Résumé.
Abstract. We give a strengthening of the description of an n-permutable category due to Carboni, Kelly and Pedicchio. This provides us with a new characterisation of the regular Mal'tsev categories from among those which are Goursat, or more generally n-permutable.
Keywords. n-permutable category, Mal'tsev category, Goursat category
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Introduction

Mal'tsev categories form an important and well-known class of categories in the study of universal algebra [2, 7]. In fact, many of their interesting properties extend to the broader class of Goursat categories [3, 5]. While most examples of Goursat categories are in fact Mal'tsev, there are as yet no simple characterisations of when this is the case. In [1], Carboni, Kelly and Pedicchio showed that both classes belong to the more general hierarchy of n-permutable categories. In this note, we give a strengthening of their original characterisation of n-permutable categories, providing us with a new condition for when a Goursat category, or more generally an n-permutable one, is Mal'tsev. The condition is a mild one, satisfied even in logically well-structured categories such as the category $\mathbf{Set}$ of sets and functions.

Throughout we work in a regular category $\mathbf{C}$, using the calculus of elements to argue as in the category $\mathbf{Set}$ of sets and functions. For objects A, B in $\mathbf{C}$, recall that a *relation* $R\colon A \to B$ is a subobject $R \rightarrowtail A \times B$. We write $R^\circ\colon B \to A$ for the converse of a relation $R\colon A \to B$. Any two relations $R\colon A \to B$ and $S\colon B \to C$ have a composite $SR\colon A \to C$ defined by

$$SR = \{(a, c) \in A \times C \mid \exists b\, R(a, b) \wedge S(b, c)\}$$

Relations are partially ordered by the usual inclusion of subobjects, and we call an endorelation $E\colon A \to A$ *reflexive* when $\mathrm{id}_A \leq E$, *symmetric* when $E = E^\circ$, *transitive* when $EE \leq E$, and an *equivalence relation* when all of these hold. Following [1], for any pair of relations $R\colon A \to B$, $S\colon B \to C$, we define a sequence of relations

$$(R,S)_1 = R,\ (R,S)_2 = RS,\ (R,S)_3 = RSR,\ (R,S)_4 = RSRS,\ \ldots$$

1. A Condition for n-Permutability

We begin by strengthening a characterisation from [1].

Theorem 1. *For a regular category* $\mathbf{C}$ *and* $n \geq 2$*, the following are equivalent:*

1. $\mathbf{C}$ *is* n-permutable*:* $(E,E')_n = (E',E)_n$ *whenever* E *and* E' *are equivalence relations on the same object;*
2. *Every relation* $R\colon A \to B$ *in* $\mathbf{C}$ *satisfies* $(R,R^\circ)_{n+1} = (R,R^\circ)_{n-1}$
3. *For every reflexive relation* $E\colon A \to A$ *in* $\mathbf{C}$*,* $(E,E^\circ)_{n-1}$ *is an equivalence relation;*
4. *For every reflexive and symmetric relation* $E\colon A \to A$ *in* $\mathbf{C}$*,* E^{n-1} *is an equivalence relation.*

Proof. See [1] for the equivalence of (1), (2) and (3). Trivially (3) $\implies$ (4). We now show that (4) $\implies$ (2). First note that by our assumption any reflexive relation E satisfies $E^n = E^{n-1}$. Indeed, always $E^{n-1} \leq E^n$, while we now have $E^n \leq E^{2(n-1)} \leq E^{n-1}$. Now for any relation $R \rightarrowtail A \times B$, we define a new relation $S \rightarrowtail R \times R$ by

$$(a,b)S(a',b') \iff aRb' \wedge a'Rb$$

noting that by definition we also have aRb and $a'Rb'$. Then S is reflexive since it is defined on R, and symmetric by definition. Hence $S^n = S^{n-1}$.

Now in fact we always have $(R^\circ,R)_{n-1} \leq (R^\circ,R)_{n+1}$, so it suffices to show the converse holds. Suppose that $b(R^\circ,R)_{n+1}a$, via a sequence of elements $(x_i)_{i=0}^{n+1}$ with $x_0 = a$, $x_{n+1} = b$ satisfying x_iRx_{i-1} and x_iRx_{i+1} for all even $i \leq n$. Then we have $(x_i, x_{i+1})S(x_{i+2}, x_{i+1})$ and $(x_{i+2}, x_{i+1})S(x_{i+2}, x_{i+3})$, for all even $i \leq n-2$. So defining $(y, z) := (b, x_n)$ if n is odd, or $(y, z) := (x_n, b)$ if n is even, we have that $(a, x_1)S^n(y, z)$.

Hence $(a, x_1)S^{n-1}(y, z)$ also. Letting $(y_0, z_0) = (a, x_1)$ and $(y_{n-1}, z_{n-1}) = (y, z)$, this means there is a sequence of pairs $(y_i, z_i)_{i=1}^{n-1} \in R$ satisfying $(y_i, z_i)S(y_{i+1}, z_{i+1})$ for all $0 \leq i \leq n-2$. In particular, for all even such i we have $y_i R z_{i-1}$ and $y_i R z_{i+1}$, and so via the sequence $a = y_0, z_1, y_2, z_3, \dots$ of length $n+1$ ending in b, we have $b(R^\circ, R)_{n-1}a$, as desired. $\square$

2. Positively Regular and Mal'tsev Categories

We now turn to classifying the Mal'tsev categories as the n-permutable categories with a special property. Let us call a relation $E\colon A \to A$ *positive* when it is of the form $E = R^\circ R$ for some relation $R\colon A \to B$. The following notion first appeared in [4].

Proposition 2. *For a regular category* $\mathbf{C}$*, the following are equivalent:*

1. *A relation* $E\colon A \to A$ *in* $\mathbf{C}$ *is positive iff it satisfies:*

$$E(a, b) \implies E(a, a) \wedge E(b, a) \qquad (*)$$

2. *Any reflexive, symmetric relation in* $\mathbf{C}$ *is positive.*

We call such a regular category positively regular[1].

Proof. For (1) $\implies$ (2), and the 'only if' in (2) $\implies$ (1), note that any reflexive, symmetric relation in a regular category automatically satisfies (*), as does any positive relation. Conversely, if (2) holds and $E\colon A \to A$ satisfies (*), define

$$I = \{a \in A \mid \exists b\, E(a, b)\} \rightarrowtail A$$

writing $i\colon I \to A$ for the obvious inclusion. Then $i^\circ E i$ is a reflexive, symmetric relation on I, and hence is positive, say equal to $T^\circ T$. Then we have $E = ii^\circ R i i^\circ = iT^\circ T i^\circ = (Ti^\circ)^\circ(Ti^\circ)$ and so E is positive. $\square$

[1]Not to be confused with the notion of a positive coherent category [6].

Example 3. **Set** is positively regular. More generally so is any regular coherent category, coming with unions of subobjects. To see this, for any relation $E \rightarrowtail A \times A$ satisfying (*), define

$$R = \{(a, (a, b)) \mid R(a, b)\} \vee \{(a, (b, a)) \mid R(a, b)\} \rightarrowtail A \times E$$

Then $E = R^\circ R$, making E positive.

Recall that a regular category is *Mal'tsev* when it is 2-permutable, and *Goursat* when it is 3-permutable. While condition (2) below is well-known, we believe the others to be new.

Corollary 4. *For a regular category* $\mathbf{C}$*, the following are equivalent:*

1. $\mathbf{C}$ *is Mal'tsev;*
2. *Every reflexive relation in* $\mathbf{C}$ *is an equivalence relation;*
3. *Every reflexive and symmetric relation in* $\mathbf{C}$ *is an equivalence relation;*
4. $\mathbf{C}$ *is Goursat and positively regular;*
5. $\mathbf{C}$ *is* n*-permutable, for some* $n \geq 2$*, and positively regular.*

Proof. The equivalence of (1), (2) and (3) is in Theorem 1, and clearly (4) implies (5). For (3) $\implies$ (4), note that any reflexive, symmetric relation E in $\mathbf{C}$ is an equivalence relation, and hence in particular positive, since $E = E^\circ E$. Hence by Proposition 2, $\mathbf{C}$ is positively regular. Further, since $\mathbf{C}$ is Mal'tsev, it is in particular Goursat.

It remains to show that (5) $\implies$ (1). Suppose that $\mathbf{C}$ is positively regular. First suppose $\mathbf{C}$ is $(2m + 1)$-permutable, for some $m \geq 1$. Let $E\colon A \to A$ be a reflexive and symmetric relation. Then by positive regularity, $E = R^\circ R$ for some relation $R\colon A \to B$, and so:

$$E^{2m} = (R^\circ, R)_{4m} = (R^\circ, R)_{(2m+2)+2(m-1)} = (R^\circ, R)_{2m} = E^m$$

where we repeatedly applied $(R^\circ, R)_{2m+2} = (R^\circ, R)_{2m}$. Hence, by condition (4) of Theorem 1, $\mathbf{C}$ is in fact $(m + 1)$-permutable.

Now if $\mathbf{C}$ is n-permutable, there is some k with $n \leq 2^k + 1$ so that $\mathbf{C}$ is $(2^k + 1)$-permutable. Then the above argument shows that $\mathbf{C}$ is in fact $2^{k-1} + 1$ permutable, and hence inductively that $\mathbf{C}$ is 2-permutable, *i.e.* Mal'tsev. $\square$

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