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Strongly dense free subgroups of semisimple algebraic groups II [☆]



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ABSTRACT

It was shown in [9] that there exist *strongly dense* free subgroups in any semisimple algebraic group over a large enough field. These are nonabelian free subgroups all of whose subgroups are either cyclic or Zariski-dense. Here we show that the same is true for as long as the transcendence degree of the field is at least 1 in characteristic 0 and at least 2 in positive characteristic. We also consider related questions for surface groups.

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1. Introduction

There is a long history regarding free subgroups of linear groups, in which J. Tits played a prominent role with his celebrated alternative [39]: every finitely generated linear group either contains a non-abelian free subgroup or a solvable subgroup of finite index. We consider here a somewhat stronger property. If G is a semisimple algebraic group over a field K , then a *strongly dense free subgroup* of G over K is a nonabelian free subgroup Γ of $G(K)$ such that every nonabelian subgroup Δ of Γ is Zariski-dense over K in G (meaning that there is no proper closed K -subgroup H over G such that $\Delta \subset H(K) \subset G(K)$).

It was shown in [9,10] that if G is a semisimple algebraic group over an algebraically closed field K of sufficiently large transcendence degree over the prime field (the degree depending upon $\dim G$), then G always contains a strongly dense subgroup. We gave applications to the Banach-Hausdorff-Tarski paradox and to generation properties for finite simple groups of Lie type of bounded rank and also used this to study Cayley graph expanders associated to finite simple groups of Lie type.

In this note, we show the existence of such subgroups over much smaller fields. In particular, we prove:

Theorem 1.1. *Let K be a transcendental extension of \mathbb{Q} and G a split semisimple algebraic group over K . Then G contains a strongly dense free subgroup over K . In fact, the set of pairs $(a, b) \in G(K) \times G(K)$ generating a strongly dense free subgroup is Zariski-dense in $G \times G$.*

Theorem 1.2. *Let K be an extension of \mathbb{F}_p of transcendence degree ≥ 2 and G a split semisimple algebraic group over K . Then G contains a strongly dense free subgroup over K . In fact, the set of pairs $(a, b) \in G(K) \times G(K)$ generating a strongly dense free subgroup is Zariski-dense in $G \times G$.*

Letting $L = K(t)$ with K a global field, we produce d -generated free subgroups Γ of $G(A)$, where A is a finitely generated K -subalgebra of L , so that if we reduce modulo certain maximal ideals \mathfrak{m}_i of A , we get strongly dense free subgroups of the K_i -points of

predetermined special subgroups H_i of G , where $K_i = A/\mathfrak{m}_i$. These H_i are images in G of products of groups of the form $\mathrm{SL}_1(D_{i,j})$, where each $D_{i,j}$ is a central division algebra of prime degree over K_i . These $\mathrm{SL}_1(D_{i,j})$ are special because all of their proper closed subgroups over K_i are close to being abelian, and it is this property that enables us to prove strong density. Then using some algebraic geometry and algebraic group theory, we see that Γ , regarded as a subgroup of $G(L)$, must be free and strongly dense in G . The strategy is not so dissimilar from that of [9] where degenerations of certain maximal rank subgroups of G were studied. However, a key feature of the proof in [9] was the use of the result of Borel [6] that word maps on algebraic groups are dominant (see also [23]). This is not used in our proof of the existence of free strongly dense subgroups, which uses the properties of special subgroups instead. A major advantage of the new strategy is that it lets us work over fields of lower transcendence degree.

A key technical ingredient, which may be of independent interest, is Theorem 4.7, relating the Zariski closure of the generic image of a family of representations of a group Γ to the Zariski closure of a member of the family.

We can also prove strong density results for certain non-free groups.

Definition 1.3. A nonabelian subgroup Γ of a semisimple algebraic group G over K is *strongly dense* (over K) if every nonabelian subgroup of Γ is Zariski-dense in G over K .

As an illustration of the definition, we prove that in a number of situations, $G(K)$ contains strongly dense surface groups.

Theorem 1.4. Let $\Sigma_g, g \geq 2$ be the fundamental group of a compact orientable surface of genus g and G a semisimple algebraic group over an algebraically closed field K . Then there are strongly dense embeddings of Σ_g in G over K if any of the following conditions holds:

1. K has infinite transcendence degree over its prime field.
2. K is transcendental over \mathbb{Q} and G is a classical group.
3. K is of characteristic $p \geq 3$ and is of transcendence degree ≥ 10 over \mathbb{F}_p , and G is a special linear group.

We also prove the existence of dense (or strongly dense) embeddings of other finitely generated groups Γ in $G(K)$ under certain assumptions on Γ , G and K . See Theorems 8.1 and 8.5.

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2. Representation varieties

Throughout this section, we fix a linear algebraic group $G = \mathrm{Spec} B$ over a field K_0 . We will recall here the definition of the *representation variety* of a discrete group with

values in G (see also [32,35]). If Γ is a d -generated group, we define \mathcal{F}_Γ to be the functor from K_0 -algebras A to sets given by

$$A \mapsto \text{Hom}(\Gamma, G(A)).$$

When Γ is the free group F_d , the functor \mathcal{F}_Γ is represented by the affine K_0 -scheme $G^d = \text{Spec } B^{\otimes d}$, and evaluation at any element γ in Γ defines a morphism e_γ of K_0 -schemes $G^d \rightarrow G$. The fiber of e_γ over the identity of G defines a closed subscheme of G^d associated to an ideal I_γ in the coordinate ring $B^{\otimes d}$. If $\Gamma = F_d/N_\Gamma$, we define

$$I_\Gamma = \sum_{\gamma \in N_\Gamma} I_\gamma.$$

We define $C_\Gamma = B^{\otimes d}/I_\Gamma$, so $\text{Spec } C_\Gamma$ is a closed subscheme of G^d . For every K_0 -algebra A , an element of $\mathcal{F}_\Gamma(A)$ corresponds to a homomorphism $\text{Hom}(\Gamma, G(A))$, i.e., a homomorphism $F_d \rightarrow G(A)$ which restricts to the identity at each $\gamma \in N_\Gamma$. Equivalently, it is an element of $G^d(A)$ which maps by e_γ to the identity, i.e., an element of $\text{Hom}_{K_0}(\text{Spec } A, G)$ whose composition with e_γ is the morphism $\text{Spec } A \rightarrow G$ which factors through the identity map $\text{Spec } K \rightarrow G$. This means that $\text{Spec } A \rightarrow G^d$ factors through $\text{Spec } B^{\otimes d}/I_\gamma$ for all $\gamma \in I_\Gamma$, or, equivalently, a morphism $\text{Spec } A \rightarrow \text{Spec } C_\Gamma$. Thus \mathcal{F}_Γ is representable by the affine scheme $\text{Spec } C_\Gamma$, which we call the *representation scheme*.

Let A_Γ denote the quotient of C_Γ by its nilradical. If we restrict to K_0 -algebras A with radical (0) , we have

$$\text{Hom}_{K_0}(C_\Gamma, A) = \text{Hom}_{K_0}(A_\Gamma, A).$$

We define $\text{Spec } A_\Gamma$ to be the *representation variety* of Γ (where G and K_0 are understood). In particular, the identity map on A_Γ defines an element of $\mathcal{F}_\Gamma(A_\Gamma)$, which we call the *universal representation*.

This representation variety is irreducible if and only if A_Γ is an integral domain. If K is any field extension of A_Γ , the universal representation defines a homomorphism $\Gamma \rightarrow G(K)$ which corresponds to a map from $\text{Spec } K$ to the representation variety whose image is the generic point. Various conditions on Γ -representations in G determine closed subsets of the representation variety defined over K_0 ; unless such a condition holds for all representations, it does not hold for $\Gamma \rightarrow G(K)$ of the kind we are discussing. Therefore, if K is any algebraically closed field of transcendence degree $\geq \dim \text{Spec } A_\Gamma$ over K_0 , there exists a homomorphism $\Gamma \rightarrow G(K)$ satisfying no proper closed condition. This can be useful, for example, in proving that there exists an injective homomorphism from Γ to $G(K)$.

3. Some groups over global fields

We first point out the following easy result (see also [15, Prop. 4.1]).

Lemma 3.1. *Let K be any field, p a prime, D a central division algebra of degree p over K , and $G = \mathrm{SL}_1(D)$ the inner form of SL_n over K associated to D . Then every proper K -subgroup of positive dimension of G is contained in the normalizer of a maximal torus of G .*

Proof. Let H denote a proper connected subgroup of G . As $G(K) \subset D^\times$, it contains no non-trivial unipotent elements, so the same must be true for H . In particular, H is reductive. If S denotes the derived group of H , it is semisimple of dimension less than $p^2 - 1$ and therefore of rank $r < p - 1$. Let T denote a maximal torus of S defined over K . Let K^s denote a separable closure of K , and $X \cong \mathbb{Z}^r$ the character group of T . Then T determines a continuous homomorphism $\mathrm{Gal}(K^s/K) \rightarrow \mathrm{Aut}(X) \cong \mathrm{GL}_r(\mathbb{Z})$ whose image I is finite. There are no elements of order p in I since the minimal polynomial of a primitive p th root of unity has degree $p - 1 > r$. Thus T splits over a Galois extension L/K such that $\mathrm{Gal}(L/K) \cong I$ is of prime to p order. Therefore the class of D does not lie in the kernel of $\mathrm{Br}(K) \rightarrow \mathrm{Br}(L)$, and $D_L := D \otimes_K L$ is a central division algebra over L .

Thus $\mathrm{SL}_1(D_L)$ is an algebraic group over L containing a non-trivial split torus T_L , i.e., an isotropic semisimple group. It follows that $\mathrm{SL}_1(D_L)$ contains non-trivial unipotents [36, 3.4 (iii)], which is absurd since D_L has no non-trivial nilpotents. \square

In fact over a global field, groups of type A_m with $m + 1$ prime are the only simple algebraic groups that have a form with the above property (called almost abelian). See [15] for more about this.

Lemma 3.2. *Let K be a global field and D_1, \dots, D_m K -central division algebras of prime degrees p_1, \dots, p_m respectively such that $\mathrm{SL}_1(D_1), \dots, \mathrm{SL}_1(D_m)$ are pairwise non-isomorphic. Then*

$$G(K) = G_1(K) \times \dots \times G_m(K) = \mathrm{SL}_1(D_1) \times \dots \times \mathrm{SL}_1(D_m)$$

contains a strongly dense free subgroup.

Proof. By the Tits alternative, there exist elements $x_i, y_i \in \mathrm{SL}_1(D_i)$ which generate a free subgroup F_2 . Let $\Gamma = \langle (x_1, \dots, x_m), (y_1, \dots, y_m) \rangle$. Since x_1 and y_1 satisfy no word relation, the same is true for (x_1, \dots, x_m) and (y_1, \dots, y_m) , so Γ is isomorphic to F_2 . Let $\Delta \subset \Gamma$ be a non-abelian subgroup. To show that Δ is Zariski-dense in G , by Goursat’s lemma ([19, 5.5.1]) and the fact that the G_i are simple as algebraic groups and pairwise non-isomorphic, it suffices to prove that the projection of Δ on each factor G_i is Zariski-dense.

Let H_i denote the Zariski closure of the image of Δ in G_i . Every proper subgroup of G_i is contained in the normalizer of a maximal torus, so we may assume H_i normalizes T_i , but as Δ is non-abelian, it is a free group on ≥ 2 generators and therefore not contained in the normalizer of a torus. \square

Corollary 3.3. *With K and p_i as above, there exists a finite separable extension L/K such that $SL_{p_1} \times \cdots \times SL_{p_m}$ contains a strongly dense free subgroup over L .*

Proof. Let L be a finite separable extension of K' which splits all the $SL_1(D_i)$. \square

4. Degenerations

In this section, we introduce a variant of the notion of *degeneration* introduced in [9]. Let K_0 be a field, A a K_0 algebra, and n a positive integer. Let P_1 and P_2 denote prime ideals of A such that $P_1 \subset P_2$, and let K_i denote the field of fractions of A/P_i . Let Γ be a finitely generated subgroup of $GL_n(A)$ and let G_1 and G_2 denote the Zariski closures over K_1 and K_2 respectively, of the images of Γ in $GL_n(K_1)$ and $GL_n(K_2)$ respectively.

Definition 4.1. In this situation, we say G_2 is a *degeneration* of G_1 or G_1 can *degenerate* to G_2 .

Note that if G_1 can degenerate to G_2 , then G_1° can degenerate to G_2° . Indeed Γ maps to $G_1(K_1) \times G_2(K_2)$, and the inverse image of $G_1^\circ(K_1) \times G_2^\circ(K_2)$ is of finite index in Γ . For $i = 1, 2$, replacing Γ by this finitely generated subgroup replaces G_i by a finite index subgroup of G_i° and therefore with G_i° itself.

Proposition 4.2. *Let $P_1 \subset P_2 \subset A$, K_i , $\Gamma \subset GL_n(A)$ a finitely generated group, and G_i be as above. Let X be a projective scheme over A endowed with an action of $(GL_n)_A$ defined over A . If $(G_1)_{\overline{K_1}}$ has a fixed point on $X_{\overline{K_1}}$, then $(G_2)_{\overline{K_2}}$ has a fixed point on $X_{\overline{K_2}}$.*

Proof. Let $\Gamma = \langle \gamma_1, \dots, \gamma_d \rangle$. The embedding of Γ in $GL_n(A)$ defines a section of $(GL_n)_A^d \rightarrow \text{Spec } A$. The image is therefore closed in $(GL_n)_A^d$. Let Y denote the inverse image of X , diagonally embedded in X^{d+1} , under the morphism $(GL_n)_A^d \times X \rightarrow X^{d+1}$ given by

$$(g_1, \dots, g_d, x) \mapsto (g_1.x, \dots, g_d.x),$$

and let Z be the image of Y under the projection map from $(GL_n)_A^d \times X$ to $(GL_n)_A^d$. As X is complete, the projection map is proper and therefore closed, so Z is a closed subset of $\text{Spec } A$. As $(G_1)_{\overline{K_1}}$ has a fixed point on $X_{\overline{K_1}}$, Z contains P_1 . As P_2 lies in the closure of P_1 , it contains P_2 as well, so choosing a $\overline{K_2}$ -point of Y which lies over P_2 , $(G_2)_{\overline{K_2}}$ has a fixed point on $X_{\overline{K_2}}$. \square

This has the following consequence:

Corollary 4.3. *With notations as above, if G_1 fixes a k -dimensional subspace of $\overline{K_1}^n$, then G_2 fixes a k -dimensional subspace of $\overline{K_2}^n$.*

Proof. This follows from the previous result using the fact that the Grassmannian variety of d -dimensional subspaces is a complete variety. \square

As K_1 and K_2 are both extensions of K_0 , we have $\text{char } K_1 = \text{char } K_2$, so we can embed both fields as subfields of a common algebraically closed field, and it then makes sense to ask whether G_2 is conjugate (in GL_n) to a subgroup of G_1 . In general, this is not the case. For instance, if $A = \mathbb{C}[t]$, $P_1 = (0)$, $P_2 = (t)$, and Γ is the (infinite cyclic) subgroup of $\text{GL}_2(A)$ generated by $\begin{pmatrix} 1 & 1 \\ 0 & 1+t \end{pmatrix}$, then $G_1 \cong \mathbf{G}_m$ and $G_2 \cong \mathbf{G}_a$.

For any linear algebraic group G , we define G^{cr} to be the (connected and reductive) quotient of the identity component G° by its unipotent radical $\text{Rad}_u(G^\circ)$. Every homomorphism of algebraic groups $\phi: G \rightarrow H$ over an algebraically closed field K determines a homomorphism $\psi: G^\circ \rightarrow H^{\text{cr}}$. As $U := \psi(\text{Rad}_u(G^\circ)) \subset H^{\text{cr}}$ is connected and unipotent, it is contained in the unipotent radical of a canonically defined parabolic subgroup $P \subset H^{\text{cr}}$ [20, §30], so $\psi(G^\circ) \subset P$. Letting $P = MN$ denote a Levi decomposition, $U \subset N$, so ψ induces a well-defined homomorphism $G^{\text{cr}} \rightarrow M$. As Levi decomposition is unique up to conjugation [20, Theorem 30.2], composition of ψ with $M \hookrightarrow H^{\text{cr}}$ gives a morphism $\phi^{\text{cr}}: G^{\text{cr}} \rightarrow H^{\text{cr}}$, which is well-defined up to conjugation. If $G^\circ \cap \ker \phi$ is connected and unipotent then the same is true of $\ker \psi$, so ϕ^{cr} is injective.

Lemma 4.4. *Let G and H be linear algebraic groups over an algebraically closed field K and $\phi: G \rightarrow H$ and $\rho: H \rightarrow \text{GL}_n$ homomorphisms defined over K . Then the representations of G^{cr} defined by $\rho^{\text{cr}} \circ \phi^{\text{cr}}$ and $(\rho \circ \phi)^{\text{cr}}$ have isomorphic semisimplifications.*

Note that the condition on semisimplifications just means that every irreducible representation of G^{cr} has the same multiplicity as a constituent of the two given representations.

Proof. It suffices to prove that the characteristic polynomials of $\rho^{\text{cr}} \circ \phi^{\text{cr}}(g^{\text{cr}})$ and $(\rho \circ \phi)^{\text{cr}}(g^{\text{cr}})$ are the same for all $g^{\text{cr}} \in G^{\text{cr}}(K)$. Let $g \in G^\circ(K)$ map to g^{cr} . Writing $\psi(g) = mn \in M(K)N(K) = P(K) \subset H^{\text{cr}}(K)$, we have $\phi^{\text{cr}}(g^{\text{cr}}) = m$.

We denote by \tilde{N} and \tilde{P} the inverse images of N and P respectively in H° . Let $h \in H^\circ(K)$ map to m , so $\phi(g) = hu$ for some $u \in \tilde{N}(K)$. We claim that $\rho(h)$ and $\rho(hu)$ have the same characteristic polynomial.

Indeed, $\rho(\tilde{N})$ is a connected unipotent subgroup of GL_n , so there exists a parabolic subgroup Q of GL_n such that $\rho(\tilde{N}) \subset \text{Rad}_u(Q)$ and $\rho(\tilde{P}) \subset Q$. Thus, $\rho(u) \in \text{Rad}_u(Q)(K)$ and $\rho(h) \in Q(K)$. So, indeed $\rho(h)$ and $\rho(h)\rho(u)$ have the same characteristic polynomial. \square

If $\rho_1: G_1 \rightarrow \text{GL}_n$ and $\rho_2: G_2 \rightarrow \text{GL}_n$ are defined over K , for a morphism $\phi: G_1^{\text{cr}} \rightarrow G_2^{\text{cr}}$ to be *compatible with ρ_i* means $\rho_2^{\text{cr}} \circ \phi$ and ρ_1^{cr} define representations of \mathbf{G}_1^{cr} which have isomorphic semisimplifications.

Lemma 4.5. *Let Γ be a group and $f_1, f_2: \Gamma \rightarrow \text{GL}_n(K)$ homomorphisms defining representations whose semisimplifications are isomorphic. Let G_1 and G_2 denote the Zariski closure of $f_1(\Gamma)$ and $f_2(\Gamma)$ respectively and ρ_1, ρ_2 the inclusion morphisms from G_1 and G_2 to GL_n . Then there exists an isomorphism between G_1^{cr} and G_2^{cr} compatible with ρ_i^{cr} .*

Proof. Passing to a finite index normal subgroup of Γ , we may assume that G_1 and G_2 are connected. As the maps ρ_i are injective, the same is true for ρ_i^{cr} .

For all $g \in \Gamma$ let g_i denote the image of $f_i(g)$ in $G_i^{\text{cr}}(K)$. Then $\rho_i^{\text{cr}}(g_i)$ has the same characteristic polynomial as $f_i(g_i)$, which is therefore the same for $i = 1$ and $i = 2$. Let G_{12} denote the Zariski closure of Γ under the diagonal map to $G_1^{\text{cr}}(K) \times G_2^{\text{cr}}(K)$. There are two morphisms from G_{12} to characteristic polynomials, one via ρ_1^{cr} and one via ρ_2^{cr} , and they must coincide.

We claim that G_{12} is the diagonal of an isomorphism between G_1^{cr} and G_2^{cr} . Otherwise, there exists a non-trivial normal subgroup H of, say, G_1^{cr} such that $H \times \{1\} \subset G_{12}$, and H maps under ρ_1^{cr} to the closed subvariety of unipotent elements. This is impossible, so H gives an isomorphism between G_1^{cr} and G_2^{cr} compatible with the ρ_i^{cr} . \square

Theorem 4.6. *Let K be a field,*

$$f: \Gamma \rightarrow \text{GL}_n(K[[t]]) \subset \text{GL}_n(K((t)))$$

a representation of a group Γ , and $\bar{f}: \Gamma \rightarrow \text{GL}_n(K)$ the reduction of $f \pmod{t}$. Let G (resp. \bar{G}) denote the Zariski closure in GL_n over $K((t))$ (resp. K) of $f(\Gamma)$ (resp. $\bar{f}(\Gamma)$), and let ρ (resp. $\bar{\rho}$) denote the inclusion $G \rightarrow \text{GL}_n$ as a morphism of algebraic groups over $K((t))$ (resp. K). Then there exists a finite extension L of $K((t))$ and an injective homomorphism $\bar{G}^{\text{cr}} \times_K L \rightarrow G^{\text{cr}} \times_{K((t))} L$ compatible with $\rho^{\text{cr}} \times_{K((t))} L$ and $\bar{\rho}^{\text{cr}} \times_K L$.

Proof. Replacing Γ with a finite index subgroup, we may assume without loss of generality that G and \bar{G} are connected.

Let $V_1 \subset V_2 \subset \dots \subset V_k = K((t))^n$ denote a flag of $K((t))$ -spaces preserved by $f(\Gamma)$ and such that $f(\Gamma)$ acts irreducibly on each V_{j+1}/V_j . Let $\Lambda_j = V_j \cap K[[t]]^n$, so that $f(\Gamma)$ preserves each Λ_j and Λ_{j+1}/Λ_j is a free $K[[t]]$ -module of rank $\dim V_{j+1}/V_j$. We may therefore fix a $K[[t]]$ -free complement M_{j+1} to Λ_j in Λ_{j+1} . We have a surjective homomorphism from the stabilizer of the flag V in $\text{GL}_n(K[[t]])$ to $\prod_{j=1}^k \text{Aut}_{K[[t]]} M_j$, and a section of this latter group whose image consists of $K[[t]]$ -linear maps preserving the direct sum decomposition given by the M_j . Replacing f by the composition with these two homomorphisms, we obtain a semisimplification of f which still lands in $\text{GL}_n(K[[t]])$ and such that $\bar{f}(g)$ has the same characteristic polynomial as $\bar{f}(g)$ for all $g \in \Gamma$. Therefore, without loss of generality, we may assume f is semisimple.

By [33, Chap. 2, §2], every finite extension of $K((t))$ is complete with respect to the unique extension of the t -adic valuation on $K((t))$. Each such extension is therefore of the form $K'((t'))$, where $K \subseteq K'$, $K[[t]] \subseteq K'[[t']]$, and t is a non-unit in $K'[[t']]$. As

every semisimple group splits over a finite extension, we may assume that G is split over $K((t))$.

Next, we claim that $f(\Gamma)$ is a bounded subgroup of $G(K((t)))$ in the sense of Bruhat-Tits [12, 4.2.19]. Indeed, we can fix a finite set of generators of the coordinate ring of G . As ρ is a closed immersion, each generator lifts to an element of the coordinate ring of GL_n . As $f(\Gamma)$ is bounded in the GL_n sense, it is bounded in the G sense as well. By [11, 3.3.1], it stabilizes the centroid of some facet of the Bruhat-Tits building $\mathcal{B}(G/K((t)))$.

By [22, 2.4], replacing $K((t))$ by a finite extension, we may assume that $f(\Gamma)$ stabilizes a hyperspecial vertex of $\mathcal{B}(G/K((t)))$. By [12, 4.6.22], there exists a split semisimple group G_0 over K and an isomorphism ι from G to $G_0 \times_K K((t))$ such that $\iota(f(\Gamma)) \subset G_0(K[[t]])$. By [38] there exists a homomorphism $\rho_0: G_0 \rightarrow \text{GL}_n$ of algebraic groups over K whose extension of scalars to $K((t))$, via ι , defines the same representation as ρ . Explicitly, this means there exists $g \in \text{GL}_n(K((t)))$ such that

$$(\rho_0 \times_K K((t))) \circ \iota = \text{Int}(g) \circ \rho$$

as $K((t))$ -morphisms $G \rightarrow \text{GL}_n$. Thus,

$$(\rho_0 \times_K K((t))) \circ \iota(f(\Gamma)) = g^{-1}f(\Gamma)g.$$

Semisimple representations with the same character are isomorphic [29, XVII, Cor 3.8], so the (mod t) reductions of $f(\Gamma)$ and $g^{-1}f(\Gamma)g$ have the same semisimplification. By Lemma 4.5, without loss of generality we may assume $g = 1$.

Replacing f by $\iota \circ f$, we have therefore reduced to the situation that G is a semisimple group over K embedded in GL_n by the K -homomorphism ρ_0 , and $f(\Gamma) \subset G_0(K((t)))$ is a Zariski dense subgroup which lies in $G_0(K[[t]])$. Reduction (mod t) commutes with ρ_0 . Thus, the Zariski-closure \bar{G} of $f(\Gamma)$ in GL_n over K is contained in $\rho_0(G)$. The Zariski closure G of $f(\Gamma)$ in GL_n over $K((t))$ is $\rho_0(G) \times_K K((t))$.

We conclude that there is an injective homomorphism of algebraic groups over $K((t))$, $\bar{G} \times_K K((t)) \rightarrow G$. By Proposition 4.4, the injective homomorphism of algebraic groups $\bar{G}^{\text{cr}} \times_K K((t)) \rightarrow G^{\text{cr}} = G$ is compatible with $\bar{\rho}^{\text{cr}}$ and ρ up to semisimplification. The theorem follows. \square

Theorem 4.7. *If G_2 is a degeneration of G_1 and ρ_i denotes the inclusion of G_i in GL_n , then there exists a field L such that after extending scalars to L , there exists an injective homomorphism $G_2^{\text{cr}} \rightarrow G_1^{\text{cr}}$ compatible with ρ_i^{cr} .*

Proof. As there exists a finite chain of prime ideals maximal among all chains connecting P_1 and P_2 , without loss of generality, we may and do assume that there is no prime ideal intermediate between P_1 and P_2 . Replacing A by the localization of A/P_1 at P_2/P_1 and Γ by its image in $\text{GL}_n((A/P_1)_{P_2/P_1})$, we may assume without loss of generality that A is a 1-dimensional local domain, $P_1 = 0$, and P_2 is a maximal ideal.

Let \tilde{A} denote the normalization of A . The morphism $\text{Spec } \tilde{A} \rightarrow \text{Spec } A$ is birational and finite [37, Tag 0BXR], so its image is closed and contains the generic point of $\text{Spec } A$. Therefore, there exists a prime ideal \tilde{P}_2 of \tilde{A} lying over P_2 . Replacing A by \tilde{A} and P_2 by \tilde{P}_2 , we may assume that A is a DVR. Replacing A by its completion at P_2 , we may assume that it is a complete DVR, P_1 is the zero-ideal, and P_2 is the maximal ideal. The fields K_1 and K_2 are respectively the fraction field and the residue field of A . By Cohen’s classification of complete equicharacteristic regular local rings [13, Theorem 15], we have isomorphisms $A \cong K_2[[t]]$, and $K_1 \cong K_2((t))$.

By the previous theorem, there exists L such that $G_2^{\text{cr}} \times_{K_2} L$ is isomorphic to a closed subgroup of $G_1^{\text{cr}} \times_{K_1} L$, compatibly with $G_i^{\text{cr}} \rightarrow \text{GL}_n$, so the theorem follows. \square

If H_1 and H_2 are closed subgroups of GL_n over K we write $H_1 \prec_{\text{GL}_n} H_2$ if and only if there exists an extension L/K and an injective homomorphism $H_1^{\text{cr}} \times_K L \hookrightarrow H_2^{\text{cr}} \times_K L$ compatible with the given n -dimensional representations of $H_1 \times_K L$ and $H_2 \times_K L$. If H_2 can degenerate to H_1 as subgroups of GL_n , then $H_1 \prec H_2$. If H_2 and H_1 are subgroups of a semisimple group G , we write $H_1 \prec H_2$ if and only if $H_1 \prec_{\text{GL}_n} H_2$ for all faithful representations $G \hookrightarrow \text{GL}_n$. If K is algebraically closed, this implies that H_1^{cr} is isomorphic to a closed subgroup of H_2^{cr} over K itself.

Our strategy for constructing strongly dense free subgroups of G over a transcendental extension K of a global field K_0 is to find *special* semisimple subgroups H_1, \dots, H_r of G and homomorphisms $\rho_i: \Gamma \rightarrow H_i(K_0)$ which are strongly dense thanks to Corollary 3.3. We then construct a curve of homomorphisms $\rho: \Gamma \rightarrow G(A)$ which specializes to ρ_1, \dots, ρ_r at different points of the curve $\text{Spec } A$. The Zariski closure H of $\rho(\Delta)$ for any non-abelian free subgroup of Γ then satisfies $H_i \prec H$ for all i , thanks to Theorem 4.7. The goal of the next section is to find choices of H_i for which these conditions imply $H = G$.

5. Special subgroups

In this section, we gather some results about subgroups of simple algebraic groups. Throughout this section, we always assume K is algebraically closed and let p be the characteristic of K when it is non-zero.

Definition 5.1. We say a closed subgroup H of a linear algebraic group G is *special* if it is the image of a homomorphism $\text{SL}_{p_1} \times \dots \times \text{SL}_{p_m} \rightarrow G$ for some sequence of primes p_1, \dots, p_m . We say it is *very special* if we can take $m = 1$ and $p_1 = 2$.

Note that if K contains a global field K_0 , every K -group of the form $\text{SL}_{p_1} \times \dots \times \text{SL}_{p_m}$ over K is obtained from a product of almost abelian simple algebraic groups of the form $\text{SL}_1(D_1) \times \dots \times \text{SL}_1(D_m)$ for some K_0 -central division algebras D_1, \dots, D_m . Moreover we may assume that each D_i ramifies over some place that no other D_j ramifies over.

Definition 5.2. A collection $\{H_1, \dots, H_k\}$ of special subgroups of G is *generating* if the only subgroup H of G satisfying $H_i \prec H$ for all i is G itself.

Note that if G has a generating collection of special (resp. very special) subgroups, then the same is true for all groups isogenous to G . Indeed, it is true for the universal covering group \tilde{G} of G since groups of the form $\mathrm{SL}_{p_1} \times \dots \times \mathrm{SL}_{p_m}$ are simply connected, so every homomorphism $\mathrm{SL}_{p_1} \times \dots \times \mathrm{SL}_{p_m} \rightarrow G$ lifts to $\mathrm{SL}_{p_1} \times \dots \times \mathrm{SL}_{p_m} \rightarrow \tilde{G}$. It is clearly true for any quotient of a group for which it is true.

Lemma 5.3. *Let k and n be positive integers with $k < n$. Then there exists a partition π of n such that no part has a prime factor greater than 3, and k is not a sum of any subset of the parts of π .*

Proof. Let n be the smallest integer for which the statement fails. As n cannot be of the form $2^a 3^b$, we may assume $n \geq 5$. Let m denote any integer of the form $2^a 3^b$ which lies in $(n/2, n)$. Thus, $m \leq n$, and $n - m < m$. If $k < n - m$ or $k > n - m$ but $k \neq m$, then by assumption, there exists a partition π' of $n - m$ such that no part of π' has a prime factor greater than 3, and k is not a sum of parts of π' . If π is the partition of n obtained by adding the part m to π' , then no part of π has a prime factor > 3 , and k is not a sum of parts of π , contrary to assumption.

Therefore, it suffices to prove that there are at least two different values of $m \in (n/2, n)$, m_1 and m_2 , neither of which has a prime factor greater than 3. We can take m_1 to be the smallest power of 2 greater than $n/2$ and m_2 to be 3 times the smallest power of 2 greater than $n/6$. \square

Proposition 5.4. *Let n be a positive integer and K a field not of characteristic 2. For any positive integer $k < n$, there exists a very special subgroup H of SL_n defined over K such that the restriction of the natural representation of SL_n to H has no k -dimensional subrepresentation.*

Proof. If K is of characteristic 0, we may take H to be the image of SL_2 under the symmetric $(n - 1)$ st power map. We therefore assume that $p > 2$.

By Lemma 5.3, there exists a partition $n = \pi_1 + \pi_2 + \dots + \pi_r$, where each π_i is of the form $2^{a_i} 3^{b_i}$ for non-negative integers a_i and b_i and k is not a sum of any subsequence of terms in the sequence π_1, \dots, π_r . Let V_2 and V_3 denote, respectively, the representation space of the natural representation of SL_2 and its symmetric square, which is irreducible since $p \geq 3$. Let

$$W_i = V_2 \otimes V_2^{(p)} \otimes \dots \otimes V_2^{(p^{a_i-1})} \otimes V_3 \otimes V_3^{(p)} \otimes \dots \otimes V_3^{(p^{b_i-1})},$$

where $\mathrm{SL}_2(K) \rightarrow \mathrm{GL}(V^{(q)})$ denotes the composition of the q -Frobenius on SL_2 with the representation $\mathrm{SL}_2 \rightarrow \mathrm{GL}(V)$. Thus W_i is an irreducible representation of SL_2 of

degree $\pi_i = 2^{a_i}3^{b_i}$. The direct sum of the W_i is therefore a semisimple determinant 1 representation of SL_2 with no k -dimensional subrepresentation. \square

Lemma 5.5. *Let n be a positive integer and K a field. There exists a special subgroup H of SL_n such that the restriction of the natural representation of SL_n to H is irreducible.*

Proof. Let $n = p_1^{e_1} \cdots p_l^{e_l}$, let

$$H = \prod_{i=1}^l SL_{p_i}^{e_i},$$

and embed H in SL_n via the external tensor product of the natural representation of the SL_{p_i} factors. \square

We shall also need a special case of a result of McLaughlin [28].

Proposition 5.6. *Suppose that $n \geq 2$, and H is a semisimple subgroup of $G = GL_n$ acting irreducibly on the natural representation V of G . If H contains a subgroup $H_1 \cong SL_2$ such that the restriction of V to H_1 is $V_0^{n-2} \oplus V_1$, then $H = Sp_n$ or $H = SL_n$.*

Proposition 5.7. *Let K be a field and $G = Sp_{2r}$ a symplectic group over K . If $\text{char } K = 3$, we further assume $r \geq 3$. Then there exist special subgroups H_1 and H_2 of G such that for all k in $[1, 2r - 1]$, there exists i such that the restriction of the natural representation of G to H_i has no k -dimensional H_i -invariant subspace. If $\text{char } K = 0$, H_1 alone suffices, and we may take it to be very special.*

Proof. Let V_i denote the symmetric i th power representation of SL_2 . If K has characteristic 0 or characteristic $\geq 2r$, then we do not need H_2 ; we may take $H_1 \subset GL_{2r}$ to be the image of SL_2 under the representation V_{2r-1} . As V_{2r-1} is irreducible and symplectic, H_1 may be taken to be a subgroup of Sp_{2r} . We therefore assume $\text{char } K = p$, where $2 \leq p \leq 2r - 1$.

If $r = p_1^{e_1} \cdots p_l^{e_l}$, then $\prod_i SL_{p_i}^{e_i}$ embeds in SL_r by the tensor product of natural representations, and thence into Sp_{2r} . Defining H_1 to be the image of this representation, the restriction of the natural representation of G to H_1 is the direct sum of two irreducible factors of dimension r . We may therefore assume, henceforth, that $k = r$.

If $\text{char } K = 2$, then tensor products of distinct Frobenius twists of V_1 give irreducible self-dual representations of SL_2 of every degree in the set $\{1, 2, 4, 8, \dots\}$ and, in particular, one whose degree lies in $[r + 1, 2r]$. Adding a trivial representation of suitable degree, we obtain a self-dual representation V of degree $2r$ which has no invariant k -dimensional subspace. Since $\text{char } K = 2$, the image H_1 of SL_2 in this representation can be taken to be in Sp_{2n} . We may therefore assume $p \geq 3$. Since we are excluding the case $(p, r) = (3, 2)$, we may assume $r \geq 3$.

We divide into cases.

Case $r \neq 8$. We know V_1 is irreducible and symplectic of degree 2, V_2 is irreducible and orthogonal of degree 3, and the tensor product of two distinct Frobenius-twists of V_1 is irreducible and orthogonal of degree 4. By tensoring suitable Frobenius twists of these three representations, we can therefore find an irreducible symplectic representation of SL_2 of any degree of the form $2^{2e+1}3^f$. Our assumptions on r guarantee there exists a number of this form in the interval $[r + 1, 2r]$. Therefore, there exists a symplectic, irreducible representation of SL_2 whose degree lies in this interval. The direct sum of this irreducible representation with a trivial representation of suitable degree is then a symplectic representation of SL_2 of degree $2r$, and defining H_2 to be the image of SL_2 in this representation, the proposition follows in this case.

Case $r = 8, p \geq 5$. Let W denote the adjoint representation of SL_3 , which is orthogonal and irreducible. Then (external) tensor product $V_1 \boxtimes W$ is a 16-dimensional irreducible symplectic representation of $\mathrm{SL}_2 \times \mathrm{SL}_3$, so we may take H_2 to be the image of this representation. (In fact, in this case, we do not need H_1 .)

Case $r = 8, p = 3$. Let W denote the semisimplification of the adjoint representation of SL_3 , which is orthogonal and the direct sum of a trivial 1-dimensional representation and an irreducible 7-dimensional representation. Then $V_1 \boxtimes W$ is a 16-dimensional symplectic representation of $\mathrm{SL}_2 \times \mathrm{SL}_3$ which decomposes into irreducible factors of degree 2 and 14, and we let H_2 denote the image of this representation. \square

Proposition 5.8. *Let K be a field not of characteristic 2 and $G = \mathrm{SO}_{2r+1}$, $r \geq 3$. Then there exist special subgroups H_1, H_2, H_3 of G such that for all k in $[1, 2r]$, there exists i such that the restriction of the natural representation of G to H_i has no k -dimensional H_i -invariant subspace. If $\mathrm{char} K = 0$, H_1 alone suffices, and we may take it to be very special.*

Proof. If K has characteristic 0 or characteristic $\geq 2r + 1$, then we do not need H_2 or H_3 . We may take $H_1 \subset \mathrm{GL}_{2r+1}$ to be the image of SL_2 in the representation V_{2r} . As V_{2r} is irreducible and orthogonal, H_1 may be taken to be a subgroup of SO_{2r+1} . We therefore assume $\mathrm{char} K = p$, where $3 \leq p \leq 2r$.

If $r = p_1^{e_1} \cdots p_l^{e_l}$, then $\prod_i \mathrm{SL}_{p_i}^{e_i}$ embeds in SL_r by the tensor product of natural representations, and thence into $\mathrm{SO}_{2r} \subset \mathrm{SO}_{2r+1}$. Defining H_1 to be the image of this representation in SO_{2r+1} , the restriction of the natural representation of G to H_1 is the direct sum of a 1-dimensional trivial factor and two irreducible factors of dimension r . We may therefore assume, henceforth, that $k \in \{1, r, r + 1, 2r\}$. There is a special subgroup of SO_{2r+1} of the form $H_2 = \mathrm{SO}_3^e \times \mathrm{SO}_4^f$, where $3e + 4f = 2r + 1$, and no composition factor of the restriction of the natural representation of G to H_2 has dimension 1, so we may assume that $k \in \{r, r + 1\}$. It therefore suffices to find H_3 such that the restriction of the natural representation to H_3 has an irreducible factor of degree $\geq r + 2$.

If $r = 3$ and $p = 3$, we let H_3 be the image of SL_3 under the (irreducible, orthogonal) quotient of the adjoint representation by its 1-dimensional trivial subrepresentation. If

$r = 3$ and $p \geq 5$, we let H_3 denote the image of SL_2 under V_4 , which is, again, irreducible and orthogonal.

If $r \geq 4$, then there exists an integer of the form $3^e 4^f$ in the interval $[r + 2, 2r + 1]$. Indeed, we take the smallest integer in the set

$$\{9 \cdot 4^f \mid f \in \mathbb{N}\} \cup \{12 \cdot 4^f \mid f \in \mathbb{N}\} \cup \{16 \cdot 4^f \mid f \in \mathbb{N}\} \cup \{27 \cdot 4^f \mid f \in \mathbb{N}\}$$

which exceeds $r + 1$. We can therefore find an irreducible, orthogonal representation of SL_2 by tensoring together e suitable Frobenius twists of V_2 and f suitable twists of V_1 . Adding a trivial factor of suitable dimension, we obtain an orthogonal representation which has no subrepresentation of degree r or $r + 1$; we take H_3 to be the image of SL_2 under such a representation. \square

Proposition 5.9. *Let K be a field and $G = SO_{2r}$, $r \geq 4$. Then there exist special subgroups H_1 and H_2 of G such that for all k in $[1, 2r - 1]$, there exists i such that the restriction of the natural representation of G to H_i has no k -dimensional H_i -invariant subspace. If $\text{char } K = 0$, we may take H_1 and H_2 to be very special.*

Proof. If $\text{char } K = 0$ or if $\text{char } K = p$ is at least $2r - 1$, then V_{2r-2} is irreducible and orthogonal, and it maps SL_2 to SO_{2r-1} and therefore to SO_{2r} . The restriction of the natural representation of SO_{2r} to the image H_1 of this homomorphism decomposes as the sum of a 1-dimensional trivial representation and an irreducible representation of dimension $2r - 1$. Let H_2 denote the image of SL_2 under the representation $V_2 \oplus V_{2r-4}$. The only dimensions of non-trivial invariant subspaces of H_2 are 3 and $2r - 3$. We may therefore assume that $2 \leq p \leq 2r - 2$.

If $r = p_1^{e_1} \cdots p_l^{e_l}$, then $\prod_i SL_{p_i}^{e_i}$ embeds in SL_r by the tensor product of natural representations, and thence into SO_{2r} . Defining H_1 to be the image of this representation in SO_{2r} , the restriction of the natural representation of G to H_1 is the direct sum of two irreducible factors of dimension r . We may therefore assume, henceforth, that $k = r$.

If $\text{char } K = 2$, there exists $2^e \in [r + 1, 2r]$, with $e \geq 2$. Taking a tensor product of e Frobenius twists of the natural representation of SL_2 , we obtain an irreducible, orthogonal representation of SL_2 of degree 2^e and therefore an orthogonal representation of SL_2 of dimension $2r$ with no r -dimensional invariant subspaces.

If $p \geq 3$, we use the construction of H_3 for SO_{2r-1} in Proposition 5.8, and define H_2 for SO_{2r} to be the image of this group under the embedding $SO_{2r-1} \subset SO_{2r}$. \square

Theorem 5.10. *Let G be a simple algebraic group over an algebraically closed field K .*

- (1) *Except in the case that G is of type C_2 and $\text{char } K = 3$, G contains a finite generating set of special subgroups.*
- (2) *If $G = SL_n$ and $\text{char } K \neq 2$, then G has a very special generating set.*
- (3) *If G is an orthogonal or symplectic group and $\text{char } K = 0$, then G has a very special generating set.*

(4) We can choose one element H_i of a generating set so that $H_i \times H_i \not\leq G$.

Proof. As the statement does not depend on isogeny class, we assume G is simply connected, except for orthogonal types, where we assume G is a special orthogonal group. In each case, we give a finite set of special subgroups H_i such that the only subgroup H of G satisfying $H_i \prec H$ for all i is G itself. The H_i are special and therefore semisimple, so if H is a reductive and $H_i \prec H$ for all i , then the same is true of the derived group of H° , so if H is reductive, we may assume it is semisimple.

We consider each of the possible types:

Case A_r , $r \geq 1$. By Lemma 5.5 and in view of Corollary 4.3, there exists a special subgroup $H_1 \subset G = \mathrm{SL}_{r+1}$ such that any subgroup $H \subset G$ with $H_1 \prec H$ acts irreducibly on the natural representation V , so it is reductive and may therefore be assumed to be semisimple. If $r = 1$, this implies $H = \mathrm{SL}_2$, so we may assume $r \geq 2$. If $r \in \{2k - 1, 2k\}$, let $H_2 = \mathrm{SL}_2^k$ with the standard embedding to G . By Theorem 4.7, the rank of any subgroup H such that $H_2 \prec H$ is at least k .

Let \tilde{H} denote the simply connected covering group of H and

$$\tilde{H} = L_1 \times \cdots \times L_l,$$

where the L_i are simple. If r_i denotes the rank of L_i , then $r_1 + \cdots + r_l \geq k$. As V is the exterior tensor product of almost faithful representations of the H_i ,

$$r + 1 \geq (r_1 + 1) \cdots (r_l + 1).$$

The last two conditions imply that either $l = 1$, or that $l = 2$ and $r_1 = 1$, $r_2 = k - 1$. If $l = 1$, H is simple and must be SO_{r+1} or Sp_{r+1} (cf. [31, Theorem 5.1 and Cor. 5.2] or [25]). If $l = 2$ and $r_1 = 1$, the first factor of H is SL_2 with the natural module. So $\dim V$ is even and $r = 2k - 1$. Also the second factor has rank $k - 1$ with an irrep of dimension k , so must be SL_k with its natural module. However $H_2 = A_1^k$ does not embed in $\mathrm{SL}_2 \times \mathrm{SL}_k$, unless $k = 2$. Therefore $r_2 = 1$, $r = 3$, and after extension of scalars, $H = \mathrm{SO}_4$.

Let $H_3 = \mathrm{SL}_3$, embedded in SL_{r+1} by any representation which is not self-dual. Then there is no homomorphism $\mathrm{SL}_3 \rightarrow \mathrm{SO}_{r+1}$ or $\mathrm{SL}_3 \rightarrow \mathrm{Sp}_{r+1}$ consistent with the embeddings of H_3 , SO_{r+1} , and Sp_{r+1} in SL_{r+1} , so the theorem holds in this case.

Case B_r , $r \geq 3$, $\mathrm{char} K \neq 2$. Let H_1 , H_2 , and H_3 be as in Proposition 5.8, and let $H_4 = \mathrm{SL}_2^r$, which embeds as a subgroup of $G = \mathrm{SO}_{2r+1}$ since G is always of the form $\mathrm{SO}_3 \times \mathrm{SO}_4^{\frac{r-1}{2}}$ or $\mathrm{SO}_5 \times \mathrm{SO}_4^{\frac{r-2}{2}}$. By Proposition 5.8, any group H satisfying $H_i \prec H$ for all $i \in \{1, 2, 3\}$ acts irreducibly on the natural representation of G , so it is reductive and we may therefore assume it is semisimple. As $H_4 \prec H$, H must have rank r . From the classification of equal rank semisimple subgroups of simple groups, this implies $H = G$.

Case C_r , $r \geq 2$, $\text{char } K \neq 3$ if $r = 2$. Let H_1 and H_2 be as in Proposition 5.7, and let $H_3 = \text{SL}_2^r$, embedded as a subgroup of $G = \text{Sp}_{2r}$ in the obvious way. By Proposition 5.7, any group H satisfying $H_1 \prec H$ and $H_2 \prec H$ acts irreducibly on the natural representation of G , so H is reductive and can be assumed semisimple. As $H_3 \prec H$, H is of rank r . If $\text{char } K \neq 2$, then the classification of equal rank subgroups of simple groups implies $H = G$. If $\text{char } K = 2$, there is an additional possibility: $H = \text{SO}_{2r} \subset \text{Sp}_{2r}$. However, $\text{SO}_{2r} \subset \text{Sp}_{2r}$ does not contain a subgroup for which the restriction of the natural representation of Sp_{2r} decomposes into r irreducible 2-dimensional factors, so this possibility is ruled out.

Case D_r , $r \geq 4$. Let H_1 and H_2 be as in Proposition 5.9. If r is even, let $H_3 = \text{SO}_4^{r/2}$, embedded as a subgroup of $G = \text{SO}_{2r}$ in the obvious way; if r is odd, let $H_3 = \text{SO}_4^{(r-1)/2}$. By Proposition 5.9, any group H satisfying $H_1 \prec H$ and $H_2 \prec H$ acts irreducibly on the natural representation of G , so we may assume H is semisimple of rank $\geq r - 1$. If H is of rank r , then $H = G$ follows from the classification of equal rank subgroups. We may therefore assume H is of rank $r - 1$, which means r is odd and therefore at least 5.

If the universal covering group \tilde{H} has simple factors of ranks $r_1 \geq \dots \geq r_l$, then

$$r_1 + \dots + r_l = r - 1$$

and

$$(2r_1) \cdots (2r_l) \leq 2r,$$

since the minimum dimension of a self-dual representation of a simple group of rank r_i is $2r_i$. The only solutions for these two conditions for $l \geq 2$ are $(r_1, r_2) = (2, 1)$ (for $l = 2$) and $r_1 = r_2 = r_3 = 1$ (for $l = 3$). However, in neither case is $n \geq 5$. Therefore H is simple, but a simple group of rank $r - 1$ cannot have a self-dual representation of dimension $2r$ for $r \geq 5$.

Case G exceptional. Suppose H_1 and H_2 are special subgroups of G with the following properties. The group H_1 is a maximal proper connected subgroup of G which is semisimple, of rank equal to $\text{rk } G$, and not isomorphic to a subgroup of any maximal proper connected subgroup of G except conjugates of itself, while the group H_2 is not isomorphic to any subgroup of H_1 . If some proper subgroup H of G satisfies $H_1 \prec H$, then H_1 is isomorphic to a quotient of H° . Therefore, the semisimple rank of H° must equal $\text{rk } G$, and by [26, Corollary 2 (ii)], that implies that H° is semisimple, so H° is conjugate to H_1 . However, H_2 is semisimple and does not embed in H_1 , so we cannot have $H_2 \prec H$.

Using [26, Table 10.3], we find such pairs (H_1, H_2) in all cases except E_7 . For E_6 , take H_1 a subgroup of type A_2^3 and H_2 a subgroup of type $A_1 \times A_4$ contained in a maximal subgroup of type $A_1 \times A_5$. For E_8 , take H_1 a subgroup of type A_1^8 and H_2 a subgroup of type A_2^4 . For F_4 , take H_1 of type A_2^2 and H_2 of type A_1^4 (contained in a maximal

subgroup of type D_4). For G_2 , take H_1 of type A_1^2 and H_2 of type A_2 . Except for case F_4 , the dimension of H_2 is greater than that of H_1 , so there is no embedding of H_2 in H_1 . For F_4 , there is no embedding of a group of type A_1^4 in a group of type A_2^2 because there is no embedding of a group of type A_1^2 in a group of type A_2 .

For E_7 , we proceed somewhat differently, letting H_1 denote a special subgroup of E_7 of type A_1^7 (contained in a maximal subgroup of type $A_1 \times D_6$) and letting H_2 be a special subgroup of type $A_2 \times A_4$ (contained in a maximal subgroup of type $A_2 \times A_5$). Suppose a proper subgroup H of G satisfies $H_1 \prec H$ and $H_2 \prec H$. Without loss of generality, we may assume H is connected. Let M be a maximal proper subgroup of G containing H_1 . Then M contains the rank 7 semisimple group H_1 , so it must be of rank 7 and therefore, by [26, Table 10.3], must be a group whose identity component is semisimple of type $A_1 \times D_6, A_7, A_2 \times A_5, A_1^3 \times D_4$, or A_1^7 . However, only a group of type $A_1 \times D_6, A_1^3 \times D_4$, or A_1^7 can contain a group of type A_1^7 , and no group of any of these types can contain a subgroup of type $A_2 \times A_4$. This finishes the proof of part (1).

For part (2), we first use Proposition 5.4 to prove $H_i \prec H$ for all i implies H° acts irreducibly on the natural representation of GL_n . We may therefore assume $n \geq 3$. To the list of H_i , we add two additional very special subgroups of SL_n , namely the image H' of SL_2 under the symplectic representation $V_0^{n-2} \oplus V_1$ and the image H'' of SL_2 under the orthogonal representation $V_0^{n-3} \oplus V_2$. Proposition 5.6 then proves that H must be Sp_n with its natural representation or all of SL_n . The Sp_n can be ruled out by $H'' \not\prec Sp_n$.

For each case in part (3), we use very special subgroups associated to two different representations of SL_2 . For type B_r , we use the representations V_{2r} and $V_0^2 \oplus V_{2r-2}$ to define very special subgroups H' and H'' of SO_{2r+1} . Thus H' is the principal SL_2 , which is known to be a maximal subgroup of SO_{2r+1} except when $r = 3$ when it is contained in the maximal subgroup G_2 of SO_7 . In general $H'' \not\prec H'$, and for $r = 3$, $H'' \not\prec G_2$, so in either case the statement of (3) holds. For type C_r , we use the representations V_{2r-1} (which gives the principal SL_2 in Sp_{2r}) and $V_{2r-3} \oplus V_1$. Since the principal SL_2 is always maximal for symplectic groups, (3) holds. For type D_r , we use the representations $V_0 \oplus V_{2r-2}$, which gives the principal SL_2 , which we denote H' , and $V_2 \oplus V_{2r-4}$, which gives another very special subgroup, H'' . Now, H' is contained in SO_{2r-1} but is in no other group intermediate between H' and SO_{2r} , while $H'' \not\prec SO_{2r-1}$.

Part (4) holds because in each of the cases above except for case A_r when r is even, one of the H_i has rank greater than half the rank of G . For the remaining case, we use the fact that SL_{2k+1} does not contain any subgroup isogenous to A_1^{2k} . \square

6. Main theorems

In this section, we prove the main theorems of the paper. We begin with a proposition.

Proposition 6.1. *Let $K_i, 1 \leq i \leq k$, be any finite separable extensions of an infinite field K , and let G be any connected linear algebraic group over K . For every element $x = (x_1, \dots, x_k) \in \prod_{i=1}^k G(K_i)$, there exists a finitely generated K -domain A*

with fraction field isomorphic to $K(t)$ as K -algebra such that x lies in the image of $G(A) \rightarrow \prod_{i=1}^k G(K_i)$.

Proof. By the theorem of the primitive element, each K_i is isomorphic as K -algebra to $K[t]/\mathfrak{m}_i$, where, as K is infinite, the \mathfrak{m}_i can be chosen to be pairwise distinct maximal ideals. Fix for each K_i an element $x_i \in G(K_i)$. Our goal is to find a polynomial $Q(t) \in K[t]$ not in any \mathfrak{m}_i and an element $x \in G(K[t, 1/Q(t)])$ which reduces to x_i under reduction (mod \mathfrak{m}_i) for all $1 \leq i \leq k$.

As G is a connected linear algebraic group over K , it is rational as a K -variety. Let $U \subset G$ denote a non-empty open K -subvariety of G which is isomorphic to an open subvariety of \mathbb{A}^n . As K is infinite, $G(K)$ is Zariski-dense in G , so there exists a translate of U by an element of $G(K)$ which contains all the closed points of G in $\{x_1, \dots, x_k\}$, so without loss of generality, G has this property. We fix an open immersion $\iota: U \rightarrow \mathbb{A}^n$ over K .

By the Chinese Remainder Theorem, the natural homomorphism $K[t] \rightarrow \prod_i K[t]/\mathfrak{m}_i$ is surjective, so there exist elements $A_1(t), \dots, A_n(t) \in K[t]$ such that $A_j(t) \pmod{\mathfrak{m}_i}$ gives the j th coordinate of $\iota(x_i)$. In other words, the n -tuple $(A_1(t), \dots, A_n(t))$ defines a morphism $\xi: \mathbb{A}^1 \rightarrow \mathbb{A}^n$ which maps the closed point associated to \mathfrak{m}_i to $\iota(x_i)$ for all i . Thus $\xi^{-1}(\iota(U))$ is an open subset of \mathbb{A}^1 . Its coordinate ring is therefore of the form $A = K[t, 1/Q(t)]$, and the restriction of ξ gives a morphism $\text{Spec } A \rightarrow U$, proving the proposition. \square

We remark that it follows immediately that every finitely generated free subgroup of $\prod_{i=1}^k G(K_i)$ lies in the image of $G(A)$ for some choice of A .

We can now finish the proofs of Theorems 1.1 and 1.2. Let K be a global field. By Corollary 3.3, for every split simple algebraic group G over K , and every special subgroup H of G , $H(K)$ contains a strongly dense free subgroup isomorphic to F_2 . We first assume that G is not of type C_2 if $\text{char } K = 3$. By Theorem 5.10, there exist almost abelian algebraic groups H_1, \dots, H_s over K such that $\{(H_1)_{\bar{K}}, \dots, (H_s)_{\bar{K}}\}$ is a generating set for $G_{\bar{K}}$.

By Lemma 3.2, we may choose injective homomorphisms $f_i: F_2 \rightarrow H_i(K)$ with strongly dense images $f_i(F_2) = \Gamma_i \subset H_i(K) \subset G(\bar{K})$ for $1 \leq i \leq s$.

By Proposition 6.1, there exist a finitely generated K -domain A with maximal ideals $\mathfrak{m}_1, \dots, \mathfrak{m}_s$ and a homomorphism $F_2 \rightarrow G(A)$ which specializes (mod \mathfrak{m}_i) to f_i for $i = 1, \dots, s$. Moreover, we may assume the fraction field of A is $K(t)$. Thus, for every non-abelian subgroup $\Delta \subset F_2$, $f_i(\Delta)$ has Zariski-closure H_i . The Zariski closure of the image of Δ in $G(K(t))$ degenerates to each of the H_i , so by Theorem 4.7, it must be all of G , as claimed.

The same method works for Sp_4 when $\text{char } K = 3$, except that instead of a generating set of special subgroups of Sp_4 , we have a generating set consisting of the image H_1 of $\text{SL}_2 \times \text{SL}_2$ in Sp_4 and the derived group H_2 of the stabilizer of a line in the natural representation. By [10, Appendix D], H_2 contains a strongly dense free subgroup F_2 ,

and we define L, A, f_i , and so on, as before. Since H_1 is a maximal connected subgroup of G , any connected subgroup H which can degenerate to H_1 is either H_1 or G , and H_1 cannot degenerate to H_2 because H_1 has an invariant 2-dimensional subspace while H_2 does not. Thus, if G is simple then $G(K(t))$ contains a strongly dense free subgroup.

Now consider the general case when G is semisimple. There is no harm in assuming that G is simply connected (or has trivial center) and so is a direct product. By Goursat’s lemma, we can assume that all simple factors are of the same type. So say $G = \underbrace{J \times \dots \times J}_k$

with J simple. By Theorem 5.10, $J_{\bar{K}}$ has a generating set $\{(H_1)_{\bar{K}}, \dots, (H_s)_{\bar{K}}\}$, and we may assume that J cannot degenerate to any group isomorphic to $H_1 \times H_1$. In fact, for each $i \in \{1, \dots, s\}$ and each $j \in \{1, \dots, k\}$ we define an almost abelian K -subgroup $H_{i,j}$ of J such that $(H_{i,j})_{\bar{K}}$ is isomorphic to $(H_i)_{\bar{K}}$ and such that each of the division algebras associated to any $H_{i,j}$ is ramified over some prime of K which none of the others is ramified over.

By Theorem 5.10 (4), we may assume that $J_{\bar{K}}$ does not have any subgroup isomorphic to $(H_1)_{\bar{K}} \times (H_1)_{\bar{K}}$. We fix for each i, j a homomorphism $f_{i,j}: F_2 \rightarrow H_{i,j}(K) \subset J(\bar{K})$, and choose $A, \mathfrak{m}_1, \dots, \mathfrak{m}_s$, and $\tilde{f}_j: F_2 \rightarrow J(A)$ which reduces to $f_{i,j}$ modulo \mathfrak{m}_i . Thus each $\tilde{f}_j(F_2)$ is Zariski-dense in $J(K(t))$. Defining $\tilde{f} = (\tilde{f}_1, \dots, \tilde{f}_k): F_2 \rightarrow J^k(K(t))$, it follows that for all non-abelian subgroups Δ of F_2 , the Zariski closure of $\tilde{f}(\Delta)$ maps surjectively to each factor J . By Goursat’s lemma to prove that $\tilde{f}(\Delta)$ is Zariski-dense, it suffices to prove that it is not contained in the graph of any isomorphism between any two factors of J .

We may therefore assume that $k = 2$ and for some $\Delta \subset F_2$, $\tilde{f}(\Delta)$ has Zariski closure isomorphic to J . However the Zariski-closure of its (mod \mathfrak{m}_1) reduction $(f_{1,1}, f_{1,2})(\Delta)$ in $J \times J$ is $H_{1,1} \times H_{1,2}$ since it maps onto $H_{1,1}$ and $H_{1,2}$, which are non-isomorphic simple algebraic groups. This is impossible since $J_{\bar{K}}$ does not contain a subgroup isomorphic to $(H_{1,1})_{\bar{K}} \times (H_{1,2})_{\bar{K}}$.

Since every transcendental field of characteristic 0 contains a subfield isomorphic to $\mathbb{Q}(t)$, taking $K = \mathbb{Q}$, this gives the first part of Theorem 1.1. Since every extension of \mathbb{F}_p of transcendence degree ≥ 2 contains a field isomorphic to $\mathbb{F}_p(s, t)$, taking $K = \mathbb{F}_p(s)$ gives the first part of Theorem 1.2.

To prove the second part of both theorems (the density of pairs generating a strongly dense subgroup), observe that this follows easily from the existence part. Indeed, if F is a strongly dense subgroup, then the set of non-commuting pairs $(x, y) \in F \times F$ is itself Zariski-dense in $G \times G$. This is because for each $x \in F \setminus \{1\}$ it contains $\{x\} \times (F \setminus C_F(x))$, whose Zariski-closure is $\{x\} \times G$.

7. Strongly dense subgroups in affine groups

Almost all the results about strongly dense subgroups have been in semisimple groups. Suppose that G is not semisimple. When can G contain a strongly dense free nonabelian subgroup Γ ?

We note:

Lemma 7.1. *Suppose that an algebraic group G over an algebraically closed field k contains a strongly dense nonabelian free subgroup Γ . Then G can be topologically generated by 2 elements, G is connected, and G is perfect.*

Proof. As Γ contains rank 2 free subgroups which are Zariski-dense, the first assertion holds. Suppose that G is not perfect. Then the Zariski closure of Γ' is contained in G' whence Γ' is not Zariski-dense. If G is not connected, then $G^\circ \cap \Gamma$ is not Zariski-dense in G . \square

In particular, this implies that $G/R_u(G)$ is semisimple. We can always replace G by $G/\Phi(G)$ where $\Phi(G)$ is the Frattini subgroup of G and so assume that $R_u(G)$ is a completely reducible G -module. The condition that G is 2-generated imposes a limit on the multiplicities of the composition factors in $R_u(G)$ (in terms of dimension and cohomology). We do note that:

Lemma 7.2. *Suppose that $R_u(G)$ is a simple G -module and k is not algebraic over a finite field. Then G is 2-generated (topologically).*

This follows easily from the fact that

$$\dim H^1(G/R_u(G), R_u(G)) < \dim R_u(G)$$

(which is an old result from [3] for finite groups and the proof for algebraic groups is much easier and also follows from the same inequality for finite groups of Lie type).

We ask whether any connected perfect algebraic group over an algebraically closed field that is topologically generated by two elements contains a strongly dense subgroup.

Here we show that certain affine groups do have this property. We could extend this but content ourselves with considering the affine groups $\text{ASL}_n(K) = V.\text{SL}(V)$ with $\dim V = n > 1$.

Theorem 7.3. *Let $K = \mathbb{Q}(t)$ or $\mathbb{F}_p(s, t)$. Let $G = \text{ASL}_n(K)$. Then G contains strongly dense free subgroups.*

Proof. If $n = 2$, this is proved in [10, Appendix D] where K is any field not algebraic over a finite field. So assume that $n > 2$. We now choose elements $x, y \in \text{ASL}_n(\mathbb{Q}[t])$ or $\text{ASL}_n(\mathbb{F}_p(s)[t])$ that specialize to strongly dense subgroups of H_1, \dots, H_r with $H_i \leq \text{SL}_n(\bar{K})$ as in the proof of the main theorems for SL_n .

Let H_0 be a 2-generated free strongly dense subgroup of $\text{ASL}_2(K)$ naturally embedded in $\text{ASL}_n(K)$. We pick x, y that also specialize to a strongly dense subgroup of H_0 . Let $S = \langle x, y \rangle$ and let T be any nonabelian subgroup of S . As we argued in the proof of our main result, we see that the Zariski closure is either G or is a complement to $V = R_u(G)$.

The rational cohomology group $H^1(\mathrm{SL}_n, V) = 0$ by [2] since if δ denotes the half sum of positive roots of SL_n and ϖ_1 is the highest weight of V , the weights $-\delta$ and $\varpi_1 - \delta$ are not in the same Weyl group orbit (mod p). Thus, T is contained in a complement to V in G if and only if T fixes a nonzero vector in the $n + 1$ dimensional representation of G (we embed $G = \mathrm{SL}_{n+1}(K)$ contained in the stabilizer of a hyperplane). Note that this is a closed condition (this is equivalent to commuting with a rank 1 idempotent whose kernel is the given hyperplane—the set of such idempotents is the set of conjugates of a single such idempotent by V). Since H_0 commutes with no such idempotent, it follows that T does not either and so the Zariski closure of T is G . \square

8. Nonfree strongly dense subgroups

In this section, we give examples of finitely generated groups which are not free but can nevertheless be embedded in groups of the form $G(K)$ as strongly dense subgroups in the sense of Definition 1.3.

Recall that a group H is called *residually free* if for every nontrivial element $h \in H$, there exists a free quotient J of H such that the image of h in J is nontrivial.

We now show that the methods of [9] and this paper can be used to prove that a large class of groups (including surface groups of genus at least 2) have strongly dense embeddings.

Theorem 8.1. *Let Γ be a finitely generated group satisfying the following conditions:*

- (i) *$\mathrm{Hom}(\Gamma, G)$ is an irreducible variety for every simply connected simple algebraic group G .*
- (ii) *Γ has trivial center and is residually free.*

If K is an algebraically closed field of infinite transcendence degree and characteristic $p \geq 0$ and G is a semisimple algebraic group over K , then there exist strongly dense embeddings of Γ into $G(K)$.

Since Γ has trivial center, it suffices to prove the theorem in the simply connected case, so we assume henceforth that G is simply connected. We do not need infinite transcendence degree to make the argument work, but the degree needed with this argument grows linearly with $\dim G$.

Let Σ_g be the fundamental group of a Riemann surface of genus g . It is well known that $\mathrm{Hom}(\Sigma_g, G)$ is an irreducible variety for any simply connected simple algebraic group G . This is shown by Simpson [34, Thm. 11.1] for the SL_n case (see also [5]) and by Liebeck and Shalev [27, Cor. 1.11(ii)] for the general case (note that they only claim the irreducibility of the top dimensional component which is of dimension $(2g - 1) \dim G$, but since $\mathrm{Hom}(\Sigma_g, G)$ is the preimage of the identity by an algebraic morphism, each

component must be top dimensional). It is also well known that $\Sigma_g, g \geq 2$, is residually free (cf. [8, Cor. 2.2]) and so:

Corollary 8.2. *There exist strongly dense embeddings of $\Sigma_g, g \geq 2$ into $G(K)$ for K an algebraically closed field of infinite transcendence degree in any characteristic and G any semisimple group over K .*

We remark that Long, Reid, and Wolff [30] use a similar strategy to show that generic Hitchin representations are strongly dense.

We begin with a lemma:

Lemma 8.3. *Let G be a simply connected semisimple algebraic group over an algebraically closed field K . There exists a countable collection of proper closed subvarieties Z_i of $G \times G$, each defined over the prime subfield, such that $\cup_i Z_i$ is the set of all $(g_1, g_2) \in Z_i$, such that group $\langle g_1, g_2 \rangle$ is not Zariski-dense in G .*

Proof. We first assume that G is simple and simply connected. Let k be the algebraic closure of the prime field. By [18, Thm. 11.7], there exists a finite set of irreducible rational G -modules (defined over k) so that no positive dimensional closed subgroup of G acts irreducibly on each of those modules. The set of pairs acting reducibly for each module is a proper closed subvariety.

Now consider the proper subvarieties $X_m = \{(g_1, g_2) \in G \times G \mid |\langle g_1, g_2 \rangle| \leq m\}$. These are defined over the prime field. The result now follows in this case by taking the Z_i to be the finite set of subvarieties given by the modules together with the countably many subvarieties X_m .

If G is simply connected but not simple, then $G = G_1 \times \dots \times G_N$ for some simply connected simple groups G_i , and every maximal subgroup of G is either the pull-back of a maximal subgroup of some G_i or the pull-back of the graph of a surjective endomorphism between the adjoint quotients of two factors, G_i and G_j . We have already dealt with the first class of subgroups. Up to the action of G , the maximal diagonal subgroups correspond to compositions of outer automorphisms and Frobenius endomorphisms and in particular there are only countably many such and each is defined over a finite extension of the prime field giving rise to countably many conjugacy classes of maximal closed diagonal subgroups D_i . For each D_i , we consider the subvariety which is the closure of $\cup_{g \in G} (D_i \times D_i)^g$. Clearly these are proper subvarieties (as generic elements are not contained in any diagonal subgroup). \square

One can also show that the complement of the union of the subvarieties above is dense as long as K is not algebraic over a finite field (if K is uncountable, this is clear). An alternate proof of the previous result can be obtained by noting that there are only countably many maximal proper closed subgroups (maximal in the category of closed subgroups) and they are all defined over the algebraic closure of the prime field. In

characteristic 0, one already knows that the set of generating pairs is a nonempty open subset (see [9, Theorem 4.1]).

We can now deduce Theorem 8.1 from the analogous result on free groups.

Proof. By condition (i), $\text{Hom}(\Gamma, G)$ cannot be written as a countable union of proper closed subvarieties. So it is enough to prove that given any pair γ_1, γ_2 of non-commuting elements in Γ , and for each closed subvariety Z_i from Lemma 8.3, the closed subvariety $W_{i, \gamma_1, \gamma_2}$ of $\text{Hom}(\Gamma, G)$ made of those representations ρ such that the pair $(\rho(\gamma_1), \rho(\gamma_2))$ belongs to Z_i is proper. By [9], there is a strongly dense free subgroup in G given say by some injective homomorphism $\pi : F_2 \rightarrow G$. By condition (ii) there is a homomorphism $\phi : \Gamma \rightarrow F_2$ such that $\phi([\gamma_1, \gamma_2]) \neq 1$. The representation $\rho := \pi \circ \phi$ is not in $W_{i, \gamma_1, \gamma_2}$, because $\langle \rho(\gamma_1), \rho(\gamma_2) \rangle$ is Zariski-dense. So $W_{i, \gamma_1, \gamma_2}$ is proper as desired. \square

We can also extend this result (essentially via the proof of [9] for free groups) to finitely generated groups Γ such that all representation varieties are irreducible and satisfy the Borel property (i.e. word maps are dominant – see [7]). This argument works aside from the case of groups involving C_2 in characteristic 3 (just as the proof in [9] did – that case was handled in [10]).

To give another application of Theorem 8.1, we introduce the following terminology. A word in F_d is (N, l) -friable if it is a juxtaposition of at least N non-empty words of length $\leq l$ in pairwise distinct variables. Recall that by the *Baumslag double* of a word $w = w(x_1, \dots, x_d) \in F_d$, we mean the one-relator group on $2d$ -generators

$$\langle x_1, \dots, x_d, y_1, \dots, y_d \mid w(x_1, \dots, x_d)w(y_1, \dots, y_d)^{-1} \rangle.$$

Theorem 8.4. *For all l there exists N such that if $w \in F_d$ is (N, l) -friable, and Γ is the Baumslag double of w , then for every semisimple group G over an algebraically closed field of infinite transcendence degree, $G(K)$ contains a strongly dense subgroup isomorphic to Γ .*

Proof. Condition (i) holds if N is large enough compared to l . By [24, Theorem 5(ii)], if N is large enough in terms of l ,

$$|\text{Hom}(\Gamma, H)| = (1 + o(1))|H|^{d-1} \tag{8.1}$$

for finite simple groups H . However, the character estimate for groups of Lie type used in the proof, namely [17, Thm 1.4] and [16], are both proved, in those papers, in the quasisimple case, and therefore also for groups of the form $\tilde{H} \cong G(\mathbb{F}_q)$, where G is simple and simply connected.

By Lang-Weil [21], if X is any variety over \mathbb{F}_q , $\limsup_{n \rightarrow \infty} \log_{q^n} |X(\mathbb{F}_{q^n})|$ is the dimension D of the highest dimensional irreducible component of X . Moreover, $\limsup_{n \rightarrow \infty} q^{-Dn} |X(\mathbb{F}_{q^n})|$ gives the number of geometric components of X of top dimension d .

Therefore, if G is a semisimple algebraic group over an algebraically closed field of characteristic $p > 0$, it follows from (8.1) that there is a unique irreducible component of $\text{Hom}(\Gamma, G)$ of dimension $D = (2d - 1) \dim G$ and no component of dimension $> D$. On the other hand, since Γ has $2d$ generators and one relation, we can regard $\text{Hom}(\Gamma, G)$ as the fiber of a morphism $G^{2d} \rightarrow G$, and it follows that every geometric component of this representation variety has dimension at least $2d \dim G - \dim G = D$. Thus $\text{Hom}(\Gamma, G)$ is irreducible and of dimension $(2d - 1) \dim G$. Since this is true in all positive characteristics, it is likewise in characteristic 0.

As long as $N > 1$, no (N, l) -friable word is a non-trivial power of another word, so Γ is residually free [4]. Therefore, the theorem follows from Theorem 8.1. \square

Up to this point, the results of this section require that the transcendence degree of K is large enough in terms of the dimension of G . However, it is also possible to use the methods of sections 3–6 of this paper to prove certain $\dim G$ -independent results.

Theorem 8.5. *Let K be a transcendental algebraically closed extension of \mathbb{Q} . Let G be a classical group over K and Γ be a finitely generated group with $\text{Hom}(\Gamma, G)$ irreducible and which admits a strongly dense embedding in $\text{SL}_2(\bar{\mathbb{Q}})$. Then Γ admits a strongly dense embedding in $G(K)$.*

Proof. Since K is algebraically closed and of characteristic zero, without loss of generality, we may assume that G comes from a split group (which we also denote G) defined over \mathbb{Q} .

In view of Theorem 5.10 (2) and (3), there exists a very special generating set $\{H_i\}$ for G and a finite collection of representations $f_i: \Gamma \rightarrow H_i(\bar{\mathbb{Q}}) \subset \text{SL}_n(\bar{\mathbb{Q}})$ such that for every non-abelian subgroup Δ of Γ , $f_i(\Delta)$ is Zariski-dense in $H_i(K)$. The f_i define points on the variety $\text{Hom}(\Gamma, G)$, which is irreducible.

For any finite set S of points on an irreducible quasi-projective variety V , there exists an irreducible curve containing S . Indeed, one can blow up the points in S , embed the resulting variety \tilde{V} in some projective space, and use Bertini's theorem to choose a linear subspace of codimension $\dim V - 1$ intersecting \tilde{V} in an irreducible curve whose image in V contains S . Applying this to the points on $V = \text{Hom}(\Gamma, G)$ corresponding to the f_i , we obtain an irreducible affine curve with coordinate ring A . The universal G -representation of Γ over V specializes to a homomorphism $\Gamma \rightarrow G(A)$ which further specializes to each of the f_i and is therefore injective. If K_0 is the field of fractions of A , as the $\{H_i\}$ form a generating set, $\Gamma \rightarrow G(K_0)$ is strongly dense. However, K_0 is of transcendence degree 1, so it embeds in K . \square

Corollary 8.6. *If K is a transcendental algebraically closed extension of \mathbb{Q} and G is a classical group over K , then for each $g \geq 2$, $G(K)$ contains a strongly dense subgroup isomorphic to a surface group of genus g .*

Proof. Every Riemann surface of genus $g \geq 2$ can be realized as a quotient of the upper half-plane by a subgroup of $\mathrm{PSL}_2(\mathbb{R})$ isomorphic to the surface group Γ . It is also well known that the injective homomorphism $\Gamma \rightarrow \mathrm{PSL}_2(\mathbb{R})$ lifts to a (necessarily injective) homomorphism $\Gamma \rightarrow \mathrm{SL}_2(\mathbb{R})$ (see, e.g., [1] for a short argument). The subset of $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{R}))$ such that the map $\Gamma \rightarrow \mathrm{PSL}_2(\mathbb{R})$ is injective with discrete image is a non-empty open subset of $\mathrm{Hom}(\Gamma, \mathrm{SL}_2(\mathbb{R}))$ ([40]). Therefore, there exists an injective homomorphism $\phi: \Gamma \rightarrow \mathrm{SL}_2(\bar{\mathbb{Q}})$. Its image is necessarily strongly dense, because proper algebraic subgroups of SL_2 are virtually solvable and Γ has no non-abelian virtually solvable subgroup. So we may apply the previous theorem. \square

We remark that, at least when $G = \mathrm{SL}_n$, the variety of G -representations of a surface group is rational [5], so one might hope to find a rational curve containing the points corresponding to the f_i . If this can be done, we can dispense with the assumption that K is algebraically closed.

Theorem 8.7. *Let $p > 2$ be a prime and $n, g \geq 2$ integers. Let K be an algebraically closed field of transcendence degree 10 over \mathbb{F}_p . Then $\mathrm{SL}_n(K)$ has a strongly dense subgroup isomorphic to the surface group of genus g .*

Proof. It suffices to prove the theorem for $g = 2$. Let $\Gamma = \Sigma_2$, and let A denote the coordinate ring of the affine variety $\mathrm{Hom}(\Gamma, \mathrm{SL}_2)$. This variety is irreducible. The coordinate ring A of this variety is of dimension 9. Let L be the fraction field of A , and let $\phi: \Gamma \rightarrow \mathrm{SL}_2(L)$ denote the composition of the universal SL_2 -representation of Γ over A with the embedding $\mathrm{SL}_2(A) \subset \mathrm{SL}_2(L)$.

For $1 \neq \gamma \in \Gamma$, then there exists a homomorphism $\Gamma \rightarrow F_d$ sending $\gamma \mapsto \bar{\gamma} \neq 1$, and there exists a homomorphism $F_d \rightarrow \mathrm{SL}_2(K)$ which is injective. Therefore, the image of γ in $\mathrm{SL}_2(L)$ is not 1, so ϕ is injective.

If $\Delta \subset \Gamma$ is a non-abelian subgroup, we would like to prove that $\phi(\Delta)$ is Zariski-dense in $\mathrm{SL}_2(L)$. If Δ is a non-abelian subgroup of Γ and Δ_1 is a subgroup of finite index, then Δ_1 is either a surface group or a non-abelian free group, so the commutator subgroup of its commutator subgroup is non-trivial. If γ is a non-trivial element in this group, then $\phi(\gamma) \neq 1$, so $\phi(\Delta_1)$ cannot be contained in a Borel subgroup of $\mathrm{SL}_2(\bar{K})$. By classification of closed subgroups of SL_2 , every proper subgroup has a finite index subgroup whose second commutator is trivial. Therefore, $\phi(\Delta)$ is indeed Zariski-dense in SL_2 .

Now we proceed as in the proof of Theorem 8.6 using part (2) of Theorem 5.10 to show that SL_n has a very special generating set and then using the connectedness of $\mathrm{Hom}(\Gamma, \mathrm{SL}_n)$ in positive characteristic to deduce that for an algebraically closed field of transcendence degree 10 over \mathbb{F}_p , there is a strongly dense subgroup isomorphic to Γ and therefore a strongly dense subgroup isomorphic to Σ_g for each $g \geq 2$. \square

With a little more work, we can use the quotient of $\mathrm{Hom}(\Gamma, \mathrm{SL}_2)$ by the action of SL_2 to reduce the transcendence degree to 7. In [14], it is proved that for $p \geq 5$, there

are faithful representations of Γ in PGL_2 over fields of transcendence degree 2. It seems possible that one could prove the same result for SL_2 and use this to reduce transcendence degree to 3.

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