating version of the story for 4d $\mathcal{N} = 1$ theories, where, instead of having "K-theoretic" twisted chiral rings, we have, conjecturally, some "quantum elliptic cohomology" of the target space, which is still very poorly understood.

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Hidden Structures in Holographic Correlators

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1. Background

The on-shell scattering program revealed many surprising properties of scattering amplitudes in flat space, which are miraculous from the Lagrangian perspective. Motivated by the AdS/CFT correspondence, it is natural to ask: do scattering amplitudes in anti de Sitter space, *i.e.*, holographic correlators, also exhibit unexpected hidden structures? Answering this question is very challenging even in the tree-level supergravity, since explicit calculations are much harder in AdS. Fortunately, modern methods have recently been developed [1, 2, 3], which allow us to bypass many difficulties by using the bootstrap philosophy. It is now possible to efficiently compute holographic four-point functions from various maximally supersymmetric backgrounds, without using explicit details of the effective Lagrangian. These new progress accumulated an abundance of new results [1, 3, 4, 2, 5, 6, 7], from which we can search for new hidden structures.

2. State of the art

This contribution discusses two such structures recently discovered in holographic four-point functions from IIB supergravity on $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times K3$, which hint for surprising hidden symmetries in ten and six dimensions.

2.1. IIB supergravity on $\mathbf{AdS_5} \times \mathbf{S^5}$

The Kaluza-Klein modes of IIB supergravity on $AdS_5 \times S^5$ are organized into one-half BPS multiplets under the $PSU(2,2|4)$ superconformal symmetry. The bottom components of the supermultiplets are scalar operators in $\mathcal{N}=4$ SYM of the form

$$\mathcal{O}_k(x,t) = trX^{\{I_1}\ldots X^{I_k\}}t_{I_1}\ldots t_{I_k}\ , \quad k=2,3,\ldots\ , \quad t^2=0\ . \tag{1}$$

These operators have protected conformal dimension $\Delta = k$. They transform in the k-fold symmetric traceless representation of the $SO(6)$ R-symmetry group, which is made automatic by contracting the R-symmetry indices with null vectors t_I. The four-point function of one-half BPS operators

Mellin space [8, 9]. The problem is formulated by imposing a number of consistency conditions on the Mellin amplitudes: *superconformal symmetry*, *Bose symmetry*, *analytic structures* and *asymptotic limit*. The problem was solved for all four-point functions, and the solution took a remarkably simple form [1, 2]. The Mellin space solution was later verified for many examples by explicit computations in position space [10, 11].

The extreme simplicity of the solution strongly suggested the existence of an elegant underlying structure, which was made precise in [12]. It was observed that all four-point functions can be packed into a generating function, which is obtained by lifting the lowest-lying $k_i = 2$ four-point function. More precisely, the $k_i = 2$ reduced correlator

$$H_{2222}(x_{ij}^2) \propto x_{13}^{-2} x_{14}^{-2} x_{34}^{-2} D_{2422} \tag{6}$$

is a function of x_{ij}^2 only, which is a consequence of the fact that it has zero R-symmetry charge. A generating function can be defined by replacing x_{ij}^2 with $x_{ij}^2 + t_{ij}$

$$\mathbf{H}(x_i, t_i) = H_{2222}(x_{ij}^2 + t_{ij}) \ . \tag{7}$$

By Taylor expanding $\mathbf{H}$ in t_{ij} and collecting all the possible monomials, one obtains the reduced correlator $H_{k_1k_2k_3k_4}$.

The appearance of $x_{ij}^2 + t_{ij}$ suggests a hidden conformal symmetry in ten dimensions, since it can be interpreted as the scalar product of two vectors in a twelve dimensional embedding space. Relatedly, the $AdS_5 \times S^5$ background is conformally flat, and can be brought to $\mathbb{R}^{9,1}$ by a Weyl transformation. Further hints for a ten dimensional conformal symmetry is given by the scattering amplitude of IIB supergravity in 10d flat space

$$\mathcal{A}_{IIB} \sim G_N^{(10)} \delta^{16}(Q) \frac{1}{stu} \ . \tag{8}$$

The object $\frac{1}{stu}$ is identified as the Mellin amplitude of $H_{k_1k_2k_3k_4}$ when the Mellin-Mandelstam variables are large. The dimensionless quantity $G_N^{(10)} \delta^{16}(Q)$ corresponds to the factor R in (3), which is a difference operator in the Mellin language. An interesting fact is that $\frac{1}{stu}$ is conformally invariant when the spacetime dimension is ten.

2.2. IIB supergravity on $\mathbf{AdS_3 \times S^3 \times K3}$

A similar hidden structure in holographic four-point functions was recently discovered in [6] for IIB supergravity on $AdS_3 \times S^3 \times K3$, which arises in the near-horizon limit of the D1-D5 system. In the supergravity limit, the size of $K3$ is very small compared to the radii of AdS_3 and S_3. IIB supergravity therefore reduces to 6d $(2,0)$ supergravity coupled to 21 tensor multiplets which are the zero modes on $K3$. Further compactification of the 6d theory on $AdS_3 \times S^3$ gives rise to infinitely many Kaluza-Klein modes which are organized into one-half BPS multiplets under the global superconformal symmetry $PSU(1,1|2) \times PSU(1,1|2)$. In [6], attention was given to the four-point functions of one-half BPS operators which descend from the 6d tensor multiplets. These operators

$$\mathcal{O}_k^I(x; v, \bar{v}) = \mathcal{O}_k^{I,\alpha_1\ldots\alpha_k,\dot{\alpha}_1\ldots\dot{\alpha}_k} v_{\alpha_1} \ldots v_{\alpha_k} \bar{v}_{\dot{\alpha}_1} \ldots \bar{v}_{\dot{\alpha}_k} \ , \quad k = 1, 2, \ldots \tag{9}$$

have conformal dimensions $(h,\bar{h})=(\frac{k}{2},\frac{k}{2})$, and $SU(2)_L\times SU(2)_R$ spins $(j,\bar{j})=(\frac{k}{2},\frac{k}{2})$. They also carry a vector index I under the flavor group $SO(21)$. We have similarly kept track of the R-symmetry indices by contracting them with two-component spinors.

The four-point functions of these one-half BPS operators

$$G^{I_1I_2I_3I_4}_{k_1k_2k_3k_4}(x_i;v_i,\bar{v}_i)=\langle\mathcal{O}^{I_1}_{k_1}(x_1;v_1,\bar{v}_1)\ldots\mathcal{O}^{I_4}_{k_4}(x_4;v_4,\bar{v}_4)\rangle \tag{10}$$

are also constrained by the superconformal symmetry, and take the form [6]

$$G^{I_1I_2I_3I_4}_{k_1k_2k_3k_4}=G^{I_1I_2I_3I_4}_{0,k_1k_2k_3k_4}+\tilde{R}\,H^{I_1I_2I_3I_4}_{k_1k_2k_3k_4}\,. \tag{11}$$

The function $G^{I_1I_2I_3I_4}_{0,k_1k_2k_3k_4}$ is protected. On the other hand, $H^{I_1I_2I_3I_4}_{k_1k_2k_3k_4}$ depends on the dynamics, and kinematically is a four-point function with $(h_i,\bar{h}_i)=(\frac{k_i+1}{2},\frac{k_i+1}{2})$ and $(j_i,\bar{j}_i)=(\frac{k_i-1}{2},\frac{k_i-1}{2})$. The factor $\tilde{R}$ is fixed by the superconformal symmetry to be§

$$\tilde{R}=t_{12}t_{34}x^2_{13}x^2_{24}(1-z\alpha)(1-\bar{z}\bar{\alpha})\,. \tag{12}$$

In [6], it was shown that these four-point functions can be computed by generalizing the bootstrap methods of [1, 2]. It further turned out that all the correlators can also be packaged into a single generating function, similar to the $AdS_5\times S^5$ case. The four-point function with lowest-lying KK modes $k_i=1$ has a reduced correlator that is independent of R-symmetry coordinates

$$H^{I_1I_2I_3I_4}_{1111}=\delta^{I_1I_2}\delta^{I_3I_4}x^{-2}_{12}D_{1122}+\delta^{I_1I_4}\delta^{I_2I_3}x^{-2}_{23}D_{2112}+\delta^{I_1I_3}\delta^{I_2I_4}x^{-2}_{13}D_{1212}\,. \tag{13}$$

By lifting this correlator,

$$\mathbf{H}^{I_1I_2I_3I_4}=H^{I_1I_2I_3I_4}_{1111}(x^2_{ij}+t_{ij})\,, \tag{14}$$

we define a generating function $\mathbf{H}^{I_1I_2I_3I_4}$, which upon Taylor expanding in t_{ij} gives all the other correlators $H^{I_1I_2I_3I_4}_{k_1k_2k_3k_4}$.

It is worth noticing the similarities with the $AdS_5\times S^5$ case. Just as in the $AdS_5\times S^5$ background, $AdS_3\times S^3$ is also conformally flat. The arguments of the generating function $x^2_{ij}+t_{ij}$ also have the interpretation as the squared distances in an eight dimensional embedding space, suggesting a six dimensional hidden conformal symmetry. Finally, the four-tensor scattering amplitude in the 6d flat space theory of $(2,0)$ supergravity coupled to 21 tensors is given by

$$\mathcal{A}_{(2,0),tensor}\sim G^{(6)}_N\delta^8(Q)\left(\frac{\delta^{I_1I_2}\delta^{I_3I_4}}{s}+\frac{\delta^{I_1I_4}\delta^{I_2I_3}}{t}+\frac{\delta^{I_1I_3}\delta^{I_2I_4}}{u}\right)\,. \tag{15}$$

After dividing by the dimensionless quantity $G^{(6)}_N\delta^8(Q)$, which is identified with $\tilde{R}$, the rest of the amplitude is just the flat space limit of the Mellin amplitude defined from $H^{I_1I_2I_3I_4}_{k_1k_2k_3k_4}$. Moreover, this quantity is conformally invariant in six dimensions.

§ Here z and $\bar{z}$ are defined in terms of the holomorphic and anti-holomorphic coordinates $z=\frac{z_{12}z_{34}}{z_{13}z_{24}}$, $\bar{z}=\frac{\bar{z}_{12}\bar{z}_{34}}{\bar{z}_{13}\bar{z}_{24}}$ where $z_{ij}=z_i-z_j$, $\bar{z}_{ij}=\bar{z}_i-\bar{z}_j$. We also used the rescaling freedom in the spinors to set $v_i=(1,y_i)$, $\bar{v}_i=(1,\bar{y}_i)$, and defined $\alpha=\frac{y_{12}y_{34}}{y_{13}y_{24}}$, $\bar{\alpha}=\frac{\bar{y}_{12}\bar{y}_{34}}{\bar{y}_{13}\bar{y}_{24}}$ where $y_{ij}=y_i-y_j$, $\bar{y}_{ij}=\bar{y}_i-\bar{y}_j$. The null vectors are four dimensional, and are constructed from the spinors via $t^\mu_i=v^\alpha_i\sigma^\mu_{\alpha\dot{\alpha}}\bar{v}^{\dot{\alpha}}_i$.

3. Outlook

The $AdS_5 \times S^5$ and $AdS_3 \times S^3 \times K3$ four-point functions exhibit interesting hidden symmetries which can be used as elegant schemes for organizing correlators. The origin of these hidden structures is still not clear, and awaits a more precise understanding. Meanwhile, it is curious to see how much of these structures can be generalized.

- For example, it would be extremely interesting to find similar structures in higher-point correlation functions. Work in this direction for $AdS_5 \times S^5$ five-point functions has been initiated in [7].
- On the other hand, since one-loop correlators are essentially determined in terms of tree-level correlators by AdS unitarity methods [13], it is promising that such hidden symmetries may also be present in a certain form at the loop level [14, 15, 16, 17, 18, 12, 19].
- Curiously, the ten dimensional symmetry for $AdS_5 \times S^5$ survives the leading string correction, and there is evidence suggesting it may even persist at the sub-leading order [20]. By contrast, the free theory correlator does not have the hidden symmetry. It would be nice to better understand when the symmetry fails in the strong coupling expansion.
- It is also interesting to find other realizations of higher dimensional hidden symmetries. For example, in $AdS_3 \times S^3 \times K3$ scalar one-half BPS four-point functions from the 6d supergravity multiplet are *not* organized by the same lifting [6]. A careful study of other correlators of the theory may reveal different hidden structures.
- Another related direction is to study backgrounds which are not conformally flat. Important backgrounds include $AdS_7 \times S^4$ and $AdS_4 \times S^7$, which are maximally supersymmetric.

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String Amplitudes from Conformal Field Theory

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1. Background

We give a pedagogical account of the computation of string theory amplitudes in curved space time from conformal field theory techniques. This lecture is mainly based on [1].

This lecture is about conformal field theory (CFT) techniques to understand scattering amplitudes in theories of gravity, including string theory. Scattering amplitudes are interesting for a variety of reasons. Ultimately, they are the observables that allow to confronting theories with experiments, but at a more formal level, they may elucidate structures and symmetries that are not obvious simply by looking at the Lagrangian of a theory.

In this lecture we will focus on the four-point scattering amplitude $\mathcal{A}(g, s, t, u)$ for the scattering of two particles with momenta p_1, p_2 into two other particles with momenta p_3, p_4. The scattering amplitude depends on the type of particles being scattered (their mass, spin, etc), on the parameters of the theory, denoted collectively by g and the Mandelstan invariants, given in terms of the momenta by

$$s = -(p_1 + p_2)^2, \quad t = -(p_1 - p_3)^2, \quad u = -(p_1 - p_4)^2 \tag{1}$$

The scattering of gravitons in general relativity (GR) has a long story. It has been found that the scattering of four gravitons suffers from UV divergences. While these divergences can be canceled by adding an appropriate counter-term at one loop, at two loops GR is non-renormalisable. UV divergences arise from the region where gravitons become very close to each other. The proposal of string theory is to replace point-like gravitons by closed strings of finite size $\sqrt{\alpha'}$, which for finite α' cannot get very close to each other. The scattering of gravitons in string theory amplitudes is then UV finite. In a small α' expansion we then recover GR plus an infinite tower of corrections suppressed by powers of α'. As the momentum of a process gets higher and higher, these corrections become important and cure the UV divergences of gravity. We say that string theory provides a UV completion of GR. We are then led to consider scattering amplitudes of graviton states in string theory. In addition to the size

α', string theory amplitudes depend on the string coupling constant g_s, which measures the probability that strings join or split. Expanding in powers of g_s corresponds to a genus expansion

$$\mathcal{A}(g_s, \alpha', s, t, u) = \mathcal{A}^{tree}(\alpha', s, t, u) + g_s^2 \mathcal{A}^{genus1}(\alpha', s, t, u) + \cdots \tag{2}$$

At tree level and in flat space the result is given by the Virasoro-Shapiro amplitude. At genus one, even in flat space, the result is very complicated, but it simplifies to each order in a α' expansion [2]. In curved space time there is no systematic prescription to compute scattering amplitudes. In this lecture we will follow an alternative route to compute these observables, based on two tools: the AdS/CFT duality and the analytic conformal bootstrap. The AdS/CFT duality dictates that closed string amplitudes in AdS are mapped to correlators of local operators in the boundary holographic conformal field theory. The genus expansion corresponds to an expansion in inverse powers of the central charge on the CFT side. We then apply methods to the analytic bootstrap in order to compute CFT correlators in the corresponding $1/c$ expansion. We will work to order $1/c^2$, corresponding to genus one string amplitudes.

2. State of the art

Here we will describe both the tools at our disposal and the results which have been already derived using them.

2.1. The tools

The first tool is the AdS/CFT duality. This states that type IIB string theory on $AdS_5 \times S^5$ is dual to four-dimensional super-conformal $\mathcal{N} = 4$ SYM with $SU(N)$ gauge group. The dictionary between the parameters on both sides reads

$$g_s \sim \frac{1}{N}, \qquad \frac{R^2}{\alpha'} = \sqrt{g_{YM}^2 N} = \sqrt{\lambda} \tag{3}$$

where R is the (common) radius of AdS_5 and S^5 and g_{YM} is the Yang-Mills coupling constant. The AdS/CFT duality also provide us with a conceptual framework to define scattering amplitudes in AdS: in compact space-times we cannot properly define asymptotic states. Instead, we identify these amplitudes with correlators of local operators at the boundary.

$$\mathcal{A}_{AdS}(g_s, \frac{\alpha'}{R^2}, s, t, u) \leftrightarrow \mathcal{G}_{CFT}(U, V) \tag{4}$$

where U, V are the conformal cross-ratios for a CFT four-point correlator defined by $U = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}, V = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$. We will make this correspondence slightly more precise below. From (4) we see that a genus expansion corresponds to a $1/N$ expansion in the CFT side, while stringy corrections to (super)gravity should be under control in a $1/\lambda$ expansion. Furthermore, we need to ask which CFT operator corresponds to the graviton in AdS. Under the duality this is mapped to the stress tensor of the CFT. In $\mathcal{N} = 4$ SYM the stress tensor lives in a short multiplet, whose superconformal primary is a scalar operator $\mathcal{O}_2(x)$ of protected dimension $\Delta = 2$. We will then consider the correlator of four such identical operators in a $1/N$ and $1/\lambda$ expansion. This is possible thanks to our second tool: the analytic bootstrap.

The object of interest is the four point correlator $\mathcal{G}(U, V)$ of four identical scalar operators at locations $x_1, \cdots, x_4$. This correlator satisfies three important properties. One is crossing symmetry

$$V^{\Delta_{\mathcal{O}}} \mathcal{G}(U, V) \underbrace{=}_{x_1 \leftrightarrow x_3} U^{\Delta_{\mathcal{O}}} \mathcal{G}(V, U) \underbrace{=}_{x_1 \leftrightarrow x_4} V^{\Delta_{\mathcal{O}}} \mathcal{G}(\frac{U}{V}, \frac{1}{V}) \tag{5}$$

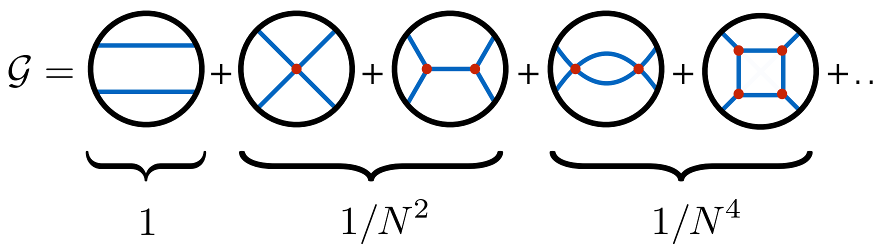
The second is that the correlator admits a decomposition in terms of conformal blocks. The third property is only manifest in the Minkowski regime. In this regime we can consider a situation where an operator is in the light cone of another operator. In terms of cross-ratios this allows to explore the region of small V with arbitrary U. As some operators became null separated the correlator develops well prescribed singularities

$$\mathcal{G}(U,V) \sim \frac{h(U)}{V^{\alpha}}, \quad as \mathrm{V} \to 0 \tag{6}$$

We will use these three properties to constraint correlators in holographic CFTs.

2.2. Correlators in holographic CFTs

Correlators in holographic CFTs admit an expansion in $\frac{1}{N^2}$ which corresponds to a loop/genus



expansion in the corresponding bulk dual, as shown in the picture. The zeroth order answer corresponds to mean field theory (MFT): the sum of three disconnected contributions. The intermediate operators correspond to double trace-operators $\mathcal{O}^n\partial^\ell\mathcal{O}$ of dimension $\Delta_{n,\ell} = 2\Delta_{\mathcal{O}} + 2n + \ell$ and spin ℓ. As we turn on $1/N$ corrections two effects will occur. First, double-trace operators will get corrected, in particular

$$\Delta_{n,\ell} = 2\Delta_{\mathcal{O}} + 2n + \ell + \frac{1}{N^2}\gamma^{(1)}_{n,\ell} + \cdots \tag{7}$$

in addition new intermediate operators may appear at higher order in $1/N$. Indeed, in holographic CFT's we always expect the stress-tensor to be present to order $1/N^2$, schematically

$$\mathcal{O} \times \mathcal{O} = 1 + [\mathcal{O},\mathcal{O}]_{n,\ell} + \frac{1}{N^2}T_{\mu\nu} \tag{8}$$

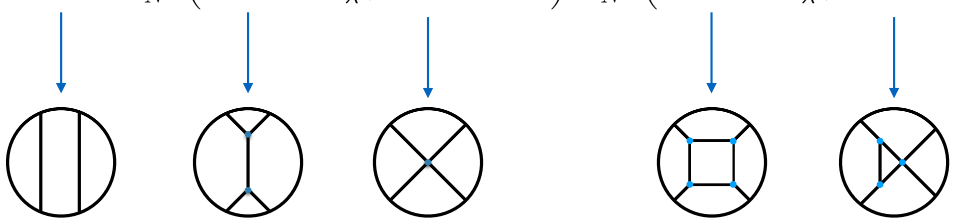
The presence of the stress-tensor gives rise to a well defined/computable singularity in the null limit. Together with crossing symmetry this is strong enough to fix the form of the solution. This solution, which we will denote by $\gamma^{(1)}_{sugra}$ corresponds to the graviton exchange diagram in AdS. In addition, we find a tower of homogenous solutions to crossing that do not produce any extra null singularity. These solutions correspond to quartic vertices in AdS and give rise to corrections $\gamma^{(1)}_{n,\ell}$ only for a finite support in the spin. We denote such solutions as $\gamma^{(1)}_{trunc}$.

To order $1/N^4$ something interesting happens: the presence of an anomalous dimensions to order $1/N^2$ together with crossing symmetry, produces a null divergence to order $1/N^4$, proportional to the square $\left(\gamma^{(1)}_{n,\ell}\right)^2$ [3]. This null divergence, which is exactly computable, leads then to $\gamma^{(2)}_{n,\ell}$. As a result, solutions to order $1/N^4$ can be schematically represented by the 'square' of the solutions to the previous order. For instance, the box diagram arises as the product of two exchange diagrams. We would like to stress that for each kind of contribution we are able to compute $\gamma^{(2)}_{n,\ell}$ fully, as well as the whole correlator.

3. Outlook

Although the discussion so far has been very general, $\mathcal{N} = 4$ SYM can be treated following the same strategy. The final result has the following structure: Namely, to order $1/N^2$ and at very large λ, we obtain the supergravity result. Corrections in $1/\lambda$, which correspond to stringy corrections, are given by a tower of quartic vertices, as shown in the figure. To order $1/N^4$ and for very large λ we obtain

$$\mathcal{G}^{(0)}(u,v) + \frac{1}{N^2}\left(\mathcal{G}^{(1)}_{grav}(u,v) + \frac{1}{\lambda^{3/2}}\mathcal{G}^{(1)}_{st}(u,v) + \cdots\right) + \frac{1}{N^4}\left(\mathcal{G}^{(2)}_{grav}(u,v) + \frac{1}{\lambda^{3/2}}\mathcal{G}^{(2)}_{st}(u,v) + \cdots\right)$$



the quantum-gravity result [4], while including $1/\lambda$ corrections corresponds to stringy corrections. We would like to stress that all contributions are exactly computable [1].

The precise relation between, the AdS scattering amplitude $\mathcal{A}(s,t,u)$ and the correlator $\mathcal{G}(U,V)$ is given by the Mellin transform:

$$\mathcal{G}(U,V) = \int_{-i\infty}^{i\infty} dsdtU^sV^t\mathcal{A}(s,t,u)\Gamma^2(2-s)\Gamma^2(2-t)\Gamma^2(2-u)$$

where $s+t+u=2$. Crossing symmetry is simply the statement that $\mathcal{A}(s,t,u)$ is completely symmetric in the variables (s,t,u). In this language the supergravity result $\mathcal{A}^{(1)}_{grav}(s,t,u)$ is simply given by a meromorphic function

$$\mathcal{A}^{(1)}_{grav}(s,t,u) = \frac{1}{(s-1)(t-1)(u-1)} \tag{9}$$

while the tower of stringy corrections to genus zero $\mathcal{A}^{(1)}_{st}(s,t,u)$ is given by symmetric polynomials, of higher and higher degree. At genus one the results have the following structure

$$\mathcal{A}^{genus\ 1}_{grav.}(s,t,u) = \sum_{m,n=2}\left(\frac{c_{mn}}{(s-2m)(t-2n)} + crossed\right)$$
$$\mathcal{A}^{genus\ 1}_{st}(s,t,u) = P(s,t,u)\psi_0\left(2-\frac{s}{2}\right) + crossed$$

Let us close with the following important remark. Given the amplitude in AdS one can consider a limit to recover the amplitude in flat space. In Mellin language this corresponds to take s,t,u large and α' small with $\alpha's, \alpha't, \alpha'u$ kept fixed. In this limit the genus zero answer has exactly the structure of the Virasoro-Shapiro amplitude, as expected. What about the genus one answer? In this limit $\mathcal{A}^{genus\ 1}_{grav.}(s,t,u)$ reduces precisely to the box function in 10D, exactly the known result for the one-loop super-gravity in flat space! Furthermore, the contributions $\mathcal{A}^{genus\ 1}_{st}(s,t,u)$ reduce precisely to the known result found in [2]. It is remarkable how, starting from a four-dimensional gauge theory, we have recovered a ten dimensional flat space scattering amplitude.

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Bootstrapping (A)dS Exchanges

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1. Introduction

It is well known that singularities and analyticity play a prominent physical role in encoding various properties of scattering processes. These elements continue to play a key role also when considering generalisation of scattering processes on curved backgrounds, for which a direct definition of S-matrix elements is not available. In these cases, it remains true that physical properties including time evolution remain deeply encoded in the singularity and analyticity structure of the boundary correlators. Such properties are expected, as in the flat space case, to be tightly related to the basic consistency requirements of causality and unitarity.

For Lorentzian CFT correlators, which encode scattering processes in anti-de Sitter space, Lorentz invariance requires such singularities to lie along lightcones so that CFT correlators are single-valued functions in the Euclidean region. The situation for purely Euclidean correlators on the other hand, which encode scattering processes in de Sitter space-time [1, 2], is still very little understood. Momentum space branch-cut singularities encode the quantum interference effects between particle creation processes and the expansion of the universe [3, 4]. These singularities map to standard conformal block singularities in position space.

It is well known that a single conformal block has spurious branch-cut singularities in the Euclidean region [5]. How to reconcile these branch-cut singularities with crossing symmetry is one of the key issues of the bootstrap approach. In particular, the appearance of such spurious singularities makes the problem of decomposing a single conformal block $g_{\tau,\ell}(u,v)$ from the crossed channel to the direct channel an ill defined question‡:

$$g_{\Delta,\ell}(v,u) \overset{?}{=} \sum_{\Delta',\ell\prime} a_{\Delta',\ell'}\, g_{\Delta',\ell'}(u,v), \tag{1}$$

where $u = \frac{x_{12}^2 x_{34}^2}{x_{13}^2 x_{24}^2}$ and $v = \frac{x_{14}^2 x_{23}^2}{x_{13}^2 x_{24}^2}$ are the usual cross ratios. It is therefore natural to ask what would be the minimal completion of a conformal block to an expression that admits a well-defined crossing decomposition. Remarkably, the solution to this problem turns out to be given unequivocally by bulk exchange diagrams [6].

While much progress has been made in the context of AdS exchanges, recent results [7] connecting them to dS exchanges naturally suggest how we might clarify the role of crossing symmetry and unitarity in dS space, especially in relation to the corresponding analyticity properties of dS exchange amplitudes. We give a brief account in this note.

2. Conformal Partial Waves

The most basic object which does admit as well-defined crossing decomposition is given by the following conformally invariant integral [8]:

$$\mathcal{F}_{\Delta,\ell}(x_j) = \int d^d x_0\, \langle \mathcal{O}_{\Delta_1}(x_1)\mathcal{O}_{\Delta_2}(x_2)\mathcal{O}_{\Delta,\ell}(x_0)\rangle^{\mu_1\cdots\mu_\ell}\langle \widetilde{\mathcal{O}}_{d-\Delta,\ell}(x_0)\mathcal{O}_{\Delta_3}(x_3)\mathcal{O}_{\Delta_4}(x_4)\rangle_{\mu_1\cdots\mu_\ell}\,, \tag{2}$$

where associated to the operator $\mathcal{O}_{\Delta,\ell}$ of scaling dimension Δ is the shadow operator $\widetilde{\mathcal{O}}_{d-\Delta,\ell}$ of scaling dimension $d-\Delta$ [9]. Both operators have spin-ℓ. (2) is the unique Eigenfunction of quadratic and quartic conformal Casimir operators which is single-valued in the Euclidean region, commonly referred to as a Conformal Partial Wave (CPW). It can also be expressed the following specific linear combination of conformal blocks:

$$\mathcal{F}_{\Delta,\ell}(u,v) = \#_1\, g_{\Delta,\ell}(u,v) + \#_2\, g_{d-\Delta,\ell}(u,v)\,. \tag{3}$$

The conformal blocks $g_{\Delta,\ell}$ and $g_{d-\Delta,\ell}$ satisfy the same Casimir equations as $\mathcal{F}_{\Delta,\ell}$ but with different boundary conditions.

The CPW is the simplest conformal invariant object possessing a well defined crossing decomposition:

$${}^{(\mathrm{t})}\mathcal{F}_{\Delta,\ell}(u,v) = \sum_{n,\ell=0}^{\infty} A_{n,\ell}\, g_{\Delta_1+\Delta_2+2n,\ell}(u,v) + \sum_{n,\ell=0}^{\infty} B_{n,\ell}\, g_{\Delta_3+\Delta_4+2n,\ell}(u,v)\,, \tag{4}$$

‡ In $d>1$, direct channel conformal blocks have branch-cut singularities for $v\sim 0$ while crossed-channel blocks are single valued. Direct channel conformal blocks therefore do not admit a crossed-channel decomposition, as it would imply it is possible to write a function that is not single-valued as an infinite sum of single valued ones.

where $\mathcal{F}^{(\mathsf{t})}_{\Delta,\ell}(u,v)$ is the t-channel CPW and $A_{n,\ell}$, $B_{n,\ell}$ are the expansion coefficients.

To go from the CPW to bulk exchange diagrams it is useful to adopt a Mellin-Barnes representation of CPWs (2) in momentum space [10, 11]. In momentum space, the CPW (2) factorises into a product of 3pt functions §:

$$\mathcal{F}_{\Delta,\ell}(\vec{k}_j;\vec{k}) = \langle \mathcal{O}_{\Delta_1}(\vec{k}_1)\mathcal{O}_{\Delta_2}(\vec{k}_2)\mathcal{O}_{\Delta,\ell}(\vec{k})\rangle^{\mu_1\cdots\mu_\ell}\langle \tilde{\mathcal{O}}_{d-\Delta,\ell}(-\vec{k}\,)\mathcal{O}_{\Delta_3}(\vec{k}_3)\mathcal{O}_{\Delta_4}(\vec{k}_4)\rangle_{\mu_1\cdots\mu_\ell}. \quad (5)$$

For scalar operators of generic scaling dimension, conformal symmetry requires CFT three-point functions in momentum space to be given (up to a coefficient) by Appell's F_4 function [12, 13]. This admits the following representation as a Mellin-Barnes integral [10, 11]:

$$\langle \mathcal{O}_{\Delta_1}(\vec{k}_1)\mathcal{O}_{\Delta_2}(\vec{k}_2)\mathcal{O}_{\Delta_3}(\vec{k}_3)\rangle = \int_{-i\infty}^{+i\infty}\prod_{j=1}^{3}\frac{ds_j}{2\pi i}\,\langle \mathcal{O}_{\Delta_1}(\vec{k}_1)\mathcal{O}_{\Delta_2}(\vec{k}_2)\mathcal{O}_{\Delta_3}(\vec{k}_3)\rangle_{s_1,s_2,s_3}, \quad (6)$$

$$\langle \mathcal{O}_{\Delta_1}(\vec{k}_1)\mathcal{O}_{\Delta_2}(\vec{k}_2)\mathcal{O}_{\Delta_3}(\vec{k}_3)\rangle_{s_1,s_2,s_3} = i\pi\delta\left(\tfrac{d}{4}-s_1-s_2-s_3\right) \times \rho_{\nu_1,\nu_2,\nu_3}(s_1,s_2,s_3)\prod_{j=1}^{3}\left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j}, \quad (7)$$

where

$$\rho_{\nu_1,\nu_2,\nu_3}\left(s_1,s_2,s_3\right) = \prod_{j=1}^{3}\frac{1}{2\sqrt{\pi}}\Gamma\left(s_j+\tfrac{i\nu_j}{2}\right)\Gamma\left(s_j-\tfrac{i\nu_j}{2}\right), \quad (8)$$

and we set $\Delta_j = \frac{d}{2}+i\nu_j$ and $k_j = |\vec{k}_j|$. This makes manifest the connection to the dual scattering process in the bulk, where the poles (8) in the Mellin variable s_j are those of the bulk-boundary propagator associated to the operator $\mathcal{O}_{\Delta_j}(\vec{k}_j)$. In the corresponding Mellin-Barnes representation for the CPW (5), which is

$$\mathcal{F}_{\nu,0}\left(\vec{k}_j,s_j;\vec{k},u,\bar{u}\right) = i\pi\delta\left(\tfrac{d}{4}-s_1-s_2-u\right)i\pi\delta\left(\tfrac{d}{4}-s_3-s_4-\bar{u}\right) \times \rho_{\nu_1,\nu_2,\nu}(s_1,s_2,u)\rho_{\nu_3,\nu_4,-\nu}(s_3,s_4,\bar{u})\left(\frac{k}{2}\right)^{-2(u+\bar{u})}\prod_{j=1}^{4}\left(\frac{k_j}{2}\right)^{-2s_j+i\nu_j}, \quad (9)$$

the Mellin variables associated to the operators $\mathcal{O}_{\Delta,\ell}(\vec{k})$ and $\mathcal{O}_{d-\Delta,\ell}(\vec{k})$, which we denote by u and $\bar{u}$, then capture the on-shell exchange of the particle in the bulk. In the above we set $\Delta = \frac{d}{2}+i\nu$. The family of poles

$$u = -\frac{i\nu}{2}-n, \qquad \bar{u} = -\frac{i\nu}{2}-\bar{n}, \qquad n,\,\bar{n}\in\mathbb{N} \quad (10)$$

generates contributions to the conformal block $g_{\Delta,\ell}$ in (3), while the family

$$u = \frac{i\nu}{2}-n, \qquad \bar{u} = \frac{i\nu}{2}-\bar{n}, \qquad n,\,\bar{n}\in\mathbb{N} \quad (11)$$

generates contributions to the shadow conformal block $g_{d-\Delta,\ell}$.

§ In (5) and in the following, momentum conserving δ-functions are left implicit.

3. From CPWs to exchange diagrams

In the previous section we saw that CPWs (2) are the unique single-valued solutions to the conformal Casimir equations. The latter equations are homogeneous equations and capture the on-shell exchange of a particle in the bulk. The full, off-shell, exchange on the other hand satisfies the quadratic Casimir equation with a source [3, 4] responsible for the contact terms. The latter is equivalent to the equation of motion for the corresponding bulk-bulk propagators, which have a Dirac delta function source.

Using Mellin-Barnes representation we can reconstruct solutions to the quadratic Casimir equation with a source from their homogeneous counterparts. In particular, the source term on the r.h.s. is generated simply by multiplying the Mellin-Barnes representation of the CPW (9) with the factor $\csc(\pi(u+\bar{u}))$ (see [7]). Different boundary conditions can then be imposed on the exchanged particle by multiplying with the projectors

$$\delta^{\pm}(u,\bar{u}) = \frac{1}{2}\sin\left(\pi\left(u \mp \tfrac{i\nu}{2}\right)\right)\sin\left(\pi\left(\bar{u} \mp \tfrac{i\nu}{2}\right)\right), \tag{12}$$

where $\delta^{+}(u,\bar{u})$ implements the Dirichlet boundary condition and $\delta^{-}(u,\bar{u})$ implements the Neumann boundary condition. In particular, the zeros of $\delta^{+}(u,\bar{u})$ overlap with the poles (11), thus projecting out the shadow conformal block $g_{d-\Delta,\ell}$. Likewise, the $\delta^{-}(u,\bar{u})$ overlap with the poles (10), projecting out the conformal block $g_{\Delta,\ell}$

This implies the following universal form for the Mellin-Barnes representation of exchanges in (A)dS$_{d+1}$:

Mellin-Barnes representation of exchange four-point functions in (A)dS$_{d+1}$

$$\mathcal{A}\left(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4\right) = \int_{-i\infty}^{i\infty}\prod_{j=1}^{4}\frac{ds_j}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{du\,d\bar{u}}{(2\pi i)^2}\,\underbrace{\csc(\pi(u+\bar{u}))}_{\text{contact terms}}\,\underbrace{\delta(u,\bar{u})}_{\text{b.c.}}\,\underbrace{\mathcal{F}_{\nu,\ell}\left(\vec{k}_j,s_j;\vec{k},u,\bar{u}\right)}_{\text{CPW}}, \tag{13}$$

where in each case the CPW is normalised so that it is consistent with on-shell factorisation – i.e. the correct 3pt functions should be plugged into (5). This expression makes manifest how exchanges in (A)dS$_{d+1}$ are fixed by a combination of symmetries, factorization and boundary conditions.

In AdS, for Dirichlet/Neumann boundary conditions on the exchanged field we have:

$$\mathcal{A}^{\pm}_{\text{AdS}}\left(\vec{k}_1,\vec{k}_2,\vec{k}_3,\vec{k}_4\right) = \int_{-i\infty}^{i\infty}\prod_{j=1}^{4}\frac{ds_j}{2\pi i}\int_{-i\infty}^{+i\infty}\frac{du\,d\bar{u}}{(2\pi i)^2}\,\csc(\pi(u+\bar{u}))\,\delta^{\pm}(u,\bar{u})\mathcal{F}_{\nu,\ell}\left(\vec{k}_j,s_j;\vec{k},u,\bar{u}\right). \tag{14}$$

In dS, the in-in exchange in the Bunch-Davies vacuum (which is a boundary condition) corresponds to a particular linear combination of projectors (12). This also implies that it can be written as a linear combination of AdS exchanges (14), and it reads [7]:

$$\mathcal{A}_{\text{dS}}^{\text{B.D.}} = \frac{1}{2}\csc\left(\tfrac{\pi}{2}(2\Delta_- - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{1+2}+\Delta_- + \ell - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{3+4}+\Delta_- + \ell - d)\right)\mathcal{A}_{\text{AdS}}^{-}$$
$$+\frac{1}{2}\csc\left(\tfrac{\pi}{2}(2\Delta_+ - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{1+2}+\Delta_+ + \ell - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{3+4}+\Delta_+ + \ell - d)\right)\mathcal{A}_{\text{AdS}}^{+}\,, \tag{15}$$

where $\Delta_{i+j} \equiv \Delta_i + \Delta_j$, $\Delta_+ = \Delta$ and $\Delta_- = d - \Delta$. Note that this is an identity which holds between AdS exchanges and the in-in exchange in the Bunch-Davies vacuum of dS, regardless of whether we are in momentum or position space. We conclude this short note with a few comments:

- Note that we can go from the full exchange (13) to the on-shell exchange in the usual way by taking the "cut" or discontinuity in the exchanged momentum $\mathsf{s} = |\vec{k}|^2$. This is made manifest by the Mellin-Barnes representation (13), since

$$\text{Disc}_{\mathsf{s}}\left[|\vec{k}|^{2(u+\bar{u})}\right] = \sin\left(\pi\left(u+\bar{u}\right)\right)|\vec{k}|^{2(u+\bar{u})}. \tag{16}$$

  The factor $\sin\left(\pi\left(u+\bar{u}\right)\right)$ then cancels the $\csc\left(\pi\left(u+\bar{u}\right)\right)$ which is responsible for generating the off-shell contact terms, leaving the on-shell exchange. The discontinuity of the AdS exchanges (14) read

$$\text{Disc}_{\mathsf{s}}\left[\mathcal{A}_{\text{AdS}}^{\pm}\right] = a_{\Delta_\pm}^2 g_{\Delta_\pm,\ell}, \tag{17}$$

  where $a_{\Delta_\pm}^2$ are the OPE coefficients. Via the identity (15), for the dS exchange we then have

$$\begin{aligned} &2\text{Disc}_{\mathsf{s}}\left[\mathcal{A}_{\text{dS}}^{\text{B.D.}}\right] \qquad (18)\\ &= \csc\left(\tfrac{\pi}{2}(2\Delta_+ - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{12}+\Delta_+ + \ell - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{34}+\Delta_+ + \ell - d)\right)a_{\Delta_+}^2\, g_{\Delta_+,\ell}\\ &+ \csc\left(\tfrac{\pi}{2}(2\Delta_- - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{12}+\Delta_- + \ell - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{34}+\Delta_- + \ell - d)\right)a_{\Delta_-}^2\, g_{\Delta_-,\ell}.\end{aligned}$$

- In flat space, the discontinuity of the (non-trivial) S-matrix $\mathcal{M}$ is related to the imaginary part via the optical theorem

$$2\text{Im}\left(\mathcal{M}\right) = \mathcal{M}^\dagger\mathcal{M}, \tag{19}$$

  which is a consequence of unitarity: $\mathcal{S}^\dagger\mathcal{S} = 1$, where $\mathcal{S} = 1 + \mathcal{M}$. In Lorentzian CFTs, dual to physics in AdS, the analogue of $\text{Im}\left(\mathcal{M}\right)$ is the double-discontinuity dDisc of the CFT correlator [14]. The double-discontinuity of a conformal block is given by:

$$\text{dDisc}\left[g_{\Delta_+,\ell}(u,v)\right] = 2\sin\left(\tfrac{\pi}{2}(\Delta_{12}+\Delta_- + \ell - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{34}+\Delta_- + \ell - d)\right)\, g_{\Delta_+,\ell}(u,v)\,,$$

  from which it follows that the double-discontinuity of the AdS exchanges (14) is

$$\text{dDisc}\left[\mathcal{A}_{\text{AdS}}^{\pm}\right] = 2a_{\Delta_\pm}^2\sin\left(\tfrac{\pi}{2}(\Delta_{12}+\Delta_\mp + \ell - d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{34}+\Delta_\mp + \ell - d)\right)g_{\Delta_\pm,\ell}.$$

Using our identity (15) we can then obtain the double-discontinuity of the dS exchange in the Bunch-Davies vacuum

$$\begin{aligned}\frac{1}{4}\text{dDisc}\left[\mathcal{A}^{\text{B.D.}}_{\text{dS}}\right] &= \sin\left(\tfrac{\pi}{2}(\Delta_{12}+\Delta_{-}+\ell-d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{34}+\Delta_{-}+\ell-d)\right)\\ &\times \sin\left(\tfrac{\pi}{2}(\Delta_{12}+\Delta_{+}+\ell-d)\right)\sin\left(\tfrac{\pi}{2}(\Delta_{34}+\Delta_{+}+\ell-d)\right)\\ &\times \mathcal{F}_{\nu,\ell}\,, \end{aligned} \tag{20}$$

where $\mathcal{F}_{\nu,\ell}$ is the conformal partial wave normalised symmetrically between shadow and non-shadow contributions,

$$\mathcal{F}_{\nu,\ell} = \frac{i\nu}{2\pi}\Gamma\left(i\nu\right)\Gamma\left(-i\nu\right)\left[a^2_{\Delta_+}\,g_{\Delta_+,\ell} - a^2_{\Delta_-}\,g_{\Delta_-,\ell}\right]\,. \tag{21}$$

The above relations should follow as a consequence of perturbative unitarity in de Sitter, relating the full exchange amplitude to its singularities and vice-versa. It would be interesting to make contact with the Cosmological Optical Theorem recently obtained in [15]. The above relations also formally extend to loop level, taking the discontinuity of each internal leg.

- The relation (15) expresses the dS exchange in the Bunch-Davies vacuum as a linear combination of AdS exchanges, which are single-valued functions of the cross-ratios in position space. The dS exchange in the Bunch-Davies vacuum is therefore itself is a single-valued function of the cross ratios in position space and admits a well-defined crossing decomposition into a single channel. Furthermore, given the role exchange amplitudes play non-perturbatively in CFTs dual to AdS physics through the newly discovered Polyakov blocks [16, 17, 18], it is tempting to suggest that the above singularity structure might have a non-perturbative role also in dS.

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