

Bistable and Multi-stable Thin-Walled Structures



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This thesis is gratefully dedicated to

my parents

for their enduring love and unconditional support

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Abstract

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This study aims to comprehend the bistable and multi-stable behaviour of flexible straws with the intention of utilising it for future engineering applications. This behaviour is achieved by the multiple inversions of conical frustum shells within the corrugation of a flexible straw. This study examined the effects of various material models, geometry variables and loading methods on the inversion of close-top and open-top conical frustum shells via experiments and FEM simulations. This thesis consists of three main parts, and the second and the third parts are complementary to each other:

First, we investigated the effects of applying a uniform vertical load to the upper rim of open-top frustum shells via FEM simulations. A reference model was simulated based on the measurements of an ordinary polypropylene flexible straw specimen, using two material models - linear elastic and elastically perfectly plastic. The effects of the interactions between frusta of the corrugated segment of a flexible straw were also studied by evaluating the difference in responses between an individual frustum and conjugated models of two or three frusta. It was found that by constraining the rotation of its bottom rim, an individual frustum can fairly reproduce the complex bistable behaviour of the shorter frustum within the corrugated part of a flexible straw. Furthermore, detailed parametric studies that focused on the effects of various geometric parameters were conducted and generalised formulas that predicted the critical force were derived. A comparison between the simulated results and the analytical model in predicting progressive inversion was made to distinguish the geometric boundaries that separate the one-off snap-through to the progressive inversion of frustum shells.

Next, the behaviour of close-top frustum shells in response to vertical point loading at various locations on the top surface was evaluated. A hyperelastic material was used to fabricate the physical specimens. During the experiments, the corresponding deformed shapes were recorded by 3D scanning in addition to measurements of the displacement and reaction force. We observed a close resemblance between the experimental and FEM simulated results, which validated the FEM models. Two local peaks were observed before the structure was fully inverted into its secondary stable state and the overall critical force of the structure was defined by the higher one of the two. The relationship between their magnitudes and the loading locations was analysed and an optimal loading location which gave the minimum critical force was proposed and verified by additional simulations. Furthermore, generalised formulas in predicting critical force were also acquired based on parametric studies. The optimal loading location was found to be constant in spite of variations in height and thickness.

The third part of this thesis discussed the effects of lateral point loading on both close-top and open-top frustum shells at various locations on the side surface and supplemented the second part. It is found that the removal of the top surface could cause the critical force to decrease if a point load was applied laterally. Moreover, we were able to fully invert the structure with a lower critical force through lateral loading in comparison to vertical loading.

Keywords: frustum shell, bistable, critical force, linear elastic, elastically perfectly plastic, hyperelastic, 3D scanning, FEM

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Nomenclature

a, b	Constants associating with the power of various parameters.
α	Angle variable between the rigid rod and horizontal plane.
α_{ini}	Initial angle between the rigid rod and horizontal plane.
c	Constant associating with the critical force.
C_1, C_2	Constants associating with the reaction force and downward displacement.
$C_{ij}, C_{10}, C_{01}, C_{20}, C_{11}, C_{02}$	Constants of a hyperelastic material in the strain energy equation, describing shear behaviour.
D	Displacement of loading.
D_i, D_1, D_2	Constants of a hyperelastic material in the strain energy equation, describing compressibility.
D_r	Distance travelled of rotational displacement.
D^*	Dimensionless displacement of loading.
E	Young's Modulus of a material.
E_0	Initial elastic Modulus of a hyperelastic material.
ϵ	Strain of a material.
ϵ_y	Yield strain of a material.
F	Horizontal force in a linear spring.
h	Height of a frustum shell.
$h_{knuckle}$	Height of the knuckle region.
h_s	Slant height of a frustum.
$\bar{I}_1, \bar{I}_2$	First and second invariants of the deviatoric strain in the strain energy equation.
IE	Total Internal energy.
J_{el}	Elastic volume ratio.

k	Stiffness of a horizontal linear spring.
L	Length of the rigid rods.
l	Width of the knuckle region.
M_0	Plastic value of the longitudinal bending moment.
M_ϕ	Longitudinal bending moment.
N	Constants of a hyperelastic material in the strain energy equation.
N_0	Plastic value of the stress resultant.
ν	Poisson ratio of a material.
N_ϕ	longitudinal stress.
N_θ	Hoop stress.
ω	Slant angle of a frustum shell.
P	Loading force.
P_{cr}, P_{cr1}, P_{cr2}	Critical force at limit point or local peak.
P^*	Dimensionless loading force.
$d\phi$	Infinitesimal angle at the knuckle region.
R	Global radius of a structure.
r	Distance between the centre of the knuckle region and the centroid line.
R_b	Larger radius of a frustum.
ρ	Density of a material.
R_t	Smaller radius of a frustum.
SE	Recoverable elastic strain energy.
σ	Stress of a material.
σ_y, Y	Yield stress of a material.
t	Thickness.
θ	Angular displacement.
U	Strain energy of a hyperelastic material.
u	Downward displacement.
H	Horizontal component of the tensile stress.
V	Vertical component of the tensile stress.
$\zeta_{knuckle}$	Elevation of the centroid above the plane of the hinge circles.

Chapter 1

Introduction

1.1 Background

Fascinating structural behaviour can be found on commonly-used objects, such as normal utensils. A drinking straw is a thin-walled tube that a drinker uses to suck a beverage from its container. Flexible straws or bendy straws showed in Figure 1.1 contain a corrugated portion that allows a kink to form and provide flexibility. It was first invented by Joseph B. Friedman in the 1930s (Wikipedia, 2015). The corrugated portion which gives the flexible straw its unique bend-and-lock behaviour is enlarged in Figure 1.2.

The flexible straw has many interesting structural features. Firstly, the corrugation is invertible under axial load. Figure 1.2a shows the initial state of the corrugation - the straw snaps at each corrugation when subjected to a longitudinal compressive force, shortening in length to a new stable configuration, Figure 1.2b. Once the direction of the force is reversed, the straw returns to its original state. The sudden change from its original shape to the current one, when subjected to a certain critical load, is known as the snap-through - which will be explained later. Secondly, if each of the corrugations is partially inverted, the inversion can easily propagate circumferentially,



Figure 1.1: Examples of typical flexible straws.

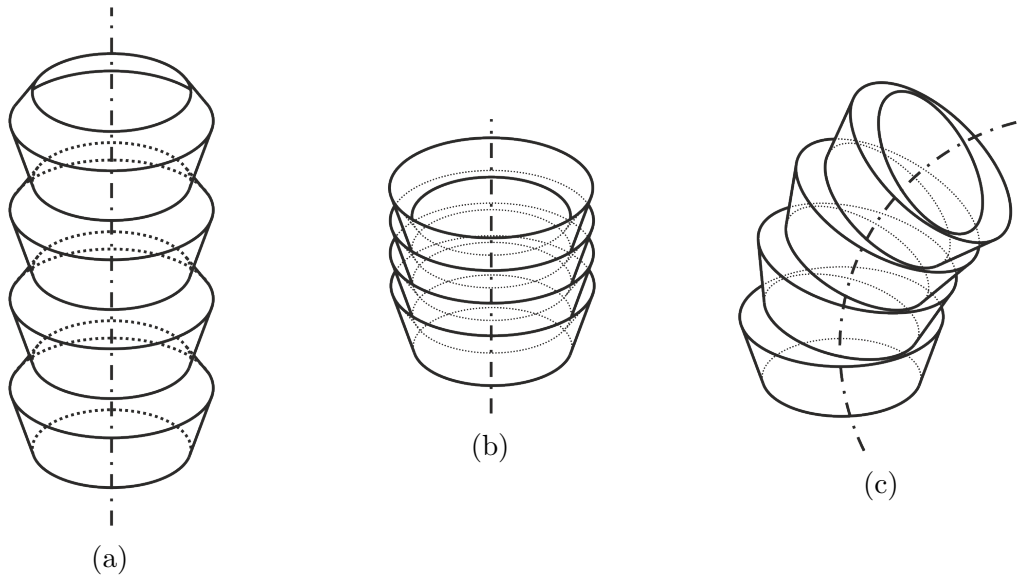


Figure 1.2: Three stable states of the corrugated portion of a flexible straw. (a) open state, (b) retracted state, and (c) bent state.



Figure 1.3: A click-clack tin with its bistable lid.



Figure 1.4: A typical example of slap bracelet, (a) unfolded and (b) folded.

Figure 1.2c, enabling the transmission of rotational force from one end of the straw to the other when the torque is rather small. All of these partially inverted shapes are stable.

The flexible straw belongs to a family of morphing shell structures. These structures adopt at least two stable positions. In the absence of large deflections, they are rigid in one configuration but can undergo substantial changes in shape to adopt another stable configuration. Other examples include the click-clack tin lid in Figure 1.3, and slap bracelet in Figure 1.4.

These multi-stable shell structures not only bring convenience to everyday life but also have tremendous potential in forming the basis for reconfigurable systems. For instance, the multi-stable structures similar to a slap bracelet have been used in aerospace systems as discrete actuators because of their multiple stable states that are absent in commonly seen continuous actuation. The discrete actuator makes latching

device redundant and therefore reduces the complexity of the system. Such features are especially beneficial in designing space antenna and boom, of which packing efficiency is crucial. Promising designs using embedded bistable tape-springs hinges (Seffen, Pellegrino, and Parks, 2000; Murphey et al., 2010) have also demonstrated the creative application of free deployment and lock-in features.

1.2 Aim, Scope, and Layout

The conical frustum shell embodies great potential in its usefulness in engineering applications but has yet to be systematically studied. Interesting features of solo or connected frustum shells - including states of bistability and multi-stability - could be used to optimise current engineering design. This study aims to comprehend the structural behaviour of the flexible straw with the intention of utilising it for future engineering applications, such as in actuators or rotational transmission devices. The focus is on computationally and experimentally modelling the bistable thin-walled conical frustum shells, in order to, firstly, understand the governing design parameters that lead to its bistability, and secondly to provide a foundation for informing the correct design of such structures.

This thesis consists of four chapters. Chapter 2 provides an encompassing review of related previous work, with foci in five areas, bistable and multi-stable structures, buckling of shells, elastic and plastic post-buckling of shells, and the numerical modelling techniques. In the first area, the definition of stability, a simple snap-through model, and bistable shell structures are reviewed. In the second and third areas, an emphasis is placed on the behaviour of spherical shells due to their similarities in geometry with conical frustum shells. In the last part, the reliability of FEM in solving multi-stable shell problems is examined, and potential solvers and methods of FEM that are suitable for this research are reviewed and discussed.

Chapter 3 examines open-top frustum shells under vertical uniform loading on the top rim. Firstly, the geometric configuration is defined, and dimensions are obtained from a flexible straw specimen and used as a reference model. Moreover, detailed FEM settings, including material models and loading methods, are given, and the simulations are verified by performing independent FEM analyses. This is followed by an investigation to evaluate the differences in responses of a singular frustum and such singular frustum in a real corrugated connection. A detailed parametric study focusing on the height, the thickness and the slant angle is conducted and presented after the latter investigation. Furthermore, a comparison between the simulated results and the modified travelling hinge model (Calladine, 1986; Deshpande and Fleck, 2003) is carried out and discussed.

Chapter 4 investigates of the response of vertical point loading on close-top frustum shells. The methods of experimental studies are described in details with respect to materials properties, specimen fabrication, and apparatus. Subsequently, equivalent FEM models are constructed and presented, thereafter, the latter findings are validated with experimental results. This is followed by detailed parametric studies that examine the height and the thickness of frustum shells.

Chapter 5 covers two cases of both open-top and close-top frustum shells under lateral point loading in individual sections. Experimental methods concerning materials properties, specimen fabrication, and apparatus used are first described in detail. Subsequently, equivalent FEM models are constructed and presented. Experimental results are then compared with FEM simulated results to validate the method. Detailed parametric studies focusing on the height and the thickness are further conducted and presented.

Chapter 6 summarises the main findings of this study, and expounds on the future directions and applications that are relevant to this work.

Chapter 2

Literature Review

2.1 Bistable and Multi-stable Structures

2.1.1 Structural Stability

Many have endeavoured to explain the concept of stability since the early days of human history, resulting in the proposal of various definitions of stability that aimed to explain this phenomenon. Aristotle defined stability as a property that is based on the motion of a perturbed system, while Archimedes considered it to be defined by the geometric state of the system after perturbation (Leipholz, 1970). This research follows the definition proposed by the theory of elastic stability of structure (Timoshenko and Young, 1951; Timoshenko and Gere, 2009; Simitses and Hutchinson, 1976; Ginsberg and Genin, 1984) - when a system experiences no acceleration (in other words, force-free), it may be said to be in a state of equilibrium. A system at its stable equilibrium state is insensitive to external ‘small’ disturbances. This means that, in a state of equilibrium, when a small external disturbance is applied to the structure, it reacts by simply performing oscillations about its equilibrium state, and the equilibrium is said to be stable (Simitses and Hutchinson, 1976). When a system is in an unstable state of equilibrium, ‘small’ disturbances cause the system to diverge from its initial state.

Additionally, there is a third state known as the neutral state of which the system reacts to the disturbances and stays in the disturbed position.

2.1.2 Snap-through Phenomenon

Snap-through is an important phenomenon that describes bistable behaviour. The inversion and snap-through behaviour of a structure is best illustrated by a 2D truss structure, shown in Figure 2.1 that consists of two identical rigid rods of length L connected with a pin joint at one end and supported by a pin joint and a roller with a horizontal linear spring of stiffness k at the other end (Thompson and Hunt, 1973). Initially, the rods are at angles α_{ini} to the horizontal, with a linear spring of stiffness k restraining axial movement of one end and fixed pin joint at the other end. As the load is applied at the middle joint, angle α starts decreasing from α_{ini} and the spring gets compressed as well. The force in the spring is

$$F = 2kL(\cos \alpha - \cos \alpha_{ini}). \quad (2.1)$$

From the force equilibrium in vertical direction, the downward force P can be written regarding α as

$$P = 4kL(\sin \alpha - \cos \alpha_{ini} \tan \alpha). \quad (2.2)$$

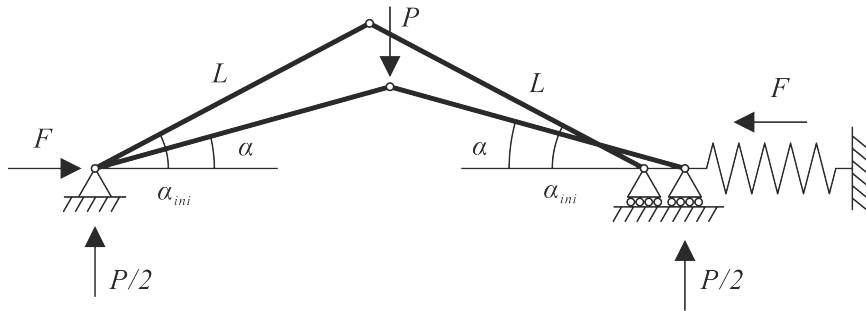


Figure 2.1: A basic 2D snap-through model.

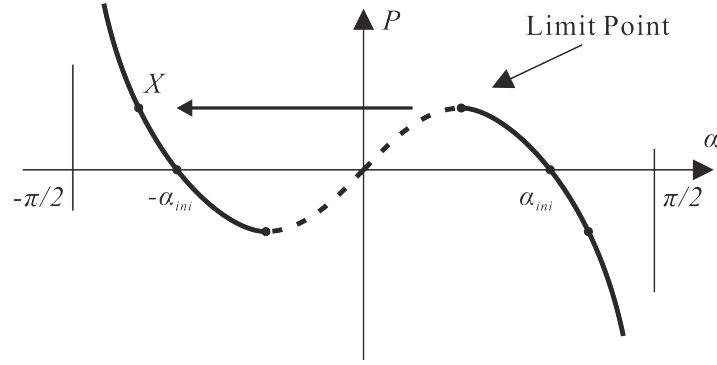


Figure 2.2: Plot of P vs. α in the snap-through model.

The graph of P vs. α is shown in Figure 2.2. Starting at $\alpha = \alpha_{ini}$, the loading path is stable until the ‘limit point’ is reached. Model snaps through to the point X . Although every point along the path (dashed line in Figure 2.2) satisfies force equilibrium, these equilibrium positions do not exist during an increase in force loading as the stability criteria cannot be met. The limit point is at $\frac{dP}{d\alpha} = 0$, so

$$P_{cr} = 4kL[1 - (\cos \alpha_{ini})^{2/3}]^{3/2}. \quad (2.3)$$

If the load is released at point X , the model does not go back to its original shape. It follows the path back to the point at $\alpha = -\alpha_{ini}$, which is the second stable state of the model.

2.1.3 Bistable Shell Structures

Bistable shell structures behave similarly to the truss shown in Figure 2.1. In general, the bistability and multi-stability of shells can be chiefly obtained by utilising nonlinear geometry, prestressing, employing composite material, or assertion of a combination of these three features.

In most planar bistable mechanisms, the point and line of action of the actuation to switch between stable states are the same. In other words, they are ‘single-port bistable mechanisms’ (Darshan et al., 2015). However, this is not the only possibility of

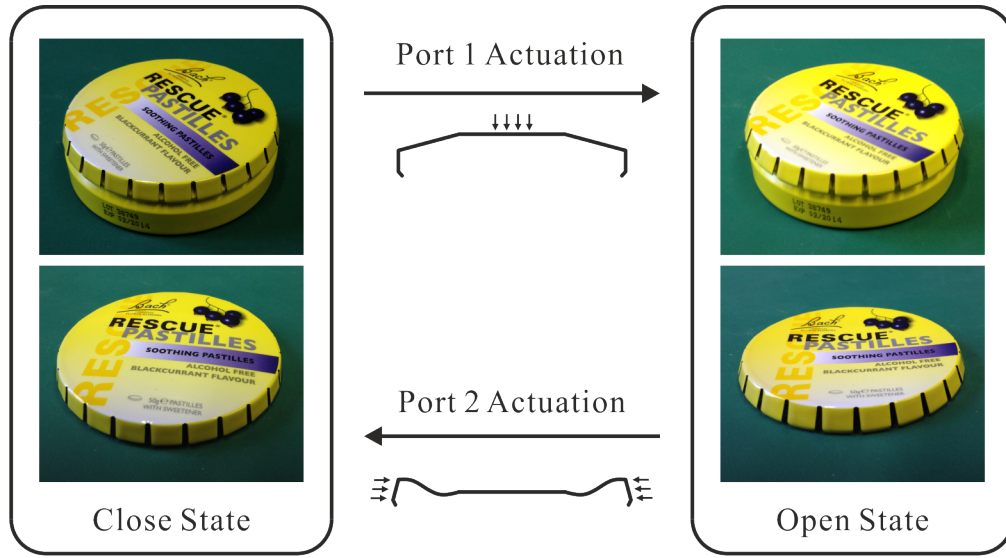


Figure 2.3: Mechanism of the bistable lid of the click-clack tin.

bistable mechanisms. The click-clack lid in Figure 1.3 clearly demonstrates a ‘two-port bistable mechanisms’ (Darshan et al., 2015). At its closed state, the lid is slightly bulged upwards (forming a shallow frustum geometry) and is secured to the bottom receptacle; by exerting a downwards compressive force on the centre of the lid allows it to be easily removed from its bottom receptacle; by exerting an inwards lateral compressive force around the skirt of the lid, the lid can in return be securely fastened to the bottom receptacle, allowing it to revert to its initial state (Figure 2.3). The force required for port 2 actuation is smaller than that for port 1 actuation. A mechanism such as this has an inherent mechanical advantage because the magnitude of the actuation force at one port is different from that at the other port. Darshan et al. (2015) utilised this feature in a planar fully-compliant two-port bistable mechanism and proposed a sit-to-stand easy chair design for older adults suffering arthritis, where a small force applied on the armrest can gently propel and allow the occupant to rise with ease. The ingenuity of this design lies in its two-port bistable mechanism which bypasses the need for electrical actuators (PHC, 2016) or complex hydraulic systems (Herman Miller, 2016), as often seen in traditional approaches.

The slap bracelet in Figure 1.4 is another example of bistable shell structures

which has been studied extensively. The slap bracelet was first invented by Stuart Anders, and soon became a major hit with children in the 1990s (Ramirez, 1990). It is a prestressed steel sheet coated with cover fabric, which is silicone rubber in this particular case. It has the following configurations that: in its unfolded state, it has a positive curvature along the short axis and zero curvature along the long axis; in its folded state, it retains a negative curvature along the long axis and zero curvature along the short axis. The transformation from the unfolded to folded state can be actuated by slapping the bracelet against a person's forearm, which allows it to be securely fastened around the forearm. Although the slap bracelet was invented in recent decades, structures of similar behaviour have been studied for a long time. The story begins with a simple measuring tape. In 1927, A. E. H. Love raised the question that why the longitudinal curvature of a spring-steel measuring tape appeared to exactly match its transverse curvature in his book *Classical Treatise on Elasticity*. An explanation was given by Rimrott (1965) only decades later, and it was subsequently reviewed and distilled into an elegant solution by Calladine (1988). Later, Iqbal and Pellegrino (2000), Iqbal, Pellegrino, and Daton-Lovett (2000), Galletly and Guest (2004b), Galletly and Guest (2004a), and Guest and Pellegrino (2006) simplified and clarified the elasticity of bistable cylindrical shells, unravelling the interplay between anisotropy and spontaneous curvature in determining the shape of bistable shells. Guest and Pellegrino (2006) used a beam model of a strip of infinite length and finite width endowed with constant spontaneous curvature along the transverse direction. Assuming inextensibility, they showed that orthotropic strips exhibited a secondary equilibrium configuration in addition to the base configuration. The stability of such a secondary minimum of the elastic strain energy depended on the relative magnitude of the bending stiffness in the longitudinal and transverse directions as well as the twisting stiffness of the strip. Furthermore, Seffen (2007) inspected the case of an extensible elliptical shell with free boundaries and non-zero, constant spontaneous

curvature and demonstrated that isotropic shells could also exhibit bistable behaviour. Moreover, other behaviours of curved elastic strips in the domain of multi-stability were discovered. Seffen’s uniform curvature model was extended by Vidoli and Maurini (2008) to show that two-dimensional orthotropic shells with initial shallow double curvature could be tristable in some range of elastic moduli and spontaneous curvatures. Seffen and Guest (2011) reported a one-parameter family of continuously variable neutrally stable shapes in the context of prestressed shells. In terms of applications, this feature has been directly implemented in novel watch wristbands design, granted patents such as (Keung, Fai, and Hong, 2013; Swartzentruber, 2003); wearable flexible screen granted patent such as (Rothkopf, Wright, and Myers, 2015) from Apple Inc.

In addition to the aforementioned structures, multi-stability has been found in many other forms of shell structures. Silverberg et al. (2015) showed that the unfoldable square twist origami structure, which had zero degrees of freedom, can nevertheless be folded by accessing bending deformation that is not explicit in the crease pattern, thus demonstrating a second stable state that is separated by an energy gap. Waitukaitis et al. (2015) found that a degree-four vertex origami could demonstrate multi-stability if torsional spring effects were considered in the folds. Moreover, Golabchi and Guest (2009) discovered two kinds of shells, doubly textured with dimples, that could have either anticlastic or synclastic bistable behaviour, depending on the direction of the plastic rolling process.

For a symmetric thin-walled shell structure, the stable state switch is often accompanied by an inversion process. Generally, it starts with a gradual deformation, followed by a snap-through from the original shape to a new one under a certain critical load. Once the load is released after the snap-through, the structure stays in its new shape, and thus a second state of stability is achieved; equivalently, it may revert when a backwards critical force is applied. Take a ping pong ball as an example. Pauchard and Rica (1998) found that a partially inverted shape would be found on a

Ping-Pong ball under a critical localised normal load. It was more or less reversible by applying an appropriate load. It has been found that the behaviour of the thin-walled shells described above not only depends on the geometry of the structure, but also on the ductility of the materials, e.g. it is impossible to push an egg, which is geometrically similar to a ping pong ball, into an inverted shape without fracturing it.

2.2 Buckling of Shells

Bushnell (1989) generalised the buckling behaviour of a conical shell into a conceptual description, which is shown in Figure 2.4. For some shells configurations and loadings, the loading may progress along a fundamental path OA. In this path, the deformed shape of the structure remains essentially the same along this path OA until point A, where the maximum load is reached. The corresponding buckling load is denoted by P_L . The path OA is called the primary equilibrium path. Normally for this behaviour, the ‘failed’ deflected shape differs from the initially deflected shape only in magnitude. Limit point buckling mode is usually related to as a snap-through phenomenon.

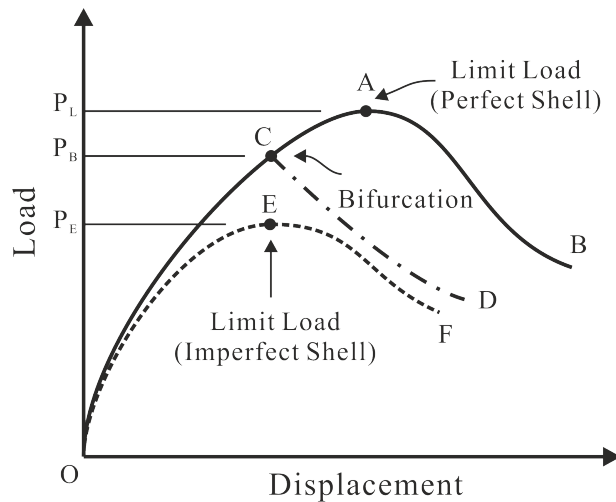


Figure 2.4: Possible load-displacement behaviour of a shell. Reproduced graph from Challa (1995).

On the other hand, for some other shell configurations and loadings, an alternative

equilibrium path is available at point C, known as the bifurcation point. After the bifurcation point, two equilibrium shapes are possible: paths CD and CA. At this point, the shell will follow the path that requires the least energy, which is path CD in this case. This behaviour is called the bifurcation buckling, the maximum load is called the bifurcation buckling load P_B , and path CD is called a secondary path. Deformations along this path differ completely from the prebifurcation path. Usually, bifurcation occurs when the deformed shape jumps from axisymmetric to asymmetric. To be clear, it is not necessary that bifurcation buckling happens only before limit point buckling, and in some unusual cases, shell structures could have two bucklings, i.e. limit point buckling happens before bifurcation buckling (Sobel and Newman, 1980) due to their configurations and loadings.

The two paths discussed previously assume that the shell is perfect, however, most real structures are imperfect, e.g., the geometry could be slightly off the proposed design, the material used in its construction could be non-homogeneous, or there would be residual stress resulting from the fabrication methods. Imperfections could dramatically reduce the buckling load. In Figure 2.4, path OEF indicates a possible loading path of an imperfect shell. The failure corresponds to the collapse load, P_E , at point E. The deformation mode at point E may include both symmetric and asymmetric components. The deformed shape at maximum load can be similar to the prebuckled shape, the perfect bifurcation buckled shape, the dominant imperfection mode, or a combination of the three.

2.2.1 Spherical Shells

The buckling of spherical shells under uniform external pressure has been investigated by many researchers. Zoelly (1915) and van der Neut (1932) used the classic theory of small deflection and solution of linear differential equations (Timoshenko and Gere, 2009), to find that the axisymmetric elastic buckling pressure p_{cr} for complete, thin and perfect spherical shells was

$$p_{cr} = \frac{2E}{\sqrt{3(1-\nu^2)}} \left(\frac{t}{R}\right)^2, \quad (2.4)$$

where t is the thickness of the spherical shell, and R is the radius of the spherical shell.

At the time they developed the Equation 2.4, no systematic experimental work was done. As more test results became available afterwards, it was shown a large discrepancy between the Zoelly-van der Neut formula's prediction and the experimental results. In fact, the value derived by means of Equation 2.4 was only applicable to a perfect shell, hence was the upper limit of the classical elastic buckling.

Metals, such as steel and aluminium alloy, were the main focus of the research due to their wider applications. Therefore, a typical Poisson's ratio ν of 0.3 was used in these studies. Karman (1941) developed a formula of lower elastic buckling load limit for such material,

$$p_{cr} = 0.36E/m^2, \quad (2.5)$$

where $m = R/t$. This formula was derived by making a number of necessary assumptions considering the effect of curvature and spherical shape imperfections, thought to be responsible for the discrepancies between the experimental results and the classic theory of elasticity predictions. This estimation was more favourable with experimental test results. In real design, Equation 2.5 is often recommended, while Equation 2.4 is only recommended if extreme manufacturing and experimental precautions were taken. The exact calculation of the load-deflection curve for a spherical segment subjected

to uniform external pressure is known to involve nonlinear terms in the equations of equilibrium, which causes substantial mathematical difficulties.

For a fully-clamped shallow spherical shell, an equation was given by Flügge (1962) as

$$p_{cr} = 0.075E(R_b/t)^{-4}\lambda_0^{4.15}e^{-0.095\lambda_0}, \quad (2.6)$$

where $\lambda_0 = 2.57(h/t)^{1/2}$ is a key parameter characterising a spherical cap, and R_b is the radius of the circle footprint of the cap. The above equation works when $R_b/t > 2.08$, otherwise, the structure would demonstrate a continuous deformation with buckling instead of an axisymmetric snap-through.

2.2.2 Close-Top Frustum Shells

Theoretical analysis (Lackman and Penzien, 1960) indicates that the critical axial load for tall close-top frustum shells, based on modifications of cylindrical shell model, can be expressed as

$$P_{cr} = \gamma \frac{2\pi Et^2 \sin^2 \omega}{\sqrt{3(1-\nu^2)}}, \quad (2.7)$$

where ω is the slant angle of the frustum, t is the thickness of the frustum, and γ is the correction factor. Lackman and Penzien (1960) gave a formula of calculation γ as

$$\gamma = \left(\frac{\sin \omega}{R_t/t + \cot \omega \cdot h/t} \right)^{0.66}, \quad (2.8)$$

where R_t is the top radius of the frustum, and h is the height of the frustum.

Equation 2.7 assumes that the buckling is happening at the side surface of the frustum asymmetrically, which is true for a tall close-top frustum vessel under uniform external pressure. However, it has been found that there is considerable disagreement between experimental loads and the loads predicted by theory. These discrepancies have been attributed to the effects of imperfections of the structure and to the edge support

conditions different from those assumed in the analysis as well as to shortcomings of the small-deflection theory used. This prediction can act as a upper bound solution to the problem.

2.3 Elastic Post-buckling of Spherical Shell

Brodland and Cohen (1987) investigated the axisymmetric deflection and snapping of spherical rubber caps, which were point loaded at the apex and simply supported at the boundary. In their research, the axisymmetric deformation of the point loaded spherical shell was characterised by two non-dimensional geometric parameters. The first of these was the shell half angle ζ_0 (For a hemispherical shell, $\zeta_0 = \pi/2$). The other was a parameter λ defined by

$$\lambda^2 \cong (2h/t)\sqrt{12(1-\nu^2)}, \quad (2.9)$$

where h is the height of the shell, t is the thickness of the shell, and ν is the Poisson's ratio of the material. Note that the approximation of λ^2 in Equation 2.9 is only valid for shallow shells, thus ζ_0 is small. The definition of 'shallow' in their research occurs when $\zeta_0 \cong h/R$, where R is the radius of the circular footprint. The parameter λ^2 is essentially a measure of shell thickness, and large λ^2 values are associated with thin shells. To exhibit axisymmetric deformation, a spherical cap has to satisfy $\lambda^2 \leq 100$ according to an experimental research (Bushnell, 1967).

A schematic graph, Figure 2.5, was given by Brodland and Cohen (1987) to illustrate the displacement-controlled response of a typical shallow spherical cap with $\zeta_0 = 22.5^\circ$ and $\lambda^2 = 0$. The spherical cap is initially at state B. State A indicates a cap under a small negative load from the initial state. In a displacement controlled loading, the load response follows the path BCD until the limit point D is reached followed by a decrease in load to zero at state E. Afterwards, the shell then snaps from state E to F, which is the behaviour known as the snap-back. Note that state F has different geometric configuration compared with state E, Figure 2.5b. As displacement continues to increase, the load response follows the path FGHJ and the structure becomes fully everted (or inverted) and load-free state at state G. Additional positive load is required

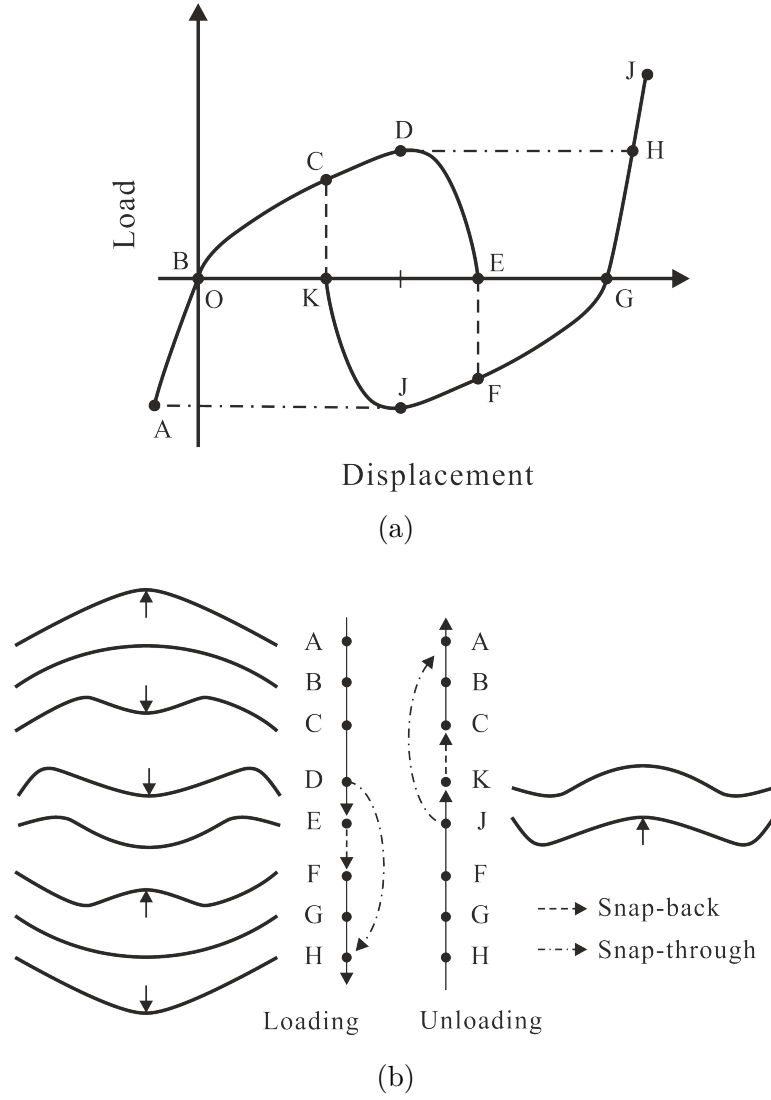


Figure 2.5: Load-displacement response of a spherical shell. (a) Load-displacement response, and (b) corresponding deformed shape in cross-section view. Modified graph based on Brodland and Cohen (1987).

to take the shell from state G to J. On the other hand, the load response follows a slightly different path in a load controlled loading. It follows the same path BCD until the limit point D is reached and then the shell snaps from state D to H, which is the classical snap-through behaviour.

In the event of unloading, the shell return from state J to its initial state B in a similar manner as loading, Figure 2.5b. In both snap-through and snap-back, some of the strain energy stored in the structure is converted to kinematic energy, which is dissipated due to damping effect of the material.

The mechanical response of a spherical cap is due to a combination of membrane effects and bending effects. Brodland and Cohen (1987) suggested that membrane effects are characterised by symmetric in load-deflection behaviour while bending effects are characterised by monotonic load-deflection behaviour. In the absence of any bending, it is reasonable to assume that a hinged spherical cap would have no residual stress in its everted shape. Hence, the load-deflection curve response of the everted cap should be identical to that of the uneverted cap, therefore a symmetry could be found in the response curve. The example in Figure 2.5 is classified as a thin shell, hence bending stresses are small and their effect is negligible. This is verified by the almost symmetric shape of the load-deflection response curve. As λ^2 decreases, bending effects start to take place so that the load-deflection response becomes monotonic: the region before the limit point starts to become less steep and progressive inversion happens, and eventually, a force-free everted state disappears if the shell is sufficiently thick. The self-equilibrating everted state is not possible in thicker caps because bending effects are too large to be overcome by membrane forces. Therefore, the edges of the cap have a tendency to roll back and restore the shell to its initial state. The membrane hoop effects, which produce inward radial forces, are to prevent bending effects from rolling the edges down and returning the cap to its initial configuration.

Madhukar et al. (2014) investigated the bistability of thick-walled axisymmetric

domes. Three categories of dome characteristics are identified: monostable, bistable, and pseudo-bistable. Monostable and bistable are self-explanatory while pseudo-bistable is a characteristic that has a second stable state but the dome reverts to its undeformed state through a swift snap-back without perturbation after a short time (around 1 to 10 s). This is because of the visco-elastic nature of the material which was used to fabricate the specimens in the paper. The pseudo-bistable characteristic of domes was studied by Brinkmeyer et al. (2012) focusing on the recovery time.

Studies presented previously discussed the axisymmetric deflection of spherical shells under axisymmetric loading. However, it has been found, in some cases, the symmetry breaks along the ridge in the form of vertices, and each vertex is linked to another by approximately straight ridges (Vaziri, 2009). It is also noted that there are several fundamental differences between the responses of spherical shells under rigid plates and point-like indentations. Under point-like indentation, the initial post-buckling shape of the spherical shell is usually manifest in the form of three vertices, except for very shallow shells ($h/t < \sim 30$), where patterns comprising of four or five vertices appear. In contrast, when a spherical shell is subjected to flat rigid plate indentation, the initial post-buckling deformation mode typically has four or five vertices. The post-buckling behaviour of the spherical cap under the two indentation conditions is also completely different. Under flat indentation, the initial asymmetric deformation mode persists over a large range of indentation. However, under point indentation, after the appearance of the asymmetric deformation mode, further increases in indentation make the system unstable and one of the vertices bifurcates into two new vertices that move away from each other. The formation of these new vertices in the shell is in general accompanied by a sudden reduction in load (Vaziri and Mahadevan, 2008).

2.4 Plastic Post-buckling of Shells

and Travelling Hinge Model

The post-buckling of perfectly plastic shells has been investigated by many researchers. An analytical model known as the *Travelling Hinge Model* was developed to describe the inversion behaviour of shells after initial buckling. In this section, this model and a variation of it will be explained and reviewed.

2.4.1 Progressive Inversion of Spherical Shells

Calladine (1986) summarised the inversion behaviour of several shell structures with large plastic deformations, which included descriptions of the spherical shell. Figure 2.6 shows the cross-section of a spherical shell with radius R , thickness t , and a downward point load P . Under the critical load, the spherical shell transforms from its original shape to a partially inverted spherical shape. A ‘knuckle’ region is formed during the snap-through and separates an inverted region from the outer portion of the sphere. After that, as the deformation proceeds, the knuckle moves outwards and the material from the outer portion enters the inverted region through the knuckle, which is connected to the two spherical portions by two travelling circumferential plastic hinge circles of radius $r \pm \frac{1}{2}l$.

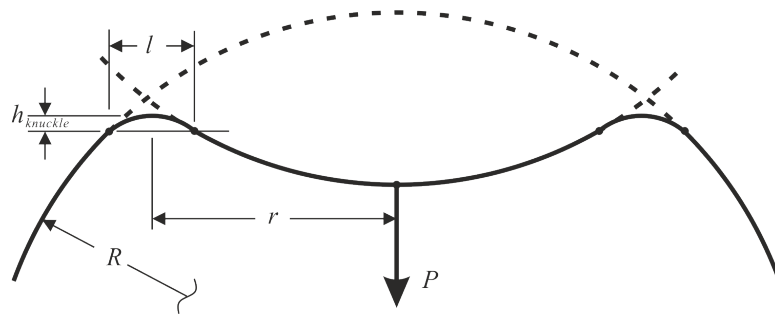


Figure 2.6: Schematic cross-section of a spherical shell with thickness t

Wasti (1964) first investigated the experimental and theoretical aspects of this problem. Subsequently, it has been interpreted by Leckie and Penny (1968), Morris and Calladine (1969), Updike (1972) and Oliveira and Wierzbicki (1982) in a variety of ways.

A sophisticated model, known as the travelling hinge model, was developed to describe the behaviour after the snap-through behaviour of an elastic-perfectly plastic material. If we allow the resultant hoop stress N_θ and longitudinal bending moment M_ϕ to describe the yield of the shell element, we arrive at the following crude circumscribing yield criteria:

$$|N_\theta| = N_o \equiv Yt, \quad (2.10)$$

$$|M_\phi| = M_o \equiv \frac{Yt^2}{4}, \quad (2.11)$$

where N_o and M_o are the plastic values of the stress resultant and the bending moment, respectively, and Y is the yield stress of the material. At yield, one or both of these equations are satisfied. A more precise combined loading yield criteria is provided by Onat (1955) in a curved, convex interaction diagram. Calladine (1986) also assumed that the longitudinal stress N_ϕ has insignificant effects on the yielding.

The meridional bending moment in the knuckle region changes between $+M_o$ at the outer plastic hinge and $-M_o$ at the inner hinge. Furthermore, it rotates between the two hinge-circles as a rigid body in the plane of the meridian because of the continuity of the longitudinal bending moment M_ϕ . Calladine (1968) assumed that the invasion and withdrawal of the hinge-circles occurred in the same plane.

It is also assumed that the toroidal knuckle has a uniform curvature. The whole toroidal knuckle rotates instantaneously as a rigid body about the outer hinge line. Every elemental hoop of the knuckle moves inwards and is displaced by a neighbour inward radial component during the inversion. Therefore, the knuckle region undergoes compressive hoop straining and the hoop stress within the knuckle region attains the

compressive yield strength.

It is assumed that $l/r \ll 1$, thus the slope of the meridian at the ends of the knuckle can be taken as $\pm r/R$. By regarding the profile of the knuckle as parabolic, its height is given as

$$h_{knuckle} = rl/4R. \quad (2.12)$$

Figure 2.7a shows the forces and stress resultants acting on the central inverted shell.

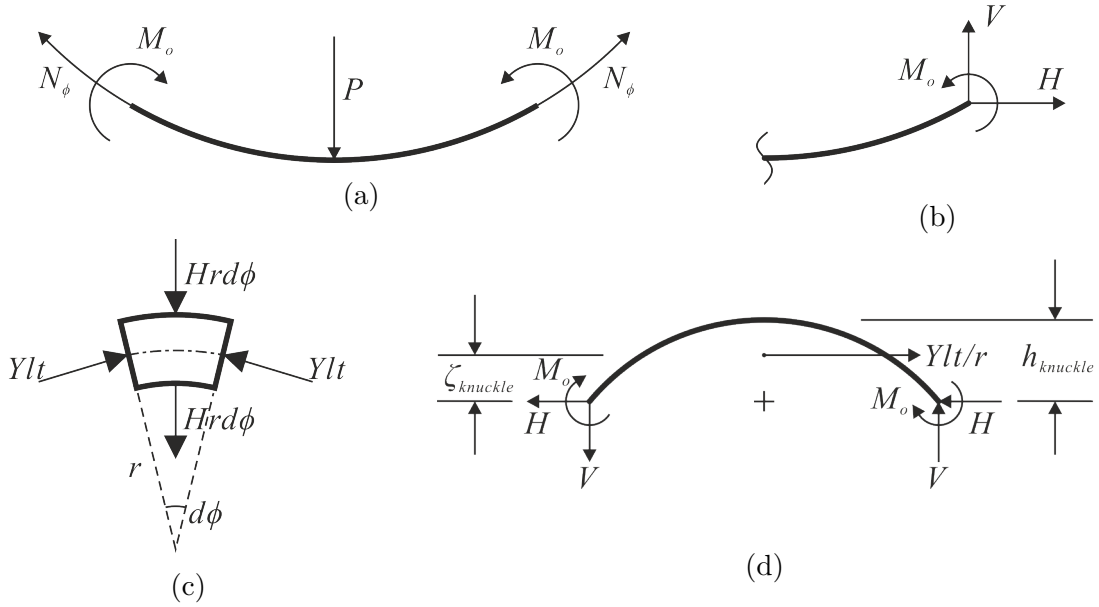


Figure 2.7: Analysis model of a spherical shell. (a) Free body diagram showing the cross-section of the inverted central region, (b) free body diagram showing the cross-section of the inverted central region by replacing the tensile stress with a vertical and a horizontal component, V and H , (c) free body diagram of the knuckle with a unit circumference at radius r , and (d) free body diagram showing the plan view of a circumferential segment of the knuckle subtending a polar angle $d\phi$ at the centre.

A bending moment $M_\phi = M_o$ and a tensile stress N_ϕ are acting at the outer edge of the central portion.

The direction of the tensile stress must be tangential to the meridian; otherwise, N_ϕ would not reach its maximum at the hinge-circle. We replace the tensile stress with a vertical and a horizontal component, V and H respectively, as shown in Figure 2.7b.

These components can be determined by P , R and r in the force equilibrium shown below,

$$V = P/2\pi r, \quad (2.13)$$

$$H = VR/r = PR/2\pi r^2. \quad (2.14)$$

The horizontal components acting at each end is both inwards and balanced by the resultant of the circumferential forces. If we investigate the radial force equilibrium of a segment of the knuckle subtending a small angle $d\phi$, as shown in Figure 2.7c, it gives the following relationship,

$$2Hrd\phi = Ytld\phi \rightarrow H = Ytl/2r. \quad (2.15)$$

Note that Calladine (1986) used a simplified expression to calculate the section area of the arc, which is lt in Equation 2.15. Using Equation 2.14, the vertical component can be deduced as well,

$$V = Ytl/2R. \quad (2.16)$$

The resultant of the circumferential forces acts at the centroid of the toroidal arc. Again, Calladine (1986) used a rough estimate for the centroid, which is at an elevation of $\zeta = 2h_{knuckle}/3$ above the plane of the hinge-circles. Figure 2.7d indicates the resultant force at its approximated location.

In order to address the moment equilibrium, we take the moment about the outer hinge with all the relevant forces,

$$Vl = 2M_o + 2h_{knuckle}Ylt/3r. \quad (2.17)$$

Substituting Equations 2.12 and 2.16 back into Equation 2.17 yields

$$V = 3M_o/l = 3Yt^2/4l. \quad (2.18)$$

Hence, l and V can be solved using Equations 2.16 and 2.18,

$$l = (3/2)^{1/2}(Rt)^{1/2} \approx 1.22(Rt)^{1/2}, \quad (2.19)$$

$$V = (3/8)^{1/2}Yt^{3/2}R^{-1/2} \approx 0.61Yt^{3/2}R^{1/2}. \quad (2.20)$$

Note that both l and V are independent of r . The downward displacement u of the apex of the sphere is given by

$$u = r^2/R. \quad (2.21)$$

From Equations 2.13 and 2.21, we get an expression for P in terms of u ,

$$P = \pi(3/2)^{1/2}Yt^{3/2}u^{1/2} \approx 3.85Yt^{3/2}u^{1/2} \quad (2.22)$$

or

$$P/4\pi M_o \approx 1.22(u/t)^{1/2}. \quad (2.23)$$

It was assumed that $l/r \ll 1$. Hence, it is clear that this model is not reliable near the origin. This travelling hinge model does not describe the initiation of the inversion of this shell structure, and therefore fails to predict the critical point load that initiates the inversion process.

2.4.2 Egg-box

Another relevant example is the egg-box. It has a lattice pattern consisting of many conical frusta with a closed-top. Deshpande and Fleck (2003) modified Calladines model (1986) to analyse a single frustum.

As shown in Figure 2.8, the dash and solid lines represent the original and deformed shapes, respectively. The frustum is compressed between two frictionless rigid plates. The yield criteria are the same as Equations 2.10 and 2.11. The height of the knuckle and the elevation of the centroid above the plane of the hinge circles are more accurately given as,

$$h_{knuckle} = R_{knuckle}(1 - \cos \omega), \quad (2.24)$$

$$\zeta_{knuckle} = R_{knuckle}(\sin \omega / \omega - \cos \omega). \quad (2.25)$$

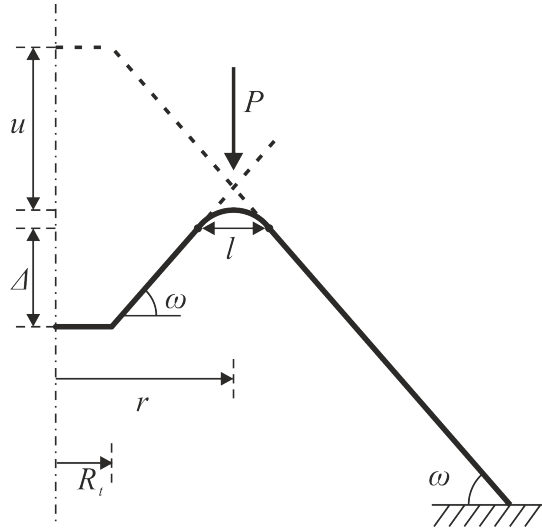


Figure 2.8: Schematic cross-sectional view of a closed-top conical frustum shell with thickness t .

Figure 2.9 provides details of the model. Since the load condition has been changed, the force equilibriums, Figures 2.9c and 2.9b, are different from those in

Figures 2.7c and 2.7d. A similar analysis as Calladine (1986)'s was carried out and the following relationship between P and u can be obtained,

$$\frac{P}{Yt^2} = \frac{2\pi \tan \omega}{(2\omega - \sin 2\omega)^{1/2}} \left[\frac{R_t}{t} + \frac{u}{t} \cot \omega \right]^{1/2}. \quad (2.26)$$

Both Equations 2.23 and 2.26 provide similar relationships between P and u , which is $P = C_1(C_2 + u)^{1/2}$, where C_1 and C_2 are constants that depend on the geometric configuration and material properties. Similar to the spherical shell, Equation 2.26 is not applicable around the centroid region of the frustum. Hence, the force that initiates the snap-through behaviour cannot be inferred using this model.

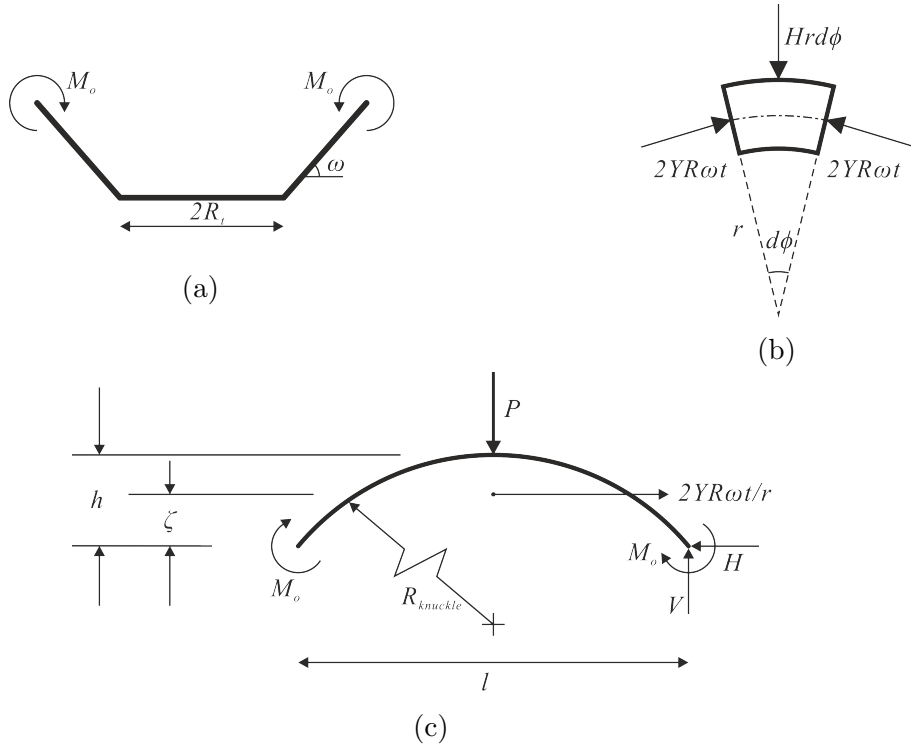


Figure 2.9: Analysis model of a closed-top conical frustum shell. (a) Free body diagram showing the cross-section of the inverted central region, (b) free body diagram showing the plan view of a circumferential segment of the knuckle subtending a polar angle $d\phi$ at the centre, and (c) free body diagram of the knuckle with a unit circumference at radius r .

2.5 Finite Element Method and Abaqus

2.5.1 Introduction

In this research, the finite element method (FEM) is used to model the bistable behaviour of frustum shells. In general, FEM is a numerical method for solving engineering problems with complex geometries, loadings, boundary conditions and material properties. It allows the user to simulate complicated engineering problems through modelling, and aids the user in designing actual experiments and deducing theoretical solutions. In order to capture the bistable and multi-stable behaviour of frusta shells, the consideration of specific constraints, non-linear structural responses and displacement-controlled loading are essential and FEM provides a robust and reliable method to involve these properties.

Schlecht, Schulte, and Hyer (1995) and Schlecht and Schulte (1999) employed FEM to predict the curved shape of asymmetrical laminates with the commercial finite element code, Marc. Giddings combined a finite element model of a Macro-Fiber Composite (MFC) with a bistable asymmetric laminate model. Both predicted the shape and the snap-through voltage of a piezo-actuated $[0/90]_T$ laminate agreed well with experimental results (Giddings et al., 2011).

In recent years, another multi-purpose finite element software package, Abaqus, has gained wide usage, and is proved to be accurate enough in its ability to perform shape predictions of bistable shell structures (Dai, Li, and Du, 2011; Mattioni et al., 2008; Mattioni, Weaver, and Friswell, 2009; Portela et al., 2008; Madhukar et al., 2014; Dai and Li, 2012). In this research, Abaqus 6.13 (2013), was employed to create FEM models of thin-walled conical frusta structures. Once the simulation of FEM has been verified, reliable simulated parametric studies can be performed without the cost of conducting a series of actual experiments.

In the following sections, various related methods and configurations of Abaqus are reviewed and compared.

2.5.2 Quasi-Static Analyses and Abaqus Solvers

In this research, we are interested in the transformation between stable states under normal conditions and investigate responses that result from static loading. This means that the load is time independent and the inertial effects are negligible in the solution. Abaqus offers two possible solvers: the implicit solver (Abaqus/Standard) and the explicit solver (Abaqus/Explicit). The implicit solver solves true static equilibrium while the explicit solver simplifies the problem as a quasi-static one, which assumes that the load is time dependent but is slow enough such that inertial effects can be ignored.

Abaqus/Standard is more efficient for solving straightforward non-linear problems but may have difficulty converging because of contact or material complexities, resulting in a significant number of iterations. Such analyses are computationally intensive in Abaqus/Standard because each iteration requires a large set of linear equations to be solved. On the other hand, Abaqus/Explicit determines the solution without iterating by explicitly advancing the kinematic state from the previous increment. Though a large number of time increments might be required in the explicit method, the analysis can still be more efficient in Abaqus/Explicit as compared with in Abaqus/Standard.

Another noticeable difference between these two solvers is that Abaqus/Explicit requires much less disk space and memory than those of Abaqus/Standard for performing the same analysis.

2.5.3 Non-linear Equilibrium Solution Methods

For the current research, incremental solution procedures are necessary to trace the non-linear equilibrium path of the model. Abaqus utilise either the Newton-Raphson Algorithm or the modified Riks Method (MRM); both of them are powerful tools for determining the non-linear response of a system.

The Newton-Raphson method is used as the default solution algorithm in Abaqus

(step option in Abaqus as ‘Static, General’). It is useful for cases that have a mild non-linear response. The Newton-Raphson method is advantageous because of its quadratic convergence rate when the approximation at a given iteration is within the radius of convergence (Abaqus, 2013). One disadvantage of this method is its inability to traverse limit points in tracing the equilibrium path that is part of the non-linear solution process. However, Abaqus offers an option called ‘Stabilize’ within the Newton-Raphson Algorithm setup, which allows the solving of highly unstable post-buckling problems.

On the other hand, MRM (step option in Abaqus as ‘Static, Riks’), which is Abaqus’s implementation of arc length methods, was commonly used to overcome the difficulties of tracing the equilibrium path in the neighbourhood of the limit points (Riks, 1979; Wempner, 1971). MRM does not suffer from this limitation and is, therefore, often used to tackle problems with significant degrees of non-linearity. MRM assumes that the load at each solution increment is related to the magnitude of the full load by a factor k and solves for the relationship between k and the displacement u . At each increment, the tangential stiffness matrix is used to find the tangent line to the $k - u$ curve at the current solution point. The tangent line traverses a distance that is related to the current arc length, which is initially specified by the user, and adaptively adjusted in Abaqus. The algorithm then looks for an equilibrium in the plane that passes through subsequent solution point that occurs perpendicularly to the tangent line. In both methods, it is assumed that the response is relatively smooth with no sudden bifurcations (Abaqus, 2013). Both methods are capable of simulating the post-buckling response.

Both the Newton-Raphson method and MRM require different sensitivity settings in their respective input parameters. Tawfik et al. (2007) systematically discussed the two methods and concluded that both were capable of accurately simulating the snap-through of bistable shells. However, Mattioni pointed out that the Stabilize

algorithm might be more suitably employed (Mattioni et al., 2008; Mattioni, Weaver, and Friswell, 2009).

2.5.4 Shell Element in Abaqus

Abaqus provides a wide range of elements types for different geometries and analysis types, such as continuum, shell, membrane, rigid, beam, and truss. Variations exist in each element that concerns the number of nodes, degrees of freedom per node, formulation, and integration methods. For shell elements, Abaqus offers both triangular and quadrilateral options with linear interpolation and the choice of using either large-strains or small-strains formulations.

S4R, S3R, and SAX1 are three types of conventional shell elements which are appropriate for most large-strain problems. S4R is a robust quadrilateral element for a general-purpose application. S3R is a triangular element that may suffer shear-locking and is not recommended for curved surfaces. Meanwhile, SAX1 is an axisymmetric element that deals with the axisymmetric responses. Furthermore, there are continuum shell elements, SC6R and SC8R, which capture through-thickness deformation. The latter are more accurate for solving complex contact problem, such as those experienced by composite laminate structures. However, these elements cannot be used with hyperelastic or hyperfoam materials. With respect to nomenclature, the presence of a letter R at the end of element names indicates reduced integration, which has one order less than the full integration. Reduced integration could cause hourglassing. To limit hourglassing effect, artificial strain energy is checked and controlled at a small level ($<1\%$ of the internal energy).

After considering the limitations of all the above-mentioned element types, S4R is the most suitable element for large-deformation nonlinear shell problems.

Chapter 3

Open-Top Frustum Shells under Vertical Uniform Loading

3.1 Geometric Configuration

The functioning part of a flexible straw contains a series of interconnected conical frustum shells; its basic element is a tandem pair of conical frustum shells with the same diameter but measures differently in height. In its open state (Figure 1.2a), the taller frustum remain relatively fixed while the shorter frustum rotates about the connected line and flips onto the taller frustum (Figure 1.2b); altogether leading to the shortening of the basic element. In the second scenario, the taller frustum remains relatively immobile again, while the shorter frustum partially inverts onto the taller frustum - this partial inversion forms a tilting angle, which allows the straw to bend towards one side (Figure 1.2c). Therefore, we must first understand the movement of the shorter frustum to decipher the overall behaviour of a flexible straw. The geometry of a frustum shell can be completely defined by slant angle ω , bottom radius R_b , top radius R_t , and shell thickness t , Figure 3.1. The height h of the frustum can be deduced

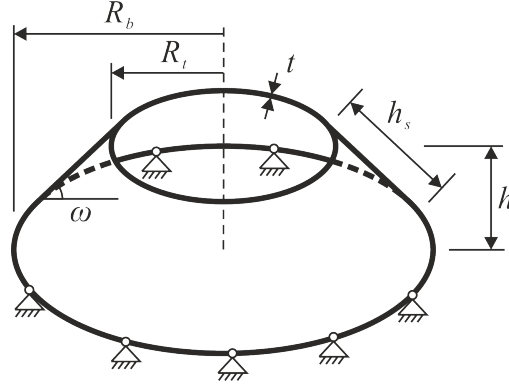


Figure 3.1: Geometric parameters of a singular conical frustum shell.

as

$$h = (R_b - R_t) \tan \omega . \quad (3.1)$$

Likewise, the slant height of the frustum h_s can be deduced as

$$h_s = (R_b - R_t) \sec \omega . \quad (3.2)$$

Here, it is assumed that R_b is always larger than R_t .

3.2 Geometric Measurements of Frusta in a Flexible Straw

A direct measurement of the frusta in a flexible straw is unlikely because of the scale of the specimen and the limitation of measuring tools available. We approach this conundrum by serially photo-capturing the flexible straw in continuous motion, in the presence of a reference rule, and using the images to gauge its geometric dimensions.

Figure 3.2 shows the corrugated portion of a typical flexible straw. It contains 9 pairs of the taller and shorter frustum shells. The total length of the retracted state of the resulting structure is 6.05 mm (Figure 3.2a) while its open state measures 14.04 mm in length (Figure 3.2b). In the close-up view of the corrugated portion (Figure 3.2c), the heights, h , of the taller and shorter frustum are measured as 1.13 mm and 0.43 mm, respectively. In Figure 3.2d, the diameters of the frusta are 4.8 mm and 5.9 mm respectively, and R_t and R_b can be respectively derived by halving these values. The angle formed at the interface between taller and shorter frusta is 103° (Figure 3.2e). The dimensions are summarised in Table 3.1, and the terms ‘shorter frustum’ and ‘taller frustum’ in the following text of this chapter will be referring these specific frusta in the table, explicitly.

Table 3.1: Configuration of the shorter and taller frustum of a flexible straw.

Variable Type	Value & Units	
	Shorter Frustum	Taller Frustum
R_b	2.95 mm	2.95 mm
R_t	2.40 mm	2.40 mm
ω	38.02°	64.05°
t	0.15 mm	0.15 mm
h	0.43 mm	1.13 mm
h_s	0.70 mm	1.26 mm

Figure 3.3a shows the bent state of the corrugated portion of the flexible straw, while Figure 3.3b highlights the bent state of a transverse section that has a 10.53°

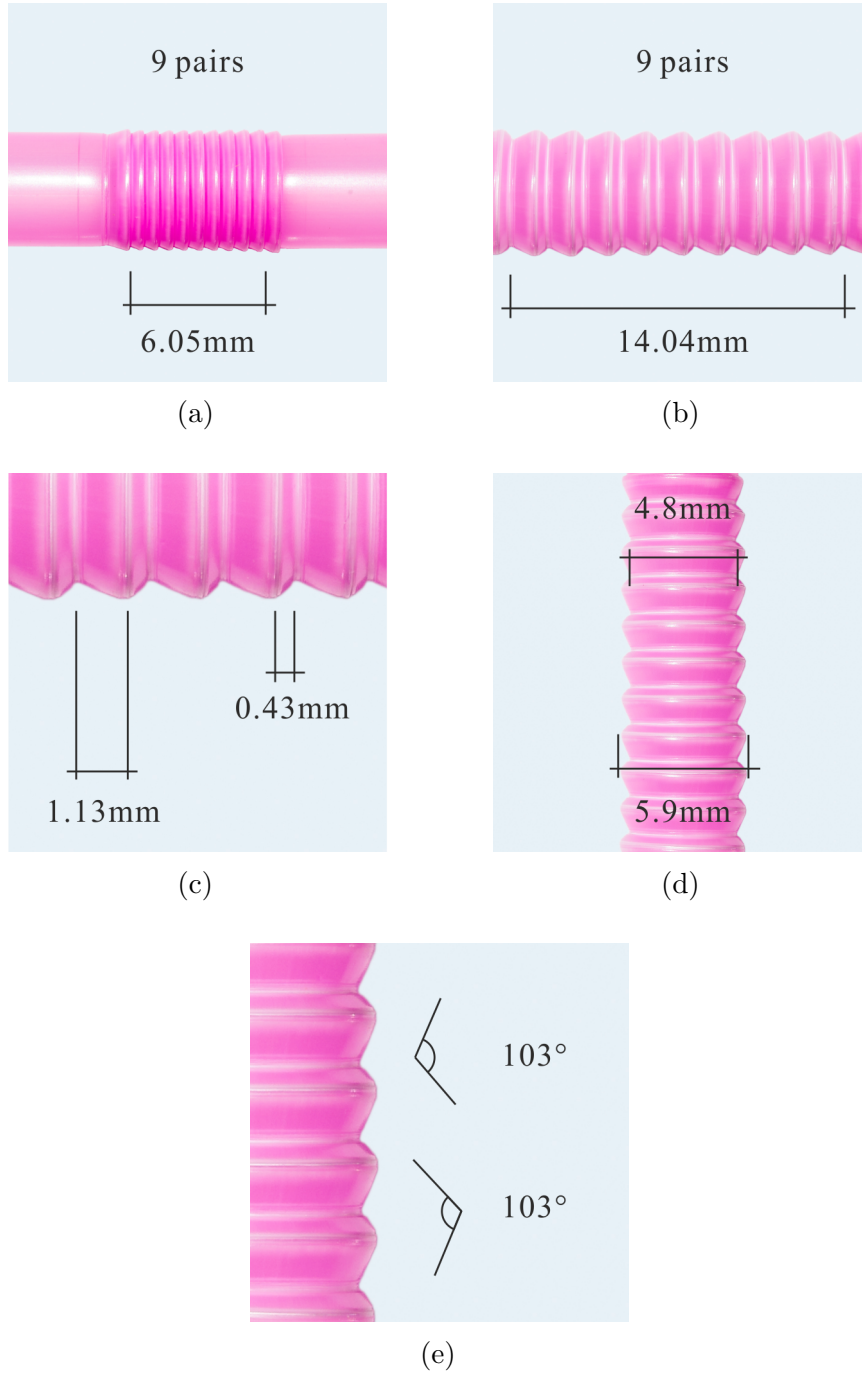


Figure 3.2: Detailed dimensions of a corrugated portion of a flexible straw. (a) Retracted state length, (b) open state length, (c) heights, (d) diameters and (e) slant angles.

kink angle. A detailed study of the behaviour of the kink angle is shown in Figure 3.4, where the average kink angle per pair fluctuates around $11.75^\circ \pm 0.15^\circ$, implying a bent state has a lesser correlation with the current configuration.

The straw is made by hot blowing a plastic cylindrical tube into a premade mould. Though the intention was to connect pairs of frustum shells together (Figure 3.5a), the final product demonstrates a transition part between each frustum shell (Figures 3.5b&c). To simplify the case, the effect of the transition would be ignored in the subsequent analysis.

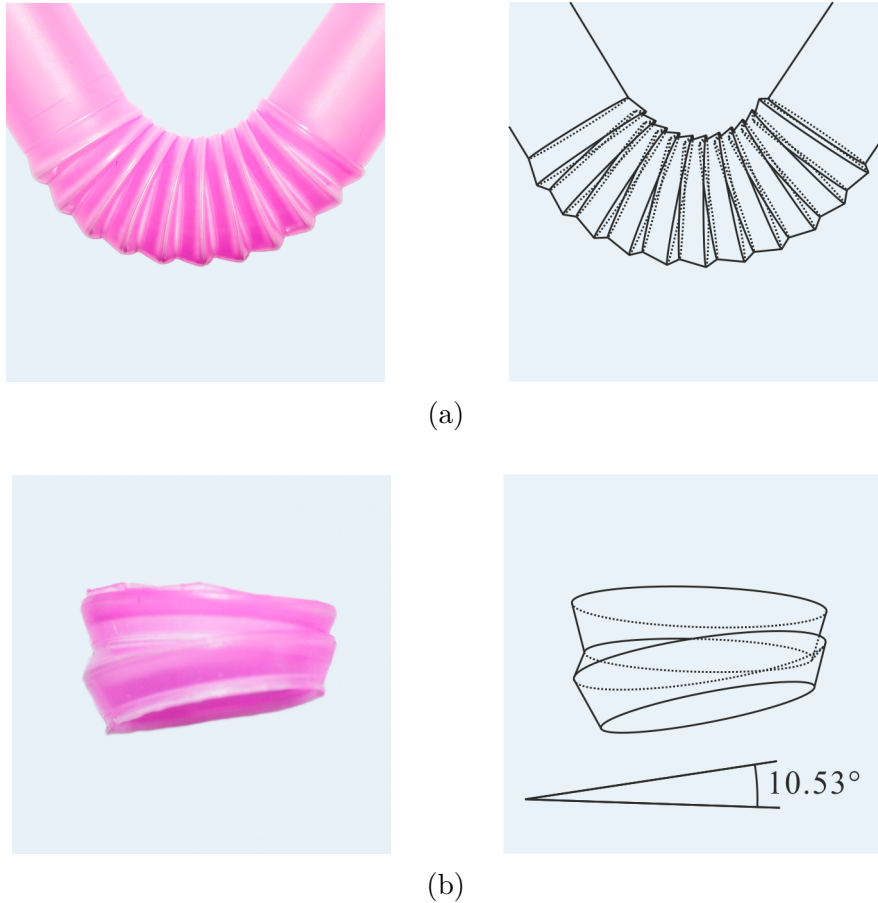


Figure 3.3: (a) The fully bended corrugated portion of a flexible straw, and (b) a cut-off component of the fully bended corrugated portion of a flexible straw.

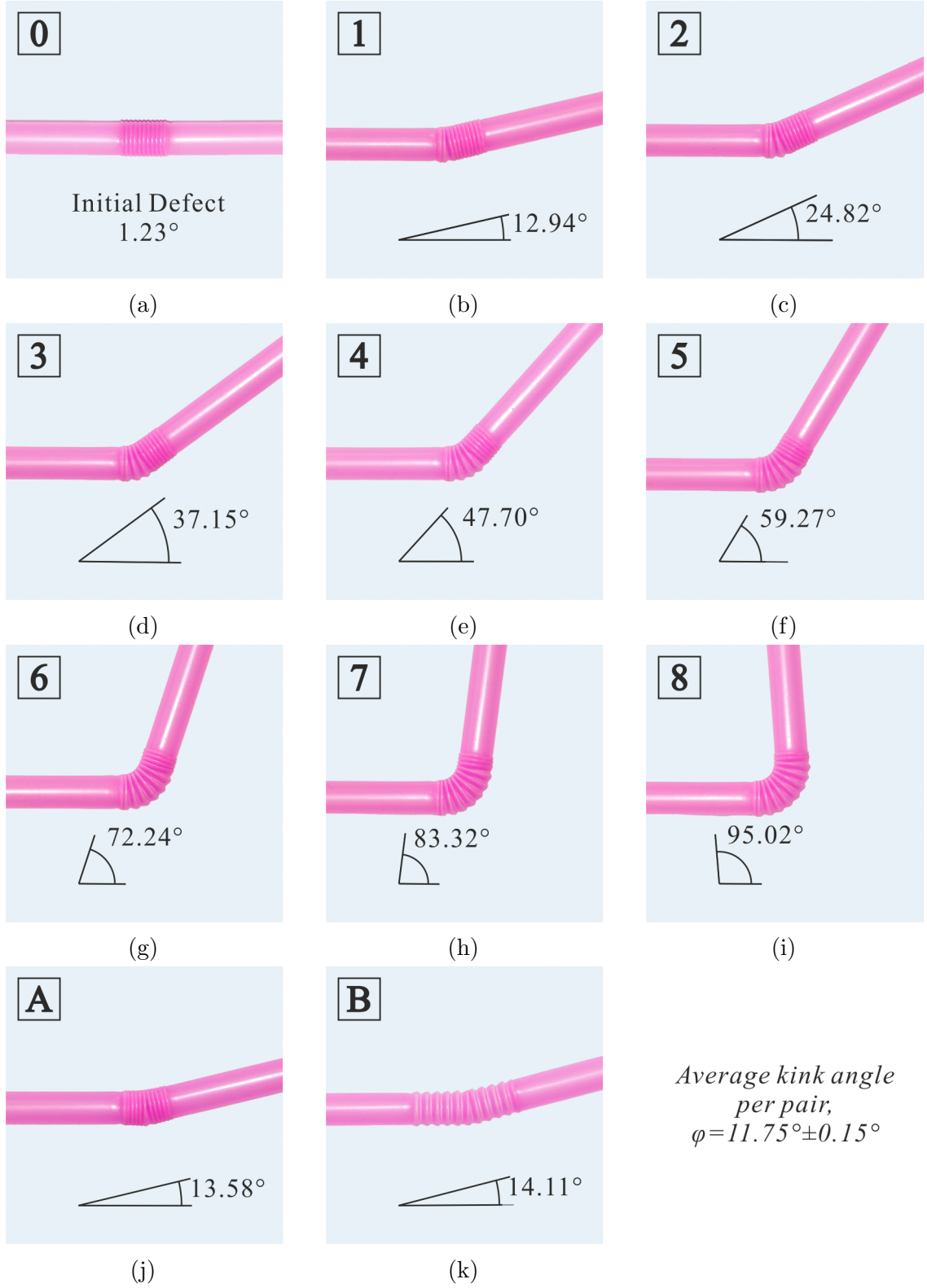
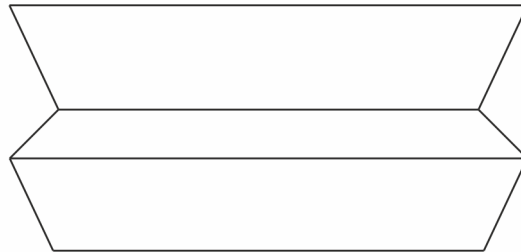
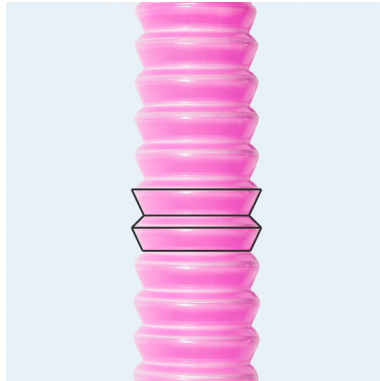
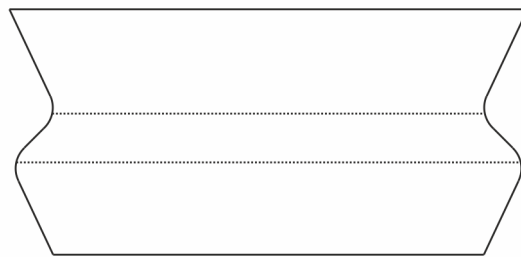


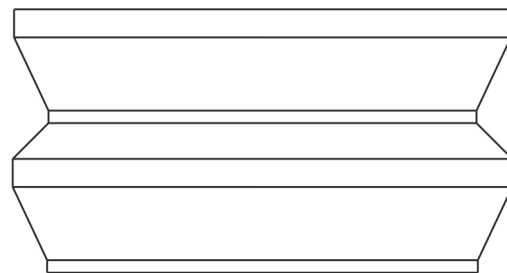
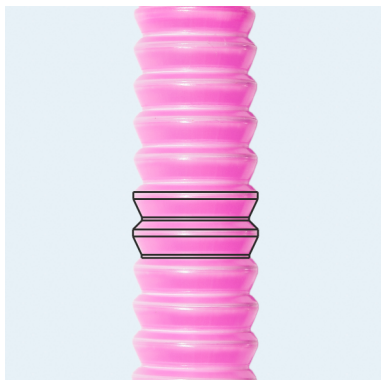
Figure 3.4: Kink angle increases as the partial inversion propagates. (a)–(i) Numbers at top left corners indicates how many pairs of frusta are partially inverted. (j)&(k) One frustum is inverted when the rest are at the retracted and open states, respectively.



(a)



(b)



(c)

Figure 3.5: Transitions connecting each pair of frusta. (a) No transition, (b) round transition, and (c) straight transition.

3.3 Abaqus Model

The finite element method (FEM) simulation was performed using the general-purpose finite element package Abaqus Ver.6.13 (Abaqus, 2013). A reference frustum and several variations were built using shell elements based the dimensions of the shorter frustum in Table 3.1 provided previously.

During each transformation from an open state to a retracted state, the shorter frustum fully inverts while the taller frustum retains most of its original form. Therefore, in setting the boundary conditions, it was assumed that the base of the frustum was constrained to have no relative displacement but was free to rotate while the top of the frustum was completely unrestricted. The outcomes of this assumption will be investigated and discussed in Section 3.5

The models were built in rotational symmetry. Therefore, the loading is symmetrically applied, with no imperfections. Detailed Abaqus settings and methods will be discussed in the following sections.

3.3.1 Loading Methods

The load is applied via the top of the frustum. In order to capture the inversion process, where a decrease in force would have to be allowed, the displacement-control loading was used in the simulations, and the corresponding reaction force caused by the displacement was monitored and recorded. Two loading methods were considered:

Ring Loading A prescribed displacement on the top circle of the conical frustum was uniformly assigned, Figure 3.6a. By using this loading method, any asymmetric buckling mode would be forbidden since the top edge of the frustum was simultaneously moving.

Plate Loading A rigid plate was placed on top of the frustum, Figure 3.6b, and the loading was applied through prescribed displacements of the plate. To avoid overlapping between the loading plate and the frustum, a tiny offset was given to the loading plate. The tangential part of the contact between the rigid plate and the frustum was modelled without frictional forces so that finite sliding was allowed and no tangential force existed in the points of contact. The normal part was modelled with the hard contact setting, which means no contact pressure until nodes are in contact and unlimited contact pressure once contact was established. The limitation of the plate-loading method is that only the positive force and corresponding energy measurements can be recorded.

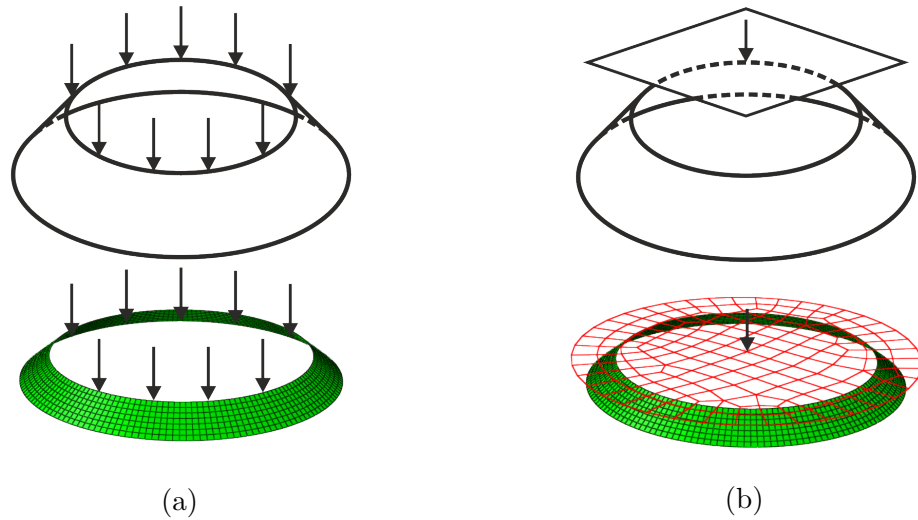


Figure 3.6: Two loading methods for the FEM simulations. (a) Ring loading, and (b) plate loading.

3.3.2 Materials Models

The flexible straw is normally made of polypropylene (PP), which is a thermoplastic polymer used in a wide variety of applications. The material properties are dependent upon the manufacture techniques and treatments. Though it was difficult to test the material properties of the specific PP material that was used in synthesising the flexible straw, typical PP's properties (Tripathi, 2002) were used, which are summarised in Table 3.2.

Table 3.2: Properties of polypropylene (PP).

Property Name	Value & Units
Density, ρ	940 kg/m ³
Poissons Ratio, ν	0.45
Yield Stress, σ_y	25 N/mm ²
Yield Strain, ϵ_y	1.9 %
Youngs Modulus, E	1300 N/mm ²

PP was treated as an isotropic material in the simulations, and two material models were used, which are outlined as follows.

Linear Elastic (LE) The material was considered to be purely linear elastic and had no yield point or hardening/softening behaviour. This material model was used to test if the inversion might occur when the material is elastic.

Elastic-Perfectly Plastic (EPP) The material was assumed to be linear elastic before the point of yielding and perfectly plastic afterwards, which means there would be no hardening/softening of the material.

3.3.3 Solving Scheme

The inversion was assumed to be quasi-static, and hence no kinetic energy was associated with the problem. Both the implicit and explicit analyses were used.

Implicit In the implicit FEM analysis, two non-linear different iterative solution methods, the Newton-Raphson method and MRM method, were used and the results were compared. The Newton-Raphson method is more suitable to the problem since the displacement is the loading factor in the simulation; however, the highly non-linear deformation and multi-buckling behaviour presents an obstacle to convergence; on the other hand, the MRM is powerful enough to maintain the simulation, but the results may diverge from the actual response since the displacement has an indirect relationship with the loading increments.

Explicit As the inversion of the flexible straw was treated as a quasi-static problem, in order to reduce the dynamic effect, the loading speed was controlled to be relatively small, and ‘smooth step’ setting was used for the amplitude to reduce the dynamic effect. The changes in kinetic energy throughout the loading process was monitored and compared with the total internal energy of the model. In this scheme, the density of the material has to be considered as the self-weight of the material will slightly affect the final results. It was also found that usage of the ‘single precision’ setting led to fluctuations in the results, potentially due to the occurrence of accumulated errors at the end of each iteration, hence, ‘double precision’ was used for all simulations.

In general, the implicit analysis was advantageous in providing accurate results; however, the explicit analysis was used when the computational time was too long or the simulation could not converge.

3.3.4 Meshing

In the simulations, the S4R shell elements were used. The size of each shell element was controlled to be approximately 0.1 mm or less on the frustum. Hence, the reference shorter frustum had a total of 1176 shell elements. In the present parametric study, the mesh had no difference from the reference frustum for the frusta with various other slant angles or thicknesses in Section 3.6. More elements of the same size were seeded for the frusta of various other heights, and the total element numbers are listed in Table 3.3. Simulations with smaller mesh sizes were carried out to ensure the current mesh fell within a range that produced stable results.

Table 3.3: No. of mesh elements of frusta with various heights.

Models	Frusta Height	No. of Elements
A1E, A1P (reference)	0.43 mm	1176
A1E_h1, A1P_h1	0.65 mm	1683
A1E_h2, A1P_h2	0.70 mm	1749
A1E_h3, A1P_h3	0.80 mm	1989
A1E_h4, A1P_h4	1.00 mm	2320
A1E_h5, A1P_h5	1.15 mm	2641
A1E_h6, A1P_h6	1.65 mm	3213
A1E_h7, A1P_h7	2.00 mm	3360

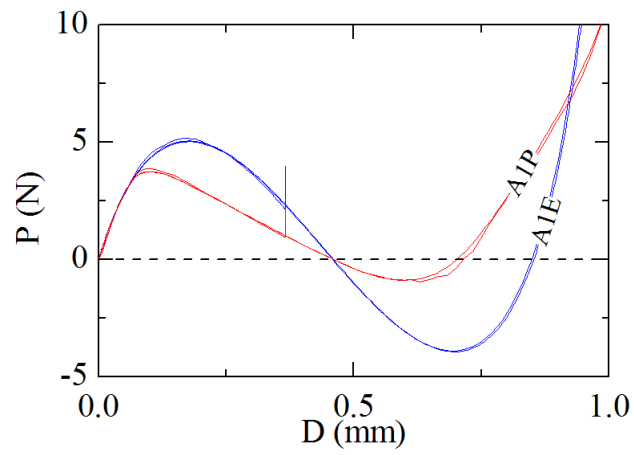
3.4 Validation of Methods

This section discusses the effects of various Abaqus settings on the simulation results of the reference shorter frustum model; four settings were investigated: meshing, loading method, material model, and the solving scheme. In order to validate the reliability of the simulation results, five simulations were carried out on the reference shorter frustum. The results are plotted in Figure 3.7. Models A1E and A1P, which respectively refer to LE and EPP, are modelled using the dimensions and material property of the shorter frustum, as given in Tables 3.1 and 3.2. The blue lines indicate simulations of A1E and the red lines indicate simulations of A1P; the simulations of models using identical material by different solvers or loading methods are plotted in the same colour as similar results were produced.

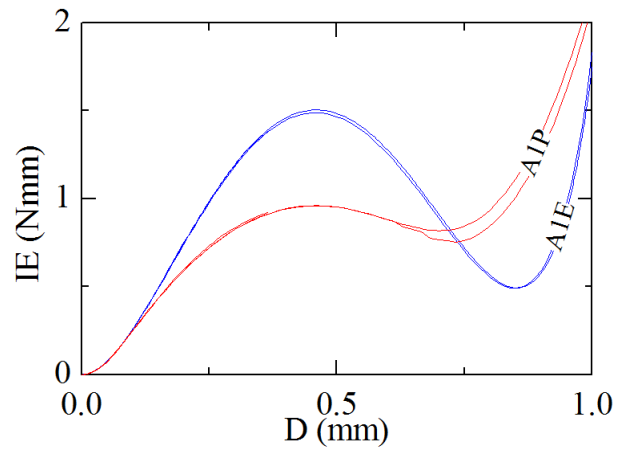
As expected, the snap-through behaviour during the inversion was captured in the analysis since the plots were quite similar to those shown in Figure 2.2. Regardless of the loading method or solving scheme, the results were similar for the same material model, which implies that the explicit solver was competent in performing the quasi-static simulations and the loading methods did not affect the response. Figure 3.7a clearly indicates that the critical force was larger for an LE material than for an EPP, given that the Young's Modulus was the same. The modelling results for LE and EPP followed the same initial trajectory but diverged when the EPP material yielded. Afterwards, the reaction forces of the EPP frustum declined while the reaction forces relating to the LE frustum continued to rise until the critical force was reached.

In the cases where the plate loading method was used, a sudden increment in force was observed and the simulations were halted when the displacement was at $0.43mm$, which was the height of the frustum. This happened when the loading plate touched the bottom plane of the frustum, which could not be penetrated according to the boundary conditions and contact settings.

Regardless of the material models, the implicit solver produced smooth curves while



(a)



(b)

Figure 3.7: Preliminary simulations results of Models A1E & A1P. (a) Vertical reaction force vs. vertical displacement, and (b) internal energy vs. vertical displacement.

the explicit solver results in slightly jagged curves after a displacement of approximately 0.6 mm, where the differences in outcomes between the implicit and explicit solver grew more divergent. These are due to the nature of the explicit solver, where errors tend to accumulate after each iteration. Nevertheless, the differences tended to be minor and did not modify the mode of the response.

Due to the existence of a gap between the loading plate and the frustum, a ‘hit’ was unavoidable in the simulation involving the plate loading method and explicit solver, even after a smooth step was used. In each of the quasi-static analyses, the kinetic energy of the structure was checked and controlled at a small level ($<1\%$ of the internal energy); otherwise, the step time would be increased until such level was achieved. In the final results, there was no obvious suggestion that the dynamic component of the results could affect the overall behaviour of the structure.

In general, consistent results have been obtained in spite of the different settings. Although we have no experimental results to compare with, the graphs in Figure 3.7 demonstrate typical snap-through responses comparing to the example reviewed in Section 2.1.2, which are expected, and the peak forces shown appear reasonable. It validates the reliability of the FEM analysis on the simulations of the bistable behaviour of the relevant shell structures.

3.5 Interaction between the Shorter Frustum and Surrounding Taller Frusta

With reference to the previous description of a typical flexible straw in Section 3.1, the shorter frustum was heavily deformed while the behaviour of the taller frustum was relatively constant during both the bending and retracting phase of a flexible straw. The taller frusta connect the shorter frusta and allow space for their inversion; while the shorter frusta account for the bistable behaviour of the flexible straw. The main objective of this section is to better understand the effects of the interaction between the shorter frustum and surrounding taller frusta; therefore, a series of simulations involving the shorter frustum, with or without its neighbouring taller frusta, were performed and results from the simulations that involved the taller frusta were compared against the simulations that excluded the latter. These comparisons would shed insight on the effect of the interaction on bistable inversion behaviour of the shorter frusta and reveal whether a simplified boundary condition, which pinned the bottom of the singular shorter frusta but allowed the top to be flexible, could be used in the future studies.

Three sets of cases, each with 4 variations, were considered (Table 3.4). The dimensions stated in Table 3.1 were used to model the frusta. These variations differed in how the boundary conditions were applied and the way in which the shorter and taller frusta were connected. In the first case (Figure 3.8a), a singular shorter frustum was modelled in Abaqus, together with variations on the freedom of the radial translation freedom of the bottom . In the second case (Figure 3.8b), a pair of both taller and shorter frustum were modelled, and variable reduction on the thickness of the crease were introduced. Similarly, the base of the taller frustum was pinned while the top of the shorter frustum was flexible. In the third case (Figure 3.8c), a shorter frustum connected by two taller frusta (overhead and supporting) was modelled, and

the thicknesses around each of the creases were variably reduced. The base of the supporting taller frustum and the top of the overhead taller frustum were pinned. These differences between the models are listed in Table 3.4.

The flexible straw was manufactured via a hot blowing process that results in an uneven thickness, particularly involving the thinning of the crease section between frusta. Furthermore, the flexible straw was manufactured at the open state but converted to the retracted state form for a compact delivery package. During this transformation, the creases experienced plastic deformation which led to the softening of the creases. In order to realistically simulate these manufacturing and packaging characteristics, which may be integral to the flexibility of the straw, we introduced variations in the thickness of the crease sections to simulate the behaviour of the system during the inversion process. Two layers of elements around the crease - one from each side with vertical height of 0.7 mm showing in Figure 3.9 - were assigned with a new thickness, 0.025 mm, (83% reduction in comparison with the global thickness 0.15 mm). Figure 3.9 clearly highlights the area of reduced thickness.

In the simulations, both loading and unloading were simulated in order to check whether the response curves were identical between various loading; or otherwise, to reveal the extent to which the critical force was affected and whether the buckling mode was altered in the process. The terms ‘loading’ and ‘unloading’ in the following results are referring to the progressive increasing and decreasing of the displacement respectively, which allow a nonlinear response of the corresponding reaction force. In the following results, the vertical displacements are presented in dimensionless forms. The dimensionless displacement was defined as D/h , and the fully inverted secondary state of the structure is expected to occur when D/h is approximately 2.

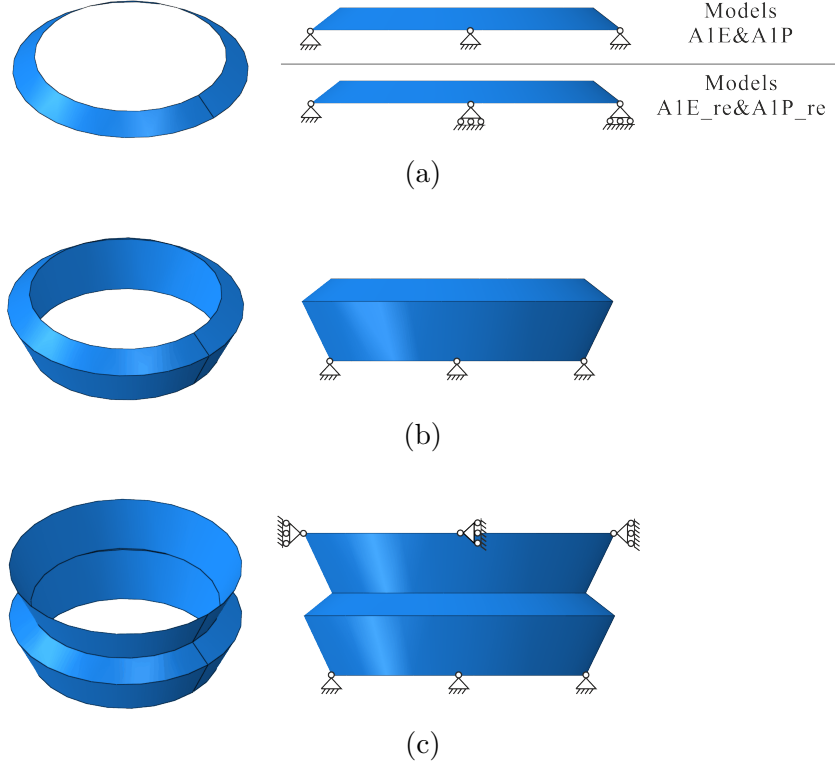


Figure 3.8: Simulated three cases of boundary conditions around the shorter frustum. (a) A singular shorter frustum with predefined boundary conditions (A1); (b) a shorter frustum connected to a taller frustum (A2); and (c) a shorter frustum connected to taller frustra both on top and at bottom (A3).

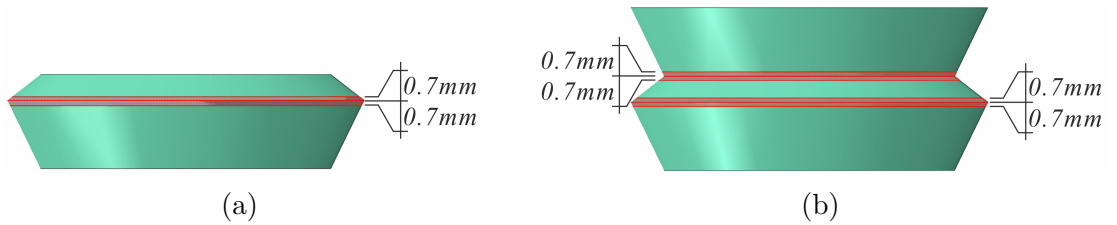


Figure 3.9: Indication of the area of reduced thickness at crease. (a) Models A2, and (b) Models A3.

Table 3.4: Boundary condition variations.

Cases	Models	Bottom Frustum Inclusive	Top Frustum Inclusive	Material	Boundary Conditions	Others
1	A1E	N	N	LE	Pinned at bottom	
	A1E_re	N	N	LE	Unconstrained Radially at bottom	
	A1P	N	N	EPP	Pinned at bottom	
	A1P_re	N	N	EPP	Unconstrained Radially at bottom	
2	A2E	Y	N	LE	Pinned at bottom	
	A2E_rt	Y	N	LE	Pinned at bottom	Reduced t at crease
	A2P	Y	N	EPP	Pinned at bottom	
	A2P_rt	Y	N	EPP	Pinned at bottom	Reduced t at crease
3	A3E	Y	Y	LE	Pinned support at bottom Unconstrained vertically at top	
	A3E_rt	Y	Y	LE	Pinned support at bottom Unconstrained vertically at top	Reduced t at crease
	A3P	Y	Y	EPP	Pinned support at bottom Unconstrained vertically at top	
	A3P_rt	Y	Y	EPP	Pinned support at bottom Unconstrained vertically at top	Reduced t at crease

3.5.1 First Case - Models A1

In this section, the results of four models, A1E, A1E_re, A1P, and A1P_re, are presented and compared, which included two material models (LE and EPP) and two boundary conditions.

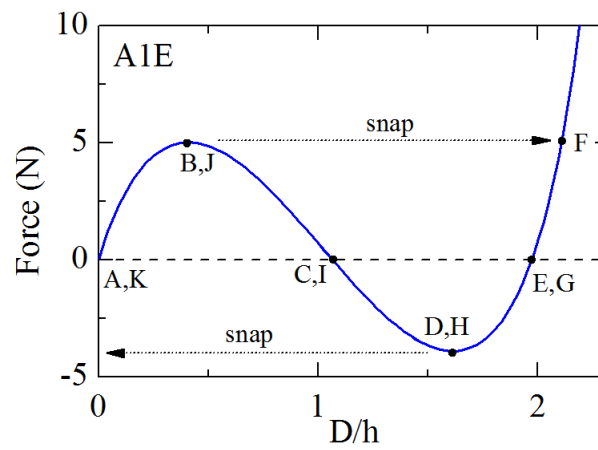
A1E

Model A1E is the basis of all LE simulations and acts as a reference for comparison with other simulations in the investigation. A singular shorter frustum (detailed dimensions in Table 3.1) was simulated by constraining the translational displacement of its bottom rim while allowing it to rotate. Figure 3.10 plots the outcome of the vertical reaction force and internal energy versus generalised vertical displacement of Model A1E. The curve is labelled with letters to indicate the chosen configurations, and the corresponding deformed shapes and stress distribution are presented in Figure 3.11. The results presents the response of displacement-controlled loading and unloading, and the response during loading is sequentially denoted by ABCDEF, while FGHIJK displays the unloading sequence, where the loading and unloading curves agree perfectly. Points B and H respectively indicate the local maximum and minimum reaction forces during loading and unloading. Points A/K and E/G represent the two stable states that occur during the loading and unloading sequences, while Point C/I refers to an unstable equilibrium state.

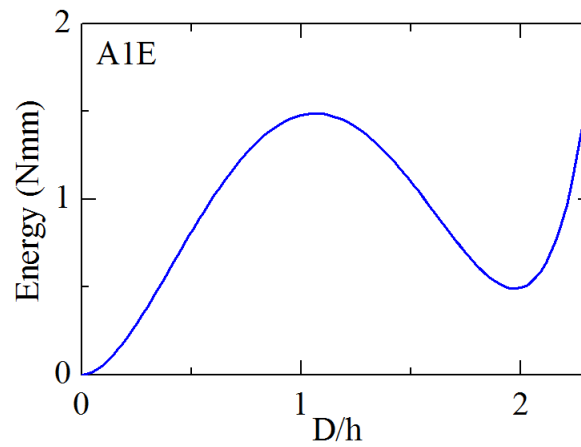
However, in a real event of a force-controlled loading, a snap-through occurs when the loading critical force was reached at Point B, and the structure would instantaneously adopt the state at Point F. As the force was released, the structure would revert to the new stable states (Point E/G). In a real event of a force-controlled unloading starting from Point F, a second snap-through would happen when the

unloading critical force was reached at Point H, and then the structure would adjust to its initial stable state at Point K (identical configuration as Point A) as the force was released.

The critical force during the loading phase was 5.01 N at Point B where $D/h = 0.42$, and the critical force during unloading was -3.91 N at Point H where $D/h = 1.60$. The new stable states occurred at Points E/G where $D/h = 1.98$. With respect to the distribution of stress, the inner edge of the frustum was subjected to the greatest stress.



(a)



(b)

Figure 3.10: Structural responses of Model A1E. (a) Vertical reaction force vs. generalised vertical displacement, and (b) internal energy vs. generalised vertical displacement.

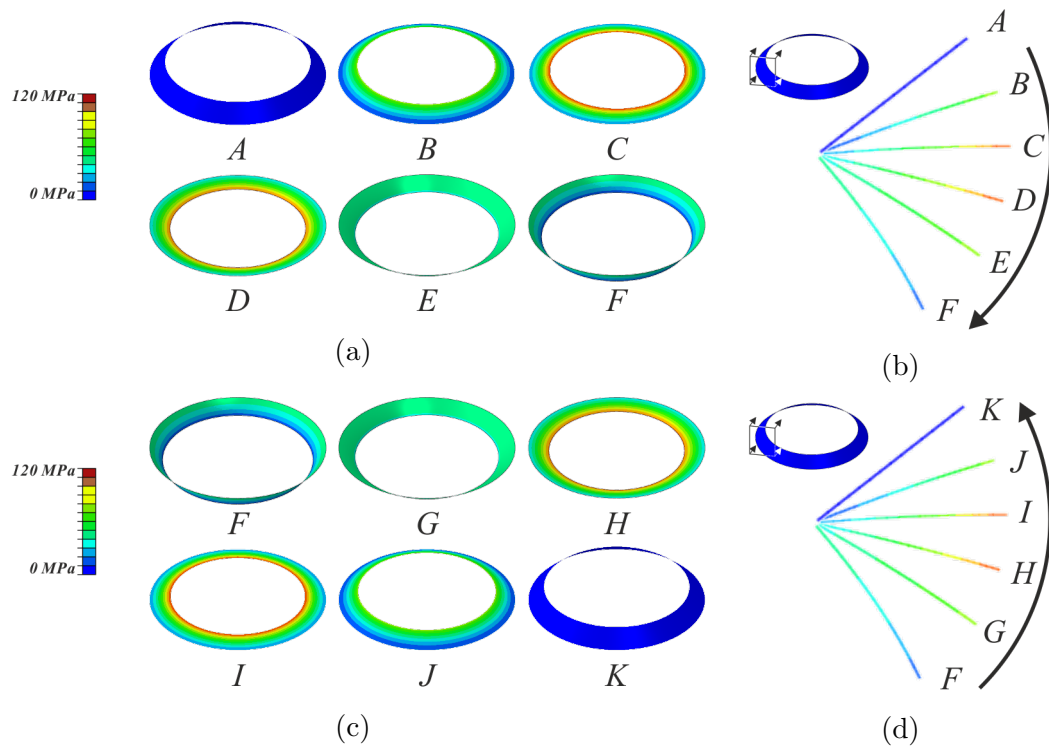


Figure 3.11: Deformed shapes and stress distribution of Model A1E. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

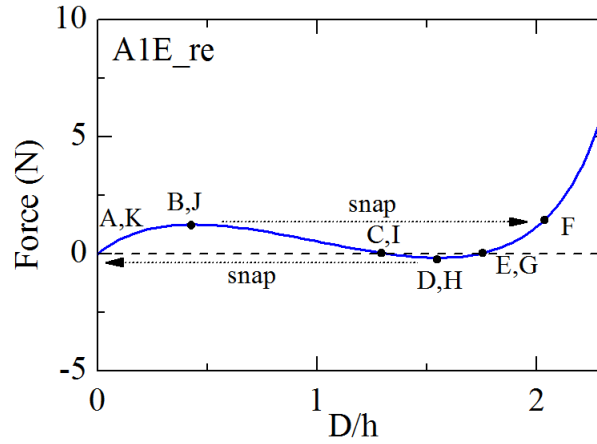
A1E_re

As compared to Model A1E, Model A1E_re allowed its bottom rim to expand or shrink radially. Figure 3.12 presents the response plots depicting (a) the vertical reaction force and (b) the strain energy versus the generalised vertical displacement. The letters labelled on the curve mark selected configurations, and the corresponding deformed shapes and stress distribution are presented in Figure 3.13. Similar to the results of Model A1E, two stable states were observed in the plots and two instances of snap-through occurred, during both the loading and unloading sequences, and the loading and unloading curves aligned with each other. In the presented results, the letters ABCDEF and FGHIJK sequentially describe the deformation patterns during loading and unloading respectively. At Points B and H, the reaction forces were at their positive maximum during loading and negative minimum during unloading, respectively. The new stable state was spotted at Point E (identical configuration as Point G), where the internal energy curve depicts a valley and the vertical reaction force was at zero. Points C/I indicates an unstable equilibrium state of the structure.

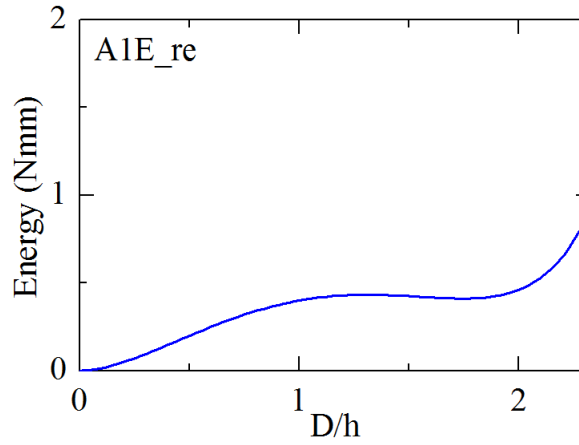
The displacement-controlled response of the frustum is predictive of the behaviour during force-controlled loading and unloading; snap-through would occur during both loading and unloading. During loading, the frustum would transit from Points B to F, and adjust to a stable state, Point E, as the force was released; correspondingly, during unloading, the frustum would snap at Point H, and revert to the original stable state, as described by Point K (identical configuration as Point A).

In comparison with Model A1E, the critical force at Point B during the loading phase was reduced by 75.0% to 1.25 N at $D/h = 0.44$ and the critical force at Point H during the unloading phase was reduced by 95.4% to -0.178 N at $D/h = 1.56$. The secondary stable state was at Point G where $D/h = 1.74$. The critical force that is required to initiate snap-through were drastically reduced by allowing the horizontal radial expansion of the base of the frustum (Model A1E_re), which was more significant

during unloading. Therefore, the radial constraint on the base of the frustum (Model A1E) offered more resistance between stable states, as compared to Model A1E_re.



(a)



(b)

Figure 3.12: Structural responses of Model A1E_re. (a) Vertical reaction force vs. generalised vertical displacement, and (b) internal energy vs. generalised vertical displacement.

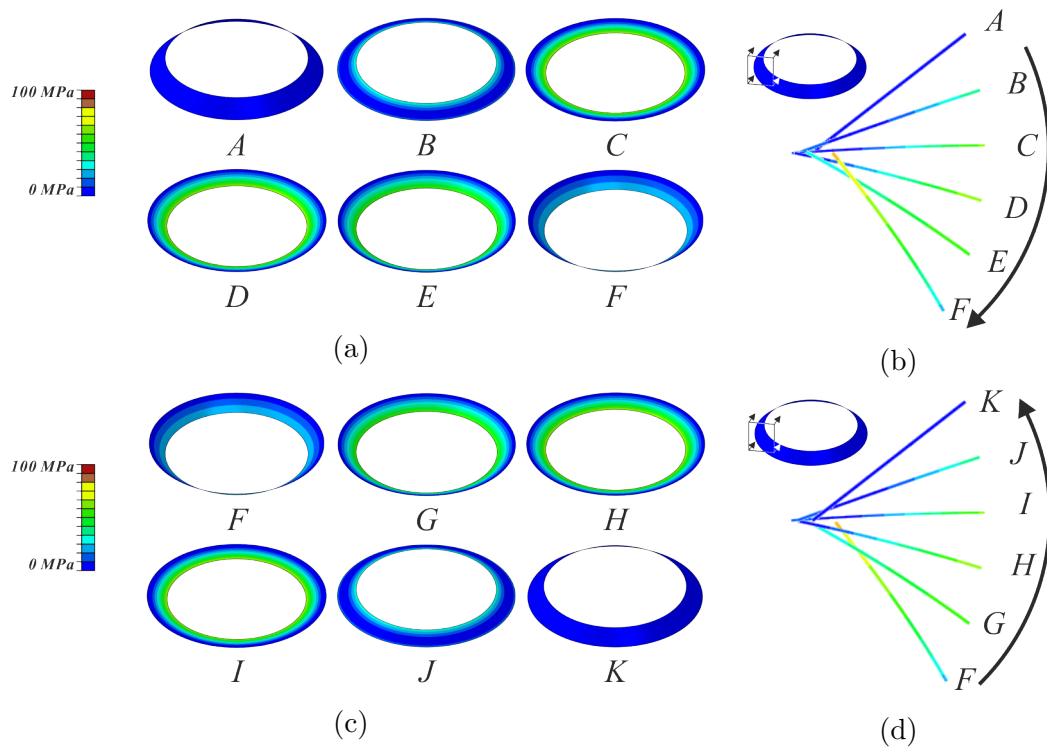


Figure 3.13: Deformed shapes and stress distribution of Model A1E_re. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

A1P

Model A1P acts as a reference for all EPP simulations. The behaviour of a singular shorter frustum (detailed dimensions in Table 3.1) was simulated by constraining the displacement of the bottom rim while allowing rotation. Figure 3.14 shows the plots of (a) the vertical reaction force, (b) the internal energy, and (c) the elastic strain energy versus the generalised vertical displacement. The letters indicate configurations of interest, and the corresponding shapes and stress distribution are presented in Figure 3.15. Similar to the LE model A1E, an incidence of snap-through was observed during both loading and unloading events, and a secondary stable state is described by Point G. In the simulated results, the behavioural progression is sequentially described in alphabetical order, ABCDEF and FGHIJK, which refers to the behaviour of the frustum during displacement-controlled loading and unloading, respectively. Locally, the reaction forces were at their maximum and minimum at Points B and H, which occur during loading and unloading, respectively. Unlike the LE models, the loading and unloading curves differ from each other due to plastic dissipation. The difference between Figures 3.14b&c indicates the energy lost to plastic dissipation. The predominantly red stress distribution graph at Point C indicates that most of the structure is at or beyond its yield point. Because of the plastic dissipation, the structure adopts a new stable state during unloading at Point J.

In events of force-controlled loading and unloading, the structure would snap from Points B to E, and from Points H to K, respectively. The new stable states are indicated by Points G and J during the loading and unloading sequence, as the load is being released.

The critical force during the loading phase was 3.72 N at Point B where $D/h = 0.23$ and the critical force during the unloading phase was -4.74 N at Point H where $D/h = 2.0$. The secondary stable state was at Point G where $D/h = 2.20$. After a cycle of loading and unloading, the initial stable state had a residual displacement

of $D/h = 0.37$ and residual plastic strain up to 3.9% around the top rim due to the unrecoverable plastic deformation of the structure. The critical force in the loading phase was reduced by 25.7%, while the critical force in the unloading phase was raised by 21.2%, as compared to the results obtained in Model A1E.

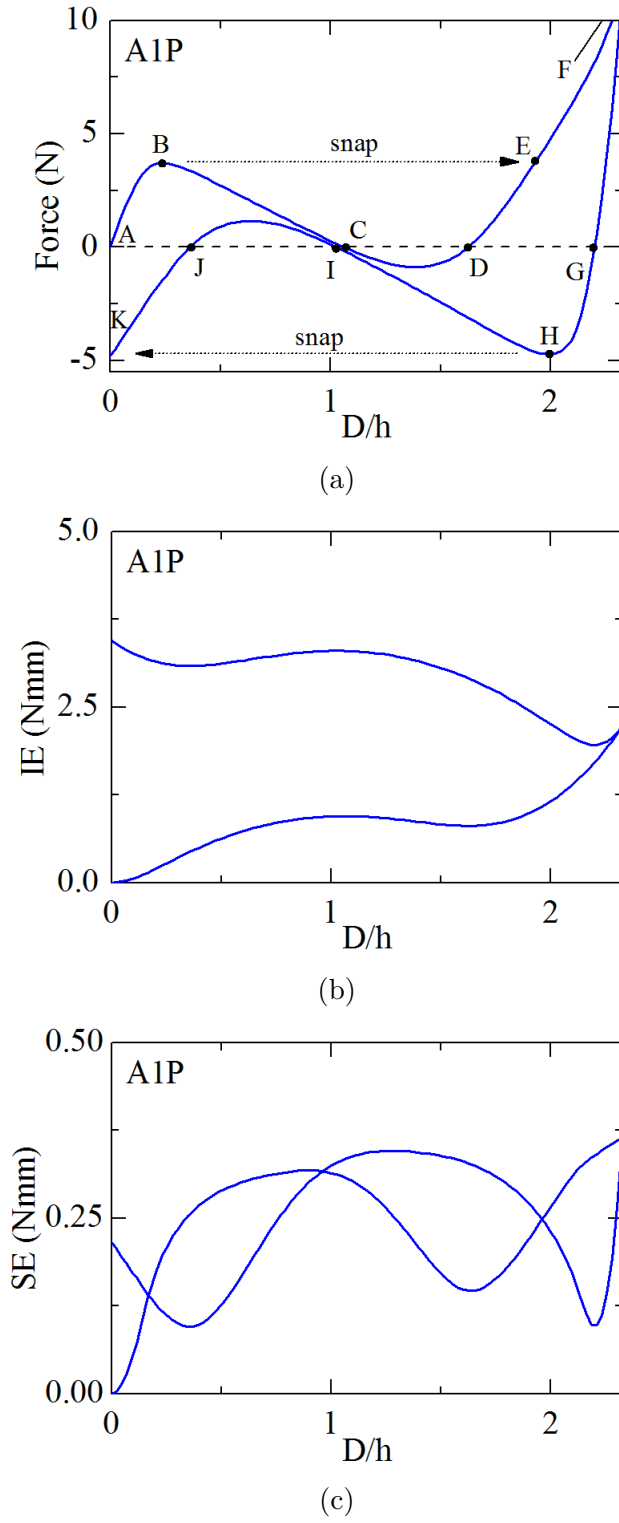


Figure 3.14: Structural responses of Model A1P. (a) Vertical reaction force vs. generalised vertical displacement, (b) internal energy vs. generalised vertical displacement, and (c) strain energy vs. generalised vertical displacement.

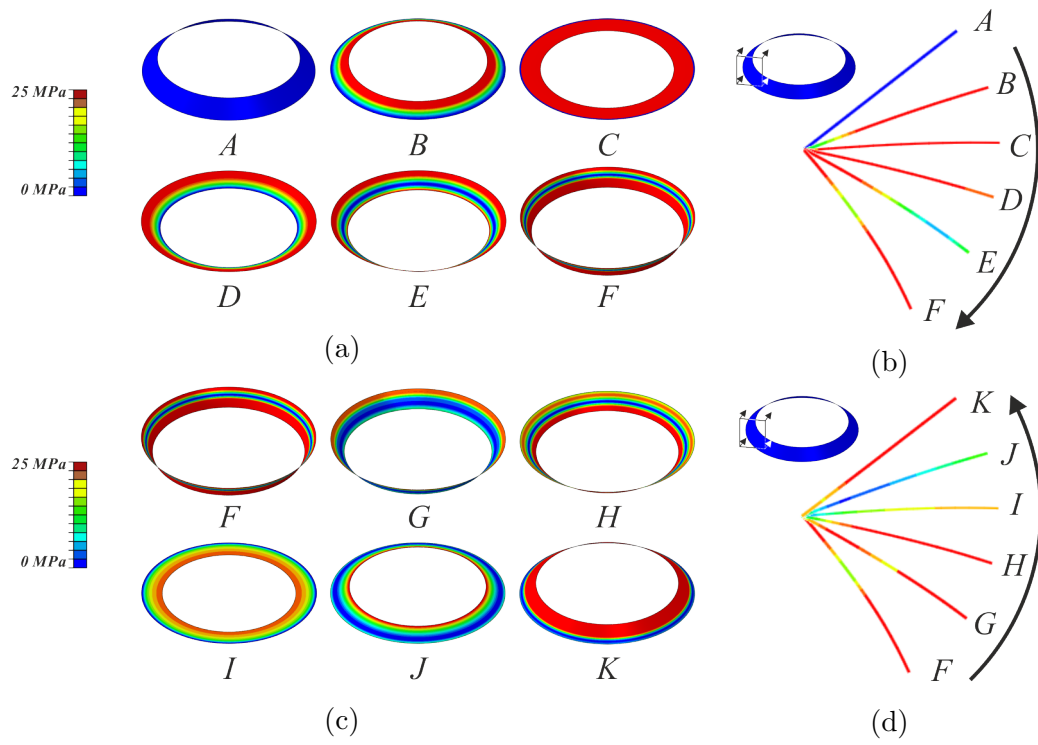


Figure 3.15: Deformed shapes and stress distribution of Model A1P. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

A1P_re

Unlike Model A1P, Model A1P_re allowed the bottom rim to shrink or expand radially. Figure 3.16 shows the plots of (a) the vertical reaction force, (b) the internal energy, and (c) the elastic strain energy versus the generalised vertical displacement of Model A1P_re. The letters on the curve indicate key configurations, and the corresponding shapes and stress distribution are revealed in Figure 3.17. Similar to the LE model A1E_re, snap-through occurs at both loading and unloading, and a secondary stable state was found; ABCDEF and FGHIJK describe the displacement-controlled loading and unloading sequence respectively. The peak positive and negative reaction forces occurred at Points B and H, during loading and unloading respectively. The loading and unloading curves differ due to plastic dissipation, which can be calculated by taking the difference between the total internal energy and the elastic strain energy, in Figures 3.15a and 3.15d respectively. Due to plastic dissipation, the initial stable state shifted to a new configuration at Point J with a residual displacement between Points J and A, after a cycle of loading and unloading. While the yield point was considerably met in the sequence of deformation shown in Figure 3.17, the affected area (shown in red) were reduced in Model A1P_re as compared to Model A1P, suggesting less plastic dissipation in the former, which results in a lower reaction force across all the corresponding stages.

The responses of force-controlled loading and unloading could be predicted from the simulated displacement-controlled results. The structure would snap from Points B to E during loading, and from Points H to K during unloading, with new stable states at Points G and J.

The critical force during the loading phase was 1.21 N at $D/h = 0.40$, and the critical force during the unloading phase was -1.28 N at $D/h = 1.74$. The secondary stable state was at $D/h = 2.07$. After a cycle of loading and unloading, the original stable state was found at Point J with a residual displacement of $D/h = 0.14$ and residual plastic strain of up to 1.6% around the top rim. The difference in the results of Models A1E_re and A1P_re is much smaller than observed between Models A1E and A1P, because the buckling occurred just after yield and less plastic deformation had taken place.

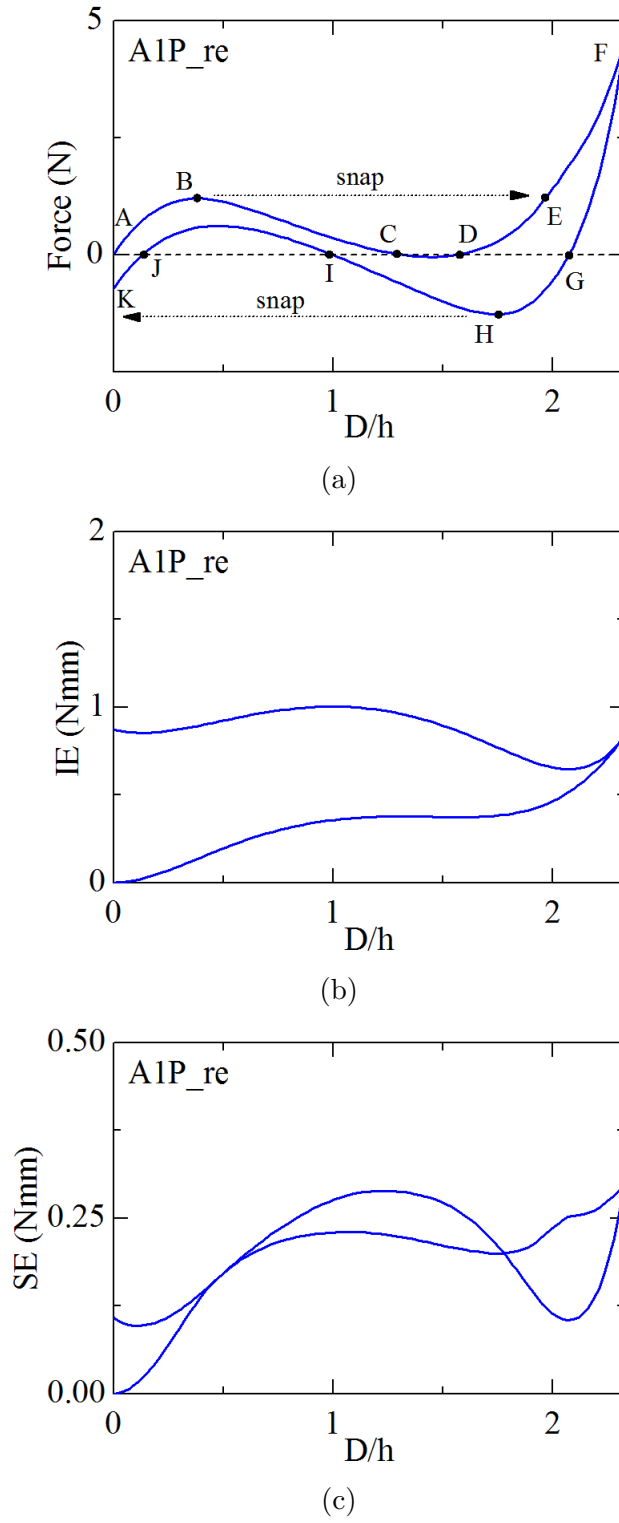


Figure 3.16: Structural responses of Model A1P_re. (a) Vertical reaction force vs. generalised vertical displacement, (b) internal energy vs. generalised vertical displacement, and (c) strain energy vs. generalised vertical displacement.

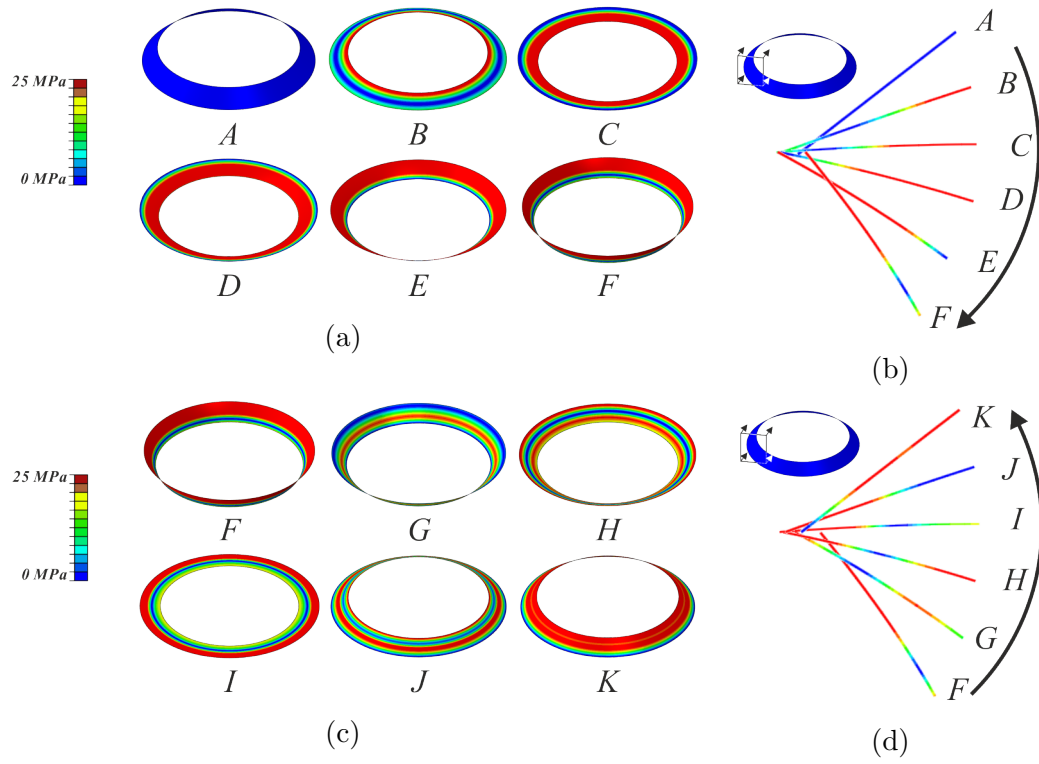


Figure 3.17: Deformed shapes and stress distribution of Model A1P.re. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

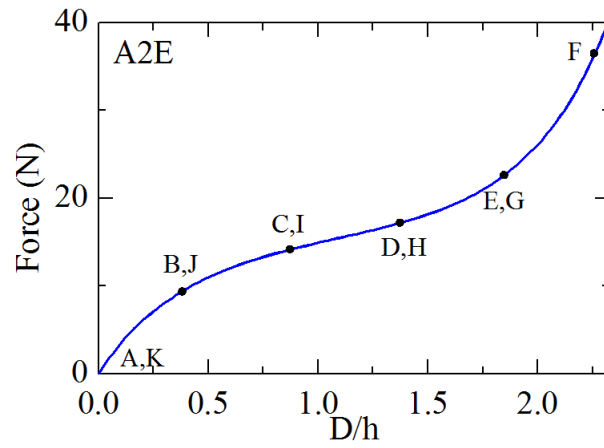
3.5.2 Second Case - Models A2

In this section, the results from four models, A2E, A2E_rt, A2P, and A2P_rt, were analysed, where a shorter frustum was inversely attached to a taller frustum at its bottom end. Two material (LE and EPP) variants of two crease thicknesses were considered here. The generalised vertical displacements presented in this section refers to the generalised relative displacement of the upper rim of the shorter frustum with respect to its lower rim. The data was presented in this format so that the results here could be directly compared to A1 models.

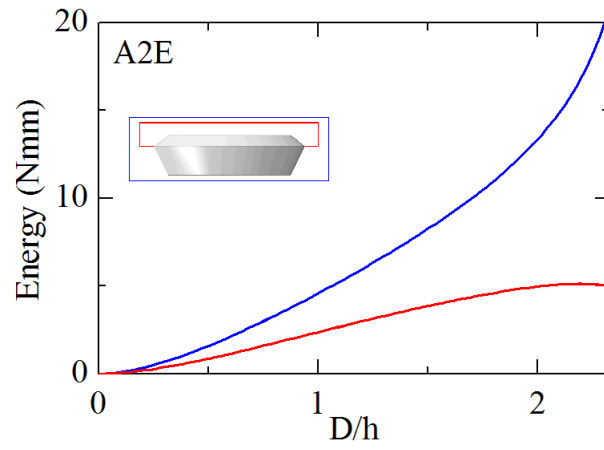
A2E

Figure 3.18 shows the plots of the vertical reaction force and the internal energy versus the generalised vertical displacement of Model A2E. The curve is labelled with letters to indicate the configurations of interest, and the corresponding deformed shapes and stress distribution are presented in Figure 3.19. The blue line in Figure 3.18b refers to the energy of the entire structure, and the red line refers to the energy of the shorter frustum. Neither snap-through nor bistable behaviour was observed for the model. While unloading, the shell reverted to its initial stable state after the force was released. In the half-sectional view in Figure 3.19b and 3.19d, it was found that most bending occurred on the faces while the angle at the crease was relatively constant during the loading and unloading (i.e. manifesting behaviour that was similar to a vertical elastic spring in a sense). The latter suggests that the bistable response is related to the hinge-like behaviour of the crease. In Figure 3.18b, the red and blue curves diverged from the beginning of the displacement graph, and the difference in energy between these two curves, which indicates the internal energy stored in the taller frustum, surpassed the values represented by the red curve itself, which amounts to the internal energy stored in the shorter frustum. Overall, the data suggests that the taller frustum was responsible for most of the deformation, which conflicts with

the actual observations concerning the flexible straw.



(a)



(b)

Figure 3.18: Structural responses of Model A2E. (a) Vertical reaction force vs. generalised vertical displacement, and (b) internal energy vs. generalised vertical displacement.

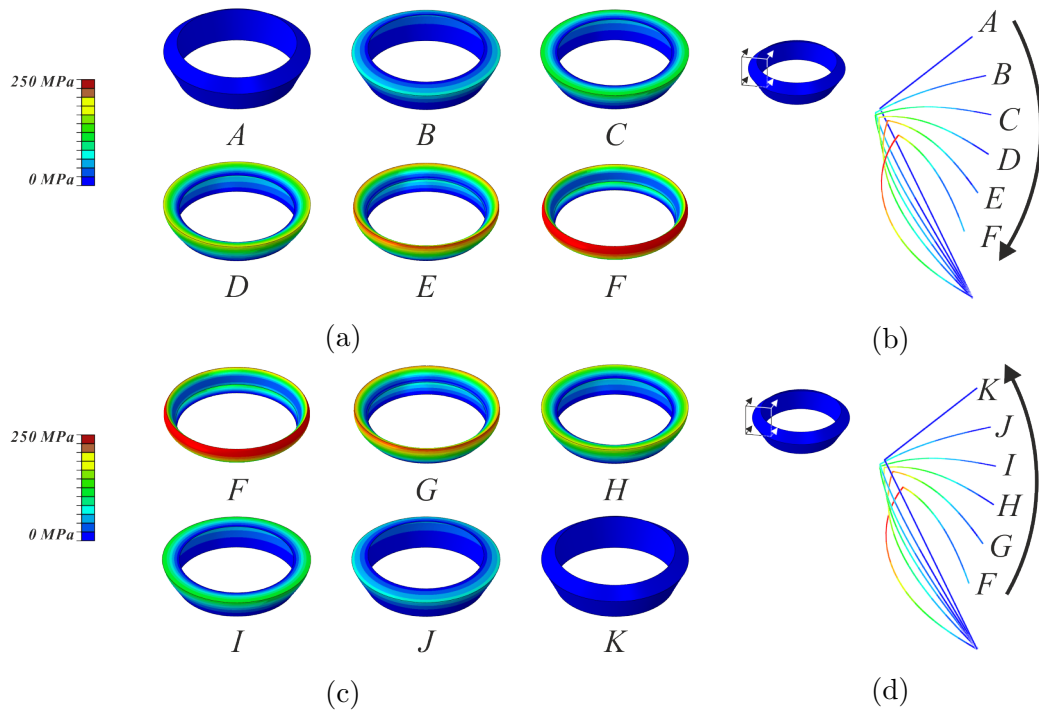


Figure 3.19: Deformed shapes and stress distribution of Model A2E. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

A2E_rt

Model A2E_rt has reduced local thickness (0.025 mm) around its crease (defined in Figure 3.9a) than Model A2E. By reducing the crease thickness of Model A2E_rt, the crease would contribute to much lower resistance to its inversion.

Figure 3.20 shows the plots of the vertical reaction force and the internal energy versus the generalised vertical displacement of Model A2E_rt. The letters labelled on the curve indicate the configurations of interest, and the corresponding deformed shapes and stress distribution are presented in Figure 3.21. In Figure 3.18b, the blue line relates to the energy of the entire structure, the red line refers to the energy of the shorter frustum, and the green line refers to the energy of the shorter frustum with the original thickness.

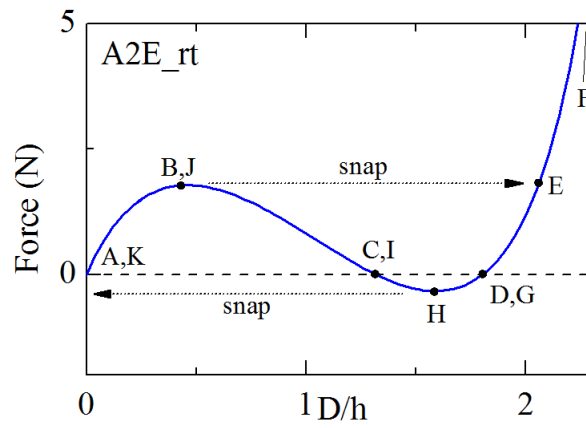
In the displacement-controlled results, the letters ABCDEF and FGHIJK sequentially describe the deformation patterns during loading and unloading respectively. The loading and unloading curves are indistinguishable from each other, as a result of fully-recoverable elastic deformation. Snap-through behaviour was observed during both loading and unloading - at Points B and H, the reaction forces were at their positive maximum during loading and negative minimum during unloading, respectively. The bistable behaviour was observed in Model A2E_rt. A secondary stable state was found at Point G (identical configuration as Point H), where the internal energy of the entire structure was at its local minimum in Figure 3.18b.

As shown in Figure 3.21, the response of Model A2E_rt was visually distinct to Model A2E (Figure 3.19): the faces of Model A2E_rt stayed relatively straight during the loading and unloading, while the angle at the crease was varied. The stress

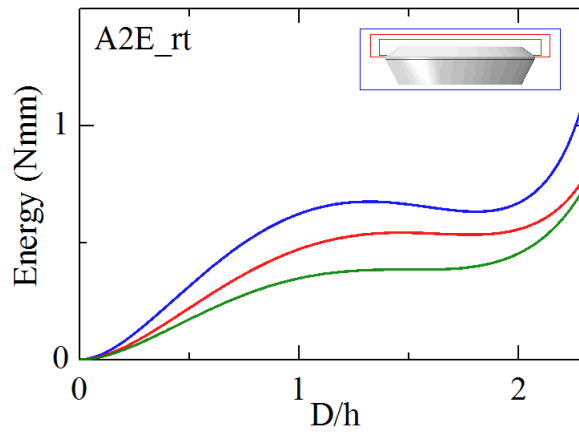
distribution is consistent with our conclusions: the energy distribution of the two taller frusta is mostly shown in blue, which indicates low stress in the stress spectrum, while the highest stress was located around the crease, especially along the bottom and top rim of the shorter frustum. The energy curves revealed that the deformation affecting the shorter frustum amounts to roughly $4/5$ of the total internal energy, while the deformation around the crease contributed to $1/5$ of the total internal energy. In comparison, most of the internal energy of Model A2E reflected deformations within the taller frustum. In the absence of a large portion of bending energy, the structure was able to demonstrate a distinguished bistable behaviour.

In an event of actual force-controlled loading, a snap-through would occur as the loading critical force was reached at Point B, and the structure instantaneously adopted the configuration shown in Point E. After the force was released, the structure would adjust itself to its secondary stable states as shown in Point D/G. In an event of actual force-controlled unloading starting from Point F, a second snap-through event would occur when the unloading critical force was reached at Point H, and then the structure would adjust to the initial stable state as shown in Point K (identical configuration as Point A) after the force was released.

The critical force during the loading phase was 1.78 N at $D/h = 0.45$ while the critical force during the unloading phase was -0.33 N at $D/h = 1.58$. The secondary stable state was at $D/h = 1.81$. In comparison to Models A1E, the critical force during the loading phase was reduced by 64.5% , and the critical force during the unloading phase was decreased by 91.6% , due to the realistic radial constraint on the lower rim of the shorter frustum by the neighbouring taller frustum.



(a)



(b)

Figure 3.20: Structural responses of Model A2E_rt. (a) Vertical reaction force vs. generalised vertical displacement, and (b) internal energy vs. generalised vertical displacement.

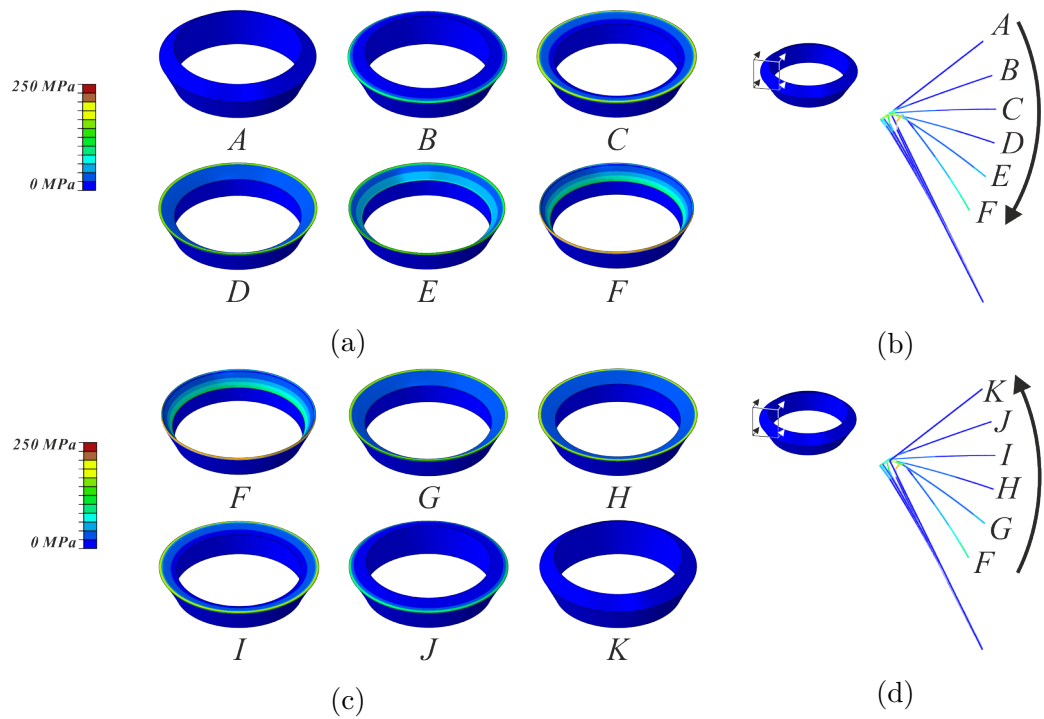


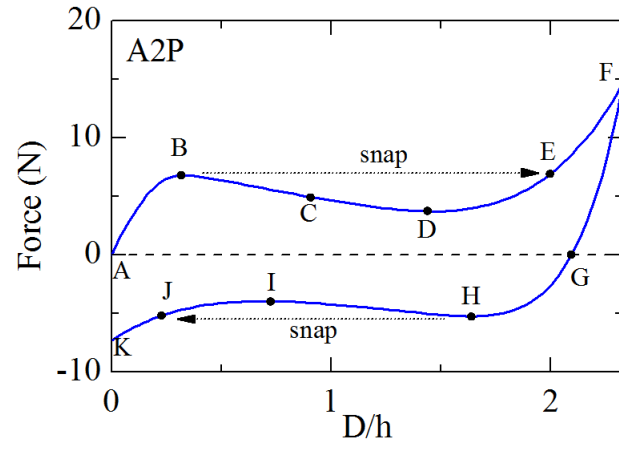
Figure 3.21: Deformed shapes and stress distribution of Model A2E_rt. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

A2P

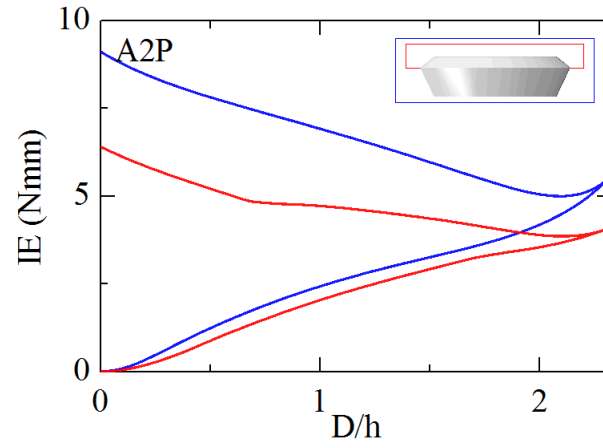
Figure 3.22 shows the plots of the vertical reaction force, the internal energy and the elastic strain energy versus the generalised vertical displacement of Model A2P. The letters indicate the configurations of interest, and the corresponding shapes and stress distribution are presented in Figure 3.23. In Figures 3.22b&c, the blue line refers to the energy of the entire structure whilst the red line reflects the energy of the shorter frustum. In the simulated results, the behavioural progression is sequentially shown in ABCDEF during displacement-controlled loading, and FGHIJK during displacement-controlled unloading. The positive local maximal reaction forces occurred when the structure was at Point B during loading, and the negative minimum was at Point H during unloading. Due to plastic dissipation during deformation, the unloading curves followed a different path from the loading curve. The associated plastic dissipation is the difference in energy between the internal energy and elastic strain energy in Figures 3.22b&c, which clearly shows that the plastic dissipation is much larger than the elastic strain energy. During the loading and unloading, the entire shorter frustum and part of taller frustum around the crease became fully yielded.

In an event of force-controlled loading, the frustum would snap from Point B, where the loading critical force occurred, to Point E. A secondary stable state would occur at Point G if the loading was performed up to Point F. Meanwhile, in an event of force-controlled unloading starting from Point F, a second snap-through would occur at Point H where the unloading critical force was reached, and then the structure would snap to Point J. However, the initial stable state did not appear to be recoverable after the unloading, and should return to its secondary stable state at Point G after the unloading force was released. In other words, the structure demonstrates a semi-bistable behaviour, which most likely results from the significant plastic deformation around the crease, and prevents the reversion of the structure.

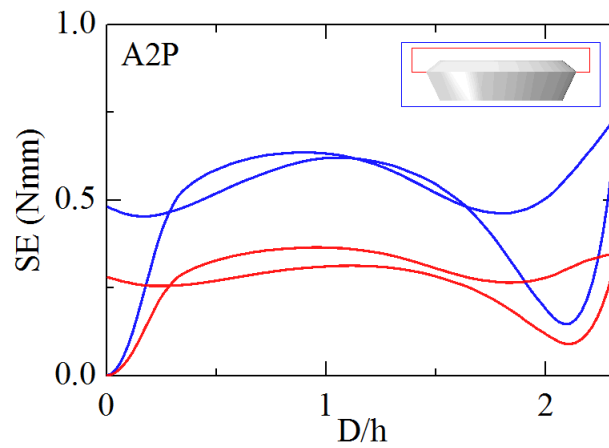
The secondary stable state was at Point G where $D/h = 2.09$. The loading critical force was 6.80 N at Point B where $D/h = 0.33$ and the unloading critical force was -5.27 N at Point H where $D/h = 1.63$.



(a)



(b)



(c)

Figure 3.22: Structural responses of Model A2P. (a) Vertical reaction force vs. generalised vertical displacement, (b) internal energy vs. generalised vertical displacement, and (c) strain energy vs. generalised vertical displacement.

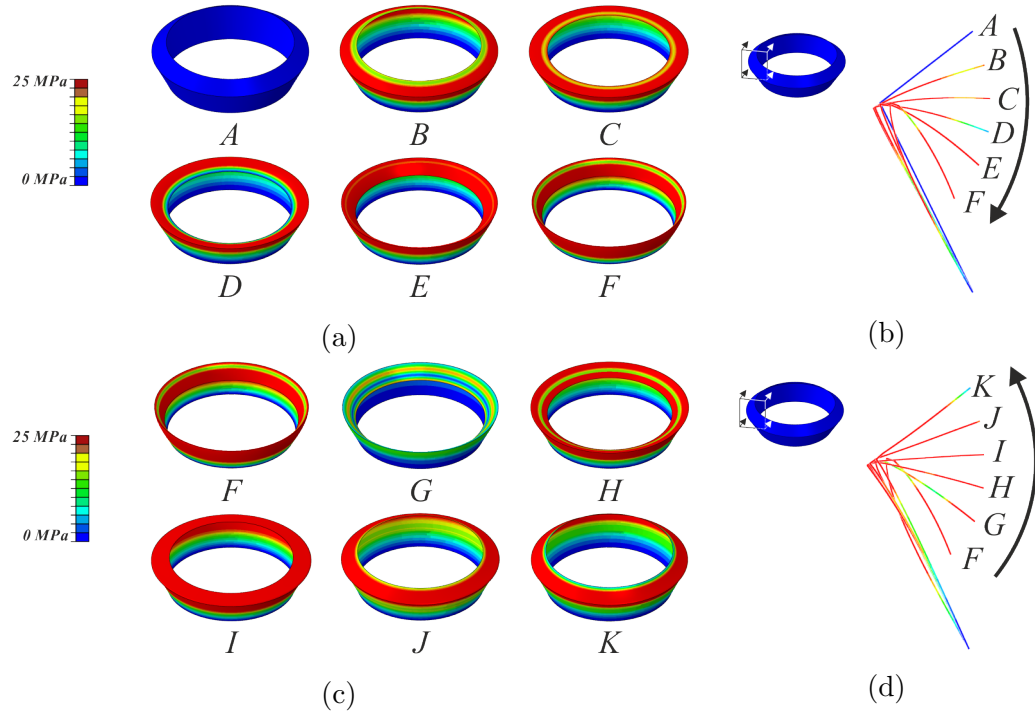


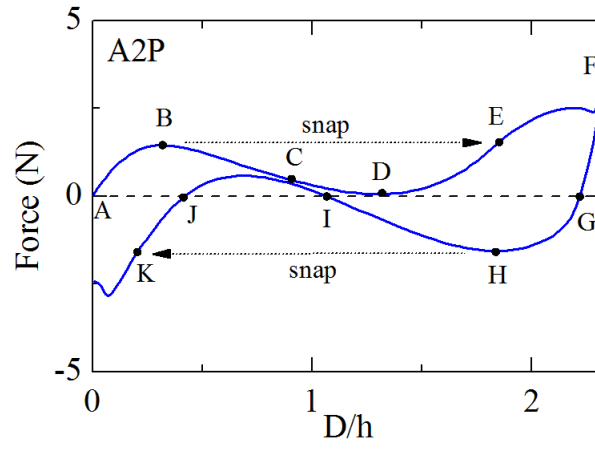
Figure 3.23: Deformed shapes and stress distribution of Model A2P. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

A2P_rt

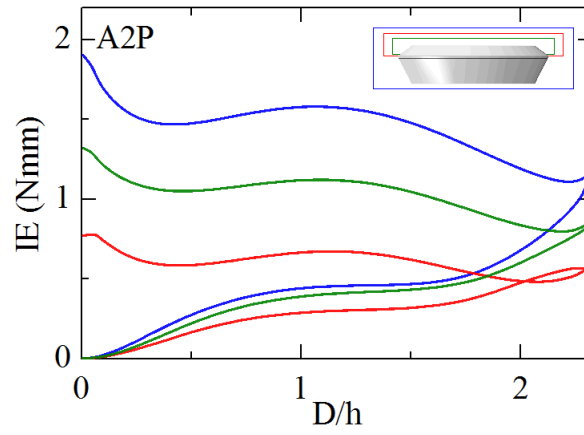
Model A2P_rt has reduced local thickness (0.025 mm) around its crease locally (defined in Figure 3.9a) as compared to Model A2P. By reducing the crease thickness, the resistance against the inversion of the structure would be dramatically reduced.

Figure 3.24 shows the response profile of Model A2P_rt, including the plots of (a) the vertical reaction force, (b) the internal energy and (c) the elastic strain energy versus the generalised vertical displacement. The curve is labelled with letters to indicate the chosen configurations, and the corresponding deformed shapes and stress distribution are presented in Figure 3.25. As shown in the legends of Figures 3.24b&c, the blue line conforms to the energy of the entire structure, the red line to the energy of the shorter frustum, and the green line to the energy of the part of the shorter frustum with original thickness. During displacement-controlled loading and unloading, the frustum deformed in the sequence ABCDEF and FGHIJK, respectively. The snap-through behaviour was observed during both the loading and unloading, arriving at the positive local maximum reaction force at Point B while loading, and the negative local minimum at Point H while unloading. In Figures 3.24b&c, the difference between the loading and unloading curves was the result of the accumulated plastic dissipation, which represents the difference between the internal energy and the elastic strain energy. During the loading and unloading, the shorter frustum was fully yielded but the tall frustum remained at a low stress level (<12 MPa in comparison to the yield stress $Y = 25$ MPa). A dip around Point F was observed. We speculate that it was the side effect of the thickness reduction around the crease since the crease became not only prone to bending but also weak in resisting stretching, resulting extra local buckling (distorted shapes at crease in Figures 3.25b&d).

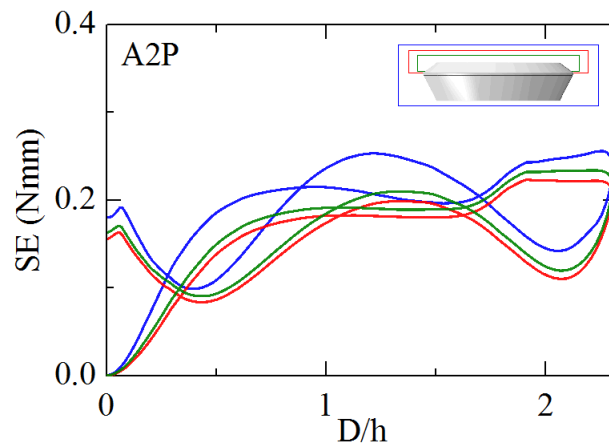
Therefore, during a force-controlled loading event, the structure snapped from Point B, where the loading critical force was, to Point E. The secondary stable state would be at Point G if the loading were carried on to Point F. Alternatively, during a



(a)



(b)



(c)

Figure 3.24: Structural responses of Model A2P_rt. (a) Vertical reaction force vs. generalised vertical displacement, (b) internal energy vs. generalised vertical displacement, and (c) strain energy vs. generalised vertical displacement.

force-controlled unloading event from Point F, a second snap-through would occur at Point H, and the structure would subsequently snap to Point K, and adapt itself to Point J, which is similar to the initial stable state but with a residual displacement between Points A and J. As compared to the semi-bistable behaviour of Model A2P, we observed the full spectrum of bistable behaviour in Model A2P_{rt}.

In summary, snap-through occurred during both loading and unloading, and a secondary stable state was found at G where $D/h = 2.21$. The loading critical force was 1.45 N at Point B where $D/h = 0.32$ and the unloading critical force was -1.56 N at Point H where $D/h = 1.86$. After a cycle of loading and unloading, the initial stable state was left was a residual displacement of $D/h = 0.42$ and a residual plastic strain up to 51.3% around two creases. A much higher residual plastic strain was observed with Model A2P_{rt} as compared to Model A1P because of the folding of the crease.

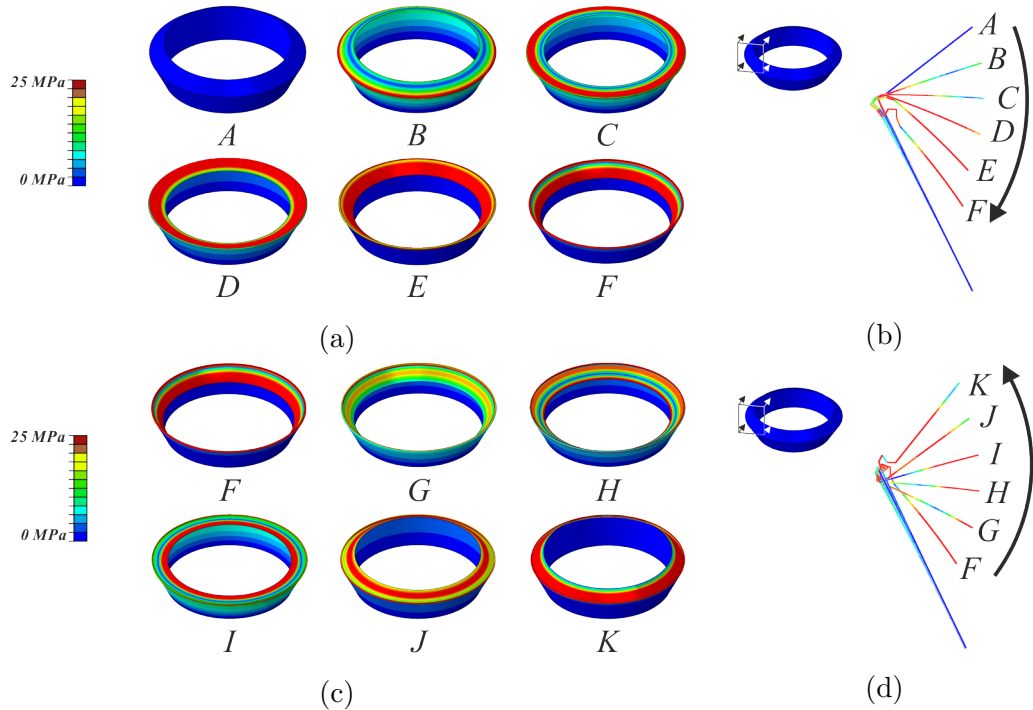


Figure 3.25: Deformed shapes and stress distribution of Model A2P_{rt}. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

3.5.3 Third Case - Models A3

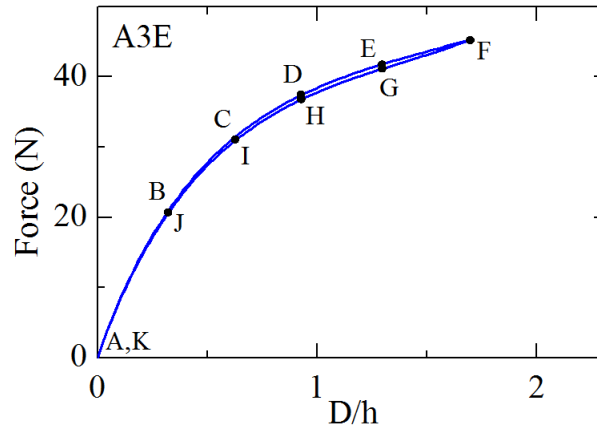
In this section, four models, A3E, A3E_rt, A3P, and A3P_rt, were analysed. In these models, the upper and lower rims of the shorter frustum were attached to two taller frusta. Two material variants of two crease thicknesses were considered here. The generalised vertical displacements presented in this section refers to the generalised relative displacement of the upper rim of the shorter frustum with respect to its lower rim. The data was presented in this format so that the results here could be directly compared to A1 and A2 Models.

A3E

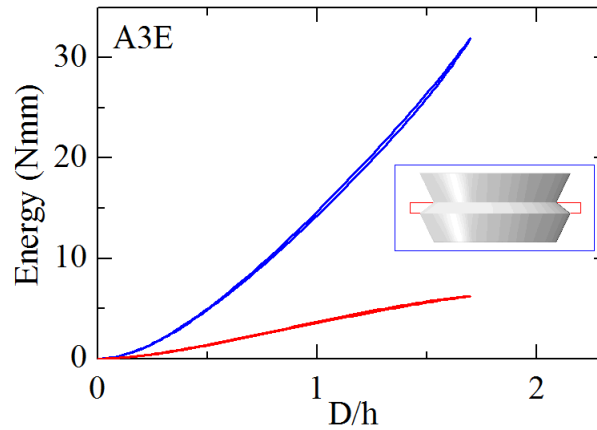
The results showing in Figure 3.26 are the plots of the vertical reaction force and the internal energy versus the generalised vertical displacement of Model A3E. The letters on the curve indicate key configurations, and the corresponding shapes and stress distribution are revealed in Figure 3.27. The blue line in Figure 3.26b refers to the energy of the entire structure, and the red line refers to the energy of the shorter frustum. Similar to the results of Model A2E, neither snap-through nor bistable behaviour occurred in Model A3E. If the force were released at any point during the loading and unloading, the structure reverted to its initial stable state. Figure 3.27b&d clearly shows that the faces of these frusta were bent and the angle at crease stayed almost the same throughout the loading and unloading. This structure acted like a vertical elastic spring, which suggests that the hinge-like behaviour of the crease has a great influence on the bistable behaviour.

The red and blue curves in Figure 3.26b diverged from the beginning of the displacement graph, and the difference in energy between the two curves, which was the internal energy stored in the two taller frusta, surpassed the values represented

by the red curve itself, which was the internal energy stored in the shorter frustum. Similar to Model A2E, this denotes that most of the deformation was occurring in the two taller frusta instead of the shorter frustum, which conflicts with the actual observations concerning the flexible straw.



(a)



(b)

Figure 3.26: Structural responses of Model A3E. (a) Vertical reaction force vs. generalised vertical displacement, and (b) internal energy vs. generalised vertical displacement.

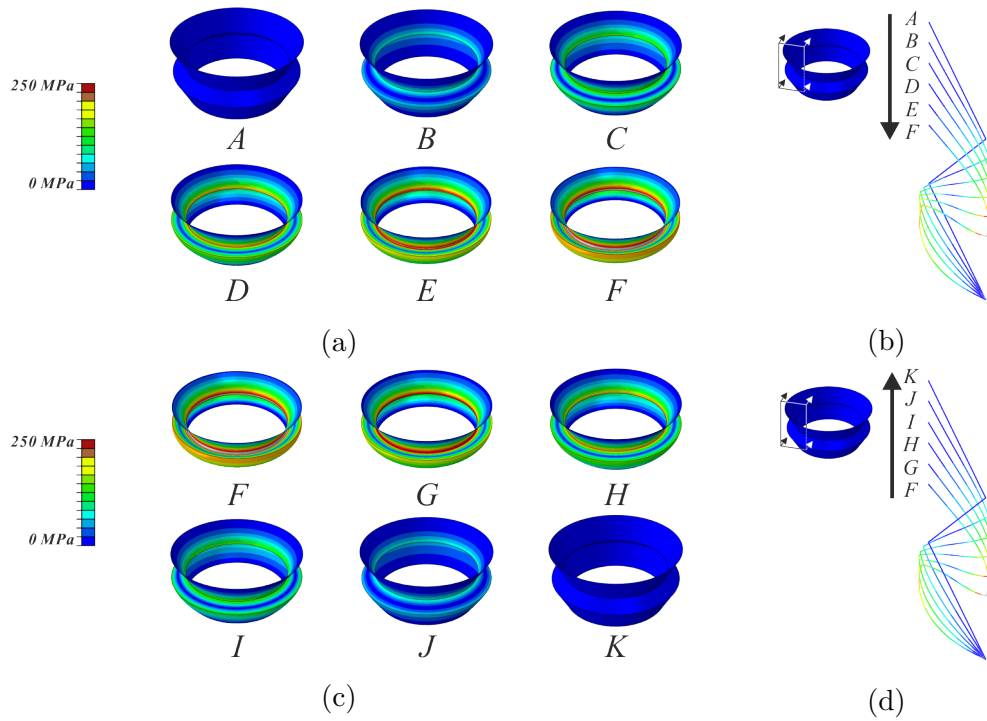


Figure 3.27: Deformed shapes and stress distribution of Model A3E. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

A3E_rt

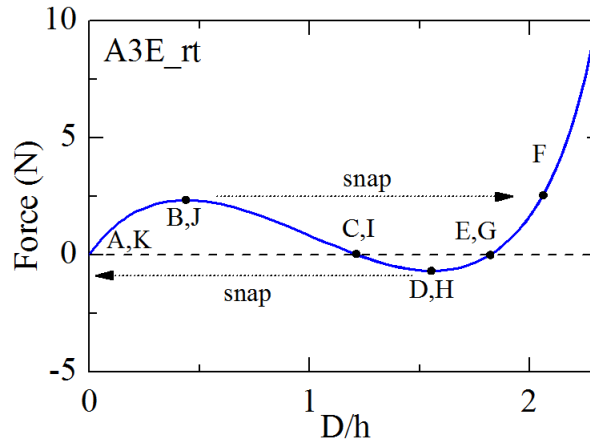
Model A3E_rt has reduced local thickness ($0.025mm$) around its creases (defined in Figure 3.9b) than Model A3E. By reducing the crease thickness of Model A3E_rt, much lower resistance against inverting would be provided by the crease.

The simulated results are showing in Figure 3.28, which are the vertical reaction force and the internal energy versus the vertical displacement of Model A3E_rt. In Figure 3.28b, the blue line refers to the energy of the entire structure, the red line refers to the energy of the shorter frustum, and the green line refers to the energy of the part of the shorter frustum with original thickness. The letters on the curve indicate the configurations of interest, and the corresponding shapes and stress distribution are revealed in Figure 3.29. In the presented displacement-controlled results, the deformation of the frustum followed the sequence of ABCDEF during loading, and FGHIJK during unloading. The loading and unloading curves coincided with each other, as a result of fully-recoverable elastic deformation. Snap-through occurred during both loading and unloading - the positive local maximum of reaction force occurred at Point B during loading, and the negative local minimum occurred at Point H during unloading. Furthermore, the bistable behaviour was observed on Model A3E_rt, and a secondary stable state was found at Point G (identical configuration as Point H), where the internal energy curve of the entire structure formed a valley.

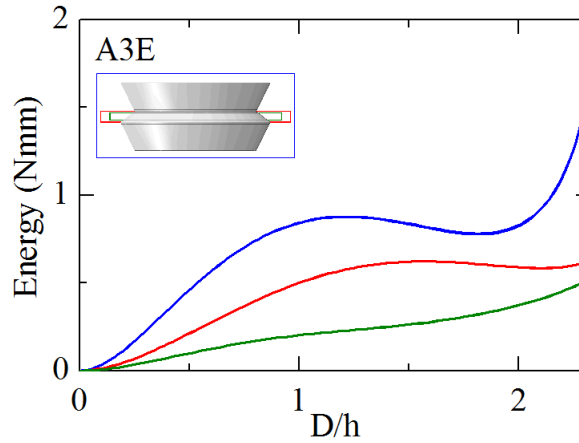
If we compare the deformed shapes between Model A3E in Figure 3.27 and Model A3E_rt in figure 3.29, a clear difference between the response of A3E_rt and A3E could be spotted: in Model A3E_rt, the angle of crease varied while the faces stayed relatively straight during both the loading and unloading; in Model A3E, there was more bending on the faces but the angles at creases remained relatively constant (90°) during the loading and unloading. In the absence of a large portion of bending energy, the structure was able to demonstrate a distinguished bistable behaviour.

In the event of actual force-controlled loading, the structure would snap at the

loading critical force at Point B to Point E, and adjust itself to the secondary stable state at Point D/G. In a similar manner with an unloading event starting from Point F, the snap-through would occur when the unloading critical force was reached at Point H, and then the structure would adjust to its initial stable state at Point K (identical configuration as Point A) after the force was released.



(a)



(b)

Figure 3.28: Structural responses of Model A3E_rt. (a) Vertical reaction force vs. generalised vertical displacement, and (b) internal energy vs. generalised vertical displacement.

The critical force during the loading phase was recorded as 2.33 N at $D/h = 0.44$ while the critical force during the unloading phase was -0.69 N at $D/h = 1.60$. The secondary stable state was at Point G where $D/h = 1.84$. In comparison with Model A1E, the loading critical force is reduced by 53.5% and the unloading critical force decreased by 82.5%, due to the realistic restricted radial movement of the upper and lower rim of the shorter frustum by the neighbouring taller frustum.

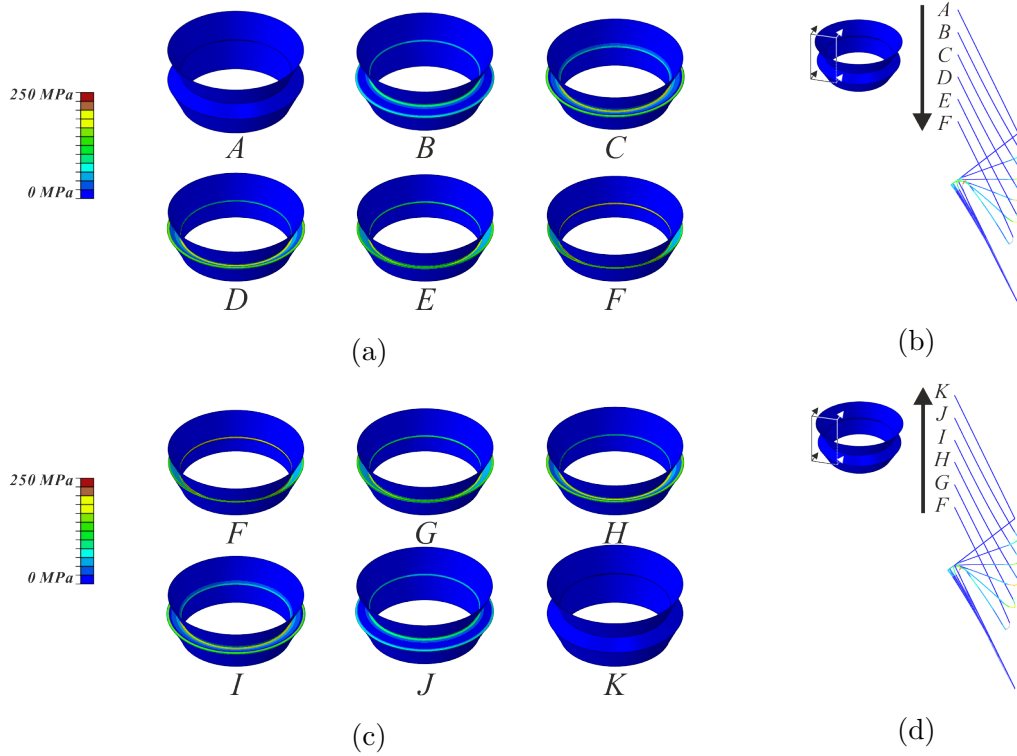
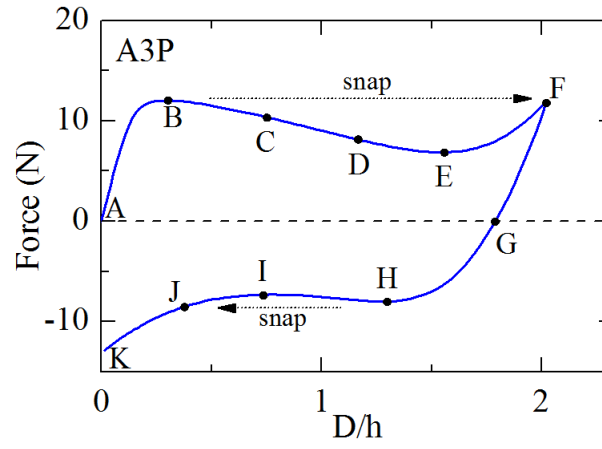


Figure 3.29: Deformed shapes and stress distribution of Model A3E_rt. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

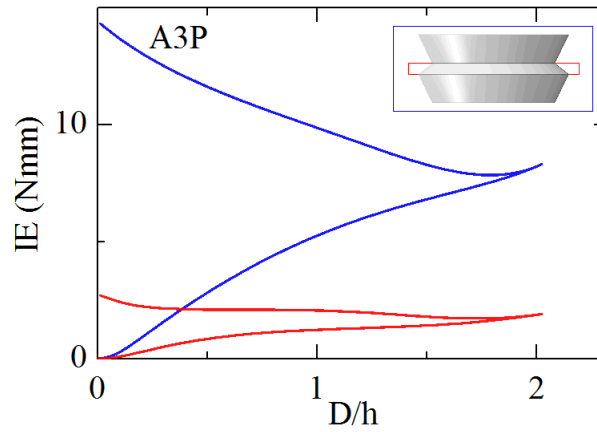
A3P

Figure 3.30 shows the plots of the vertical reaction force and strain energy versus the generalised vertical displacement of Model A3P. The letters on the curve indicate the key configurations, and the corresponding shapes and stress distribution are revealed in Figure 3.31. In Figures 3.30b&c, the blue line reflects the energy of the entire structure whilst the red line reflects the energy of the shorter frustum. In the simulation, Model A3P was only loaded up to $D/h = 2$ in comparison with $D/h = 2.3$ of Model A2P's. Beyond $D/h = 2$, no stable convergent results could be obtained in the simulation of Model A3P, due to the increased complexity of the geometry and highly non-linear nature of the problem.

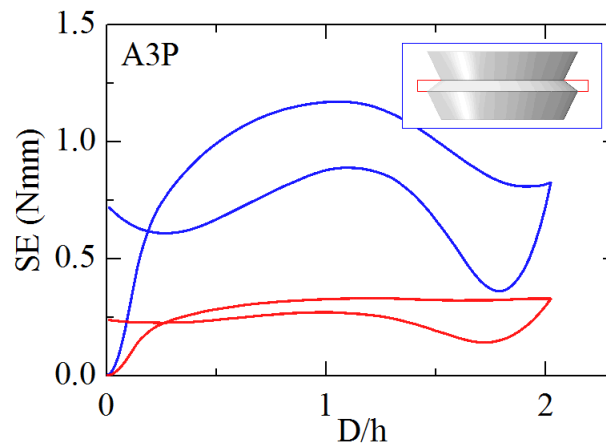
In general, the response of Model A3P shared great similarity as compared to Model A2P. They all demonstrated the same semi-bistable behaviour: the response curves were comparable but different in magnitude. The new stable state of Model A3P occurred at Point G where $D/h = 1.79$. The critical force during the loading phase was 12.04 N at Point B where $D/h = 0.30$ and the critical force during the unloading phase was -8.00 N at Point H where $D/h = 1.28$.



(a)



(b)



(c)

Figure 3.30: Structural responses of Model A3P. (a) Vertical reaction force vs. generalised vertical displacement, (b) internal energy vs. generalised vertical displacement, and (c) strain energy vs. generalised vertical displacement.

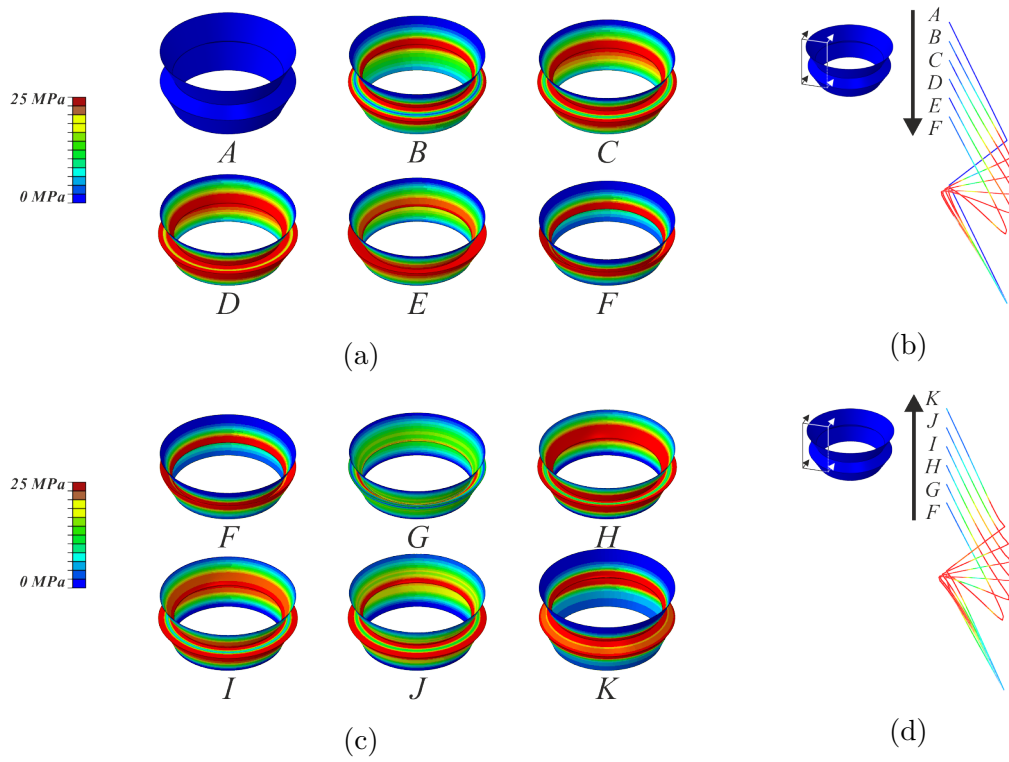


Figure 3.31: Deformed shapes and stress distribution of Model A3P. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

A3P_{rt}

Model A3P_{rt} has locally reduced thickness (0.025 mm) around its crease (defined in Figure 3.9b) in comparison with Model A3P. By reducing the crease thickness, the resistance against the inverting of structure would be dramatically limited.

Figure 3.32 shows the plots of the vertical reaction force, the internal energy, and the strain energy against the vertical displacement of Model A3P_{rt}. The letters on the curve indicate the key configurations, and the corresponding shapes and stress distribution are revealed in Figure 3.33. In Figures 3.32b&c, the blue line refers to the energy of the entire structure while the red line to the energy of the shorter frustum, and the green line to the energy of the part of the shorter frustum with original thickness.

To a great extent, the response of Model A3P_{rt} was similar to Model A2P_{rt}. Bistable behaviour was observed in both models, and the response curves were comparable but different in magnitude.

The secondary stable state of Model A3P_{rt} was at Point G where $D/h = 2.28$. The critical force during the loading phase was 1.72 N at Point B where $D/h = 0.28$ and the critical force during the unloading phase was -2.17 N at Point H where $D/h = 2.23$. After a cycle of loading and unloading, the initial stable state had a residual displacement of $D/h = 0.88$ and residual plastic strain up to 39.7% around two creases. A much higher residual plastic strain of Model A3P_{rt} than that of Model A1P was because of the substantial folding of the creases.

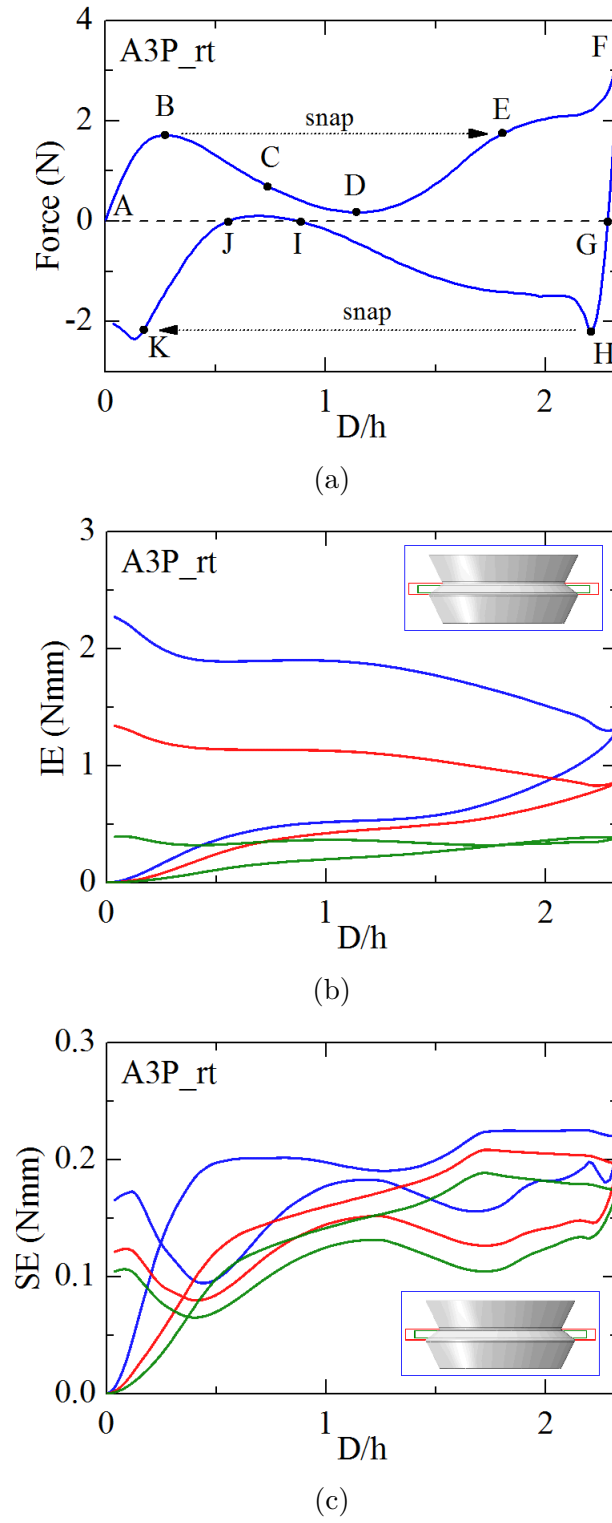


Figure 3.32: Structural responses of Model A3P_rt. (a) Vertical reaction force vs. generalised vertical displacement, (b) internal energy vs. generalised vertical displacement, and (c) strain energy vs. generalised vertical displacement.

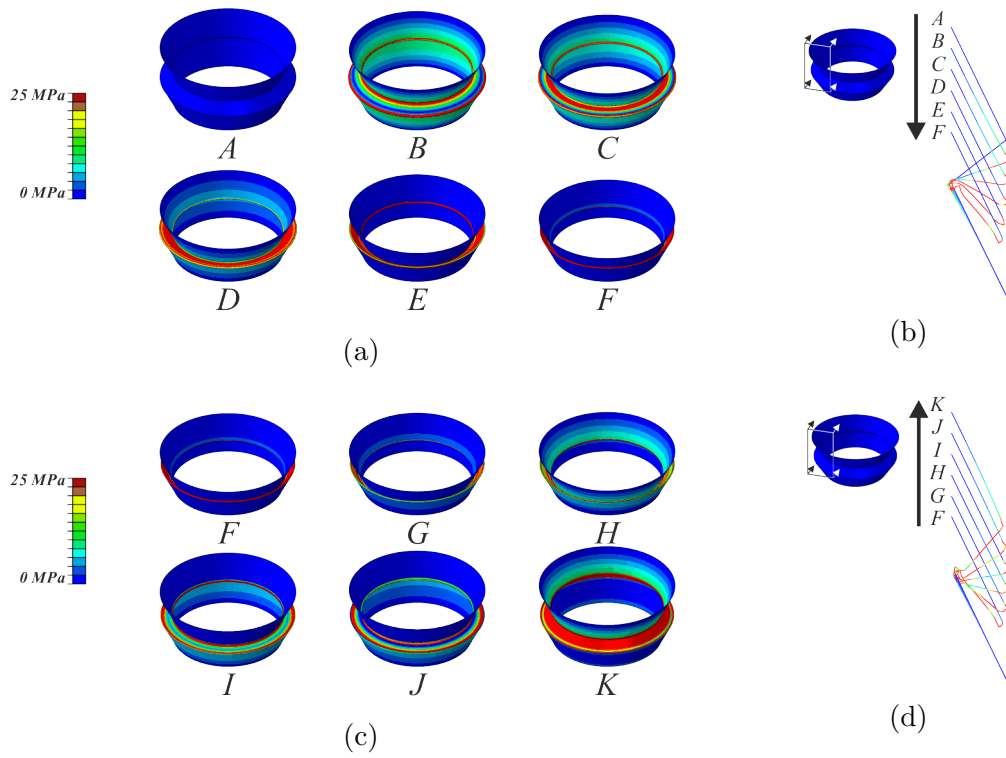


Figure 3.33: Deformed shapes and stress distribution of Model A3P_rt. (a) Isometric view (loading), and (b) half sectional view (loading); (c) isometric view (unloading), and (d) half sectional view (unloading).

3.5.4 Discussion

The results of the investigation are summarised in Table B.1 in Appendix. One of the objectives of this investigation is to obtain valid reference models for parametric studies. The reference models should be able to describe the bistable behaviour of frusta and yet conform to simple geometries that allow the performance of reliable FEM simulations.

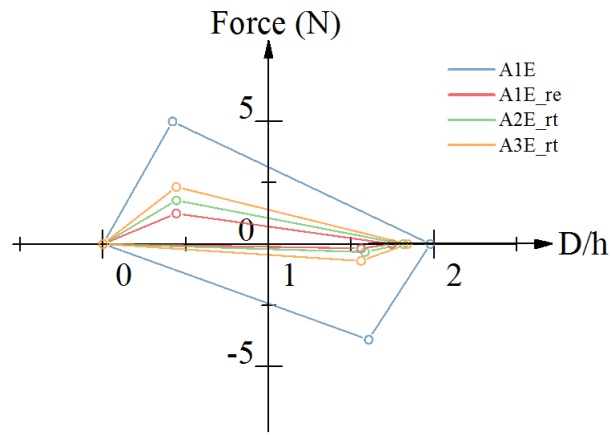
The structural response could be described in four important stages of the entire loading and unloading: the critical points during loading and unloading, and the initial and the secondary stable states. In some cases, a residual displacement is observed after a cycle of loading and unloading in the reversion to the initial stable state. In Figure 3.34, the forces and the corresponding displacement of these stages were plotted and the responses form an envelope defined by these four or five points. A lack of either snap-through or bistable behaviour was observed for Models A2E and A3E, and have accordingly been omitted in the Figures.

1. The responses of LE frusta are summarised in Figure 3.34a. Among the results of all the LE Models, Model A1E gave the outer envelope and Model A1E_re gave the inner envelope to the problem. Model A1E's bottom rim was constrained from expanding, which suggests that its results could be slightly overestimated. On the other hand, Model A3E_rt had the most realistic boundary condition set-ups among all models however its results should be slightly underestimated because its thicknesses around the creases were reduced so that less bending would be provided by the surface of the shorter frustum. Both Models A1E and A3E_rt showed bistable behaviour during both loading and unloading, and its responses were similar to a typical straw segment. However, the computer resources to run Model A3E_rt would be much more than one to run Model A1E. Furthermore there would be risks of non-convergent solutions if variations of Model A3E_rt were used in later parametric studies because of its complexity

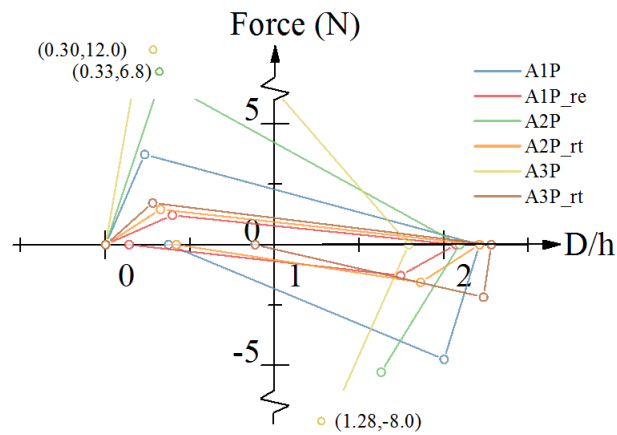
of geometry. Considering all the factors, Model A1E is the best valid choice of reference LE model in the parametric studies.

2. The simulations of EPP frusta also revealed that the minimum reaction forces were higher in value during unloading phase than during loading phase. This is because the excessive plastic deformation has increased the bending capacity of the shell. It is best illustrated on the stress profile through the thickness in Figure 3.35.
3. The responses of EPP frusta are summarised in Figure 3.34b. Among all the results, Model A3P yielded the outer envelope while Model A1P_{-re} resulted in the inner envelope. Semi-bistable behaviour was predicted by Models A2P&A3P as noted from the absence of recovery to the initial stable state after a cycle of loading and unloading; bistable behaviour was found on the remaining models with various residual displacement between the recovered and initial stable states after a cycle of loading and unloading. Model A1P's bottom rim was constrained from expanding, which suggests that its results could be slightly overestimated. On the other hand, Model A3P_{-rt} had the most realistic setups in simulating the problem among all models; however, the forces are most likely to be slightly underestimated because its thicknesses around the creases were reduced such that less bending would be provided by the surface of the shorter frustum. By following the same logic in choosing the reference LE model, Model A1P was found to be the most suitable as the reference EPP model in the parametric studies.

In conclusion, Models A1E and A1P offer conservative estimations of the critical force. In the following section on parametric studies, the variants are all based on Models A1E and A1P, and likewise provide conservative solutions as well.



(a)



(b)

Figure 3.34: Summary of response profiles in the interaction investigation. (a) LE models, and (b) EPP models.

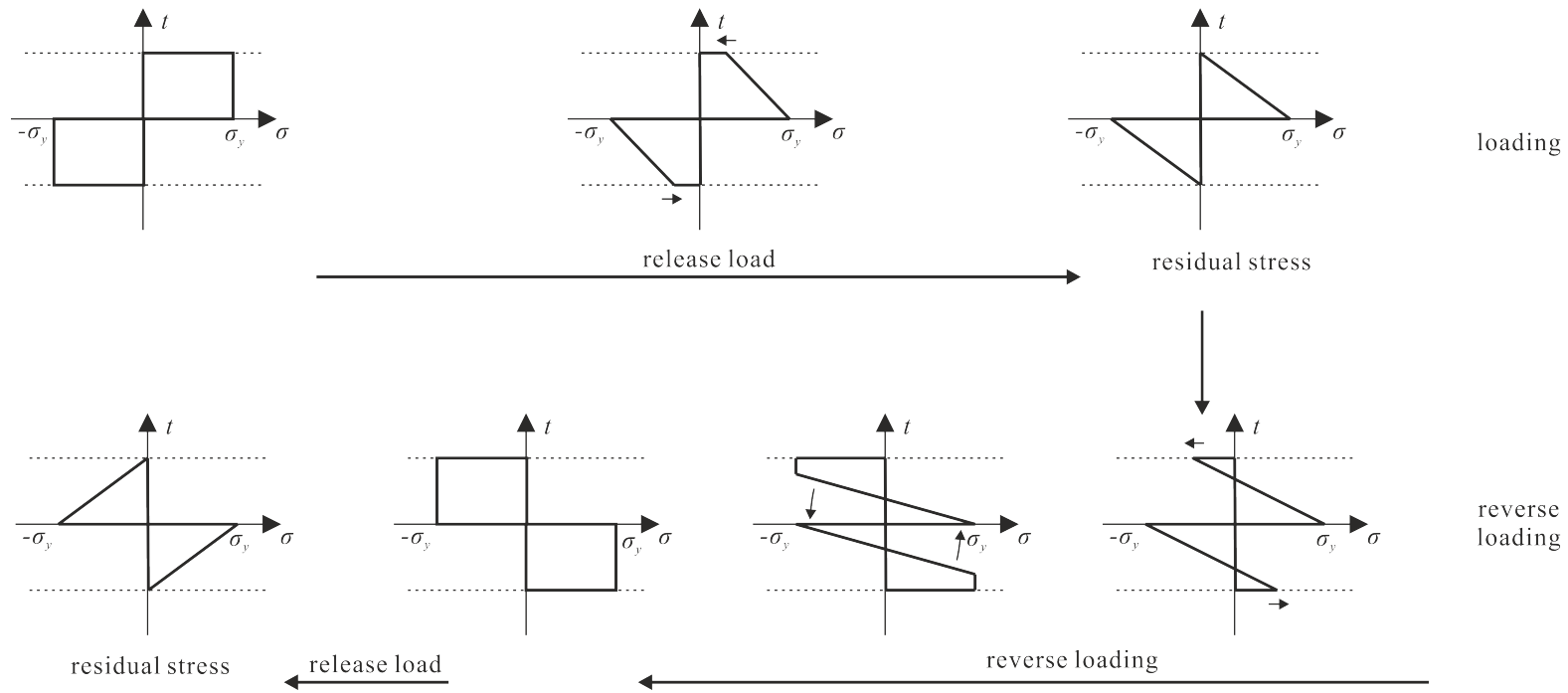


Figure 3.35: Stress profile of plastic loading and reverse unloading.

3.6 Parametric Study

3.6.1 Configuration of Parametric Variations

According to the previously reviewed work on the buckling and post-buckling of spherical shells, the buckling force is dependent on their geometric and material properties (Zoelly, 1915; van der Neut, 1932). With respect to material factors, the critical force is proportional to $E/\sqrt{1-\nu^2}$, while for geometric factors, the critical force depends on the thickness t , the thickness to span ratio t/R , and how shallow the spherical shell is. A frustum shell closely resembles the geometry of a spherical shell, so the critical force that triggers its symmetrical buckling is likely to be governed by similar mathematical relationships. Although Lackman and Penzien (1960) theoretically calculated the buckling force of the frustum shells and proposed a formula (Equation 2.7) that has a similar mathematical form to the buckling of the spherical shell, the formula lacks agreement with experimental results and acts as a conservative derivation of the critical force. In this parametric study, we endeavour to find more reliable formulas to describe the bistable behaviour of the frustum shells.

We simplified the equations that were derived for the spherical shell to retain the following dimensionless terms but allowed the order of their relationship to be determined by the simulation results. The dimensionless displacement, D^* is defined as:

$$D^* = D/h, \quad (3.3)$$

where D is the displacement of the loading and h is the height of the structure. In this definition, $D/h = 2$ is the displacement of the mirror position about the bottom rim of the frustum.

The dimensionless load, P^* is defined as:

$$\begin{aligned} P^* &= P \frac{\sqrt{1 - \nu^2}}{ER_b^2(t/R_b)^a\omega^b} \\ &= P \frac{\sqrt{1 - \nu^2}}{ER_b^{2-a}t^a\omega^b}, \end{aligned} \tag{3.4}$$

where a and b are constants to be derived from the simulation results, ω in radians is in itself a dimensionless term, t/R_b is a dimensionless term describing the thickness-to-span ratio, and R_b^2 relates to the vertical projected area and cancels out the units.

Three sets of variations of the shorter frustum listed below were considered:

Height Variation: The height h was increased from $0.43mm$ and to a maximum of $2.0mm$ while the bottom radius, the tilt angle and the thickness were left unchanged. The top radius R_t was varied according to the change in h .

Thickness Variation: Only the thickness t was altered.

Slant Angle Variation: The slant angle ω was varied around 38.02° while the slant height l_s , the bottom radius r_B and the thickness t were left unchanged from those of the reference frustum.

The relationship between P and P^* in Equation 3.4 allows us to simulate values for a and b to until the closest alignment in dimensionless critical forces is obtained despite changes in h .

3.6.2 Height Variation

Including the reference configuration, frusta with eight height values were analysed, as given in Table 3.5. Other dimensions were obtained from those shown in Table 3.1.

Table 3.5: Parametric study: height variation.

Models	Height, h	Material
A1E (reference)	0.43 mm	LE
A1E_h1	0.65 mm	LE
A1E_h2	0.70 mm	LE
A1E_h3	0.80 mm	LE
A1E_h4	1.00 mm	LE
A1E_h5	1.15 mm	LE
A1E_h6	1.65 mm	LE
A1E_h7	2.00 mm	LE
A1P (reference)	0.43 mm	EPP
A1P_h1	0.65 mm	EPP
A1P_h2	0.70 mm	EPP
A1P_h3	0.80 mm	EPP
A1P_h4	1.00 mm	EPP
A1P_h5	1.15 mm	EPP
A1P_h6	1.65 mm	EPP
A1P_h7	2.00 mm	EPP

LE Material

The results are plotted in Figure 3.36, and both the value of the critical forces and the curve shape of the loading response varied with increments in h . Initially, the reference frustum produced a sine-like curve; and as the height is increased, P begins to plateau more extensively around $D/h = 1$ and appear more monotonic, resulting in a progressive inversion when $h > 1.00$ mm. Figure 3.37 shows the deformed shapes and stress distribution at $D/h = 1$. For the reference model, with shortest h , the stress is highest within the inner circumference; and as h increases, the stress is emphasised within a ‘knuckle’ region that begins to form during the snap-through and separates the inverted region from the outer circumference of the frustum. The highest stress

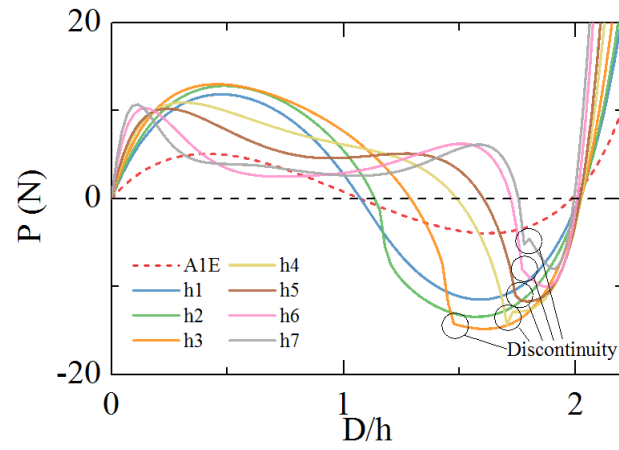
zone shifted to the knuckle region. The critical force increases from 5 N (Model A1E) to 13 N (Model A1E_h3) where the knuckle region was first observed; thereafter, the critical force hovers around 11 N for the subsequent models but the snap-through limit point progressively shifted to the left of the D/h axis (0.5 to 0.2). Although the original geometric shape was similar to the frustum studied by Deshpande and Fleck (2003), the latter does not cater to the purely linear elastic material. In our models, the formation and intensification of the knuckle region with increasing height introduces monotonic bending effects that strengthen to the extent where monotonic responses started to become more dominant than non-monotonic membrane effects in the overall response. Interestingly among all LE models of various heights, a secondary stable state was found at $D/h = 2$ where the structure was fully inverted. For models that involved knuckle formation, a discontinuity (labelled in Figure 3.36a) was observed just before the resolution of the secondary stable state. This was due to a second buckling that completes the inversion that progresses with increasing D/h to achieve its secondary stable state (Figure 3.38). At that point, the radial curvature of the frusta inverts from positive to negative. This occurs when the new configuration is more energetically favourable which can be seen on Figure 3.36b where a sudden drop of energy happened on Model A1E_h3-7.

EPP Material

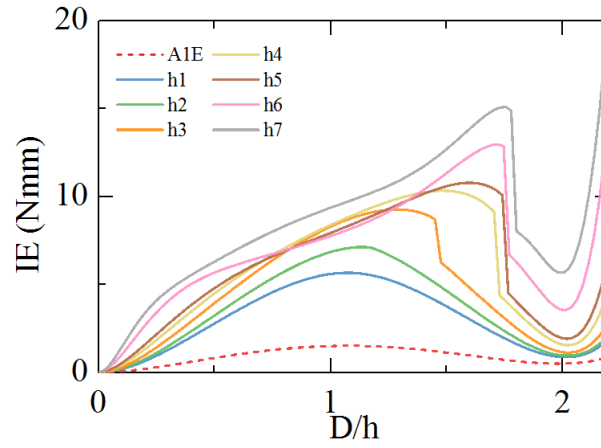
The results using EPP are shown in Figure 3.39. The overall behaviour of the EPP models is similar to A1E models with varying height. The critical force increases from 3.71 N in Model A1P to 6.58 N in Model A1P_h3, and the formation of a knuckle region was observed in subsequent models as the height increases, where a lowering of the initial peak was observed in subsequent models between A1P_h3 (6.58 N) and A1P_h7 (3.54 N); in the latter models, we began to observe a secondary peak around $D/h = 1.5$ that begins to eclipse the initial peak with increasing h . Behaviourally, we

started to observe a profile with two snaps - with a first occurrence near the beginning and a second towards the end of the graph - instead of the original profile depicting a single snap; but with a similar progressive inversion in the middle of the deformation profile.

Figure 3.40 shows the deformed shapes and stress distribution at $D/h = 1$. The knuckle regions are the toroidal red regions on the deformed shapes, which indicates the fully yielded regions. Figure 3.41 shows the formation of the knuckle region of Model A1P_h7 was formed. When juxtaposed with Deshpande and Fleck (2003)'s model, our FEM results agree best in models where the progressive inversion is particularly pronounced. A detailed comparison between the FEM results and Deshpande and Fleck (2003)'s analyses is given in Section 3.7 of this Chapter.



(a)



(b)

Figure 3.36: Structural responses of Models A1E&A1E_h1-7. (a) Vertical reaction force P vs. D/h , and (b) internal energy IE vs. D/h .

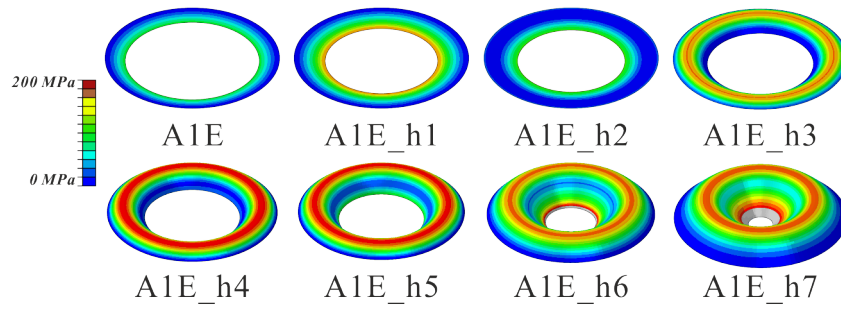


Figure 3.37: Deformed shapes and stress distribution of Model A1E and its height variants at $D/h = 1$.

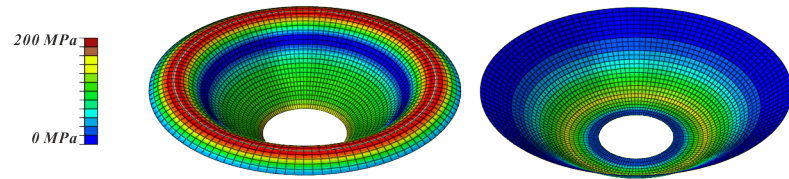


Figure 3.38: The second buckling of Model A1E_h6, which caused the discontinuity labelled in Figure 3.36a

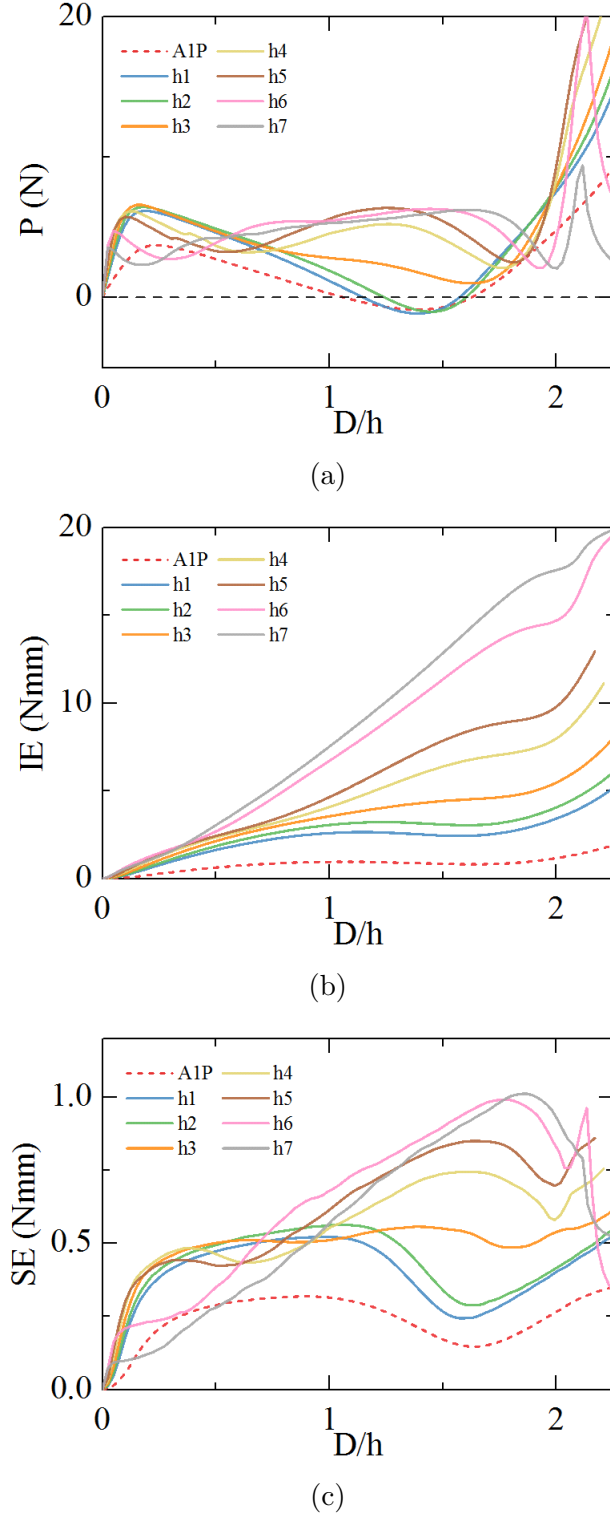


Figure 3.39: Structural responses of Models A1P&A1P_h1-7. (a) Vertical reaction force P vs. D/h , (b) internal energy IE vs. D/h , and (c) Elastic strain energy SE vs. D/h .

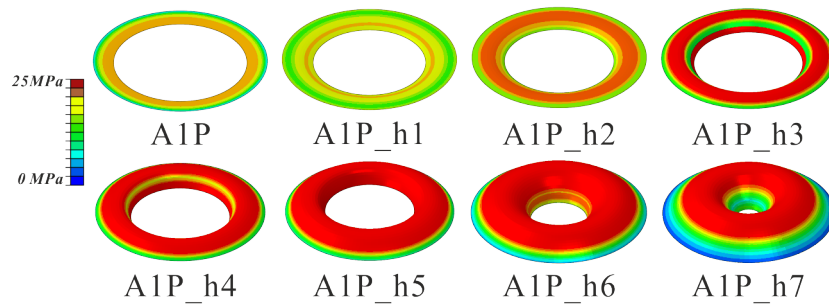


Figure 3.40: Deformed shapes and stress distribution of Model A1P and its height variants at $D/h = 1$.

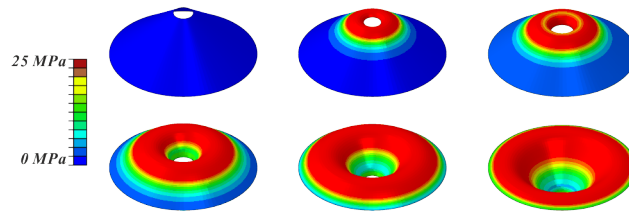


Figure 3.41: The progressive inversion of Model A1P_h7.

3.6.3 Thickness Variation

Including the reference configuration, frusta of seven thickness variations were analysed (Table 3.6).

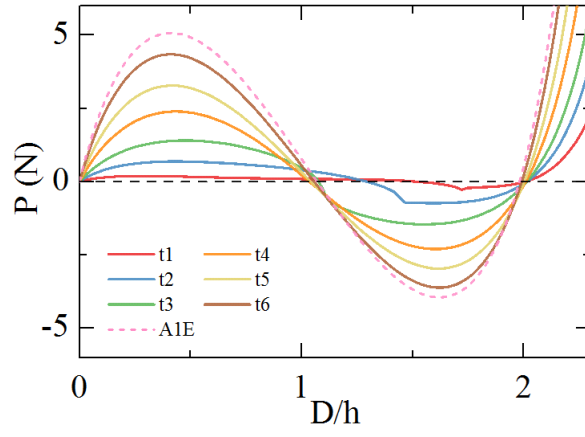
Table 3.6: Parametric study: thickness variation.

Models	Thickness, t	Material
A1E_t1	0.02 mm	LE
A1E_t2	0.035 mm	LE
A1E_t3	0.05 mm	LE
A1E_t4	0.075 mm	LE
A1E_t5	0.1 mm	LE
A1E_t6	0.13 mm	LE
A1E (reference)	0.15 mm	LE
A1P_t1	0.05 mm	EPP
A1P_t2	0.035 mm	EPP
A1P_t3	0.05 mm	EPP
A1P_t4	0.075 mm	EPP
A1P_t5	0.1 mm	EPP
A1P_t6	0.13 mm	EPP
A1P (reference)	0.15 mm	EPP

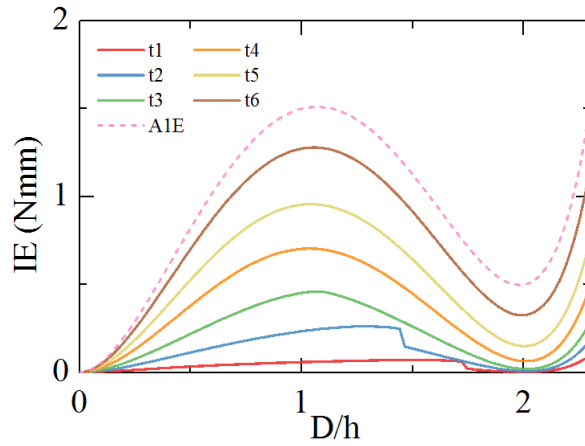
LE Material

The results are shown on Figure 3.42.

As the thickness varies, we expected the simulated critical forces to increase, as demonstrated by Figure 3.42a. In this figure, curve shapes of A1E and A1E_t4-6 are comparable, but for graphs t1-3, the graph increasingly plateaus with decreasing t before $D/h = 1.5$. We observed discontinuities in Models A1E_t1&2, at which the radial curvature of the frusta inverts from positive to negative, showing in Figure 3.43. The latter occurs when the configuration of negative curvature is more energetically favourable, as shown in Figure 3.42b where a momentous drop in internal energy was observed. In conclusion, although the snap-through behaviour was observed on all simulated frusta, variations in the thickness of the LE frustum impact the collapse



(a)



(b)

Figure 3.42: Structural responses of Models A1E&A1E_t1-6. (a) Vertical reaction force P vs. D/h , and (b) internal energy IE vs. D/h .

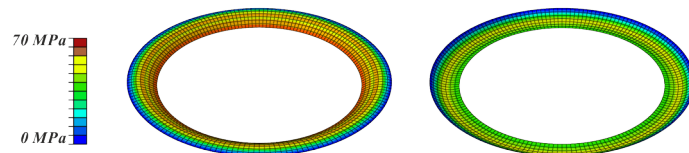


Figure 3.43: Second buckling of Model A1E_t2.

mechanism, but does not alter the D/h value where the secondary stable states occurred.

EPP Material

The simulated results with EPP material are shown in Figure 3.44, and the results are comparatively similar to A1E models with varying thickness. On the P graph (Figure 3.44a), Models A1P and A1P_t4-6 retain the positive correlation between t and P . Models A1P_t1-3 had decreasing critical force as thickness decreases. All models demonstrated the bistable behaviour.

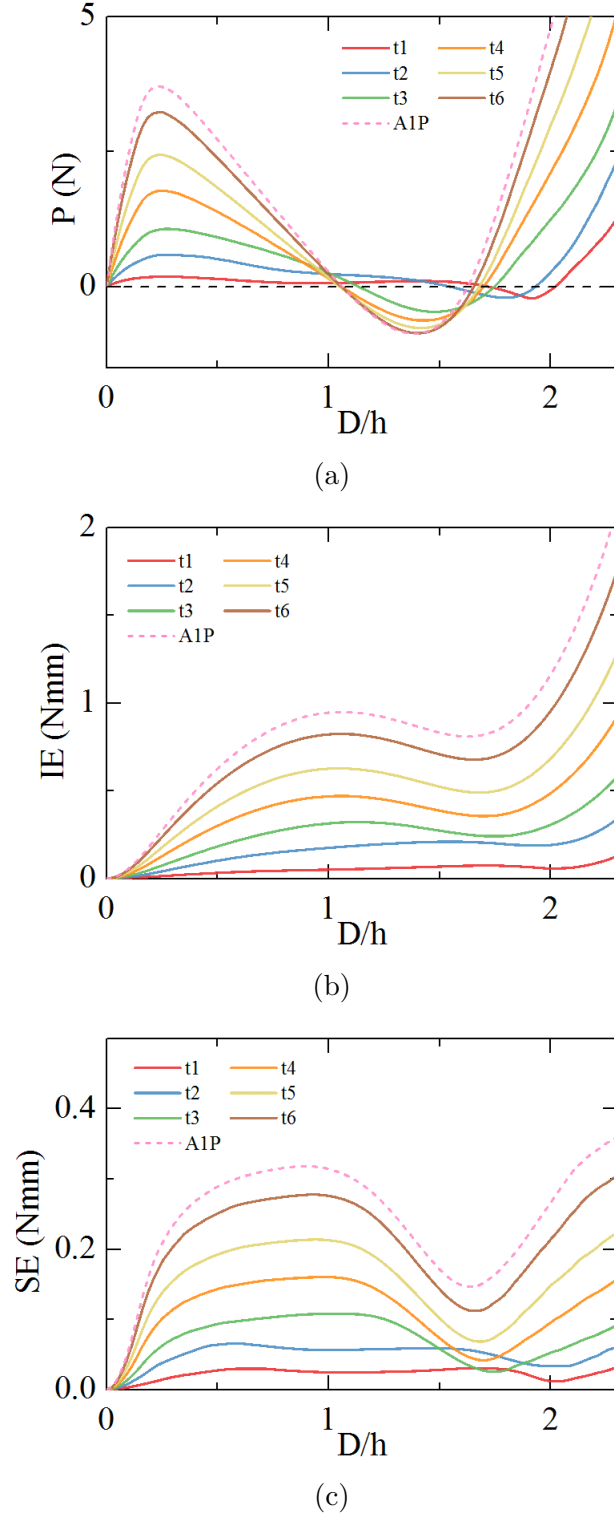


Figure 3.44: Structural responses of Models A1P&A1P_t1-6. (a) Vertical reaction force P vs. D/h , (b) internal energy IE vs. D/h , and (c) elastic strain energy SE vs. D/h .

3.6.4 Slant Angle Variation

Including the reference configuration, frusta with nine different slant angles were analysed. These are listed in Table 3.7. We limited the range of slant angles to be $< 90^\circ$, as beyond this angle, frusta would undergo cylindrical shell buckling when placed under an axial compressive load, and the latter has been investigated by many researchers (Alexander, 1960; Abramowicz and Jones, 1986; Abramowicz and Jones, 1984) and has seen widespread use in industry.

Table 3.7: Parametric study: slant angle variation.

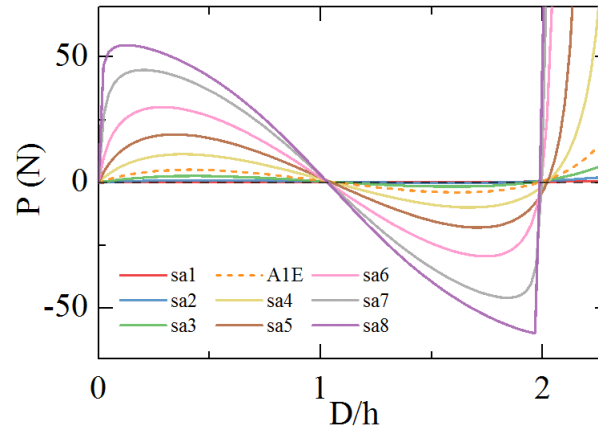
Models	Slant Angle, ω	Material
A1E_sa1	10 °	LE
A1E_sa2	20 °	LE
A1E_sa3	30 °	LE
A1E (reference)	38.02 °	LE
A1E_sa4	50 °	LE
A1E_sa5	60 °	LE
A1E_sa6	70 °	LE
A1E_sa7	80 °	LE
A1E_sa8	85 °	LE
A1P_sa1	10 °	EPP
A1P_sa2	20 °	EPP
A1P_sa3	30 °	EPP
A1P (reference)	38.02 °	EPP
A1P_sa4	50 °	EPP
A1P_sa5	60 °	EPP
A1P_sa6	70 °	EPP
A1P_sa7	80 °	EPP
A1P_sa8	85 °	EPP

LE Material

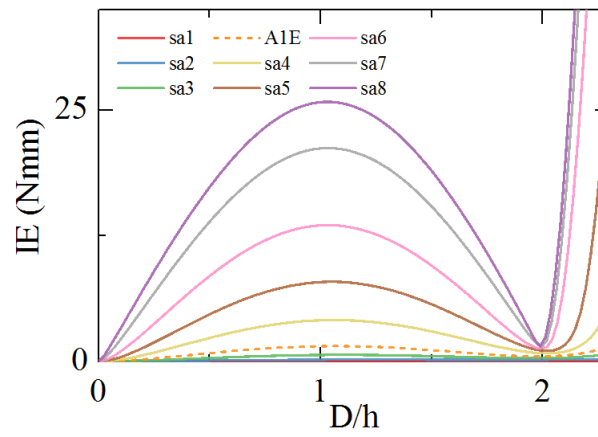
Figure 3.45a shows that the critical forces increases with the slant angle and snap-through behaviour was found on all simulated frusta except Model A1E_sa1. We also found that critical force is positively related to ω , which is shown in the plot of P versus D/h in Figure 3.45a. In addition, P_{cr} was observed to vary with ω , as the critical point shifts from $D/h = 0.70$ to $D/h = 0.14$ as ω increases from 10° to 85° , and these shifts in P_{cr} were not observed in the parametric studies that varied h and t .

EPP Material

Figure 3.46a reveals a clear trend of increasing critical force with incremental changes in the slant angle. The critical force was also observed to be positively related to ω , which is shown in the plot of P versus D^* in Figure 3.46a. The bistable behaviour was found on all frusta expect Model A1P_sa1. In a trend that is similar to our results with LE models, the critical point shifts from $D/h = 0.70$ to $D/h = 0.02$ as ω increases from 10° to 85° .



(a)



(b)

Figure 3.45: Structural responses of Models A1E&A1E_sa1-8. (a) Vertical reaction force P vs. D/h , and (b) internal energy IE vs. D/h .

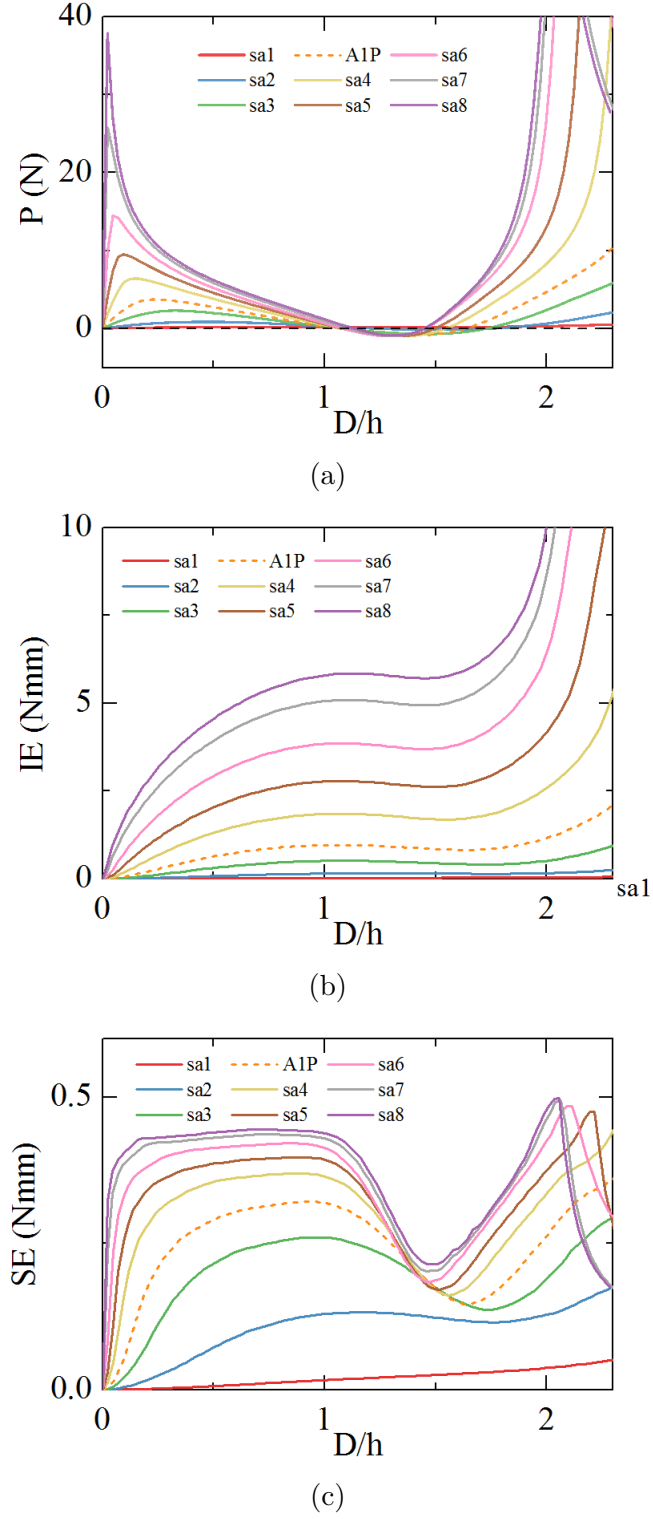


Figure 3.46: Structural responses of Models A1P&A1P_sa1-8. (a) Vertical reaction force P vs. D/h , (b) internal energy IE vs. D/h , and (c) elastic strain energy SE vs. D/h .

3.6.5 Determining a and b in Equation 3.4

According to the previous results, P_{cr} was observed to increase with t and ω while h had limited effects on the magnitude of P_{cr} . In Equation 3.4, P_{cr} is directly proportional to t^a and ω^b . In order to determine suitable values for a and b , we logarithmically drew scatter graphs of P_{cr} of both LE and EPP material against the corresponding t and ω in Figure 3.47, as well as a corresponding linear trend line in each of the figures. The gradient of these trend lines reveals the following information about the values of a and b ,

$$a = 1, \quad b = \begin{cases} 3 & \text{for LE material} \\ 2.25 & \text{for EPP material.} \end{cases} \quad (3.5)$$

Subsequently, we calculated P^* in Equation 3.4 with these values of a and b , and plotted the results of all three parametric studies, with the results of LE material in Figure 3.48 and the results of EPP material in Figure 3.49. With these a and b values, P^* plots revealed a close alignment of the critical forces despite changes in the parameter of interest.

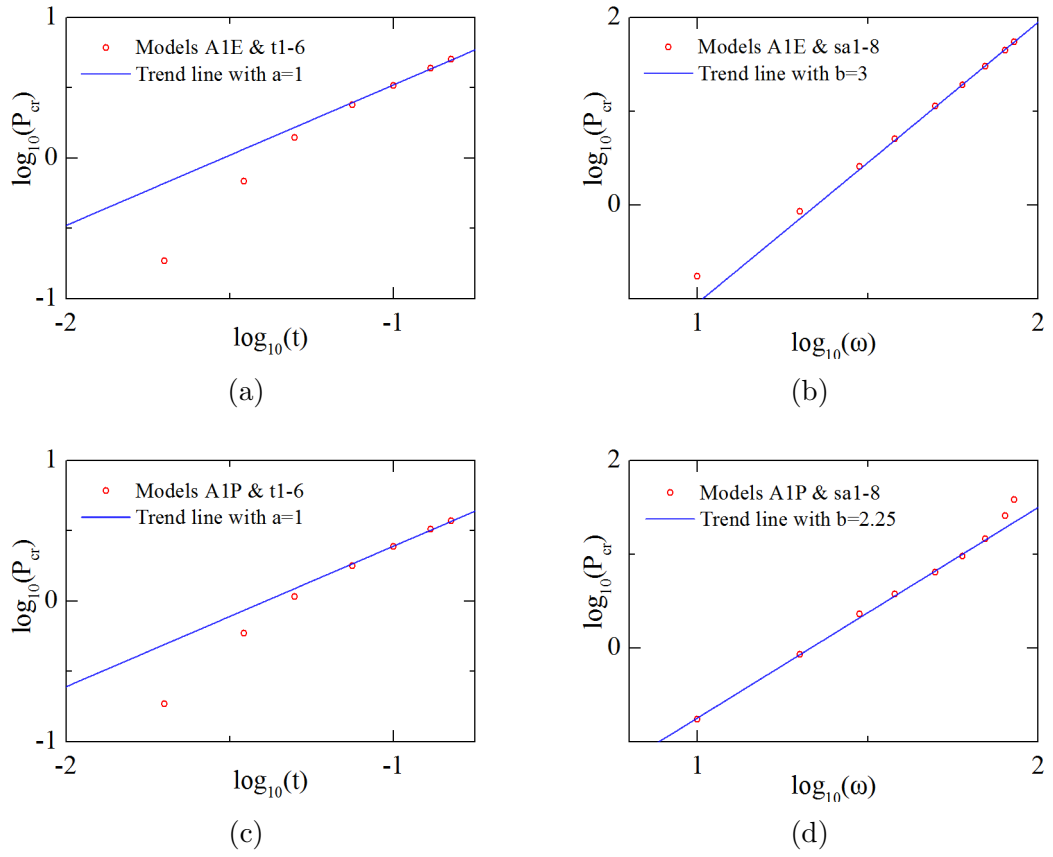
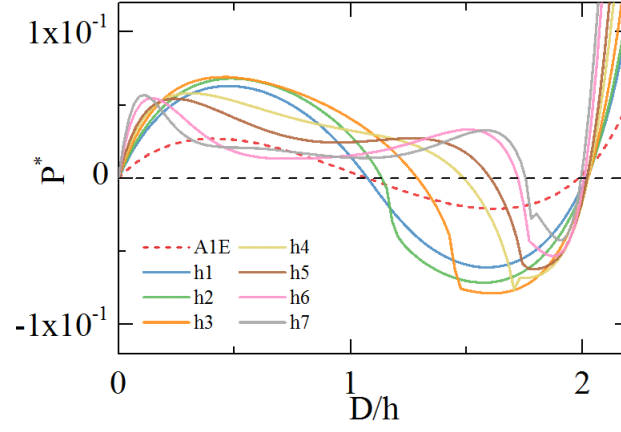
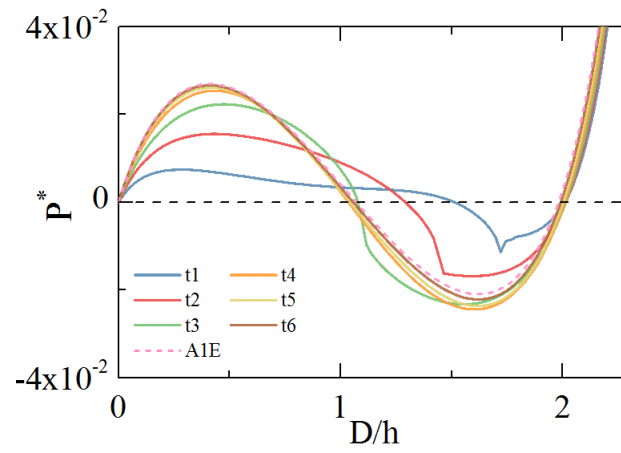


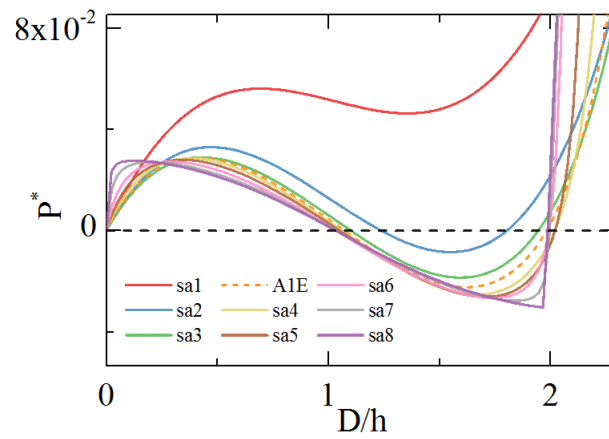
Figure 3.47: Logarithmically plots of P_{cr} vs. t and ω , with trend lines. (a) and (b) LE material, and (c) and (d) EPP material.



(a)

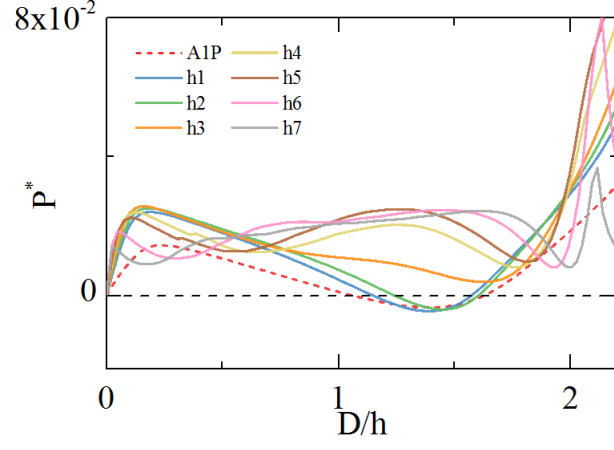


(b)

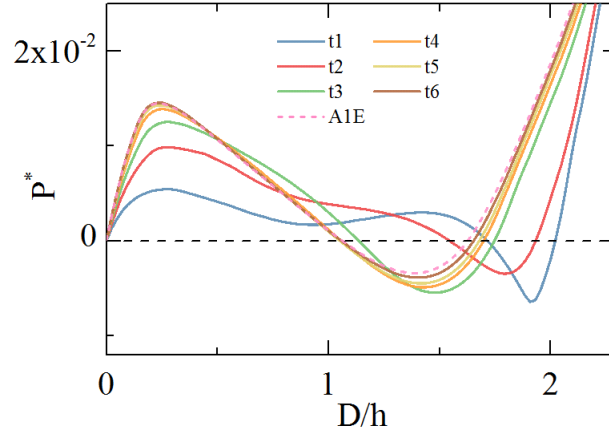


(c)

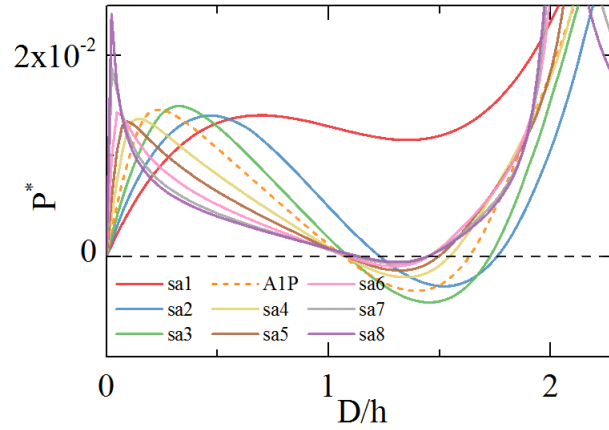
Figure 3.48: Generalised response profiles of LE Model A1E, with (a) h variants, h_1 to h_7 , (b) t variants, t_1 to t_6 , and (c) ω variants, sa_1 to sa_8 .



(a)



(b)



(c)

Figure 3.49: Generalised response profiles of EPP Model A1P, with (a) h variants, h1h7, (b) t variants, t1t6, and (c) ω variants, sa1sa8.

3.6.6 Discussion

In this parametric study, we have demonstrated how the change of three geometric parameters, h , t , and ω , could affect the bistable response of a frustum shell. We can summarise the findings as following.

1. In the LE material models, the critical force P_{cr} can be described by the following formula,

$$P_{cr} = P_{cr}^* \cdot \frac{ER_b t \omega^3}{\sqrt{1 - \nu^2}}, \quad (3.6)$$

where P_{cr}^* is P^* at its local peak.

Figure 3.50 summarises the peak forces in dimensionless terms P_{cr}^* with respect to its parametric variables (h , t , and ω), which allows us to determine a constant, c , to represent a P_{cr}^* value that is stable within a range of parametric values.

- Concerning height h , P_{cr}^* was relatively stable in Models A1E_h4-7 in Figure 3.50a, which gives $c = 5.5 \times 10^{-2}$ in the region $h = [1.0 \text{ mm}, 2.0 \text{ mm}]$ where ω and t are at their reference configuration. Brodland and Cohen (1987) suggested that for spherical shells, membrane effects are characterised by symmetric in load-deflection behaviour while bending effects are characterised by monotonic load-deflection behaviour; a similar logic could apply to the frustum shell. In the flat region, the bending effects were stronger than the membrane effects, as suggested by the extensive monotonic responses shown in Figure 3.48a.
- For thickness t and slant angle ω , c was determined to be approximately 2.6×10^{-2} in the region of $\omega = [30^\circ, 85^\circ]$ and $t = [0.075 \text{ mm}, 0.15 \text{ mm}]$ but the height is at its reference configuration $h = 0.43 \text{ mm}$. In these range of values, the membrane effects were stronger than the bending effects as the response appeared more symmetric than monotonic.

- We may speculate that for $h = [0.65 \text{ mm}, 2.0 \text{ mm}]$, $c = 5.5 \times 10^{-2}$ when $\omega = [30^\circ, 85^\circ]$ and $t = [0.075 \text{ mm}, 0.15 \text{ mm}]$ because of the likely unchanged response. However further simulations are needed to confirm this hypothesis.

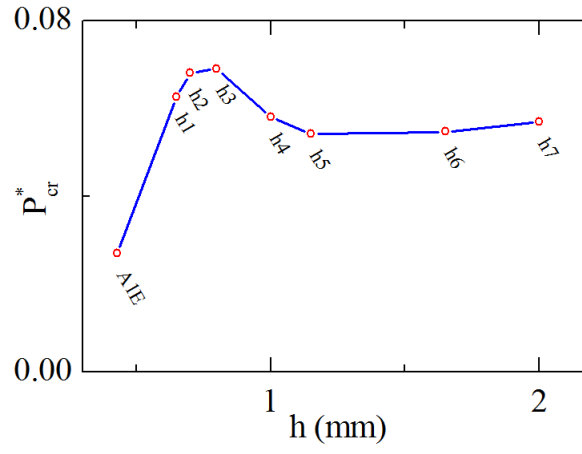
2. In the EPP material models, the critical force P_{cr} can be described by the following formula,

$$P_{cr} = P_{cr}^* \cdot \frac{ER_b t \omega^{2.25}}{\sqrt{1 - \nu^2}}, \quad (3.7)$$

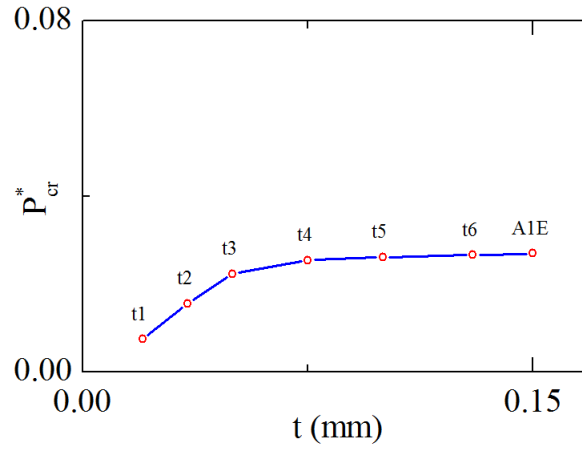
where P_{cr}^* is the P^* at its local peak.

Similarly, Figure 3.51 plots the dimensionless peak force P_{cr}^* against the respective parametric factors (h , t , and ω), and the flat regions shown in these plots where P_{cr}^* values remain relatively constant in a certain parametric range is likewise denoted by c .

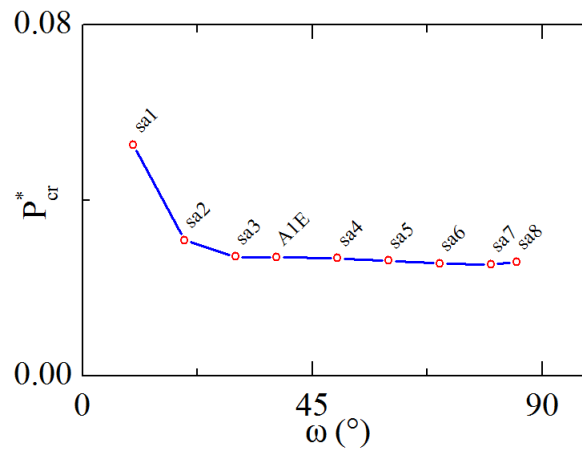
- The parametric study of h reveals interesting details on how the buckling and post-buckling behaviour results in changed in P_{cr} values with varying h values, and c could not be well-defined. P_{cr}^* reached its peak of 2.5×10^{-2} at $h = 0.8 \text{ mm}$, which marks the point where plastic progressive inversion occurs instead of a one-off snap-through. Here, the bending effects are dominant over the membrane effects, as suggested by the predominant monotonic responses in Figure 3.49a. In comparison, the responses of Model A1P were more symmetric than monotonic, suggesting that membrane effects were dominant over bending effects. In the region beyond $h = 0.8 \text{ mm}$, P_{cr} decreases with the occurrence of plastic progressive inversion, in scenarios where Deshpande and Fleck (2003)s egg-box formula more accurately describes (Equation 2.26) the frusta's behaviour. A more detailed comparison will be given in Section 3.7.
- For thickness t and slant angle ω , it was found that $c = 1.45 \times 10^{-2}$ in



(a)



(b)



(c)

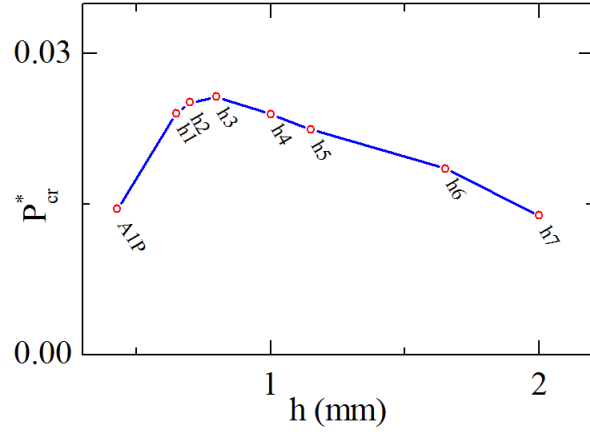
Figure 3.50: Summary of the dimensionless critical forces of LE models with respect to the corresponding parametric factors. (a) Height, h , (b) thickness, t , and (c) slant angle, ω .

the region $\omega = [10^\circ, 70^\circ]$ and $t = [0.075 \text{ mm}, 0.15 \text{ mm}]$ but the height is at its reference configuration $h = 0.43 \text{ mm}$. In these parametric ranges, the membrane effects were stronger than the bending effects as the response were more symmetric than monotonic.

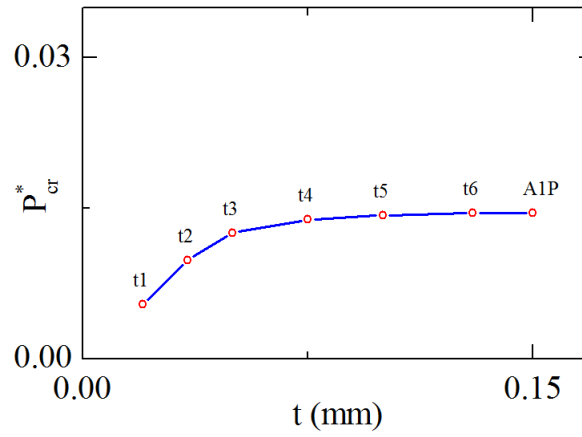
- A similar conjecture that was made for LE materials could be made for EPP materials concerning how P_{cr} behaves in broader dimensional perspectives: the local gradients of P_{cr}^* with respect to t and ω would still be zero when $h = [0.65 \text{ mm}, 1.15 \text{ mm}]$, as observed for the reference case when $h = 0.43 \text{ mm}$ in Figures 3.51b&c, and the flat zones with zero gradients would also be in the roughly similar region of $\omega = [10^\circ, 70^\circ]$ and $t = [0.075 \text{ mm}, 0.15 \text{ mm}]$. However, more simulations are required to confirm this.

3. By directly comparing the LE and EPP material models, we observed that the yield stress σ_y affects the critical force. Figure 3.52 shows the percentage reductions in critical forces from LE to EPP models with respect to the correlative parametric factors - h , t , and ω .

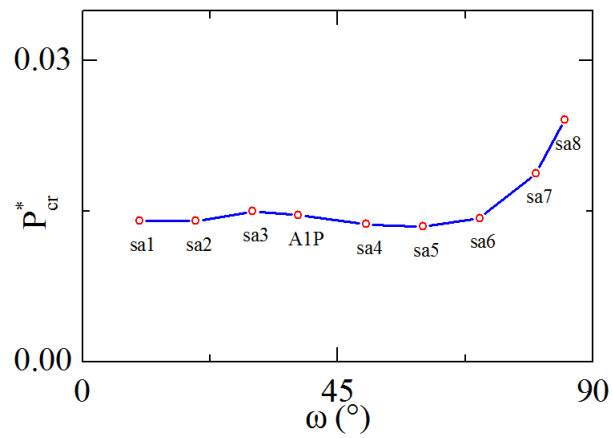
- A comparison between two reference models (A1E and A1P) gives 25.7% reduction in critical forces compared with Mode A1E.
- It was discussed previously that the buckling and post-buckling modes were largely affected by changes in h , which therefore affects P_{cr}^* . Therefore it is expected that Figure 3.52a shows no obvious relationship between critical force reduction and height. In spite of shifts in the percentage reduction which was caused by the changing of modes, the percentage reduction generally increases as h increases, which could be due to the presence of a larger yield zone.
- Figure 3.52b shows no critical force reduction between LE and EPP models at $t = 0.02 \text{ mm}$ (t1) but the value increases to 25% at $t = 0.05 \text{ mm}$ (t3), and



(a)



(b)

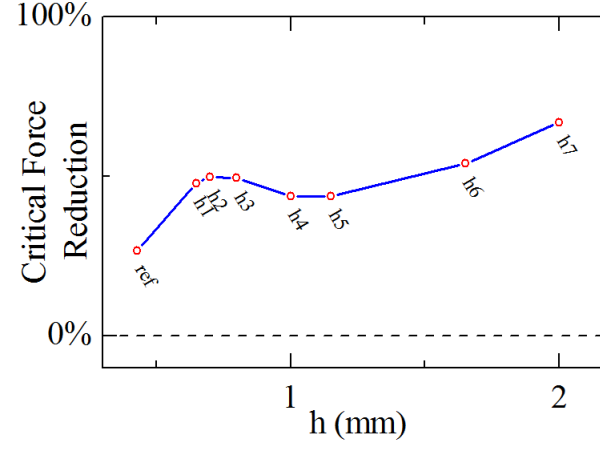


(c)

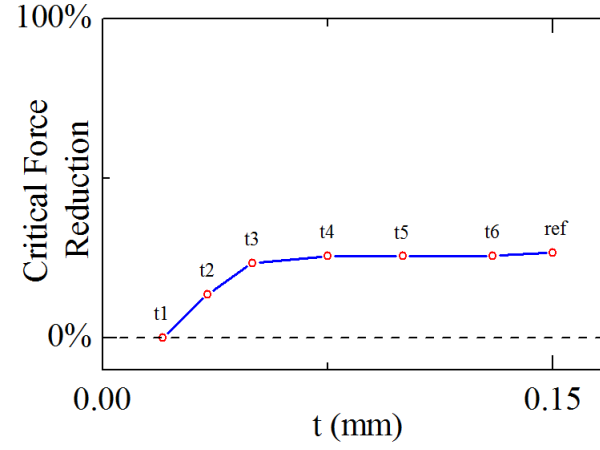
Figure 3.51: Summary of dimensionless critical forces of EPP models with respect to the corresponding parametric factors. (a) Height, h , (b) thickness, t , and (c) slant angle, ω .

is maintained at this level even when t is increased to 0.15 mm (reference model). Beyond $t = 0.15$ mm, the frusta are no longer thin-walled by definition, and is, therefore, not analysed and discussed here. When t is less than 0.035 mm, the EPP material showed no signs of yielding and its response is identical to its LE counterpart up to the limit point (for instance, Model A1E_t1 & A1P_t1). As the thickness is increased, the structure yields before the limit point is reached, lead to a greater percentage reduction in critical forces, until a stable value of 25% is achieved when the buckling mode no longer changes.

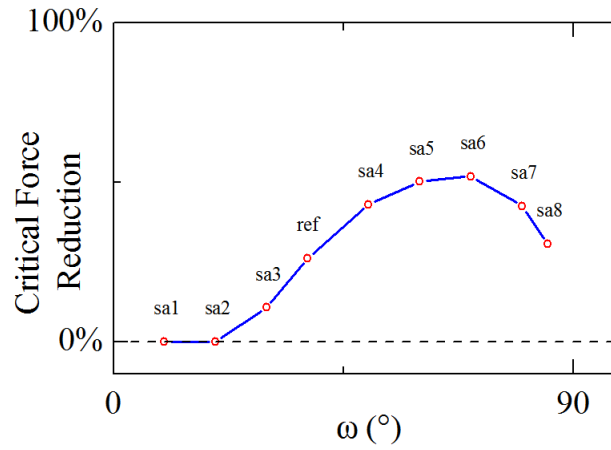
- Figure 3.52c displays the influence of slant angle, ω , on the difference in critical forces between LE and EPP models: when ω is small (sa1 & sa2), there were no differences in the critical forces; with subsequent increments of ω , the percentage reduction in critical forces rose to a maximum of 51.7% at sa6 where $\omega = 70^\circ$; thereafter, the percentage difference begin to decline, and the buckling mode gradually becomes more similar to the buckling of cylinders.



(a)



(b)



(c)

Figure 3.52: Reduction percentage of critical forces of EPP models in comparison with LE models. (a) Height, h , (b) thickness, t , and (c) slant angle, ω . ‘ref’ indicates the comparison between the reference models A1E&A1P.

3.7 Comparison with Deshpande & Fleck's Model

As previously reviewed, Deshpande and Fleck (2003) proposed a sophisticated analytical solution to describe the progressive inversion (post-buckling) of a close-top EPP frustum shell after its snap-through (buckling), based on Calladine (1986)'s travelling hinge model on spherical shells. They predicted the load in terms of vertical displacement (Equation 2.26) and their models assumed that $l/r \ll 1$, where l indicates the radius from the knuckle ring and r indicates the width of the knuckle ring. Hence, their model assumes that there is enough material on the surface for knuckle formation, which is not always the case, as observed for a shorter frustum. Moreover, their model precludes the contribution of membrane effects, and the equation was solely deduced by solving equilibriums that involve bending effects. Yet, our data revealed situations where membrane effects contribute significantly to the bistable behaviour of the frustum shell. Here, we compare our models with the predictions of Deshpande and Fleck (2003) in describing the post-buckling behaviour of the frustum shell and discuss its suitability over a wide range of h values that result in progressive inversion (Figure 3.53).

In these plots, the blue line correlates to Abaqus simulations results while the red line correlates to the analytical prediction. Generally, our simulations began to approximate the analytical prediction of Deshpande and Fleck (2003) as h is increased. As shown in Figures 3.53d-f, this trend becomes more apparent as h continues to increase, and the response becomes less symmetric and more monotonic in the central region of the graph; with increasing h , the monotonic region expands in a pattern that more closely resembles the analytical prediction. As shown in Figures 3.53g&h, there are extensive similarities between the red and blue graphs, and we approach a threshold where we consider Deshpande and Fleck (2003)'s model to be acceptable in matching the simulated result, i.e. analytical prediction becomes a valid method to describe the majority of its response. However, it is worth noting that the simulation results were slightly lower than analytical predictions in Models A1P_h6&7. This could

be due to two reasons: 1. the lack of consideration of membrane effects in Deshpande and Fleck (2003)'s model and 2. the Abaqus models were slightly less constrained than the analytical model - the top rim of the Abaqus models were radially unconstrained while ones in Deshpande and Fleck (2003)'s model were naturally constrained by the close-top geometry. This difference is so small that Deshpande and Fleck (2003)'s prediction can still act as a conservative solution to the problem.

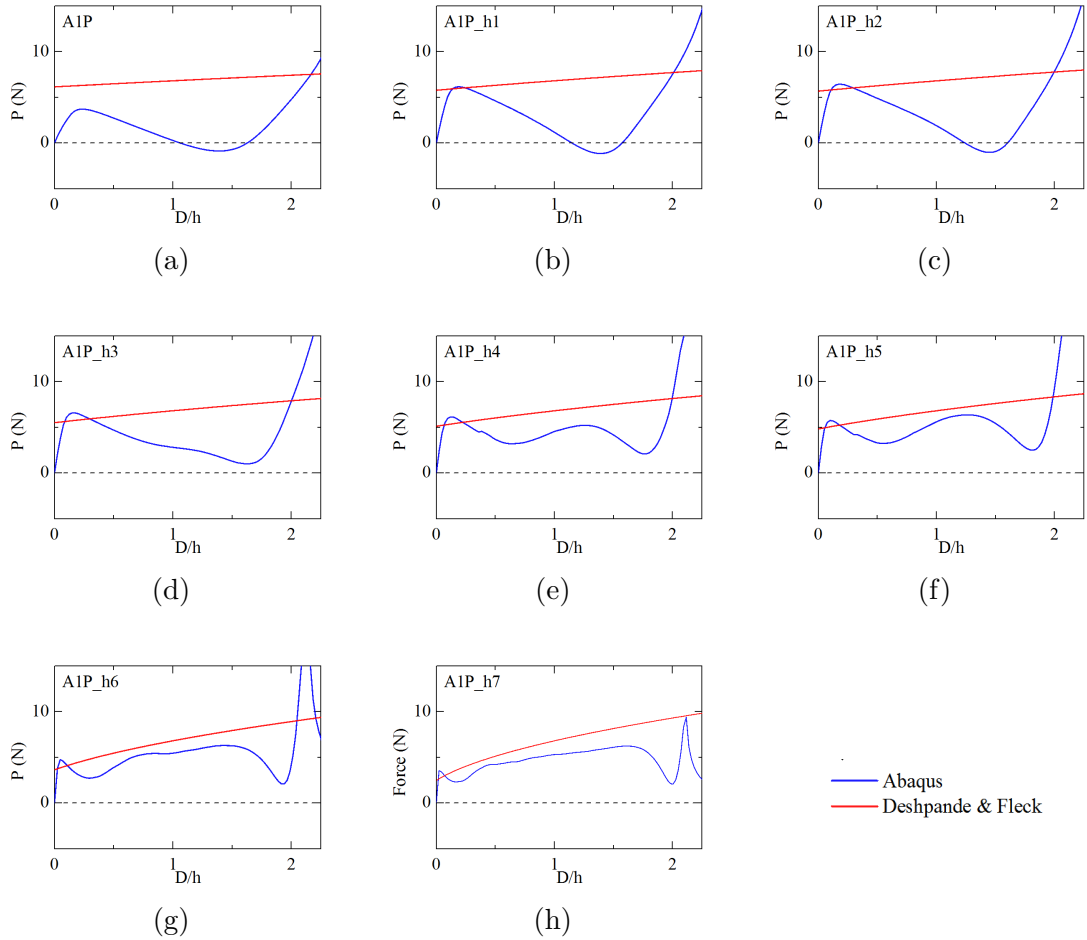


Figure 3.53: Comparison of the Abaqus results and Deshpande and Fleck (2003)'s predictions.

3.8 Summary

By presenting a series of interrelated FEM simulation results, this chapter has rendered a comprehensive picture of the behaviour of a close-top conical frustum shell under vertical uniform loading. The following conclusions could be drawn from the previous results and discussions.

1. In the boundary condition investigation, various models were simulated and results were compared. It was found that a singular frustum with its bottom rim restrained in rotation can fairly accurately reproduce the complex bistable behaviour of a shorter frustum within the corrugated part of a flexible straw. Although it is worth noting that the critical force tends to be an overestimation of realistic boundary conditions, it is less computationally demanding and is probabilistically favourable in yielding convergent results, and was decidedly the ideal candidate for use in simulated parametric studies.
2. We have carried out sophisticated parametric studies by focusing on three geometric parameters, h , t , and ω , which define the key variables of a frustum's geometry. It was shown that an increase in t or ω correlated with an increase in P_{cr} . Meanwhile, the influence of h on P_{cr} is more multifaceted since it affects the failure modes. Furthermore, fitted equations were obtained from the parametric simulations and shown in Equations 3.6 (LE material) and 3.7 (EPP material) to predict the magnitude of the critical force that was required to trigger snap-through that resulted in a secondary stable state; and limitations in its scope of implementation were also stated in the discussion.
3. In general, a degree of reduction in critical forces between EPP models and its LE counterparts were consistent for most of the cases. However, we observed exceptions where t and ω were reduced to a certain extent but did not contribute to yielding before the limit point.

4. We compared our simulation results with Deshpande and Fleck (2003)'s analytic prediction. In this comparison, we have demonstrated the limited scope of their analytical solution - which has inconsistencies and limitations in estimating critical forces within the parametric spectrum.
5. In my opinion, a 'useful' bistable structure should satisfy several criteria: firstly, it should be easily inverted by a small critical force; secondly, its secondary stable state is distinguished and the structure can be locked up properly at this state; thirdly, it allows as little plastic deformation as possible, i.e. it possesses strong cyclic reliability; fourthly, no neutral state (monotonic-like response) or second buckling should occur between the transition of two stable states to avoid uncertainties in its operation. Sometimes, the first and second criteria may directly conflict with each other, requiring one to be sacrificed at the expense of the other depending on the application. Considering these, for a bistable structure with $R_b = 2.95$ mm, it is recommended to limit its design to the following constraints: $h = [0.43 \text{ mm}, 0.70 \text{ mm}]$, $t = [0.075 \text{ mm}, 0.015 \text{ mm}]$, and $\omega = [30^\circ, 80^\circ]$ for LE material and $h = [0.43 \text{ mm}, 0.70 \text{ mm}]$, $t = [0.075 \text{ mm}, 0.015 \text{ mm}]$, and $\omega = [20^\circ, 40^\circ]$ for EPP material with $\sigma_y = 25 \text{ N/mm}^2$. The correlative critical force can be obtained by calculation of Equations 3.6&3.7, or reading of Figures 3.50a&3.51a.

Chapter 4

Close-Top Frustum Shells under Vertical Point Loading

In the chapter, we will perform an in-depth analysis of the close-top frustum shell under vertical point loading. The load will be applied onto various locations of the top face, and the critical forces and bistable behaviour will be compared. Experimental tests will be conducted and its results will then be used to validate Abaqus simulations. Furthermore, parametric studies of height and thickness will be performed on Abaqus.

4.1 Experiments

The experimentation relies on several crucial decisions: the choice of material, manufacturing conditions of the testing specimens, and the experimental set-up. These aspects will be explained in detail in the following sections.

4.1.1 Material

The inversion of a frustum shell involves a large deformation in its geometry. In order to reproduce a specimen that can demonstrate multi-stable behaviour within

the material's elastic range, hyperelastic material silicone rubber provides an ideal candidate.

Chemists at Corning Glass and General Electric were investigating heat-resistant materials for use as resinous binders when they synthesized the first silicone polymers. They found that silicone rubbers were functional and could be produced commercially (Colas and Curtis, 2004). Since then, it has infiltrated all areas of our life. Today the production of silicone rubbers involve a huge range of companies, resulting in a wide range of material properties. In this research, *addition cure silicone rubber* from *easycomposites.co.uk* was used to make frustum specimens. The addition cure silicone rubber and the catalyst arrived in separate packages. The original use of this product was in mould making, which takes advantages of its excellent dimension reproduction, low shrinkage, suitable working time, and flexibility after being cured. Together with its hyperelastic material properties, it is an ideal material for the frustum model-making in this research. However, in order to model it accurately in Abaqus, the knowledge of its detailed material properties is necessary but was not available in the product specifications (easycomposites, 2015). Therefore, we conducted material tests to obtain this information. Five sheets of silicone rubber plates were cast and five pairs of dogbone-shaped specimens were cut out of these sheets according to the general code of tensile testing (detailed dimensions in Appendix A). The finished products are shown in Figure 4.1. The presence of tiny bubbles in a finished product of silicone rubber frusta is inevitable; therefore, pairs 3&5 were intentionally made to contain a large number of bubbles by reducing the duration of the vacuuming process. The dimensions of each dogbone specimen were measured and are shown in Appendix B.2.

The tests were performed using an Instron Universal Testing Machine with a 500 N load cell. After loading a dogbone specimen on the machine, a separate elongation gauge was attached to the specimen to avoid using the inbuilt displacement measurement equipment, which is unreliable due to the elastic nature of the material. The samples

were preloaded in a slightly tense state to reduce the amount of noise in the subsequent measurements. In addition, we use a default value of $\nu = 0.475$ (Abaqus, 2013) for the simulation which is typical for these type of materials (Fathi et al., 2012; Matsuzaki et al., 2008; Gonca and Shvabs, 2011).

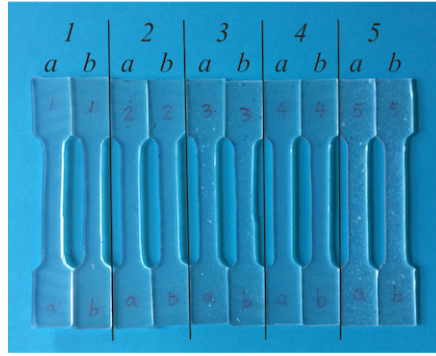


Figure 4.1: Silicone rubber dogbone specimens

Figure 4.2 shows the processed results; the slight difference in results obtained from different dogbone specimens was considered to be acceptable as they are likely to have resulted from dimensional variations during manufacturing. From Figure 4.2, we also observed that the effects of tiny bubbles on the specimen were negligible in affecting the stress-strain response.

In the Abaqus evaluation, we identified a line which corresponded most closely to the average trend and fitted the values into a hyperelastic polynomial model (Equation 4.2), which yields the key parameters that are later used in the Abaqus simulation. The biaxial test, planar test and volumetric test data were not available due to the limitation in resources. However, the uniaxial test results show a typical hyperelastic material response (Duncan and Dean, 2003) that is sufficient to perform the Abaqus Evaluation. The red line in Figure 4.2 shows the Abaqus evaluation result, which fits well into the trend of the test data. Table 4.1 lists the key parameters of the polynomial model form this specific silicone rubber material.

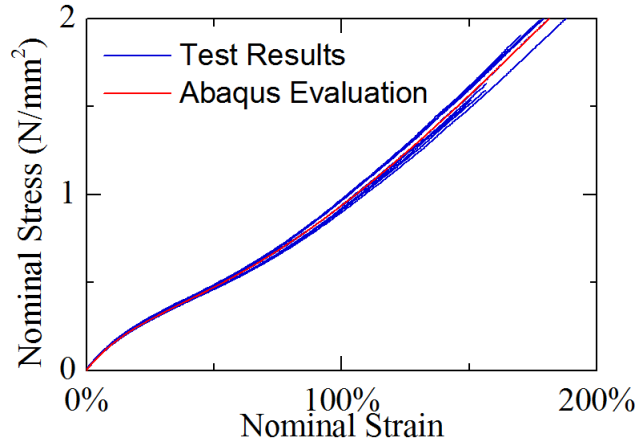


Figure 4.2: Material uniaxial testing results and Abaqus evaluation.

These parameters are explained as following. For a linear elastic material, like the one in previous chapter, the strain energy can be exactly calculated by determining the area under the stress-strain curve, which is represented by:

$$U = \frac{1}{2} \sigma \epsilon. \quad (4.1)$$

A hyperelastic material adopts the behaviour of an elastic material (reverts to its original shape when all affecting forces have been removed), and is also Cauchy-elastic, which means that the stress is determined by the current state of deformation, and not the path or history of its deformation. The latter differs from a typical elastic material, in that the stress-strain relationship is derived from a strain energy density function. Furthermore, the loading and unloading stress-strain curve of an elastomer

Table 4.1: Parameters of polynomial model (strain energy potential order = 2) for hyperelastic materials.

C_{10}	C_{01}	C_{20}	C_{11}	C_{02}
-0.201846798	0.515007704	-0.032320963	0.178762852	-0.048375306

D_1	D_2
0.129455944	0

material, such as silicone rubber is influenced by factors such as time, type of loading (static or dynamic), loading frequency. However, this viscous behaviour is ignored if the hyperelastic material model is used for description.

The description of the strain energy density U is much more complex than a linear elastic material, where the stress is a linear function of strain. For typical hyperelastic material models, the phenomenological form of the strain energy density (Seibert and Schoche, 2000) is given by

$$U = \sum_{i+j=1}^N C_{ij}(\bar{I}_1 - 3)^i(\bar{I}_2 - 3)^j + \sum_{i=1}^N \frac{1}{D_i}(J_{el} - 1)^{2i}, \quad (4.2)$$

where U is the strain energy; J_{el} is the elastic volume ratio; $\bar{I}_1$ and $\bar{I}_2$ are the first and second invariants of the deviatoric strain; and N , C_{ij} and D_i are material constants. C_{ij} describes the shear behaviour of the material, and D_i introduces compressibility. The strain energy function is a polynomial function. Abaqus evaluate the uniaxial test results and fits the data into a polynomial form with an order of 2, which is represented by,

$$\begin{aligned} U = & C_{10}(\bar{I}_1 - 3) + C_{01}(\bar{I}_2 - 3) + \frac{1}{D_1}(J_e - 1)^2 + C_{20}(\bar{I}_1 - 3)^2 \\ & + C_{02}(\bar{I}_2 - 3)^2 + \frac{1}{D_2}(J_e - 1)^4 + C_{11}(\bar{I}_1 - 3)(\bar{I}_2 - 3). \end{aligned} \quad (4.3)$$

The initial elastic modulus E_0 is $1.76N/mm^2$. According to the model, a hyperelastic material shows fairly linear behaviour up to 20% strain. In all of the following simulations, a maximum of 15% strain is observed. Therefore, E_0 is used in later evaluations.

4.1.2 Specimens

In the previous chapter, the simulated frustum shells were relatively small, which may be difficult to validate by conducting same-scale experiments. Therefore, in the subsequent experiments, we used dimensions that could be reasonably tested in actual experiments (Table 4.2). We have fabricated a series of frustum shells with fixed R_b and R_t but with variable h . The thickness, t , was not predetermined as it could be adjusted after the mould is fixed.

After taking into account the working time and the viscosity of the silicone rubber before curing, the following moulds were made (Figure 4.3). Each set of mould consists of a female and male parts. Two $\varnothing 6$ mm holes were drilled into the rims of both parts of the mould for future alignment. Two $\varnothing 10$ mm holes were drilled into the male part to allow the silicone rubber to be poured into the mould, together with a $\varnothing 6$ mm hole to allow the expulsion of air. The thickness can be controlled by offsetting the male part from the female part by inserting plates with a predetermined thickness between them. Detailed drawings of the moulds are presented in Appendix A.

During the fabrication process, addition cure silicone rubber was mixed with 10% catalyst. The thoroughly-mixed viscous liquid was then degassed by using a vacuum chamber. As we have discussed previously, the presence of small amount of tiny bubbles inside the structure had limited effects on the material behaviour; moreover, it is impractical to degas every single air bubble in the mixture within the given working

Table 4.2: Silicone Frustum Specimen Dimensions.

Variable Type	Value & Units		
	B20C & B20O	B40C & B40O	B60C & B60O
Bottom Radius, R_b	75 mm	75 mm	75 mm
Top Radius, R_t	30 mm	30 mm	30 mm
Slant Angle, ω	23.96 °	41.63 °	53.13 °
Height, h	20 mm	40 mm	60 mm
Slant Height, h_s	49.2 mm	60.2 mm	75 mm

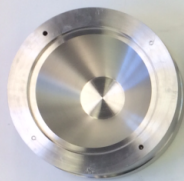
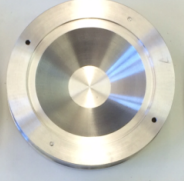
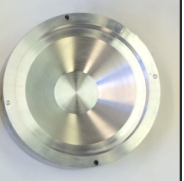
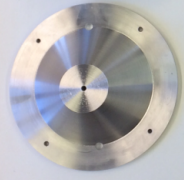
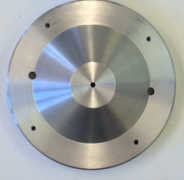
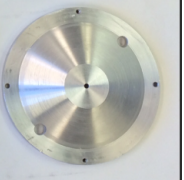
	<i>Frustum Height</i>		
	<i>60mm</i>	<i>40mm</i>	<i>20mm</i>
<i>Female Part</i>			
<i>Male Part</i>			

Figure 4.3: Silicone rubber frustum moulds.

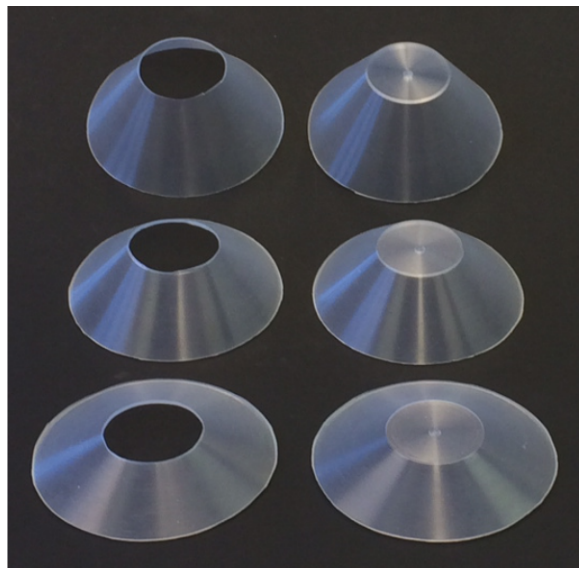


Figure 4.4: Silicone rubber frustum specimens.

time of 1.5 hours. The degassing procedure was considered complete when bubbles were no longer visible within the mixture. After 24 hours of curing, the fully set silicone was removed from the mould. Excess material was trimmed off so that the remaining structure was just like designed. Figure 4.4 shows a series of final products of frustum specimens made with various height and thickness. We measured the dimensions of the final product to ensure that they matched the initial design. It was observed that when specimens were too thin, they were unsuitable for testing as the specimen would collapse as a result of its self-weight. By casting several frustum shells, a suitable model, B40C, with 40 mm height was chosen, which had a side surface thickness of 1.72 mm and a top surface thickness of 2.30 mm.

4.1.3 Apparatus

In the experiments, we were not only interested in the reaction force and corresponding displacement but also the structural deformations at each of the recorded points. The deformation results were directly compared with the Abaqus simulations - a comparison method that is more accurate than simply comparing the reaction force and displacement, as the actual deformation behaviour is more complex and cannot be solely described by the reaction force and displacement.

A dock was used in the experiments to constrain the bottom rim of the frustum during testing (Figure 4.5a), and was made by removing a $\varnothing 150$ mm circular piece from a square plate. Figure 4.5b shows a loading dock and the fixated specimen. Although the dock restricts the expansion or downward movement of the bottom rim of the specimen, it does not prevent the infolding or the upward movement. The dock also physically limits the maximum allowable loading displacement.

During testing, vertical point loads were individually applied on three point location - 0 mm, 15 mm and 29 mm as measured from the centre of the structure (Figure 4.6). These locations were chosen to determine how the critical force would vary with the location of the load. We choose 29 mm instead of 30 mm as the latter was at the inner circumference of the frustum.

A digital force gauge with a built-in displacement measurement was used to record the reaction force and the displacement (Figure 4.7b). The force gauge was



Figure 4.5: Loading docks. (a) Loading docks without specimen on, and (b) loading dock with specimen on.

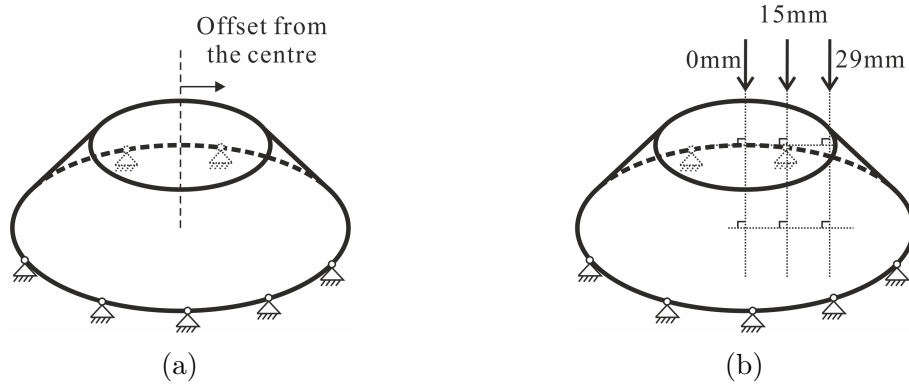


Figure 4.6: Schematic graphs of loading location definition

calibrated before performing the tests and has a 5 N capacity and 0.001 N accuracy. The force gauge could be affixed with several different modules, and we selected a rod-shaped probe with a pointed tip to perform point loading (visible in Figure 4.8). The manufacturer also provided a sturdy supporting dock, which stabilises the force gauge (Figure 4.7b) during its application and houses a loading handle that is used to execute loading.

A TenYoun OKIO-H-400 3D scanner (Figure 4.7a) was used to capture the 3D shape of the specimen throughout multiple loading events. OKIO-H-400 is a non-contact scanner consisting of a projector in the middle and two optical cameras along each side. As the scanner does not work well with a reflective or transparent material, the specimens and the dock are coated with DPT-5 Dye penetrant agent before experiments; the dye treatment creates a matte and white surface that has a negligible impact on other material properties. Furthermore, the loading dock was affixed with stickers that were randomly positioned to allow the computer to recognise the orientation of the dock (Figure 4.7c). These stickers provided reference points that aided the computer in assembling the different segments that were generated by capturing multiple perspectives into a global 3D image.

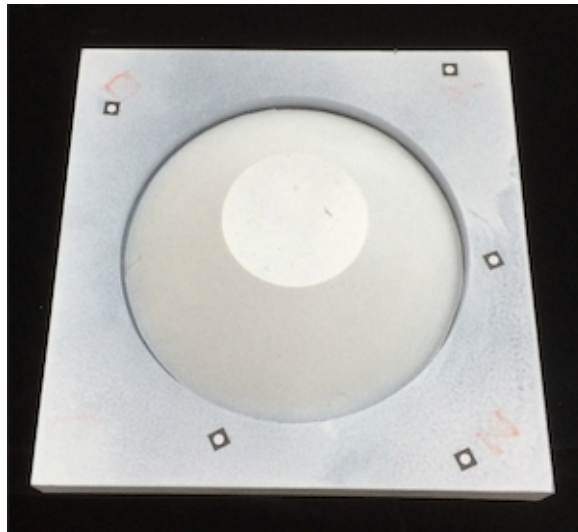
During scanning, the centre projector illuminates the object of interest with various patterns of light (Figure 4.8). Simultaneously, two optical cameras that flank the



(a)



(b)



(c)

Figure 4.7: Scanning equipments and accessory. (a) TenYoun OKIO-H-400 3D scanner (TenYoun, 2015), (b) digital force gauge with build-in displacement measurement (installed on a stand), and (c) mark stickers randomly positioned on a loading dock after DPT treatment.

projector captures the pattern which is deformed by the object and calculate the distance of every point to the camera in the field of view. This information was then used for an exact geometric reconstruction of the surface shape. Each scan would produce an uncompleted 3D segment of the structure from a specific perspective, and multiple perspectives are recorded to capture all the surface information required to reconstruct the completed 3D image. These segments would then be sewed together with the aid of the reference points to reconstruct the structure. Occasionally, certain areas were not scanned due to a limitation of the scanner and the experimental setup; a repair function was used to produce continuous second-order polynomial surfaces that anneal the gaps.

During the loading test, the load was applied, the corresponding reaction force was measured, and the deformed shape was captured. Several sets of data were recorded during each loading.

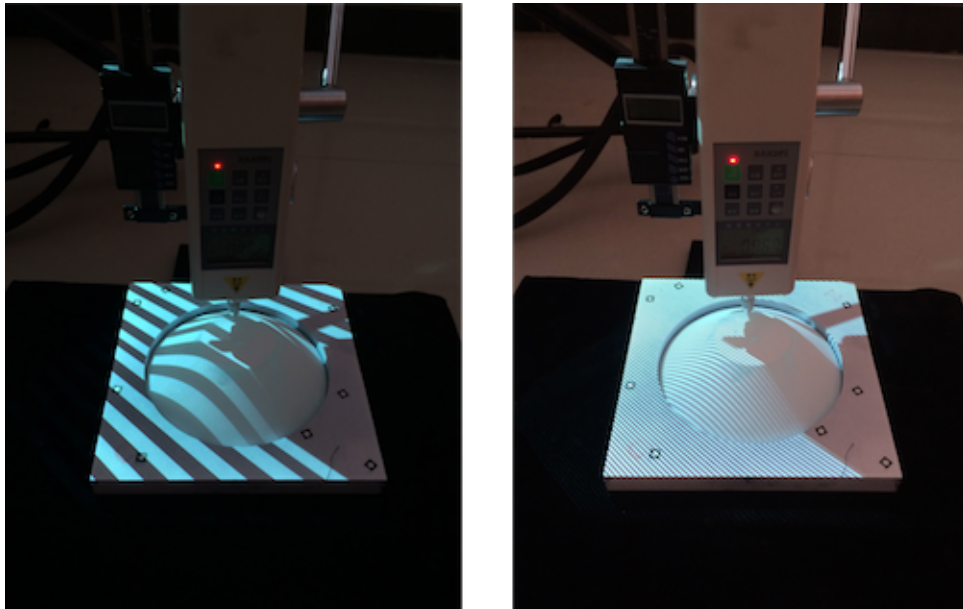


Figure 4.8: Scanning in progress.

4.2 Abaqus Simulations

The FEM simulation was performed using the general-purpose finite element package Abaqus Ver. 6.13 (Abaqus, 2013). The methodology behind the Abaqus model reconstructions was similar to the method used in the previous chapter. A reference frustum B40C (detailed dimensions in Table 4.2) and several height and thickness variants were built using shell element S4R. The mesh size was set at around 2.5 mm. The implicit solver (“Static, General” option in Abaqus) was used to incorporate aspects of quasi-static loading, non-linear responses, and large deformation. The automatic stabilisation function was used to ensure the analysis would persist in the event of dramatic buckling. The application of a sharp point load could cause localised stress concentration and result in a convergence problem during the simulation, and a small hemispherical probe that circumvents this issue was used in the simulation to apply the displacement-controlled loading. We used the centre of the hemisphere as the reference point, which required the direction of the reaction force to intersect the reference point. The tangential part of the contact point between the probe and the frustum was modelled as frictionless, which allowed the existence of finite sliding but not tangential force between the bodies in contact; while the normal part was modelled as hard contact, which means the absence of contact pressure until nodes were in contact, where unlimited contact pressure was permitted. In this setup, only positive force can be recorded.

4.3 Comparison between Simulated and Experimental Results

In this section, results of both Abaqus simulation and experiments of Model B40C under vertical point loading will be presented and compared. As mentioned previously for the apparatus loading method, there were physical constraints that limited the maximum displacement introduced by the probe, whereas Abaqus is capable of simulating the structural changes beyond the experimental data.

4.3.1 Model B40C under a Vertical Point Load with 0 mm Offset from the Centre

In this case, axisymmetric ring mode was found in the simulation results if no imperfection were introduced, as shown in Figure 4.9. However it is not the case in a real experimental set-up since the imperfection could be avoided and an asymmetric mode is preferred. The imperfection could be introduced by seeding an asymmetric mesh or offsetting the load location slightly. A study addressing the sensitivity of the structure's response to various imperfections were carried out, and the results are shown in Figure 4.10. Relative similar response were found among all variations. 'Mesh 1' were used in the future simulations.

Figure 4.11 shows the structural responses of Model B40C under a vertical point load with 0 mm offset from the centre. Generally, the Abaqus results fit well with the experimental results, but the latter tends to reveal a slightly larger reaction force after the first limit point as compared with the Abaqus simulations. Figure 4.11a shows two local peaks before the frustum was fully inverted, and the first and second peak forces were 3.65 N at 17.5 mm and 1.74 N at 40.5 mm, respectively. Furthermore, there is a close visual agreement between the structures that were captured at each point in the deformation history in both the results from Abaqus simulations and experimental 3D scanning in Figure 4.11b.

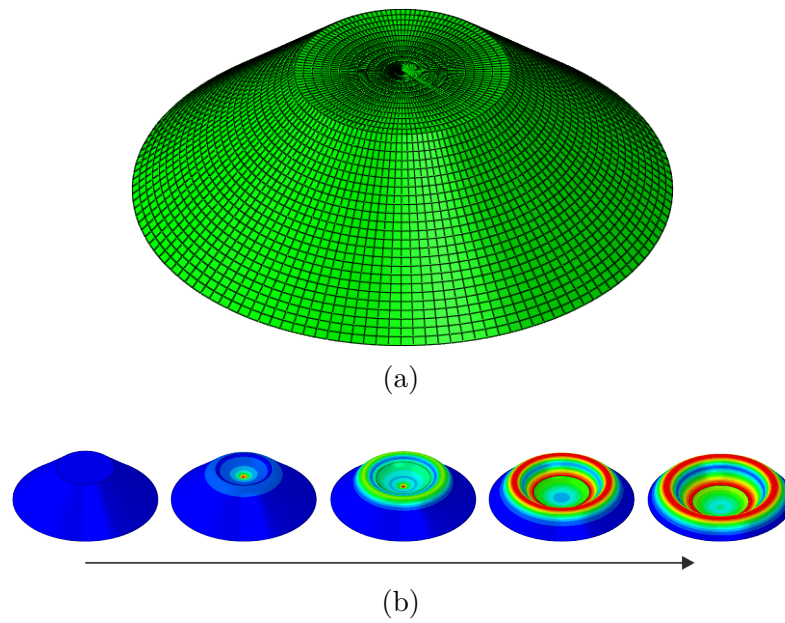


Figure 4.9: Axisymmetric response if no imperfection was introduced. (a) Mesh with axisymmetric sweep pattern, and (b) ring mode.

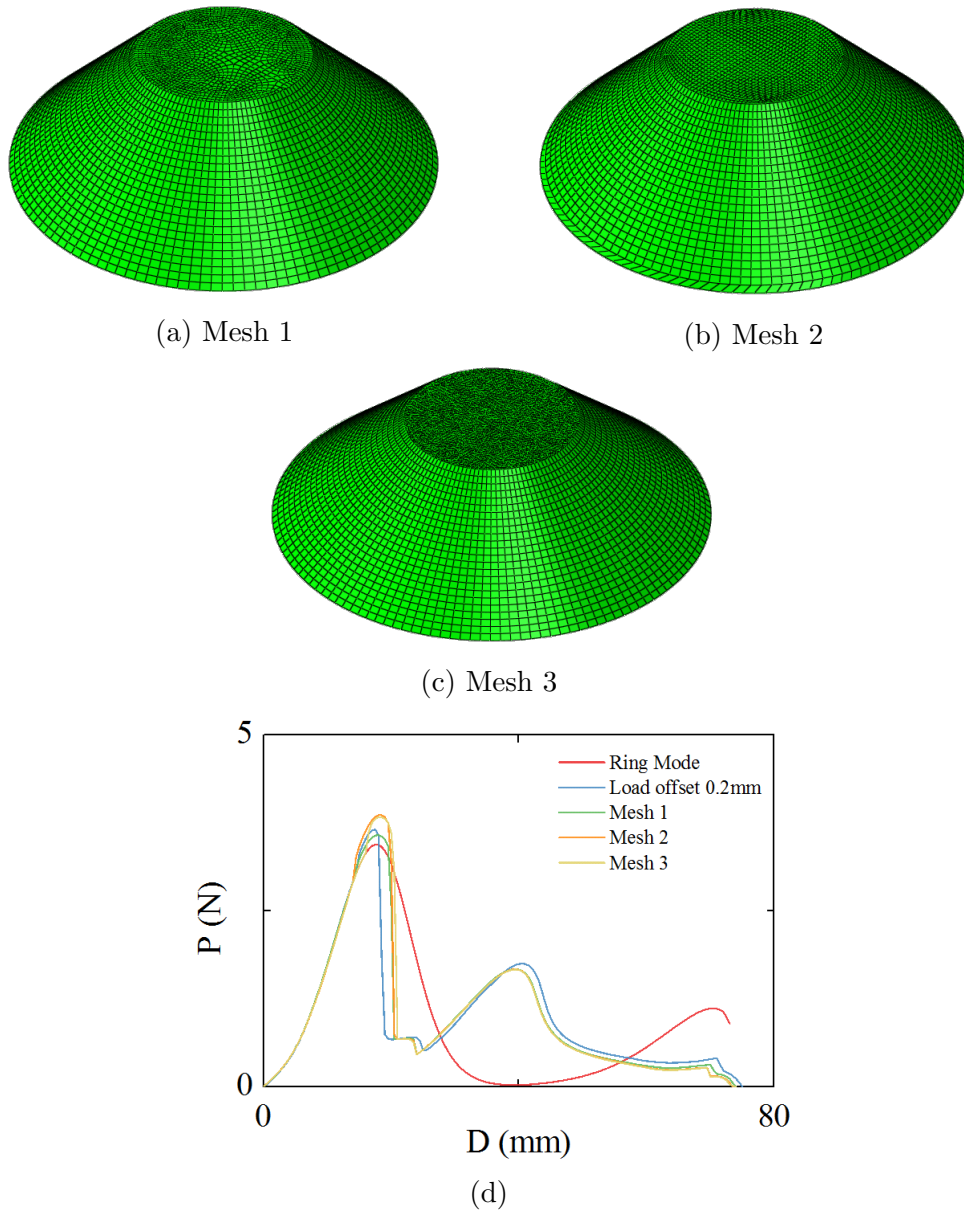


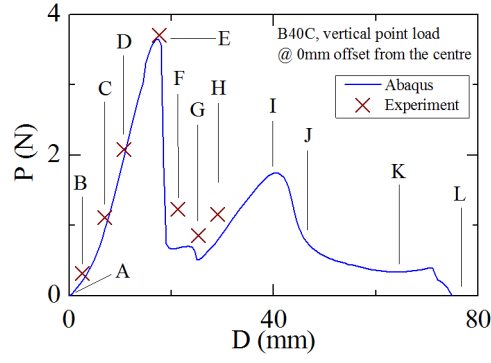
Figure 4.10: Imperfection study.

4.3.2 Model B40C under a Vertical Point Load with 15 mm Offset from the Centre

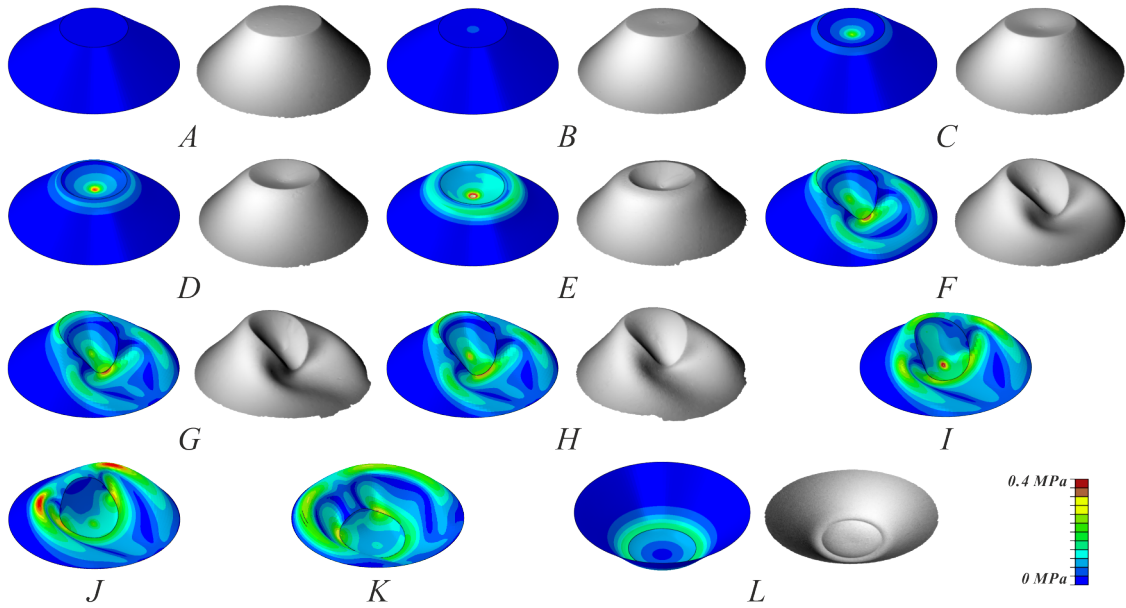
Figure 4.12 shows the structural response of Model B40C under a vertical point load with 15 mm offset from the centre. Similar to the 0 mm offset case, the Abaqus results fit well with the experimental results. The experiments reveal a slightly larger reaction force after first limit point as compared to the Abaqus simulation. Figure 4.12a shows two local peaks before the frustum was fully inverted as well. The first peak force is 1.56 N at 10.0 mm and the second one is 2.16 N at 48.0 mm. Furthermore, the deformed shapes at each stage show acceptable agreement between Abaqus simulations and Experimental 3D scanning in Figure 4.12b.

4.3.3 Model B40C under a Vertical Point Load with 29 mm Offset from the Centre

Figure 4.13 shows the structural response of Model B40C under a vertical point load with 29 mm offset from the centre. Similar to the 0 mm offset case, Abaqus results fits well with experimental results. Experiments show slightly larger reaction force after first limit point comparing with Abaqus simulation. Figure 4.13b shows two local peaks before the frustum became fully inverted. The first critical force is 1.07 N at 3.5 mm and the second one is 2.69 N at 53.0 mm. Moreover, the deformed shapes at each stage show acceptable agreement between Abaqus simulations and Experimental 3D scanning in Figure 4.13b.



(a)



(b)

Figure 4.11: Structural response of Model B40C under vertical point load with 0 mm offset from the centre. (a) Vertical reaction force P vs. D , and (b) deformed shapes and stress distribution. Coloured graphs are Abaqus simulated results while grey graphs are 3D scanned shapes.

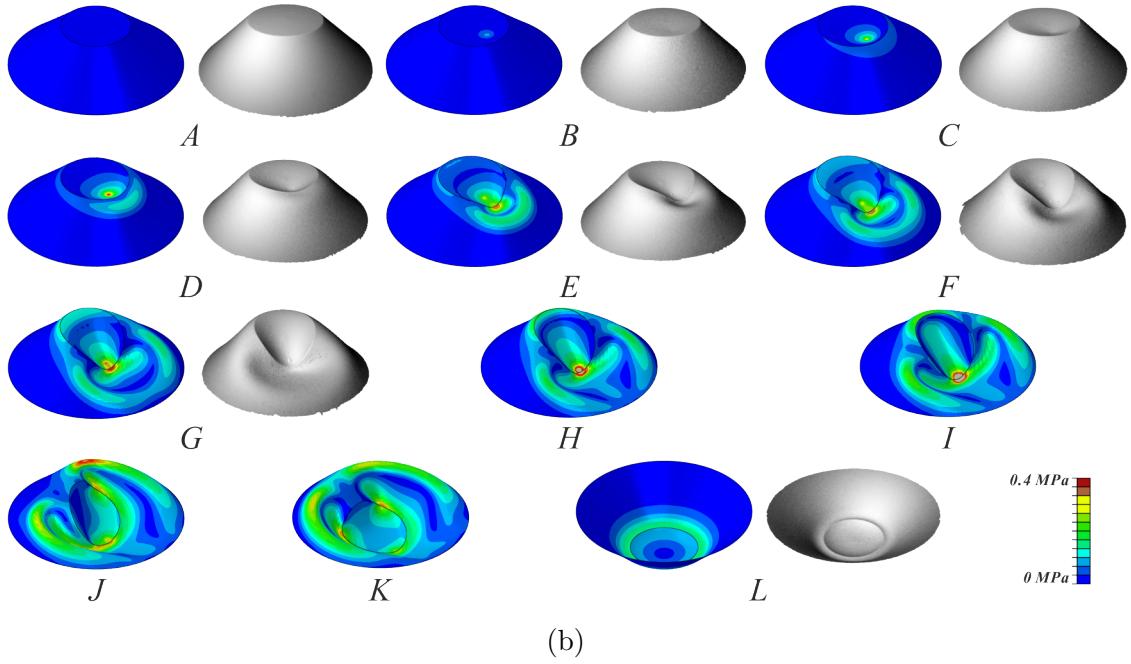
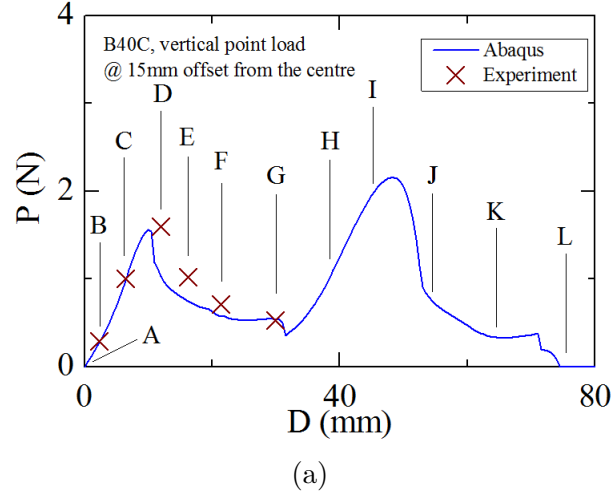


Figure 4.12: Structural response of Model B40C under vertical point load with 15 mm offset from the centre. (a) P vs. D , and (b) deformed shapes and stress distribution. Coloured graphs are Abaqus simulated results while grey graphs are 3D scanned shapes.

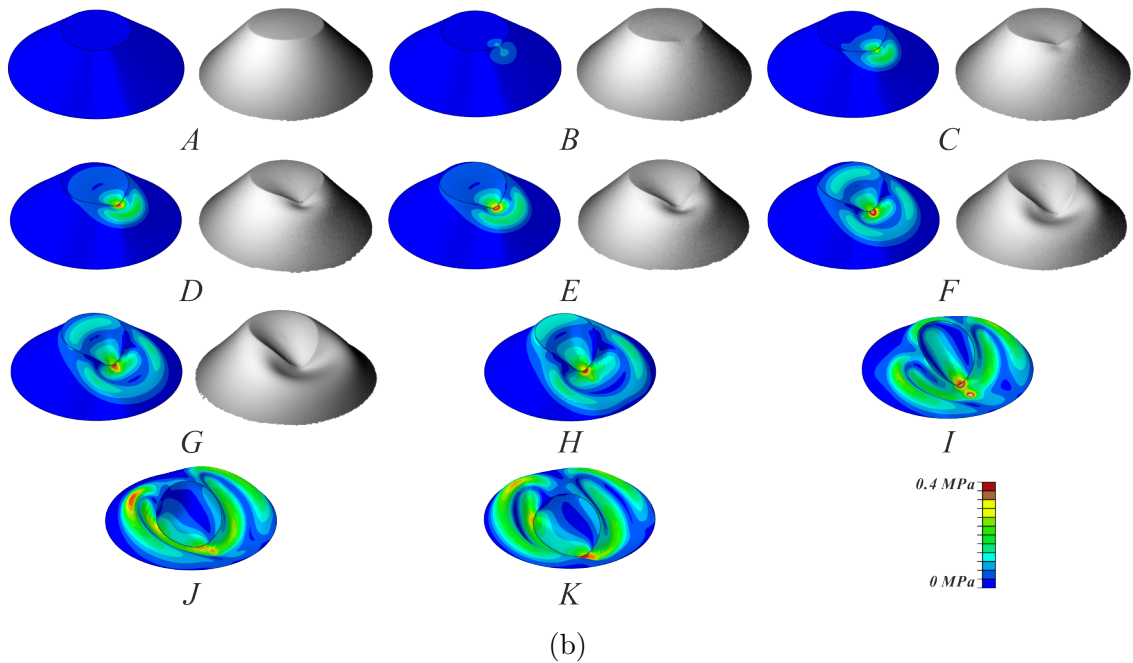
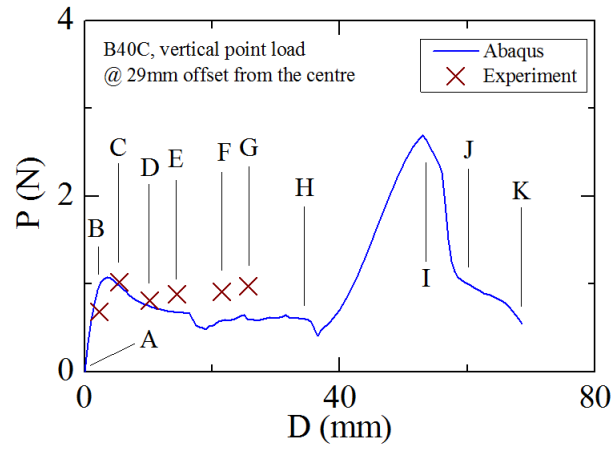


Figure 4.13: Structural response of Model B40C under vertical point load with 29 mm offset from the centre. (a) P vs. D , and (b) Deformed shapes and stress distribution. Coloured graphs are Abaqus simulated results while grey graphs are 3D scanned shapes.

4.3.4 Loading Point for a Minimum Inversion Force

As reviewed in the previous sections, the Abaqus results provide a good approximation of the experimental results and validate the chosen settings that were used in the Abaqus simulations. We are confident that the FEM utilised by the Abaqus software is useful in simulating the large deformations of close-top frustum shells during vertical point loading.

In all three loading scenarios, two local peak forces were observed before the frusta were fully inverted. The following observations were consistent in Figures 4.11b, 4.12b, and 4.13b: the stretching of the top surface and bending of the side surface were the dominant behaviour before the first peak (Point E in Figure 4.11b, Points D in Figure 4.12b, and Points C in Figure 4.13b) was reached; after which, the top rim buckled and became partially inverted; the inversion propagated both downwards and circumferentially along the side surface as loading continued, and arrived at edge of the bottom rim, leading to a slight dip in reaction force (Point G, Point G, and Point H, respectively); thereafter, the reaction force started to increase again until a second peak was reached (Points Is); upon the maximum propagation of the circumferential inversion, a bifurcation occurred, and the force dropped to zero (Points Ls); finally, a new stable state was established where the structures were fully inverted.

As we located the loading point further from the centre, a consistent trend of how the two peak forces changed was observed: there was a noticeable decline in the first peak force while the second peak force increased as the loading point was shifted outwards. This is because the corresponding lever arms (connecting the position between the load and the snaps) were altered by the location of the load, which influences the force that required to trigger the snaps. In a realistic force-controlled loading, the critical force is defined by the highest peak force, and since one of the major objectives of this research is to discover an efficient route of inverting a frustum shell, we aimed to determine the optimal loading location where inversion can occur

with the lowest load, i.e. where the two peaks forces have equal values. In order to derive the intersection between two peaks, the first and second peak forces of each of the three loading scenarios were plotted against the loading location in Figure 4.14; polynomial functions were used in the curve fitting and the curves intersected at (10.5 mm, 2.02 N). This argument is supported by the additional Abaqus simulations results in Figure 4.15, and we may provisionally conclude that applying a point load at 10.5 mm from the centre of the top surface of the frustum with the stipulated configurations is most efficient in achieving fully inversion.

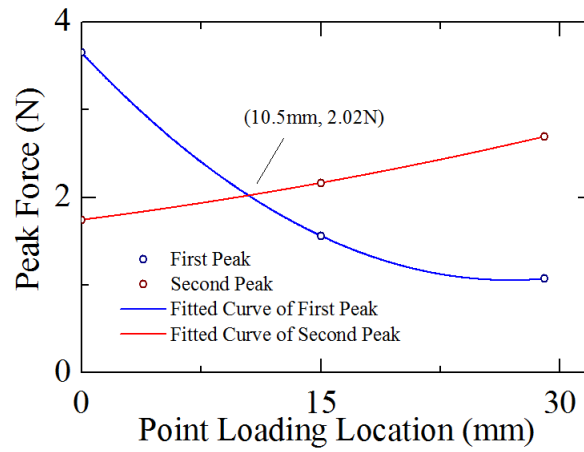


Figure 4.14: First and second peak forces against point loading location from the centre.

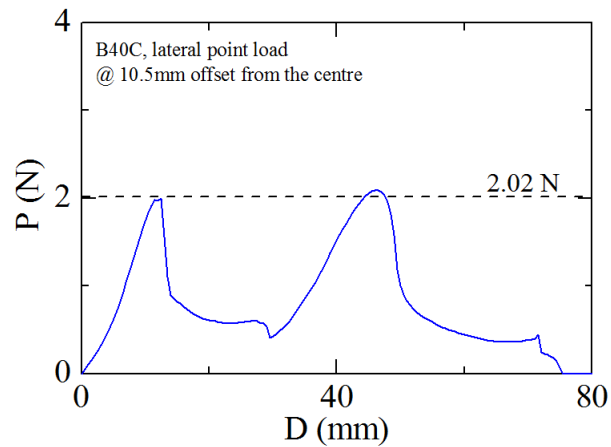


Figure 4.15: Structural response of Model B40C under vertical point load at proposed location, which is 10.0 mm offset from the centre.

4.4 Parametric Study

4.4.1 Height

In this parametric study, we would like to evaluate the effects of height on the structural response of the close-top frustum shell. Two additional height variants, Models B20C and B60C (detailed dimensions are in Table 4.2) were simulated in Abaqus. Unlike the parametric study of height in Chapter 3, the slant angles did not stay constant but were increased with the height. The thicknesses of Models B20C and B60C were same as the ones of Model B40C, which were 1.72 mm of the side surface and 2.30 mm of the top surface. The heights of Models B20C, B40C and B60C are 20 mm, 40 mm and 60 mm, respectively.

For all height variants, their reaction forces P versus dimensionless displacement D/h were plotted in Figures 4.16a, 4.17a & 4.18a, and the dimensionless force P^* versus D/h are shown in Figures 4.16b, 4.17b & 4.18b. Convergent solutions up to $D/h = 2$ were not obtained in some of the simulations but the results are reliable since equilibrium was achieved in each step of the simulation using implicit solvers.

The results in Figures 4.16a, 4.17a, and 4.18a show a substantial influence of h on the response. As h increased, the peak forces increased. The displacement D/h of where the peaks took places was affected by h as well: the first peak shifted from $D/h = 0.61$ to $D/h = 0.33$, $D/h = 0.38$ to $D/h = 0.2$, and $D/h = 0.16$ to $D/h = 0.06$ in Figures 4.16a, 4.17a, and 4.18a, respectively; although some simulations were terminated before the second peak, the trend suggests that the second peak would occur at lower D/h as h increased. For Model B60C, a 29 mm off-centre loading point resulted in contact between the loading tip and the side surface of the frustum after the first peak, and this may account for any differences in its response profile.

Similar dimensionless group setups as the ones in Equation 3.4 of Chapter 3 were

used. The dimensionless displacement is defined as

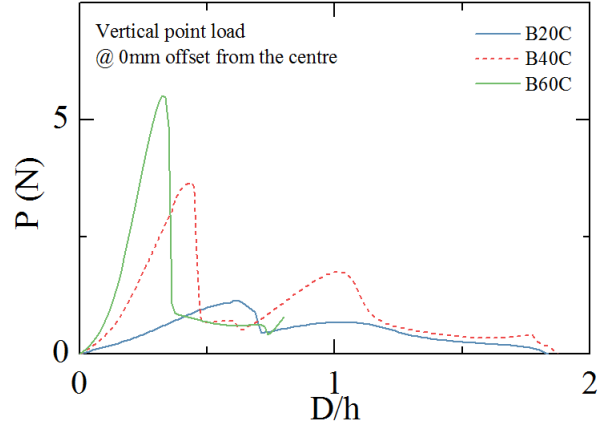
$$D^* = D/h; \quad (4.4)$$

The dimensionless force P^* is defined as

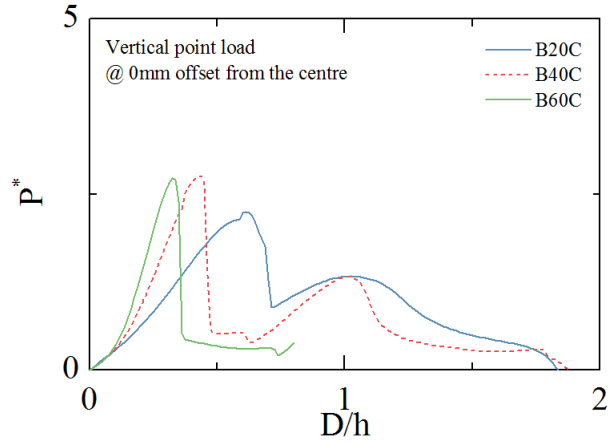
$$P^* = P \frac{\sqrt{1-\nu^2}}{E_0 R_b^{2-a} t^a \omega^b}, \quad (4.5)$$

where E_0 is the initial elastic modulus of the material, and a and b are constants yet to be found through simulation results.

According to the definition of P^* , only b is directly related to the change of ω , hence the change of h in the current setting. Using the same method in Section 3.6.5, it is found that $b = 1.75$ gave the best fitting between result. It is noted that $a = 2.25$ was used based on the future finding in the next section.

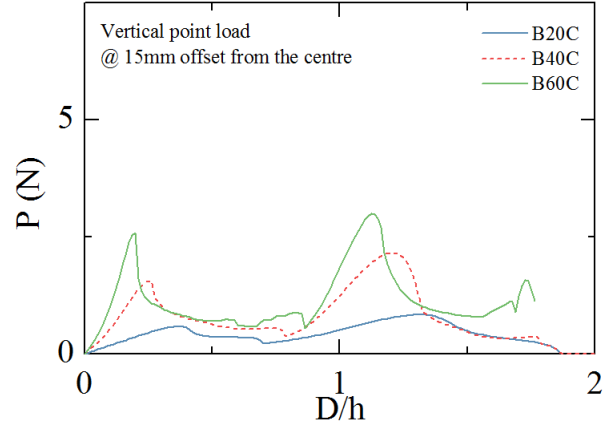


(a)

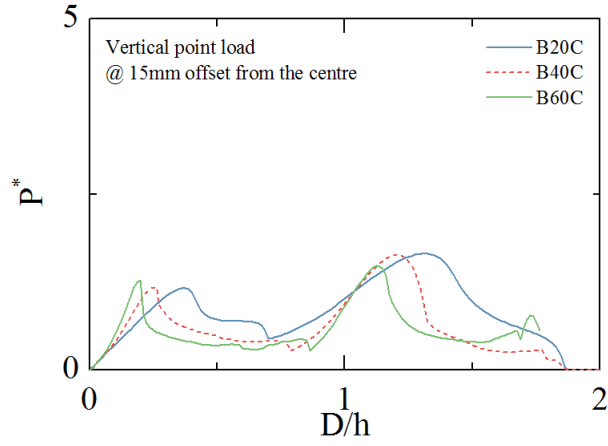


(b)

Figure 4.16: Comparison of structural responses of Model B40C and its height variants Models B20C&B60C under vertical point loading with 0 mm offset from the centre. (a) P vs. D/h , (b) P^* vs. D/h . Dimensionless load P^* is defined in Equation 4.5 with $a = 2.25$ and $b = 1.75$.

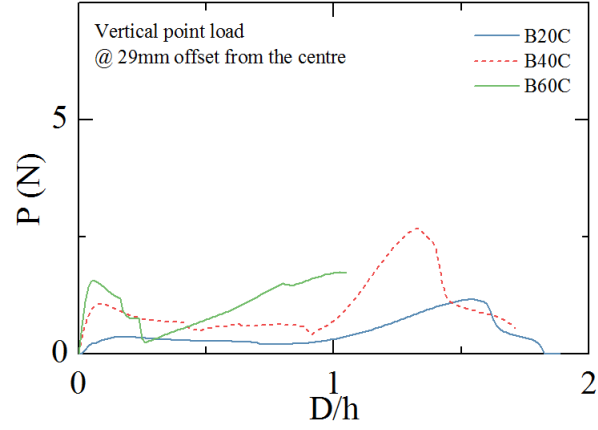


(a)

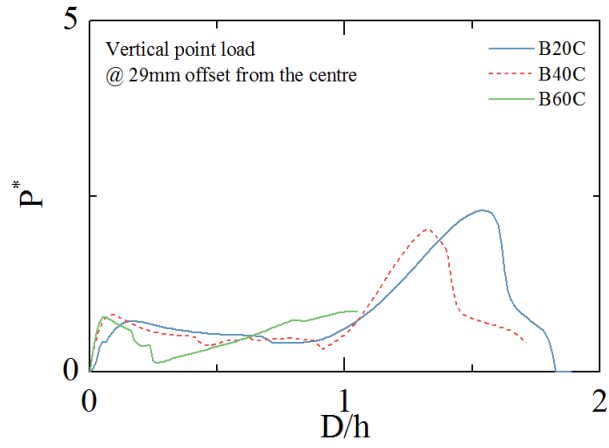


(b)

Figure 4.17: Comparison of structural responses of Model B40C and its height variants Models B20C&B60C under vertical point loading with 15 mm offset from the centre. (a) P vs. D/h , (b) P^* vs. D/h . Dimensionless load P^* is defined in Equation 4.5 with $a = 2.25$ and $b = 1.75$.



(a)



(b)

Figure 4.18: Comparison of structural responses of Model B40C and its height variants Models B20C&B60C under vertical point loading with 29 mm offset from the centre. (a) P vs. D/h , (b) P^* vs. D/h . Dimensionless load P^* is defined in Equation 4.5 with $a = 2.25$ and $b = 1.75$.

4.4.2 Thickness

In this section, the effects of parameter thickness on the close-top frustum shell are investigated, and the results are presented in Figures 4.19, 4.20, and 4.21. Including the reference configuration Model B40C, frusta with six different thicknesses were analysed. They are listed in Table 4.3. These thickness variants were chosen such that the ratio of thicknesses of the top surface and the side surface would not change among all cases. During the fabrication of the silicone frustum, the structure collapsed under its own weight if the thickness was reduced below certain values. However, in this study, the self-weight of the structure has not been considered in any of the Abaqus simulations. Similarly, some of the simulations did not result in convergent solutions but were still reliable since equilibrium was achieved during each step of simulation using implicit solvers.

For each loading scenario, a consistent trend is observed where the two peak forces increased proportionally with the thickness in Figures 4.19a, 4.20a, and 4.21a, and no fundamental change in buckling mode were observed between all three models.

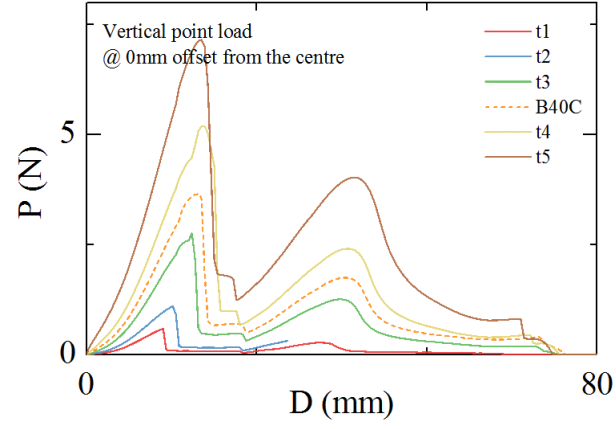
The same dimensionless terms (D^* and P^* in Equations 4.4 and 4.5, respectively) that we used in the parametric study of the height were used here. The constant a is associated with t by the definition of P^* . It is found that, by using $a = 2.25$, P^* for various ts align neatly with each other, as plotted in Figures 4.19b, 4.20b, and 4.21b. It is noted that $b = 1.75$ was used based on the results in the previous section.

Figures 4.19c, 4.20c, and 4.21c summarise the two dimensionless peak forces with respect to the changing parameter t . The lack of large variations in P_{cr}^* suggests a suitable choice of a value. Noted that dashed lines and points on these plots are extrapolated values assuming the flat trend continued.

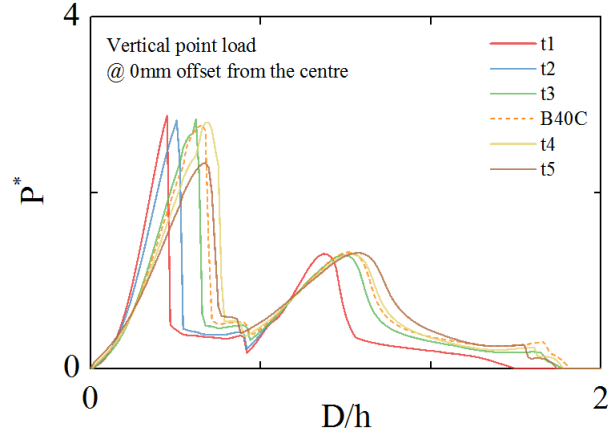
Table 4.3: Parametric study: thickness variation.

Models	Thickness, t	
	Side Surface	Top Surface
B40C_t1	0.75 mm	1 mm
B40C_t2	1 mm	1.34 mm
B40C_t3	1.5 mm	2 mm
B40C (reference)	1.72 mm	2.3 mm
B40C_t4	2 mm	2.67 mm
B40C_t5	2.5 mm	3.34 mm
B40O_t1	0.75 mm	1 mm
B40O_t2	1 mm	1.34 mm
B40O_t3	1.5 mm	2 mm
B40O (reference)	1.72 mm	2.3 mm
B40O_t4	2 mm	2.67 mm
B40O_t5	2.5 mm	3.34 mm

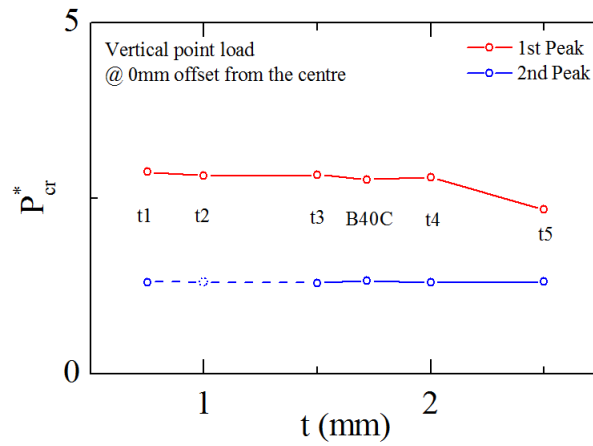
CHAPTER 4. CLOSE-TOP FRUSTUM SHELLS UNDER VERTICAL POINT LOADING



(a)

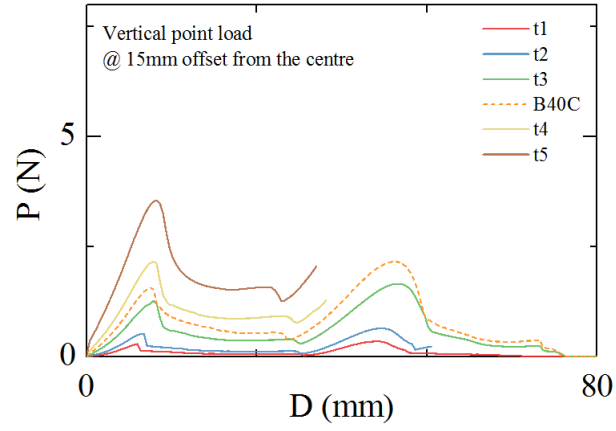


(b)

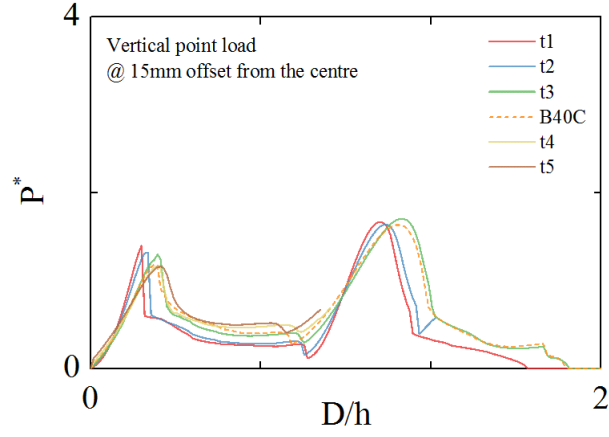


(c)

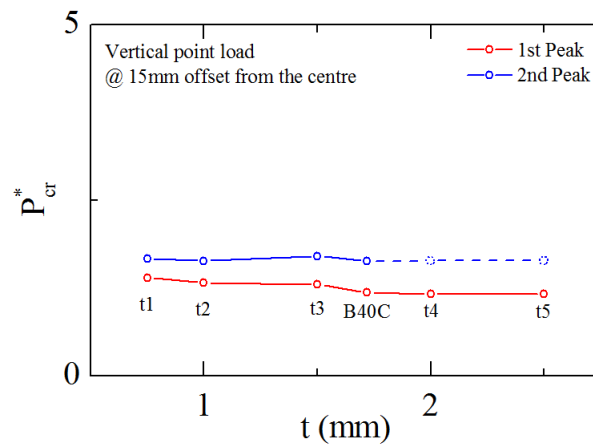
Figure 4.19: Comparison of structural responses of Model B40C and its thickness variants, Models B40C_t1-5 under vertical point loading with 0 mm offset from the centre. (a) P vs. D/h , (b) P^* vs. D/h , and (c) P_{cr}^* vs. t . Dimensionless load P^* is defined in Equation 4.5 with $a = 2.25$ and $b = 1.75$.



(a)

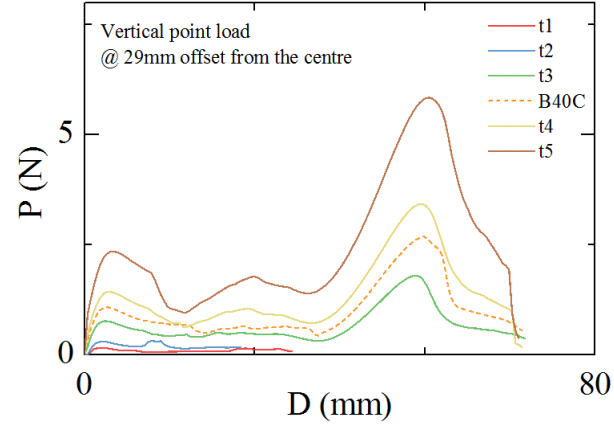


(b)

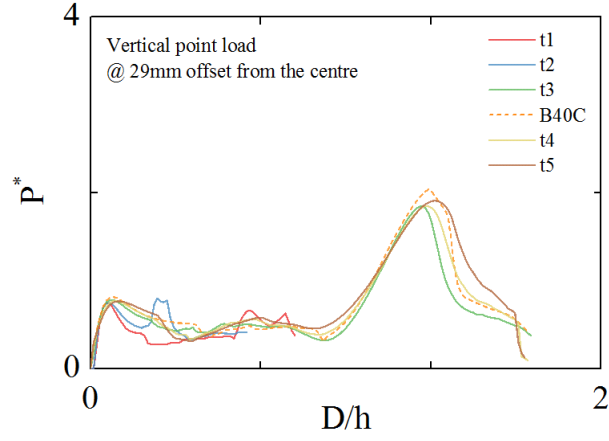


(c)

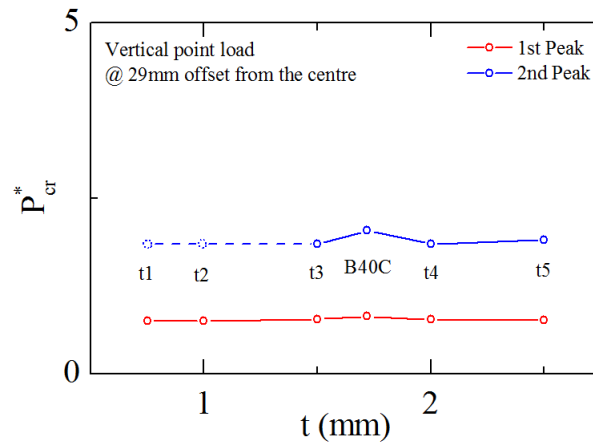
Figure 4.20: Comparison of structural responses of Model B40C and its thickness variants, Models B40C.t1-5 under vertical point loading with 15 mm offset from the centre. (a) P vs. D/h , (b) P^* vs. D/h , and (c) P_{cr}^* vs. t . Dimensionless load P^* is defined in Equation 4.5 with $a = 2.25$ and $b = 1.75$.



(a)



(b)



(c)

Figure 4.21: Comparison of structural responses of Model B40C and its thickness variants, Models B40C.t1-5 under vertical point loading with 29 mm offset from the centre. (a) P vs. D/h , (b) P^* vs. D/h , and (c) P_{cr}^* vs. t . Dimensionless load P^* is defined in Equation 4.5 with $a = 2.25$ and $b = 1.75$.

4.5 Discussion

The previous parametric studies were carried out by focusing on two parameters - height and thickness. It is found that $a = 2.25$ and $b = 1.75$ in Equation 4.5 give the best fitting. Therefore, the two peak force P_{cr1} and P_{cr2} could be calculated using the following formulas,

$$P_{cr1} = P_{cr1}^* \cdot \frac{E_0 R_b^{-0.25} t^{2.25} \omega^{1.75}}{\sqrt{1 - \nu^2}}, \quad P_{cr2} = P_{cr2}^* \cdot \frac{E_0 R_b^{-0.25} t^{2.25} \omega^{1.75}}{\sqrt{1 - \nu^2}}, \quad (4.6)$$

where P_{cr1}^* and P_{cr2}^* are the corresponding peak values of P^* for the two peaks. These specific values were showing in Table 4.4. It is worth mentioning that Equation 4.6 is a modified formula that was originally derived for the buckling force of perfect elastic shells; differently, the second peak P_{cr2} only exists in the post-buckling during the deformation. However, it is found that P_{cr2} shows consistent outcomes in calculating P_{cr2}^* using Equation 4.6 with the same a and b values.

Table 4.4: Summary of P_{cr1}^* and P_{cr2}^* of vertical point loading.

Load Location	P_{cr1}^*	P_{cr2}^*
0 mm offset from centre	2.80	1.30
15 mm offset from centre	1.20	1.65
29 mm offset from centre	0.75	1.90

As shown in Section 4.3.4, our method of deriving the optimal loading location was verified by the simulation results. We could extend that method to frusta with a broad range of parameters with confidence. The dimensionless peak forces in Table 4.4 were plotted with respect to their point loading locations and Polynomial fitted curves were also calculated and presented in Figure 4.22. The overall critical force P_{cr} is the maximum value of the two peak forces. Therefore, the P_{cr}^* can be found as the green line in Figure 4.22. The intersection point at (10.5 mm, 1.56) suggests the lowest critical forces. Therefore, the generalised formula of predicting the critical force could

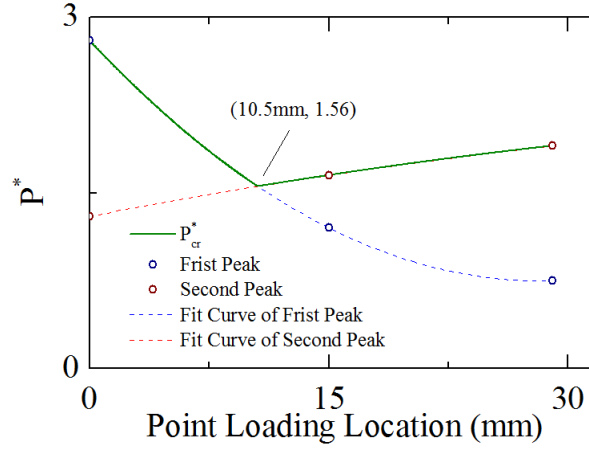


Figure 4.22: Dimensionless critical force prediction. P^* was calculated with $a = 2.25$ and $b = 1.75$ in Equation 4.5

be given as,

$$P_{cr} = c \cdot \frac{E_0 R_b^{-.25} t^{2.25} \omega^{1.75}}{\sqrt{1 - \nu^2}}, \quad (4.7)$$

where c could be calculated using the following fitted formulas of the green line in Figure 4.22,

$$c = \begin{cases} 2.8 - 1.4521 \times 10^{-1}x + 2.57 \times 10^{-2}x^2 & \text{for } 0 \leq x < 10.53754 \\ 1.3 + 2.617 \times 10^{-2}x - 1.88834 \times 10^{-4}x^2 & \text{for } 10.53754 \leq x \leq 30, \end{cases} \quad (4.8)$$

where x indicates the distance of the loading point from the centre. For close-top frustum shell with $R_b = 75$ mm that are made of the same hyperelastic silicone rubber material, Equations 4.7&4.8 are applicable in the range of $t = [0.75 \text{ mm}, 2.5 \text{ mm}]$ of side surface, $h = [20 \text{ mm}, 60 \text{ mm}]$, and $\omega = [24^\circ, 53^\circ]$. The lowest critical force of vertical point loading can be calculated using Equation 4.7 with $c = 1.56$, where loading is offset at 10.5 mm from the centre.

4.6 Summary

In this chapter, the structural responses of close-top frustum shells under various vertical point loadings were investigated using the FEM method. Experimental results were used to validate the FEM simulations. In preparing the experiments, a particular silicone rubber material was chosen to manufacture frustum specimens. The material properties have been axially tested and fitted with the classical polynomial model for hyperelastic material. The silicone frustum specimens were cast using bespoke moulds and the experiments were performed by measuring reaction force and displacement at various loading locations. In addition, we applied an innovative method of 3D recording the structural deformities that evolved during the experiments. These 3D images showed the deformation history visually and were compared with Abaqus simulations to verify the suitability of the latter; the Abaqus simulations revealed reasonable concordance with the experimental results.

An overview of the findings in this chapter is listed below:

1. 3D scanning techniques were used to record 3D shapes at selected stages during the experiments were showcasing along with the corresponding simulated deformed shapes. It provides an intuitive comparison between experimental and simulation results. Along with the comparison of displacements and loads between experimental and simulation results, Abaqus models were validated with great confidence.
2. During displacement-controlled vertical point loading on the top surface of the frustum, two local peak forces would occur before the structure get fully inverted to its secondary stable state. This was demonstrated in both experimental and simulation results, which generally agree on how the magnitudes of these two peak forces were affected by the location of loading.
3. The overall critical force is the maximum of the two peak forces. By applying

polynomial fitting of the two peaks with respect to their corresponding locations, an optimal loading location was proposed to be at 10.5 mm from the centre, which gave the lowest critical force of 2.02 N during a force-controlled loading. The corresponding lowest critical force was verified by additional Abaqus simulation.

4. Parametric studies relating the geometric parameters (h , ω , and t) were conducted. By examining their effects on the structural responses of frustum shells, generalised fitted formulas were proposed in Equations 4.7 and 4.8 in predicting the overall critical forces.
5. The most efficient method of inverting the frusta with the stipulated properties into its secondary stable state should be achieved by the application of the lowest critical force. The lowest critical force when the reference close-top frustum shell and its variations were point loaded vertically was found at a location which is offset from the centre by 10.5 mm. The corresponding critical force could then be calculated using Equation 4.7 with $c = 1.56$.

Chapter 5

Frustum Shells

under Lateral Point Loading

In the previous chapter, we have investigated the structural behaviour of the case of close-top frustum shell under vertical point loading; in this chapter, we will explore the case of close-top frustum shell under lateral point loading. The application of lateral point loading is described in two scenarios involving either the close-top or the open-top frustum shells. In a similar methodology to the previous chapter, the load was applied at multiple locations of the side surface and the critical forces and bistable behaviour was compared. Furthermore, parametric studies on height and thickness were performed with Abaqus.

5.1 Open-Top Frustum

under Lateral Point Loading

5.1.1 Experiments

Experimental tests have similar setups as compared to the ones in Chapter 4.1.

Material

The same silicone rubber material has been used in the experiments and simulations as in Chapter 4.1.

Specimen

Specimens were prepared in the same manner as discussed in Chapter 4.1. The specimens Model B40O in this section have the same dimensions as Model B40C, which were listed in Table 4.2, but lack a top surface. Excess material including the top surface was carefully trimmed so that the remaining structure matched the proposed dimensions. The final products were shown in Figure 4.4.

Apparatus

Various lateral point loadings were applied during the test. The location of each load application was defined relative to the top rim of the side surface (Figure 5.1a). We investigated the responses on two representative locations (4 mm and 20 mm normal to the top rim in Figure 5.1b) in this study. The 4 mm offset allows enough space for the loading probe to operate while simulating a case of load application at the rim; the 20 mm offset presents a comparison point where the load is applied approximately one third from the upper edge of the side surface. The experimental and simulated results were compared to determine any changes in the critical force or the buckling mode. The load was applied in a straight line due to the limitation of equipment (Figure 5.1c).

During the test, the digital force gauge was placed on a tilting stand so that the probe is perpendicular to the side surface of the frustum specimen. Figure 5.2a shows a tilting stand equipped with a digital force gauge and Figure 5.2b presents a snapshot of the experiment with the modified setup.

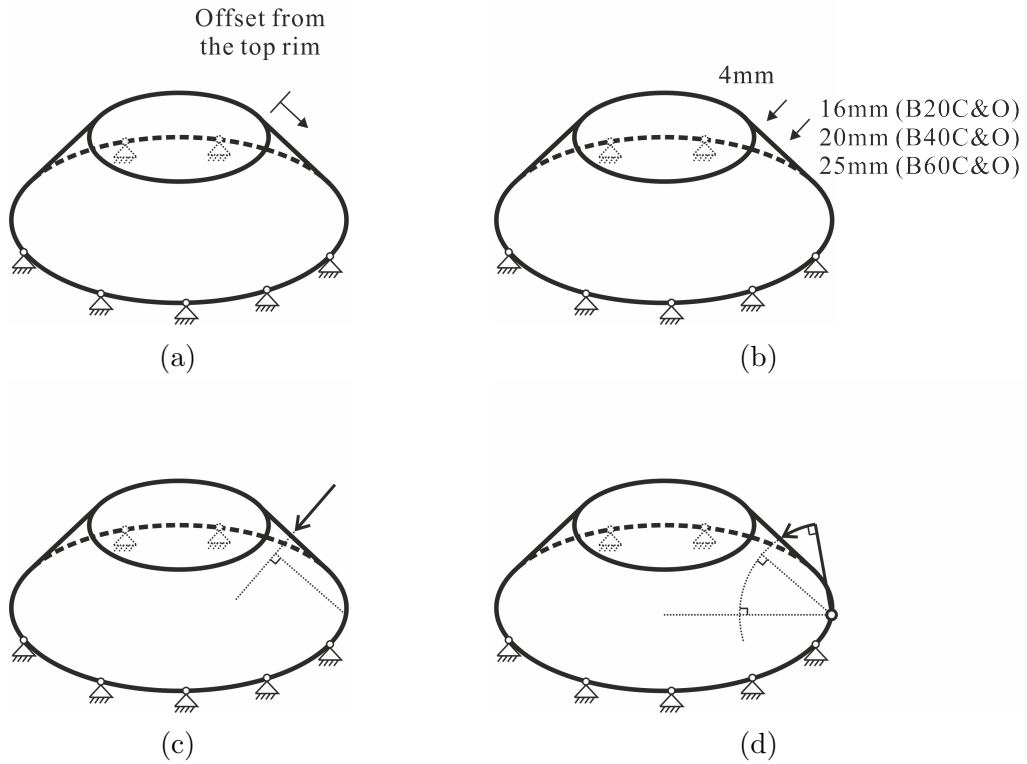


Figure 5.1: Schematic graphs of loading location and method definition

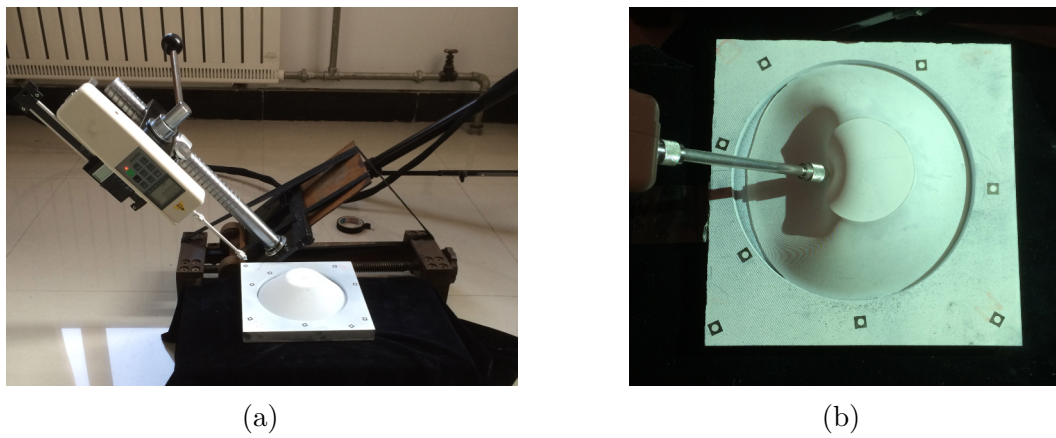


Figure 5.2: Loading Plates. (a) A tilting stand with the scanning equipment installed, and (b) scanning while the equipment is tilted.

5.1.2 Abaqus Simulations

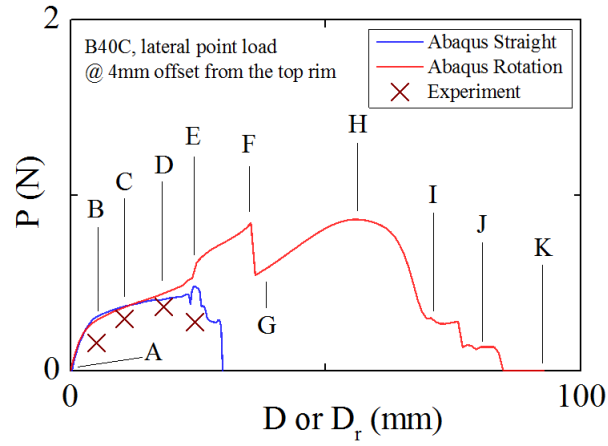
The finite element simulation was performed using the general-purpose finite element package ABAQUS Ver.6.13 (Abaqus, 2013). A reference open-top frustum B40O (detailed dimensions in Table 4.2) and several height and thickness variants were built using shell element S4R. The mesh size was controlled at around 2.5 mm. The analysis involved quasi-static loading, non-linear response, and large deformations; hence, the implicit solver, ‘Static, General’, was used. The automatic stabilisation function was turned on to ensure that the analysis would persist in the event of dramatic buckling. The application of a sharp point load could cause localised stress concentration that can potentially result in convergence problem during the simulation. In order to avoid the latter, a small hemispherical probe was used in the simulation to apply displacement-controlled loading. The reference point of that probe was at the centre of that hemisphere which allows the direction of reaction force pointing to the reference point. The tangential part of the contact between the probe and the frustum was modelled as frictionless, which means finite sliding is allowed and no tangential force between the bodies in contact; while the normal part was modelled as hard contact, which means no contact pressure until nodes are in contact and unlimited contact pressure once contact has been established. In this set-up, only positive force can be recorded. In the experiment, the load was applied in a straight line due to limited conditions (Figure 5.1c). However, this setup does not allow the frustum to be fully inverted during the experiment. In the simulations, which included straight-line loading, a second loading method was introduced where the load was applied via an arm rotating about the bottom rim of the frustum (Figure 5.1d). In the result sections, the trajectory of the loading point using the latter method is denoted as D_r . The term ‘Abaqus straight’ refers to the simulation of the loading method used in the experiments (shown in Figure 5.1c) while the term ‘Abaqus rotation’ relates to rotational loading (shown in Figure 5.1d). We expect similar results in the initial response profiles in both scenarios, but the rotational loading allows the structure to be fully inverted.

5.1.3 Comparison between Simulated and Experimental Results

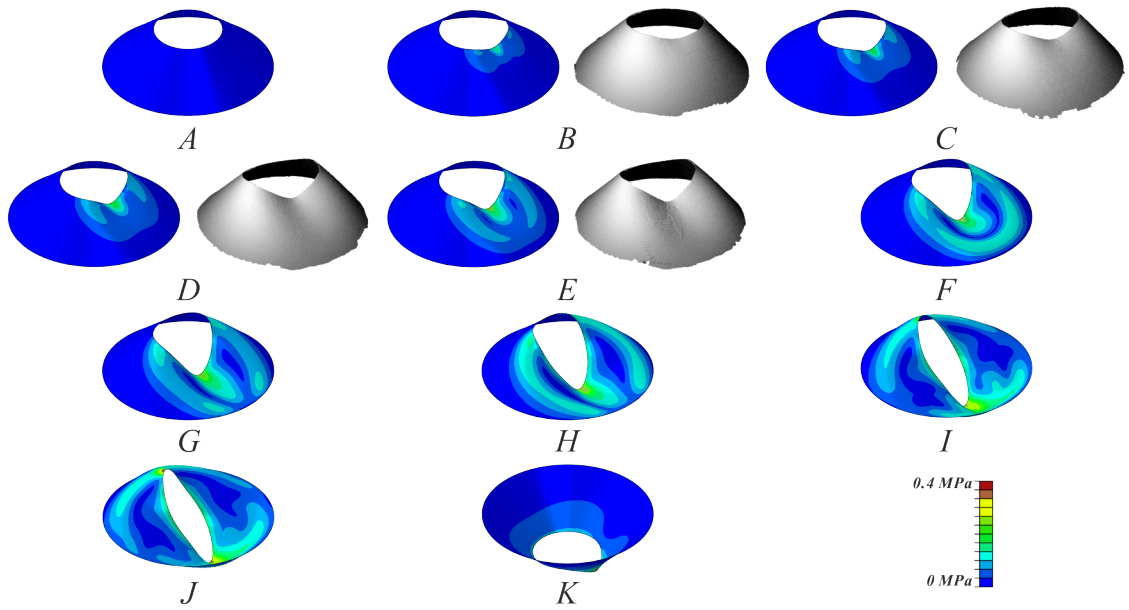
In this section, the results of lateral point loading from both Abaqus simulations and experiments with Model B40O will be presented and compared. ‘Abaqus straight’ indicates the same loading method as the experiments while ‘Abaqus rotation’ is the loading method introduced in the simulation, which is discussed in the previous section.

Model B40O under lateral point loading with 4 mm offset from the top rim

Figure 5.3 shows the structural response of Model B40O to a lateral point load that is applied with a 4 mm offset from the top rim. The reaction force of ‘Abaqus rotation’ is calculated by dividing the corresponding moment by the length of the lever arm between the contact point and the bottom rim, which is $60.2\text{mm} - 4\text{mm} = 56.2\text{mm}$. Generally, the Abaqus results fit well with the experimental results. The experiments reveal a slightly smaller reaction force in comparison to either scenario in Abaqus. The ‘Abaqus rotation’ response curve is initially aligned with ‘Abaqus straight’ but departs around Point D; ‘Abaqus straight’ terminates shortly after this point due to loading constraints while ‘Abaqus rotation’ continues to simulate the results of continuous loading until the frustum is fully inverted. Furthermore, the shapes at each state show visual agreement in the extent of deformation between Abaqus simulations and experimental 3D scanning.



(a)

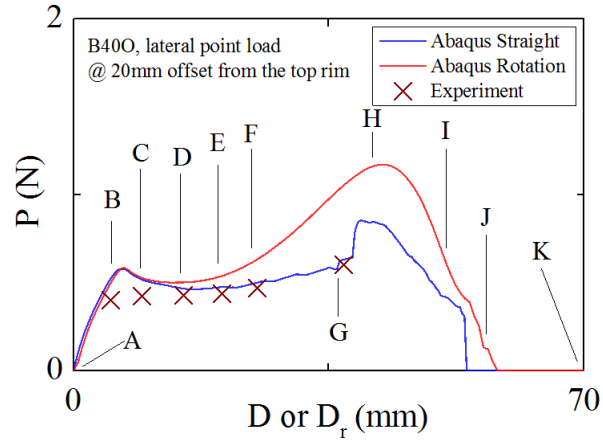


(b)

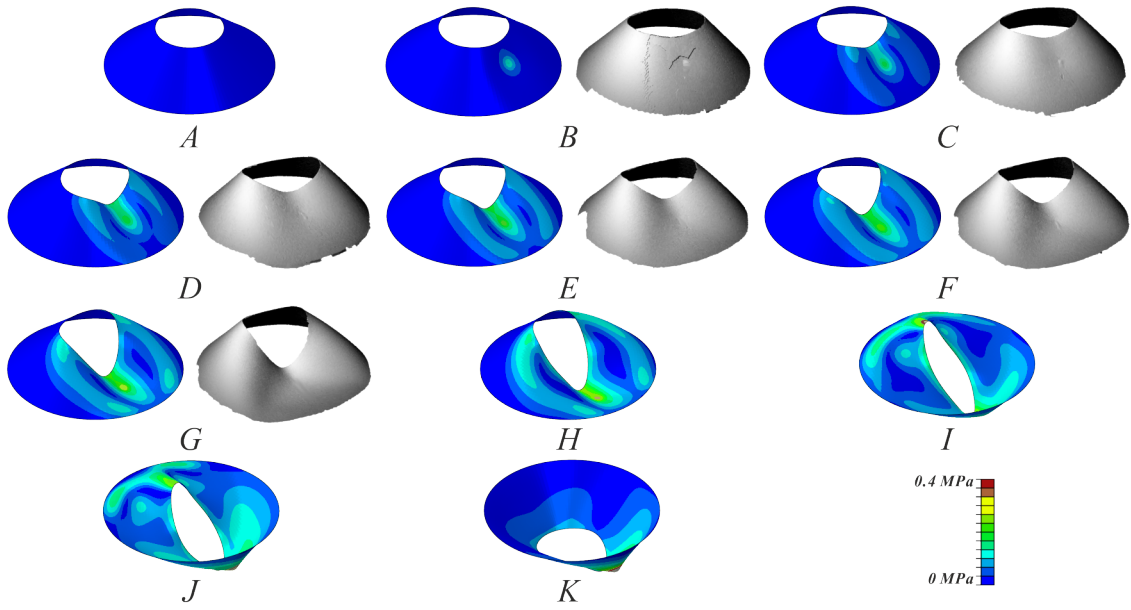
Figure 5.3: Structural response of Model B40O under lateral point load with 4 mm offset from the top rim. (a) Vertical reaction force P vs. displacement of the probe point D or D_r , and (b) deformed shapes and stress distribution. Coloured graphs are Abaqus simulated results while grey graphs are 3D scanned shapes.

**Model B40O under lateral point loading
with 20 mm offset from the top rim**

Figure 5.4 shows the structural response of Model B40O to a lateral point load that is applied with a 20 mm offset from the top rim. The reaction force of ‘Abaqus rotation’ is calculated by dividing the corresponding moment by the length of the lever arm between the contact point and the bottom rim, which is $60.2\text{mm} - 20\text{mm} = 40.2\text{mm}$. The Abaqus results fit well with the experimental results. The experiments reveal a slightly smaller reaction force as compared to the Abaqus simulation and the absence of the first local peak force. The results of ‘Abaqus straight’ reveals a similar trend as experimental results while the results of ‘Abaqus rotation’ coincides well with ‘Abaqus straight’ during initial loading but the curves begin to depart from each other as the loading continues. Furthermore, there is a visual agreement between the respective shapes at each stage of deformation related to both the Abaqus simulations and experimental 3D scanning. However, a sewing error was present for the 3D recorded shape at stage B, which is possibly caused by environmental disturbance during the scanning process.



(a)



(b)

Figure 5.4: Structural response of Model B40O under lateral point load with 20 mm offset from the top rim. (a) P vs. D or D_r , and (b) deformed shapes and stress distribution. Coloured graphs are Abaqus simulated results while grey graphs are 3D scanned shapes.

Discussion

For both comparisons, the Abaqus results show fair agreement with the experimental results, which validates the ability of Abaqus to simulate large deformation problems in this context. Due to limitations in the apparatus setup, the loading was originally applied in a straight line in experiments as showing in Figure 5.1c, which limits the extent of inversion. By introducing a second loading method (Abaqus rotation) in the Abaqus simulations, this limitation was circumvented. The results of ‘Abaqus straight’ and ‘Abaqus rotation’ align closely at the beginning but gradually diverged with progressive loading and ‘Abaqus straight’ terminates shortly after the divergence. Therefore, ‘Abaqus rotation’ provides the only alternative that allows the structure to be fully inverted and will be used in later parametric studies.

By reviewing the deformed shapes, we arrive at a clear insight on how the structure was deformed. For the 4 mm case, the material close to the loading on the side surface became partially inverted as loading commenced; the indentation enlarged until it reached the bottom rim (Point E) and this was where the first peak force took place, followed by a sudden drop in reaction force at Point G; the inversion dragged the top rim downwards and the reaction force restarted to grow until the second peak force happened at Point H; after that, the reaction force dropped to zero and the structure inverted. The first peak force is 0.84 N at 35.3 mm displacement and the second one is 0.86 N at 55.6 mm displacement.

For the 20 mm case, the side surface was indented immediately after the loading began; the first peak force took place when such indentation propagated to the top rim, and the top rim became partially inverted at Point C; similarly to the 4 mm case, the inversion dragged the top rim downwards and the reaction force resumed to grow to the second peak at Point H; afterwards, the reaction force dropped to zero and the structure became fully inverted. The first peak force is 0.58 N at 6.94 mm displacement and the second one is 1.17 N at 42.3 mm displacement.

We expect that the peak forces would be inversely proportional to the corresponding lever arms about the rotating point at the bottom rim. It is exactly the case for the second peak forces, since $40.2 \text{ mm}/56.2 \text{ mm} \cong 0.86 \text{ N}/1.17 \text{ N}$, while this argument is not applicable to the first peak forces since the causes of the latter between two cases were different.

5.1.4 Parametric Study - Height

The parametric study was set up in a similar manner as shown in Section 4.4.1. In this parametric study, the effects of height on the structural response of the open-top frustum shell were investigated. Two height variants, Models B20O and B60O (detailed dimensions are in Table 4.2) were simulated in Abaqus.

It is worth noting that because the slant angles did not remain constant but increased with the height, it differs from the study mentioned in Chapter 3. The thicknesses of Models B20C and B60C were identical to Model B40C, where the side surface was 1.72 mm thick and the top surface was 2.30 mm thick. The heights of Models B20C, B40C and B60C are 20 mm, 40 mm and 60 mm, respectively. The loads were applied via an arm rotating about the bottom rim of the frustum, which is the loading method in Figure 5.1d. As discussed in Section 5.1.3, the rotation loading method resulted in a response profile that revealed precise alignment with both the experimental and simulations using straight loading methods, but more importantly, also allowed the structure to be fully inverted.

Dimensionless groups were defined according to the same philosophy as outlined in Chapter 3. The dimensionless displacement is defined as:

$$D^* = \theta/\omega, \quad (5.1)$$

where θ is the rotation of the probe, and ω is the slant angle of the frustum. Although it is different from D/h which was used previously, they all represent the same thing - $\theta/\omega = 2$ is the displacement of the mirror position about the bottom rim of the frustum, just like $D/h = 2$.

The dimensionless load P^* is defined as:

$$P^* = P \frac{\sqrt{1-\nu^2}}{E_0 R_b^{2-a} t^a \omega^b}, \quad (5.2)$$

where E_0 is the initial elastic modulus of the material, and a and b are constants yet to be found through simulation results.

The results are displayed in Figures 5.5&5.6. Convergent solutions up to $\theta/\omega = 2$ were not available for some simulations, but the available results are reliable since equilibrium was achieved in every step of the implicit Abaqus simulation. In Equation 5.2, b is associated with the change of ω . By using the same method in Section 3.6.5, it was found that each result in this parametric study fit the best when $b = 1.75$, especially at the beginning of the loading. Additionally, the value of $a = 2.75$ based on the fitting relating to the parametric study of t in the next section was used in calculating P^* .

In the case where the load was applied next to the top rim, Figure 5.5a shows the reaction force P versus the dimensionless displacement θ/ω while Figure 5.5b show the dimensionless force P^* versus θ/ω . The choice of $a = 2.75$ and $b = 1.75$ gave great fitting at the beginning. It is found that Model B20O only had one peak force before the structure get fully inverted.

In the case where the load was applied at 1/3 of the side surface from the top rim, Figure 5.6a shows the plot of P versus θ/ω while Figure 5.6b shows the plot of P^* versus θ/ω . The chosen P^* with $a = 2.75$ and $b = 1.75$ resulted in a good alignment in the magnitude of the first peak force while not of the tested a and b values were suitable for the alignment of the second peak. In explaining the latter, we rationalise that the formation of the second peak might be driven by different buckling mechanism, or that different buckling mechanisms may be involved in the behaviour of the different models.

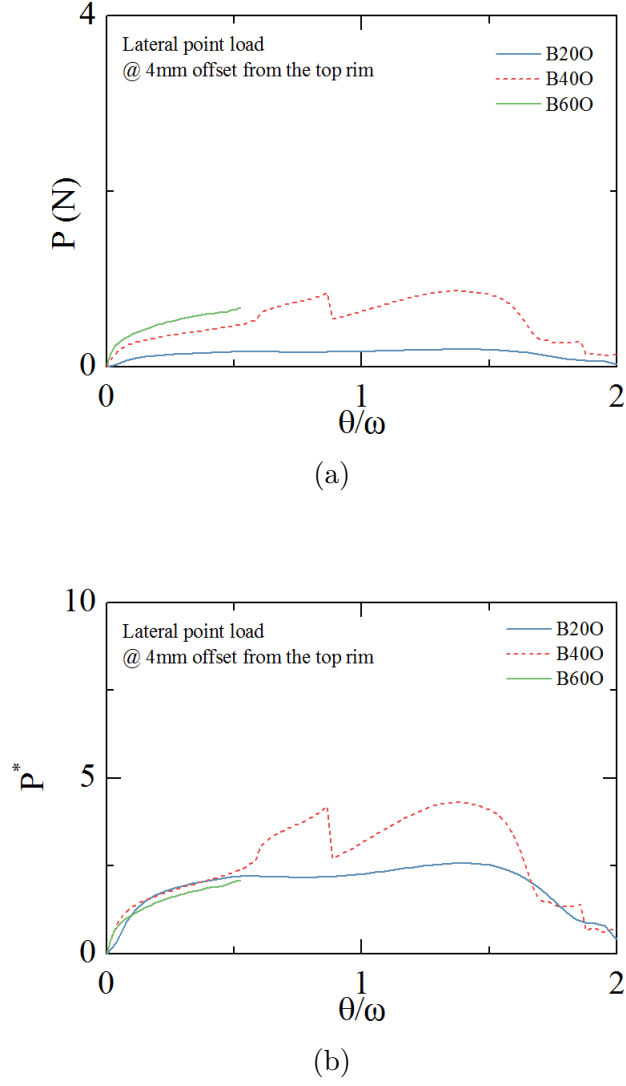


Figure 5.5: Comparison of structural response of Model B40O and its height variants Models B20O&B60O under lateral point loading with 4 mm offset from the top rim. (a) P vs. dimensionless displacement θ/ω , (b) P^* vs. θ/ω . P^* is defined in Equation 5.2 with $a = 2.75$ and $b = 1.75$.

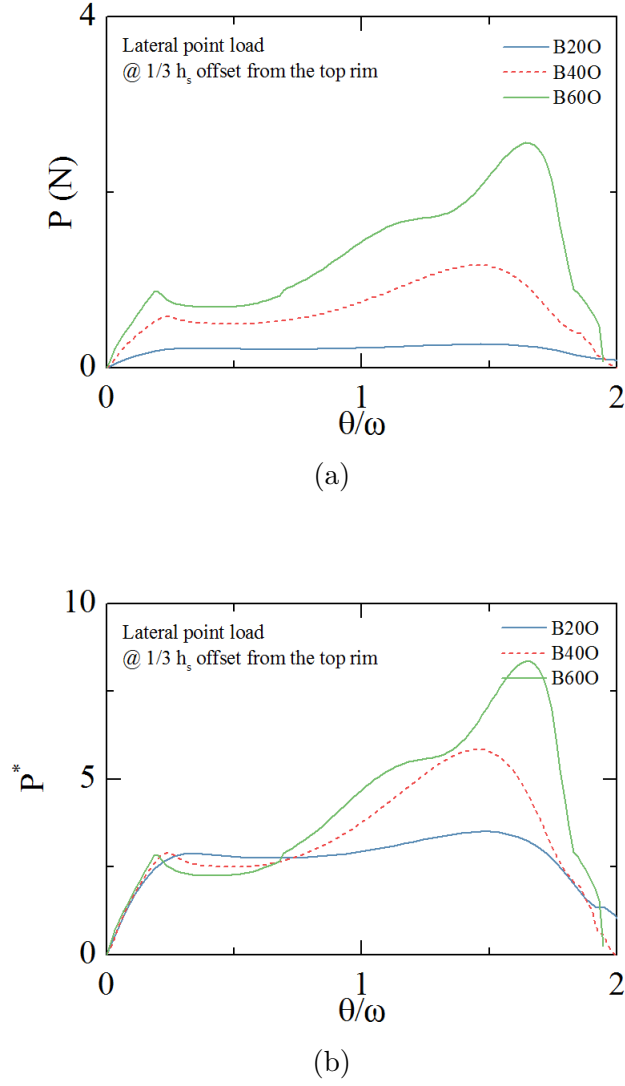


Figure 5.6: Comparison of structural response of Model B40O and its height variants Models B20O & B60O under lateral point loading with $1/3h_s$ offset from the top rim. (a) P vs. θ/ω , (b) P^* vs. θ/ω . P^* is defined in Equation 5.2 with $a = 2.75$ and $b = 1.75$.

5.1.5 Parametric Study - Thickness

In this parametric study, we would like to evaluate the effects of thickness on the structural response of the close-top frustum shell. Including the reference configuration Model B40C, frusta of six thicknesses were analysed (Table 4.3). In this study, the self-weight of the structure has not been considered in any Abaqus simulation. Other dimensions were kept the same as the ones in Table 4.2. Similarly, convergent solutions were not derived for some of the simulations but the available results are reliable since equilibrium is achieved during each step of the simulation using implicit solvers.

The same dimensionless groups (D^* and P^* in Equations 5.1 and 5.2, respectively) were used as the ones in the previous section. The constant a is associated with t by the definition of P^* . It is found that, by using $a = 2.75$, the first peak of P^* for various t align neatly with each other while no suitable a and b values were useful in fitting the second peak. This suggests that the buckling mode behind the second peak could be multifactorial and changes with thickness. It is noted that $b = 1.75$ was used based on the results in the previous section.

In the case where the load was applied next to the top rim, Figure 5.7a shows the plot of P versus D_r , while Figure 5.7b show the plot of P^* versus θ/ω . Under the current choice of a and b , P^* showed great fitting with each other before the first peak. With the exception of Model B40O, which revealed only one peak, all other models revealed two local peak forces. We found great intermodel differences between the shapes and locations of the peaks in the curves, suggesting that the buckling mode is very sensitive to changes in the thickness configuration. In the case where the point load was applied at approximately 1/3 of the length of the side surface from the top rim, Figure 5.8a shows P versus D_r , while Figure 5.8b shows P^* versus θ/ω . Two peaks were observed during the loading.

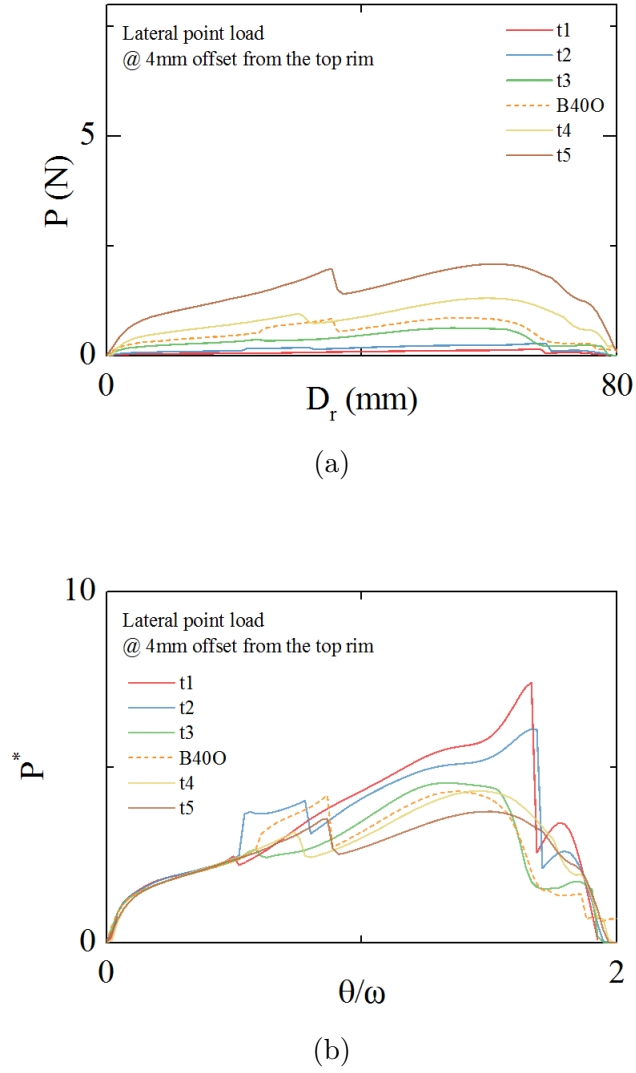


Figure 5.7: Comparison of structural response of Models B40O and its thickness variants Models B40O.t1-5 under lateral point loading with 4 mm offset from the top rim. (a) P vs. θ/ω , (b) P^* vs. θ/ω . P^* is defined in Equation 5.2 with $a = 2.75$ and $b = 1.75$.

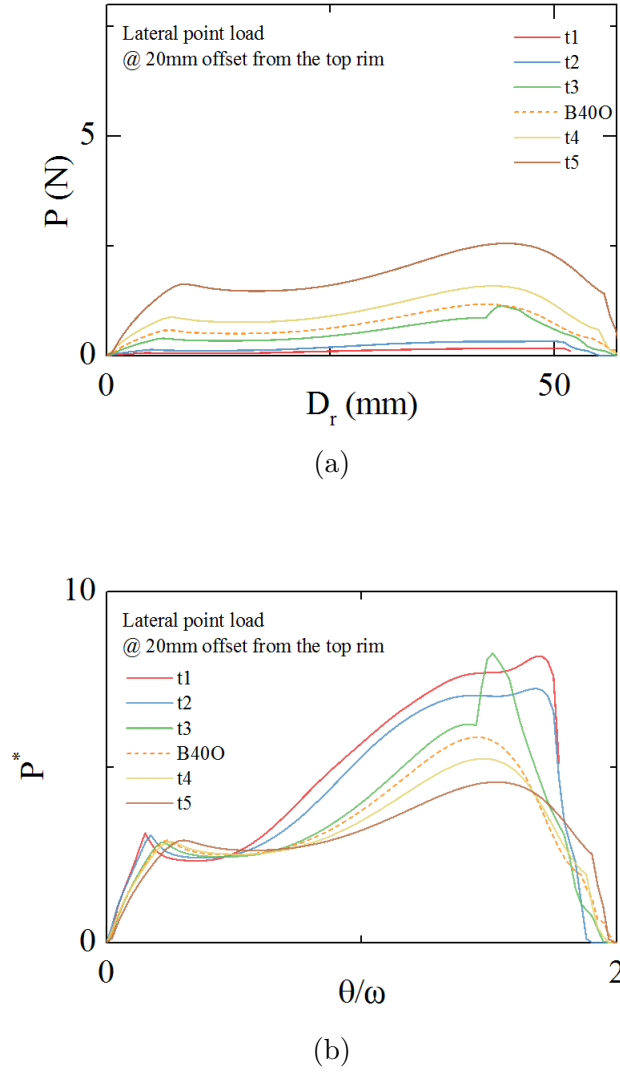


Figure 5.8: Comparison of structural response of Models B40O and its thickness variants Models B40O.t1-5 under lateral point loading with $1/3h_s$ (20 mm) offset from the top rim. (a) P vs. θ/ω , (b) P^* vs. θ/ω . P^* is defined in Equation 5.2 with $a = 2.75$ and $b = 1.75$.

5.2 Close-Top Frustum

under Lateral Point Loading

5.2.1 Experiments

Experimental tests have similar setups as those described in Chapter 4.1.

Material

The same silicone rubber material were used in the experiments and simulations as described in Chapter 4.1.

Specimen

Specimens were prepared in the same manner as discussed in Chapter 4.1. The excess material was carefully trimmed so that the remaining structure matched the proposed dimensions. The final products are shown in Figure 4.4.

Apparatus

Various lateral point loadings were applied during the test. The location of each loading application on the side surface was defined relative to the top rim (Figure 5.1a). The same two locations as those in the previous section were chosen (4 mm and 20 mm normal to the top rim in Figure 5.1b). The experimental and simulations results of these cases will be compared to check if the critical force and the buckling mode be changed. The load was applied in a straight line due to the limitation of the equipment (Figure 5.1c).

5.2.2 Abaqus Simulations

Abaqus simulations were set up in a similar manner as described in Section 5.1.2. A reference close-top frustum B40C (detailed dimensions in Table 4.2) and several height and thickness variants were built using shell element S4R. In the experiment, the load was applied in a straight line due to limited conditions. However, this setup does not allow the frustum to be fully inverted during the experiment. In the simulations, which included straight-line loading, a second loading method was introduced where the load was applied via an arm rotating about the bottom rim of the frustum (Figure 5.1d). In the result sections, the trajectory of the loading point using the latter method is denoted as D_r . The term ‘Abaqus straight’ refers to the simulation of the loading method used in the experiments (Figure 5.1c) while the term ‘Abaqus rotation’ refers to the simulations of the introduced rotational loading (Figure 5.1d). We expect similar results in the initial response profiles in both scenarios, but the rotational loading allows the structure to be fully inverted.

5.2.3 Comparison between Simulated and Experimental Results

In this section, the results of lateral point loading from both Abaqus simulations and experiments of Model B40C will be presented and compared.

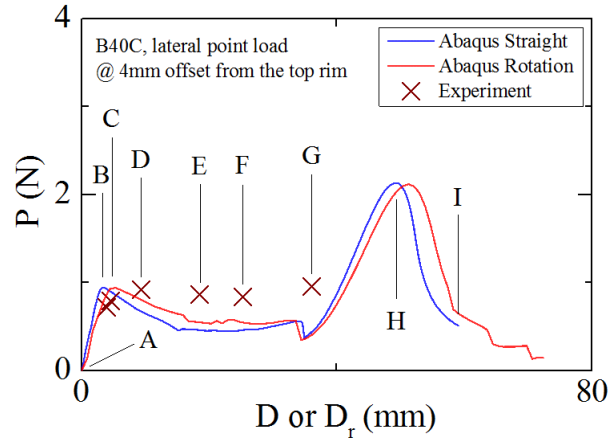
Model B40C under lateral point loading with 4 mm offset from the top rim

Figure 5.9 shows the response of Model B40C under a lateral point load that was applied with a 4 mm offset from the top rim. The reaction force of ‘Abaqus rotation’ is calculated by dividing the corresponding moment by the length from the contact point to the bottom rim, which is $60.2\text{mm} - 4\text{mm} = 56.2\text{mm}$.

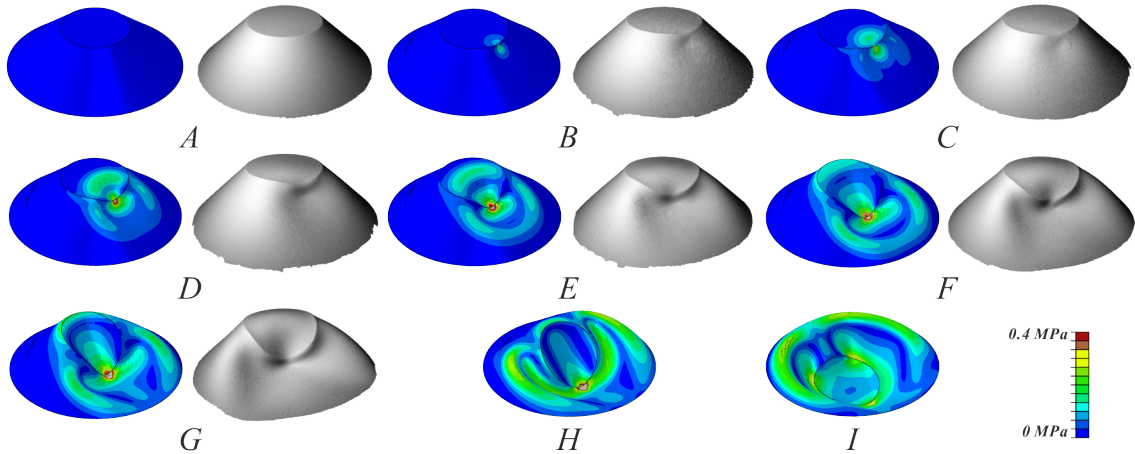
The Abaqus results fit well with the experimental results. The results of ‘Abaqus straight’ reveals a similar trend as experimental results while the results of ‘Abaqus rotation’ coincides well with ‘Abaqus straight’ but with a small phase difference. Furthermore, there is visual agreement between the respective shapes at each stage of deformation related to both the Abaqus simulations and experimental 3D scanning, except some differences after Point D. This could be caused by the inward fold of the bottom rim in the experiments while the bottom rim was constrained from any radial displacement in the simulations. However, this did not affect the reaction force by a lot.

**Model B40C under lateral point loading
with 20 mm offset from the top rim**

Figure 5.10 shows the structural response of Model B40C when placed under a lateral point load that was applied with a 20 mm offset from the top rim. The reaction force of ‘Abaqus rotation’ is calculated by dividing the corresponding moment by the length from the contact point to the bottom rim, which is $60.2\text{mm} - 20\text{mm} = 40.2\text{mm}$. Generally, the experimental and simulated data concur with the occurrence of only one local peak force before the full inversion of the frustum shown in Figure 5.10a. Moreover, there is a visual agreement between the respective shapes at each stage of deformation related to both the Abaqus simulations and experimental 3D scanning. A sewing error was present for the 3D recorded shape at stage B, which is possibly caused by environmental disturbance during the scanning process. Nevertheless, this error did not affect the subsequent loading since each scanning was individually performed.



(a)



(b)

Figure 5.9: Structural response of Model B40C under lateral point load with 4 mm offset from the top rim. (a) P vs. D or D_r , and (b) Deformed shapes and stress distribution. Coloured graphs are Abaqus simulated results while grey graphs are 3D scanned shapes.

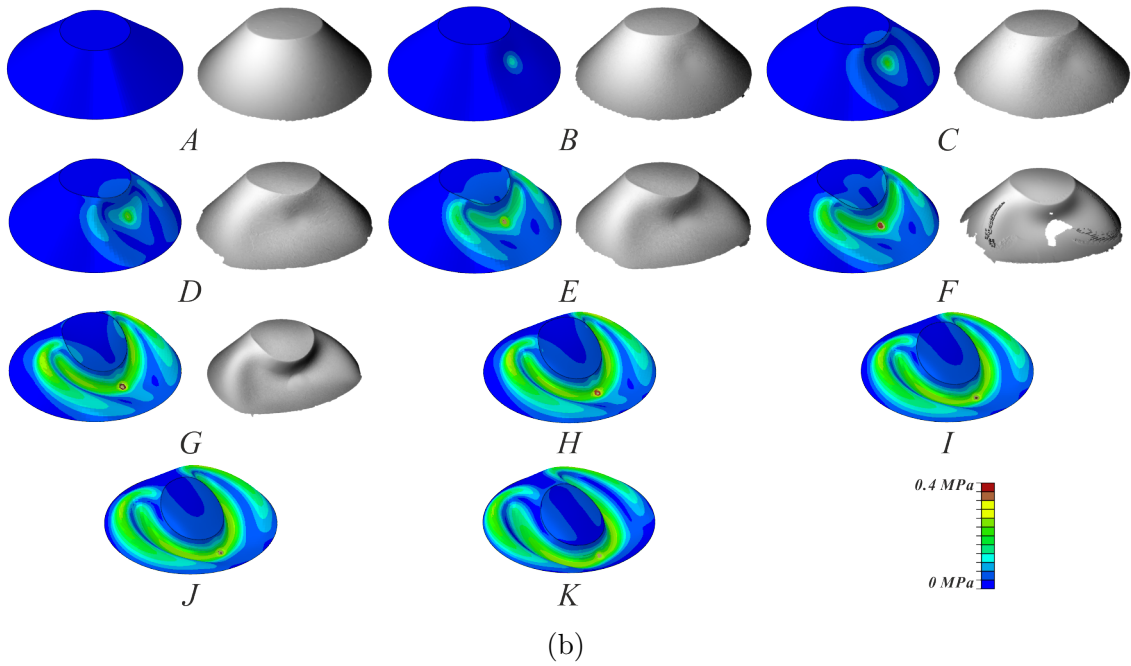
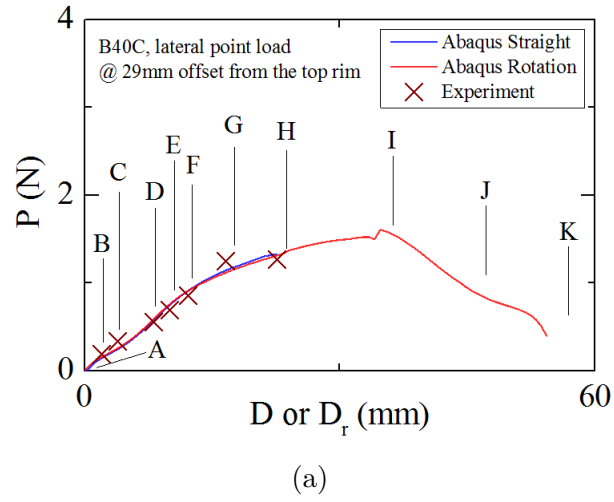


Figure 5.10: Structural response of Model B40C under lateral point load with 20 mm offset from the top rim. (a) P vs. D or D_r , and (b) deformed shapes and stress distribution. Coloured graphs are Abaqus simulated results while grey graphs are 3D scanned shapes.

Discussion

The response profile obtained with the experiments and simulations of the close-top frustum under later point loading are generally well aligned. This validated the Abaqus models for future parametric studies. The second loading method ('Abaqus rotation') was also verified to be a suitable and better substitute than the original loading method ('Abaqus straight') in revealing the inversion process. Therefore, 'Abaqus rotation' method will be used in the following parametric studies.

In all our data, we see an enormous influence of the loading location on the responses. In the 4 mm case, two peak forces were observed before the frustum was fully inverted: the structure was immediately indented by the loading, and the indentation broadened until the first peak was reached at Point B, where the top rim was buckled and the top and side surfaces became partially inverted; the inversion propagated both downwardly and circumferentially along the side surface; when the inversion reached the bottom rim, the structure snapped with a small drop of reaction force at Point G; after that, the structures readjusted and reaction force started to increase again until the second peak was reached (Point H); when the circumferential inversion was fully propagated, a bifurcation occurred which caused the load to suddenly drop, and subsequently, the structure became fully inverted. The first peak force was 0.94 N at 5.3 mm displacement and the second one was 2.12 N at 51.2 mm displacement. It is also worth noting that the response here is very similar to the response of the same frustum B40C when subjected a vertical point load that was applied with a 29 mm offset from the centre in Figure 4.13 in Chapter 4.3.3. The application of both loadings was close to each other in location but different in direction.

In the 20 mm case, only one peak force at Point I was established, which was considered to be equivalent to the second peak force in the 4 mm case. The presence of only one peak force has possibly resulted from locating the load further away from the top rim, which did not cause the top surface to buckle at all during loading. The

peak force was 1.60 N at 34.7 mm displacement.

Although the 20 mm case has shorter lever arms about the rotating point, we observed a smaller critical force than the 4 mm case of Model B40C, as energy was not lost to the incidence of buckling of the top surface in the event of the former.

5.2.4 Parametric Study - Height

The setup of this parametric study and the comparison of results were performed with the same methodology as the analysis of the open-top frusta, with the exception that close-top models were used, i.e. Model B40C and its two height variants, Models B20C & B60C (detailed dimensions are in Table 4.2). Models B20C and B60C has the same thickness as Model B40C - 1.72 mm on the side surface and 2.30 mm on the top surface. The heights of Models B20C, B40C and B60C are 20 mm, 40 mm and 60 mm, respectively. Unlike the parametric study in Chapter 3, their slant angles are not constant but increase with the height. The loading location is chosen to be at 1/3 of the side surface from the top rim. The loads were applied via an arm rotating about the bottom rim of the frustum, which is the loading method in Figure 5.1d.

Similarly, dimensionless displacement D^* and dimensionless force P^* are defined in Equations 5.1&5.2, with a and b yet to be determined via this and the following parametric studies. The results are displayed in Figures 5.11&5.12. Convergent solutions up to $\theta/\omega = 2$ were not available for some simulations, but the available results are reliable since equilibrium was achieved in every step of the implicit Abaqus simulation. In Equation 5.2, b is associated with the change of ω . By using the same method in Section 3.6.5, it was found that each result in this parametric study fit the best when $b = 1.75$, especially at the beginning of the loading. Additionally, the value of $a = 2.5$ based on the fitting relating to the parametric study of t in the next section was used in calculating P^* .

Figure 5.11 refers to data where the lateral point load was applied close to the top rim, Figure 5.11a shows P versus the predefined dimensionless displacement θ/ω while Figure 5.11b show P^* versus θ/ω . As shown in Figure 5.11b, the curves are well-matched in their predictions of the first peak force.

Figure 5.12 shows the results of the case where the lateral point load was applied at 1/3 of the side surface from the top rim: Figure 5.12a shows P versus θ/ω while

Figure 5.12b shows P^* versus θ/ω . The buckling mode of Model B20C is different from what we see for Models B40C & B60C: as the height increases, we begin to observe the occurrence of an additional peak force and a possible explanation for this change could be that the steeper side surfaces of Models B40C and B60C reinforce the structure, and prevent the top surface from buckling. In comparing the two peaks shown in Figures 5.11a&5.12a, we find that the location of the loading could also affect the number of peaks.

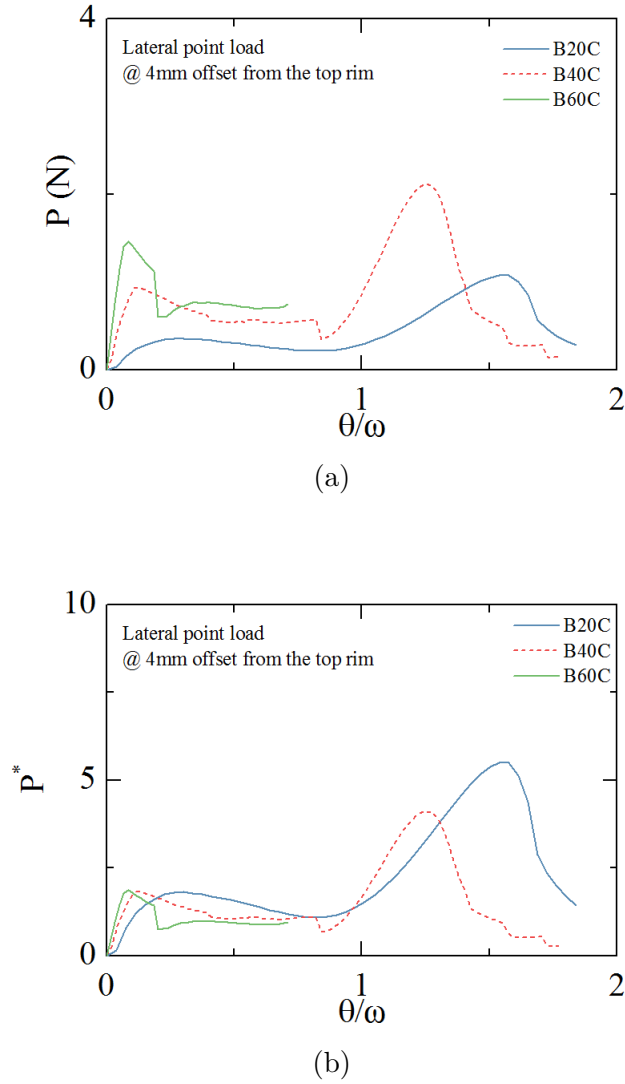


Figure 5.11: Comparison of structural response of Model B40C and its height variants Models B20C&B60C under lateral point loading with 4 mm offset from the top rim. (a) P vs. θ/ω , (b) P^* vs. θ/ω . P^* is defined in Equation 5.2 with $a = 2.5$ and $b = 1.75$.

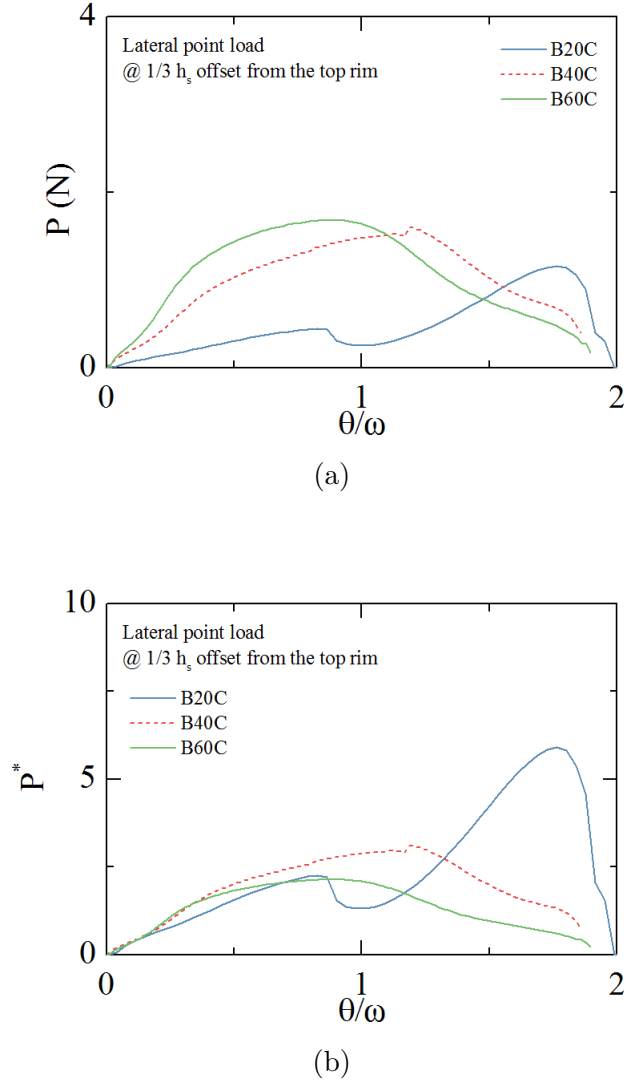


Figure 5.12: Comparison of structural response of Model B40C and its height variants Models B20C&B60C under lateral point loading with $1/3h_s$ offset from the top rim. (a) P vs. θ/ω , (b) P^* vs. θ/ω . P^* is defined in Equation 5.2 with $a = 2.5$ and $b = 1.75$.

5.2.5 Parametric Study - Thickness

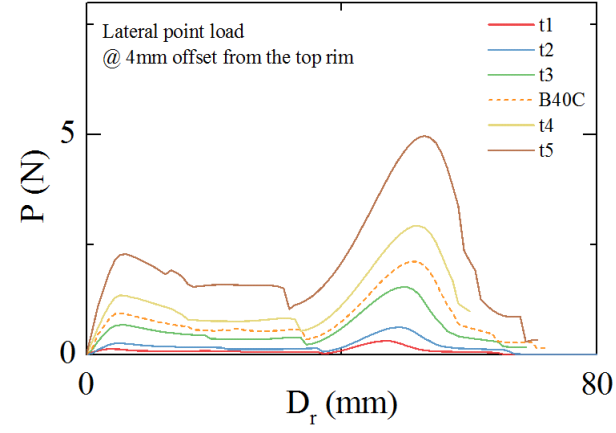
The parametric study of t is presented in this section. Including the reference configuration Model B40C, frusta with six different thicknesses were analysed, which are listed in Table 4.3. While making the silicone frustum specimens, we observed that the structure collapsed from its own weight when the thickness was reduced below a certain value. In this study, the effects of self-weight of the structure were not considered in the Abaqus simulation. Similarly, convergent solutions were not derived for some of the simulations but the available results are reliable since equilibrium is achieved during each step of the simulation using implicit solvers.

The same dimensionless groups (D^* and P^* in Equations 5.1 and 5.2, respectively) were used as the ones in previous section. The constant a is associated with t by the definition of P^* . It is found that, by using $a = 2.5$, the first peak of P^* for various t align neatly with each other. It is noted that $b = 1.75$ was used based on the results in the previous section.

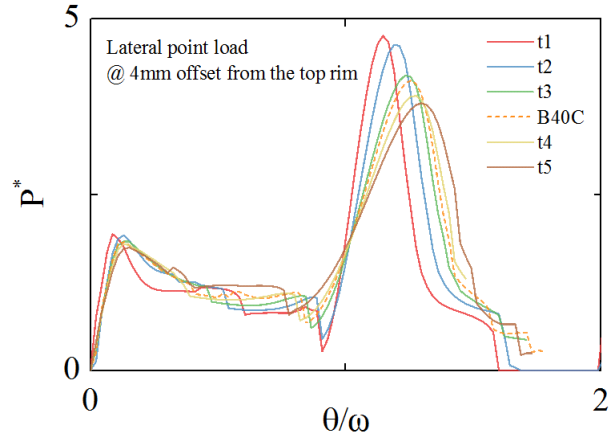
Figure 5.13 presents the results where the load application occurred close to the top rim (4 mm offset): Figure 5.13a shows the plot of P versus D_r , Figure 5.13b shows the plot of P^* versus θ/ω , and Figure 5.13c summarises the two dimensionless peak forces with respect to the changing parameter t . In Figure 5.13b, the curves are reasonably well-aligned. Two-peak response were observed in all t variants. The chosen a and b values resulted in the closer alignment of the first peak then the second, indicating compositional changes in the contribution of membrane and bending effects to the behaviour of the two peaks.

Figure 5.14 show the behaviour when the point load was applied at at 1/3 of the side surface from the top rim: Figure 5.14a shows the plot of P versus θ/ω and Figure 5.14b shows the plot of P^* versus θ/ω . With the exception of Model B40C_t5, which indicated a two-peak response, a single peak responses were observed for all the models. P_{cr}^* (excluding Model B40C_t5) are also plotted with respect to its varying

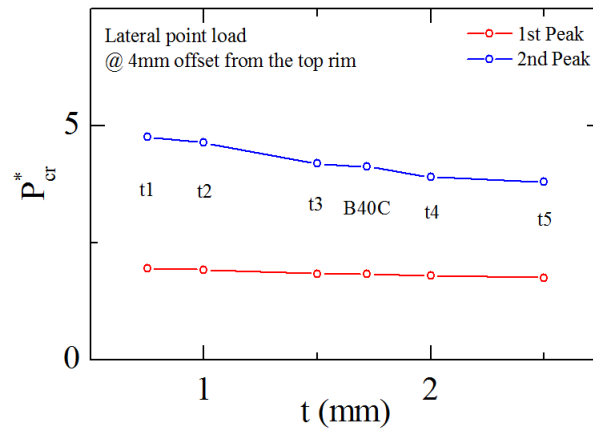
parameter t in Figure 5.14c. Fairly flat lines in both Figures 5.13c&5.14c suggest good choice of a value.



(a)

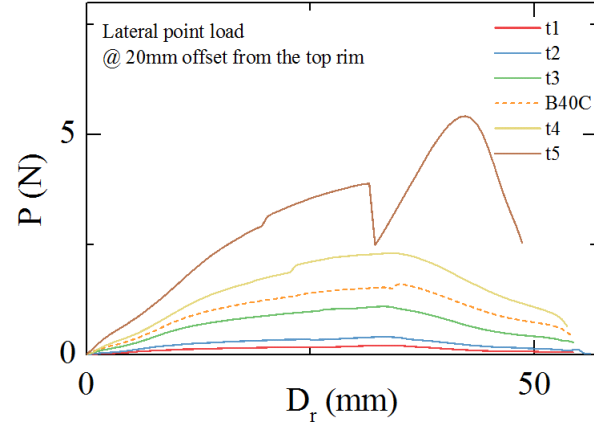


(b)

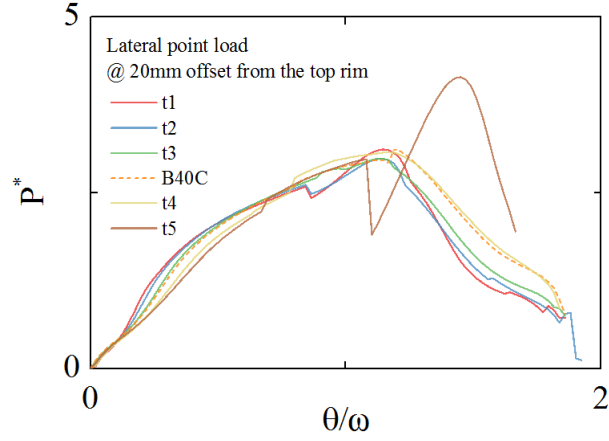


(c)

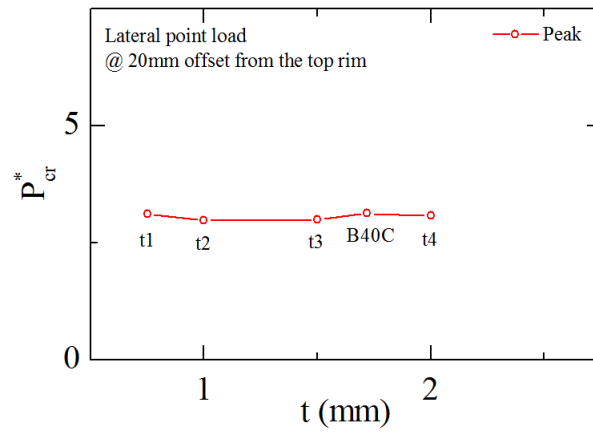
Figure 5.13: Comparison of structural response of Models B40C and its thickness variants Models B40C.t1-5 under lateral point loading with 4 mm offset from the top rim. (a) P vs. θ/ω , (b) P^* vs. θ/ω . P^* is defined in Equation 5.2 with $a = 2.5$ and $b = 1.75$.



(a)



(b)



(c)

Figure 5.14: Comparison of structural response of Models B40C and its thickness variants Models B40C_t1-5 under lateral point loading with $1/3h_s$ (20 mm) offset from the top rim. (a) P vs. θ/ω , (b) P^* vs. θ/ω . P^* is defined in Equation 5.2 with $a = 2.5$ and $b = 1.75$.

5.3 Discussion and Summary

In this chapter, we have investigated the response of conical frustum shells under lateral point loading at various locations. The frusta were configured geometrically and materially exactly as the ones in the previous chapter. Two cases - close-top and open-top frusta - were analysed and discussed. This chapter is a supplement to the previous chapter, and these two chapters as a whole provided a whole picture on how a frustum would respond under point loading. The additional findings are listed as following.

1. In this chapter, the simulated parametric studies focused on two parameters, height and thickness, were conducted for both open-top and close-top frustum shells. It was found that $a = 2.75$ & $b = 1.75$ for open-top frustum shells and $a = 2.5$ & $b = 1.75$ for close-top frustum shells give the best possible fitting on generalising the response using Equation 5.2. In comparison, the previous chapter concludes that $a = 2.25$ & $b = 1.75$ offered the best fitting for close-top under vertical point loading. However, no unified equation on predicting the critical forces P_{cr} could be given at this stage because their post-buckling behaviours were rather complicated and various modes were observed for the models.
2. Without the presence of the top surface of the frustum shells, we have seen a decrease in the critical force, despite the fact that the buckling modes were not the same.
3. We found similar responses with close-top frustum shells regardless of whether the point load was vertically applied to the edge of the top surface or laterally at the upper edge of the side surface.
4. In the previous chapter, we have found an optimal location (10.5 mm offset from the centre of the top surface) for vertical loading on the top surface, with the

lowest critical force of 2.02 N. However, we observed that for the exact same close-top frustum, the lateral application of a point load on the side surface, 20 mm offset from the top rim, resulted in a lower critical force (1.60 N). This is possibly caused by the lack of buckling of the top rim in the latter case. However, in our thickest model, Model B40C_t5 in Figure 5.14, we observed a transition from a singular peak response to a two-peak response, along with a significant increase in critical force. Therefore, further parametric studies are suggested for a comprehensive understanding of such change.

Chapter 6

Conclusions

The initial motivation behind the study presented here was to expand our limited understanding of the bistable behaviour of conical frustum shells. This thesis has bridged the gap in understanding with reference to a variety of specific factors, including material, geometric variables and loading methods. This research remains mostly computationally, performed by the FEM software, Abaqus, but also complemented by experimental evidence. A novel 3D scanning technique was employed to record deformities at various stages during experimental loading, which yielded evidence that intuitively validated the simulation models. The major findings and potential future works are summarised in the following sections.

6.1 Summary of Findings

Open-Top Frustum Shells under Vertical Uniform Loading

Chapter 3 refers to the FEM analyses on open-top frustum shells when subjected to vertical uniform loading on the top rim. A reference configuration of frustum shells was obtained from a typical flexible straw, and its simulated results were crosschecked independently by using multiple solver schemes and methods in Abaqus to verify FEM's

reliability. An investigation was carried out to evaluate the differences in responses of a singular frustum or multiple frusta in a corrugated configuration. Our results have shown that a singular frustum with pinned bottom can fairly reproduce the bistable behaviour of a flexible straw with conservative estimation of the critical force. A detailed parametric study based this boundary condition setup were carried out while focusing on the influences of key geometric parameters (h , t , and ω). We generalised the problem by proposing fitted equations that predicted the critical force of the inversion for both a LE and an EPP material, and the scope of their implementations were also stated. Moreover, we were able to distinguish a geometrical boundary of the parameter h between a one-off snap-through behaviour and a progressive inversion via a comparison between the simulated results and the predictions of Deshpande and Fleck (2003)'s model. Furthermore, we suggested how frustum shells could be optimally designed to satisfy the bistable behaviour.

Close-Top Frustum Shells under Vertical Point Loading

Chapter 4 investigated of the response of vertical point loading at various locations of the top surface on close-top frustum shells. In preparing these experiments, a specific silicone rubber material was chosen to manufacture the frustum specimens. The material properties have been axially tested and fitted with the classical polynomial model for hyperelastic material. The silicone frustum specimens were cast using bespoke moulds and the experiments were performed by measuring the reaction force and displacement at various loading locations. In addition, we 3D scanned the structural deformities that evolved during the experiments. The visual deformation history along with the measured reaction force and displacement were compared with Abaqus simulations to verify the suitability of the latter. We observed two peak forces before the structure was fully inverted to its secondary stable state and the overall critical force is defined by the maximum of the two. By applying polynomial fitting of the two peaks with

respect to their corresponding locations, an optimal loading location was proposed at 10.5 mm from the centre, which gave the lowest critical force of 2.02 N during a force-controlled loading. Furthermore, a parametric study relating h , t , and ω , were conducted to evaluate these parameters on the structural responses of the frustum. Generalised fitted formulas were proposed in predicting the critical forces, and the optimal location was proved to be applicable in a broad scope.

Frustum Shells under Lateral Point Loading

Chapter 5 presented two cases of the responses of both open-top and close-top frustum shells under lateral point loading and complemented Chapter 4 in providing a whole picture on how a frustum would respond under point loading. Experiments and Abaqus simulations were set up in a similar manner as Chapter 4. Similarly, the Abaqus results provide a good approximation of the experimental results and validate the choice of settings that were adopted in the Abaqus simulations. A parametric study relating h , t , and ω , were conducted to evaluate these parameters on the structural responses of the frustum. We observed complex responses with both open-top and close-top frustum shells and no unified equation that reliably predicted the critical forces could be given, which requires further investigation. In comparing the results of both Chapter 4 and 5, it was found that the presence of the top surface of the frustum shells yielded a lower critical force, despite the fact that the buckling modes were dissimilar; comparable responses with close-top frustum shells were obtained, regardless of whether the point load was vertically applied to the edge of the top surface or laterally at the upper edge of the side surface. We also observed that the lateral application of a point load on the side surface, 20 mm offset from the top rim, resulted in a lower critical force (1.60 N) as compared to the lowest critical force of vertical point loading on the top surface.

6.2 Future Work

In the section, a number of possibilities for future investigation concerning the bistability and multi-stability of frustum shells will be suggested.

6.2.1 Research

In this research, the parametric studies were carried out based on variations of individual parameters of a reference models. Extensive parametric studies considering variations of multiple parameters are suggested to render a more general picture of the structure's performance and to confirm speculations we made in the previous discussion.

In comparison with Chapter 4 and 5, Chapter 3 considered conical frustum shells with different scale and material properties. It is suggested that future researchers could unify the parameters so that a direct comparison could be made between the three loading scenarios which were represented by the three chapters of this thesis.

We only considered the cases of displacement-controlled quasi-static loading in this research. Since snap-through is a dynamic process in a real loading event, force-controlled loading tests and simulations are recommended in a future study.

The 3D images captured during the experiments provide an intuitive evidence in validating the Abaqus simulations. For more precise comparison between experiments and simulations in the future research, grid lines based on the mesh of Abaqus models could be draw on the specimens.

6.2.2 Applications

Sealed Bistable Actuation Device

The implementation of bistable frustum shells may be beneficial to the design of an actuation device in a hydraulic system, of which the internal modules are contained in pressurised hydraulic fluid. The main advantage of a hydraulic system is that it allows a small force to be transmitted into a much larger force which can be used to control heavy machinery. Such a system requires a high level of impermeability, and every module which interacts with the external environment can be a potential source of leakage, and lubricated o-rings have been adopted as the conventional solutions to the problem. A novel conceptual design is proposed to combine the seal function and the actuation function into a single device by introducing a frustum shell (Figure 6.1). This design utilise the bistability of frustum shells, allowing the switch to be locked at its on or off position without a constant input of load. This design could reduce the complexity of the system without sacrificing the impermeability. Research in this direction involves analysing the effects of extra internal or external pressure on the bistable behaviour of conical frustum shells.

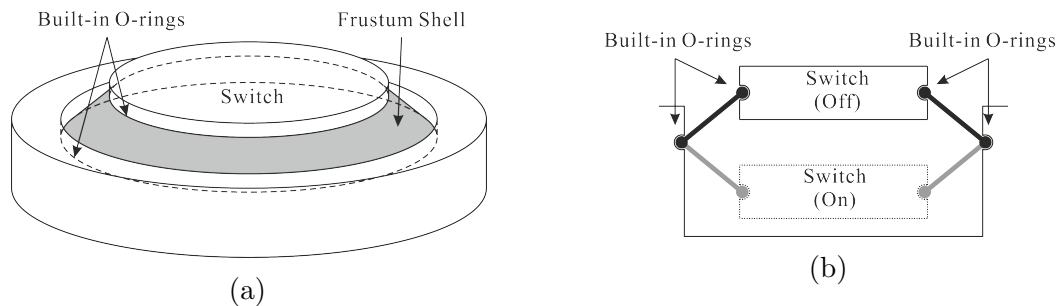


Figure 6.1: Sealed bistable switch concept design. (a) General view, and (b) cross-sectional view.

Frustum shells connected in a serial setup

In this research, we focused on the bistable response of a singular conical frustum under various loading scenarios. In addition, we briefly studied the interaction between frusta within the corrugated part of a flexible straw in Chapter 3 by observing the behaviour of two or three frusta that were serially linked to each other. This research remains preliminary since the number of connected frusta that we have simulated is limited. Possible setups include tandem pairs with the same diameter (for instance, flexible straws), or frustum shells separated by alternating cylinder shells with increasing diameters (such as a retractable funnel, shown in Figure 6.2). In order to understand its behaviour, we need to address questions relating to how the bistability of a singular frustum contributes to its global multi-stability, or whether the multi-stability can be triggered by changes in the pressure, etc. This structural form could be used in the designs requiring features such as easily deployable, self-sustainable after deployed.



Figure 6.2: A retractable funnel, with frustum and cylindrical shells that are alternately connected.

Half-inverted Frustum Shells under Rotational Loading

Although the bent state of a flexible straw was not considered in this research, a flexible straw in the latter conformation can transmit the rotation from one end to the other with relative ease. In its bent state, multiple shorter frusta within the corrugated part of a flexible straw were half-inverted as shown in Figure 1.2c. However, it has been observed that the rotational transmission became less applicable when a significant number of shorter frusta becomes half-inverted since the corrugated part becomes increasingly twisted. A rotational-transmission device that adopts this structural form has the potential to reduce the energy loss while allowing the angle between the input and output to be adjustable. Therefore, future work are suggested to firstly simulate the bent state with Abaqus, secondly derive the maximum number of half-inverted frusta with which the system is still able to deliver robust rotational transmission, thirdly evaluate the advantage of this structural form in reducing energy loss as compared to other related devices.

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Appendix A

Technical Drawings

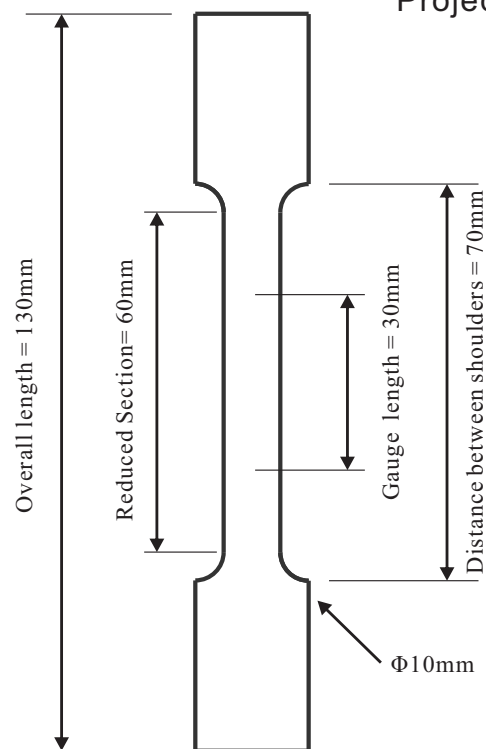
Frustum Dogbone Specimen

Material: Silicone Rubber

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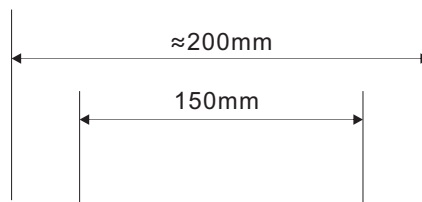
Superviosr: Dr. Zhong You
zhong.you@eng.ox.ac.uk

Project Code: DFRXRYO



Frustum Loading Plate

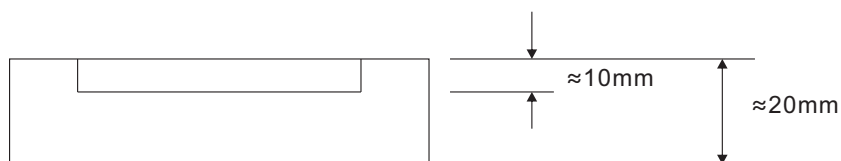
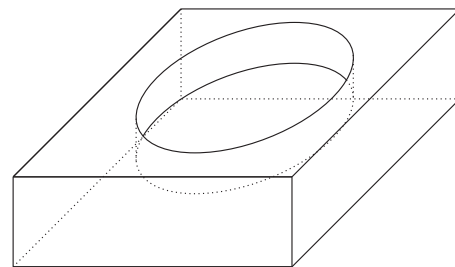
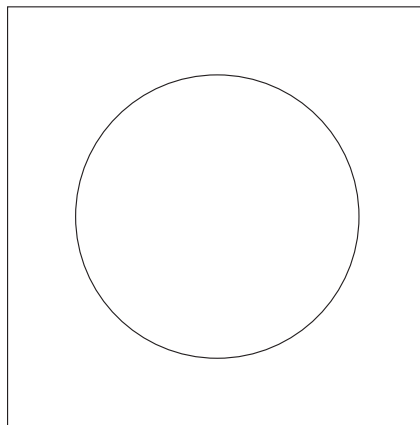
Material: Plastic Block



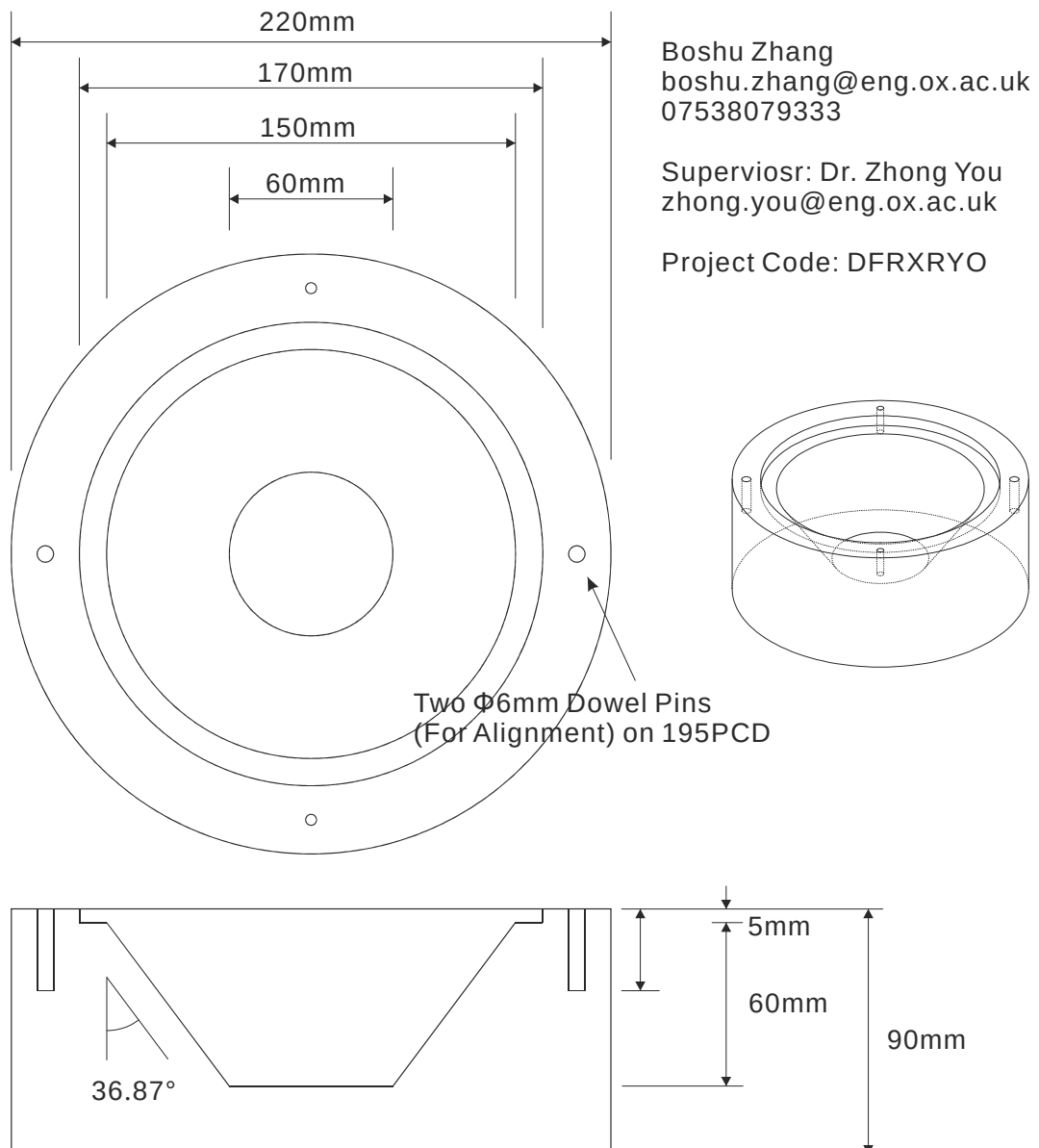
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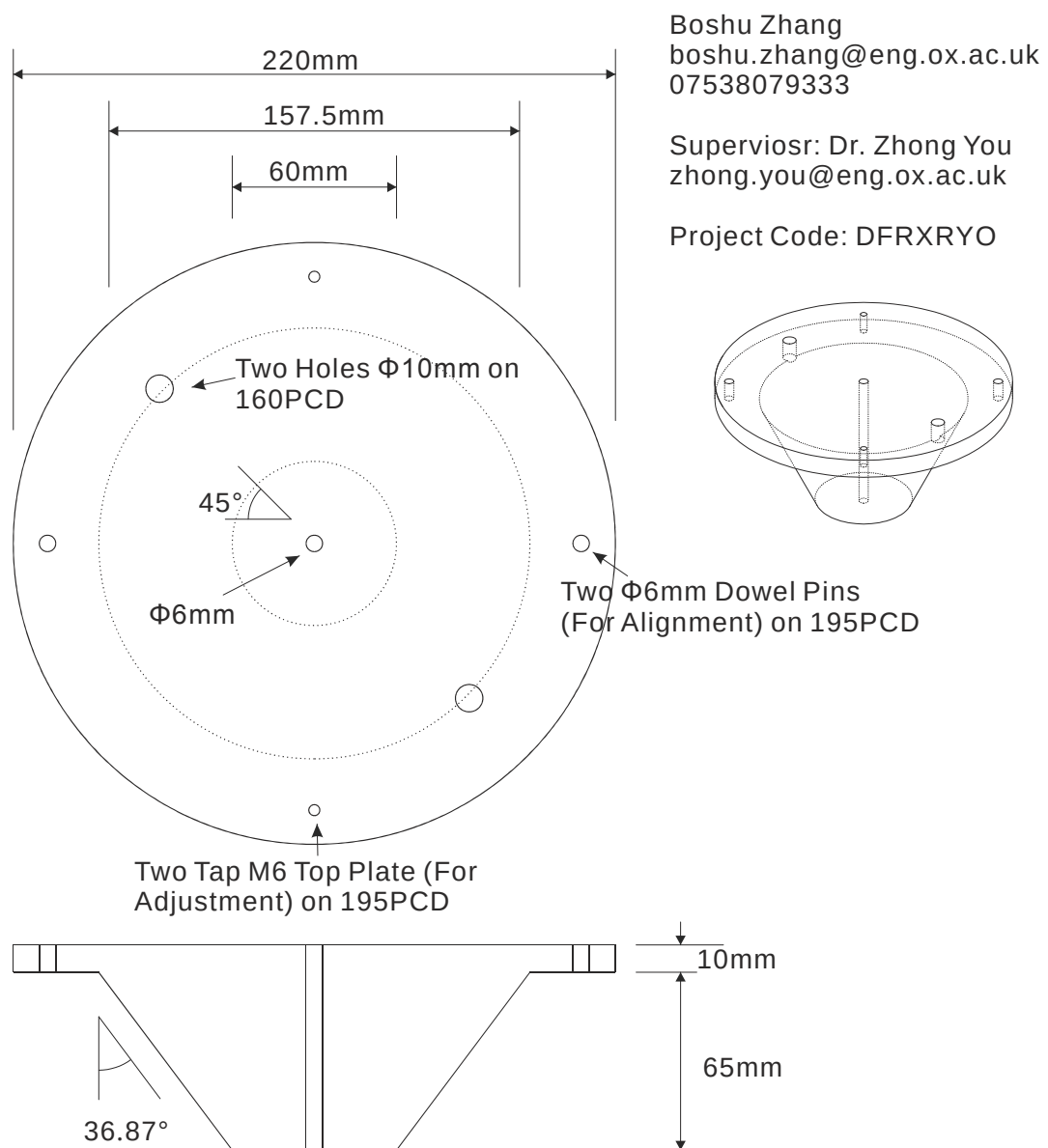
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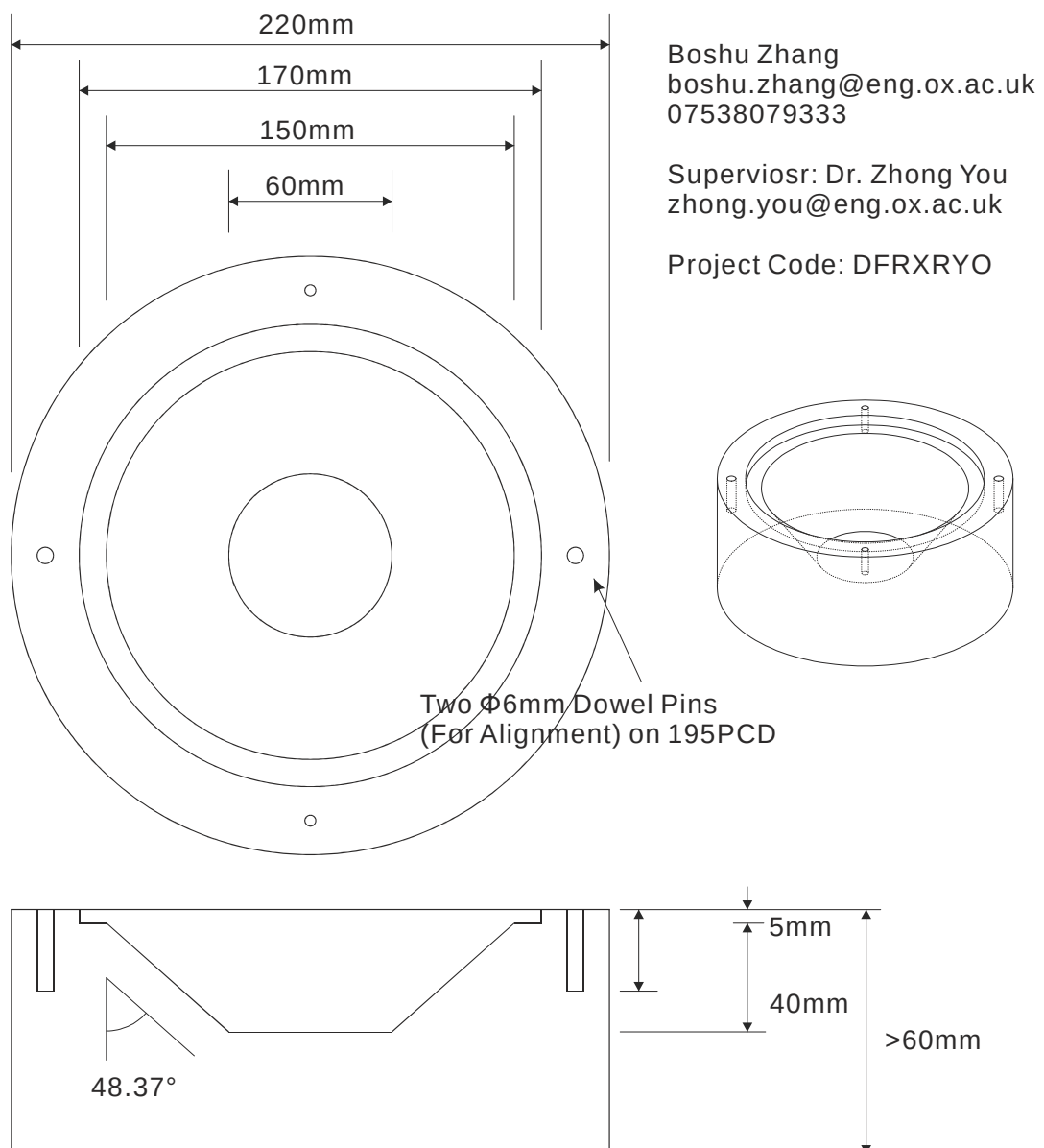
Silicone Rubber Casting Mould, Female Part Frustum Height 60mm, Material: Al Alloy



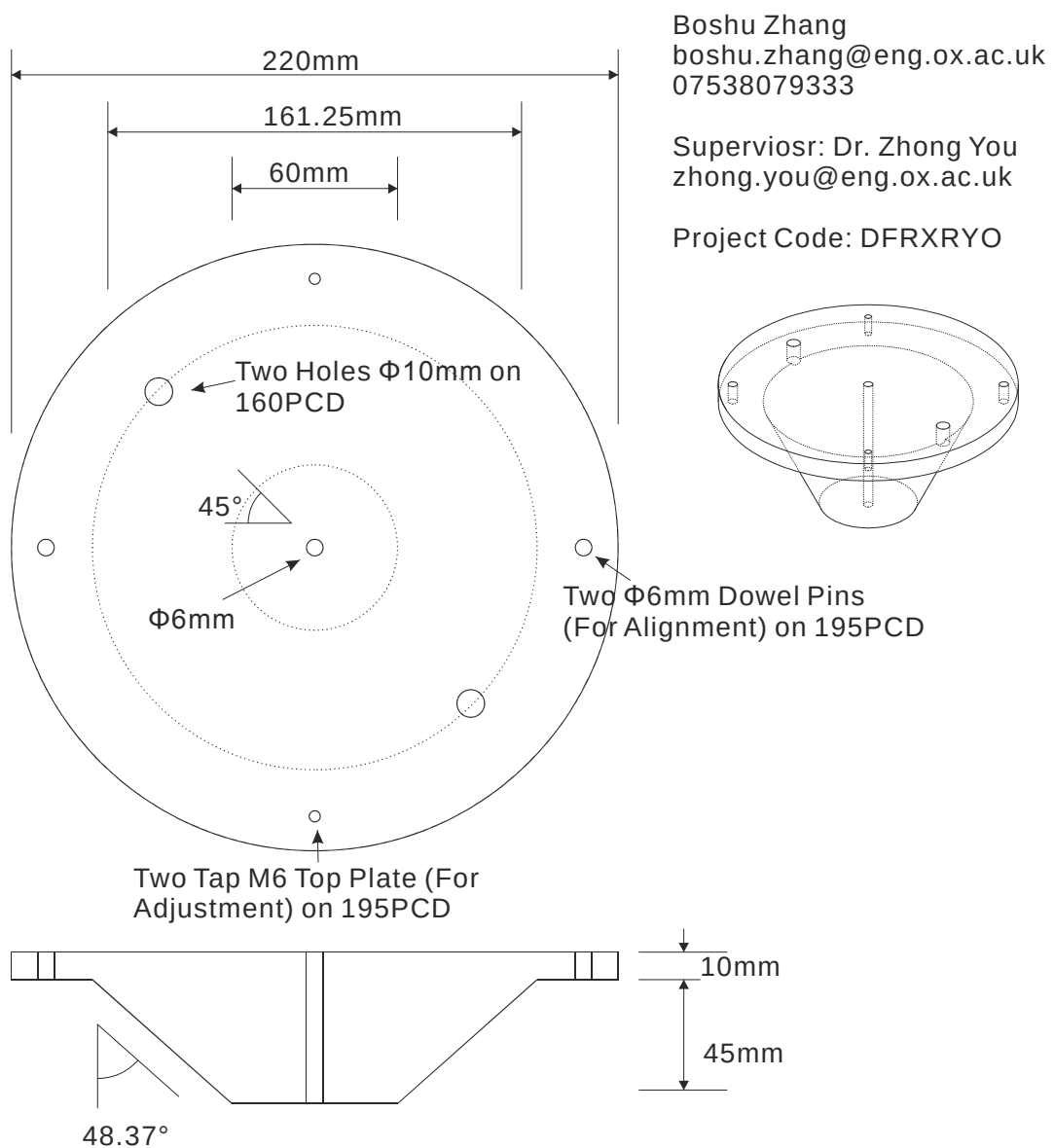
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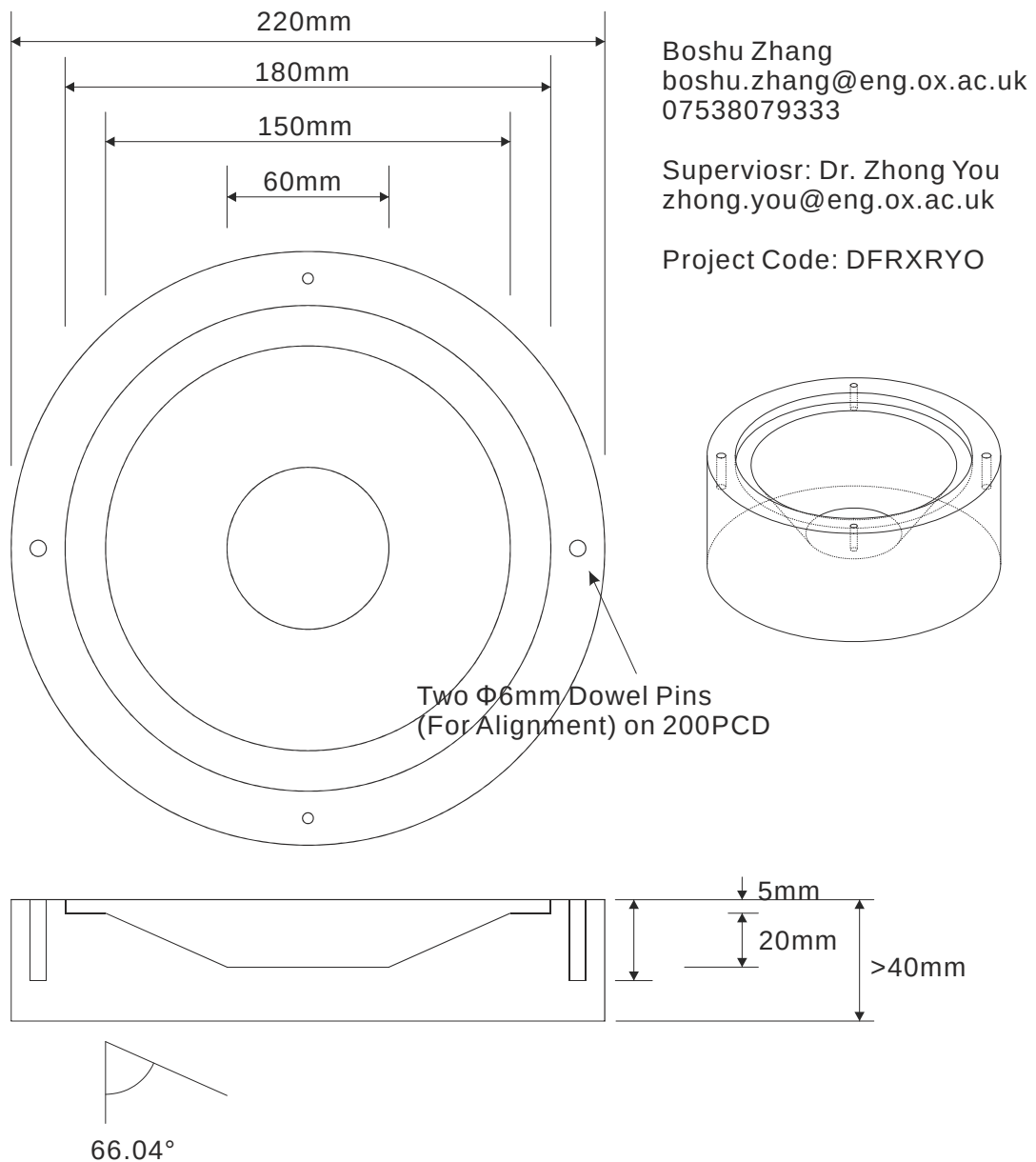
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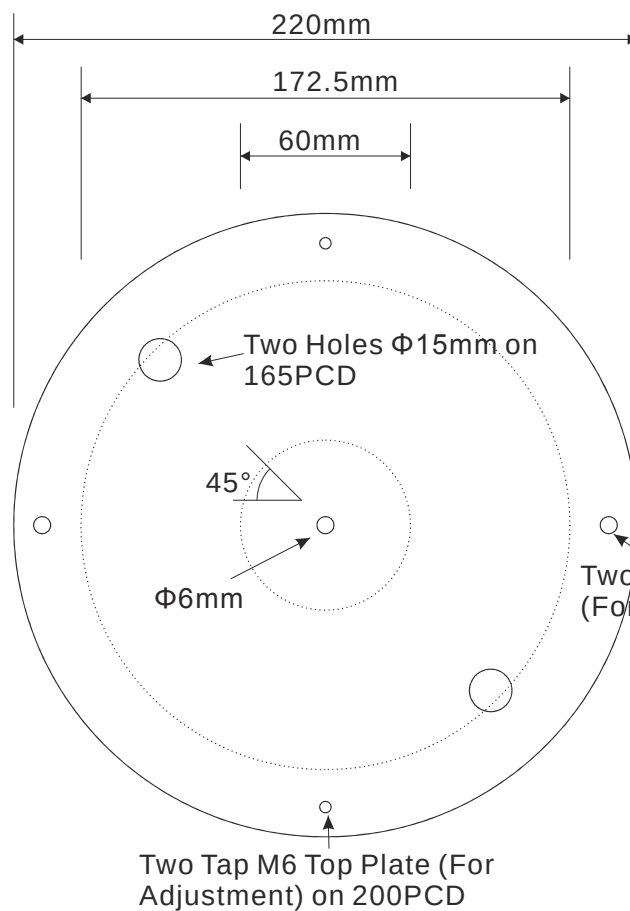
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Silicone Rubber Casting Mould, Female Part Frustum Height 20mm, Material: Al Alloy



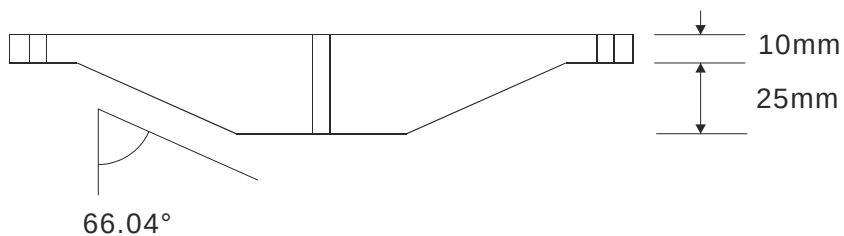
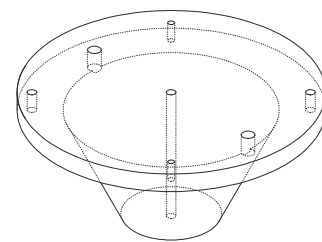
Silicone Rubber Casting Mould, Male Part Frustum Height 20mm, Material: Al Alloy



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Appendix B

Raw Data

Table B.1: Summary of Boundary Condition Investigation Findings.

Models	Bistability	Critical Force		2nd Stable State	New 1st Stable State
		Loading	Unloading		
A1E	Y	5.01 N (0.18 mm)	−3.91 N (0.69 mm)	0.85 mm	0 mm (same)
A1E_re	Y	1.25 N (0.19 mm)	−0.18 N (0.67 mm)	0.75 mm	0 mm (same)
A1P	Y	3.72 N (0.10 mm)	−4.74 N (0.86 mm)	0.95 mm	0.16 mm
A1P_re	Y	1.21 N (0.17 mm)	−1.28 N (0.75 mm)	0.89 mm	0.06 mm
A2E	N	-	-	-	-
A2E_rt	Y	1.78 N (0.19 mm)	−0.33 N (0.68 mm)	0.78 mm	-
A2P	Semi	6.80 N (0.14 mm)	−5.27 N (0.70 mm)	0.90 mm	None
A2P_rt	Y	1.45 N (0.14 mm)	−1.56 N (0.80 mm)	0.95 mm	0.18 mm
A3E	N	-	-	-	-
A3E_rt	Y	2.33 N (0.19 mm)	−0.69 N (0.67 mm)	0.79 mm	0 mm (same)
A3P	Semi	12.04 N (0.13 mm)	−8.00 N (0.55 mm)	0.77 mm	None
A3P_rt	Y	1.72 N (0.12 mm)	−2.17 N (0.96 mm)	0.98 mm	0.38 mm

Table B.2: Silicone Rubber Dogbone Specimen Measurements

Specimens	Width (mm)				Thickness (mm)			
	1	2	3	Average	1	2	3	Average
1a	9.798	9.984	9.845	9.88	3.386	3.35	3.505	3.414
1b	9.042	8.699	8.629	8.79	3.1	3.323	3.313	3.245
2a	9.789	9.998	9.968	9.92	3.128	3.077	3.098	3.101
2b	9.747	9.827	9.891	9.82	3.259	3.125	3.162	3.182
3a*	9.583	9.519	9.671	9.59	2.522	2.574	2.624	2.573
3b*	9.835	9.253	9.082	9.39	2.715	2.703	2.755	2.724
4a	9.924	10.253	10.296	10.16	2.976	2.956	2.974	2.969
4b	10.229	9.661	9.507	9.80	2.926	2.972	2.921	2.940
5a*	10.283	10.255	10.149	10.23	2.619	2.586	2.764	2.656
5b*	10.014	10.181	10.279	10.16	2.571	2.565	2.527	2.554

*Noted that Specimens 3a&b and 5a&b contained a large number of bubbles.

