

Lie symmetries of nonlinear systems with unknown inputs

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Abstract

It has been revealed that a dynamical system being unobservable or unidentifiable for a given set of observations is fundamentally related to the existence of Lie symmetries. Lie symmetries thus have the potentiality of providing useful tools to analyze and improve the observability and identifiability properties of dynamical systems from a fundamental perspective. This work proposes a computational framework for finding general Lie symmetries of nonlinear systems in the presence of unknown inputs. The occurring framework does not rely on mathematical Ansatz as the typical setting in the existing methods, and it is therefore capable of computing all the groups of Lie symmetries admitted by the system of interest. To alleviate the computation burden of the general framework, an alternative method is developed which is based on a priori assumptions on the functional forms of Lie symmetries to be calculated. The two proposed computation methods can be used as complementary tools to handle real engineering systems efficiently and robustly. Furthermore, utilization of the calculated Lie symmetries to detect and improve observability properties (with identifiability being the observability of unknown model parameters) is systematically discussed. Effective strategies of changing sensor placement, transforming system model and adding model assumptions are introduced to improve observability, and optimally render an unobservable system observable. The

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concept, computation and application of Lie symmetries are illustrated through several examples of linear, nonlinear and non-smooth dynamical systems.

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1. Introduction

System identification techniques [1] have gained increasing attention in recent years for their usefulness in estimating dynamic states, unknown parameters and unmeasured inputs of dynamical systems, and in minimizing the gaps
5 between real engineering systems and their mathematical models. Whether the unknown quantities of a dynamical system can be successfully estimated from a given set of input-output measurements is associated with the observability properties of the system [2]. If the system is theoretically observable, the set of solutions of the unknown quantities is unique (at least locally) to satisfy the sys-
10 tem model and input-output data, which could lead to a practically successful system identification campaign. Otherwise if the system is unobservable, there then exist infinitely many sets of incorrect solutions for the quantities to equally well fulfill the input-output mapping. It is impossible to properly estimate the unobservable states, parameters and/or inputs using any identification method.
15 These infinitely many sets of incorrect solutions can be implied by the Lie symmetries of the system [3]. Obtaining the knowledge of Lie symmetries allows engineers to gain deeper insights into which of and why the unknown quantities of a system are unobservable. More importantly, the results of Lie symmetries can provide systematic guidance for engineers to render an unobservable system
20 observable, through either introducing model assumptions, transforming system model and/or changing sensor placement as will be demonstrated in this paper.

Methods for testing whether a nonlinear system is theoretically observable for a given setup of sensors have been explored in a large amount of literature. Hermann and Krener [4] introduced the well-known Observability Rank Con-
25 dition (ORC) which is a method based on rank analysis of the observability

matrix computed from successive Lie derivatives of the system. Diop and Fliess [5] proposed a differential-algebra definition based on the existence of algebraic relations between states, parameters and the derivatives of input-outputs. Other approaches exist for various system nonlinearities and concepts of observability/identifiability [6, 7, 8, 9], where it should be noted that identifiability is an alternative term used to describe the observability of unknown parameters. Observability in specific engineering applications has also been studied in numerous works, such as the global identifiability of shear-type structures [10], the observability properties of non-smooth systems [2] and modally reduced order models [11] among others. To address the significant computation cost of the implementation of the ORC, an efficient algorithm was developed in [12] enabling the method to examine large linear systems with unknown parameters. Research has recently been oriented towards investigating systems with unknown inputs, as it is often the case in practice that measuring all inputs or excitations of an engineering system is too difficult, expensive or even impossible. Martinelli [13] provided an extension of the ORC, namely the EORC, to account for the observability of input-affine nonlinear systems with unknown inputs. A further extension of the EORC was given in [14] where the presence of direct feedthrough in outputs was taken into consideration. Shi and Chatzis [15] relaxed the EORC's requirement for systems being affine in inputs, and proposed a robust algorithm for maximizing the applicability of the method to real-world engineering systems that are often large and complex.

It was not until the recent decades that a system being unobservable was found to be fundamentally related to the Lie symmetries of the system, and Lie symmetry computation drew attention. For a given nonlinear system, existing methods are typically oriented towards finding specific types of Lie symmetries with known forms, while often inputs are required to be fully measured and known. Ürgüplü [16] and Anguelova [17] established a framework for robust computation of translation, scaling, affine and quadratic types of Lie symmetries. The computed Lie symmetries were further employed in [17] for determining minimal sets of outputs to ensure the identifiability of a system. Merkt [18]

restricted the view on rational nonlinearities, and extended the framework with the aim of finding higher-order polynomial symmetries that can be encountered in certain biological systems. Sedoglavic [19] initiated a basic idea to calculate general Lie symmetries by making use of the observability matrix with polynomial coefficients. As the existing works have the limitations on either full measurement of inputs and/or non-generality of calculated Lie symmetries, there is still a lack of focus on defining and computing general Lie symmetries in the case of unknown inputs.

This work introduces the concept of Lie symmetries of nonlinear systems with unknown inputs, and proposes a novel framework for computing general Lie symmetries based on the kernel of a Jacobian matrix with symbolic elements. The structure of the paper is organized as follows. Section 2 defines the considered category of systems and the admitted groups of Lie symmetries, and describes the key properties associated with the Lie symmetries. Section 3 presents the detailed derivation of the general framework, which stems from an augmented form of the considered system. Section 4 develops a complementary method which requires assumptions on the forms of Lie symmetries to be computed but is computationally efficient and robust. Section 5 discusses the applications of Lie symmetries to detect the observability properties of the considered system and improve the observability if it is an unobservable system. The section introduces effective strategies and the corresponding mathematical conditions which should be fulfilled to fix unobservability. Section 6 uses examples of linear, nonlinear and non-smooth mass-spring systems and a biochemical model to demonstrate the performance and capability of the proposed methods.

2. Concept of Lie symmetries

This work considers nonlinear systems with unknown inputs which can be generally written in the following continuous-time state-space representation:

$$\begin{aligned}\dot{\mathbf{x}}_t(t) &= \mathbf{F}(\mathbf{x}_t(t), \boldsymbol{\theta}, \mathbf{u}(t), \mathbf{w}(t)), \quad \dot{\boldsymbol{\theta}} = \mathbf{0} \\ \mathbf{y}(t) &= \mathbf{H}(\mathbf{x}_t(t), \boldsymbol{\theta}, \mathbf{u}(t), \mathbf{w}(t))\end{aligned}\tag{1}$$

where $\mathbf{x}_t = [x_{t,1}, \dots, x_{t,n_t}]^T \in \mathbb{R}^{n_t}$ is the vector of dynamic states, $\boldsymbol{\theta} =$
85 $[\theta_1, \dots, \theta_{n_\theta}]^T \in \mathbb{R}^{n_\theta}$ is the vector of unknown time-invariant parameters, $\mathbf{u} =$
 $[u_1, \dots, u_l]^T \in \mathbb{R}^l$ is the vector of measured and known inputs, $\mathbf{w} = [w_1, \dots, w_m]^T \in$
 \mathbb{R}^m is the vector of unmeasured and unknown inputs, and $\mathbf{y} = [y_1, \dots, y_p]^T \in \mathbb{R}^p$
is the output vector. \mathbf{F} and \mathbf{H} are vector-valued analytic nonlinear functions of
their arguments. It is possible to absorb both the dynamic states and unknown
90 parameters into a common state vector, $\mathbf{x} = [x_1, \dots, x_n]^T \in \mathbb{R}^n$, such that:

$$\mathbf{x}(t) = \begin{bmatrix} \mathbf{x}_t(t) \\ \boldsymbol{\theta} \end{bmatrix}, \quad \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), \mathbf{w}(t)) = \begin{bmatrix} \mathbf{F}(\mathbf{x}_t(t), \boldsymbol{\theta}, \mathbf{u}(t), \mathbf{w}(t)) \\ \mathbf{0}_{n_\theta \times 1} \end{bmatrix}, \quad n = n_t + n_\theta \quad (2)$$

and the system described in Eq. (1) can then be re-written with respect to \mathbf{x}
as:

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\ \mathbf{y} &= \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \end{aligned} \quad (3)$$

where the dependence on time t is notationally omitted.

The detailed exposition of Lie symmetry analysis of general differential and
95 non-differential equations can be found in [20, 21, 22]. Under the context, the
specified concept of Lie symmetries is introduced for the considered system in
Eq. (3) as follows, by regarding Eq. (3) as a system of first order differential and
non-differential equations with dependent variables \mathbf{x} and \mathbf{w} and independent
variable t . The action of Lie symmetries will be on the dependent variables only,
100 as \mathbf{u} and \mathbf{y} are measured and known variables and t is a known and absolute
variable. It is assumed that the system in Eq. (3) contains r ($r \in \mathbb{N}^*$) groups
of Lie symmetries, and the i^{th} ($1 \leq i \leq r$) group is defined as a one-parameter
group of transformations, denoted as $\{^i\phi_{\mathbf{x}}, ^i\phi_{\mathbf{w}}\}$:

$$\begin{aligned} ^i\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_i) &= \begin{bmatrix} ^i\phi_{x,1}(\mathbf{x}, \mathbf{w}, \epsilon_i) & \dots & ^i\phi_{x,n}(\mathbf{x}, \mathbf{w}, \epsilon_i) \end{bmatrix}^T \\ ^i\phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, \epsilon_i) &= \begin{bmatrix} ^i\phi_{w,1}(\mathbf{x}, \mathbf{w}, \epsilon_i) & \dots & ^i\phi_{w,m}(\mathbf{x}, \mathbf{w}, \epsilon_i) \end{bmatrix}^T \end{aligned} \quad (4)$$

where the group of transformations depends on a single constant parameter ϵ_i
105 which belongs to a continuous interval $S \subseteq \mathbb{R}$. The j^{th} ($1 \leq j \leq n$) component
of $^i\phi_{\mathbf{x}}$, i.e. $^i\phi_{x,j}$, corresponds to the Lie symmetry of the j^{th} component of \mathbf{x} ,

i.e. x_j , and is generally a function of all the unknown quantities of the system and the constant parameter. Similarly, the j^{th} ($1 \leq j \leq m$) component of ${}^i\phi_{\mathbf{w}}$, i.e. ${}^i\phi_{w,j}$, corresponds to the Lie symmetry of the j^{th} component of \mathbf{w} , i.e. w_j ,
110 and is generally a function of \mathbf{x} , \mathbf{w} and ϵ_i . Throughout this paper, $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\}$ is also referred to as a one-parameter Lie group of symmetries (can be a specific type) or transformations.

A fundamental property of the group of Lie symmetries admitted by the system in Eq. (3) is that, for any value of ϵ_i in S and any known input \mathbf{u} ,
115 $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\}$ satisfies the state and output equations of the system equally well as $\{\mathbf{x}, \mathbf{w}\}$, leaving the output \mathbf{y} invariant. Mathematically, that is:

$$\begin{aligned}\frac{d^i\phi_{\mathbf{x}}}{dt} &= \mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}}) \\ \mathbf{y} &= \mathbf{h}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})\end{aligned}\tag{5}$$

where the functions \mathbf{f} and \mathbf{h} between Eqs. (3) and (5) remain the same. This property implies that, in a system identification campaign, if one uses the measurements of \mathbf{u} and \mathbf{y} to estimate the unknown quantities \mathbf{x} and \mathbf{w} , the attempt
120 would never be successful. There exist infinitely many sets of solutions, which correspond to $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\}$ for infinite possible values of ϵ_i , to equally well fit the input-output measurements. Based on the definition of observability as in [4] the system is unobservable, unless $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\} \equiv \{\mathbf{x}, \mathbf{w}\}$. This property indicates why a system being unobservable is basically relevant to the existence of
125 Lie symmetries.

The other important properties of Lie symmetries are listed in the following:

- (i) Let D be a domain such that $\{\mathbf{x}, \mathbf{w}\} \in D \subseteq \mathbb{R}^{n+m}$. For each value of $\epsilon_i \in S$, the group of transformations $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\} : D \times S \rightarrow D$ is one-to-one.
- (ii) S equipped with the composition law μ is a group with an identity element
130 zero, and $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\}|_{\epsilon_i=0} = \{\mathbf{x}, \mathbf{w}\}$ with $0 \in S$.
- (iii) The following transformations hold for any $a, b \in S$:

$$\begin{aligned}{}^i\phi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, b), {}^i\phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, b), a) &= {}^i\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \mu(a, b)) \\ {}^i\phi_{\mathbf{w}}({}^i\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, b), {}^i\phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, b), a) &= {}^i\phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, \mu(a, b))\end{aligned}\tag{6}$$

where μ is an analytic function of a and b . In addition, a one-parameter Lie group of transformations of any other group of the system gives a two-parameter Lie group of symmetries, which still fulfills the state and output equations of the system. This property allows one to combine all the r one-parameter Lie groups of symmetries to form an r -parameter Lie group of symmetries, as will be discussed in detail in Section 5.

(iv) $\{^i\phi_x, ^i\phi_w\}$ is smooth, i.e. infinitely differentiable, with respect to $\{x, w\}$, and is a set of analytic functions of ϵ_i . The property of analyticity in ϵ_i enables one to express $\{^i\phi_x, ^i\phi_w\}$ as a convergent Taylor/power series expansion in powers of ϵ_i , as will be discussed in detail in the next section.

(v) Based on Lie's First Fundamental Theorem, $\{^i\phi_x, ^i\phi_w\}$ is equivalent to the solution of the initial value problem for the following system of first-order differential equations:

$$\begin{aligned}\frac{\partial^i\phi_x}{\partial\epsilon_i} &= {}^i\xi_x(^i\phi_x, ^i\phi_w), \quad {}^i\phi_x(x, w, 0) = x \\ \frac{\partial^i\phi_w}{\partial\epsilon_i} &= {}^i\xi_w(^i\phi_x, ^i\phi_w), \quad {}^i\phi_w(x, w, 0) = w\end{aligned}\tag{7}$$

When evaluated at $\epsilon_i = 0$,

$$\left. {}^i\xi_x(^i\phi_x, ^i\phi_w) \right|_{\epsilon_i=0} = {}^i\xi_x(x, w), \quad \left. {}^i\xi_w(^i\phi_x, ^i\phi_w) \right|_{\epsilon_i=0} = {}^i\xi_w(x, w)\tag{8}$$

where the components of $\{{}^i\xi_x(x, w), {}^i\xi_w(x, w)\}$ are defined as the infinitesimals of Eq. (4). First Fundamental Theorem of Lie ensures that the infinitesimals contain the necessary information for characterizing a one-parameter Lie group of symmetries.

3. Computation of general Lie symmetries

With the definition of Lie symmetries, this section provides a computational framework for finding arbitrary Lie symmetries existing in a system characterized by Eq. (3). More specifically, the aim is to compute the expressions of $^i\phi_x$ and $^i\phi_w$ for $i = 1, \dots, r$ without the need to assume their forms or types. The basic idea of the framework was initially presented in the conference paper

[3] by the authors, where detailed mathematical derivation was omitted; this section completes the derivation and presents the approaches to obtain analytic and power series solutions of Lie symmetries. To derive the framework, an augmented form of the system in Eq. (3) and the corresponding groups of Lie symmetries are first introduced.

3.1. Augmented system

Assuming the unknown input \mathbf{w} is differentiable with respect to time up to order k ($k \in \mathbb{N}$), \mathbf{w} and its time derivatives up to order k can be treated as the additional states of the system in Eq. (3) and thus the state vector can be augmented as:

$$\mathbf{x}^k = \begin{bmatrix} \mathbf{x}^T & \mathbf{w}^T & \dot{\mathbf{w}}^T & \dots & \mathbf{w}^{(k)T} \end{bmatrix}^T \quad (9)$$

where $\mathbf{w}^{(k)} = \frac{d^k \mathbf{w}}{dt^k}$. Consequently, the state equation can be re-written with respect to \mathbf{x}^k , resulting in an equivalent augmented system:

$$\dot{\mathbf{x}}^k = \begin{bmatrix} \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \\ \dot{\mathbf{w}} \\ \ddot{\mathbf{w}} \\ \vdots \\ \mathbf{w}^{(k+1)} \end{bmatrix} = \mathbf{F}^k(\mathbf{x}^k, \mathbf{u}, \mathbf{w}^{(k+1)}) \quad (10)$$

$$\mathbf{y} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})$$

Such a means of system augmentation was previously used to investigate the observability properties of systems with unknown inputs as in [13, 14, 15]. It should be noted that the augmented system in Eq. (10) still contains m unknown inputs which now coincide with the $(k+1)^{th}$ order time derivative of the original unknown inputs, i.e. the components of $\mathbf{w}^{(k+1)}$.

To proceed, the following formulation introduces the Lie derivatives of the vector of output functions, also referred to as the extended Lie derivatives as in [23]:

$$L_{\mathbf{f}}^j \mathbf{h} = \frac{\partial L_{\mathbf{f}}^{j-1} \mathbf{h}}{\partial \mathbf{x}} \mathbf{f} + \sum_{q=1}^j \frac{\partial L_{\mathbf{f}}^{j-1} \mathbf{h}}{\partial \mathbf{w}^{(q-1)}} \mathbf{w}^{(q)} + \sum_{q=1}^j \frac{\partial L_{\mathbf{f}}^{j-1} \mathbf{h}}{\partial \mathbf{u}^{(q-1)}} \mathbf{u}^{(q)} \quad (11)$$

where the known input \mathbf{u} is assumed to be time-differentiable up to order j . $\mathbf{L}_f^j \mathbf{h}$ is defined as the j^{th} order (extended) Lie derivative of \mathbf{h} along the vector field \mathbf{f} , which can be calculated recursively given the zero-order $\mathbf{L}_f^0 \mathbf{h} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})$. Throughout this paper, the operation $\frac{\partial \mathbf{V}}{\partial \mathbf{S}}$ for a column vector $\mathbf{V}(\mathbf{S}) = [v_1(\mathbf{S}), \dots, v_n(\mathbf{S})]^T$ with respect to a column or row vector $\mathbf{S} = [s_1, \dots, s_m]$ refers to the Jacobian:

$$\frac{\partial \mathbf{V}}{\partial \mathbf{S}} = \begin{bmatrix} \frac{\partial v_1}{\partial s_1} & \frac{\partial v_1}{\partial s_2} & \cdots & \frac{\partial v_1}{\partial s_m} \\ \frac{\partial v_2}{\partial s_1} & \frac{\partial v_2}{\partial s_2} & \cdots & \frac{\partial v_2}{\partial s_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial v_n}{\partial s_1} & \frac{\partial v_n}{\partial s_2} & \cdots & \frac{\partial v_n}{\partial s_m} \end{bmatrix} \quad (12)$$

Using the chain rule, it can be further deduced that the j^{th} order Lie derivative is equivalent to:

$$\mathbf{L}_f^j \mathbf{h} = \frac{d\mathbf{L}_f^{j-1} \mathbf{h}}{dt} = \frac{d^j \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{dt^j} \quad (13)$$

Analogous to the original system, the augmented system in Eq. (10) also contains r one-parameter Lie groups of symmetries, and the i^{th} group can be denoted as $\{^i \phi_{\mathbf{x}^k}, ^i \phi_{\mathbf{w}^{(k+1)}}\}$:

$$\begin{aligned} ^i \phi_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)}, \epsilon_i) &= \left[^i \phi_{\mathbf{x}}^T \quad ^i \phi_{\mathbf{w}}^T \quad ^i \phi_{\mathbf{w}}^T \quad \cdots \quad ^i \phi_{\mathbf{w}^{(k)}}^T \right]^T \\ ^i \phi_{\mathbf{w}^{(k+1)}}(\mathbf{x}^k, \mathbf{w}^{(k+1)}, \epsilon_i) &= \left[^i \phi_{\mathbf{w}^{(k+1)},1} \quad \cdots \quad ^i \phi_{\mathbf{w}^{(k+1)},m} \right]^T \end{aligned} \quad (14)$$

where $^i \phi_{\mathbf{w}^{(k)}}$ is given by:

$$^i \phi_{\mathbf{w}^{(k)}} = \frac{d^i \phi_{\mathbf{w}^{(k-1)}}}{dt} = \frac{d^k(^i \phi_{\mathbf{w}})}{dt^k} \quad (15)$$

Similarly due to the fundamental property of Lie symmetries, for any value of ϵ_i in S and any known input \mathbf{u} , $\{^i \phi_{\mathbf{x}^k}, ^i \phi_{\mathbf{w}^{(k+1)}}\}$ satisfies the state and output equations of the augmented system equally well as $\{\mathbf{x}^k, \mathbf{w}^{(k+1)}\}$, leaving the output \mathbf{y} invariant:

$$\begin{aligned} \frac{d^i \phi_{\mathbf{x}^k}}{dt} &= \mathbf{F}^k(^i \phi_{\mathbf{x}^k}, \mathbf{u}, ^i \phi_{\mathbf{w}^{(k+1)}}) \\ \mathbf{y} &= \mathbf{h}(^i \phi_{\mathbf{x}}, \mathbf{u}, ^i \phi_{\mathbf{w}}) \end{aligned} \quad (16)$$

where the functions \mathbf{F}^k and \mathbf{h} between Eqs. (10) and (16) remain the same. Further based on First Fundamental Theorem of Lie, the following system of first-order differential equations hold with initial values:

$$\begin{aligned}\frac{\partial^i \phi_{\mathbf{x}^k}}{\partial \epsilon_i} &= {}^i \xi_{\mathbf{x}^k}({}^i \phi_{\mathbf{x}^k}, {}^i \phi_{\mathbf{w}^{(k+1)}}), \quad {}^i \phi_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)}, 0) = \mathbf{x}^k \\ \frac{\partial^i \phi_{\mathbf{w}^{(k+1)}}}{\partial \epsilon_i} &= {}^i \xi_{\mathbf{w}^{(k+1)}}({}^i \phi_{\mathbf{x}^k}, {}^i \phi_{\mathbf{w}^{(k+1)}}), \quad {}^i \phi_{\mathbf{w}^{(k+1)}}(\mathbf{x}^k, \mathbf{w}^{(k+1)}, 0) = \mathbf{w}^{(k+1)}\end{aligned}\quad (17)$$

195 and the corresponding infinitesimals are given by $\{{}^i \xi_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)}), {}^i \xi_{\mathbf{w}^{(k+1)}}(\mathbf{x}^k, \mathbf{w}^{(k+1)})\}$.

3.2. Analytic computation

This subsection achieves analytic computation of the expression of ${}^i \phi_{\mathbf{x}^k}$ for $i = 1, \dots, r$, i.e. the one-parameter Lie groups of symmetries of the augmented system. Once ${}^i \phi_{\mathbf{x}^k}$ is determined, it is observed from Eq. (14) that ${}^i \phi_{\mathbf{x}}$ and
200 ${}^i \phi_{\mathbf{w}}$ can be directly obtained as the components of ${}^i \phi_{\mathbf{x}^k}$, i.e. the groups of Lie symmetries of the original system are obtained. The computation method stems from the treatment of Eq. (16). Differentiating both sides of the output equation in Eq. (16) with respect to time at order j gives:

$$\mathbf{y}^{(j)} = \frac{d^j \mathbf{h}({}^i \phi_{\mathbf{x}}, \mathbf{u}, {}^i \phi_{\mathbf{w}})}{dt^j} \quad (18)$$

Since Eq. (18) holds for all possible values of ϵ_i in S , further differentiating Eq.
205 (18) with respect to ϵ_i leads to:

$$\frac{d\mathbf{y}^{(j)}}{d\epsilon_i} = \mathbf{0} = \frac{d \frac{d^j \mathbf{h}({}^i \phi_{\mathbf{x}}, \mathbf{u}, {}^i \phi_{\mathbf{w}})}{dt^j}}{d\epsilon_i} \quad (19)$$

For any j where $0 \leq j \leq k$, the chain rule can be applied to Eq. (19) and by using Eq. (17), the equation amounts to the following:

$$\frac{\partial \frac{d^j \mathbf{h}({}^i \phi_{\mathbf{x}}, \mathbf{u}, {}^i \phi_{\mathbf{w}})}{dt^j}}{\partial {}^i \phi_{\mathbf{x}^k}} \frac{\partial {}^i \phi_{\mathbf{x}^k}}{\partial \epsilon_i} = \frac{\partial \frac{d^j \mathbf{h}({}^i \phi_{\mathbf{x}}, \mathbf{u}, {}^i \phi_{\mathbf{w}})}{dt^j}}{\partial {}^i \phi_{\mathbf{x}^k}} {}^i \xi_{\mathbf{x}^k}({}^i \phi_{\mathbf{x}^k}, {}^i \phi_{\mathbf{w}^{(k+1)}}) = \mathbf{0} \quad (20)$$

It is feasible to evaluate Eq. (20) at $\epsilon_i = 0$, resulting in:

$$\frac{\partial \frac{d^j \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{dt^j}}{\partial \mathbf{x}^k} {}^i \xi_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)}) = \mathbf{0} \quad (21)$$

Using Eq. (13), Eq. (21) can then be written in terms of the Lie derivatives as:

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$$\frac{\partial L_f^j \mathbf{h}}{\partial \mathbf{x}^k} {}^i \boldsymbol{\xi}_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)}) = \mathbf{0} \quad (22)$$

Taking all $j = 0, \dots, k$ into consideration, one attains from Eq. (22) the following equation with matrix multiplication:

$$\begin{bmatrix} \frac{\partial L_f^0 \mathbf{h}}{\partial \mathbf{x}^k} \\ \frac{\partial L_f^1 \mathbf{h}}{\partial \mathbf{x}^k} \\ \vdots \\ \frac{\partial L_f^k \mathbf{h}}{\partial \mathbf{x}^k} \end{bmatrix} {}^i \boldsymbol{\xi}_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)}) = \mathbf{0} \quad (23)$$

Eq. (23) allows one to express the vector of infinitesimals as the kernel of a Jacobian matrix:

$${}^i \boldsymbol{\xi}_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)}) = \ker(d\boldsymbol{\Omega}^k) \quad (24)$$

215 where $d\boldsymbol{\Omega}^k$ is the Jacobian of the Lie derivatives with respect to the augmented state vector:

$$d\boldsymbol{\Omega}^k = \begin{bmatrix} \frac{\partial L_f^0 \mathbf{h}}{\partial \mathbf{x}^k} \\ \frac{\partial L_f^1 \mathbf{h}}{\partial \mathbf{x}^k} \\ \vdots \\ \frac{\partial L_f^k \mathbf{h}}{\partial \mathbf{x}^k} \end{bmatrix} \quad (25)$$

Eq. (24) suggests that ${}^i \boldsymbol{\xi}_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)})$ for $i = 1, \dots, r$ can be calculated from the kernel of the matrix $d\boldsymbol{\Omega}^k$: at a chosen order k , $\ker(d\boldsymbol{\Omega}^k)$ results in the basis of the kernel given by \mathbf{B} :

$$\mathbf{B} = \left\{ {}^1 \boldsymbol{\xi}_{\mathbf{x}^k} \quad {}^2 \boldsymbol{\xi}_{\mathbf{x}^k} \quad \dots \quad {}^r \boldsymbol{\xi}_{\mathbf{x}^k} \right\} \quad (26)$$

220 where it assumes that \mathbf{B} does not exclude the zero vector. To compute $d\boldsymbol{\Omega}^k$, the following algorithm based on the recursion of Lie derivatives in Eq. (11) can be used:

Algorithm I

Initialization: Set $j = 0$, $\mathbf{x}^j = [\mathbf{x}^T, \mathbf{w}^T]^T$, $L_f^j \mathbf{h} = \mathbf{h}(\mathbf{x}, \mathbf{u}, \mathbf{w})$, $d\boldsymbol{\Omega}^j = \frac{\partial L_f^j \mathbf{h}}{\partial \mathbf{x}^j}$.

225 1. Set $j = j + 1$;

2. Set $\mathbf{x}^j = [\mathbf{x}^{j-1T}, \mathbf{w}^{(j)T}]^T$;
3. Compute $\mathbf{L}_f^j \mathbf{h} = \frac{\partial \mathbf{L}_f^{j-1} \mathbf{h}}{\partial \mathbf{x}} \mathbf{f} + \sum_{q=1}^j \frac{\partial \mathbf{L}_f^{j-1} \mathbf{h}}{\partial \mathbf{w}^{(q-1)}} \mathbf{w}^{(q)} + \sum_{q=1}^j \frac{\partial \mathbf{L}_f^{j-1} \mathbf{h}}{\partial \mathbf{u}^{(q-1)}} \mathbf{u}^{(q)}$;
4. Compute and arrange $d\mathbf{\Omega}^j = \begin{bmatrix} d\mathbf{\Omega}^{j-1} & \mathbf{0} \\ \frac{\partial \mathbf{L}_f^j \mathbf{h}}{\partial \mathbf{x}^j} \end{bmatrix}$;
5. End if $j = k$. Otherwise go to step 1;

230 It should be noted that $d\mathbf{\Omega}^k$ attains the same structure as the observability matrix derived in [14, 15].

Different choices of order k may lead to different results of the basis of the kernel. The expression of ${}^i\xi_{\mathbf{x}^k}$ should be determined at $k = k_0$ for which the dimension of \mathbf{B} and the expressions of ${}^i\xi_{\mathbf{x}}$ and ${}^i\xi_{\mathbf{w}}$ remain invariant for
 235 any larger $k > k_0$, where ${}^i\xi_{\mathbf{x}}$ and ${}^i\xi_{\mathbf{w}}$ are the components of the vector ${}^i\xi_{\mathbf{x}^k}$ as can be deduced from Eqs. (14) and (17). This is associated with the k -row observability properties detected from the observability matrix $d\mathbf{\Omega}^k$ being invariant as defined in [14, 15], and a practical guideline was provided in [14, 15] to determine the point of convergence k_0 . It should be noted that although [15]
 240 focused on rational nonlinearities to facilitate computation, the proposed theory there applies to general analytic nonlinear systems with unknown inputs.

With the calculated expressions of infinitesimals, the one-parameter Lie groups of symmetries of the augmented system can readily be obtained by solving the differential equation in Eq. (17) as an initial value problem. It should be
 245 further noted that $d\mathbf{\Omega}^k$ consisting of the Lie derivatives up to order k depends on \mathbf{x}^k but not on $\mathbf{w}^{(k+1)}$ as can be deduced from Eq. (11), and thus the kernel of the matrix is a function of \mathbf{x}^k but not of $\mathbf{w}^{(k+1)}$. As a result, it is feasible to write ${}^i\xi_{\mathbf{x}^k}(\mathbf{x}^k, \mathbf{w}^{(k+1)}) = {}^i\xi_{\mathbf{x}^k}(\mathbf{x}^k)$ and the differential equation becomes:

$$\frac{\partial {}^i\phi_{\mathbf{x}^k}}{\partial \epsilon_i} = {}^i\xi_{\mathbf{x}^k}({}^i\phi_{\mathbf{x}^k}), \quad {}^i\phi_{\mathbf{x}^k}|_{\epsilon_i=0} = \mathbf{x}^k \quad (27)$$

Analytically solving Eq. (27) allows to obtain the exact solution of ${}^i\phi_{\mathbf{x}^k}$ for
 250 $i = 1, \dots, r$, and as aforementioned, ${}^i\phi_{\mathbf{x}}$ and ${}^i\phi_{\mathbf{w}}$ are given as the components of ${}^i\phi_{\mathbf{x}^k}$. The previous computations of Jacobian matrix, kernel and the resolution of differential equations should be implemented symbolically, i.e. symbols are used as generic representations of the involved variables, which can rely on a

variety of software with symbolic computing tools such as MATLAB [24], Maple
 255 [25] and Mathematica [26].

3.3. Power series solution

Complicated functions of ${}^i\xi_{\mathbf{x}^k}$ often create mathematical difficulties in solving Eq. (27) analytically. Under such circumstances, attention can be placed on finding the power series solution of ${}^i\phi_{\mathbf{x}^k}$, instead of its exact solution, for
 260 further application and numerical studies of Lie symmetries. Based on property (iv) in Section 2, the group of Lie symmetries of the augmented system can be expanded as a convergent power/Taylor series in ϵ_i , i.e.:

$${}^i\phi_{\mathbf{x}^k} = a_0 + a_1\epsilon_i + a_2\frac{\epsilon_i^2}{2!} + \dots = \sum_{q=0}^{\infty} a_q \frac{\epsilon_i^q}{q!} \quad (28)$$

where a_0 is the initial value of ${}^i\phi_{\mathbf{x}^k}$ at $\epsilon_i = 0$:

$$a_0 = {}^i\phi_{\mathbf{x}^k} \Big|_{\epsilon_i=0} = \mathbf{x}^k \quad (29)$$

and the coefficient of power series a_q is given by:

$$a_q = \left. \frac{d^q({}^i\phi_{\mathbf{x}^k})}{d\epsilon_i^q} \right|_{\epsilon_i=0} \quad (30)$$

265 The chain rule and Eq. (27) lead Eq. (30) to:

$$a_q = \left. \frac{\partial \frac{d^{q-1}({}^i\phi_{\mathbf{x}^k})}{d\epsilon_i^{q-1}}}{\partial {}^i\phi_{\mathbf{x}^k}} \frac{\partial {}^i\phi_{\mathbf{x}^k}}{\partial \epsilon_i} \right|_{\epsilon_i=0} = \left. \frac{\partial \frac{d^{q-1}({}^i\phi_{\mathbf{x}^k})}{d\epsilon_i^{q-1}}}{\partial {}^i\phi_{\mathbf{x}^k}} {}^i\xi_{\mathbf{x}^k}({}^i\phi_{\mathbf{x}^k}) \right|_{\epsilon_i=0} \quad (31)$$

and consequently:

$$a_q = \frac{\partial a_{q-1}}{\partial \mathbf{x}^k} {}^i\xi_{\mathbf{x}^k}(\mathbf{x}^k) \quad (32)$$

With the calculated infinitesimals, Eq. (32) enables one to update the coefficients of power series in a recursive manner up to a desired order. The following algorithm summarizes the procedure to compute the power series solution of

270 ${}^i\phi_{\mathbf{x}^k}$:

Algorithm II

Initialization: Set $q = 0$, $a_q = \mathbf{x}^k$, ${}^i\phi_{\mathbf{x}^k} = a_q$, desired order = z .

1. Set $q = q + 1$;

2. Compute $a_q = \frac{\partial a_{q-1}}{\partial \mathbf{x}^k} {}^i \boldsymbol{\xi}_{\mathbf{x}^k}(\mathbf{x}^k)$, and ${}^i \phi_{\mathbf{x}^k} = {}^i \phi_{\mathbf{x}^k} + a_q \frac{\epsilon_i^q}{q!}$;
- 275 3. End if $q = z$, and ${}^i \phi_{\mathbf{x}^k} = {}^i \phi_{\mathbf{x}^k} + O(\epsilon_i^{z+1})$. Otherwise go to step 1;

Furthermore if ${}^i \boldsymbol{\xi}_{\mathbf{x}^k}$ is a vector of rational functions, i.e. any fractional function whose numerator and denominator are both polynomial functions, Newton's iteration [27, 28] can be used to solve Eq. (27) efficiently for the power series solution of ${}^i \phi_{\mathbf{x}^k}$. In an algorithm of Newton's method with quadratic convergence property, only a small number of iterations are needed to result in a power series with a large number of terms. Once the power series expansion of ${}^i \phi_{\mathbf{x}^k}$ is determined, similarly, the power series expansions of ${}^i \phi_{\mathbf{x}}$ and ${}^i \phi_{\mathbf{w}}$ can be obtained as its elements.

285 4. Robust computation of specific Lie symmetries

The general method presented in Section 3 can suffer from the issues of high computational cost due to its symbolic nature. As discussed in [15, 29], symbolic computation of the Jacobian matrix $d\boldsymbol{\Omega}^k$ as well as the evaluation of kernel can require a significant amount of physical memory in computer, especially when implemented for large and complex systems. Often a large order k is expected so as to reach the convergence of the expressions of ${}^i \boldsymbol{\xi}_{\mathbf{x}}$ and ${}^i \boldsymbol{\xi}_{\mathbf{w}}$, which further increases the burden of calculation. This section presents a second method for more efficiently and robustly computing specific types of Lie symmetries. The key feature of the method is that a priori assumptions must be made for the types or functional forms of Lie symmetries to be computed, e.g. assuming $\{{}^i \phi_{\mathbf{x}}, {}^i \phi_{\mathbf{w}}\}$ as a commonly-encountered type of group of Lie symmetries and then determining the unknown coefficients related to it. The a priori assumption is mathematically referred to as *Ansatz*, which is a widely-adopted means for finding Lie symmetries of differential equations. Nevertheless, an obvious limitation of the method is that it is not capable of finding more general Lie symmetries with unpredictable forms, and thus often not able to calculate all the existing groups of Lie symmetries in a system. With the advantage in effi-

ciency of computation but also the limitation in non-generality, the method is developed herein to be used as a complementary tool for the general method.

305 The basic idea of this second method was initially presented in the conference paper [30] by the authors, where detailed mathematical derivation was omitted and the discussion was limited to translation and scaling symmetries; this section completes the derivation and presents the full framework.

4.1. Computational framework

310 Instead of relying on the augmented system, computational framework of the second method is derived directly from Eq. (5). Since the state equation in Eq. (5) holds for all possible values of ϵ_i in S , differentiating both sides of the equation with respect to ϵ_i yields:

$$\frac{d \frac{d^i \phi_{\mathbf{x}}}{dt}}{d\epsilon_i} = \frac{d\mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{d\epsilon_i} \quad (33)$$

Considering $\frac{d}{d\epsilon_i}(\frac{d^i \phi_{\mathbf{x}}}{dt}) = \frac{d}{dt}(\frac{d^i \phi_{\mathbf{x}}}{d\epsilon_i})$, Eq. (33) can be expressed as:

$$\frac{d^i \xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}})}{dt} = \frac{d\mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{d\epsilon_i} \quad (34)$$

315 Applying the chain rule to Eq. (34) leads to:

$$\begin{aligned} & \frac{\partial^i \xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}})}{\partial^i \phi_{\mathbf{x}}} \frac{d^i \phi_{\mathbf{x}}}{dt} + \frac{\partial^i \xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}})}{\partial^i \phi_{\mathbf{w}}} \frac{d^i \phi_{\mathbf{w}}}{dt} \\ &= \frac{\partial \mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{\partial^i \phi_{\mathbf{x}}} \frac{\partial^i \phi_{\mathbf{x}}}{\partial \epsilon_i} + \frac{\partial \mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{\partial^i \phi_{\mathbf{w}}} \frac{\partial^i \phi_{\mathbf{w}}}{\partial \epsilon_i} \end{aligned} \quad (35)$$

Using Eqs. (5) and (7), Eq. (35) becomes:

$$\begin{aligned} & \frac{\partial^i \xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}})}{\partial^i \phi_{\mathbf{x}}} \mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}}) + \frac{\partial^i \xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}})}{\partial^i \phi_{\mathbf{w}}} \frac{d^i \phi_{\mathbf{w}}}{dt} \\ &= \frac{\partial \mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{\partial^i \phi_{\mathbf{x}}} {}^i\xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}) + \frac{\partial \mathbf{f}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{\partial^i \phi_{\mathbf{w}}} {}^i\xi_{\mathbf{w}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}) \end{aligned} \quad (36)$$

In a similar fashion, both sides of the output equation in Eq. (5) can be differentiated with respect to ϵ_i :

$$\frac{d\mathbf{y}}{d\epsilon_i} = \mathbf{0} = \frac{d\mathbf{h}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{d\epsilon_i} \quad (37)$$

Applying the chain rule to Eq. (37) gives:

$$\mathbf{0} = \frac{\partial h({}^i\phi_x, \mathbf{u}, {}^i\phi_w)}{\partial {}^i\phi_x} {}^i\xi_x({}^i\phi_x, {}^i\phi_w) + \frac{\partial h({}^i\phi_x, \mathbf{u}, {}^i\phi_w)}{\partial {}^i\phi_w} {}^i\xi_w({}^i\phi_x, {}^i\phi_w) \quad (38)$$

It is then feasible to evaluate the obtained Eqs. (36) and (38) at $\epsilon_i = 0$, resulting in the following system of differential equations:

$$\begin{cases} \frac{\partial {}^i\xi_x(\mathbf{x}, \mathbf{w})}{\partial \mathbf{x}} \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w}) + \frac{\partial {}^i\xi_x(\mathbf{x}, \mathbf{w})}{\partial \mathbf{w}} \frac{d\mathbf{w}}{dt} = \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{x}} {}^i\xi_x(\mathbf{x}, \mathbf{w}) + \frac{\partial \mathbf{f}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{w}} {}^i\xi_w(\mathbf{x}, \mathbf{w}) \\ \frac{\partial h(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{x}} {}^i\xi_x(\mathbf{x}, \mathbf{w}) + \frac{\partial h(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{w}} {}^i\xi_w(\mathbf{x}, \mathbf{w}) = \mathbf{0} \end{cases} \quad (39)$$

Compared to the general framework in Eq. (23), derivation of the above framework does not rely on higher order derivatives of the output equation and not involve higher order derivatives of \mathbf{w} as the augmented states, which significantly simplifies the expression. As a sacrifice however, solving Eq. (39) for the analytical solutions of ${}^i\xi_x$ and ${}^i\xi_w$ ($i = 1, \dots, r$) is in general challenging or mathematically intractable.

4.2. Assumption and computation

The proposed strategy herein is to assume $\{{}^i\phi_x, {}^i\phi_w\}$ ($i = 1, \dots, r$) as a certain type of Lie symmetries, such as a one-parameter Lie group of translation, scaling, inversion, affine, quadratic, polynomial symmetries, etc., which are commonly encountered in practice and admitted by a wide range of civil, mechanical and biochemical systems. These Lie symmetries are in explicit forms but with unknown coefficients to be determined based on Eq. (39). To be taken as examples, Table 1 lists the expressions of translation, scaling and inversion symmetries and their corresponding expressions of infinitesimals, where $\alpha_{i,1}, \dots, \alpha_{i,n+m}$ are the involved unknown constant coefficients.

As revealed by Table 2, once the type of $\{{}^i\phi_x, {}^i\phi_w\}$ is assumed, a condensed form of Eq. (39) can be obtained by substituting the corresponding ${}^i\xi_x$ and ${}^i\xi_w$ into the equation. Then the remaining task is to utilize Eq. (39) to determine the values of the unknown coefficients and thus the ultimate expression of $\{{}^i\phi_x, {}^i\phi_w\}$. In the cases of, but not limited to, translation, scaling and inversion symmetries, let $\boldsymbol{\alpha}_i = [\alpha_{i,1}, \dots, \alpha_{i,m+n}]^T$ denote the vector of coefficients.

Assumed type	$\{^i\phi_x, ^i\phi_w\}$	Corresponding $\{^i\xi_x, ^i\xi_w\}$
Translation	$^i\phi_{x,1} = x_1 + \alpha_{i,1}\epsilon_i$	$^i\xi_{x,1} = \left. \frac{\partial^i\phi_{x,1}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,1}$
	\vdots	\vdots
	$^i\phi_{x,n} = x_n + \alpha_{i,n}\epsilon_i$	$^i\xi_{x,n} = \left. \frac{\partial^i\phi_{x,n}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n}$
	$^i\phi_{w,1} = w_1 + \alpha_{i,n+1}\epsilon_i$	$^i\xi_{w,1} = \left. \frac{\partial^i\phi_{w,1}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n+1}$
	\vdots	\vdots
	$^i\phi_{w,m} = w_m + \alpha_{i,n+m}\epsilon_i$	$^i\xi_{w,m} = \left. \frac{\partial^i\phi_{w,m}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n+m}$
Scaling	$^i\phi_{x,1} = e^{\alpha_{i,1}\epsilon_i} x_1$	$^i\xi_{x,1} = \left. \frac{\partial^i\phi_{x,1}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,1}x_1$
	\vdots	\vdots
	$^i\phi_{x,n} = e^{\alpha_{i,n}\epsilon_i} x_n$	$^i\xi_{x,n} = \left. \frac{\partial^i\phi_{x,n}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n}x_n$
	$^i\phi_{w,1} = e^{\alpha_{i,n+1}\epsilon_i} w_1$	$^i\xi_{w,1} = \left. \frac{\partial^i\phi_{w,1}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n+1}w_1$
	\vdots	\vdots
	$^i\phi_{w,m} = e^{\alpha_{i,n+m}\epsilon_i} w_m$	$^i\xi_{w,m} = \left. \frac{\partial^i\phi_{w,m}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n+m}w_m$
Inversion	$^i\phi_{x,1} = \frac{x_1}{1 - \alpha_{i,1}\epsilon_i x_1}$	$^i\xi_{x,1} = \left. \frac{\partial^i\phi_{x,1}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,1}x_1^2$
	\vdots	\vdots
	$^i\phi_{x,n} = \frac{x_n}{1 - \alpha_{i,n}\epsilon_i x_n}$	$^i\xi_{x,n} = \left. \frac{\partial^i\phi_{x,n}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n}x_n^2$
	$^i\phi_{w,1} = \frac{w_1}{1 - \alpha_{i,n+1}\epsilon_i w_1}$	$^i\xi_{w,1} = \left. \frac{\partial^i\phi_{w,1}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n+1}w_1^2$
	\vdots	\vdots
	$^i\phi_{w,m} = \frac{w_m}{1 - \alpha_{i,n+m}\epsilon_i w_m}$	$^i\xi_{w,m} = \left. \frac{\partial^i\phi_{w,m}}{\partial\epsilon_i} \right _{\epsilon_i=0} = \alpha_{i,n+m}w_m^2$

Table 1: One-parameter Lie groups of translation, scaling and inversion symmetries and their infinitesimals

Assumed type	$\{^i\phi_{\mathbf{x}}, ^i\phi_{\mathbf{w}}\}$	Corresponding Eq. (39)
Translation	$^i\phi_{x,1} = x_1 + \alpha_{i,1}\epsilon_i$	$\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1} \\ \vdots \\ \alpha_{i,n} \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \dots & \frac{\partial f_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial w_1} & \dots & \frac{\partial f_n}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1} \\ \vdots \\ \alpha_{i,n+m} \end{bmatrix} = \mathbf{0}$
	$^i\phi_{x,n} = x_n + \alpha_{i,n}\epsilon_i$	
	$^i\phi_{w,1} = w_1 + \alpha_{i,n+1}\epsilon_i$	$\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \dots & \frac{\partial h_p}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1} \\ \vdots \\ \alpha_{i,n} \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial w_1} & \dots & \frac{\partial h_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial w_1} & \dots & \frac{\partial h_p}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1} \\ \vdots \\ \alpha_{i,n+m} \end{bmatrix} = \mathbf{0}$
	$^i\phi_{w,m} = w_m + \alpha_{i,n+m}\epsilon_i$	
Scaling	$^i\phi_{x,1} = e^{\alpha_{i,1}\epsilon_i} x_1$	$\begin{bmatrix} \alpha_{i,1} & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & \alpha_{i,n} \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1}x_1 \\ \vdots \\ \alpha_{i,n}x_n \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \dots & \frac{\partial f_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial w_1} & \dots & \frac{\partial f_n}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1}w_1 \\ \vdots \\ \alpha_{i,n+m}w_m \end{bmatrix} = \mathbf{0}$
	$^i\phi_{x,n} = e^{\alpha_{i,n}\epsilon_i} x_n$	
	$^i\phi_{w,1} = e^{\alpha_{i,n+1}\epsilon_i} w_1$	$\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \dots & \frac{\partial h_p}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1}x_1 \\ \vdots \\ \alpha_{i,n}x_n \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial w_1} & \dots & \frac{\partial h_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial w_1} & \dots & \frac{\partial h_p}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1}w_1 \\ \vdots \\ \alpha_{i,n+m}w_m \end{bmatrix} = \mathbf{0}$
	$^i\phi_{w,m} = e^{\alpha_{i,n+m}\epsilon_i} w_m$	
Inversion	$^i\phi_{x,1} = \frac{x_1}{1 - \alpha_{i,1}\epsilon_i x_1}$	$\begin{bmatrix} 2\alpha_{i,1}x_1 & \dots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \dots & 2\alpha_{i,n}x_n \end{bmatrix} \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1}x_1^2 \\ \vdots \\ \alpha_{i,n}x_n^2 \end{bmatrix} + \begin{bmatrix} \frac{\partial f_1}{\partial w_1} & \dots & \frac{\partial f_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial w_1} & \dots & \frac{\partial f_n}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1}w_1^2 \\ \vdots \\ \alpha_{i,n+m}w_m^2 \end{bmatrix} = \mathbf{0}$
	$^i\phi_{x,n} = \frac{x_n}{1 - \alpha_{i,n}\epsilon_i x_n}$	
	$^i\phi_{w,1} = \frac{w_1}{1 - \alpha_{i,n+1}\epsilon_i w_1}$	$\begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \dots & \frac{\partial h_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial x_1} & \dots & \frac{\partial h_p}{\partial x_n} \end{bmatrix} \begin{bmatrix} \alpha_{i,1}x_1^2 \\ \vdots \\ \alpha_{i,n}x_n^2 \end{bmatrix} + \begin{bmatrix} \frac{\partial h_1}{\partial w_1} & \dots & \frac{\partial h_1}{\partial w_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_p}{\partial w_1} & \dots & \frac{\partial h_p}{\partial w_m} \end{bmatrix} \begin{bmatrix} \alpha_{i,n+1}w_1^2 \\ \vdots \\ \alpha_{i,n+m}w_m^2 \end{bmatrix} = \mathbf{0}$
	$^i\phi_{w,m} = \frac{w_m}{1 - \alpha_{i,n+m}\epsilon_i w_m}$	

Table 2: One-parameter Lie groups of translation, scaling and inversion symmetries and the corresponding Eq. (39)

Eq. (39) can be converted to a linear-in-coefficient system:

$$\mathbf{M}\boldsymbol{\alpha}_i = \mathbf{0} \quad (40)$$

where \mathbf{M} is a matrix of functions of \mathbf{x} , \mathbf{w} and \mathbf{u} , which is symbolically expressed by:

$$\mathbf{M} = \frac{\partial \mathbf{P}}{\partial \boldsymbol{\alpha}_i}, \quad \mathbf{P} = \begin{bmatrix} \frac{\partial^i \boldsymbol{\xi}_x}{\partial \mathbf{x}} \mathbf{f} - \frac{\partial \mathbf{f}}{\partial \mathbf{x}}^i \boldsymbol{\xi}_x - \frac{\partial \mathbf{f}}{\partial \mathbf{w}}^i \boldsymbol{\xi}_w \\ \frac{\partial \mathbf{h}}{\partial \mathbf{x}}^i \boldsymbol{\xi}_x + \frac{\partial \mathbf{h}}{\partial \mathbf{w}}^i \boldsymbol{\xi}_w \end{bmatrix} \quad (41)$$

Eq. (40) suggests that $\boldsymbol{\alpha}_i$ can be calculated from the basis of the kernel of \mathbf{M} , i.e.:

$$\boldsymbol{\alpha}_i = \ker(\mathbf{M}) \quad (42)$$

Based on the fact of Eq. (42), a robust approach was presented in [16] (see details in Section 4.2.3) to efficiently determine $\boldsymbol{\alpha}_i$ using multiple random realizations of the variables in \mathbf{M} and stacking the sampled matrices of \mathbf{M} . The numerical algorithm there releases the framework from symbolic computations of high complexity. Consequently, if the a priori assumption is correct, i.e. the system in Eq. (3) indeed contains the assumed type of Lie symmetries (for instance, translation, scaling or inversion), the calculated coefficients $\boldsymbol{\alpha}_i$ would be a vector of non-zero constant values; otherwise, $\boldsymbol{\alpha}_i = \mathbf{0}$ and a different type of Lie symmetries may be searched.

5. Usage of Lie symmetries

Given all the r groups of Lie symmetries computed from the two complementary methods in Sections 3 and 4, this section investigates the use of Lie symmetries to detect and improve the observability and identifiability properties of nonlinear systems. The relationship between the combined multi-parameter Lie group of symmetries and the observability of individual unknown quantities (dynamic states/parameters/inputs) of a system is discussed in detail. More importantly, it demonstrates that the computed Lie symmetries can be used to improve observability, and at best render an unobservable system become

observable, in systematic manners which are namely to ‘destroy’ the Lie symmetries via changing *sensor placement*, *model transformation* and introducing *model assumptions*.

370 5.1. From Lie symmetries to observability

The obtained r one-parameter Lie groups of symmetries can be treated separately, or alternatively as indicated in property (iii) in Section 2, they can be combined to be an r -parameter Lie group of symmetries. This is achieved by successively applying the Lie groups to transform the states and inputs, which
375 could be mathematically represented by:

$$\begin{aligned}\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \boldsymbol{\epsilon}) &= {}^r\phi_{\mathbf{x}}(\dots {}^2\phi_{\mathbf{x}}({}^1\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_1), {}^1\phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, \epsilon_1), \epsilon_2) \dots, \epsilon_r) \\ \phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, \boldsymbol{\epsilon}) &= {}^r\phi_{\mathbf{w}}(\dots {}^2\phi_{\mathbf{w}}({}^1\phi_{\mathbf{x}}(\mathbf{x}, \mathbf{w}, \epsilon_1), {}^1\phi_{\mathbf{w}}(\mathbf{x}, \mathbf{w}, \epsilon_1), \epsilon_2) \dots, \epsilon_r)\end{aligned}\quad (43)$$

where $\{\phi_{\mathbf{x}}, \phi_{\mathbf{w}}\}$ denotes the r -parameter Lie group of symmetries with $\boldsymbol{\epsilon} = [\epsilon_1, \dots, \epsilon_r]$. $\{\phi_{\mathbf{x}}, \phi_{\mathbf{w}}\}$ inherits the fundamental property of Lie symmetries: for any value of $\boldsymbol{\epsilon}$ in S^r and any known input \mathbf{u} , $\{\phi_{\mathbf{x}}, \phi_{\mathbf{w}}\}$ satisfies the state and output equations of the system in Eq. (3) equally well as $\{\mathbf{x}, \mathbf{w}\}$, leaving the
380 output \mathbf{y} invariant, i.e.:

$$\begin{aligned}\frac{d\phi_{\mathbf{x}}}{dt} &= \mathbf{f}(\phi_{\mathbf{x}}, \mathbf{u}, \phi_{\mathbf{w}}) \\ \mathbf{y} &= \mathbf{h}(\phi_{\mathbf{x}}, \mathbf{u}, \phi_{\mathbf{w}})\end{aligned}\quad (44)$$

where the functions \mathbf{f} and \mathbf{h} between Eqs. (3) and (44) remain the same. Eq. (44) builds the ultimate bridge between Lie symmetries and observability properties of the system in Eq. (3). If the j^{th} ($1 \leq j \leq n$) component of $\phi_{\mathbf{x}}$, i.e. $\phi_{x,j}$, is not identically equal to x_j , i.e. $\phi_{x,j} \not\equiv x_j$, then $\phi_{x,j}$ must
385 be a function of $\boldsymbol{\epsilon}$ and thus x_j is unobservable. There exist infinitely many solutions of x_j that cannot be distinguished from each other by observing the input-outputs. Similarly if $\phi_{w,j} \not\equiv w_j$ ($1 \leq j \leq m$), then w_j is unobservable. The remaining states and unknown inputs are (at least locally weakly as defined in [4]) observable. The system is observable, i.e. all of its states and unknown
390 inputs are observable, if and only if $\{\phi_{\mathbf{x}}, \phi_{\mathbf{w}}\} \equiv \{\mathbf{x}, \mathbf{w}\}$.

As above, Lie symmetries provide an alternative path to analyze the observability properties of nonlinear systems, compared to the existing method in [14, 15] based on rank testing of the observability matrix. A simple calculation as follows checks that the two approaches indeed agree with each other. [14, 15] proposed a criterion that the system in Eq. (3) can be concluded to be observable if and only if the observability matrix is of full-rank. According to Section 3.2, the matrix $d\Omega^k$ in Eq. (25) attains exactly the same structure as the observability matrix; if $d\Omega^k$ is of full-rank, its kernel is zero and thus ${}^i\xi_{\mathbf{x}^k} = \mathbf{0}$ for $i = 1$. Subsequently solving Eq. (27) gives ${}^i\phi_{\mathbf{x}^k} \equiv \mathbf{x}^k$ for $i = 1$, which reveals that the system is observable.

5.2. Improving observability

5.2.1. Sensor placement

To improve the observability properties for an unobservable system, an effective means is to change sensor placement and thus the output measurements of the system. Suppose that the original sensor configuration remains unchanged, and additional sensors are placed to obtain the following output measurements:

$$\mathbf{y}_{new} = \mathbf{h}_{new}(\mathbf{x}, \mathbf{u}, \mathbf{w}) \quad (45)$$

The vector of output functions \mathbf{h}_{new} adequately accounts for the number, types and locations of the newly introduced sensors. To improve observability, a number of one-parameter Lie groups of symmetries must not satisfy the new output equation, that is said to destroy these groups of Lie symmetries by the new outputs. For the i^{th} group where $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\} \not\equiv \{\mathbf{x}, \mathbf{w}\}$, it is equivalent to:

$$\mathbf{y}_{new} \neq \mathbf{h}_{new}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}}) \quad (46)$$

Differentiating Eq. (46) with respect to ϵ_i , and applying the chain rule lead to:

$$\frac{d\mathbf{y}_{new}}{d\epsilon_i} = \mathbf{0} \neq \frac{\partial \mathbf{h}_{new}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{\partial {}^i\phi_{\mathbf{x}}} {}^i\xi_{\mathbf{x}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}) + \frac{\partial \mathbf{h}_{new}({}^i\phi_{\mathbf{x}}, \mathbf{u}, {}^i\phi_{\mathbf{w}})}{\partial {}^i\phi_{\mathbf{w}}} {}^i\xi_{\mathbf{w}}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}) \quad (47)$$

Eq. (47) can be evaluated at $\epsilon_i = 0$ and as a result,

$$\frac{\partial \mathbf{h}_{new}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{x}}^i \boldsymbol{\xi}_{\mathbf{x}}(\mathbf{x}, \mathbf{w}) + \frac{\partial \mathbf{h}_{new}(\mathbf{x}, \mathbf{u}, \mathbf{w})}{\partial \mathbf{w}}^i \boldsymbol{\xi}_{\mathbf{w}}(\mathbf{x}, \mathbf{w}) \neq \mathbf{0} \quad (48)$$

Eq. (48) thus provides a mathematical condition which must be fulfilled by \mathbf{h}_{new} so as to destroy the i^{th} group of Lie symmetries. Optimally, if all the groups of Lie symmetries (except $\{\mathbf{x}, \mathbf{w}\}$) can be destroyed, i.e. Eq. (48) is true for all i (except $\{^i\phi_{\mathbf{x}}, ^i\phi_{\mathbf{w}}\} \equiv \{\mathbf{x}, \mathbf{w}\}$), the unobservable system can become fully observable due to the newly introduced sensors.

Based on the condition in Eq. (48), the most straightforward way to destroy the i^{th} group is to measure an unobservable dynamic state, i.e. x_j ($1 \leq j \leq n$), whose Lie symmetry $^i\phi_{x,j} \neq x_j$ if practically possible. In civil and mechanical applications, a dynamic state is often the displacement or velocity of a degree of freedom. It can be easily checked that the measurement indeed fulfills the condition, by substituting $\mathbf{h}_{new}(\mathbf{x}, \mathbf{u}, \mathbf{w}) = x_j$ into Eq. (48):

$$\frac{\partial x_j}{\partial \mathbf{x}}^i \boldsymbol{\xi}_{\mathbf{x}}(\mathbf{x}, \mathbf{w}) = ^i\xi_{x,j}(\mathbf{x}, \mathbf{w}) = \frac{\partial ^i\phi_{x,j}}{\partial \epsilon_i} \neq \mathbf{0} \quad (49)$$

where $^i\xi_{x,j}$ is the j^{th} component of $^i\boldsymbol{\xi}_{\mathbf{x}}$. However it should be noted that if x_j is an unmeasurable dynamic state the approach could not apply, and if x_j is an unidentifiable parameter the case will be discussed in Section 5.2.3. Similarly, the additional measurement can also be an unobservable unknown input to destroy the i^{th} group of Lie symmetries, i.e. $\mathbf{h}_{new}(\mathbf{x}, \mathbf{u}, \mathbf{w}) = w_j$ ($1 \leq j \leq m$):

$$\frac{\partial w_j}{\partial \mathbf{w}}^i \boldsymbol{\xi}_{\mathbf{w}}(\mathbf{x}, \mathbf{w}) = ^i\xi_{w,j}(\mathbf{x}, \mathbf{w}) = \frac{\partial ^i\phi_{w,j}}{\partial \epsilon_i} \neq \mathbf{0} \quad (50)$$

where $^i\xi_{w,j}$ is the j^{th} component of $^i\boldsymbol{\xi}_{\mathbf{w}}$, but note that unknown inputs are often difficult or infeasible to measure in practice.

5.2.2. Model transformation

An alternative means to destroy the i^{th} group of Lie symmetries, so that to improve observability, is to combine or transform the unobservable states and unknown inputs to form a new vector of variables. The vector, denoted by \mathbf{x}_T , consists of the originally observable states and inputs and the transformed

440 variables, while the dimension of \mathbf{x}_T is reduced to at most $n+m-1$. The model of the unobservable system can then be reformulated with respect to \mathbf{x}_T , leaving the measured input and output unchanged. Let \mathbf{T} denote the combination or transformation function,

$$\mathbf{x}_T \equiv \mathbf{T}(\mathbf{x}, \mathbf{w}) \quad (51)$$

In this case, the i^{th} group of Lie symmetries of \mathbf{x}_T is identically equivalent to:

$${}^i\phi_{\mathbf{x}_T} \equiv \mathbf{T}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}) \quad (52)$$

445 which means that ${}^i\phi_{\mathbf{x}_T}$ can be obtained by applying the same function to the Lie symmetries of \mathbf{x} and \mathbf{w} [21]. To improve the observability of the transformed system with respect to \mathbf{x}_T , ${}^i\phi_{\mathbf{x}_T}$ must not be a function of ϵ_i and hence there would not be infinitely many solutions of \mathbf{x}_T due to the i^{th} group; in other words, the function \mathbf{T} eliminates the existing ϵ_i between the components of
450 $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\}$. Mathematically, \mathbf{T} must fulfill the following condition:

$$\frac{d\mathbf{T}({}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}})}{d\epsilon_i} = \mathbf{0} \quad (53)$$

for any value of ϵ_i in S . Applying the chain rule to Eq. (53) and evaluating it at $\epsilon_i = 0$ yield:

$$\frac{\partial \mathbf{T}(\mathbf{x}, \mathbf{w})}{\partial \mathbf{x}} {}^i\boldsymbol{\xi}_{\mathbf{x}}(\mathbf{x}, \mathbf{w}) + \frac{\partial \mathbf{T}(\mathbf{x}, \mathbf{w})}{\partial \mathbf{w}} {}^i\boldsymbol{\xi}_{\mathbf{w}}(\mathbf{x}, \mathbf{w}) = \mathbf{0} \quad (54)$$

The choice of the function \mathbf{T} to meet the condition in Eq. (54) is not unique and often requires human intelligence and engineering judgement. On one hand, the
455 system should be reformulatable with respect to the corresponding \mathbf{x}_T leaving \mathbf{u} and \mathbf{y} unchanged, and on the other hand, it would be more beneficial if the states in \mathbf{x}_T do not lose physical meanings.

5.2.3. Model assumptions

Model assumptions herein refer to the assumptions or constraints applied
460 to model parameters so as to improve observability/identifiability. It should be recalled that identifiability is a term particularly used to account for the observability of time-invariant parameters of a system, i.e. $\boldsymbol{\theta}$. As has been proven in

Section 5.2.1, to destroy the i^{th} group of Lie symmetries, it is sufficient to additionally measure an unobservable dynamic state, i.e. x_j ($1 \leq j \leq n$), whose Lie symmetry ${}^i\phi_{x,j} \not\equiv x_j$. When x_j is an unidentifiable parameter which is however unmeasurable, assuming x_j to be known or properly constraining x_j can reach the same goal of destroying the i^{th} group. A typical example is the problem in Operational Modal Analysis [31], where structural response under ambient excitations is measured to identify the modal parameters of a structure, including natural frequencies, damping ratios and mode shapes. The mode shape parameters without constraints are unidentifiable, which are found to admit groups of scaling symmetries, although it was not noted as ‘Lie symmetries’ in the literature. To be able to identify the mode shapes and other relevant parameters, an effective strategy is to constrain the mode shapes to have unit norm without losing their physical meanings.

5.2.4. Remark

The strategies of sensor placement, model transformation and model assumptions can be used in combination with each other to destroy the groups of Lie symmetries as many as possible. Parts of the unobservable states and unknown inputs can be rendered observable if they are only related to the destroyed Lie symmetries; as long as all the groups of Lie symmetries (except $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\} \equiv \{\mathbf{x}, \mathbf{w}\}$) can be destroyed, the unobservable system can be rendered fully observable. The detailed use of the strategies will be illustrated in the following examples of engineering and biochemical systems.

6. Illustrative examples

6.1. A linear mass-spring system

This example uses a simple linear mass-spring system with 2 degrees of freedom (DOFs), as shown in Fig. 1, for the purpose of illustrating the computation, concept and application of Lie symmetries. For the considered system, let x_1 and v_1 denote the displacement and velocity of the first mass respectively, and

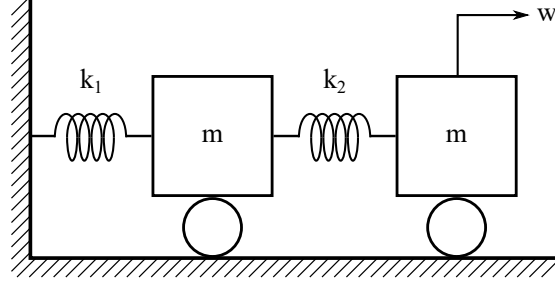


Figure 1: A 2 DOFs linear mass-spring system

x_2 and v_2 denote those of the second mass. It is assumed that the two mass parameters are equal which are denoted by m . The spring element connecting the first mass to the fixed support has the stiffness of k_1 , and the stiffness of the spring connecting the two masses is k_2 . The dynamical system is driven by an
495 unknown input w applied at the second mass. The state equation of the system is therefore given by:

$$\dot{\mathbf{x}}_t = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \frac{-k_1 x_1 + k_2 (x_2 - x_1)}{m} \\ \frac{k_2 (x_1 - x_2) + w}{m} \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} = \frac{d}{dt} \begin{bmatrix} k_1 \\ k_2 \\ m \end{bmatrix} = \mathbf{0} \quad (55)$$

$$\mathbf{x} = [x_1 \quad x_2 \quad v_1 \quad v_2 \quad k_1 \quad k_2 \quad m]^T$$

where x_1 , x_2 , v_1 and v_2 are treated as the dynamic states, and k_1 , k_2 and m are the unknown time-invariant parameters of the system. Suppose two sensors are placed to measure the displacement of the first mass and the acceleration of
500 the second mass, and thus the output equation is given by:

$$\mathbf{y} = \begin{bmatrix} x_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ \frac{k_2 (x_1 - x_2) + w}{m} \end{bmatrix} \quad (56)$$

It would be of interest to investigate whether the dynamic states, parameters and the unknown input can be successfully identified based on the measurement of \mathbf{y} , i.e. observability of the considered system.

The general method presented in Section 3 is first employed to compute

505 the Lie symmetries of the system. The matrix $d\Omega^k$ is symbolically calculated up to order $k = 5$ using Algorithm I. The kernel of $d\Omega^k$ then results in a 2-dimensional basis $\mathbf{B} = \{{}^1\xi_{\mathbf{x}^k}, {}^2\xi_{\mathbf{x}^k}\}$, where ${}^1\xi_{\mathbf{x}^k} = \mathbf{0}$. Solving the differential equation (27) for $i = 1$ gives ${}^1\phi_{\mathbf{x}^k} = \mathbf{x}^k$, which indicates that the first one-parameter Lie group of symmetries $\{{}^1\phi_{\mathbf{x}}, {}^1\phi_{\mathbf{w}}\} = \{\mathbf{x}, w\}$. It should be noted
510 that in fact, $\{{}^i\phi_{\mathbf{x}}, {}^i\phi_{\mathbf{w}}\} = \{\mathbf{x}, w\}$ is always one of the groups of Lie symmetries for any system and it is only useful when characterizing an observable system, and therefore for the sake of brevity, it will be omitted in the later examples. The expression of ${}^2\xi_{\mathbf{x}^k}$ is determined as:

$$\begin{aligned} {}^2\xi_{\mathbf{x}^k} &= \begin{bmatrix} {}^2\xi_{\mathbf{x}}^T & {}^2\xi_{\mathbf{w}}^T & \dot{w} & w^{(2)} & w^{(3)} & w^{(4)} & w^{(5)} \end{bmatrix}^T \\ {}^2\xi_{\mathbf{x}} &= \begin{bmatrix} 0 & 0 & 0 & 0 & k_1 & k_2 & m \end{bmatrix}^T, \quad {}^2\xi_{\mathbf{w}} = w \end{aligned} \quad (57)$$

where the expressions of ${}^2\xi_{\mathbf{x}}$ and ${}^2\xi_{\mathbf{w}}$ remain invariant for larger order k . Using
515 the expression of ${}^2\xi_{\mathbf{x}^k}$, Eq. (27) becomes the following differential equation:

$$\frac{\partial^2 \phi_{\mathbf{x}^k}}{\partial \epsilon_2} = \frac{\partial}{\partial \epsilon_2} \begin{bmatrix} {}^2\phi_{x,1} \\ {}^2\phi_{x,2} \\ {}^2\phi_{x,3} \\ {}^2\phi_{x,4} \\ {}^2\phi_{x,5} \\ {}^2\phi_{x,6} \\ {}^2\phi_{x,7} \\ {}^2\phi_w \\ {}^2\phi_{\dot{w}} \\ \vdots \\ {}^2\phi_{w^{(5)}} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ {}^2\phi_{x,5} \\ {}^2\phi_{x,6} \\ {}^2\phi_{x,7} \\ {}^2\phi_w \\ {}^2\phi_{\dot{w}} \\ \vdots \\ {}^2\phi_{w^{(5)}} \end{bmatrix}, \quad \left. \begin{bmatrix} {}^2\phi_{x,1} \\ {}^2\phi_{x,2} \\ {}^2\phi_{x,3} \\ {}^2\phi_{x,4} \\ {}^2\phi_{x,5} \\ {}^2\phi_{x,6} \\ {}^2\phi_{x,7} \\ {}^2\phi_w \\ {}^2\phi_{\dot{w}} \\ \vdots \\ {}^2\phi_{w^{(5)}} \end{bmatrix} \right|_{\epsilon_2=0} = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ k_1 \\ k_2 \\ m \\ w \\ \dot{w} \\ \vdots \\ w^{(5)} \end{bmatrix} \quad (58)$$

Analytically solving Eq. (58) gives the result of ${}^2\phi_{\mathbf{x}^k}$, and the second one-parameter Lie group of symmetries $\{{}^2\phi_{\mathbf{x}}, {}^2\phi_{\mathbf{w}}\}$ can be obtained as the components of ${}^2\phi_{\mathbf{x}^k}$:

$$\begin{aligned} {}^2\phi_{\mathbf{x}} &= \begin{bmatrix} x_1 & x_2 & v_1 & v_2 & e^{\epsilon_2} k_1 & e^{\epsilon_2} k_2 & e^{\epsilon_2} m \end{bmatrix}^T \\ {}^2\phi_{\mathbf{w}} &= e^{\epsilon_2} w \end{aligned} \quad (59)$$

It is straightforward to verify the results by substituting ${}^2\phi_x$ and ${}^2\phi_w$ into Eqs. (55) and (56) to check that they certainly satisfy the state and output equations of the system.

For the purpose of comparison, the second method presented in Section 4 is also used to compute Lie symmetries. When assuming $\{{}^2\phi_x, {}^2\phi_w\}$ to be a group of scaling symmetries as in Tables 1 and 2, i.e.:

$$\begin{aligned} {}^2\phi_x &= \begin{bmatrix} e^{\alpha_{2,1}\epsilon_2} x_1 & e^{\alpha_{2,2}\epsilon_2} x_2 & e^{\alpha_{2,3}\epsilon_2} v_1 & e^{\alpha_{2,4}\epsilon_2} v_2 & e^{\alpha_{2,5}\epsilon_2} k_1 & e^{\alpha_{2,6}\epsilon_2} k_2 & e^{\alpha_{2,7}\epsilon_2} m \end{bmatrix}^T \\ {}^2\phi_w &= e^{\alpha_{2,8}\epsilon_2} w \end{aligned} \quad (60)$$

the occurring Eq. (42) gives the following values of unknown coefficients:

$$\begin{aligned} \alpha_2 &= \begin{bmatrix} \alpha_{2,1} & \alpha_{2,2} & \alpha_{2,3} & \alpha_{2,4} & \alpha_{2,5} & \alpha_{2,6} & \alpha_{2,7} & \alpha_{2,8} \end{bmatrix}^T \\ &= \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}^T \end{aligned} \quad (61)$$

Eq. (61) thus shows an absolute agreement between the results from the general method and the second method. In this case, computation of the second method is more efficient than the general method, while it should be noted that this is a consequence of the a priori assumption on scaling. Moreover, by using the second method only, it cannot be concluded that the other types of Lie symmetries do not exist when obtaining the scaling symmetries. This is as said a main limitation of the second method as well as the existing Lie symmetry computation methods in the literature.

Based on Section 5, the obtained Lie symmetries can be applied to analyze and improve the observability properties of the system. Combining the two groups of Lie symmetries as in Eq. (43) leads to:

$$\begin{aligned} \phi_x &= {}^2\phi_x({}^1\phi_x, {}^1\phi_w, \epsilon_2) \\ &= \begin{bmatrix} x_1 & x_2 & v_1 & v_2 & e^{\epsilon_2} k_1 & e^{\epsilon_2} k_2 & e^{\epsilon_2} m \end{bmatrix}^T \\ \phi_w &= {}^2\phi_w({}^1\phi_x, {}^1\phi_w, \epsilon_2) \\ &= e^{\epsilon_2} w \end{aligned} \quad (62)$$

According to Section 5.1, k_1 , k_2 and m are unidentifiable parameters since their

Lie symmetries are not identically equal to themselves, and similarly w is an un-observable input. For instance, $e^{\epsilon_2} k_1$ for any value of ϵ_2 in a continuous interval
540 can be a solution of the first stiffness which perfectly fits the measurement of \mathbf{y} , and this is also the same for k_2 , m and w . It is hence impossible to correctly estimate k_1 , k_2 , m and w from the output measurement using any system identification algorithm. On the contrary, the dynamic states x_1 , x_2 , v_1 and v_2 are observable since $\phi_{x,j} \equiv x_j$ for $j = 1, \dots, 4$.

545 To make the system observable it is straightforward to adopt the strategy of model assumptions, assuming that the unknown input is difficult or infeasible to measure. As in Section 5.2.3, if any of the parameters k_1 , k_2 , m is assumed to be known, then $\{{}^2\phi_{\mathbf{x}}, {}^2\phi_{\mathbf{w}}\}$ can be destroyed and thus all the one-parameter Lie groups of symmetries of the system, except $\{{}^1\phi_{\mathbf{x}}, {}^1\phi_{\mathbf{w}}\} = \{\mathbf{x}, w\}$, are destroyed.
550 In practice, it is often more appropriate to use an assumption on m , since mass parameters output from, e.g. finite element model or theoretical calculation, are often more accurate than stiffness parameters and they therefore rely less on numerical identification from measurements. As a result of assuming m to be known, k_1 , k_2 and w can become fully observable at the same time.

555 Alternatively, the strategy of model transformation can be applied to improve the observability of the considered system. As in Section 5.2.2, a function \mathbf{T} is determined as the following to transform the states and the unknown input to a vector of variables $\mathbf{x}_{\mathbf{T}}$:

$$\mathbf{x}_{\mathbf{T}} = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ k_{1,m} \\ k_{2,m} \\ w_m \end{bmatrix} \equiv \mathbf{T}(\mathbf{x}, w) = \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ \frac{k_1}{m} \\ \frac{k_2}{m} \\ \frac{w}{m} \end{bmatrix} \quad (63)$$

The idea behind the way of determining \mathbf{T} is to ensure that, the group of Lie
560 symmetries of $\mathbf{x}_{\mathbf{T}}$, equivalent to $\mathbf{T}({}^2\phi_{\mathbf{x}}, {}^2\phi_{\mathbf{w}})$ based on Eq. (52), eliminates

ϵ_2 between the components of $\{{}^2\phi_{\mathbf{x}}, {}^2\phi_{\mathbf{w}}\}$ and thus is not a function of ϵ_2 . It can be checked that $\mathbf{T}(\mathbf{x}, w)$ certainly fulfills the mathematical condition in Eq. (54) for $i = 2$ and thus $\{{}^2\phi_{\mathbf{x}}, {}^2\phi_{\mathbf{w}}\}$ is successfully destroyed. Consequently, the system can be reformulated with respect to \mathbf{x}_T in a reduced form:

$$\begin{aligned} \dot{\mathbf{x}}_t = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} &= \begin{bmatrix} v_1 \\ v_2 \\ -k_{1,m}x_1 + k_{2,m}(x_2 - x_1) \\ k_{2,m}(x_1 - x_2) + w_m \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} = \frac{d}{dt} \begin{bmatrix} k_{1,m} \\ k_{2,m} \end{bmatrix} = \mathbf{0} \\ \mathbf{y} = \begin{bmatrix} x_1 \\ \dot{v}_2 \end{bmatrix} &= \begin{bmatrix} x_1 \\ k_{2,m}(x_1 - x_2) + w_m \end{bmatrix} \end{aligned} \quad (64)$$

where w_m is the unknown input to the transformed system, which is equivalent to the original input scaled by $1/m$. $k_{1,m}$ and $k_{2,m}$ are the model parameters of the system, which can be related to modal parameters, i.e. natural frequencies and mode shapes, through the eigenvalue equation. Most importantly, the transformed system in Eq. (64) is now fully observable for the given output measurement, that is, its dynamic states, parameters and the unknown input are all observable.

6.2. A mass-spring system with a nonlinear element

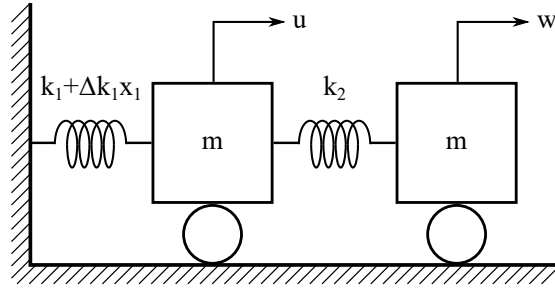


Figure 2: A 2 DOFs mass-spring system with a polynomial nonlinear element

Analyzing the linear system in Example 1 may not necessarily rely on the tools of Lie symmetries due to its simplicity, while Lie symmetries indeed provide more systematic manners of analysis than using human intelligence. This

example demonstrates the capability of the proposed methods through a dynamical system with a polynomial nonlinear spring element. As shown in Fig. 2, the system of interest is a 2 DOFs mass-spring system whose first spring has the stiffness of $k_1 + \Delta k_1 x_1$, where k_1 and Δk_1 are the stiffness parameters and x_1 is the displacement of the first mass. In addition to the unknown input w applied at the second mass, a known input or excitation u is applied at the first mass to drive the system. Similarly as in Example 1, let x_1, x_2 and v_1, v_2 denote the displacements and velocities of the two masses respectively, m denote the mass parameters, and k_2 denote the stiffness of the second spring. The state equation of the system is therefore given by:

$$\dot{\mathbf{x}}_t = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \frac{-(k_1 + \Delta k_1 x_1)x_1 + k_2(x_2 - x_1) + u}{m} \\ \frac{k_2(x_1 - x_2) + w}{m} \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} = \frac{d}{dt} \begin{bmatrix} k_1 \\ \Delta k_1 \\ k_2 \\ m \end{bmatrix} = \mathbf{0} \quad (65)$$

$$\mathbf{x} = [x_1 \quad x_2 \quad v_1 \quad v_2 \quad k_1 \quad \Delta k_1 \quad k_2 \quad m]^T$$

where x_1, x_2, v_1 and v_2 are the dynamic states, $k_1, \Delta k_1, k_2$ and m are the unknown model parameters of the system. Suppose two accelerometers are placed to measure the accelerations of the two masses, and thus the output equation is given by:

$$\mathbf{y} = \begin{bmatrix} \dot{v}_1 \\ \dot{v}_2 \end{bmatrix} = \begin{bmatrix} \frac{-(k_1 + \Delta k_1 x_1)x_1 + k_2(x_2 - x_1) + u}{m} \\ \frac{k_2(x_1 - x_2) + w}{m} \end{bmatrix} \quad (66)$$

The aim is to investigate whether the dynamic states, parameters and the unknown input can be successfully estimated based on the measurement of \mathbf{y} , i.e. observability properties of the system.

To compute the Lie symmetries of the system using the general method in Section 3, the matrix $d\boldsymbol{\Omega}^k$ is symbolically calculated up to order $k = 6$ using Algorithm I, and its kernel results in a basis $\mathbf{B} = \{{}^1\boldsymbol{\xi}_{x^k}\}$ (omitting ${}^2\boldsymbol{\xi}_{x^k} = \mathbf{0}$ as

aforementioned), where the expression of ${}^1\xi_{x^k}$ is given by:

$$\begin{aligned} {}^1\xi_{x^k} &= \begin{bmatrix} {}^1\xi_x^T & {}^1\xi_w^T & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ {}^1\xi_x &= \begin{bmatrix} 1 & \frac{k_1+k_2}{k_2} & 0 & 0 & -2\Delta k_1 & 0 & 0 & 0 \end{bmatrix}^T, \quad {}^1\xi_w = k_1 \end{aligned} \quad (67)$$

where ${}^1\xi_x$ and ${}^1\xi_w$ remain invariant for larger order k . With the expression of infinitesimals, Eq. (27) for $i = 1$ can be solved analytically and as a result, the one-parameter Lie group of symmetries $\{{}^1\phi_x, {}^1\phi_w\}$ is obtained as:

$${}^1\phi_x = \begin{bmatrix} x_1 + \epsilon_1 \\ x_2 + \frac{k_1+k_2}{k_2}\epsilon_1 - \frac{\Delta k_1}{k_2}\epsilon_1^2 \\ v_1 \\ v_2 \\ k_1 - 2\Delta k_1\epsilon_1 \\ \Delta k_1 \\ k_2 \\ m \end{bmatrix}, \quad {}^1\phi_w = w + k_1\epsilon_1 - \Delta k_1\epsilon_1^2 \quad (68)$$

600 In terms of solving the differential equation (27), it might be easier to calculate the power series solution using Algorithm II, which gives exactly the same result of $\{{}^1\phi_x, {}^1\phi_w\}$.

In this example, the second computational method in Section 4 is not capable of handling the system, because the Lie symmetries of the system do not belong
605 to any regular type of Lie symmetries and thus proper a priori assumption is infeasible to make. To detect the observability properties of the system, the one-parameter Lie groups are combined to be:

$$\phi_x = {}^1\phi_x, \quad \phi_w = {}^1\phi_w \quad (69)$$

Following the deduction described in Section 5.1, x_1 , x_2 and k_1 are unobservable states, w is an unobservable input, and v_1 , v_2 , Δk_1 , k_2 and m are observable
610 states for the given output measurement. In terms of the observability results, it fully agrees with what was reported in [15] where rank testing of the observability matrix was used for the same system.

To fix the unobservability, a direct strategy that can be utilized is to place additional sensors so as to destroy the group of Lie symmetries, supposing that
 615 it is difficult or infeasible to measure the unknown input. As in Section 5.2.1, if any of x_1 and x_2 (or both) can be measured, then $\{{}^1\phi_x, {}^1\phi_w\}$ is destroyed and consequently the system of interest can be rendered observable. In other words, the newly introduced sensor(s) is expected to obtain the following output measurement:

$$\mathbf{y}_{new} = x_1 \text{ or } \mathbf{y}_{new} = x_2 \text{ or } \mathbf{y}_{new} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (70)$$

620 It can be verified that the corresponding \mathbf{h}_{new} in Eq. (70) certainly meets the mathematical condition in Eq. (48) for $i = 1$, and as a consequence of destroying the Lie symmetries, the original unobservable states and unknown input x_1 , x_2 , k_1 and w become fully observable.

6.3. A mass-spring system with a Bouc-Wen element

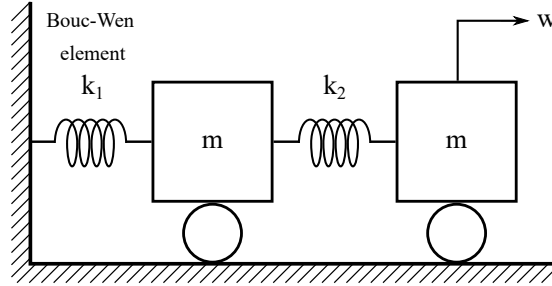


Figure 3: A 2 DOFs mass-spring system with a Bouc-Wen element

625 This example considers a non-smooth nonlinear dynamical system whose Lie symmetries and observability properties vary with respect to different realizations of its dynamic states. As shown in Fig. 3, the system is a 2 DOFs mass-spring system where the behaviour of the first spring is formulated by a Bouc-Wen hysteretic model. The observability of similar systems with Bouc-
 630 Wen elements was studied in the literature [2, 14] based on rank testing methods,

and reparameterization of the Bouc-Wen model was proposed using human intelligence and engineering judgement in order to improve its identifiability. The procedure of study is herein illustrated from the point of view of Lie symmetries, which offers the possibility of conducting similar procedures in a more organized and automated way. For the Bouc-Wen element, let β , γ and ν denote the hysteretic model parameters, r denote the elastic displacement of the spring and k_1 denote its stiffness. Similarly as in the previous examples, x_1 , x_2 and v_1 , v_2 are the displacements and velocities of the two masses m respectively, k_2 is the stiffness of the second spring, and the system is driven by an unknown input at the second mass while no known input is applied in this case. The state equation of the system is given by:

$$\dot{\mathbf{x}}_t = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ r \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ \frac{-k_1 r + k_2(x_2 - x_1)}{m} \\ \frac{k_2(x_1 - x_2) + w}{m} \\ v_1 - \beta|v_1||r|^{\nu-1}r - \gamma v_1|r|^\nu \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} = \frac{d}{dt} \begin{bmatrix} k_1 \\ k_2 \\ m \\ \beta \\ \gamma \\ \nu \end{bmatrix} = \mathbf{0} \quad (71)$$

$$\mathbf{x} = [x_1 \quad x_2 \quad v_1 \quad v_2 \quad r \quad k_1 \quad k_2 \quad m \quad \beta \quad \gamma \quad \nu]^T$$

where x_1 , x_2 , v_1 , v_2 and r are the dynamic states, and k_1 , k_2 , m , β , γ and ν are the unknown model parameters to be identified. Suppose two sensors are placed to measure the acceleration of the first mass and the displacement of the second mass, and thus the output equation is given by:

$$\mathbf{y} = \begin{bmatrix} \dot{x}_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{-k_1 r + k_2(x_2 - x_1)}{m} \\ x_2 \end{bmatrix} \quad (72)$$

Since the system in Eq. (71) is non-smooth and non-analytic due to the presence of absolute value operators, it is not allowed to use the tools of observability and Lie symmetries directly for analysis. Instead, it was suggested in [2] to consider the system as a piecewise-smooth one, and divide it into several smooth branches with respect to different realizations of the dynamic states. To

achieve this, the differential equation of r can be written as:

$$\begin{aligned}
A : \dot{r} &= v_1 - \beta v_1 r^\nu - \gamma v_1 r^\nu, \text{ for } v_1 > 0, r > 0 \\
B : \dot{r} &= v_1 + \beta v_1 r^\nu - \gamma v_1 r^\nu, \text{ for } v_1 < 0, r > 0 \\
C : \dot{r} &= v_1 + \beta v_1 (-r)^\nu - \gamma v_1 (-r)^\nu, \text{ for } v_1 > 0, r < 0 \\
D : \dot{r} &= v_1 - \beta v_1 (-r)^\nu - \gamma v_1 (-r)^\nu, \text{ for } v_1 < 0, r < 0
\end{aligned} \tag{73}$$

while the differential equations of the remaining dynamic states are the same as in Eq. (71). Consequently, each of the smooth branches A , B , C and D without absolute value operators can be analyzed separately as an independent analytic subsystem.

The one-parameter Lie groups of symmetries found by the general and second computational methods are in complete alignment. The overall system contains 3 groups of Lie symmetries in total, while each of the branches admits 2 groups of them:

$$\begin{aligned}
\begin{bmatrix} {}^1\phi_x \\ {}^1\phi_w \end{bmatrix} &= \begin{bmatrix} x_1 & x_2 & v_1 & v_2 & r & e^{\epsilon_1} k_1 & e^{\epsilon_1} k_2 & e^{\epsilon_1} m & \beta & \gamma & \nu & e^{\epsilon_1} w \end{bmatrix}^T \\
\begin{bmatrix} {}^2\phi_x \\ {}^2\phi_w \end{bmatrix} &= \begin{bmatrix} x_1 & x_2 & v_1 & v_2 & r & k_1 & k_2 & m & \beta + \epsilon_2 & \gamma - \epsilon_2 & \nu & w \end{bmatrix}^T \\
\begin{bmatrix} {}^3\phi_x \\ {}^3\phi_w \end{bmatrix} &= \begin{bmatrix} x_1 & x_2 & v_1 & v_2 & r & k_1 & k_2 & m & \beta + \epsilon_3 & \gamma + \epsilon_3 & \nu & w \end{bmatrix}^T
\end{aligned} \tag{74}$$

where $\{{}^1\phi_x, {}^1\phi_w\}$ is a group of scaling symmetries, and $\{{}^2\phi_x, {}^2\phi_w\}$ and $\{{}^3\phi_x, {}^3\phi_w\}$ are groups of translation symmetries. In particular, branch A or D admits $\{{}^1\phi_x, {}^1\phi_w\}$ and $\{{}^2\phi_x, {}^2\phi_w\}$, and branch B or C admits $\{{}^1\phi_x, {}^1\phi_w\}$ and $\{{}^3\phi_x, {}^3\phi_w\}$. Combining the one-parameter Lie groups for each of the branches leads to:

$$\begin{bmatrix} \phi_x \\ \phi_w \end{bmatrix} = \begin{cases} \begin{bmatrix} \mathbf{x}_t^T & e^{\epsilon_1} k_1 & e^{\epsilon_1} k_2 & e^{\epsilon_1} m & \beta + \epsilon_2 & \gamma - \epsilon_2 & \nu & e^{\epsilon_1} w \end{bmatrix}, & \text{for } A, D \\ \begin{bmatrix} \mathbf{x}_t^T & e^{\epsilon_1} k_1 & e^{\epsilon_1} k_2 & e^{\epsilon_1} m & \beta + \epsilon_3 & \gamma + \epsilon_3 & \nu & e^{\epsilon_1} w \end{bmatrix}, & \text{for } B, C \end{cases} \tag{75}$$

The results of two-parameter Lie groups of symmetries reveal that, the dynamic states and ν are observable, while k_1 , k_2 , m , β , γ are unidentifiable parameters

and w is an unobservable input in any of the four branches under the measurement of \mathbf{y} .

In this example, the strategy of model transformation is adopted to improve the observability of the unobservable system. A function \mathbf{T} is determined as the following to transform the states and the unknown input to a vector of variables \mathbf{x}_T :

$$\mathbf{x}_T = \begin{bmatrix} \mathbf{x}_t \\ k_{1,m} \\ k_{2,m} \\ \Delta_1 \\ \Delta_2 \\ \nu \\ w_m \end{bmatrix} \equiv \mathbf{T}(\mathbf{x}, w) = \begin{bmatrix} \mathbf{x}_t \\ \frac{k_1}{m} \\ \frac{k_2}{m} \\ \beta + \gamma \\ \beta - \gamma \\ \nu \\ \frac{w}{m} \end{bmatrix} \quad (76)$$

The function $\mathbf{T}(\mathbf{x}, w)$, with $\beta + \gamma$ for branches A and D and with $\beta - \gamma$ for branches B and C , fulfills the mathematical condition in Eq. (54) for $i = 1, 2, 3$, and thus all the three groups of Lie symmetries in Eq. (74) can be destroyed.

With respect to \mathbf{x}_T , the system can be reformulated as the following form:

$$\begin{aligned} \dot{\mathbf{x}}_t = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ v_1 \\ v_2 \\ r \end{bmatrix} &= \begin{bmatrix} v_1 \\ v_2 \\ -k_{1,m}r + k_{2,m}(x_2 - x_1) \\ k_{2,m}(x_1 - x_2) + w_m \\ G(v_1, r, \Delta_1, \Delta_2, \nu) \end{bmatrix}, \quad \dot{\boldsymbol{\theta}} = \frac{d}{dt} \begin{bmatrix} k_{1,m} \\ k_{2,m} \\ \Delta_1 \\ \Delta_2 \\ \nu \end{bmatrix} = \mathbf{0} \\ \mathbf{y} = \begin{bmatrix} \dot{v}_1 \\ x_2 \end{bmatrix} &= \begin{bmatrix} -k_{1,m}r + k_{2,m}(x_2 - x_1) \\ x_2 \end{bmatrix} \end{aligned} \quad (77)$$

where

$$\begin{aligned} A : G(v_1, r, \Delta_1, \Delta_2, \nu) &= v_1 - \Delta_1 v_1 r^\nu, \text{ for } v_1 > 0, r > 0 \\ B : G(v_1, r, \Delta_1, \Delta_2, \nu) &= v_1 + \Delta_2 v_1 r^\nu, \text{ for } v_1 < 0, r > 0 \\ C : G(v_1, r, \Delta_1, \Delta_2, \nu) &= v_1 + \Delta_2 v_1 (-r)^\nu, \text{ for } v_1 > 0, r < 0 \\ D : G(v_1, r, \Delta_1, \Delta_2, \nu) &= v_1 - \Delta_1 v_1 (-r)^\nu, \text{ for } v_1 < 0, r < 0 \end{aligned} \quad (78)$$

where w_m is a scaled unknown input to the transformed system, and $k_{1,m}$, $k_{2,m}$, Δ_1 and Δ_2 are new model parameters which are the combinations of original model parameters. In consequence, Δ_1 is identifiable within branches A and D , Δ_2 is identifiable within branches B and C , and the other states and the input of the transformed system are observable within the four branches; the transformed system in Eq. (77) is therefore overall observable for the given output measurement. To successfully identify the system in practice, however, the system is required to be sufficiently excited so that its behaviour travels through at least one combination of branches which allows to estimate both Δ_1 and Δ_2 , i.e. A and B , A and C , B and D , or C and D . Interestingly, the original hysteretic parameters β and γ can also be obtained by solving the equations: $\Delta_1 = \beta + \gamma$ and $\Delta_2 = \beta - \gamma$ based on the estimated values of Δ_1 and Δ_2 .

6.4. A biochemical model

This example presents a biochemical model taken from [19], which exhibits more complicated Lie symmetries that can be handled by the proposed tools of this work. It is a pharmacokinetic (PK) model used to describe the time course of drug concentrations when a drug is administered to a living organism. The state and output equations of the model are given by:

$$\begin{aligned} \dot{\mathbf{x}}_t = \frac{d}{dt} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} &= \begin{bmatrix} w - (\theta_1 + \theta_2)x_1 \\ \theta_1 x_1 - (\theta_3 + \theta_6 + \theta_7)x_2 + \theta_5 x_4 \\ \theta_2 x_1 + \theta_3 x_2 - \theta_4 x_3 \\ \theta_6 x_2 - \theta_5 x_4 \end{bmatrix} \\ \dot{\boldsymbol{\theta}} = \frac{d}{dt} \begin{bmatrix} \theta_1 & \theta_2 & \theta_3 & \theta_4 & \theta_5 & \theta_6 & \theta_7 & \theta_8 & \theta_9 \end{bmatrix}^T &= \mathbf{0} \\ \mathbf{y} &= \begin{bmatrix} \theta_8 x_3 & \theta_9 x_2 & x_1 \end{bmatrix}^T \end{aligned} \quad (79)$$

where x_1 , x_2 , x_3 and x_4 are the dynamic states, θ_1 , θ_2 , \dots , θ_9 are the unknown model parameters, and w is assumed to be an unknown input.

Applying the general computational method, the matrix $d\boldsymbol{\Omega}^k$ is symbolically calculated up to order $k = 5$ using Algorithm I. The kernel of $d\boldsymbol{\Omega}^k$ results in a

700 2-dimensional basis $B = \{{}^1\xi_{\mathbf{x}^k}, {}^2\xi_{\mathbf{x}^k}\}$, where ${}^1\xi_{\mathbf{x}^k}$ is expressed as:

$$\begin{aligned} {}^1\xi_{\mathbf{x}^k} &= \begin{bmatrix} {}^1\xi_{\mathbf{x}}^T & {}^1\xi_{\mathbf{w}}^T & 0 & 0 & 0 & 0 & 0 \end{bmatrix}^T \\ {}^1\xi_{\mathbf{x}} &= \begin{bmatrix} 0 & x_2 & -\frac{\theta_1 x_3}{\theta_2} & x_4 & {}^1\xi_{\boldsymbol{\theta}}^T \end{bmatrix}^T, \quad {}^1\xi_{\mathbf{w}} = 0 \\ {}^1\xi_{\boldsymbol{\theta}} &= \begin{bmatrix} \theta_1 & -\theta_1 & -\frac{(\theta_1+\theta_2)\theta_3}{\theta_2} & 0 & 0 & 0 & \frac{(\theta_1+\theta_2)\theta_3}{\theta_2} & \frac{\theta_1\theta_8}{\theta_2} & -\theta_9 \end{bmatrix}^T \end{aligned} \quad (80)$$

and ${}^2\xi_{\mathbf{x}^k}$ is expressed as:

$$\begin{aligned} {}^2\xi_{\mathbf{x}^k} &= \begin{bmatrix} {}^2\xi_{\mathbf{x}}^T & {}^2\xi_{\mathbf{w}}^T & \theta_2 \dot{x}_1 & \theta_2 x_1^{(2)} & \theta_2 x_1^{(3)} & \theta_2 x_1^{(4)} & \theta_2 x_1^{(5)} \end{bmatrix}^T \\ {}^2\xi_{\mathbf{x}} &= \begin{bmatrix} 0 & 0 & x_3 & 0 & {}^2\xi_{\boldsymbol{\theta}}^T \end{bmatrix}^T, \quad {}^2\xi_{\mathbf{w}} = \theta_2 x_1 \\ {}^2\xi_{\boldsymbol{\theta}} &= \begin{bmatrix} 0 & \theta_2 & \theta_3 & 0 & 0 & 0 & -\theta_3 & -\theta_8 & 0 \end{bmatrix}^T \end{aligned} \quad (81)$$

where ${}^1\xi_{\mathbf{x}}$, ${}^1\xi_{\mathbf{w}}$, ${}^2\xi_{\mathbf{x}}$ and ${}^2\xi_{\mathbf{w}}$ remain invariant for larger order k . Due to the complicated expression of ${}^1\xi_{\mathbf{x}^k}$, solving Eq. (27) for $i = 1$ analytically is mathematically difficult, and thus Algorithm II is employed to compute the

705 power series solution of $\{{}^1\phi_{\mathbf{x}}, {}^1\phi_{\mathbf{w}}\}$. As a result,

$$\begin{aligned} {}^1\phi_{\mathbf{x}} &= \begin{bmatrix} x_1 \\ x_2 + x_2\epsilon_1 + \frac{x_2}{2}\epsilon_1^2 \\ x_3 - \frac{\theta_1 x_3}{\theta_2}\epsilon_1 - \frac{\theta_1 x_3}{2\theta_2}\epsilon_1^2 \\ x_4 + x_4\epsilon_1 + \frac{x_4}{2}\epsilon_1^2 \\ {}^1\phi_{\boldsymbol{\theta}} \end{bmatrix} + O(\epsilon_1^3), \quad {}^1\phi_{\boldsymbol{\theta}} = \begin{bmatrix} \theta_1 + \theta_1\epsilon_1 + \frac{\theta_1}{2}\epsilon_1^2 \\ \theta_2 - \theta_1\epsilon_1 - \frac{\theta_1}{2}\epsilon_1^2 \\ \theta_3 - \frac{(\theta_1+\theta_2)\theta_3}{\theta_2}\epsilon_1 + \frac{(\theta_1+\theta_2)\theta_3}{2\theta_2}\epsilon_1^2 \\ \theta_4 \\ \theta_5 \\ \theta_6 \\ \theta_7 + \frac{(\theta_1+\theta_2)\theta_3}{\theta_2}\epsilon_1 - \frac{(\theta_1+\theta_2)\theta_3}{2\theta_2}\epsilon_1^2 \\ \theta_8 + \frac{\theta_1\theta_8}{\theta_2}\epsilon_1 + \frac{\theta_1\theta_8(2\theta_1+\theta_2)}{2\theta_2^2}\epsilon_1^2 \\ \theta_9 - \theta_9\epsilon_1 + \frac{\theta_9}{2}\epsilon_1^2 \end{bmatrix} \\ {}^1\phi_{\mathbf{w}} &= w \end{aligned} \quad (82)$$

where for the sake of brevity, the power series is calculated only up to the second order. Further exploring higher order power series can provide the possibility for one to deduce the closed form of $\{{}^1\phi_{\mathbf{x}}, {}^1\phi_{\mathbf{w}}\}$ asymptotically. On the other hand, analytic solution of $\{{}^2\phi_{\mathbf{x}}, {}^2\phi_{\mathbf{w}}\}$ can be obtained by solving Eq. (27) for

710 $i = 2$:

$$\begin{aligned} {}^2\phi_{\mathbf{x}} &= \begin{bmatrix} x_1 & x_2 & e^{\epsilon_2}x_3 & x_4 & \theta_1 & e^{\epsilon_2}\theta_2 & e^{\epsilon_2}\theta_3 & \theta_4 & \theta_5 & \theta_6 & (1 - e^{\epsilon_2})\theta_3 + \theta_7 & e^{-\epsilon_2}\theta_8 & \theta_9 \end{bmatrix}^T \\ {}^2\phi_w &= (e^{\epsilon_2} - 1)\theta_2x_1 + w \end{aligned} \quad (83)$$

Similarly as in Example 2, the second computational method is not able to apply since the Lie symmetries of the model do not belong to any common types. Combining the two one-parameter Lie groups of symmetries, the result of two-parameter Lie group of symmetries shows that, $x_2, x_3, x_4, \theta_1, \theta_2, \theta_3, \theta_7,$
715 θ_8 and θ_9 are unobservable states, w is an unobservable input, and x_1, θ_4, θ_5 and θ_6 are observable states for the given output measurement. In this example, additional sensor placement could bring the chance to improve the observability properties of the model. Following Section 5.2.1, if the newly introduced sensors measure any of the following combinations of dynamic states or unknown input:

$$720 \quad \mathbf{y}_{new} = \begin{bmatrix} x_2 \\ x_3 \end{bmatrix} \text{ or } \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \text{ or } \begin{bmatrix} x_2 \\ w \end{bmatrix} \text{ or } \begin{bmatrix} x_3 \\ w \end{bmatrix} \text{ or } \begin{bmatrix} x_4 \\ w \end{bmatrix} \quad (84)$$

the both groups $\{{}^1\phi_{\mathbf{x}}, {}^1\phi_w\}$ and $\{{}^2\phi_{\mathbf{x}}, {}^2\phi_w\}$ would be destroyed and the original unobservable states and unknown input would become all observable.

7. Conclusions

This paper establishes a general framework to compute the admitted groups
725 of Lie symmetries of nonlinear systems with unmeasured and unknown inputs. The computation is based on the kernel of a symbolic Jacobian matrix which consists of the Lie derivatives of an augmented form of the considered system. As a sequel to finding the basis of kernel, the differential equation based on First Fundamental Theorem of Lie is solved leading to analytic or power series
730 solutions of Lie symmetries. Symbolic computation of the Jacobian matrix and its kernel is expensive and requires high physical memory when used for large and complicated systems, and hence a second computationally efficient method is proposed which is complementary to the general method. The second method

assumes commonly-encountered types of Lie symmetries for the considered system, and a framework is derived with the aim of determining the unknown coefficients involved in the expressions of the assumed Lie symmetries. The limitation associated with the second method is however that it may only be capable of calculating partial groups of Lie symmetries and it is not able to find more general Lie symmetries whose forms cannot be foreseen; therefore the use of the general method is unavoidable to ensure the completeness of calculated products.

As the usage of Lie symmetries, the observability and identifiability properties of the system of interest can be detected based on the relationship to the combined multi-parameter Lie group of symmetries. If the set or any subset of the dynamic states, unknown parameters and unmeasured inputs is unobservable, the unobservable model of the system can be further repaired by destroying the calculated one-parameter Lie groups of symmetries. To achieve this, this paper introduces three useful strategies and their mathematical conditions: 1) placing additional sensors to obtain new output measurements, 2) combining or transforming the states and unknown inputs to reduce the dimension of model, and 3) applying assumptions or constraints to unidentifiable model parameters. The introduced strategies can be used in combinations to destroy the groups of Lie symmetries as many as possible; if optimally all of them are destroyed, i.e. the Lie symmetries no longer satisfy the state and output equations of the system, the system becomes completely observable.

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