

Plane incomplete contact problems subject to bulk stress with a varying normal load

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Abstract

For a general symmetric incomplete contact, if we can deduce the partial slip solution when it is subject first to a normal load (held constant) and then to a monotonically increasing bulk tension, we show, here, how to obtain the solution when the normal load and bulk tension vary with time in an arbitrary way. The procedure is demonstrated for a Hertzian contact where the constant normal load solution is known in closed form. The procedure reveals the extent of slip and the shear traction distribution at all points within the contact at any instant. It is found that the size of the permanent stick zone, for a given cyclic loading trajectory, is unique in the steady state. Steady state is established after one cycle of loading: that is, it is independent of the transient loading before the steady state cycle is reached, and this same property is observed for the case of varying normal and shear loading ($P - Q$). An example is given to illustrate the type of behaviour that is to be expected.

1. Introduction

Fretting fatigue is an area of significant interest since it is a cause of premature failure in many engineering components. In order to predict the life of components in contact, a detailed knowledge of the local state of stress under service loads is required. We start by categorising contacts into four idealised types, each exhibiting a different behaviour to applied loads [1]. Here, we look at incomplete contacts, where the contact area increases with normal load. We also specialise our analysis to instances where both contacting bodies are uncoupled. That is, the normal and transverse loads only change the normal and shear tractions respectively. The first incomplete contact problem to be solved was by Hertz [2] who looked at pressing two frictionless spheres together. Subsequent analysis has been to incorporate transverse loads and friction to mimic better real service loads. The first steps towards this were made by Cattaneo [3] and independently Mindlin [4] who found the shear traction distribution for a frictional Hertzian contact subject to a shear force. This result was generalised by Ciavarella [5] and Jäger [6] to all incomplete contacts. Barber et al. [7] derived the shear traction distribution for any incomplete contact subject to any combination of normal and shear loads.

In many real contacts, bulk tension acting parallel to the surfaces of the contacting bodies may be present and vary with time, and this, too induces interfacial shear tractions. Progress was made on this front by Nowell and Hills [8], they solved, numerically, the problem of applying the normal load first, holding it constant, and then subjected it to a combination of shear force and bulk tension. Ciavarella and Macina [9] built on this by solving, in closed form, the problem of applying the normal load first then imposing bulk tension only.

Here, we present a generalisation which allows us to use the solution for applying the normal load first then imposing bulk tension to solve the case where the normal load is varying.

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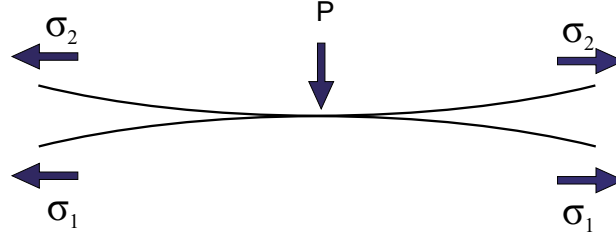


Figure 1: An incomplete contact subject to bulk tension

Figure 1 shows an incomplete contact where each body is subject to bulk tension. It is the difference between the tensions which is responsible for inducing shear tractions $\sigma = \sigma_1 - \sigma_2$. We begin by looking at the loading required to keep the whole contact adhered.

2. Condition for full stick

The condition for full stick of an incomplete contact capable of idealisation by a half-plane and subject to bulk tension, σ , was given by Hills et al. [10]. It depends on the *change* in normal load and bulk tension and is given by

$$\frac{\partial \sigma}{\partial P} < \frac{4f}{\pi a} \quad (1)$$

where f is the coefficient of friction and a is the half-width of the contact. In the formulation developed, the contact half width, a , gives a cleaner measure of the normal load than P itself, and we can rewrite the full-stick condition as

$$\frac{\partial \sigma}{\partial a} < \frac{4f}{\pi a} \frac{dP}{da} \quad (2)$$

In the $P - Q$ problem considered by Barber et al. [7] the lines $Q = \pm fP$ set the bounds for gross sliding, in the case of bulk tension alone, there is no equivalent behaviour. Since the slip is in opposite directions, the contact cannot slide.

3. General loading solution procedure

Given the solution to the sequential loading problem, where we apply the normal load first, hold it constant, and introduce a differential bulk tension, we can solve the problem of general loading trajectories. We track out the problem in $a - \sigma$ space for simplicity of algebra. Consider a general point, a_X, σ_X where the contact is under a state of partial slip. We want to determine the size of the stick zone and the shear tractions in the contact at this loading point. Only bulk tension is present and hence we expect to have a central stick zone with slip zones of opposite sign at either edge of the contact. In the slip zones we expect the following traction distribution,

$$q_X(x) = \pm f p_X(x) \operatorname{sgn}(x); \quad b < |x| < a_X \quad (3)$$

where b is the half width of the stick zone. We use the shorthand notation $p_X(x)$ to denote the contact pressure associated with the normal load P_X and distributed over the interval $-a_X < x < a_X$, where the contact law is $a = a(P)$. Now, consider a previous point in the load trajectory which is fully stuck, (a_Y, σ_Y) . We choose this point such that a_Y is the size of the stick zone at X ($b = a_Y$). This may be seen graphically in Figure 2.

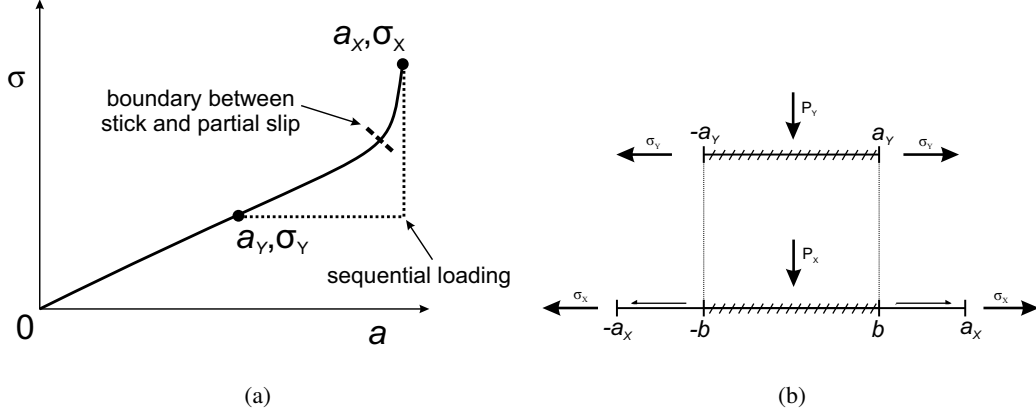


Figure 2: Transformation of the general loading problem

We can consider the problem as two phases of loading. First, from no loading to Y which is fully stuck, and the second phase from Y to X which is in partial slip. During the first phase strain differences are locked into the contact as the material is laid down and subject to differential tensions at the same time. During the second phase the strain difference already locked in is preserved despite additional tension being applied, so that

$$\left. \frac{\partial(u_1 - u_2)}{\partial x} \right|_X = \left. \frac{\partial(u_1 - u_2)}{\partial x} \right|_Y; \quad |x| < a_Y. \quad (4)$$

In order to preserve the strain difference between Y and X there is a condition on the shear traction *change*, within the stick zone, $q_X(x) - q_Y(x)$, from this phase, viz. [8]

$$\left. \frac{\partial(u_1 - u_2)}{\partial x} \right|_{X-Y} = 0 = -\frac{1}{\pi E^*} \int_{-a_X}^{a_X} \frac{q_X(\xi) - q_Y(\xi)}{x - \xi} d\xi - \frac{\sigma_X - \sigma_Y}{E^*}; \quad |x| < a_Y \quad (5)$$

where $E^* = \frac{E}{4(1-\nu^2)}$ is the composite elastic modulus in plane strain. The boundary conditions given by Equations 3 and 5 are the same as those in the bulk stress phase of the sequential loading problem.

4. Specialisation to a Hertzian contact

By writing the full stick condition in terms of $a(P)$ instead of P , the condition for full stick is independent of a for a Hertzian contact. This means that the mean load value in the steady state doesn't affect the gradient ($\frac{d\sigma}{da}$) of loading required to initiate or terminate slip. Therefore, the steady state solution is independent of the mean load. A similar transformation may be made for other types of contact based on the form of the contact law.

We specialise from this point onwards to a Hertzian contact, since for this class of contact we have a closed form solution for the sequential loading problem. We can now use the Hertzian contact law to make explicit the general equations given in previous sections. The contact law is

$$P = \frac{\pi E^* a^2}{4R}, \quad (6)$$

where R is the radius of curvature, and P the applied normal contact load. Using the contact law, the condition for full stick (Equation 2) may be written as

$$\left| \frac{d\bar{\sigma}}{d\bar{a}} \right| < f \quad (7)$$

where

$$\bar{\sigma} = \frac{\sigma}{E^*}; \quad \bar{a} = \frac{2a}{R}; \quad \bar{b} = \frac{2b}{R}. \quad (8)$$

4.1. Sequential application of the normal load and bulk tension

A Hertzian contact is formed, the normal force is subsequently held constant and a differential bulk tension is applied to the bodies, which results in the formation of slip zones of opposite sign attached to the contact edges. The derivation of the extent of the slip zone and the tractions within the stuck region is presented in [9].

The underlying argument for the solution procedure used in [8, 9] is that, when we form an incomplete contact in the absence of bulk tension, equal surface strains ($\epsilon_{xx}(x)$) develop in each body, so that as points on the surfaces of the bodies are brought into contact they remain opposite each other. At this point the surface shear tractions are zero everywhere. When remote tensions are imposed the *total* surface strains in the stick zones, have (a) (equal) contributions from the normal load, together with further contributions from (b) the remote tensions applied, and (c) the (initially partly unknown) interfacial shearing tractions, so that

$$F(x) \equiv \frac{\partial u_1}{\partial x} - \frac{\partial u_2}{\partial x} = -\frac{1}{\pi E^*} \int_{-\bar{a}}^{\bar{a}} \frac{q(\xi)}{x - \xi} d\xi - \frac{\sigma}{E^*}. \quad (9)$$

We now set, the strain difference to be zero within the stick region

$$F(x) = 0 \quad -\bar{b} < x < \bar{b}, \quad (10)$$

where \bar{b} is the stick zone half width, and note that, in the slip zones

$$q(x) = f p(x) \text{sgn}(x) \quad \bar{b} < |x| < \bar{a}. \quad (11)$$

Equations 10 and 11 define an integral equation of the first kind with a Cauchy kernel, in terms of the shear traction in the stick zone and where the additional consistency condition due to the shear tractions being bounded at both ends $x = \pm\bar{b}$ of the stick zone enables the value of \bar{b} to be found. The solution is

$$K(k) - E(k) = \frac{\pi\sigma}{8fp_0}, \quad (12)$$

where

$$\frac{\bar{b}}{\bar{a}} = \sqrt{1 - k^2}, \quad (13)$$

and where $K(k)$, $E(k)$ are, respectively, the complete elliptic integrals of the first and second kind. A full definition of the elliptic integrals used and some useful relations are given in Appendix B. We also define $p_0 = \frac{2P}{\pi a}$ as the the peak pressure in the contact for a given normal load. Figure 3 shows the size of the stick zone for a given magnitude of bulk tension. Note that at the limit, an infinite tension is required to achieve a zero stick zone size. We are also able to write down the shear traction distribution within the stuck region, $q_s(x; a, b)$, from Ciavarella and Macina [9] who wrote

$$q_s(x; \bar{a}, \bar{b}) = \frac{2}{\pi} f p_0 \frac{x}{\bar{a}} \sqrt{\frac{\bar{b}^2 - x^2}{\bar{a}^2}} \left[\Pi\left(\sqrt{\frac{\bar{a}^2 - \bar{b}^2}{\bar{a}^2 - x^2}}, \sqrt{1 - \frac{\bar{b}^2}{\bar{a}^2}}\right) - K\left(\sqrt{1 - \frac{\bar{b}^2}{\bar{a}^2}}\right) \right]; \quad |x| < \bar{b} \quad (14)$$

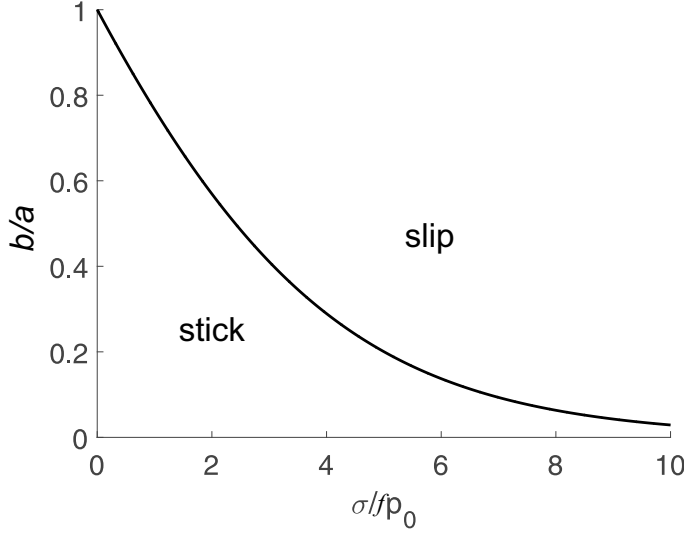


Figure 3: Slip and stick regions for a Hertzian contact with bulk stress

where $\Pi(m, n)$ is the complete elliptic integral of the third kind. Note that the term outside the square brackets is the fundamental function for a square root bounded solution, and which clearly goes to zero at either end of the interval, whereas the terms within the square bracket shows local square root singular behaviour at the ends of the interval, such that the product of the two is finite.

Expanding the elliptic integral for points near $x = \pm b$ and taking the limit of the resulting shear traction as these points are approached, we find that the traction tends to a constant which is continuous with the slip tractions in $|x| > b$.

$$q_s(\bar{b}; \bar{a}, \bar{b}) = \frac{2}{\pi \bar{a}} f p_0 \frac{\bar{b}}{\bar{a}} \sqrt{\bar{b}^2 - x^2} \left[\frac{\pi \bar{a}}{2 \bar{b}} \frac{\sqrt{\bar{a}^2 - \bar{b}^2}}{\sqrt{\bar{b}^2 - x^2}} + \dots \right] \approx \pm f p_0 \sqrt{1 - \frac{\bar{b}^2}{\bar{a}^2}}; \quad |x| \rightarrow \bar{b}^-. \quad (15)$$

5. Periodic loading

The process outlined in the following sections is an extension to the method developed by Barber et al. [7] which considers incomplete contacts under varying shear and normal loads. In the $Q-P$ case, the process was more straightforward due to the direct application of the Ciavarella-Jäger theorem [5, 6]. The size of the stick zone was found by drawing a straight line of gradient $\pm f$ from the current load state, until it meets a previous point in the load cycle, utilising arguments based on equilibrium. In the case of bulk tension, we require curvilinear lines to define the corrective shear traction in the stick region and its size which for a Hertzian contact are defined by Equations 22 (forward slip) and 25 (reverse slip), both derived from the sequential loading problem. If we want to consider any other symmetric incomplete contact, we just need to find the analogous equations for that geometry. The procedure may then be carried out in an identical fashion.

We consider a general elliptic loading path in $\bar{a} - \bar{\sigma}$ space, which is centered on a point $(\bar{a}_0, \bar{\sigma}_0)$. The initial loading to get to the ellipse ensures full stick. Thus it must not violate the inequality given in Equation 7.

The general parametric equation of an ellipse is

$$\bar{a} = \bar{a}_0 + \bar{a}_c \sin(t); \quad \bar{\sigma} = \bar{\sigma}_0 + \bar{\sigma}_c \sin(t - \phi). \quad (16)$$

When the phase lag, ϕ , is zero the ellipse collapses to a straight line. In this instance, the gradient of the line will determine if we have slip during the cycle. If there is a non zero phase lag, there will be slip present during the cycle, at least when the gradient is negative and we have $dP < 0$ and so $da < 0$.

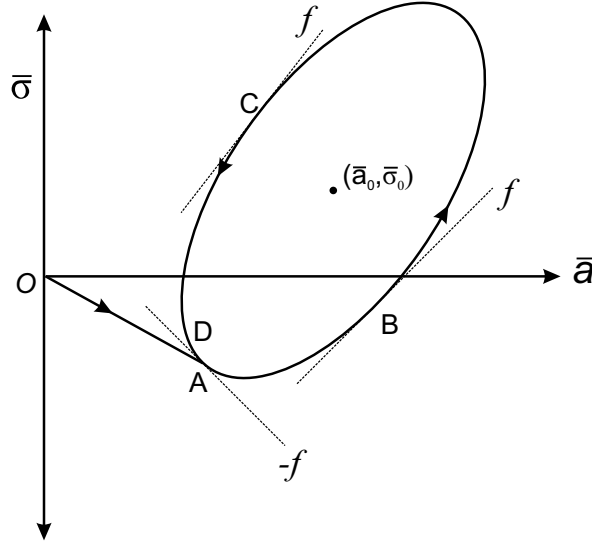


Figure 4: The loading cycle

We can freely switch between using a or P as the variable representing the normal loading:

$$P = \frac{\pi E^*}{4R} (a_0 + a_c \sin(t))^2. \quad (17)$$

Throughout the solution procedure, X represents the point on the load cycle (a_X, σ_X) for which we want a solution. The point $Y (a_Y, \sigma_Y)$ denotes the point during the immediately preceding loading cycle at which the contact area was equal to the instantaneous stick area at loading point X .

6. Transient cycle

6.1. Full stick (OB)

The initial loading and the sector of the ellipse from A to B is fully stuck as can be seen in Figure 4. To find the shear traction distribution incorporated during this loading, we begin with the contribution from an increment of bulk loading

$$\Delta q = \frac{-x \Delta \bar{\sigma}}{4 \sqrt{\bar{a}^2 - x^2}}. \quad (18)$$

Assuming the increment is small, we can take a derivative with respect to \bar{a} .

$$\frac{\partial q}{\partial \bar{a}} = \frac{-x}{4 \sqrt{\bar{a}^2 - x^2}} \frac{\partial \bar{\sigma}}{\partial \bar{a}}(\bar{a}). \quad (19)$$

Integrating Equation 19 gives the fully stuck traction distribution at a general point X which lies between $O \rightarrow A \rightarrow B$,

$$q_X(x) = \int_0^{\bar{a}_X} \left[\frac{-x}{4 \sqrt{\bar{a}^2 - x^2}} \frac{d\bar{\sigma}}{d\bar{a}}(\bar{a}) \right] d\bar{a}. \quad (20)$$

6.2. Forward slip (BC)

The contact enters a state of partial slip from point B (refer to Figure 5), when the gradient of the cycle crosses, f , violating the full stick condition. We know that the addition of bulk tension will cause slip zones of opposite sign at the edges of the contact with a central stick zone. Since we know the normal traction, we can write down the shear traction distribution for a point X (see Figure 5), in the slip zones

$$q_X(x) = f p_X(x) \text{sgn}(x); \quad \bar{b} < |x| < \bar{a}_X, \quad (21)$$

where, b is the size of the stick zone. To find the size of the stick zone, we draw a line

$$\bar{\sigma}(\bar{a}) = \bar{\sigma}_X - \frac{2f\bar{a}_X}{\pi} \left[K \left(\sqrt{1 - \left(\frac{\bar{a}}{\bar{a}_X} \right)^2} \right) - E \left(\sqrt{1 - \left(\frac{\bar{a}}{\bar{a}_X} \right)^2} \right) \right] \quad (22)$$

from X on the load cycle to locate a previous point in the load cycle, Y , which was fully stuck. An example line of this kind, marked in red can be seen in Figure 5.

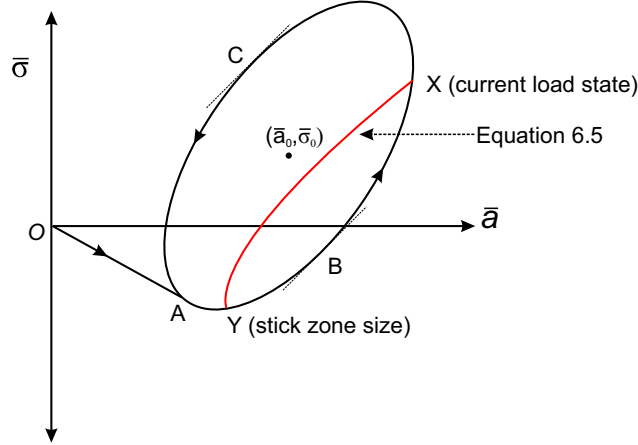


Figure 5: Finding the size of the stick zone

As X moves along the cycle between B and C the predicted size of Y becomes smaller, so throughout this phase the absolute size of the stick zone recedes although the contact becomes initially larger, then smaller. Initially, the point Y will be mapped onto the cycle itself, but as Y becomes smaller than A , it intersects the initial loading (OA). Therefore, in the first cycle, the size of the stick zone depends on the initial loading. For all subsequent cycles, the initial loading only affects the locked in shear tractions in regions that have never slipped. The traction distribution within the stick zone at a general point X between B and C , Figure 5 is given by

$$q_X(x) = q_S(x; \bar{a}_X, \bar{a}_Y) + \int_0^{\bar{a}_Y} \left[\frac{-x}{4\sqrt{\bar{a}^2 - x^2}} \frac{d\bar{\sigma}}{d\bar{a}}(\bar{a}) \right] d\bar{a}; \quad |x| < \bar{a}_Y. \quad (23)$$

where $q_S(x; a, b)$ is defined by Equation 14. The tractions are made up of two parts, which originate from the way we have segmented the loading. First, the fully stuck traction distribution from the initial loading stage (up to point Y) which locks in displacements into the zone that will remain stuck. The next stage introduces tractions which ensure that the locked in displacements are preserved. We know the traction distribution that does this, since we solved for it in the sequential loading problem.

We locate C , the point where slip no longer advances, to be when the the gradient of the line used to determine Y is equal to the gradient of the cycle, i.e.

$$\left. \frac{d\bar{\sigma}(\bar{a})}{d\bar{a}} \right|_{\bar{a}=\bar{a}_C} = \frac{2f}{\pi} K(0) = f \quad (24)$$

which shows us that C may be found when the gradient of the cycle is equal to the coefficient of friction. At C , the contact becomes instantaneously stuck.

6.3. Reverse slip (CD)

Reverse slip begins as soon as X is past C . We retain the notation that X is a point on the cycle where the solution is required and Y is the location of the point determining the size of the stick zone (and the shear traction distribution within it). The location of Y is determined by drawing a line from X given by

$$\bar{\sigma}(\bar{a}) = \bar{\sigma}_X + \frac{2f\bar{a}_X}{\pi} \left[K\left(\sqrt{1 - \frac{\bar{a}^2}{\bar{a}_X^2}}\right) - E\left(\sqrt{1 - \frac{\bar{a}^2}{\bar{a}_X^2}}\right) \right] \quad (25)$$

(Note the change in sign from Equation 22) and finding its intersection with Equation 22 when $X \rightarrow C$. Refer to Figure 6 which shows the line used to find the size of the stick zone. The method is the same as in the forward slip case, except we look for the intersection with the line from YC to C and not the load cycle or initial path see Figure 6. We are looking to conserve the locked in displacements in the stick region defined by a new point Y , which can again be done with the same form of solution as the sequential loading case.

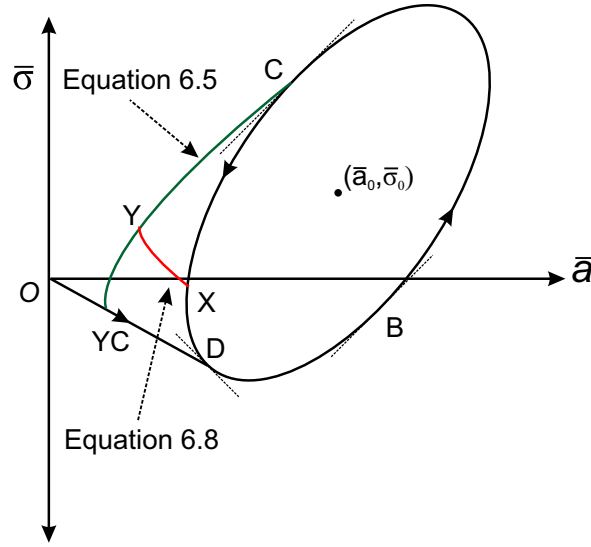


Figure 6: Finding the size of the stick zone in reverse slip

The shear traction is a superposition of the tractions at Y and those of the equivalent sequential loading from Y to X . We consider the tractions at Y to be those found from unloading along the line $C \rightarrow YC$, this ensures that in the new stick zone, the tangential strains locked in at C are preserved. The tractions in the slip zone are

$$q_X(x) = -fp_X(x)\text{sgn}(x); \quad \bar{a}_Y < |x| < \bar{a}_X. \quad (26)$$

Moving inwards from the edges, in the region which is now stuck but experienced forward slip just before C , the tractions are given by

$$q_X(x) = fp_Y(x)\text{sgn}(x) - q_S(x; \bar{a}_X, \bar{a}_Y); \quad \bar{a}_{YC} < |x| < \bar{a}_Y. \quad (27)$$

If the current stick zone becomes smaller or equal to the size of the stick zone just before C , this region will be null. Moving closer to the center of the contact, in the region which has always remained stuck, the traction due to the fully stuck initial loading is preserved:

$$q_X(x) = -q_S(x; \bar{a}_X, \bar{a}_Y) + q_S(x; \bar{a}_Y, \bar{a}_{YC}) + \int_0^{\bar{a}_{YC}} \left[\frac{-x}{4\sqrt{\bar{a}^2 - x^2}} \frac{d\bar{\sigma}}{d\bar{a}}(\bar{a}) \right] d\bar{a}; \quad |x| < \bar{a}_{YC}. \quad (28)$$

Reverse slip continues until D , where the contact becomes fully stuck, because the cycle no longer violates the full stick inequality given by Equation 7. Thus, D is found as the point when the gradient of the cycle is given by $-f$.

7. The second cycle

Much like the first cycle, we start off fully stuck, until we reach B . The traction distribution at a general point, X , between D and B is given by:

$$q_X(x) = q_D(x) + \int_{\bar{a}_D}^{\bar{a}_X} \left[\frac{-x}{4\sqrt{\bar{a}^2 - x^2}} \frac{d\bar{\sigma}}{d\bar{a}}(\bar{a}) \right] d\bar{a} \quad (29)$$

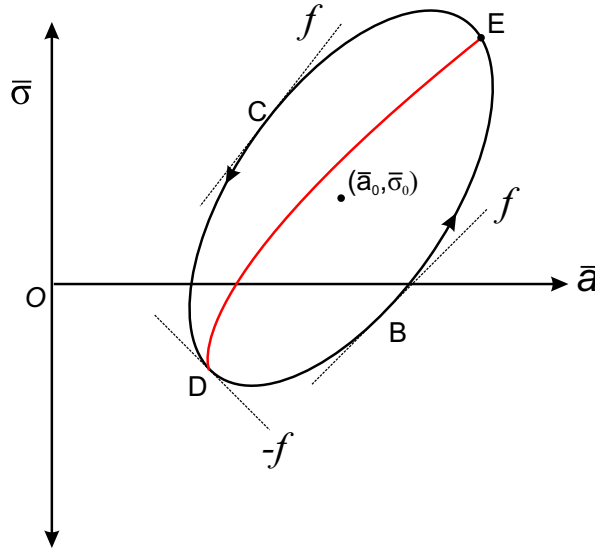


Figure 7: The second cycle

Forward slip begins from the point B where the gradient of the cycle crosses f . The size of the stick zone is given by the same procedure as in the first cycle. The traction distributions in the second cycle may also be found in an identical fashion to the first cycle, the superposition of the tractions at Y and the sequential loading from Y to X .

This continues until point E (see Figure 7), where the size of the stick zone is equal to that defined by point D . As we move beyond E , the stick zone becomes smaller than D , we no longer have an obvious corrective solution as the cycle was accumulating reverse slip before D . Once we pass E on the second cycle, the slip zone starts to erase the reverse slip tractions locked in at D and a new solution is necessary.

7.1. The phase EC

Instead of looking at the intersection of Equation 22 with the initial load path (which has now been erased) or the loop itself, we need the point where it intersects the line used to find the size of the stick zone from point D when the contact was in reverse slip. This is given by the Equation 25 as $\bar{a}_X \rightarrow D$. This can be seen in Figure 8 as the intersection of the red and green lines denoted by the point Y .

At C , the contact again becomes instantaneously stuck. Just before C , the size of the stick zone is given by F (intersection of green and blue lines in Figure 8). Moving beyond C , the reverse slip solution is found in an identical

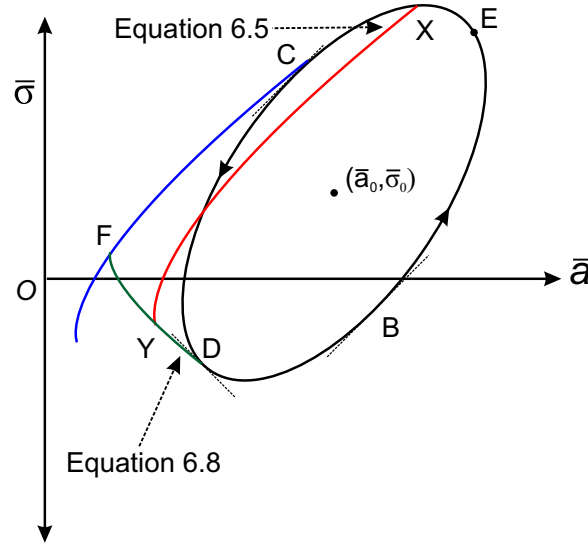


Figure 8: Stick zone size in the second cycle

fashion to the first cycle until D . All further cycles will follow the same solution and so steady state has been attained after one cycle of loading.

8. The permanent stick zone

The permanent stick zone size is denoted by the point F in Figure 8. It is found from the intersection of Equation 22 where $X \rightarrow C$ and Equation 25 where $X \rightarrow D$. The size of the permanent stick zone is dependent wholly on the steady state loading cycle and independent of the initial loading. The locked in tangential strains in the stick zone are however from the initial loading. What we find is that in the steady state, the locked in tangential strains have no effect on the size of the slip zone. We also know that these tractions have no effect on the slip displacements in the steady state [11]. This is a very useful result for fretting fatigue calculations. We do not need to concern ourselves with the transient problem when looking at the steady state behaviour.

9. Example

To illustrate the procedure we look at an example loading cycle, defined by:

$$a = 0.5 + 0.07 \sin(t); \quad \sigma = 0.05 + 0.1 \sin(t - 0.9) \quad (30)$$

Figure 9 shows the size of the contact and the slip-stick boundary as we move around the cycle twice. There are four different types of behaviour that may be seen for a particle in the contact region. In the first cycle we see the smallest stick zone, this is because the size is determined by the initial loading. It can be seen that the second cycle has attained steady state and all further cycles will have the same solution. The initial loading only contributes to the locked in shear tractions of the permanent stick zone and has no effect on the size of the stick zone in the steady state cycle.

1. Near the edge, particles only experience forward slip, they lift off and move back to their original positions when not in contact.
2. Moving closer to the center, some particles experience a combination of forward slip, some reverse slip and lift off and return to their original location.

3. Closer still, the particles are subject to equal amounts of forward and reverse slip.
4. The innermost region is in a state of full stick throughout the loading cycle.

The normalised shear tractions at significant points in the load cycle are given in Figure 10 for the transient cycle and Figure 11 for the steady state. The blue dashed lines are the normalised normal tractions ($\text{sgn}(x)p(x)/p_0(a_0)$) at the respective point. We can see that the transient loading has locked in tangential strains in the stick zone which are preserved in subsequent cycles.

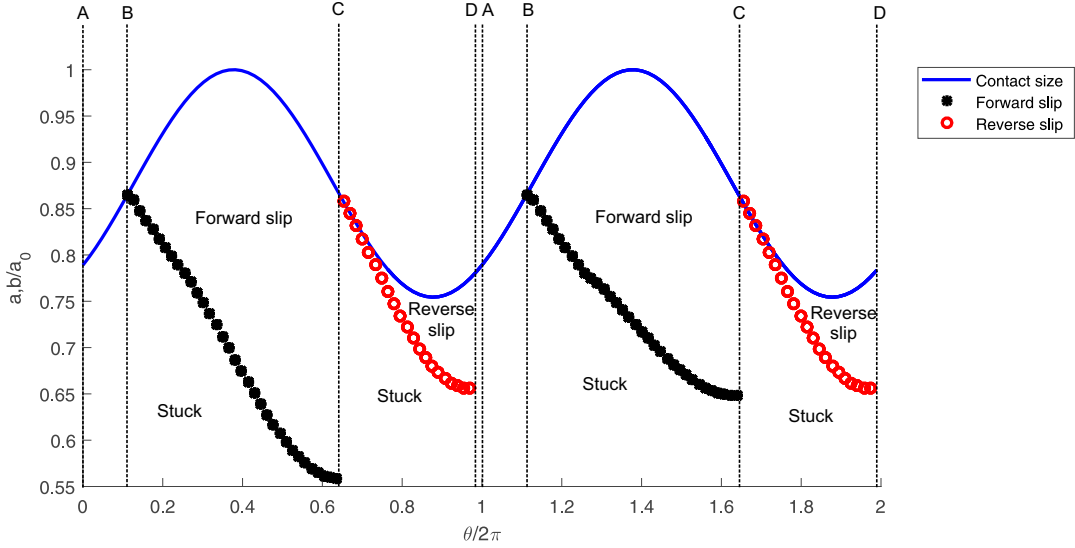


Figure 9: Size of contact and slip-stick boundary ($f = 0.8$)

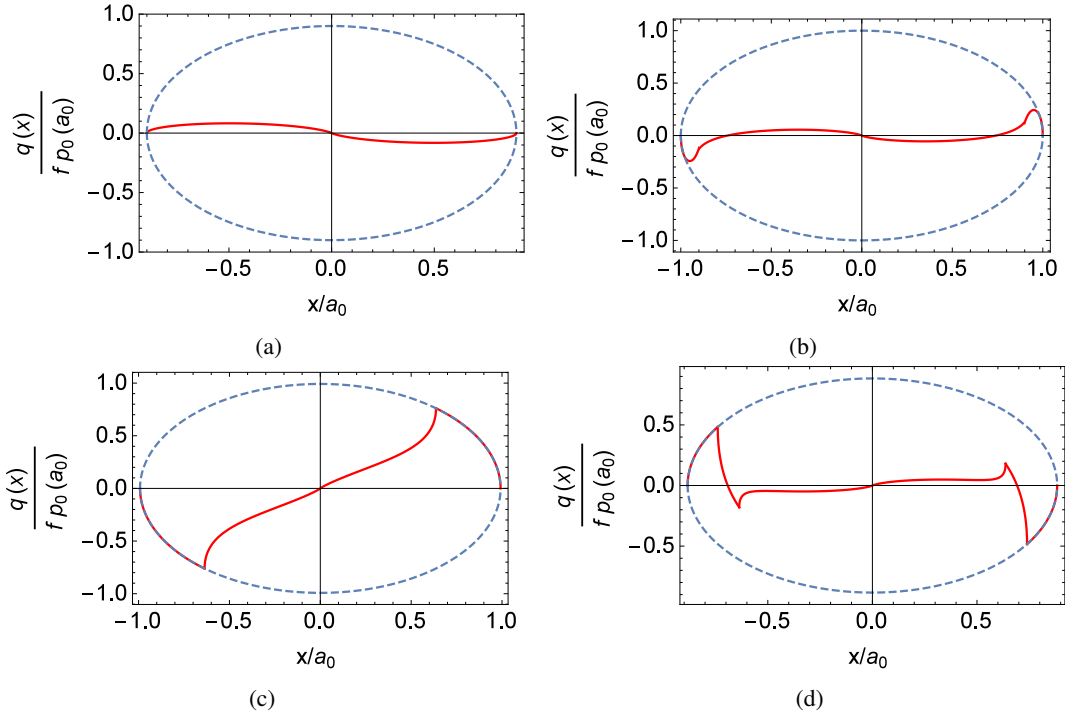


Figure 10: Traction in the first cycle

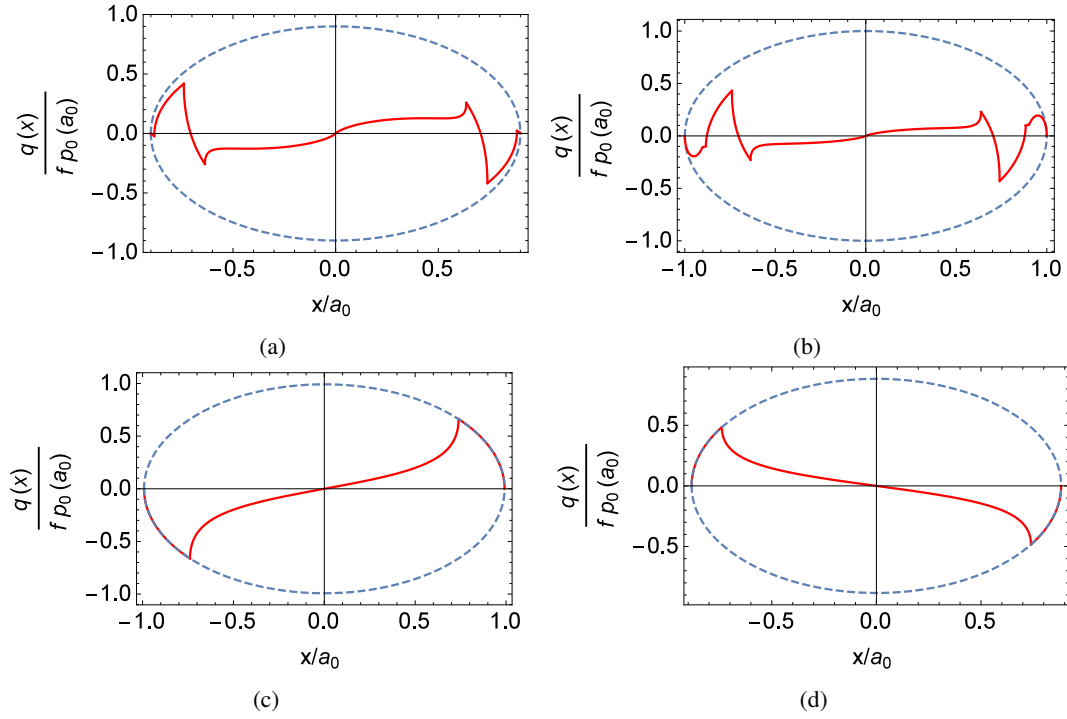


Figure 11: Tractions in the steady state cycle

10. Conclusion

The shear traction distribution and the size of stick zones in an incomplete contact subject to varying normal load and bulk tension solution may be found using the superposition principles adduced. We illustrate the procedure for a Hertzian contact taken around an elliptic load path in $a - \sigma$ space. There are four distinct regions we find in the contact region as we move around a cycle. It is not obvious which region will have the greatest energy dissipation. We find the size of the permanent stick zone is independent of the initial loading trajectory, which was also found to be the case for the constant bulk stress, varying $P - Q$ problem.

References

- [1] R. Ramesh and D. A. Hills. Recent progress in understanding the properties of elastic contacts. *J. Mech. Eng. Sci.*, 229(12):2117 – 2126, 2014.
- [2] H. Hertz. On contact between elastic bodies. *Journal fur die Reine und Angewandte Mathematic, (In German)*, pages 156–171, 1882.
- [3] C. Cattaneo. Sul contatto di due corpi elastici. *Atti Accad. naz. Lincei Rc*, 27:342–348, 434–436, 474–478, 1938.
- [4] R. D. Mindlin. Compliance of elastic bodies in contact. *J. Appl. Mech. Trans. ASME*, 16:249 – 268, 1949.
- [5] M. Ciavarella. The generalized Cattaneo partial slip plane contact problem. i - theory. *Int. J. Solids Struct.*, 35(18):2349 – 2362, 1998.
- [6] J. Jäger. A new principle in contact mechanics. *J. Tribol.*, 120(4):677 – 684, 1998.
- [7] J. R. Barber, M. Davies, and D. A. Hills. Frictional elastic contact with periodic loading. *Int. J. Solids and Structures*, 48:2041 – 2047, 2011.
- [8] D. Nowell and D. A. Hills. Mechanics of fretting fatigue tests. *Int. J. Mechanical Sciences*, 29(5):335 – 365, 1987.
- [9] M. Ciavarella and G. Macina. New results for the fretting-induced stress concentration on hertzian and flat rounded contacts. *Int. J. Mech. Sci.*, 45:449– 467, 2003.
- [10] D. A. Hills, M. Davies, and J. R. Barber. An incremental formulation for half-plane contact problems subject to varying normal load, shear and tension. *J. Strain Analysis*, 46(6):436 – 443, 2011.
- [11] L. E. Andersson, J. R. Barber, and A. R. S. Ponter. Existence and uniqueness of attractors in frictional systems with uncoupled tangential displacements and normal tractions. *Int. J. Solids Struct.*, 51:3710 – 3714, 2014.

Appendix A. Elliptic integrals

The complete elliptic integral of the first kind is defined as

$$K(k) = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \quad (\text{A.1})$$

The complete elliptic integral of the second kind is defined as

$$E(k) = \int_0^1 \frac{\sqrt{1-k^2t^2}}{\sqrt{1-t^2}} dt \quad (\text{A.2})$$

The complete elliptic integral of the third kind is defined as

$$\Pi(n, k) = \int_0^{\frac{\pi}{2}} \frac{d\theta}{(1 - n \sin^2 \theta) \sqrt{1 - k^2 \sin^2 \theta}} \quad (\text{A.3})$$

The derivatives of the integrals are given by

$$\frac{d}{dk} K(k) = \frac{E(k)}{k(1-k^2)} - \frac{K(k)}{k} \quad (\text{A.4})$$

$$\frac{d}{dk} E(k) = \frac{E(k) - K(k)}{k} \quad (\text{A.5})$$