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Wronskians form the inverse system of the arcs of a double point

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Abstract
The ideal of the arc scheme of a double point or, equivalently, the differential ideal generated by the ideal of a double point is a primary ideal in an infinite-dimensional polynomial ring supported at the origin. This ideal has a rich combinatorial structure connecting it to singularity theory, partition identities, representation theory, and differential algebra. Macaulay inverse system is a powerful tool for studying the structure of primary ideals which describes an ideal in terms of certain linear differential operators. In the present paper, we show that the inverse system of the ideal of the arc scheme of a double point is precisely a vector space spanned by all the Wronskians of the variables and their formal derivatives. We then apply this characterization to extend our recent result on Poincaré-type series for such ideals.

Keywords Differential algebra · Macaulay inverse system · Arc spaces

1 Introduction

The main object studied in this paper is the arc scheme of a double point. Let k be a field of characteristic zero (all the fields in the paper will be assumed to be of characteristic zero). In algebraic geometry, for a k-variety X, the points of the arc space correspond to the Taylor coefficients of the $k[[t]]$-points of X. Thus, the arc space of X can be viewed as an infinite-order generalization of the tangent bundle or, in other words, the space of formal trajectories on the variety [12, 18]. For a variety defined by an ideal $I \subset k[\mathbf{x}]$, where $\mathbf{x} = (x_1, \ldots, x_n)$, the defining ideal I^{Arc} of the arc space belongs to

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the ring of formal derivatives of $\mathbf{x}$

$$k[\mathbf{x}^{(\infty)}] := k[x_i^{(j)} \mid 1 \leqslant i \leqslant n,\ 0 \leqslant j].$$

If we define $x_i(t) := \sum_{j=0}^{\infty} x_i^{(j)} t^j$, then I^{Arc} is generated by the coefficients of $f(\mathbf{x}(t))$ with respect to t for all $f \in I$.

We will be interested in the case when X is a double point, that is, is defined by an ideal

$$\mathcal{I}_n = \langle x_i x_j \mid 1 \leqslant i, j \leqslant n \rangle \subset k[x_1, \ldots, x_n]. \tag{1}$$

The ideal $\mathcal{I}_n^{\text{Arc}}$ defining its arc space corresponds only to one point over k, the origin, but exhibits a rich multiplicity structure which was initially studied in the context of differential algebra [8, 27, 30, 31], appeared in representation theory [23], and recently attracted attention because of its connections (especially in the case $n = 1$) to partition identities [1–4, 9, 13, 29].

Example 1.1 Consider $\mathcal{I}_2 = \langle x_1^2, x_2^2, x_1 x_2 \rangle \subset k[x_1, x_2]$. For $1 \leqslant i \leqslant 2$, let $x_i(t) = \sum_{j=0}^{\infty} x_i^{(j)} t^j$. Then $x_i(t)^2 = \sum_{j=0}^{\infty} \left(\sum_{r=0}^{j} x_i^{(r)} x_i^{(j-r)} \right) t^j$, and $x_1(t) x_2(t) = \sum_{j=0}^{\infty} \left(\sum_{r=0}^{j} x_1^{(r)} x_2^{(j-r)} \right) t^j$. Hence,

$$\mathcal{I}_2^{\text{Arc}} = \langle \sum_{r=0}^{j} x_1^{(r)} x_1^{(j-r)}, \sum_{r=0}^{j} x_2^{(r)} x_2^{(j-r)}, \sum_{r=0}^{j} x_1^{(r)} x_2^{(j-r)} \mid j \in \mathbb{N} \rangle.$$

$\triangle$

One classical approach to study such $\mathfrak{m}$-primary ideals of the maximal ideal $\mathfrak{m}$ at the origin is via considering linear differential operators with constant coefficients called *Noetherian equations*. These operators map the given primary ideal to a subspace of $\mathfrak{m}$. This approach goes back to Macaulay and Gröbner (see [7] for a survey) and can be viewed as a far-reaching generalization of the characterization of the root multiplicities of univariate polynomials through the vanishing of the derivatives. Noetherian equations provide an alternative representation for the corresponding ideals, and a generalization of this construction has been recently employed, for example, for numerical primary decomposition [14] and for using commutative algebra to solve linear PDEs [6, 17].

Since $\mathcal{I}_n^{\text{Arc}}$ is a primary ideal of the maximal ideal at the origin (although in the infinite-dimensional polynomial ring), a natural question is to find the Noetherian equations (or, equivalently, the Macaulay inverse system) for $\mathcal{I}_n^{\text{Arc}}$. The main result of this paper is a concise and surprising answer to this question: (see Theorem 4.11 below)

The Macaulay inverse system of the arc scheme of a double point is spanned by the Wronskians of finite subsets of $\mathbf{x}^{(\infty)}$.

One intriguing aspect of the proof is that it borrows some ideas from a seemingly unrelated characterization of Null Lagrangians [10] from the calculus of variations.

In the finite-dimensional case, a natural invariant of a primary ideal contained within a maximal ideal is its dimension. The dimension of $\mathcal{I}_n^{\mathrm{Arc}}$ is infinite, but one consider instead the dimensions of the truncations $\dim k[\mathbf{x}^{(\leqslant h)}]/(\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{(\leqslant h)}])$, where $k[\mathbf{x}^{(\leqslant h)}]:=k[x_i^{(j)} \mid 1 \leqslant i \leqslant n,\ 0 \leqslant j \leqslant h]$ (see [5] for details). As an application, we use the computed Macaulay inverse system for $\mathcal{I}_n^{\mathrm{Arc}}$ and the recently proved Kolchin-Schmidt conjecture [15, 21, 22] to compute these dimensions for the case of double points thus extending the main result of [5]. Specifically, we show that (see Theorem 6.1 below)

$$\dim k[\mathbf{x}^{(\leqslant h)}]/(\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{(\leqslant h)}]) = (n+1)^{h+1}. \tag{2}$$

The rest of the paper is structured as follows: Section 2 contains necessary preliminaries and notation. In Sect. 3, we state the main results of the paper. In Sect. 4 we prove the characterization of the inverse system of $\mathcal{I}_n^{\mathrm{Arc}}$ in terms of Wronskians (Theorem 3.1). Then, we use it to characterize the inverse systems of the elimination ideals $\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{(\leqslant h)}]$ in Sect. 5. This characterization is then used in Sect. 6 to establish the dimension result (2).

2 Preliminaries

Notation 2.1 *(Formal derivatives)* Let x be a variable in a polynomial ring. Then we will consider $x, x', x'', x^{(3)}, \ldots$ as independent polynomial variables, and, for any $h \in \mathbb{Z}_{\geqslant 0}$, we introduce the following notation

$$x^{(<h)}:=(x, x', \ldots, x^{(h-1)}) \quad \text{and} \quad x^{(\infty)}:=(x, x', x'', \ldots).$$

The symbol $x^{(\leqslant h)}$ is defined analogously. Similarly, for a tuple $\mathbf{x} = (x_1, \ldots, x_n)$ of variables we have

$$\mathbf{x}^{(<h)}:=(x_1^{(<h)}, \ldots, x_n^{(<h)}) \quad \text{and} \quad \mathbf{x}^{(\infty)}:=(x_1^{(\infty)}, \ldots, x_n^{(\infty)}).$$

The symbols $\mathbf{x}^{(h)}$ and $\mathbf{x}^{(\leqslant h)}$ are defined analogously.

Definition 2.2 *(Arc space [18])* Let $I \subset k[\mathbf{x}]$ (where $\mathbf{x} = (x_1, \ldots, x_n)$) be an ideal defining a variety X. Then the defining ideal I^{Arc} of the *arc space* of X belongs to the polynomial ring $k[\mathbf{x}^{(\infty)}]$ in formal derivatives of $\mathbf{x}$. If we denote $\mathbf{x}(t):=\sum_{j=0}^{\infty} \mathbf{x}^{(j)} t^j$, then I^{Arc} is generated by the coefficients of powers of t in $f(\mathbf{x}(t))$ for all $f \in I$.

For defining inverse systems, we will restrict ourselves to ideals with the support at the origin in order to keep things simple. For the purposes of the present paper, we will give a definition valid for polynomial rings in infinitely many variables.

Definition 2.3 *(Inverse system* [7]*)* Let $\mathbf{x} = \{x_\lambda\}_{\lambda\in\Lambda}$ is a (possibly infinite) set of variables indexed by a set Λ. Let $p, q \in k[\mathbf{x}]$ be polynomials. We define the polynomial $p \bullet q \in k[\mathbf{x}]$ as follows:

$$p \bullet q(\mathbf{x}) = p(\mathbf{x})|_{x_\lambda = \frac{\partial}{\partial x_\lambda}} q(\mathbf{x}),$$

where $p(\mathbf{x})|_{x_\lambda = \frac{\partial}{\partial x_\lambda}}$ denotes the result of substituting $\frac{\partial}{\partial x_\lambda}$ for x_λ for all $\lambda \in \Lambda$. Let $I \subset k[\mathbf{x}]$ be an $\mathfrak{m}$-primary ideal, where $\mathfrak{m}$ is the maximal ideal generated by $\mathbf{x}$. Then the *inverse system* (also called *Macaulay inverse system*) $I^\perp \subset k[\mathbf{x}]$ is a vector space defined by

$$I^\perp := \{f \in k[\mathbf{x}] \mid \forall p \in I : (f \bullet p) \in \mathfrak{m}\}.$$

Example 2.4 Let $I = \langle x^2, y^2, z - x - y\rangle \subseteq k[x, y, z]$, which is $\langle x, y, z\rangle$-primary. Then $I^\perp$ is the k-vector space generated by $\{1, x - y, x + z, xy + yz + xz + z^2\}$. In fact, since $x^2, y^2 \in I$, then for every $P \in I^\perp$, $\deg_x(P) \leqslant 1$ and $\deg_y(P) \leqslant 1$. Let $P = P_0(x, y) + P_1(x, y)z + \cdots + P_n(x, y)z^n \in I^\perp$. After applying $\frac{\partial}{\partial z} - \frac{\partial}{\partial x} - \frac{\partial}{\partial y}$ to P we get that for all $0 \leqslant i \leqslant n - 1$,

$$(i + 1)P_{i+1} = \frac{\partial P_i}{\partial x} + \frac{\partial P_i}{\partial y}. \tag{3}$$

Since $\deg_x(P_0), \deg_y(P_0) \leqslant 1$, then by (3) we have $\deg(P_1) \leqslant 1$ which also implies that P_2 is a constant and $\deg(P) \leqslant 2$. Suppose that $\deg(P) = 1$, given by $P = a + bx + cy + dz$. Then by (3) we have, $b + c = d$ which implies that P belongs to the vector space generated by $\{1, x - y, x + z\}$. Suppose that P is homogeneous of degree 2 given by $P = axy + (bx + cy)z + dz^2$. Then by (3) we have, $b = c = a = d$. Therefore, we conclude that $I^\perp$ is the vector space generated by $\{1, x - y, x + z, xy + yz + xz + z^2\}$.$\triangle$

Proposition 2.5 *Let $\mathbf{x} = (x_1, \ldots, x_n)$. For every ideal I of $k[\mathbf{x}]$ the following characterization of $I^\perp$ holds:*

$$I^\perp = \{P \in k[\mathbf{x}] \mid \forall f \in I, f \bullet P = 0\}.$$

Proof For $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n$ we abbreviate $\mathbf{x}^{\boldsymbol{\alpha}} = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Note that, for every two monomials $\mathbf{x}^{\boldsymbol{\alpha}}$ and $\mathbf{x}^{\boldsymbol{\beta}}$ we have:

$$\mathbf{x}^{\boldsymbol{\beta}} \bullet \mathbf{x}^{\boldsymbol{\alpha}} = \begin{cases} \frac{\alpha!}{(\alpha-\beta)!}\mathbf{x}^{\boldsymbol{\alpha}-\boldsymbol{\beta}} & \text{if } \beta_i \leqslant \alpha_i \ \forall 1 \leqslant i \leqslant n \\ 0, & \text{otherwise} \end{cases}$$

where $\boldsymbol{\alpha}! := \alpha_1! \cdots \alpha_n!$. Let $P = a_{u_0}\mathbf{x}^{u_0} + \ldots + a_{u_k}\mathbf{x}^{u_k} \in k[\mathbf{x}]$ and $f = \sum_{u\in\mathbb{N}^n} f_u \mathbf{x}^u \in I$, such that $f \bullet P = 0$. This implies that,

$$f \bullet P = \sum_{v \leqslant u_0} a_{u_0} f_v \frac{u_0!}{(u_0 - v)!}\mathbf{x}^{u_0 - v} + \ldots + \sum_{v \leqslant u_k} a_{u_k} f_v \frac{u_k!}{(u_k - v)!}\mathbf{x}^{u_k - v} = 0.$$

Thus, the coefficient of $\frac{1}{u!}\mathbf{x}^u$ in the last equation is $a_{u_0} f_{u_0-u} u_0! + \cdots + a_{u_k} f_{u_k-u} u_k!$ where $f_{u_i-u} = 0$ if $u_i - u \in \mathbb{Z}^n \setminus \mathbb{N}^n$. Note also that, $a_{u_0} f_{u_0-u} u_0! + \cdots + a_{u_k} f_{u_k-u} u_k!$ is equal to the remainder of $P \bullet (f \cdot \mathbf{x}^u)$ modulo $\mathfrak{m}$. Therefore $f \bullet P = 0$ if and only if, $P \bullet (f \cdot \mathbf{x}^u) \in \mathfrak{m}$ for all monomial $\mathbf{x}^u$. i.e., $P \bullet f \in \mathfrak{m}$ for all $f \in I$ if and only if $f \bullet P = 0$ for all $f \in I$. □

Notation 2.6 *(Wronskians)* We define a derivation on $k[\mathbf{x}^{(\infty)}]$ by $(x_i^{(j)})' := x_i^{(j+1)}$ for every $1 \leqslant i \leqslant n$ and $j \geqslant 0$. Then, for polynomials $f_1, \ldots, f_\ell \in k[\mathbf{x}^{(\infty)}]$, we define the *Wronskian determinant* $\mathrm{Wr}(f_1, \ldots, f_\ell)$ as follows:

$$\mathrm{Wr}(f_1, \ldots, f_\ell) := \begin{vmatrix} f_1 & f_2 & \ldots & f_\ell \\ f_1' & f_2' & \ldots & f_\ell' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(\ell-1)} & f_2^{(\ell-1)} & \ldots & f_\ell^{(\ell-1)} \end{vmatrix}. \tag{4}$$

3 Main results

Our first main result characterizes the inverse system of the arc space of a double point defined by the ideal $\mathcal{I}_n$ (see (1)).

Theorem 3.1 *For every integer $n > 0$, the inverse system $(\mathcal{I}_n^{\mathrm{Arc}})^\perp$ of the arc space of a double point is spanned by the Wronskians $\mathrm{Wr}(S)$, where S ranges over all finite subsets $S \subset \mathbf{x}^{(\infty)}$.*

Example 3.2 Consider $\mathcal{I}_1^{\mathrm{Arc}} \subset k[x^{(\infty)}]$ which is a $\mathfrak{m}$-primary ideal for $\mathfrak{m} = \langle x^{(\infty)} \rangle$. Let us determine the elements of $(\mathcal{I}_1^{\mathrm{Arc}})^\perp \subseteq k[x^{(\infty)}]$ of order and degree at most 2. The generators of $\mathcal{I}_1^{\mathrm{Arc}}$ involving a monomial in $k[x, x', x'']$ are $G = \{x^2, xx', 2xx'' + (x')^2, xx^{(3)} + x'x'', 2xx^{(4)} + 2x'x^{(3)} + (x'')^2\}$. Consider the following differential operator

$$P = c_0 + c_1 \frac{\partial}{\partial x} + c_2 \frac{\partial}{\partial x'} + c_3 \frac{\partial}{\partial x''} + c_4 \frac{\partial^2}{\partial x^2} + c_5 \frac{\partial^2}{\partial (x')^2} + c_6 \frac{\partial^2}{\partial (x'')^2} + c_7 \frac{\partial^2}{\partial x x'} + c_8 \frac{\partial^2}{\partial x x''} + c_9 \frac{\partial^2}{\partial x' x''}.$$

By applying P to the polynomials in G we realize that the resulting polynomials belongs to $\mathfrak{m}$ if and only if

$$c_4 = c_6 = c_7 = c_9 = 0 \text{ and } c_5 + c_8 = 0.$$

Therefore the vector space $(\mathcal{I}_1^{\mathrm{Arc}})^\perp \cap k[x, x', x'']_{\leqslant 2}$ is generated by $\{1, x, x', x'', xx'' - (x')^2\}$. These generators are the minors of the matrix matrix $\begin{pmatrix} x & x' \\ x' & x'' \end{pmatrix}$, which are themselves Wronskian determinants. In fact, one could even show that it is also equal to $(\mathcal{I}_1^{\mathrm{Arc}})^\perp \cap k[x, x', x'']$. △

We use Theorem 3.1 to derive the following Poincare-type series for $\mathcal{I}_n^{\mathrm{Arc}}$ extending [5, Theorem 3.1].

Theorem 3.3 *For every positive integer n, we have*

$$\sum_{h=0}^{\infty} \dim \left(k[\mathbf{x}^{(\leqslant h)}]/(\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{(\leqslant h)}])\right) t^h = \frac{n+1}{1-(n+1)t}.$$

4 Proof of theorem 3.1

Notation 4.1 *(Evaluating differential polynomials)* For a polynomial $P(\mathbf{x}) \in k[\mathbf{x}^{(\infty)}]$ with $\mathbf{x} = (x_1, \ldots, x_n)$ and a tuple $\mathbf{a} \in R^n$, where R is a ring with a specified derivation, the expression $P(\mathbf{a})$ means *differential evaluation* of P at $\mathbf{a}$, that is, the result of substituting $x_i^{(j)}$ with $a_i^{(j)}$ for every $1 \leqslant i \leqslant n$ and $j \geqslant 0$.

We introduce new transcendental constants $\xi, \alpha_1, \ldots, \alpha_n$ (and denote $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n)$), and consider the following differential operator with coefficients in $k[\xi, \boldsymbol{\alpha}]$

$$D_{\xi,\boldsymbol{\alpha}} := \sum_{j=1}^{n} \sum_{i=0}^{\infty} \alpha_j \xi^i \frac{\partial}{\partial x_j^{(i)}}.$$

The key property of this operator is given by the lemma as follows:

Lemma 4.2 *Let $P \in k[\mathbf{x}^{(\infty)}]$, then $P \in (\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ if and only if $D_{\xi,\boldsymbol{\alpha}}^2 P = 0$.*

Proof We will consider $Q = D_{\xi,\boldsymbol{\alpha}}^2(P)$ as a polynomial in $\xi, \boldsymbol{\alpha}$. Since $D_{\xi,\boldsymbol{\alpha}}$ is linear in $\boldsymbol{\alpha}$, the monomials that may appear in Q will be of the form $\alpha_i \alpha_j \xi^\ell$ for $1 \leqslant i \leqslant j \leqslant n$ and $\ell \geqslant 0$. Then direct computation shows that the coefficient in front of this monomial is

$$C \sum_{s=0}^{\ell} \frac{\partial^2}{\partial x_i^{(s)} \partial x_j^{(\ell-s)}} P, \tag{5}$$

where $C = 1$ if $i = j$ and is equal to 2 otherwise. This expression can be rewritten as $C(\sum_{s=0}^{\ell} x_i^{(s)} x_j^{(\ell-s)}) \bullet P$, and the polynomial $\sum_{s=0}^{\ell} x_i^{(s)} x_j^{(\ell-s)}$ is the coefficient at t^ℓ in $x_i(t)x_j(t)$. Therefore, vanishing of (5) for all $1 \leqslant i \leqslant j \leqslant n$ and $\ell \geqslant 0$ is equivalent by Proposition 2.5 to $P \in (\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$, and this finishes the proof. ◻

Lemma 4.2 implies that the elements of $(\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ satisfy the following "linearity under exponential perturbations" property.

Lemma 4.3 *Let $P \in k[\mathbf{x}^{(\infty)}]$, then $P \in (\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ if and only if the following equality holds in the ring $k[\mathbf{x}^{(\infty)}, \boldsymbol{\alpha}][[\xi, t]]$ with the derivation extended by $t' = 1$:*

$$P(\mathbf{x} + \boldsymbol{\alpha} e^{\xi t}) = P(\mathbf{x}) + e^{\xi t} Q(\xi, \boldsymbol{\alpha}, \mathbf{x}).$$

where $Q \in k[\xi, \boldsymbol{\alpha}, \mathbf{x}^{(\infty)}]$ *and* $P(\mathbf{x} + \boldsymbol{\alpha} e^{\xi t})$ *is understood as differential evaluation (see Notation 4.1). Furthermore, if such an equality holds, then* $Q = D_{\xi,\boldsymbol{\alpha}} P(\mathbf{x})$.

Proof First we observe that, for every $P \in k[\mathbf{x}^{(\infty)}]$, we have

$$\frac{\partial}{\partial(e^{\xi t})} P(\mathbf{x} + \boldsymbol{\alpha} e^{\xi t}) = D_{\xi,\boldsymbol{\alpha}} P(\mathbf{x} + \boldsymbol{\alpha} e^{\xi t}).$$

Therefore, we can write the Taylor expansion of $P(\mathbf{x} + \boldsymbol{\alpha} e^{\xi t})$ with respect to $e^{\xi t}$ using the operator $D_{\xi,\boldsymbol{\alpha}}$:

$$P(\mathbf{x} + \boldsymbol{\alpha} e^{\xi t}) = P(\mathbf{x}) + e^{\xi t} D_{\xi,\boldsymbol{\alpha}} P(\mathbf{x}) + \frac{e^{2\xi t}}{2} D^2_{\xi,\boldsymbol{\alpha}} P(\mathbf{x}) + \ldots.$$

The statement of the lemma follows from the expansion above and Lemma 4.2. □

Corollary 4.4 *Let* $P \in (\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ *be homogeneous of degree* $d+1$. *Then, for any* $\xi_1, \ldots, \xi_d \in k$ *and* $\boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_d} \in k^n$, *we have*

$$P(\boldsymbol{\alpha_1} e^{\xi_1 t} + \ldots + \boldsymbol{\alpha_d} e^{\xi_d t}) = 0.$$

Proof We consider $\xi_1, \ldots, \xi_d, \boldsymbol{\alpha_1}, \ldots, \boldsymbol{\alpha_d}$ as unknowns. Let $\mathbf{x}^* = \boldsymbol{\alpha_1} e^{\xi_1 t} + \ldots + \boldsymbol{\alpha_{d-1}} e^{\xi_{d-1} t}$ By Lemma 4.3 we have

$$P(\mathbf{x}^* + \boldsymbol{\alpha_d} e^{\xi_d t}) = P(\mathbf{x}^*) + e^{\xi_d t} D_{\xi_d,\boldsymbol{\alpha_d}} P(\mathbf{x}^*).$$

This implies that $e^{\xi_d t}$ appears in $P(\boldsymbol{\alpha_1} e^{\xi_1 t} + \ldots + \boldsymbol{\alpha_d} e^{\xi_d t})$ with degree at most 1. Therefore, for every $1 \leqslant j \leqslant d$, we have $e^{\xi_j t}$ appearing with degree at most 1. However, P is homogeneous of degree $d+1$, so it must vanish on $\boldsymbol{\alpha_1} e^{\xi_1 t} + \ldots + \boldsymbol{\alpha_d} e^{\xi_d t}$. □

From this, we will deduce that every element of $(\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ of degree d vanishes on solutions of low order linear differential equations.

Proposition 4.5 *Let* $V \subset k[[t]]$ *be a subspace of dimension d closed under differentiation. Then every homogeneous* $P \in (\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ *of degree* $d+1$ *vanishes on any point of* V^n.

Proof We fix a space V and a polynomial P as in the statement of the proposition. By the cyclic vector theorem [16], there exists $f \in k[[t]]$ satisfying a linear differential equation of order d such that $V = \langle f, f', \ldots, f^{(d-1)} \rangle$. Let $\mu_1, \ldots, \mu_\ell$ and $e_1, \ldots, e_\ell$ be the roots and their multiplicities of the characteristic polynomial of the minimal linear equation for f. Then the minimality implies that $\sum e_i = d$.

If $e_1 = \ldots = e_\ell = 1$, then $\ell = d$ and every element of V is a linear combination $e^{\mu_1 t}, \ldots, e^{\mu_\ell t}$. By Corollary 4.4, P vanishes on any n-tuple of linear combinations of these exponentials.

Now we consider the case when at least one of e_i's is greater than one. In this case, any element of V is a quasi-polynomial of the form $\sum_{i=1}^{\ell}\sum_{j=0}^{e_i-1} a_{i,j}t^j e^{\mu_i t}$. We will first consider the case $k=\mathbb{C}$, and we claim that any such function can be represented as a limit of linear combinations of at most d exponentials. Indeed, using the Taylor expansion we can write $e^{(\mu_i+\frac{r}{N})t}$ for every $0 \leqslant j \leqslant e_i - 1$ and for every $r \leqslant j$ as:

$$e^{(\mu_i+\frac{r}{N})t} = e^{\mu_i t} + \frac{r}{N}te^{\mu_i t} + \ldots + \frac{r^j}{j!N^j}t^j e^{\mu_i t} + C_r(t,N), \tag{6}$$

where $C_r(t,N) = \mathrm{o}(\frac{1}{N^j})$ as $N \to \infty$. Furthermore, by differentiating (6), we show that $\frac{\partial}{\partial t^s}C_r(t,N) = \mathrm{o}(\frac{1}{N^j})$ for every s. Formulas (6) for $0 \leqslant r \leqslant j$ written in a matricial form allow us to express $t^j e^{\mu_i t}$ as follows:

$$t^j e^{\mu_i t} = \begin{pmatrix}1 & 0 & \ldots & 0\end{pmatrix} \begin{pmatrix} \frac{1}{j!}\left(\frac{j}{N}\right)^j & \frac{1}{(j-1)!}\left(\frac{j}{N}\right)^{(j-1)} & \ldots & 1 \\ \frac{1}{j!}\left(\frac{j-1}{N}\right)^j & \frac{1}{(j-1)!}\left(\frac{j-1}{N}\right)^{(j-1)} & \ldots & 1 \\ \vdots & \ddots & \ddots & \vdots \\ \frac{1}{j!}\left(\frac{1}{N}\right)^j & \frac{1}{(j-1)!}\left(\frac{1}{N}\right)^{(j-1)} & \ldots & 1 \\ 0 & 0 & \ldots & 1 \end{pmatrix}^{-1} \begin{pmatrix} e^{(\mu_i+\frac{j}{N})t} - C_j \\ e^{(\mu_i+\frac{j-1}{N})t} - C_{j-1} \\ \vdots \\ e^{(\mu_i+\frac{1}{N})t} - C_1 \\ e^{\mu_i t} \end{pmatrix}. \tag{7}$$

The matrix in (7) can be factored as

$$M_j \cdot \operatorname{diag}\left(\frac{1}{j!N^j},\ \frac{1}{(j-1)!N^{j-1}},\ \ldots,\ \frac{1}{N},\ 1\right),$$

where M_j is a Vandermonde matrix (thus, invertible) not depending on N. The inverse of this matrix appearing in (7) is therefore a matrix with the entries being polynomials in N of degree at most j. Therefore, (7) can be written as

$$t^j e^{\mu_i t} = \left(\sum_{r=0}^{j} \lambda_r(N) e^{(\mu_i + r/N)t}\right) + C(t,N),$$

where $C(t,N) = \mathrm{o}(1)$ when $N \to \infty$ as well as all its derivatives with respect to t.

Therefore, the whole linear combination $\sum_{i=1}^{\ell}\sum_{j=0}^{e_i-1} a_{i,j}t^j e^{\mu_i t}$ can be written as a limit $N \to \infty$ of linear combinations of $\{e^{(\mu_i+j/N)t} \mid 1 \leqslant i \leqslant \ell,\ 0 \leqslant j < e_i\}$, and the convergence is uniform on a bounded domain in t for the functions and their derivatives up to any order. We take a tuple $\mathbf{x}^* \in V^n$ and approximate it by such a sequence of linear combinations of exponentials $\mathbf{x}_N^* \to \mathbf{x}^*$. Since the derivatives converge as well, $P(\mathbf{x}_N^*) \to P(\mathbf{x}^*)$, so the latter is equal to zero.

We finish by extending the proof for the case when at least one of e_i is greater than one to an arbitrary ground field k of characteristic zero. We observe that the property $P \in (\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ is defined by a system of linear equations over $\mathbb{Q}$ on the coefficients of P. Therefore, we can express P as a linear combination of elements of $(\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ with the coefficients in $\mathbb{Q}$, so we will further assume that P has coefficients in $\mathbb{Q}$. If we consider μ_i's and $a_{i,j}$'s as variables and plug expressions of the form $\sum_{i=1}^{\ell}\sum_{j=0}^{e_i-1} a_{i,j}t^j e^{\mu_i t}$ to $P(\mathbf{x})$, the result will be a combinations of different products of the exponentials with the coefficients being polynomials in μ_i's and $a_{i,j}$'s. These polynomials vanish over $\mathbb{C}$, so they are identically zero and, thus, vanish over any field k. ◻

Definition 4.6 For any positive integers n and h, we define the infinite Hankel matrix as follows:

$$\mathbf{H}_{n,h} := \begin{pmatrix} x_1 & x_2 & \cdots & x_n & x_1' & x_2' & \cdots\cdots \\ x_1' & x_2' & \cdots & x_n' & x_1'' & x_2'' & \cdots\cdots \\ x_1'' & x_2'' & \cdots & x_n'' & x_1^{(3)} & x_2^{(3)} & \cdots\cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\ \cdots \\ x_1^{(h-1)} & x_2^{(h-1)} & \cdots & x_n^{(h-1)} & x_1^{(h)} & x_2^{(h)} & \cdots\cdots \end{pmatrix} \tag{8}$$

Since $\mathbf{H}_{n,h}$ is a submatrix of $\mathbf{H}_{n,h+1}$, an infinite-by-infinite matrix $\mathbf{H}_{n,\infty}$ can be defined by taking $h \to \infty$.

Proposition 4.7 *Every minor of the infinite Hankel matrix* (8) *belongs to* $(\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$.

Proof Consider one such $\ell \times \ell$-minor. It can be written as $\det(x_{s_j}^{(a_i+b_j)})_{i,j}$, where $(a_1, \dots, a_\ell)$, $(b_1, \dots, b_\ell)$, and $(s_1, \dots, s_\ell)$ are tuples of integers. We denote this determinant by $W(\mathbf{x}) \in k[\mathbf{x}^{(\infty)}]$. Since $(\mathbf{x} + \boldsymbol{\alpha} e^{\xi t})^{(k)} = \mathbf{x}^{(k)} + \xi^k \boldsymbol{\alpha} e^{t\xi}$ we can write $W(\mathbf{x}+\boldsymbol{\alpha} e^{t\xi})$ (understood as differential evaluation, see Notation 4.1) using the linearity of the determinant as follows:

$$W(\mathbf{x} + \boldsymbol{\alpha} e^{t\xi}) = W(\mathbf{x}) + e^{t\xi} S_1 + e^{2t\xi} S_2 + \dots,$$

where S_1 is a sum over j of determinants given by replacing a column

$${}^{\mathrm{t}}\left(x_{s_j}^{(b_j+a_1)},\ x_{s_j}^{(b_j+a_2)},\ \cdots,\ x_{s_j}^{(b_j+a_\ell)}\right)$$

of $W(\mathbf{x})$ by

$${}^{\mathrm{t}}\left(\alpha_{s_j}\xi^{b_j+a_1},\ \alpha_{s_j}\xi^{b_j+a_2},\ \cdots,\ \alpha_{s_j}\xi^{b_j+a_\ell}\right),$$

and S_2 is a sum of determinants given by performing two such replacements in $W(\mathbf{x})$, and so on for all S_k where $k \geqslant 1$. Since two columns of the form

$$^{\mathsf{t}}\left(\alpha_{s_j}\xi^{b_j+a_1}, \alpha_{s_j}\xi^{b_j+a_2}, \cdots, \alpha_{s_j}\xi^{b_j+a_\ell}\right), \ ^{\mathsf{t}}\left(\alpha_{s_k}\xi^{b_k+a_1}, \alpha_{s_k}\xi^{b_k+a_2}, \cdots, \alpha_{s_k}\xi^{b_k+a_\ell}\right),$$

are proportional, we have $S_k = 0$ for $k > 1$ and, thus

$$W(\mathbf{x} + \boldsymbol{\alpha} e^{\xi t}) = W(\mathbf{x}) + e^{\xi t} S_1.$$

Hence by Lemma 4.3 we conclude that $W(\mathbf{x}) \in (\mathcal{I}_n^{\mathrm{Arc}})^\perp$. □

We will now recall the concept of 1–generic matrices which we will use in the proof of Lemma 4.10. For more details see [19, 20, 24]).

Definition 4.8 We say that a matrix M with the entries being linear forms in some variables is 1–generic if for every nonzero vectors v_1 and v_2 over the ground field:

$$^{\mathsf{t}}v_1 M v_2 \neq 0.$$

Remark 4.9 Hankel matrices are examples of 1–generic matrices [20, p. 549] over the algebraic closure of the ground field. Therefore, the maximal minors of a 1–generic matrix form a prime ideal [19, Theorem 1].

Lemma 4.10 *Consider the infinite Hankel matrix* $\mathbf{H}_{n,h}$ *and let* $\mathbf{J}_{n,h}$ *the ideal generated by the* $h \times h$ *minors of* $\mathbf{H}_{n,h}$. *Then* $\mathbf{J}_{n,h}$ *is a prime differential ideal.*

Proof For every $k \geq h - 1$ the following matrix:

$$\mathbf{H}_{n,h,k} := \begin{pmatrix} x_1 & x_2 & \cdots & x_n & x_1' & x_2' & \cdots & x_n' & \cdots & x_1^{(k)} & x_2^{(k)} & \cdots & x_n^{(k)} \\ x_1' & x_2' & \cdots & x_n' & x_1'' & x_2'' & \cdots & x_n'' & \cdots & x_1^{(k+1)} & x_2^{(k+1)} & \cdots & x_n^{(k+1)} \\ x_1'' & x_2'' & \cdots & x_n'' & x_1^{(3)} & x_2^{(3)} & \cdots & x_n^{(3)} & \cdots & x_1^{(k+2)} & x_2^{(k+2)} & \cdots & x_n^{(k+2)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_1^{(h-1)} & x_2^{(h-1)} & \cdots & x_n^{(h-1)} & x_1^{(h)} & x_2^{(h)} & \cdots & x_n^{(h)} & \cdots & x_1^{(h-1+k)} & x_2^{(h-1+k)} & \cdots & x_n^{(h-1+k)} \end{pmatrix}$$

is 1–generic since $\mathbf{H}_{n,h,k}$ is block matrix of Hankel matrices (up to exchanging columns). Therefore the maximal minors of $\mathbf{H}_{n,h,k}$ form a prime ideal $\mathbf{J}_{n,h,k}$. Hence the ideal $\mathbf{J}_{n,h} = \bigcup_k \mathbf{J}_{n,h,k}$ is a prime ideal. Furthermore, since a derivative of every maximal minor of $\mathbf{H}_{n,h}$ is a linear combination of maximal minors of the same matrix, this ideal is a differential ideal. □

Now we are ready to proof Theorem 3.1.

Theorem 4.11 (cf. Theorem 3.1) *Let* $P \in k[\mathbf{x}^{(\infty)}]$ *such that* $P \in (\mathcal{I}_n^{\mathrm{Arc}})^\perp$. *Then* P *is a linear combination of Wronskians* $\mathrm{Wr}(S)$, *where* S *ranges over all finite subsets* $S \subset \mathbf{x}^{(\infty)}$.

Proof Let $P \in (\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$. Since the generators of $\mathcal{I}_n^{\mathrm{Arc}}$ are homogeneous, every homogeneous component of P must also belong to $(\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$. Thus, without loss of generality we can suppose that P is homogeneous of degree d. Let K be an uncountable algebraically closed field containing k. Let $\mathbf{x}(t):=(x_1(t), \dots, x_n(t))$ be an n-tuple of power series from $K[[t]]$ which belongs to $V(\mathbf{J}_{n,d})$. The fact that $\mathbf{x}(t)$ satisfies every equation in $V(\mathbf{J}_{n,d})$ implies that the dimension of the vector space spanned by $x_1(t), \dots, x_n(t)$ and their derivatives is less than d by the standard property of the Wronskian [25, Chapter II, Section 1, Theorem 1]. Then, by Proposition 4.5, P vanishes at $\mathbf{x}(t)$.

The power series solutions of a differential ideal are in a bijective correspondence with points in an infinite-dimensional space over the ground field via the evaluation of the power series at $t = 0$ (see [32, Section 3.2]). Then we apply Hilbert's Nullstellensatz for polynomials in countably many variables [26] (and use that we working over an uncountable field K) to $\mathbf{J}_{n,d}$ and P, and conclude that P belongs to $\mathbf{J}_{n,d}$. Since P and the generators of $\mathbf{J}_{n,d}$ are homogeneous of degree d, P is a linear combination of the generators, that is, of the maximal minors of $\mathbf{H}_{n,d}$. □

Proposition 4.7 and Theorem 4.11 imply:

Corollary 4.12 *The vector space $(\mathcal{I}_n^{\mathrm{Arc}})^{\perp}$ is equal to the vector space spanned by the minors of $\mathbf{H}_{n,\infty}$.*

5 Inverse system and elimination

Lemma 5.1 *Let $J \subset k[\mathbf{x}]$ be a $\mathfrak{m}$-primary ideal where $\mathfrak{m}$ is the maximal ideal at the origin. Then for every $m < n$ we have:*

$$(J \cap k[x_1, \dots, x_m])^{\perp} = \left\{ P|_{\{x_s=0,\, \forall\, m<s\leqslant n\}} \;:\; P \in J^{\perp} \right\}.$$

Proof See [28, Proposition 3.2] □

Remark 5.2 In this section, we will sometimes consider power series in infinitely many variables. By this we will mean the algebra of formal infinite linear combination of all finite-degree monomials, for a formal definition see [11, Ch.3, §2.11]. The action of partial derivatives (and thus more general differential operators) is naturally defined for such linear combination.

Lemma 5.3 *Consider the polynomial ring $k[x_{i,j} \mid i \in \mathbb{N}, 1 \leqslant j \leqslant n]$, and let J an $\mathfrak{m}$-primary ideal where $\mathfrak{m}:=\langle \{x_{i,j} \mid i \in \mathbb{N}, 1 \leqslant j \leqslant n\}\rangle$, such that J is homogeneous with respect to the grading $\operatorname{gdeg}(x_{i,j}) = i + 1$. Then for every $\ell \in \mathbb{N}$ we have:*

$$(J \cap k[x_{0,j} \dots, x_{\ell,j} \,|\, 1 \leqslant j \leqslant n])^{\perp} = \left\{ P|_{\{x_{s,j}=0,\, \forall\, s>\ell\}} \;:\; P \in J^{\perp} \right\}.$$

Proof One inclusion follows immediately from the fact that if $\forall f \in J,\ f \bullet P = 0$, then $\forall f \in J \cap k[x_{0,j} \dots, x_{\ell,j} \,|\, 1 \leqslant j \leqslant n],\ f \bullet P|_{\{x_{s,j}=0,\, \forall\, s>\ell\}} = 0$. Let us prove the opposite inclusion:

$$(J \cap k[x_{0,j} \ldots, x_{\ell,j} \,|\, 1 \leqslant j \leqslant n])^{\perp} \subseteq \left\{ P|_{\{x_{s,j}=0,\, \forall s>\ell\}} \,:\, P \in J^{\perp} \right\}.$$

Let $P \in (J \cap k[x_{0,j} \ldots, x_{\ell,j} \,|\, 1 \leqslant j \leqslant n])^{\perp}$, then by Lemma 5.1 there exists $P_1 \in (J \cap k[x_{0,j} \ldots, x_{\ell+1,j} \,|\, 1 \leqslant j \leqslant n])^{\perp}$ such that $P_1|_{\{x_{\ell+1,j}=0\}} = P$. By repeating this, one can construct a power series $\widetilde{P} \in k[\![x_{i,j} \,|\, i \in \mathbb{N}, 1 \leqslant j \leqslant n]\!]$ such that $\widetilde{P}|_{\{x_{s,j}=0,\, \forall s>\ell\}} = P$, and

$$f \bullet \widetilde{P} = 0 \text{ for all } f \in J.$$

For every $d \in \mathbb{N}$, let $\widetilde{P}_d$ be the homogeneous summand of $\widetilde{P}$ of graded degree d. Since there are only finitely many monomials of this graded degree, $\widetilde{P}_d$ is a polynomial. Since J is a homogeneous polynomial ideal we have:

$$\forall d \in \mathbb{N},\ \forall f \in J :\ f \bullet \widetilde{P}_d = 0.$$

Hence, $\forall d \in \mathbb{N},\ \widetilde{P}_d \in J^{\perp}$.

Let $m \in \mathbb{N}$ such that, $\forall 0 \leqslant i \leqslant \ell, \forall 1 \leqslant j \leqslant n : x_{i,j}^m \in J$, then $\deg_{x_{i,j}}(\widetilde{P}) < m$. Let us introduce

$$Q = \sum_{d \leqslant d_m} \widetilde{P}_d$$

where $d_m := \frac{(m-1)(\ell+1)(\ell+2)n}{2}$. We would like to prove that $Q|_{\{x_{s,j}=0,\, \forall m<s\}} = P$. Since the monomial $\prod_{1 \leqslant j \leqslant n}(x_{0,j}^{m-1} \cdots x_{\ell,j}^{m-1})$ has graded degree d_m , every monomial summand of $\widetilde{P}$ whose graded degree bigger than d_m is involving $x_{s,j}$ for some $s > \ell$, and $1 \leqslant j \leqslant n$. Therefore $\widetilde{P}|_{\{x_{s,j}=0,\, \forall s>\ell\}} = Q|_{\{x_{s,j}=0,\, \forall s>\ell\}} = P$. Hence,

$$P \in \left\{ P|_{\{x_{s,j}=0,\, \forall s>\ell\}} \,:\, P \in J^{\perp} \right\}.$$

□

Theorem 5.4 *For every non negative integer h, we have*

$$(\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{\leqslant h}])^{\perp} = \left\{ P|_{\{\mathbf{x}^{(s)}=0,\, \forall s>h\}} \,:\, P \in (\mathcal{I}_n^{\mathrm{Arc}})^{\perp} \right\}.$$

Proof Since $\mathcal{I}_n^{\mathrm{Arc}}$ is homogeneous with respect to the graded degree $\operatorname{gdeg}(x_j^{(i)}) = i+1$, this follows immediately from Lemma 5.3 □

We can make this statement more explicit using Corollary 4.12.

Notation 5.5 For a differential variable x and positive integer h, we denote by $\mathbf{T}_h(x)$ the upper-triangular matrix of order $h+1$ with $x^{(h-i)}$ on the i-th diagonal above the main one. That is:

$$\mathbf{T}_h(x) = \begin{pmatrix} x^{(h)} & x^{(h-1)} & x^{(h-2)} & \dots & x \\ 0 & x^{(h)} & x^{(h-1)} & \dots & x' \\ \vdots & \vdots & x^{(h)} & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & x^{(h-1)} \\ 0 & 0 & 0 & \dots & x^{(h)} \end{pmatrix}.$$

We define $\mathbf{T}_{n,h}$ to be the $(h+1) \times n(h+1)$ matrix obtained by concatenation of $\mathbf{T}_h(x_1), \dots, \mathbf{T}_h(x_n)$.

Corollary 5.6 *The vector space* $(\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{\leqslant h}])^{\perp}$ *is equal to the vector space generated by the minors of* $\mathbf{T}_{n,h}$.

6 Poincaré-type series for $\mathcal{I}_n^{\mathbf{Arc}}$

The goal of this section is to use the obtained characterization of $(\mathcal{I}_n^{\mathrm{Arc}} \cap k[x^{\leqslant h}])^{\perp}$ in order to prove the following dimension result (cf. [5, Theorem 3.1]).

Theorem 6.1 *For every* $h \in \mathbb{Z}_{\geqslant 0}$ *we have* $\dim k[\mathbf{x}^{(\leqslant h)}]/(\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{(\leqslant h)}]) = (n+1)^{h+1}$.

Notation 6.2 For a differential variable x and positive integer h, we denote by $\mathbf{S}_h(x)$ the upper-triangular matrix of order $h+1$ with $x^{(i+1)}/i!$ on the i-th diagonal above the main one. That is:

$$\mathbf{S}_h(x) = \begin{pmatrix} x & x' & \frac{x''}{2} & \dots & \frac{x^{(h)}}{h!} \\ 0 & x & x' & \dots & \frac{x^{(h-1)}}{(h-1)!} \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \vdots & \ddots & x' \\ 0 & 0 & 0 & \dots & x \end{pmatrix}.$$

We define $\mathbf{S}_{n,h}$ to be the $(h+1) \times n(h+1)$ matrix obtained by concatenation of the matrices $\mathbf{S}_h(x_1), \dots, \mathbf{S}_h(x_n)$.

Lemma 6.3 *For every* n, h, *the dimensions of the spaces of the minors of* $\mathbf{T}_{n,h}$ *and* $\mathbf{S}_{n,h}$ *coincide.*

Proof The bijective map between these spaces is defined by $x^{(i)} \to x^{(h-i)}/(h-i)!$. □

Lemma 6.4 *For every* n, h, *the space of the minors of* $\mathbf{S}_{n,h}$ *coincides with the space of maximal minors of* $\mathbf{S}_{n+1,h}|_{x_{n+1}=1}$.

Proof We observe that $\mathbf{S}_h(x_{n+1})|_{x_{n+1}=1}$ is simply the $(h+1)$-dimensional identity matrix. Therefore, every $\ell \times \ell$-minor of $\mathbf{S}_{n,h}$ can be completed to a maximal minor of $\mathbf{S}_{n+1,h}|_{x_{n+1}=1}$ by adding the rows of $\mathbf{S}_{n,h}$ missing in this minor and then adding $h-\ell+1$ distinct columns, each containing just one nonzero element (equal to one) on one of the added rows. In the other direction, every maximal minor of $\mathbf{S}_{n+1,h}|_{x_{n+1}=1}$ can be reduced to a minor of $\mathbf{S}_{n,h}$ by discarding the columns in the $\mathbf{S}_h(x_{n+1})$-part and the rows in which these columns contain ones. □

Definition 6.5 A differential polynomial $P \in k[\mathbf{x}^{(\infty)}]$ of degree d is called *differentially homogeneous* if, for a differential indeterminate y, the equality $P(y \cdot \mathbf{x}) = y^d P(\mathbf{x})$ holds.

Lemma 6.6 *The dimension of the space of maximal minors of* $\mathbf{S}_{n+1,h}|_{x_{n+1}=1}$ *is equal to* $(n+1)^{h+1}$.

Proof By [21, Theorem 2.5.3 and Proposition 2.1.1] the space of maximal minors of $\mathbf{S}_{n+1,h}$ is exactly the space of differentially homogeneous polynomials of degree $h+1$, and its dimension is equal to $(n+1)^{h+1}$. Furthermore, by [33, Proposition 1.3], the restriction of the map $P \to P|_{x_{n+1}=1}$ on the space of differentially homogeneous polynomials of fixed degree is injective. Therefore, the dimension of the space of maximal minors of $\mathbf{S}_{n+1,h}|_{x_{n+1}=1}$ is also equal to $(n+1)^{h+1}$. □

Proof of Theorem 6.1 The concatenation of Lemmas 6.3, 6.4, and 6.6 implies that the dimension of the space of minors of $\mathbf{T}_{n,h}$ is equal to $(n+1)^{h+1}$. Then Corollary 5.6 implies that $\dim(\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{\leqslant h}])^{\perp} = (n+1)^{h+1}$. Moreover, [28, Proposition 5.5] implies that

$$\dim k[\mathbf{x}^{\leqslant h}]/(\mathcal{I}_n^{\mathrm{Arc}} \cap k[\mathbf{x}^{\leqslant h}]) = (n+1)^{h+1}.$$

□

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