Globally Valued Fields

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Contents

0.1 Outline

The material of this thesis is composed out of three articles [9, 71, 28] each constituting a separate chapter. At the beginning of each chapter the author's contribution and the status of the corresponding paper is described. In this outline we describe how these parts fit together and what are the main contributions of this thesis into the theory of *globally valued fields*. For more detailed descriptions of results in respective chapters we refer the reader to introductions therein.

Definition 0.1.1. Let F be a field. A *global height* on F is a collection of functions $h : \mathbb{P}^n(F) \to \mathbb{R}_{\geq 0}$, for every natural n, satisfying the following axioms.

Height of one:		$h(1:1) = 0$
Invariance:	$\forall x \in \mathbb{P}^n(F)$, $\forall \sigma \in \mathrm{Sym}_{n+1}$,	$h(\sigma x) = h(x)$
Additivity:	$\forall x \in \mathbb{P}^n(F)$, $\forall y \in \mathbb{P}^m(F)$,	$h(x \otimes y) = h(x) + h(y)$
Monotonicity:	$\forall x \in \mathbb{P}^n(F)$, $\forall y \in \mathbb{P}^m(F)$,	$h(x) \leq h(x : y)$
Triangle inequality:	$\forall x, y \in F^{n+1}, x + y \neq 0$,	$h(x + y) \leq h(x : y) + e$

for some real number $e \geq 0$ which is called the Archimedean error, and $(x_i)_i \otimes (y_j)_j = (x_i y_j)_{i,j}$ is the Segre embedding. A *globally valued field* is a field equipped with a global height.

These structures were originally defined in a different (equivalent) way by Ben Yaacov and Hrushovski in [10]. They form a first-order class GVF_e in unbounded continuous logic from [5]. In any GVF, the Archimedean error can be taken to be any number bigger or equal $h(2:1)$. Thus, when talking about a fixed GVF, we will typically not mention an Archimedean error. The reason why e appears in the definition is that if one puts $h(2:1)$ instead in the above triangle inequality, the obtained class is not anymore first-order. The main examples of globally valued fields are *global fields*, i.e., number fields and function fields of irreducible curves over finite fields. In this case, the heights can be defined using the product formula. The fundamental motivating question is: what structural properties explain the similarities between number fields and function-field type global fields?

When faced with fields equipped with an extra structure (for example empty structure, derivation or an automorphism), in mathematical practice we usually build 'geometry' dual to the algebra imposed by the extras. For example algebraic geometry is dual to commutative algebra. Or, Kolchin varieties are dual to differential algebra. Model theory provides a toolkit in analysing classes of enriched fields and the associated geometries. In the above mentioned cases it leads to the understanding of universal domains (through the notion of existential closedness) and to questions about classification of certain special varieties/equations (stably embedded sets, strongly minimal equations, modular forking geometry). For example, in the case of fields with an automorphism, one characterises existentially closed models as a first order class ACFA and there is a rich geometric theory on difference varieties in ACFA developed in [21]. Moreover, in this case, there are classical structures that are models of ACFA, namely the ultraproducts of algebraic closures of finite fields with Frobenius automorphisms [44].

The main motivation of this thesis was to understand a piece of this model-theoretic toolkit for globally valued fields, namely identifying 'classical' existentially closed models, and studying the question of axiomatizability of all of them. This first part of this goal was achieved through Theorem 0.1.3, however, the existence of the model companion for globally valued fields is still conjectural. Moreover, in this work we connect various 'geometries' to the algebra of fields with heights. For example we use Arakelov geometry to prove Theorem 0.1.3 and the theory of adelic curves [24] to prove Theorem 0.1.5. Let us now go more thoroughly through the core statements proved in this thesis.

The main result of Chapter 1 is the following theorem (joint with Ben Yaacov, Destic and Hrushovski) giving many equivalent definitions of globally valued fields, including one given in terms of a (measure theoretic) product formula.

Theorem 0.1.2. [9] There is a bijective correspondence between the following structures on a field F:

1. Local terms satisfying the product formula,
2. Global heights,
3. Global functionals,
4. Global families of local measures,
5. Renormalisation classes of global admissible measures,
6. Equivalence classes of global lattice valuations.

For a description of all these equivalent structures, see Introduction 1.1. The significance of this theorem is that it unifies various approaches in studying global structures on fields, e.g. [10, 24, 36], as well as allows tools from various areas of mathematics to be used in their study [1, 18, 27, 60] (see Table 1.1). Indeed, in Chapter 2 we use Arakelov geometry to prove the main result of this thesis, resolving a conjecture of Hrushovski from [43, Conjecture 4.1].

Theorem 0.1.3. [71] The field $\overline{\mathbb{Q}}$ equipped with the Weil projective height is an existentially closed globally valued field.

This means that if X is a variety over $\overline{\mathbb{Q}}$ and the function field of X is equipped with a global height h satisfying $h(2:1) = \log 2$ then there is a generic sequence of points $x_n \in X(\overline{\mathbb{Q}})$ such that for any tuple of rational functions $f_0, \dots, f_k$ on X, we have

$$\lim_n \mathrm{ht}(f_0(x_n) : \dots : f_k(x_n)) = h(f_0 : \dots : f_k),$$

where on the left hand side ht is the Weil projective height on $\overline{\mathbb{Q}}$. This result is an arithmetic analogue of the main result of [10] where Ben Yaacov and Hrushovski show that the algebraic closure of $\mathbb{C}(t)$ is existentially closed as a globally valued field (in that case there is a unique GVF structure that is zero on $\mathbb{C}$ and satisfies $h(t:1) = 1$). For some applications of Theorem 0.1.3, see Introduction 2.1.

Let us note that even though the paper [71] was written before [9], in this thesis we incorporate Theorem 0.1.2 to simplify a part of the proof of Theorem 0.1.3. Another place where Theorem 0.1.2 is important, is Chapter 3. There, we use the comparison between globally valued fields and adelic curves [24] to translate the adelic intersection theory developed by Chen and Moriwaki [25] to the GVF context. In elementary terms, it means that if X is a variety over a field F equipped with a GVF structure, and we are given embeddings $j_i : X \to \mathbb{P}^{n_i}$ defined over F (for $i = 0, \dots, d = \dim X$), then we can associate to this data a real number $\widehat{\deg}(j_0, \dots, j_d)$ which measures the arithmetic complexity of the embeddings, relative to the GVF structure on F. The adelic intersection product $\widehat{\deg}(j_0, \dots, j_d)$ can be defined as follows. There is a unique (up to a multiplicative scalar from F) polynomial $R(\lambda_0, \dots, \lambda_d)$ of multidegree $(\delta_0, \dots, \delta_d)$ where δ_j is the intersection number $\deg\left(\prod_{i\neq j} c_1(j_i^*\mathcal{O}(1))\right)$, such that

$$R(\lambda_0, \dots, \lambda_d) = 0 \iff (\exists x \in X)(\forall i = 0, \dots, d)(j_i(x) \subset \{\lambda_i = 0\}),$$

for λ_i being a tuple of coordinates of a linear form on $\mathbb{P}^{n_i}$. Call R the resultant of the family $j_0, \dots, j_d$. Let us denote by R_n the resultant of the family of embeddings $j_0', \dots, j_d'$, where we replace each j_i by the post-composition with the n'th Veronese map on the corresponding projective space. Then

$$\widehat{\deg}(j_0, \dots, j_d) := \lim_n \frac{1}{n^{d+1}} h(R_n),$$

where by $h(R_n)$ we mean the global height of the tuple of the coefficients of R_n. For a more detailed definition of the adelic intersection product $\widehat{\deg}(j_0, \dots, j_d)$, see Section 3.3 and [25]. To state the next result we need one more notation.

Definition 0.1.4. Let S be a variety over a globally valued field F. We denote by S_{GVF} the space of quantifier-free types extending S over F. In other words, as a set

$$S_{\mathrm{GVF}} = \{(x, h_x) : x \in S \text{ and } h_x \text{ is a global height on } \kappa(x) \text{ extending the one on } F\},$$

where by $\kappa(x)$ we denote the residue field of x over F, and x varies over scheme-theoretic points of S. The topology is defined as the weakest for which Zariski open sets $U \subset S$ induce open subsets $U_{\mathrm{GVF}} \subset S_{\mathrm{GVF}}$ and for which a morphism $f : U \to \mathbb{P}^n$ induces a continuous real-valued function $(x, h_x) \mapsto h_x(f(x))$ on U_{GVF}. With this topology S_{GVF} becomes a locally compact, Hausdorff topological space.

One of the main results of Chapter 3 (joint with Pablo Destic and Nuno Hultberg) is the following.

Theorem 0.1.5. [28] Let $\mathcal{X} \to S$ be a flat projective morphism of finite type schemes over a globally valued field F, of relative dimension d. Let $j_i : \mathcal{X} \to \mathbb{P}^{n_i}_S$ be closed embeddings over S, for $i = 0, \dots, d$. For $s = (x, h_x) \in S_{\mathrm{GVF}}$, let us write $\widehat{\deg}(j_0, \dots, j_d | \mathcal{X}_s)$ for the adelic intersection product of pullbacks of j_i's with respect to the morphism $x \to S$ and the global height h_x on $\kappa(x)$. Then, the map

$$\begin{aligned} S_{\mathrm{GVF}} &\to \mathbb{R} \\ s &\mapsto \widehat{\deg}(j_0, \dots, j_d | \mathcal{X}_s) \end{aligned}$$

is continuous. In other words, the adelic intersection product is quantifier-free definable in families.

This can probably be deduced from the existence and regularity of the Deligne pairing, however, we provide a proof using resultants and [25] directly. As an application, we prove a conjecture of Gualdi [32] about heights of complete intersections in toric varieties. The precise statement is below.

Theorem 0.1.6. [28] Let $f_1, \dots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K and let T be a proper toric variety with torus $\mathbb{T} = \mathbb{G}_m^n \subset T$. Denote by V_i the hypersurface defined by f_i and by ρ_i its Ronkin function. Let $(\zeta_{1,j}, \dots, \zeta_{m,j})_j$ be a generic sequence of tuples of roots of unity in $\mathbb{T}^m$ and let $\overline{D}_0, \dots, \overline{D}_{n-m}$ be semipositive toric adelic divisors on T with associated local roof functions $\theta_{0,v}, \dots, \theta_{n-m,v}$. Then,

$$\begin{aligned} &\lim_{j \to \infty} \widehat{\deg}(\overline{D}_0, \dots, \overline{D}_{n-m} \mid \zeta_{1,j} V_1 \cap \dots \cap \zeta_{m,j} V_m) \\ &= \sum_{v \in M_K} n_v \, \mathrm{MI}_M(\theta_{0,v}, \dots, \theta_{n-m,v}, \rho_1^\vee, \dots, \rho_m^\vee). \end{aligned}$$

In particular, if $V_1, \dots, V_n$ are hypersurfaces in $T = \mathbb{P}^n$ over K and $(\zeta_{1,j}, \dots, \zeta_{n,j})_j$ is a generic sequence of tuples of roots of unity in $\mathbb{T}^n$, then this gives a formula for

$$\lim_{j \to \infty} \mathrm{ht}(\zeta_{1,j} V_1 \cap \cdots \cap \zeta_{n,j} V_n)$$

in terms of mixed integrals of Ronkin functions associated to Laurent polynomials defining the V_i's. Thus the computation of the above arithmetic intersection numbers is reduced to a (global) convex geometric problem.

Summarising, in this thesis, we make the following contributions to the theory of globally valued fields:

- Theorem 0.1.2 explains *what* globally valued fields really are, building a dictionary between various definitions.
- Theorem 0.1.3 shows that studying universal problems over all globally valued fields is equivalent to studying them over $\overline{\mathbb{Q}}$ uniformly in parameters. In other words, it shows that $\overline{\mathbb{Q}}$ is a kind of *universal* GVF, in the sense that every GVF extending $\mathbb{Q}$ embeds into an ultrapower of $\overline{\mathbb{Q}}$.
- Theorem 0.1.5 says that heights of cycles are quantifier-free expressible in the GVF language. As a corollary, this leads to a better understanding of *equidistribution* and cycles defined over an equidistributing sequence.
- Theorem 0.1.6 is an instance of the previous bullet, where the *Bilu equidistribution* is understood as a convergence in $\mathbb{P}^1_{\mathrm{GVF}}$.

To use tools from model theory in order to study globally valued fields it would be desirable to characterise all existentially closed models. Theorem 0.1.5 was originally motivated by the search of geometric axioms for the (conjecturally existing) model companion of GVF. However, even without the model companion, the use of geometric stability methods is possible in this landscape. For example, Ben Yaacov and Hrushovski show [11] that the theory of globally valued fields is quantifier-free stable. Another exciting direction of development of GVF would be the study of adeles and adelic lattices over a field with a global height. This is related to the study of adelic vector spaces e.g. in [13, 24] and is a part of the current research program of the author.

Chapter 1

Foundations

This chapter is based on the paper [9] joint with Itaï Ben Yaacov, Pablo Destic and Ehud Hrushovski. The author's most notable contributions to this project consist of establishing the connections with 'Riemann-Zariski-Berkovich' spaces, fixing some proofs from [11], an abstract characterisation of the lattice of adelic divisors, and an introduction to the 'baby unbounded continuous logic' at the end. The paper contains some references to the author's work [71], which are changed here to references to Chapter 2 (mostly in Section 1.8). The paper has been submitted to a journal.

1.1 Introduction

A global field K, i.e., either a number field, or a finite extension of the field $k(t)$ (for some finite base-field k), satisfies a *product formula*, which means that there is a natural choice of absolute values $(|\cdot|_i)_{i\in I}$ on K such that

$$\prod_{i\in I} |a|_i = 1 \text{ for all } a \in K^\times. \tag{1.1}$$

In fact, if one of the absolute values has either discrete value group with finite residue field or is Archimedean, the product formula characterises this class of fields, see the Artin-Whaples theorem [2, Theorem 3]. Geometrically, the product formula can be interpreted as the fact that every 'function' $a \in K^\times$ has the same number of zeros and poles, counted with multiplicities. Indeed, by taking $-\log$ of the previous formula, we get

$$\sum_{i\in I} -\log |a|_i = 0, \tag{1.2}$$

where we can think of $-\log |a|_i$ as of the order of vanishing of a at $i \in I$. This point of view and the analogy between function fields and number fields is very useful, see [39, Chapter I.6], and [79, Chapter IV].

However, despite Artin-Whaples, there are classical fields with valuations satisfying 'product formulas', that do not come from number fields or function fields of curves over finite fields. First, note that function fields of curves over any field give such examples, as if C is a smooth projective curve over any field, then non-zero rational functions on C have the same number of zeroes and poles (counted with multiplicities). Even more generally, if X is a one-dimensional proper regular Noetherian scheme, it may happen that rational functions on X satisfy the product formula with respect to orders of vanishing at closed points of X. A spectacular example of this behavior is when X is the Fargues-Fontaine curve, see [54]. Also, in the world of analytic functions, Jensen's formula

$$\sum_{0<|a|<r} \operatorname{ord}_a(f) \log \frac{r}{|a|} + \frac{1}{2\pi} \int_0^{2\pi} -\log |f(re^{i\theta})| d\theta = -\log |f(0)| \tag{1.3}$$

can be thought of as a product formula, with an error $-\log |f(0)| = O_r(1)$. This observation is a part of the Vojta's dictionary between diophantine approximation and value distribution theory, see [76]. Moreover, by using the above Jensen's formula for $r = 1$ together with p-adic Jensen's formulas [48, Remark 2.8], one can deduce that for $f \in \mathbb{Q}(z)^\times$ we have

$$\sum_p -\log \|f\|_p + \frac{1}{2\pi} \int_0^{2\pi} -\log |f(e^{i\theta})| d\theta + \sum_{x \in \mathbb{P}^1_{\mathbb{Q}}} \operatorname{ord}_x(f) \cdot \operatorname{ht}(x) = 0, \tag{1.4}$$

where $\|f\|_p$ is the p-adic Gauss norm of f (i.e., the supremum of the p-adic norm of f on the p-adic unit disc, if $f \in \mathbb{Q}[z]$) and for $x \in \mathbb{P}^1_{\mathbb{Q}}$ the number $\operatorname{ht}(x)$ is the Weil height of x.

Equation (1.4) is a natural example of a product formula with a continuous measure (at least over Archimedean absolute values), in contrary to the counting measure case in Equation (1.2). Moreover, even if we restrict to number fields, while each has a discrete space of places, when we pass to $\overline{\mathbb{Q}}$, the space becomes locally compact but not discrete. This suggests that a more general notion of 'global fields' should include not only counting measures on the space of absolute values on a field. There are a few candidates of such notion in the literature. First, Gubler in [36] defined an *M-field*, which is a field K together with a measure space (M, μ) and μ-almost everywhere defined maps

$$(v \in M) \mapsto (|x|_v \in \mathbb{R}_{\geq 0})$$

for $x \in K$, such that μ-almost everywhere $|\cdot|_v$ is an absolute value, and $\log|z|_v \in L^1(M,\mu)$ for $z \in K^\times$. This formulation allows errors like in Equation (1.3), and if we want the product formula to hold, we need to additionally assume that $\log|z|_v \in L^1_0(M,\mu)$ (integral must be zero). The consideration of such more general product formulas found applications soon after the appearance of [36], as Moriwaki [56] used height functions on finitely generated fields over $\mathbb{Q}$, to recover the Raynaud's theorem (Manin-Mumford conjecture). Later, Chen and Moriwaki introduced *(proper) adelic curves* in [24], and Ben Yaacov and Hrushovski introduced *(globally or) multiply valued fields* in [10]. Both of these notions are defined similarly to M-fields (satisfying the product formula). Let us also mention the work [83] in which Yuan and Zhang extend the theory of Moriwaki developed in [56, 57].

In this paper we present the basics of the theory of globally valued fields, compare it to adelic curves, and describe how to see those as models of an unbounded continuous logic theory GVF_e (where $e \geq 0$). The logic interpretation of globally valued fields allows to define various operations (e.g. ultraproducts) typical to model theory, in the context of global fields. In particular this implies some transfer principles from number fields to function fields, or in Nevanlinna theory, see Example 1.11.11 and Example 1.11.12 respectively. Moreover, the logic treatment implies automatic effectiveness given finiteness in Corollary 1.12.9. The connection to stability in model theory would be desirable, especially because it has been useful in other algebro-geometric contexts, like in the solution of the geometric Mordell-Lang conjecture [42].

This paper is partially based on the unpublished notes [11], however, some definitions are adjusted, and some new results are obtained. We hope that it will provide a useful reference for globally valued fields in the future.

Let us describe the main definitions and results of this text in more detail. Recall that an adelic curve is a field F together with a measure space $(\Omega, \mathcal{A}, \nu)$ and a map $\phi : \omega \mapsto |\cdot|_\omega$ from Ω to the space of absolute values M_F on F, such that for all $a \in F^\times$ the function

$$\omega \mapsto \log|a|_\omega$$

is integrable (and measurable) on Ω. Given such data, we can define functions $h : F^n \to \mathbb{R} \cup \{-\infty\}$ given by

$$h(a_1, \ldots, a_n) = \int_\Omega \max(\log|a_1|_\omega, \ldots, \log|a_n|_\omega) d\nu(\omega). \tag{1.5}$$

Denote by $\mathbb{A}(F) := \bigcup_{n\in\mathbb{N}} F^n$ and $\mathbb{P}(F) := \bigcup_{n\in\mathbb{N}} \mathbb{P}^n(F)$. We use the symbol $\otimes$ to denote both Segre products $\mathbb{A}(F) \times \mathbb{A}(F) \to \mathbb{A}(F)$ and $\mathbb{P}(F) \times \mathbb{P}(F) \to \mathbb{P}(F)$. It is easy to check that the above functions form a 'height' on F, which we define below.

Definition 1.1.1. A *height* on F is a function $h : \mathbb{A}(F) \to \mathbb{R} \cup \{-\infty\}$ satisfying the following axioms, for some *Archimedean error* $e \geq 0$.

Height of zero:	$\forall x \in F^n$,	$h(x) = -\infty \Leftrightarrow x = 0$
Height of one:		$h(1,1) = 0$
Invariance:	$\forall x \in F^n$, $\forall \sigma \in \mathrm{Sym}_n$,	$h(\sigma x) = h(x)$
Additivity:	$\forall x \in F^n$, $\forall y \in F^m$,	$h(x \otimes y) = h(x) + h(y)$
Monotonicity:	$\forall x \in F^n$, $\forall y \in F^m$,	$h(x) \leq h(x,y)$
Triangle inequality:	$\forall x, y \in F^n$,	$h(x+y) \leq h(x,y) + e$

A height h is called *global* if $h \restriction_{F^\times} \equiv 0$. It then induces a function $\mathbb{P}(F) \to \mathbb{R}_{\geq 0}$ and we write $\mathrm{ht}(x) := h[x : 1]$ for $x \in F$.

The main theorem of this text says that the data of (global) heights on F is equivalent to a few other types of data, each of different flavour. We first state it and then explain the nature of various structures appearing therein.

Theorem 1.1.2. (Theorem 1.7.7) There is a bijective correspondence between the following structures on F:

1. Local terms (Definition 1.7.1),
2. Heights (Definition 1.1.1),
3. Positive functionals (Definition 1.6.3),
4. Families of local measures (Definition 1.6.6),
5. Renormalisation classes of admissible measures (Definition 1.6.7),
6. Equivalence classes of lattice valuations (Definition 1.4.11).

Moreover, this correspondence respects the 'global' ones, i.e., the ones satisfying the product formula.

We define a *multiply valued field* (resp. *globally valued field*) to be a field with any one of these equivalent structures (resp. that are global); thus if K is a globally valued field, we may freely talk about the Weil heights, the local terms, the valuation into the L^1-lattice, or about a renormalization class of a globalizing measure. We use the abbreviations MVF and GVF respectively.

If we require a specific presentation, we will simply say "a GVF, presented via local terms" (or via Weil heights etc.) A product formula field is another synonym that has been used, especially for the local term presentation.

Note that "local" is a slightly misleading shorthand; "mean local terms" would be a better description, as they arise by integration over the various localities (and similarly for Weil heights over MVFs).

We note that while MVFs can be useful for local or "mean-local" work, they do not form an axiomatizable class (in the sense of unbounded continuous logic). Any axiomatizable class containing all MVFs would allow non-zero elements whose height is $-\infty$. We can avoid this by adding an axiom insisting on a lower bound for the heights of non-zero elements; but any such bound will actually imply that 0 is a lower bound, leading precisely to GVFs.

The above theorem says that GVF structures on a field can be seen as coming from seemingly different worlds.

- First, one can define them through heights (points (1) and (2)), which easily fits into the realm of unbounded continuous logic.
- Second, on a countable characteristic zero field, point (3) specialises to the data of a linear functional $\mathrm{APic}(F) \to \mathbb{R}$ which is non-negative on the effective cone, where $\mathrm{APic}(F)$ is the colimit of the spaces of adelic divisors on arithmetic varieties with function field F. Also, if F is finitely generated over a subfield k (of any characteristic), then global functionals trivial on k correspond to linear functionals $\mathrm{NS}(F/k) \to \mathbb{R}$ that are non-negative on the effective cone, where $\mathrm{NS}(F/k)$ is the colimit of the Néron–Severi groups of proper varieties over k with function field F. We give details on this approach in Section 1.8.
- Third, GVF structures can come from measures on the space of pseudo-absolute values like in Equation (1.4), or from adelic curves. These correspond to points (4) and (5). Another measure theoretic way of presenting a GVF structure on a field is a global 'valuation' valued in a Banach L^1-lattice as in point (6), see Definition 1.4.11 or [1] for an example.

We summarise this dictionary in the table below.

Weil heights	Algebraic cycles	Measure theory	Banach lattices
Global heights as in Definition 1.1.1	$\mathrm{APic}(F) \to \mathbb{R}$ or $\mathrm{NS}(F/k) \to \mathbb{R}$ positive on the effective cone [71] or [10, 27]	Proper adelic curves, M-fields satisfying the product formula [24, 36]	$v : F^\times \to \Gamma \cup \{\infty\}$ 'valuation' valued in an L^1-lattice Γ, see Definition 1.4.11 and [1]

In particular, if the field F is countable, it follows from the proof of Theorem 1.1.2 that every (global) height on F actually comes from a (proper) adelic curve structure, as in Equation (1.5).

Corollary 1.1.3 (Corollary 1.7.11)**.** Any GVF structure on a countable field F has a proper adelic curve representation with $\Omega = M_F$.

This means that GVF structures on a countable field F can be seen as equivalence classes of adelic structures on F with respect to the equivalence relation defined by inducing the same height via Equation (1.5). This corollary is important for the future development of globally valued fields, as it allows to use vast existing literature on adelic curves e.g. from [25, 26, 30, 52, 53, 72] to study model theory of GVF structures. For example, the Hilbert-Samuel theorem in [26] may be useful in proving amalgamation theorems over some globally valued fields (see [6, 7] for more details). However, it is natural to treat h in Definition 1.1.1 as a continuous logic predicate, so by 'model theory' we mean the model theory in *unbounded continuous logic* introduced in [5]. Axioms from Definition 1.1.1 define an unbounded continuous logic theory GVF_e of globally valued fields with Archimedean error e. Another reason for considering continuous logic, is that for example $\overline{\mathbb{Q}}$ with Weil heights considered in discrete logic (with an additional sort for values of heights) is undecidable, as it interprets[1] the totally real algebraic integers (known to be undecidable by [66]).

We provide a survey of the unbounded continuous logic (in a simplified setting) in Section 1.11. In Section 1.12, we give some background on existential closedness in this logic and state the following problem.

Conjecture 1.1.4. (Conjecture 1.12.7) The theory GVF_e has a model companion, for every Archimedean error $e \geq 0$.

This conjecture asks whether we can axiomatize (in a first order way) existentially closed globally valued fields. Intuitively, a GVF F is existentially closed, if all polynomial equations together with height inequalities that are consistent (solved in an extension of F), have a solution in F. Existential closedness of GVFs implies existence of small sections of certain adelic vector bundles, see e.g. Minkowskian line bundles in [26, Section 8.9]. This weaker condition may be equivalent to existential closedness, but in any case forms an important part and is in itself axiomatizable.

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[1] By the formula: $v(x) = 0$ for almost all non-Archimedean v, and $v(i - x/i + x) = 0$ for almost all Archimedean v.

In [10, 71] respectively (the latter corresponds to Chapter 2 here), the globally valued fields $\overline{k(t)}$ (with $\mathrm{ht}(t) = 1$, $\mathrm{ht}|_k = 0$) and $\overline{\mathbb{Q}}$ (equipped with Weil projective heights) have been proven to be existentially closed. Over $\overline{\mathbb{Q}}$ it implies for example, that a finiteness statement (e.g. Bogomolov) is true in every GVF extension of $\overline{\mathbb{Q}}$ if and only if it is true in $\overline{\mathbb{Q}}$ uniformly in parameters. Conjecture 1.12.7 would imply that all existentially closed globally valued fields are elementary equivalent to one of these two kinds of GVFs. In particular, the problem of approximating essential infimum of an adelic line bundle (up to some $\varepsilon > 0$) would be decidable.

In course of proving Theorem 1.1.2 we introduce a few gadgets associated to F and prove some results that may be of independent interest. We call a function $|\cdot| : F \to [0, \infty]$ a *pseudo-absolute value*, if $|xy| = |x| \cdot |y|$, $|x + y| \leq |x| + |y|$, $|0| = 0, |1| = 1$ (with $0 \cdot \infty$ and $\infty \cdot 0$ undefined). By an *absolute value* we mean a pseudo-absolute value valued in $[0, \infty)$. We provide a Riemann-Zariski-Berkovich characterisation of the space of pseudo-absolute values Θ_F on F with the pointwise convergence topology. Let us remark that a similar result (in the case of a field extension while fixing the absolute value on the smaller field) was independently obtained by Antoine Sédillot in [73, Birational approach I.11.2.2].

Proposition 1.1.5. (Proposition 1.2.7) Let F be of characteristic zero. Then there is a homeomorphism $\theta : \Theta_F \to \mathcal{M}(F)$ between the space of pseudo-absolute values on F, and

$$\mathcal{M}(F) := \varprojlim \mathcal{M}(\mathcal{X}),$$

where $\mathcal{X}$ are in the system of arithmetic varieties over $\mathrm{Spec}(\mathbb{Z})$ with the data of an embedding of their function field into F, and $\mathcal{M}$ is the Berkovich analytification functor (so $\mathcal{M}(\mathrm{Spec}(A))$ is the set of all multiplicative real-valued seminorms on A).

Proposition 1.1.6. (Remark 1.2.10) In any characteristic, if F/k is either finitely generated, or F is countable, then there is a homeomorphism

$$\theta : \Theta_{F/k} \to \varprojlim X^{\mathrm{an}},$$

where $\Theta_{F/k}$ is the space of pseudo-absolute values on F that restrict to the trivial absolute value on k, and X are normal projective varieties over k together with an embedding of $k(X)$ to F over k. Here $(-)^{\mathrm{an}}$ is the Berkovich analytification with respect to the trivial absolute value on k (so $\mathrm{Spec}(A)^{\mathrm{an}}$ is the set of all multiplicative real-valued seminorms on a k-algebra A that restrict to the trivial absolute value on k).

This description is useful, because using the results about Berkovich spaces over $\mathbb{Z}$ from [51] we easily get the following result (related to [83, Lemma 3.1.1]).

Corollary 1.1.7. (Corollary 1.2.9 and Corollary 1.2.11) The set of absolute values on a countable field F is dense in the space of pseudo-absolute values Θ_F.

Also, if F/k is finitely generated (or both are countable), then the set of absolute values on F that are trivial on k is dense in $\Theta_{F/k}$.

We use this corollary to identify positive elements of the *universal lattice of* $F^\times$, i.e., the divisible hull of the Grothendieck group of the lattice of finite subsets of $F^\times$ (with $+$ being the multiplication of sets and $\vee$ being the union). We denote this lattice by $\mathrm{ULat}_{\mathbb{Q}}(F)$, see Definition 1.4.4 for more details. Let $|\cdot|$ be a pseudo-absolute value on F and $\alpha = \bigvee_{x \in I} \widehat{\mathrm{div}}(x) - \bigvee_{y \in J} \widehat{\mathrm{div}}(y) \in \mathrm{ULat}_{\mathbb{Q}}(F)$. Using the notation $v(x) := -\log|x|$ we define $v(\alpha) = \max_{x \in I} \min_{y \in J} v(x/y)$. We call $\alpha \in \mathrm{ULat}_{\mathbb{Q}}(F)$ *positive* if for all v as above, $v(\alpha) \geq 0$. The following characterisation of positive elements is used to get equivalence of points (2) and (3) in Theorem 1.1.2.

Theorem 1.1.8. (Corollary 1.5.15) For each $\alpha \in \mathrm{ULat}_{\mathbb{Q}}(F)$, the following conditions are equivalent:

(i) For every pseudo-absolute value $|\cdot|$ on F, $v(\alpha) \geq 0$.

(ii) For every $\varepsilon \in \mathbb{R}_{>0}$, there is some $m \in \mathbb{N}$ such that $m\alpha \in \Gamma_{m\varepsilon}$.

In this statement, $\Gamma_\varepsilon \subset \mathrm{ULat}_{\mathbb{Q}}(F)$ (for any $\varepsilon > 0$) is the set of elements of the form

$$\bigwedge_{i=1}^{p} \widehat{\mathrm{div}} \left(\sum_{j=1}^{q} m_{i,j} a_j \right) - \bigwedge_{j=1}^{q} \widehat{\mathrm{div}}(a_j),$$

where $a_1, \dots, a_q \in F^\times$, $p, q \in \mathbb{N}^*$, and $(m_{i,j})_{\substack{1 \leq i \leq p \\ 1 \leq j \leq q}} \in \mathbb{Z}^{pq}$ are such that for all $i \leq p$ we have $\sum_{j=1}^{q} |m_{i,j}| < 2^\varepsilon$. Moreover, if F is countable, in (i) one can only consider absolute values. The significance of this theorem is that it gives a syntactic criterion for positivity (which is a semantic property).

Furthermore, we study the quotient $\mathrm{LDiv}_{\mathbb{Q}}(F)$ of $\mathrm{ULat}_{\mathbb{Q}}(F)$ by the kernel ideal I of the pairing $(v, \alpha) \mapsto v(\alpha)$. The following characterisations are provided:

- if F is countable of characteristic zero, then $\mathrm{LDiv}_{\mathbb{Q}}(F)$ can be seen as a sublattice of the space of Arakelov divisors on F-models, see Corollary 1.5.17;

- if F/k is a finitely generated extension, then $\mathrm{LDiv}_{\mathbb{Q}}(F)$ divided by the ideal generated by $\mathrm{LDiv}_{\mathbb{Q}}(k)$ is isomorphic to the space of b-divisors on the Riemann-Zariski space of F/k, i.e., to the colimit of rational Cartier divisors on birational models of F over k, see Corollary 1.8.4.

The lattice structure on the space of Arakelov divisors on F-models from the first bullet has been introduced in [77, 67] in the language of (generalised or b-) Weil functions. We use the a different description of this lattice from Chapter 2. The second bullet has been discovered (in a slightly different language) also in [47, Theorem 5.2]. Interestingly, already André Weil in the 1950's was aware of this interpretation of (b-)divisors, see [78, Theorem 13].

The *positive functional* appearing in the statement of Theorem 1.1.2 is by definition a linear functional $l : \mathrm{LDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$ which is non-negative on the positive cone $\{\alpha \geq 0\} \subset \mathrm{LDiv}_{\mathbb{Q}}(F)$.

Let us elaborate on the meaning of point (5) in Theorem 1.1.2 (point (4) is similar). We introduce a locally compact topological space Ω_F° which is defined as the set of *pseudo-valuations* $v : F \to [-\infty, \infty]$, i.e., functions of the form $-c \log |\cdot|$ for $c \in \mathbb{R}_{>0}$ and non $\{0, 1, \infty\}$-valued pseudo-absolute values $|\cdot|$ on F. Thus, one can think of Ω_F° as an additive version of $\mathcal{M}(F)$ (see the difference between Equation (1.1) and Equation (1.2)). For technical reasons, we sometimes need to work with a slightly bigger space $\Omega_F^\circ \subset \Omega_F$, but in practice one can restrict attention to pseudo-valuations, see Definition 1.2.4 and Remark 1.6.8. The following representation theorem follows from equivalence of (1) and (5) in Theorem 1.1.2.

Corollary 1.1.9. Let F be any field equipped with a GVF structure. Then there exists a measure μ on Ω_F° such that the heights on F are given by the formula

$$h(a_1, \dots, a_n) = \int_{\Omega_F^\circ} - \min(v(a_1), \dots, v(a_n)) d\mu(v).$$

The minus sign appears so that the formula is consistent with Equation (1.5). There are two differences between Corollary 1.1.3 and Corollary 1.1.9: the latter works also for uncountable fields, and the ambient topological space Ω_F° does not vary. In fact, the previous corollary follows from the latter. In general, the flexibility of choosing Ω to be any measurable space in the definition of adelic curves is useful, as it allows to compare different metrics over the same absolute value, see for example Harder-Narasimhan filtrations in [24, 26, 52].

After proving Theorem 1.1.2 we give some examples and focus on the relative situation when $K \subset F$ is an extension of globally valued fields. Using a disintegration type result, we get a relative version of Corollary 1.1.9, namely Corollary 1.9.2 and Corollary 1.9.4. This is interesting in light of recent applications of families of measures on Berkovich spaces, for example in [61, 62].

In Section 1.10 we prove uniqueness (up to scaling) of the GVF structure on classical global fields (Lemma 1.10.1 and Lemma 1.10.2), and the following.

Proposition 1.1.10. (Proposition 1.10.5) Let K be a GVF and consider a finite Galois field extension $K \subset F$. Then there is a unique GVF structure on F extending the one on K that is invariant under the action of $\mathrm{Aut}(F/K)$.

It is worth mentioning that using representations by adelic curves, the existence (but not uniqueness) follows from [24, Section 3.4]. Also, under an additional assumption on the GVF structure, this result was independently obtained by Antoine Sédillot in [73, Propositions II.5.3.1 and II.5.4.1].

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1.2 Valuations

Let F be a field. We will later need an enlargement of the space of valuations on F to be able to consider Archimedean valuations even if F has cardinality bigger than continuum. That is why we introduce the following definition (partially appearing already in [78]).

Definition 1.2.1.

- A *pseudo-absolute value* $|\cdot|$ on F is a function $|\cdot| : F \to \mathbb{R}_{>0} \cup \{0, \infty\}$ satisfying $|x + y| \leq |x| + |y|, |xy| = |x||y|, |0| = 0, |1| = 1$ with $0 \cdot \infty$ undefined.

- We call a pseudo-absolute value *trivial*, if it attains values only from $\{0, 1, \infty\}$. Otherwise, we call it a non-trivial pseudo-absolute value.

- A *pseudo-valuation* v on F is a map $v : F \to \mathbb{R} \cup \{\pm\infty\}$ attaining at least one value outside of the set $\{-\infty, 0, \infty\}$ and such that $v(xy) = v(x) + v(y), v(0) = \infty, v(1) = 0, v(2) \neq -\infty, v(x+y) \geq \min(v(x), v(y)) + \min(v(2), 0)$, with $-\infty+\infty$ undefined.

- An *abstract pseudo-valuation* v on F is a map $v : F \to \Gamma \cup \{\pm\infty\}$, where Γ is a divisible ordered abelian group, which satisfies the above axioms of a pseudo-valuation (but with addition and minima interpreted in Γ).

Lemma 1.2.2. Let $v : F \to \Gamma \cup \{\pm\infty\}$ be an abstract pseudo-valuation on F and let $\Delta \subset \Gamma$ be a convex subgroup such that $v(2) \not< \Delta$. Consider the function $u(a) = v(a) \mod \Delta$ and

$$w(a) = \begin{cases} \infty & \text{if } v(a) > \Delta, \\ v(a) & \text{if } v(a) \in \Delta, \\ -\infty & \text{if } v(a) < \Delta. \end{cases}$$

Then u, w satisfy the axioms of abstract pseudo-valuations on F, possibly without the axiom asserting that they attain at least one value outside of the set $\{-\infty, 0, \infty\}$.

Proof. Straightforward. ☐

We omit the prefix "pseudo" in Definition 1.2.1, if the map does not attain $\pm\infty$ at non-zero elements. Note that this gives a slightly non-standard definition of a *valuation* on F, as we exclude the $\{0, \infty\}$-valued ones from the usual definition, and include some that do not satisfy the ultrametric inequality without the correction $\min(v(2), 0)$. The set of valuations on F is denoted by Val_F and the set of absolute values is denoted by M_F. We topologize those spaces with the pointwise convergence topology. Note that for a pseudo-absolute value $|\cdot|$ on F, the set of elements $\mathcal{O} := \{|x| < \infty\}$ is valuation ring with the maximal ideal $\mathfrak{m}$ consisting of elements with $|x| = 0$. The quotient $K = \mathcal{O}/\mathfrak{m}$ is then a field with an absolute value. Thus we get an induced surjective map $p : F \to K \cup \{\infty\}$ where the preimage of infinity is a complement of a valuation ring. Such a map is called a place. Summarising, the following holds.

Remark 1.2.3.

1. If $|\cdot|$ is a pseudo-absolute value on F, then there exists a place $p : F \to K \cup \{\infty\}$ into a field K with an absolute value $|\cdot|_K$ such that $|x| = |p(x)|_K$.

2. If $|\cdot|$ is a non-trivial pseudo-absolute value on F and $c \in \mathbb{R}_{>0}$, then $v(x) = -c\log|x|$ is a pseudo-valuation on F.

3. If v is a pseudo-valuation on F, then there is a non-trivial pseudo-absolute value $|\cdot|$ on F and a constant $c \in \mathbb{R}_{>0}$ such that $v(x) = -c\log|x|$ for all $x \in F$.

Proof. The proof of (1) is outlined above. For (2) note that by (1) we can assume that $|\cdot|$ is a norm. By passing to the completion, without loss of generality $|\cdot|$ is complete on F. Then either $|\cdot|$ is a non-Archimedean absolute value on F in which case the statement follows, or (by the classification of Archimedean normed complete fields) $(F, |\cdot|) \subset (\mathbb{C}, \|\cdot\|^t)$, where $\|\cdot\|$ is the standard Euclidean absolute value on $\mathbb{C}$ and $t \in (0, 1]$. The direct inspection of the statement on $(\mathbb{C}, \|\cdot\|^t)$ finishes the proof of (2).

To see (3), put $a = -\min(v(2), 0)$. The case $a = 0$ is clear, so assume that $a > 0$. We define $|x| = 2^{-v(x)/a}$ and check that $|\cdot|$ is a pseudo-absolute value. First note that $|x + y| \leq 2\max(|x|, |y|)$ implies by induction that

$$\Big|\sum_{i=1}^{2^n} x_i\Big| \leq 2^n \max_i |x_i|.$$

In particular $|2^n| \leq 2^n$ and for any integer N we get $|N| \leq N\log(N)$ using the binary expansion of N. Replacing N with N^k and taking limit over $k \to \infty$ we get $|N| \leq N$. Hence

$$|x + y|^{2^n} = |(x + y)^{2^n}| \leq 2^{n+1} \max_{0 \leq i \leq 2^n} \binom{2^n}{i} |x|^i |y|^{2^n - i} \leq 2^{n+1}(|x| + |y|)^{2^n}.$$

Taking 2^n'th roots and the limit $n \to \infty$ we get $|x + y| \leq |x| + |y|$ which finishes the proof. □

We can drop the prefix "pseudo" everywhere in the above remark and points 2. and 3. remain true. Note that non-trivial pseudo-absolute values and pseudo-valuations represent almost the same data, the difference being that we can multiply pseudo-valuations by an arbitrary positive number and we cannot always increase a non-trivial pseudo-absolute value to a positive number, in the space of non-trivial pseudo-absolute values.

Definition 1.2.4. Let A be the set of functions $v : F \to [-\infty, \infty]$ satisfying

$$v(xy) = v(x) + v(y), v(0) = \infty, v(1) = 0, v(x + y) \geq \min(v(x), v(y)) + \min(v(2), 0),$$

and $v(2) \neq -\infty$. We define the space Ω_F as $\mathrm{cl}(A) \setminus \{-\infty, 0, \infty\}^F$, so it is an open subset of the compact space $\mathrm{cl}(A) \subset [-\infty, \infty]^F$. Hence, it is a locally compact, Hausdorff topological space. Moreover, we denote the subset of pseudo-valuations by $\Omega_F^\circ := \{v(2) \neq -\infty\} \subset \Omega_F$ which equivalently can be defined as $A \setminus \{-\infty, 0, \infty\}^F$.

Example 1.2.5. Consider $F = \mathbb{Q}(x)$ and let $|\cdot|_n$ be the absolute value on F coming from the embedding $\mathbb{Q}(x) \subset \mathbb{C}$ sending x to $1 + \frac{\pi}{n}$. Then for $v_n = -n \log |\cdot|_n$ we have $\lim_n v_n(x) = -\pi$ and $\lim_n v_n(2) = -\infty$. Hence any limit point v of $(v_n)_{n \in \mathbb{N}}$ is in Ω_F, but is not a pseudo-valuation.

We call an element $v \in \Omega_F$ non-Archimedean, if $v(2) \geq 0$ and Archimedean in the other case. Not every element $v \in \Omega_F$ is a non-trivial pseudo-absolute value on F, however, the set of non-trivial pseudo-absolute values is dense in Ω_F. We choose to work with Ω_F instead of the set of non-trivial pseudo-absolute values because of nicer topological properties, see Lemma 1.2.13 and the line under it.

Note that if F is countable, then

$$\mathrm{Val}_F = \bigcap_{a \in F^\times} \{v \in \Omega_F : v(a) \neq \pm\infty\}$$

is a Borel subset of Ω_F and in that case we equip Val_F with the σ-algebra of its Borel subsets. Denote by Θ_F the space of pseudo-absolute values equipped with pointwise convergence topology - note that it is a compact Hausdorff space. Overall, we get the following diagram of spaces associated to F

$$\begin{tikzcd}
M_F \arrow[r, hook] \arrow[d, dashed] & \Theta_F \arrow[d, dashed] \\
\mathrm{Val}_F \arrow[r, hook] & \Omega_F
\end{tikzcd}$$

where the dashed arrows are $-\log$'s defined only on non-trivial (pseudo-)absolute values.

Our goal now is to prove Corollary 1.2.9 and Corollary 1.2.11. Since these corollaries hold under different assumptions, let us restrict our attention to characteristic zero F until Remark 1.2.10. Call a normal, projective, generically smooth, integral scheme $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$ an *F-submodel*, if the function field of $\mathcal{X}$ is equipped with an embedding to F (this definition will also appear in a different context in Chapter 2, Remark 2.3.19). For an F-submodel $\mathcal{X}$ define its Berkovich analytification as

$$\mathcal{M}(\mathcal{X}) := \{(p, |\cdot|) \text{ with } p \in \mathcal{X} \text{ and } |\cdot| \text{ is an absolute value on } \kappa(p)\}.$$

For a point $x = (p, |\cdot|)$ if $f \in \mathcal{O}_{\mathcal{X}}(U)$ where $p \in U \subset \mathcal{X}$, then we write $|f(x)|$ for $|f(p)|$ where $f(p) \in \kappa(p)$ is the evaluation of f at p. The topology on $\mathcal{M}(\mathcal{X})$ is the coarsest such that the projection $q : \mathcal{M}(\mathcal{X}) \to \mathcal{X}$ is continuous, and for every f as above, the function $x \mapsto |f(x)|$ is continuous on $q^{-1}(U)$. This makes $\mathcal{M}(\mathcal{X})$ a compact Hausdorff topological space. For a thorough treatment of these spaces, see [51, 60]. We define

$$\mathcal{M}(F) := \varprojlim \mathcal{M}(\mathcal{X})$$

to be the inverse limit over (the directed system of) all F-submodels of F. If X is a proper scheme over a complete valued field, we write X^{an} for the Berkovich analytification, i.e., the subset of $\mathcal{M}(X)$ where absolute values on residue fields extend the one on the base-field.

Construction 1.2.6. Let $|\cdot| \in \Theta_F$. Denote by $\mathcal{O} \subset F$ the valuation subring consisting of elements with $|x| < \infty$, by $\mathfrak{m}$ its maximal ideal, and by k the residue field. Denote the induced absolute value on k also by $|\cdot|$. Fix an F-submodel $\mathcal{X}$. Consider the diagram

$$\begin{tikzcd}
\operatorname{Spec}(F) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spec}(\mathcal{O}) \arrow[r] \arrow[ur, dashed, "\phi"] & \operatorname{Spec}(\mathbb{Z})
\end{tikzcd}$$

where the top horizontal arrow comes from the embedding of the function field of $\mathcal{X}$ into F and the diagonal arrow exists by projectivity of $\mathcal{X}$ over $\mathbb{Z}$. Let $p = \phi(\mathfrak{m})$ and consider the induced field extension $\kappa(p) \subset k$. We can restrict $|\cdot|$ to $\kappa(p)$ to get an absolute value inducing a point $x \in \mathcal{M}(\mathcal{X})$. Summarising, we constructed a map

$$\theta : \Theta_F \to \mathcal{M}(F).$$

Proposition 1.2.7. The map θ is a homeomorphism.

Proof. First let us check that for an F-submodel $\mathcal{X}$ the composed map

$$\Theta_F \to \mathcal{M}(F) \to \mathcal{M}(\mathcal{X}) \to \mathcal{X}$$

is continuous with respect to the Zariski topology on the codomain. Let $\operatorname{Spec}(A) \subset \mathcal{X}$ be an open affine subset and let $A \subset F$ be the induced embedding of A. If we denote by $p : \Theta_F \to \mathcal{X}$ the composition above, then

$$p^{-1}(\operatorname{Spec}(A)) = \{|\cdot| \in \Theta_F : A \subset \{|x| < \infty\}\}.$$

However, since A is finitely generated over $\mathbb{Z}$, if we pick its generators $a_1, \dots, a_n$, then this set can be written as $\{|\cdot| \in \Theta_F : \max_i |a_i| < \infty\}$ which is an open subset of Θ_F. Thus p is continuous. Hence, the continuity of θ follows, by the definition of topology on $\mathcal{M}(\mathcal{X})$. Since both domain and codomain of θ are compact Hausdorff spaces, it is enough to show that θ is a bijection. The classical Riemann-Zariski space construction shows that valuation subrings of F are in bijection with the inverse limit of the system of F-submodels, see e.g. [74, Corollary 3.4.7]. More precisely, if $\mathcal{O} \subset F$ is a valuation ring, then there exists a unique sequence of points $x(i) \in \mathcal{X}_i$, where $(\mathcal{X}_i)_{i\in I}$ is the system of F-submodels, such that $\mathcal{O} = \varinjlim \mathcal{O}_{\mathcal{X}_i, x(i)}$. Then absolute values on the residue field k of $\mathcal{O}$ correspond to consistent families of absolute values on $\kappa(x(i))$ for $i \in I$, which finishes the proof. □

Proposition 1.2.8. Let $\mathcal{X}$ be an F-submodel. Then the generic fiber of the projection $q : \mathcal{M}(\mathcal{X}) \to \mathcal{X}$ is dense in $\mathcal{M}(\mathcal{X})$.

Proof. First note that it is enough to prove that for a nonempty open $\mathcal{U} \subset \mathcal{X}$ the set $q^{-1}(\mathcal{U})$ is dense. Indeed, this is because $\mathcal{M}(\mathcal{X})$ is a Baire space, and $\mathcal{X}$ has a countable basis of open subsets, whose intersection is exactly the generic point of $\mathcal{X}$.

Fix open non-empty subsets $\mathcal{U} \subset \mathcal{X}$ and $V \subset \mathcal{M}(\mathcal{X})$. We need to show that $V \cap q^{-1}(\mathcal{U}) \neq \varnothing$. By [51, Proposition 6.4.1 and Proposition 6.6.10] the map $\mathcal{M}(\mathcal{X}) \to \mathcal{M}(\mathbb{Z})$ is open. Hence, by the description of $\mathcal{M}(\mathbb{Z})$ e.g. in [60, Corollaire 3.1.12], there is $x \in V$ with its image $x|_{\mathbb{Q}}$ in $\mathcal{M}(\mathbb{Z})$ being an absolute value on $\mathbb{Q}$. Since the problem is invariant under putting an absolute value to a positive power, without loss of generality $x|_{\mathbb{Q}}$ is either a p-adic absolute value on $\mathbb{Q}$ (for some prime p), or the Euclidean one. Let K be the completion of $\mathbb{Q}$ with respect to this absolute value.

Let X be the generic fiber of $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$, so it is a smooth, projective scheme over $\mathbb{Q}$ with the function field F. Denote by U the intersection of $\mathcal{U}$ with X and assume that the point $x \in V$ corresponds to an absolute value $|\cdot|$ on $\kappa(x_0)$ for $x_0 \in X$. We consider two cases:

(1) $x|_{\mathbb{Q}}$ is non-Archimedean. In that case $K = \mathbb{Q}_p$ and we consider the Berkovich analytification X_K^{an} of X_K with projection $\rho : X_K^{\mathrm{an}} \to X_K$. We get the pullback diagram

$$\begin{tikzcd}
X_K^{\mathrm{an}} \arrow[r, hook, "j"] \arrow[d] & \mathcal{M}(\mathcal{X}) \arrow[d] \\
x|_{\mathbb{Q}} \arrow[r, hook] & \mathcal{M}(\mathbb{Z})
\end{tikzcd}$$

As $U_K \subset X_K$ is everywhere dense open, by [12, Corollary 3.4.5] the subset $\rho^{-1}(U_K)$ is everywhere dense in X_K^{an}. Thus, as V intersects the fiber over $x|_{\mathbb{Q}}$ witnessed by $x \in V$, the set $V \cap \rho^{-1}(U_K)$ is non-empty. But $\rho^{-1}(U_K) = j^{-1}(q^{-1}(\mathcal{U}))$, so if $y \in \rho^{-1}(U_K)$, then $j(y) \in V \cap q^{-1}(\mathcal{U})$, which finishes the proof in this case.

(2) $x|_{\mathbb{Q}}$ is Archimedean. In that case $K = \mathbb{R}$ and we look at the complex analytification $X_{\mathbb{C}}^{\mathrm{an}}$ which is a smooth complex variety with the conjugation map c. Then the fiber of $\mathcal{M}(\mathcal{X})$ over $x|_{\mathbb{Q}}$ is $X_{\mathbb{C}}^{\mathrm{an}}/c$, and it is enough to prove that non-empty Zariski open subsets of X yield everywhere dense open subsets of $X_{\mathbb{C}}^{\mathrm{an}}$. This is classical, and can be easily deduced e.g. from the resolution of singularities. □

We use the above proposition to get the following corollary.

Corollary 1.2.9. Let F be a countable field of characteristic zero. The set M_F is a dense subset of Θ_F. In particular, the set Val_F is a dense subset of Ω_F.

Proof. Note that $M_F = \bigcap_{a \in F^\times} \{|\cdot| \in \Theta_F : |a| < \infty\}$. By the fact that Θ_F is a Baire space, and F is countable, it is enough to show that for $a \in F^\times$, the set $V = \{|\cdot| \in \Theta_F : |a| < \infty\}$ is dense.

Using the homeomorphism from Proposition 1.2.7, it is enough to check that for an F-submodel $\mathcal{X}$ and an open subset $U \subset \mathcal{M}(\mathcal{X})$, if we denote by $s : \Theta_F \to \mathcal{M}(\mathcal{X})$ the natural projection, $V \cap s^{-1}(U) \neq \varnothing$. By possibly replacing the F-submodel $\mathcal{X}$ by a one over it, we may assume that there is an open subscheme $\mathrm{Spec}(A) \subset \mathcal{X}$ such that $a \in A \subset F$. By Proposition 1.2.8, there is a point $x \in U \subset \mathcal{M}(\mathcal{X})$ such that its first coordinate (if we write is as a pair) is in $\mathrm{Spec}(A) \subset \mathcal{X}$.

Now, if $|\cdot| \in \Theta_F$ is such that $s(|\cdot|) = x$, then $|\cdot| \in V \cap s^{-1}(U)$, which gives the desired non-emptiness. Since maps in the system of Berkovich analytifications of F-submodels are surjective, such pseudo-absolute value $|\cdot|$ exists, which finishes the proof. □

Remark 1.2.10. One can perform the above constructions also in the relative context, where we fix a base-field $k \subset F$ and F is either finitely generated over k of arbitrary characteristic, or countable. Indeed, let $\Theta_{F/k} \subset \Theta_F$ be the set of pseudo-absolute values that are 1 on $k^\times$ and let $M_{F/k}$ be the set of absolute values on F that are trivial on k. We then get (using the same proof as in Proposition 1.2.7)

$$\Theta_{F/k} \simeq \varprojlim X^{\mathrm{an}},$$

where the inverse limit is taken over normal projective varieties X over k together with an embedding of $k(X)$ to F over k. The Berkovich analytification X^{an} is with respect to the trivial absolute value on k. The generic fiber of the natural projection $\pi : X^{\mathrm{an}} \to X$ is dense, by [12, Section 3.5].

In the case F is countable, we can copy the proof of Corollary 1.2.9 to get density of $M_{F/k}$ in $\Theta_{F/k}$. In the case F is finitely generated over k, we can write

$$\Theta_{F/k} \simeq \varprojlim_{k(X)\simeq F} X^{\mathrm{an}},$$

where in the limit we only take varieties over k with an isomorphism of their function field with F (over k). If $X' \to X$ is a birational map in the system, we get a commutative diagram

$$\begin{tikzcd}
X'^{\mathrm{an}} \arrow[r] & X^{\mathrm{an}} \\
M_{F/k} \arrow[u, hook] \arrow[r, "\mathrm{id}"'] & M_{F/k} \arrow[u, hook]
\end{tikzcd}$$

with the vertical maps come from isomorphisms $k(X), k(X') \simeq F$. Taking inverse limits, we also get density in that case. Summarising, we get the following.

Corollary 1.2.11. Let $k \subset F$ be either a finitely generated extension, or an extension of countable fields. Then the set $M_{F/k}$ is a dense subset of $\Theta_{F/k}$.

Let us come back to the context of a general field F. Define $\overline{\Omega}_F$ to be the quotient of Ω_F by the multiplicative action of $\mathbb{R}_{>0}$ (with the quotient topology) and call two elements of Ω_F *equivalent* if they lie in a single orbit of this action. We use the symbol $\sim$ to denote this equivalence relation. For $a \in F^\times$ we define sets

$$\Omega_F(a) := \{v \in \Omega_F : v(a) = 1\},$$

and $\overline{\Omega}_F(a) := \pi(\Omega_F(a))$ for the projection $\pi : \Omega_F \to \overline{\Omega}_F$. Note that $\overline{\Omega}_F(a)$ is the set of classes of $v \in \Omega_F$ with $0 < v(a) < \infty$. Moreover, $\Omega_F(a)$ is compact and π defines a bijection between $\Omega_F(a)$ and $\overline{\Omega}_F(a)$. We also have

$$\overline{\Omega}_F = \bigcup_{a \in F^\times} \overline{\Omega}_F(a).$$

Remark 1.2.12. In general $\overline{\Omega}_F$ is not Hausdorff. For example, if $F = \mathbb{Q}(x)$, then the relation of being equivalent in Ω_F is not closed. Indeed, consider the elements $0, x, x - p^{-1}$ in F and let $|\cdot|$ be an appropriate power of the Gauss absolute value (with respect to the p-adic absolute value on $\mathbb{Q}$) so that we have

$$|x| = 2, |x - p^{-1}| = 2, |p^{-1}| = 1 + \frac{1}{n}.$$

Let v_n be the corresponding valuation on F and note that $v_n \sim n \cdot v_n$. By taking limits (possibly we need to pass to a subnet in $[-\infty, \infty]^F$), we see that v_n converges to a pseudo-valuation v where the above triangle has sides $(2,2,1)$, while $n \cdot v_n$ converges to a pseudo-valuation w where the same triangle has sides (∞,∞,e). Note that these values are not contained it $\{-\infty,1,\infty\}$, so $v,w \in \Omega_F$. Moreover, v is not equivalent to w.

Even though the full space $\overline{\Omega}_F$ may not be Hausdorff, we have the following lemma.

Lemma 1.2.13. The sets $\overline{\Omega}_F(a)$ are Hausdorff topological subspaces, for $a \in F^\times$.

Proof. Consider $\pi^{-1}(\overline{\Omega}_F(a)) = \{v \in \Omega_F : 0 < v(a) < \infty\}$. We show that the relation of being equivalent is closed when restricted to this subset. Let $f_b : \pi^{-1}(\overline{\Omega}_F(a)) \to [-\infty,\infty]$ be a function given by $f_b(v) = \frac{v(b)}{v(a)}$, for $b \in F$. Note that it these are continuous functions and for $v,w \in \pi^{-1}(\overline{\Omega}_F(a))$ we have $v \sim w$ if and only if for all $b \in F$ the equality $f_b(v) = f_b(w)$ holds, which is a closed condition. □

In particular, because restricted quotients $\pi : \Omega_F(a) \to \overline{\Omega}_F(a)$ are continuous, they are homeomorphisms between compact Hausdorff spaces.

Lemma 1.2.14. Let $v \in \Omega_F$. Then, there exists $v' \in \Omega_{F'}$ such that $v'|_F = v$.

Proof. Let $a \in F$ be such that $0 < v(a) < \infty$. Without loss of generality, we can assume that $v(a) = 1$. Consider the restriction map $\Omega_{F'}(a) \to \Omega_F(a)$. It is a continuous map between compact Hausdorff spaces, so to prove it is surjective it is enough to prove it has dense image. Thus, we can assume that $v(2) \neq -\infty$. By Remark 1.2.3 we can assume that $v = -\log|\cdot|$ for some non-trivial pre-absolute value $|\cdot|$.

Let $\mathcal{O} \subset F$ be the valuation ring of elements with $|x| \neq \infty$ and let k and $\mathfrak{m}$ be its residue field and maximal ideal respectively. By Chevalley's theorem on extending valuations, there is a valuation ring $\mathcal{O}' \subset F'$ with maximal ideal $\mathfrak{m}'$ and residue field k' such that $\mathcal{O}' \cap F = \mathcal{O}$ and $\mathfrak{m}' \cap \mathcal{O} = \mathfrak{m}$. The non-trivial pre-absolute value $|\cdot|$ induces an absolute value $|\cdot|$ on k and we are reduced to proving that the corresponding valuation extends to a pre-valuation on k'.

Let K be a complete, algebraically closed, normed field, such that the absolute value $|\cdot|$ on k comes from an embedding $k \subset K$. Such exists, as we can extend the induced absolute value on the completion of k to its algebraic closure. Take an ultrapower K^* of K such that k' embeds to K^* over k. The ultrapower of the valuation on K gives an abstract pre-valuation v^* on K^* with the value group Γ. Let $\Delta \subset \Gamma$ be the convex hull of $\mathbb{R}$ and $\Delta_0 \subset \Delta$ be the convex subgroup of infinitesimal elements.

We use Lemma 1.2.2 two times: first restricting v^* to Δ, and next quotienting Δ by Δ_0. Identifying Δ/Δ_0 with $\mathbb{R}$, we get a (non-abstract) pseudo-valuation u on K^* extending v on k. By Remark 1.2.3, we can write $u = -\log|\cdot|_u$ for some pseudo-absolute value $|\cdot|_u$ that extends $|\cdot|$ on k (without loss of generality we can take $c = 1$ for the constant from Remark 1.2.3). By restricting $|\cdot|_u$ to k', we are done. $\square$

1.3 Seminorms

In this section we introduce a submultiplicative variant of (pseudo-)absolute values and show how to obtain (pseudo-)absolute values from them.

Definition 1.3.1. A *pseudo-seminorm* $|\cdot|$ on a ring R is a function $|\cdot| : R \to \mathbb{R}_{\geq\nvdash} \cup \{+\infty\}$ satisfying $|x+y| \leq |x| + |y|$, $|xy| \leq |x| \cdot |y|$, $|0| = 0$, $|\pm 1| = 1$, with $0 \cdot \infty$ undefined. A pseudo-seminorm $|\cdot|$ is called *power-multiplicative* if $|x^n| = |x|^n$ for all $x \in R$ and $n \in \mathbb{N}$. It is a pseudo-absolute value if $|xy| = |x||y|$ for all $x, y \in R$. We define a partial order on the set of pseudo-seminorms of F by

$$|\cdot|_1 \leq |\cdot|_2 \Leftrightarrow \forall x \in F,\ |x|_1 \leq |x|_2.$$

Note that a pseudo-seminorm on a ring R is simply the data of a subring $R' \subseteq R$ (characterized by $R' = \{x \in R : |x| \neq \infty\}$) and a non-zero seminorm on R'.

Lemma 1.3.2. Let $|\cdot|$ be a minimal pseudo-seminorm on a field R. Then $|\cdot|$ is a pseudo-absolute value.

Proof. Let $R' = \{|x| < \infty\}$. Note that if there was a seminorm on R' smaller than the restriction of $|\cdot|$ to R', then one could extend it to a pseudo-seminorm on R by taking the value ∞ at $R \setminus R'$, which would contradict minimality of $|\cdot|$. Hence, the restriction of $|\cdot|$ is a minimal seminorm on R'. Consider its spectral radius norm

$$\rho(x) = \begin{cases} \lim\limits_{n\to+\infty} |x^n|^{1/n} & \text{if } |x| \neq \infty \\ \infty & \text{if } |x| = \infty. \end{cases}$$

It is a classical result (see [12, Corollary 1.3.3]) that the spectral radius of a seminorm is a power-multiplicative seminorm. By minimality, we get $|\cdot| = \rho$, which shows that $|\cdot|$ is power multiplicative on R'.

Note that if $|x| = 0$ for $x \in R'$, then for any $y \in R'$, we have $|xy| \leq |x||y| = 0$, so in fact $|xy| = |x||y|$ in this case. Take on the other hand $x \in R'$ that satisfies $|x| \neq 0$. Let us define the following function on R':

$$p(z) = \inf_{n\in\mathbb{N}} \frac{|x^n z|}{|x|^n}.$$

By the power-multiplicativity of $|\cdot|$ on R', $p(1) = 1$ and it is easy to check that p is subadditive, and submultiplicative on R'. Moreover, looking at $n = 0$ one gets $p \leq |\cdot|$ on R'. By minimality, this means that $p = |\cdot|$, so $|y| = p(y) \leq \frac{|xy|}{|x|}$, which proves that $|xy| = |x||y|$. Thus, $|\cdot|$ is an absolute value on R'.

Let $\mathfrak{p} = \{|x| = 0\} \subset R'$ be the prime zero ideal. By the Chevalley's extention theorem, there is a valuation subring $\mathcal{O} \subset R$ with a maximal ideal $\mathfrak{m}$ satisfying $R' \cap \mathfrak{m} = \mathfrak{p}$. By Lemma 1.2.14, there is a pseudo-absolute value v' on $\mathcal{O}/\mathfrak{m}$ extending the norm $|\cdot|$ on $R'/\mathfrak{p}$. Extending this pseudo-absolute value by ∞ on $R \setminus \mathcal{O}$, we get that there is a pseudo-absolute value v on R extending the seminorm $|\cdot|$ on R'. By minimality (as a pseudo-seminorm on R) of $|\cdot|$, we get that $|\cdot| = v$, i.e., that $|\cdot|$ is a pseudo-absolute value. $\square$

Corollary 1.3.3. Let $|\cdot|$ be a pseudo-seminorm on F. Then, there exists a pseudo-absolute value $\|\cdot\|$, such that $\|\cdot\| \leq |\cdot|$.

Proof. It is clear that if S is a totally ordered set of pseudo-seminorms of F, the lower bound $|\cdot|_{\inf}$ defined by

$$\forall x \in F,\ |x|_{\inf} = \inf_{|\cdot|_s \in S} |x|_s.$$

is again a pseudo-seminorm. Thus, by Zorn's lemma, there exists a minimal pseudo-seminorm on F which is smaller than $|\cdot|$. By Lemma 1.3.2, it is a pseudo-absolute value. $\square$

Lemma 1.3.4. Let $f : R \to F$ be a ring homomorphism from a ring to a field, and let $|\cdot|$ be a pseudo-seminorm on R such that

$$\forall x \in R,\ f(x) = 1 \Rightarrow |x| \geq 1.$$

Then, there exists a pseudo-absolute value $\|\cdot\|$ on F such that

$$\forall x \in R,\ \|f(x)\| \leq |x|.$$

Proof. Define a function $\|\cdot\|_0 : F \to \mathbb{R}_{\geq 0} \cup \{+\infty\}$ by

$$\|y\|_0 = \inf_{x \in f^{-1}(\{y\})} |x|.$$

(thus $\|y\|_0 = +\infty$ if y is not in the image of f). The condition $f(x) = 1 \Rightarrow |x| \geq 1$ implies that $\|1\|_0 = 1$, hence $\|\cdot\|_0$ is a pseudo-seminorm. By Corollary 1.3.3, there exists a pseudo-absolute value $\|\cdot\|$ on F such that $\|\cdot\| \leq \|\cdot\|_0$. By construction, it satisfies the lemma's conclusion. $\square$

1.4 Lattices

In order to encode global structures on F we use tropical polynomials.

Definition 1.4.1. A $\mathbb{Q}$-*tropical polynomial* is a term in the language of divisible ordered abelian groups, i.e., in $+, \max, 0, \alpha \cdot x$ for $\alpha \in \mathbb{Q}$. We also use the symbol $\vee$ for maximum and $\wedge$ for minimum (which is defined by $a \wedge b := -((-a) \vee (-b))$ for any a, b). For example $t(x, y) = \min(\frac{1}{2}\max(x, y + x), \frac{1}{3}y)$ is a $\mathbb{Q}$-tropical polynomial. We call t a $\mathbb{Z}$-*tropical polynomial*, if it only uses $\alpha \in \mathbb{Z}$.

Certain universal lattices in $F^\times$ play an important role in this thesis. Before we define those, we recall the definitions of a lattice monoid and a lattice group.

Definition 1.4.2. A *lattice monoid (group)* is a commutative monoid (group) $(\Gamma, +)$ equipped with a *join* operation $\vee$ which satisfies the following axioms.

- Idempotence: $\forall x \in \Gamma,\ x \vee x = x$
- Commutativity: $\forall x, y \in \Gamma,\ x \vee y = y \vee x$
- Associativity: $\forall x, y, z \in \Gamma,\ (x \vee y) \vee z = x \vee (y \vee z)$
- Translation-invariance: $\forall x, y, z \in \Gamma,\ (x + z) \vee (y + z) = (x \vee y) + z$

A *morphism* of lattice monoids (groups) is a map from a lattice monoid (group) to another which preserves addition and join. We call a lattice group *divisible*, if the ambient abelian group is divisible. Every lattice monoid has a natural partial order defined by

$$x \geq y \iff x \vee y = x.$$

If it does not cause confusion, we may write $|x|$ for $\max\{x, -x\}$.

It is easy to see that this partial order is compatible with the group law of the lattice group, and that the operations $\vee$ and $+$ are increasing with respect to this order. Lattice groups have the following property which may not hold in lattice monoids (for example in the lattice monoid of formal joins in F, see Definition 1.4.4).

Lemma 1.4.3. Let Γ be a lattice group. Then for any $x, y \in \Gamma$ and $n \in \mathbb{N}$ we have

$$n(x \vee y) = (nx) \vee (ny).$$

Proof. By writing $x = x^+ + x^-$, where $x^+ = x \vee 0, x^- = x \wedge 0$, and same with y, we are reduced to show that $(nx)^\pm = nx^\pm$. This follows from the calculation:

$$(nx)^+ + nx^+ = (nx) \vee 0 + \bigvee_{k=0}^{n} (kx) = \bigvee_{k=0}^{2n} (kx) = 2nx^+,$$

and similarly for $(nx)^-$. $\square$

The forgetful functor from the category of lattice groups to the category of abelian groups has a left adjoint. Moreover, by the above lemma, the forgetful functor from the category of divisible lattice groups to the category of lattice groups also has a left adjoint. By composing those we get a functor from a category of abelian groups to the category of divisible group lattices (left adjoint to the forgetful functor), which we call the universal lattice group functor.

Let us describe the lattice group structure on the universal lattice group of $F^\times$. When we want to write the product in $F^\times$ additively, we use the notation $\widehat{\mathrm{div}}(a)$ for an element $a \in F^\times$. For example $\widehat{\mathrm{div}}(a) + \widehat{\mathrm{div}}(b) = \widehat{\mathrm{div}}(ab)$ and $\widehat{\mathrm{div}}(1) = 0$.

Definition 1.4.4. A *formal join in* $F^\times$ is an expression of the form $\bigvee_{x\in I} \widehat{\mathrm{div}}(x)$, where I is a finite subset of $F^\times$. The set of such expressions forms a lattice monoid with operations defined as follows:

$$\bigvee_{x\in I} \widehat{\mathrm{div}}(x) \vee \bigvee_{y\in J} \widehat{\mathrm{div}}(y) = \bigvee_{z\in I\cup J} \widehat{\mathrm{div}}(z),$$

$$\bigvee_{x\in I} \widehat{\mathrm{div}}(x) + \bigvee_{y\in J} \widehat{\mathrm{div}}(y) = \bigvee_{(x,y)\in I\times J} \widehat{\mathrm{div}}(xy),$$

where on the right hand sides we skip the repeats. In other words, the formal joins form a lattice monoid isomorphic to the lattice monoid of finite subsets of $F^\times$ with join being the sum and plus being the set of multiples of pair of elements from the corresponding sets.

The divisible hull of the Grothendieck group of the lattice monoid of formal joins in $F^\times$ satisfies the universal property of the universal lattice group of $F^\times$. Equivalently, one could also take the Grothendieck group of the lattice monoid of formal joins in $F^\times_{\mathbb{Q}} = F^\times \otimes \mathbb{Q}$. To shorten the notation, we thus refer to this divisible group lattice as the *universal lattice of* $F^\times$ and denote it by $\mathrm{ULat}_{\mathbb{Q}}(F)$.

In other words, elements of the universal lattice of $F^\times$ can be represented by differences

$$\bigvee_{x\in I} \widehat{\mathrm{div}}(x) - \bigvee_{y\in J} \widehat{\mathrm{div}}(y),$$

for finite $I, J \subset F_{\mathbb{Q}}^{\times}$. It follows that if t is a $\mathbb{Q}$-tropical polynomial and $a = (a_1, \dots, a_n)$ is a tuple of elements from $F^{\times}$, then $t(\widehat{\operatorname{div}}(a)) = t(\widehat{\operatorname{div}}(a_1), \dots, \widehat{\operatorname{div}}(a_n))$ can be written in the above form in $\mathrm{ULat}_{\mathbb{Q}}(F)$ with I and J consisting of multiples of rational powers of a_i's (and hence in any divisible group lattice with a homomorphism $\widehat{\operatorname{div}} : F^{\times} \to \Gamma$). More generally, using the same argument the following holds.

Lemma 1.4.5. Let $t(x)$ be a $\mathbb{Q}$-tropical polynomial with $x = (x_1, \dots, x_n)$. Then there are terms $\alpha_i(x) = \sum_{k=1}^{n} q_{ik} \cdot x_k$ and $\beta_j(x) = \sum_{k=1}^{n} r_{jk} \cdot x_k$ (with $q_{ik}, r_{jk} \in \mathbb{Q}$) over some finite index sets $i \in I, j \in J$ such that in any divisible group lattice Γ, the equality

$$t(x) = \max_{i \in I} \alpha_i(x) - \max_{j \in J} \beta_j(x) = \max_{i \in I} \min_{j \in J} (\alpha_i(x) - \beta_j(x))$$

holds, as functions on Γ^n.

Using the above observations, we define a pairing between Ω_F and $\mathrm{ULat}_{\mathbb{Q}}(F)$.

Definition 1.4.6. We define a pairing

$$\mathrm{ev} : \Omega_F \times \mathrm{ULat}_{\mathbb{Q}}(F) \to [-\infty, \infty],$$

by sending a pair $(v, t(\widehat{\operatorname{div}}(a)))$ to $v\big(t(\widehat{\operatorname{div}}(a))\big) := t(v(a))$, where a is a tuple. More precisely, this formula works in the case where v is a valuation. In general, we define it in the following way. Fix an element $v \in \Omega_F$ and consider a quotient group $F_{\mathbb{Q}}^{\times}/\{v = 0\}$, where we quotient by elements x with $v(x) = 0$ (note that v has a natural extension to $F_{\mathbb{Q}}^{\times}$). We define a linear order there by

$$\overline{x} \geq \overline{y} \iff v(x/y) \geq 0,$$

where $\overline{x}, \overline{y}$ are classes of x, y in the quotient. Now consider an element

$$\alpha = \bigvee_{x \in I} \widehat{\operatorname{div}}(x) - \bigvee_{y \in J} \widehat{\operatorname{div}}(y) \in \mathrm{ULat}_{\mathbb{Q}}(F).$$

Let $\overline{i}, \overline{j}$ be the classes of the biggest elements (with respect to $\geq$) of I, J respectively. We put $\mathrm{ev}(v, \alpha) = v(\alpha) := v(i/j) \in [-\infty, \infty]$ and leave to the reader the fact that it doesn't depend on the choice of a presentation of α in $\mathrm{ULat}_{\mathbb{Q}}(F)$. Equivalently, one could define

$$v(\alpha) = \max_{x \in I} \min_{y \in J} v(x/y).$$

Moreover, the same definition makes sense if v is an abstract pseudo-valuation on F.

Remark 1.4.7. Let t be a $\mathbb{Q}$-tropical polynomial and a be a tuple of elements from $F^\times$. From now on, when we write $t(v(a))$, we mean $v\big(t(\widehat{\mathrm{div}}(a))\big)$, for any abstract pseudo-valuation v on F or $v \in \Omega_F$. Note that the relation $t(v(a)) = v(b)$ is uniformly definable in any two sorted structure $(K, \Gamma \cup \{\pm\infty\}, v)$ where K is a field, Γ is a divisible ordered abelian group and $v : K \to \Gamma \cup \{\pm\infty\}$ is an abstract pseudo-valuation.

Lemma 1.4.8. The pairing $\mathrm{ev} : \Omega_F \times \mathrm{ULat}_{\mathbb{Q}}(F) \to [-\infty, \infty]$ has the following properties:

- $v(\alpha \vee \beta) = \max(v(\alpha), v(\beta))$,
- $v(\alpha + \beta) = v(\alpha) + v(\beta)$,

for all $v \in \Omega_F$, $\alpha, \beta \in \mathrm{ULat}_{\mathbb{Q}}(F)$, and in the second equation the right hand side is undefined if the terms are $-\infty, \infty$. Moreover, for $\alpha \in \mathrm{ULat}_{\mathbb{Q}}(F)$, the corresponding map $\mathrm{ev}_\alpha : \Omega_F \to [-\infty, \infty]$ is continuous.

Proof. Continuity follows from the continuity of max and min on $[-\infty, \infty]$. The rest is omitted. $\square$

Definition 1.4.9. By an *ideal* I in a group lattice Γ we mean a subgroup having the following property:

$$(\forall \alpha \in I)(|\alpha| \in I) \text{ and } (\forall \beta \in \Gamma)(\forall \alpha \in I)(0 \le \beta \le \alpha \implies \beta \in I).$$

Note that an ideal in a divisible lattice group is automatically itself divisible. The quotient group Γ/I admits a natural group lattice structure.

Now, we define the notion of L^1-lattices.

Definition 1.4.10. An L^1*-lattice* is a lattice group Γ which is also an $\mathbb{R}$-vector space, together with a norm $\|\cdot\|$ such that

- $(\Gamma, \|\cdot\|)$ is a Banach space.
- For all $\alpha \in \Gamma$, $\|\alpha\| = \|\alpha^+\| + \|\alpha^-\|$.

If $(\Gamma, \|\cdot\|)$ is an L^1-lattice, then we can define a continuous linear form $\int : \Gamma \to \mathbb{R}$ by

$$\int \alpha := \|\alpha^+\| - \|\alpha^-\|.$$

Let us mention that every L^1-lattice is isomorphic to a one of the form $L^1(\mu)$ for some measure μ on some measure space, see [55].

Definition 1.4.11. Let F be a field. A *lattice valuation* on F is the data of an L^1-lattice Γ and a map $\underline{v} : F \to \Gamma \cup \{\infty\}$ satisfying:

$$\underline{v}(xy) = \underline{v}(x)+\underline{v}(y), \quad \underline{v}(x+y) \geq \underline{v}(x)\wedge\underline{v}(y)+\underline{v}(2)\wedge 0, \quad \underline{v}(1) = 0, \quad \underline{v}(x) = \infty \Leftrightarrow x = 0,$$

and such that Γ is generated as an L^1-lattice by values of $\underline{v}$ (i.e., the divisible sublattice generated by those values is dense in Γ). Such a lattice valuation is called *global* if for all $x \in F^\times$, $\int \underline{v}(x) = 0$.

There is a natural notion of equivalence of two lattice valuations $\underline{v} : F \to \Gamma\cup\{\infty\}$, $\underline{v}' : F \to \Gamma' \cup \{\infty\}$, namely an isomorphism of L^1-lattices $\Gamma \to \Gamma'$ such that the following diagram commutes

$$\begin{tikzcd}
 & \Gamma' \cup \{\infty\} \\
F \arrow[ur, "\underline{v}'"] \arrow[r, "\underline{v}"] & \Gamma \cup \{\infty\} \arrow[u, "\simeq"']
\end{tikzcd}$$

Note that we could have not imposed the condition of being generated by values of $\underline{v}$, but we do it to have a bijective correspondence in Theorem 1.7.7.

1.5 Positivity

Here we study when the evaluation pairing is non-negative, for all $v \in \Omega_F$.

Definition 1.5.1. Consider the subset $I \subset \mathrm{ULat}_{\mathbb{Q}}(F)$ consisting of those α such that for all $v \in \Omega_F$ we have $\mathrm{ev}(v, \alpha) = 0$. It is an ideal by Lemma 1.4.8. We define a divisible group lattice $\mathrm{LDiv}_{\mathbb{Q}}(F) := \mathrm{ULat}_{\mathbb{Q}}(F)/I$ and call it the space of *lattice divisors on* F. In particular, $\alpha \in \mathrm{LDiv}_{\mathbb{Q}}(F)$ is *positive* (written $\alpha \geq 0$) if and only if, for all $v \in \Omega_F$ we have $\mathrm{ev}(v, \alpha) \geq 0$.

Remark 1.5.2. The ideal I is in general not trivial. For example, if $F = \mathbb{Q}$ one can consider an element

$$\alpha = \widehat{\mathrm{div}}(pq) \vee 0 - \widehat{\mathrm{div}}(p) \vee \widehat{\mathrm{div}}(q) \vee 0,$$

for some primes $p \neq q$. Then, by using Ostrowski's theorem and direct inspection, for any pseudo-valuation v on $\mathbb{Q}$ we have $v(\alpha) = 0$. However, if $\alpha = 0$ in $\mathrm{ULat}_{\mathbb{Q}}(\mathbb{Q})$, this would mean that there is a finite subset $A \subset \mathbb{Q}^\times$ such that

$$A \cup pqA = A \cup pA \cup qA,$$

but looking at the element of the highest Euclidean norm in A, we get that this is impossible.

Definition 1.5.3. We call lattice divisors of the form $\widehat{\operatorname{div}}(f)$, for $f \in F_{\mathbb{Q}}^{\times}$, *principal.* The quotient of $\mathrm{LDiv}_{\mathbb{Q}}(F)$ by the subspace of principal lattice divisors is called the space of *lattice line bundles on* F and is denoted by $\mathrm{LPic}_{\mathbb{Q}}(F)$. The *effective cone* of $\mathrm{LPic}_{\mathbb{Q}}(F)$ is defined as the set of classes of positive lattice divisors on F.

Definition 1.5.4. Consider pairs $(\phi, t) = (\phi(x, y), t(x))$, where x, y are tuples of variables, $\phi(x, y)$ is a quantifier-free ring language formula implying that none of variables of x are zero, and $t(x)$ is a $\mathbb{Q}$-tropical polynomial. We call such a pair *positive*, if

$$\mathrm{VF} \models (\forall a, b)(\phi(a, b) \implies t(v(a)) \geq 0).$$

By VF we mean a common theory of triples $(K, \Gamma \cup \{\pm\infty\}, v)$, where v is an abstract pseudo-valuation on a field K and Γ is the corresponding divisible ordered abelian group.

Below we prove a syntactic criterion on positivity of a lattice divisor. Existence of such a criterion also follows from Corollary 1.5.15 but we provide an elementary proof here, that only uses the compactness theorem.

Proposition 1.5.5. Let a be a tuple of elements in $F^{\times}$ and let $t(x)$ be a $\mathbb{Q}$-tropical polynomial of the corresponding arity. Then the following are equivalent:

1. $t(\widehat{\operatorname{div}}(a)) \geq 0$,
2. for every $n \in \mathbb{N}$ there is a quantifier-free ring formula $\phi_n(x, y_n)$ such that $F \models \phi_n(a, b_n)$ for some tuple b_n in F and $(\phi_n(x, y_n), t(x) - \frac{1}{n}v(2)^-)$ is a positive pair, where $v(2)^- = \min(v(2), 0)$.

Proof. First, note that by definition of Ω_F and continuity of the evaluation function at $t(\widehat{\operatorname{div}}(a))$, the first condition is equivalent to $(\forall v \in \Omega_F^{\circ})(t(v(a)) \geq 0)$.

Assume that we have positive pairs $(\phi_n(x, y_n), t(x) - \frac{1}{n}v(2)^-)$ and tuples b_n in F such that $F \models \phi_n(a, b_n)$ for each $n \in \mathbb{N}$ and fix a pseudo-valuation $v \in \Omega_F^{\circ}$. Note that $(F, v) \models \mathrm{VF}$, so $t(v(a)) - \frac{1}{n}v(2)^- \geq 0$ for each $n \in \mathbb{N}$. Thus in fact $t(v(a)) \geq 0$, as $v(2) \neq -\infty$. By the above observation, we get $t(\widehat{\operatorname{div}}(a)) \geq 0$.

On the other hand, assume that for some $n \in \mathbb{N}$ there is no positive pair $(\phi_n(x, y_n), t(x) - \frac{1}{n}v(2)^-)$ with quantifier-free $\phi_n(x, y_n)$ and no tuple b_n in F, such that $F \models \phi_n(a, b_n)$. Consider the following theory in the VF language with constants for F:

$$T = \mathrm{D}_{\mathrm{at}}(F) \cup \{v \text{ is an abstract pseudo-valuation}\} \cup \{t(v(a)) < \frac{1}{n}v(2)^-\},$$

where $\mathrm{D}_{\mathrm{at}}(F)$ is the quantifier-free diagram of F (so the quantifier-free theory of F in the language of rings enriched with constants from F). If it was inconsistent, there would be a quantifier-free formula $\psi(a,b)$ in $\mathrm{D}_{\mathrm{at}}(F)$ such that $\psi(a,b) \wedge (t(v(a)) < \frac{1}{n}v(2)^-)$ is inconsistent with VF. But then $(\psi(x,y), t(x) - \frac{1}{n}v(2)^-)$ would be a positive pair such that $F \models \psi(a,b)$ which can not happen by our assumption.

Let $(K, \Gamma \cup \{\pm\infty\}, v)$ be a model of T and consider the natural embedding $F \subset K$ with $\beta := t(v(a)) < \frac{1}{n}v(2)^-$. First consider the case $\beta = -\infty$. Write $t(\widehat{\mathrm{div}}(a)) = q \cdot (\bigwedge_i \widehat{\mathrm{div}}(x_i) - \bigwedge_j \widehat{\mathrm{div}}(y_j))$ for some $q \in \mathbb{Q}_{>0}$ and x_i, y_j from $F^\times$. Let $\mathcal{O} = \{x \in F : v(x) \neq -\infty\}$ be the valuation ring of $v|_F$. Choose i_0 and j_0 such that for all i and j we have $v(x_{i_0}/x_i) \geq 0, v(y_{j_0}/y_j) \geq 0$. In particular $x_{i_0}/x_i \in \mathcal{O}, y_{j_0}/y_j \in \mathcal{O}$, but $x_{i_0}/y_{j_0} \notin \mathcal{O}$. Let $u : F^\times \to F^\times/\mathcal{O}^\times =: \Sigma$ and note that it is a non-Archimedean abstract valuation. Let β' be the class of x_{i_0}/y_{j_0} in Σ. Define $\Delta \subset \Sigma$ as the smallest convex subgroup containing β' and let $\Delta_0 \subset \Delta$ be the biggest convex subgroup of Δ without β'. Then by truncating u to Δ and taking the quotient by Δ_0 we get an abstract pseudo-valuation on F where the value group Δ/Δ_0 can be embedded to $\mathbb{R}$. Hence, there exists a pseudo-valuation u' on F with $t(u'(a)) < 0$. That is because the relations $u'(x_{i_0}/x_i) \geq 0, u'(y_{j_0}/y_j) \geq 0$ and $u'(x_{i_0}/y_{j_0}) < 0$ hold by construction. This finishes the proof in this case.

Now assume that $\beta \neq -\infty$. Let $\Delta \subset \Gamma$ be the smallest convex subgroup containing β and let $\Delta_0 \subset \Delta$ be the biggest convex subgroup of Δ with $\beta \notin \Delta_0$. If we choose an embedding $\Delta/\Delta_0 \subset \mathbb{R}$, we can define a function w on F by sending a to $v(a) \bmod \Delta_0$ if $v(a) \in \Delta$, to ∞ if $v(a) > \Delta$, and to $-\infty$ if $v(a) < \Delta$. Note that $v(2) \not< \Delta$ because $\beta < \frac{1}{n}v(2)^-$, so by Lemma 1.2.2 the function w is a pseudo-valuation on F. Moreover, by construction $t(w(a)) = \overline{\beta} < 0$, where $\overline{\beta}$ is the class of β modulo Δ_0 in $\mathbb{R}$. Hence, $t(\widehat{\mathrm{div}}(a)) \not\geq 0$, which finishes the proof. $\square$

Let us consider a relative situation where F is a field extending k. Using Lemma 1.2.14 we get the following corollaries.

Corollary 1.5.6. The natural morphism $\mathrm{ULat}_{\mathbb{Q}}(k) \to \mathrm{ULat}_{\mathbb{Q}}(F)$ induces an embedding $\mathrm{LDiv}_{\mathbb{Q}}(k) \subset \mathrm{LDiv}_{\mathbb{Q}}(F)$ respecting the non-negative cone.

Corollary 1.5.7. Let A be a domain with $\mathrm{Frac}(A) = F$ and fix an absolute value $|\cdot|$ on $\kappa(p)$ for a point $p \in \mathrm{Spec}(A)$. There is a pseudo-valuation v on F such that for all $a \in A$ we have $v(a) = -\log|a \bmod p|$.

Now we prove a relative variant of Proposition 1.5.5.

Proposition 1.5.8. Let k be a subfield of F and $\alpha \in \mathrm{LDiv}_{\mathbb{Q}}(F)$. The following are equivalent:

1. for every $v \in \Omega_F$ that restricts to the trivial valuation on k, we have $v(\alpha) \leq 0$;
2. there are finitely many $a_1, \ldots, a_n \in k^\times$ such that $\alpha \leq \bigvee_{i=1}^n \widehat{\mathrm{div}}(a_i)$.

Proof. The implication from (2) to (1) follows from Lemma 1.4.8. To prove the other one, assume for a contradiction that one cannot find finitely many elements in k that satisfy (2). This means that for every $a_1, \ldots, a_n \in k^\times$, there is a pseudo-valuation $v \in \Omega_F$ that satisfies $v(\alpha) > \max_i v(a_i)$. Consider the following first order theory:

$$T = \mathrm{D}_{\mathrm{at}}(F) \cup \{v \text{ is an abstract pseudo-valuation}\} \cup \{v(\alpha) > v(b) : b \in k^\times\}.$$

This theory is consistent by the assumption, so there is an abstract pseudo-valuation $v_0 : F \to \Gamma \cup \{\pm\infty\}$ such that for all $b \in k^\times$ we have $v_0(\alpha) > v_0(b)$. Now we proceed as in the proof of Proposition 1.5.5, to get a pseudo-valuation v_1 on F that restricts to the trivial valuation on k and satisfies $v_1(\alpha) > 0$, which contradicts (1). □

Corollary 1.5.9. Let $k \subset F$ be a finitely generated extension. Then the ideal

$$I_k = \{\alpha \in \mathrm{LDiv}_{\mathbb{Q}}(F) : \forall v \in \mathrm{Val}_{F/k} \ \ v(\alpha) = 0\}$$

is the smallest ideal containing $\mathrm{LDiv}_{\mathbb{Q}}(k)$ in $\mathrm{LDiv}_{\mathbb{Q}}(F)$.

Here, the embedding $\mathrm{LDiv}_{\mathbb{Q}}(k) \subset \mathrm{LDiv}_{\mathbb{Q}}(F)$ is the one from Corollary 1.5.6 and by $\mathrm{Val}_{F/k}$ we denote valuations on F that are trivial on k. Let us also denote by $\Omega_{F/k}$ pseudo-valuations on F that restrict to the trivial valuation on k.

Proof. By Corollary 1.2.11 we can replace $\mathrm{Val}_{F/k}$ in the definition of I_k by $\Omega_{F/k}$. Then the statement follows from Proposition 1.5.8. □

Next, we want describe the set of positive lattice divisors in more detail, see Corollary 1.5.15. Let us start with the following lemma.

Lemma 1.5.10. For every tuple $a = (a_1, \ldots, a_n) \in (F^\times)^n$ and $\varepsilon \in \mathbb{R}_{>0}$, the following are equivalent:

(i) For every pseudo-valuation v on F, $v\left(\bigwedge_{i=1}^n \widehat{\mathrm{div}}(a_i)\right) \leq 0$ if v is non-Archimedean, and $v\left(\bigwedge_{i=1}^n \widehat{\mathrm{div}}(a_i)\right) < -\varepsilon v(2)$ if v is Archimedean.

(ii) There exists a family $(m_s)_{s\in\mathbb{N}^n}$ of integers with $m_s = 0$ except for a finite number of indices s, such that

$$\sum_{s\in\mathbb{N}^n} |m_s| 2^{-\varepsilon|s|} < 1,$$

and

$$\sum_{s\in\mathbb{N}^n} m_s a^s = 1.$$

Proof. First assume that (ii) is satisfied, and that there is a $v \in \Omega_F$ such that $v\left(\bigwedge_{i=1}^{n} \widehat{\operatorname{div}}(a_i)\right) > -\varepsilon v(2)^-$. Let $(m_s)_{s\in\mathbb{N}^n}$ be a family satisfying (ii). Then:

- If v is non-Archimedean, we have $\min_{i\leq n} v(a_i) > 0$, so $v(a_i) > 0$ for all $i \leq n$. Thus, the equality $\sum_{s\in\mathbb{N}^n} m_s a^s = 1$ implies $v(1) > 0$.
- If v is Archimedean, we may, up to multiplication by a positive constant, write $v = -\log\|\cdot\|$, where $\|\cdot\|$ is a pseudo-absolute value on F with $\|2\| = 2$. We then have $\min_{i\leq n} v(a_i) > -\varepsilon v(2) = \varepsilon\log(2)$, so $\|a_i\| < 2^{-\varepsilon}$. Thus,

$$1 = \|1\| = \left\|\sum_{s\in\mathbb{N}^n} m_s a^s\right\| \leq \sum_{i=1}^{n} |m_s| 2^{-\varepsilon|s|} < 1.$$

Either way, we obtain a contradiction, which shows that (ii) implies (i).

Now, assume that (ii) is not satisfied. Consider the sub-norm $\|\cdot\|_\varepsilon$ defined on $\mathbb{Z}[X_1, \dots, X_n]$ by

$$\left\|\sum_{s\in\mathbb{Z}^n} a_s X^s\right\|_\varepsilon = \sum_{s\in\mathbb{N}^n} |a_s| 2^{-\varepsilon|s|},$$

and the ring homomorphism $f : \mathbb{Z}[X_1, \dots, X_n] \to F$ sending X_i to a_i for all $i \leq n$. Then, $\|\cdot\|_\varepsilon$ and f satisfy the conditions of Lemma 1.3.4, hence there exists a pseudo-absolute value $\|\cdot\|$ of F such that for all $P \in \mathbb{Z}[X_1, \dots, X_n]$, $\|f(P)\| \leq \|P\|_\varepsilon$. In particular $\|a_i\| \leq 2^{-\varepsilon}$ for all $i \leq n$.

Assume that $\|\cdot\|$ is a non-trivial pseudo-absolute value. Then by Remark 1.2.3, $v = -\log\|\cdot\|$ is a pseudo-valuation on F. But v satisfies $\min_{1\leq i\leq n} v(a_i) \geq \varepsilon\log(2)$, which contradicts (i) since, if v is Archimedean, then $v(2) = -\log\|2\| \geq -\log(2)$.

On the other hand, if $\|\cdot\|$ is a trivial pseudo-absolute value, then $\|a_i\| \leq 2^{-\varepsilon} < 1$ implies $\|a_i\| = 0$ for all $1 \leq i \leq n$. Let $\mathcal{O} = \{x \in F : \|x\| < \infty\}$ be the valuation ring associated to $\|\cdot\|$, and let $v' : F \to \Sigma \cup \{+\infty\}$ the corresponding abstract valuation, where $\Sigma := F^\times/\mathcal{O}^\times$. Then $\beta := \min_{1\leq i\leq n} v'(a_i) > 0$ in Σ. Let $\Delta \subset \Sigma$ be the smallest convex subgroup of Σ containing β, and Δ_0 be the biggest convex subgroup not containing β. Then, by truncating u to Δ and taking the quotient by Δ_0, we obtain an abstract non-Archimedean pseudo-valuation v whose value group Δ/Δ_0 can be embedded into $\mathbb{R}$. Hence, v is a non-Archimedean pseudo-valuation on F and by construction satisfies $\min_{1\leq i\leq n} v(a_i) > 0$, which contradicts (i). $\square$

Definition 1.5.11. For every $\varepsilon \in \mathbb{R}_{>0}$, let $\Gamma_\varepsilon \subseteq \mathrm{ULat}_{\mathbb{Q}}(F)$ be the set of elements of the form

$$\bigwedge_{i=1}^{p} \widehat{\mathrm{div}}\left(\sum_{j=1}^{q} m_{i,j}a_j\right) - \bigwedge_{j=1}^{q} \widehat{\mathrm{div}}(a_j),$$

where $a_1, \dots, a_q \in F^\times$, $p, q \in \mathbb{N}^*$, and $(m_{i,j})_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} \in \mathbb{Z}^{pq}$ is such that for all $i \leq p$, $\sum\limits_{j=1}^{q} |m_{i,j}| < 2^\varepsilon$.

Remark 1.5.12. For all $\varepsilon > 0$, Γ_ε is closed under $\wedge$. Indeed, if two elements of $\mathrm{ULat}_{\mathbb{Q}}(F)$ are written as $\bigwedge\limits_{1\leq i\leq m} \widehat{\mathrm{div}}(b_i) - \bigwedge\limits_{1\leq i\leq n} \widehat{\mathrm{div}}(a_i)$ and $\bigwedge\limits_{1\leq i\leq p} \widehat{\mathrm{div}}(b'_i) - \bigwedge\limits_{1\leq i\leq q} \widehat{\mathrm{div}}(a'_i)$ respectively, then their meet is equal to:

$$\left(\bigwedge_{\substack{1\leq i\leq m\\ 1\leq j\leq q}} \widehat{\mathrm{div}}(b_i a'_j)\right) \wedge \left(\bigwedge_{\substack{1\leq i\leq n\\ 1\leq j\leq p}} \widehat{\mathrm{div}}(b'_i a_i)\right) - \bigwedge_{\substack{1\leq i\leq p\\ 1\leq j\leq q}} \widehat{\mathrm{div}}(a_i a'_j).$$

Thus, if both elements of $\mathrm{ULat}_{\mathbb{Q}}(F)$ are of the form described in Definition 1.5.11, then their meet is also of this form.

Lemma 1.5.13. For any $\alpha \in \mathrm{ULat}_{\mathbb{Q}}(F)$ and $\varepsilon \in \mathbb{R}_{>0}$, the following are equivalent:

(i) For every pseudo-valuation v on F, $v(\alpha) \geq 0$ if v is non-Archimedean, and $v(\alpha) > \varepsilon v(2)$ if v is Archimedean.

(ii) There exists an integer $m \in \mathbb{N}$ such that $m\alpha \in \Gamma_{m\varepsilon}$.

Proof. (ii) $\Rightarrow$ (i) Assume that $m\alpha \in \Gamma_{m\varepsilon}$ for some $m \in \mathbb{N}$. Then, $m\alpha$ can be written as $\bigwedge_{i=1}^{p} \widehat{\mathrm{div}}(b_i) - \bigwedge_{j=1}^{q} \widehat{\mathrm{div}}(a_j)$ where $a_1, \dots, a_q, b_1, \dots, b_p \in F^\times$, and for all $i \leq p$, there is a tuple $(m_{i,1}, \dots, m_{i,q}) \in \mathbb{Z}^q$ such that $b_i = \sum\limits_{j=1}^{q} m_{i,j}a_j$, and $\sum_{j=1}^{q} |m_{i,j}| < 2^{m\varepsilon}$. Then, for every pseudo-valuation v and $i \leq p$:

- if v is non-Archimedean, $\min\limits_{j=1}^{q} v(a_j/b_i) \leq 0$,
- if v is Archimedean, $\min\limits_{j=1}^{q} v(a_j/b_i) < -m\varepsilon v(2)$.

Using the fact that $m\cdot v(\alpha) = v(m\alpha) = \min_{i=1}^{p} \max_{j=1}^{q} v(b_i/a_j)$, we conclude that α satisfies (i).

(i) $\Rightarrow$ (ii) Let $\alpha \in \mathrm{ULat}_{\mathbb{Q}}(F)$ and assume that α satisfies (i). Since any element of $\mathrm{ULat}_{\mathbb{Q}}(F)$ can be written as a meet of elements of the form $-\bigwedge_{i=1}^{n} \widehat{\mathrm{div}}(a_i)$, with $a = (a_1, \ldots, a_n) \in (F^\times)^n$, and by Remark 1.5.12, it is enough to prove (i) $\Rightarrow$ (ii) for elements of this form.

Let $\|\cdot\|_\varepsilon$ be the ring norm on $\mathbb{Z}[X_1, \ldots, X_n]$ defined by

$$\left\| \sum_{s\in\mathbb{N}^n} m_s X^s \right\| = \sum_{s\in\mathbb{N}^n} |m_s| 2^{-\varepsilon|s|}.$$

Then, by Lemma 1.5.10, there exists a polynomial $P \in \mathbb{Z}[X_1, \ldots, X_n]$ with $\|P\|_\varepsilon < 1$, such that $P(a) = 1$.

Claim 1.5.14. We have $\alpha \in \Gamma_\delta$, with $2^\delta = \deg P \left(\frac{4^\varepsilon}{2^\varepsilon - 1} + n\right)$.

Proof. We define the (finite) subset E of $\mathbb{Z}[X_1, \ldots, X_n]$ of homogeneous polynomials H of degree $\deg H \leq \deg P$, such that $\|H\|_\varepsilon \leq 1$ and $H(a) \neq 0$. Then, a may be written as

$$\bigwedge_{H\in E} \widehat{\mathrm{div}}(H(a)) - \bigwedge_{\substack{1\leq i\leq n \\ H\in E}} \widehat{\mathrm{div}}(a_i H(a)).$$

We now show that $\alpha \in \Gamma_\delta$ by writing each $H \in E$ as a sum of at most 2^δ terms of the form $a_i H'$ with $1 \leq i \leq n$, $H' \in E$.

- If $\deg H \geq 1$, then we may write $H = \sum_{i=1}^{n} X_i H_i$ with $H_1, \ldots, H_n$ homogeneous polynomials of degree $\deg H - 1$ such that $\|H\|_\varepsilon = 2^{-\varepsilon} \sum_{i=1}^{n} \|H_i\|_\varepsilon$. Then, for each $1 \leq i \leq n$, the absolute values of the coefficients of H_i sum to $2^{\varepsilon(\deg P - 1)} \|H_i\|_\varepsilon$. By splitting the coefficients adequately, we can write each H_i as a sum of $N_i = \left\lceil \frac{2^{\varepsilon(\deg H-1)}\|H_i\|_\varepsilon}{t} \right\rceil$ (homogeneous) polynomials such that the absolute values of the coefficients in each of them sum to at most $t = \lfloor 2^{\varepsilon(\deg H - 1)} \rfloor$. Thus, H_i may be written as a sum of N_i homogeneous polynomials of degree $\deg H - 1$ and of ε-norm at most $t 2^{\varepsilon(1-\deg H)} \leq 1$. Hence, $H(a)$ is a sum of at most $N = N_1 + \ldots + N_n$ terms of the required form, with

$$N \leq \sum_{i=1}^{n} \left(1 + \frac{2^{\varepsilon(\deg H-1)}\|H_i\|_\varepsilon}{t}\right) = n + \frac{2^{\varepsilon \deg H}\|H\|_\varepsilon}{t} \leq n + \frac{4^\varepsilon}{2^\varepsilon - 1},$$

  since $\|H\|_\varepsilon \leq 1$, and $t \geq 2^{\varepsilon(\deg H - 2)}(2^\varepsilon - 1)$.

- If $\deg H = 0$, then up to a sign change, $H = 1$, and by assumption

  $$H(a) = 1 = P(a) = \sum_{d=1}^{\deg P} P_d(a),$$

  where $P_1, \ldots, P_{\deg P}$ are the homogeneous components of P. As seen above, each P_d may be written as a sum of at most $n + \frac{4^\varepsilon}{2^\varepsilon - 1}$ terms of the required form, which concludes the proof of Claim 1.5.14.

□

Now, by Lemma 1.5.10, α also satisfies (i) for some $0 < \varepsilon' < \varepsilon$. But, for all $m \in \mathbb{N}$, we have

$$m\alpha = -\bigwedge_{P} \widehat{\mathrm{div}}(P(a)),$$

where the meet ranges over all homogeneous monomials of degree m in $X_1, \ldots, X_n$, whose number is $\binom{n+m-1}{n-1}$. Thus, claim 1.5.14 applies, replacing α by $m\alpha$, ε by $m\varepsilon'$, P by P^m and n by $\binom{n+m-1}{n-1}$, showing that $m\alpha \in \Gamma_\delta$ with $2^\delta = m \deg P \left(\frac{4^{m\varepsilon'}}{2^{m\varepsilon'} - 1} + \binom{n+m-1}{n-1} \right)$. Since $\varepsilon' < \varepsilon$, for large enough m, we have $\delta < m\varepsilon$, which concludes the proof. □

Corollary 1.5.15. For each $\alpha \in \mathrm{ULat}_{\mathbb{Q}}(F)$, the following conditions are equivalent:

(i) For every pseudo-valuation v on F, $v(\alpha) \geq 0$.

(ii) For every abstract valuation v on F, $v(\alpha) \geq 0$.

(iii) For every abstract pseudo-valuation v on F, $v(\alpha) \geq 0$.

(iv) For every $v \in \Omega_F$, $v(\alpha) \geq 0$.

(v) For every $\varepsilon \in \mathbb{R}_{>0}$, there is some $m \in \mathbb{N}$ such that $m\alpha \in \Gamma_{m\varepsilon}$.

Proof. It is clear that (iii) ⇒ (ii) ⇒ (i), and (i) ⇔ (iv) since the set of pseudo-valuations is a dense subset of Ω_F. Additionally, (i) ⇔ (v) is a consequence of Lemma 1.5.13. We then prove (v) ⇒ (iii) as in the proof of (ii) ⇒ (i) in Lemma 1.5.13. □

If the ambient field is countable, positivity of an element $\alpha \in \mathrm{LDiv}_{\mathbb{Q}}(F)$ can be checked by pairing with not necessarily all pseudo-valuations.

Lemma 1.5.16. Assume that F is countable and let $\alpha = t(\widehat{\mathrm{div}}(a)) \in \mathrm{LDiv}_{\mathbb{Q}}(F)$. Then $\alpha \geq 0$ if and only if for all $v \in \mathrm{Val}_F$ we have $t(v(a)) \geq 0$.

Proof. By Lemma 1.4.8 the evaluation function $v \mapsto t(\widehat{\mathrm{div}}(a))$ is continuous. Thus the result follows from Corollary 1.2.9 and Corollary 1.2.11. $\square$

Corollary 1.5.17. Let F be a countable field of characteristic zero. Then for $\alpha \in \mathrm{LDiv}_{\mathbb{Q}}(F)$ we have $\alpha = 0$ if and only if for all $v \in \mathrm{Val}_F$ we have $v(\alpha) = 0$.

1.6 Measures

Recall that for a locally compact Hausdorff topological space X, positive functionals on the space of compactly supported continuous functions on X correspond to Radon measures on X (this correspondence sometimes serves as a definition of a Radon measure).

Definition 1.6.1. Let $\alpha \in \mathrm{LDiv}_{\mathbb{Q}}(F)$ be a non-zero, positive ($\alpha \geq 0$) element. We define a compact space

$$\Omega_F(\alpha) = \{v \in \Omega_F : v(\alpha) = 1\},$$

and its projection to $\overline{\Omega}_F$ denoted $\overline{\Omega}_F(\alpha)$. Moreover, let

$$\langle \alpha \rangle := \{\beta \in \mathrm{LDiv}_{\mathbb{Q}}(F) : (\exists n \in \mathbb{N})(|\beta| \leq n \cdot \alpha)\}.$$

Note that $\langle \alpha \rangle$ is a sublattice of $\mathrm{LDiv}_{\mathbb{Q}}(F)$. We call it the set of elements *dominated by* α.

Construction 1.6.2. Fix $\alpha \geq 0$ from $\mathrm{LDiv}_{\mathbb{Q}}(F)$ and consider the restricted evaluation product

$$\Omega_F(\alpha) \times \langle \alpha \rangle \to \mathbb{R}.$$

It defines a natural map

$$i : \langle \alpha \rangle \to C(\Omega_F(\alpha)),$$

given by $\beta \mapsto (v \mapsto v(\beta))$. Note that by max-closed/tropical Stone-Weierstrass theorem (see e.g. [40, Theorem 7.29]), the image of i is a dense subset with respect to the uniform convergence topology.

Definition 1.6.3. A *positive functional on* F is a $\mathbb{Q}$-linear map $l : \mathrm{LDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$ which takes non-negative lattice divisors to non-negative numbers. We call l a *global functional*, if additionally it takes principal lattice divisors (see Definition 1.5.3) to zero. Note that it equivalently can be described as a linear map $l : \mathrm{LPic}_{\mathbb{Q}}(F) \to \mathbb{R}$ non-negative on the effective cone. Furthermore, if F is of characteristic zero we say that l is *normalised* if $l(\widehat{\mathrm{div}}(2)^+) = \log 2$.

Now, the following corollary follows from applying the Riesz-Markov-Kakutani representation theorem to the pairing from Construction 1.6.2.

Corollary 1.6.4. Let l be a positive functional on F. Then for every non-zero $\alpha \geq 0$ in $\mathrm{LDiv}_{\mathbb{Q}}(F)$, there is a unique finite Radon measure μ_α on $\Omega_F(\alpha)$ such that for all $\beta \in \langle \alpha \rangle$ we have

$$l(\beta) = \int_{\Omega_F(\alpha)} v(\beta) d\mu_\alpha(v).$$

We also use the notation μ_α for the pushforward of this measure to $\overline{\Omega}_F(\alpha)$.

Hence, a positive functional l on F induces a family of measures μ_α on $\overline{\Omega}_F(\alpha)$ for non-zero, positive $\alpha \in \mathrm{LDiv}_{\mathbb{Q}}(F)$.

Lemma 1.6.5. For any non-zero, positive $\alpha, \gamma \in \mathrm{LDiv}_{\mathbb{Q}}(F)$ the local measures μ_α, μ_γ associated to a positive functional l on F satisfy

$$\frac{d\mu_\alpha}{d\mu_\gamma} = \frac{v(\alpha)}{v(\gamma)},$$

when restricted to $\overline{\Omega}_F(\alpha) \cap \overline{\Omega}_F(\gamma)$, where $\frac{v(\alpha)}{v(\gamma)}$ is a function taking the class of v to $\frac{v(\alpha)}{v(\gamma)}$ (which does not depend on the representative).

Proof. First, note that we have

$$l(\gamma) \geq l(\gamma \wedge n\alpha) = \int_{\Omega_F(\alpha)} v(\gamma \wedge n\alpha) d\mu_\alpha(v) \geq n \cdot \mu_\alpha(\{v \in \Omega_F(\alpha) : v(\gamma) = \infty\}).$$

Since n is an arbitrary natural number here, we get that

$$\mu_\alpha(\{v \in \Omega_F(\alpha) : v(\gamma) = \infty\}) = 0.$$

Thus, for $\beta \in \langle \alpha \rangle \cap \langle \gamma \rangle$ we can write

$$l(\beta) = \int_{\Omega_F(\alpha)} v(\beta) d\mu_\alpha(v) = \int_{\overline{\Omega}_F(\alpha) \cap \overline{\Omega}_F(\gamma)} \frac{v(\beta)}{v(\alpha)} d\mu_\alpha(v).$$

Symmetrically, we have

$$l(\beta) = \int_{\Omega_F(\gamma)} v(\beta) d\mu_\gamma(v) = \int_{\overline{\Omega}_F(\alpha) \cap \overline{\Omega}_F(\gamma)} \frac{v(\beta)}{v(\gamma)} d\mu_\gamma(v).$$

This finishes the proof. □

For $a \in F^\times$ we denote by μ_a the measure on $\overline{\Omega}_F(a)$ coming from $\alpha = \widehat{\mathrm{div}}(a)^+$ (it depends on the choice of a positive functional on F). Note that with this definition $\overline{\Omega}_F(a) = \overline{\Omega}_F(\alpha)$. According to the following definition, a positive functional on F induces a family of local measures on F. Later we will see that every such family determines uniquely a positive functional on F.

Definition 1.6.6. By a *family of local measures on* F we mean a family of finite Radon measures μ_a on $\overline{\Omega}_F(a)$ for $a \in F^\times$, such that for any $a, b \in F^\times$ we have

$$\frac{d\mu_a}{d\mu_b} = \frac{v(a)}{v(b)},$$

on $\overline{\Omega}_F(a) \cap \overline{\Omega}_F(b)$ and $\mu_a\{v \in \overline{\Omega}_F(a) : v(b) = \infty\} = 0$. We call such a family *global*, if in addition, for every $a \in F^\times$, we have

$$\mu_a(\overline{\Omega}_F(a)) = \mu_{a^{-1}}(\overline{\Omega}_F(a^{-1})).$$

Now we pass to a global description of GVF functionals.

Definition 1.6.7. An *admissible measure on* F is a Radon measure μ on Ω_F which satisfies the following conditions.

1. For every $a \in F^\times$ the function $\mathrm{ev}_a : v \mapsto v(a)$ is μ-integrable.
2. There is a family of Borel sections $s_a : \overline{\Omega}_F(a) \to \pi^{-1}(\Omega_F(a))$ of π, such that μ restricted to $\pi^{-1}(\overline{\Omega}_F(a))$ is equal to $s_{a*}\mu_a$ for $\mu_a := \pi_*(\mu|_{\pi^{-1}(\Omega_F(a))})$.

Note that the Borel sections s_a are uniquely determined, up to a set of μ_a-measure zero. Moreover, we call μ a *global measure on* F (or a GVF measure), if in addition for all $a \in F^\times$ the product formula holds:

$$\int_{\Omega_F} v(a) d\mu(v) = 0.$$

Remark 1.6.8. Note that the condition (1) in the above definition implies in particular that $\mu\{v(2) = -\infty\} = 0$, hence μ can be seen as a measure on the set of pseudo-valuations Ω_F°. We choose to work with the bigger space Ω_F to ensure that the sets $\Omega_F(a)$ are compact.

Definition 1.6.9. Let μ, μ' be two admissible measures on F. We say that μ *can be renormalised to* μ', if for every $a \in F^\times$ and Borel sections $s_a, s'_a : \overline{\Omega}_F(a) \to \pi^{-1}(\overline{\Omega}_F(a))$ associated to measures μ, μ' respectively, the measures μ_a, μ'_a obtained as pushforwards of μ, μ' through $\pi : \Omega_F(a) \to \overline{\Omega}_F(a)$ respectively, satisfy:

$$\frac{d\mu_a}{d\mu'_a} = \frac{s'_a}{s_a}.$$

Lemma 1.6.10. Renormalisation is an equivalence relation among admissible measures on F that respects being a global measure.

Proof. The first part is clear. The second part follows from Lemma 1.7.2. $\square$

We now describe how to construct an admissible measure on F from a family of local measures.

Construction 1.6.11. Fix a family of local measures $(\mu_a)_{a \in F^\times}$. Well order $F^\times$ as $(a_\alpha)_{\alpha<\kappa}$ for some ordinal κ. We inductively build sections $s_\alpha : \overline{\Omega}_F(a_\alpha) \to \Omega_F(a_\alpha)$ such that for $\alpha, \beta < \kappa$ we have

$$s_\alpha|_{\overline{\Omega}_F(a_\alpha)\cap\overline{\Omega}_F(a_\beta)} = s_\beta|_{\overline{\Omega}_F(a_\alpha)\cap\overline{\Omega}_F(a_\beta)}$$

almost everywhere with respect to μ_{a_α} or μ_{a_β} equivalently. Assume that we have already built such s_γ for $\gamma < \alpha$ and let $a = a_\alpha$. Pick a countable sequence of $\gamma_n < \alpha$ such that for $a_n = a_{\gamma_n}$ the set $\bigcup_n \overline{\Omega}_F(a) \cap \overline{\Omega}_F(a_n)$ has maximal μ_a-measure. Define $s = s_\alpha : \overline{\Omega}_F(a) \to \Omega_F(a)$, at least up to μ_a-measure zero, as $s_n = s_{\gamma_n}$ on $\overline{\Omega}_F(a)\cap\overline{\Omega}_F(a_n)$ (this makes sense by the inductive assumption about s_n's) and on the rest define it to be the inverse of the projection $\pi : \Omega_F(a) \to \overline{\Omega}_F(a)$. Let $\nu_\alpha := g_\alpha \cdot s_{\alpha*}\mu_\alpha$, where $g_\alpha(v) = 1/s_\alpha(\pi(v))(a_\alpha)$. Next, define a measure μ on Ω_F to be the unique Radon measure which coincides with ν_α on $\pi^{-1}(\overline{\Omega}_F(a_\alpha))$. This makes sense, as every compact subset of Ω_F is contained in a finite union of $\pi^{-1}(\overline{\Omega}_F(a))$'s and the choice of g_α's ensures that ν_α's agree on overlaps because of the following calculation (on $\pi^{-1}(\overline{\Omega}_F(a_\alpha)) \cap \pi^{-1}(\overline{\Omega}_F(a_\beta))$):

$$\nu_\alpha = g_\alpha \cdot s_{\alpha*}\mu_\alpha = g_\alpha \cdot s_{\alpha*}\left(\frac{v(a_\alpha)}{v(a_\beta)}\mu_\beta\right) = \frac{1}{s_\alpha(\pi(v))(a_\alpha)} \cdot \frac{s_\alpha(\pi(v))(a_\alpha)}{s_\alpha(\pi(v))(a_\beta)} \cdot s_{\alpha*}\mu_\beta$$

$$\frac{1}{s_\alpha(\pi(v))(a_\beta)} \cdot s_{\alpha*}\mu_\beta = \frac{1}{s_\beta(\pi(v))(a_\beta)} \cdot s_{\beta*}\mu_\beta = g_\beta \cdot s_{\beta*}\mu_\beta = \nu_\beta.$$

Since

$$\nu_\alpha = g_\alpha \cdot s_{\alpha*}\mu_\alpha = s_{\alpha*}\left(\frac{1}{s_\alpha(\overline{v})(a_\alpha)}\mu_\alpha\right),$$

where $\overline{v} \in \overline{\Omega}_F(a_\alpha)$, the measure μ satisfies the second condition in the definition of an admissible measure. Moreover, by construction for every $a = a_\alpha \in F^\times$ (with $s_a = s_\alpha, \nu_a = \nu_\alpha$) we have

$$\int_{\Omega_F} v(a)^+ d\mu(v) = \int_{\pi^{-1}(\overline{\Omega}_F(a))} v(a) d\mu(v) = \int_{\pi^{-1}(\overline{\Omega}_F(a))} v(a) d\nu_a(v)$$

$$= \int_{\pi^{-1}(\overline{\Omega}_F(a))} \frac{v(a)}{s_a(\pi(v))(a)} ds_{a*}\mu_a(v) = \int_{\overline{\Omega}_F(a)} d\mu_a = \mu_a(\overline{\Omega}_F(a)) < \infty.$$

Thus, μ also satisfies the first condition from the definition of an admissible measure, and a similar calculation for $v(a)^-$ shows that a global family of local measures induces a global measure on F.

Remark 1.6.12. Note that in the above construction, if we start with $a_0 = 1/2$ in the characteristic zero case, then we can assume that the obtained measure μ on Ω_F restricted to Archimedean $v \in \Omega_F$ is concentrated on v's with $v(2) = -\log 2$. In particular, if F is a countable field of characteristic zero, then by Lemma 1.2.3 we can assume that the measure comes from a pushforward of some measure via $-\log : M_F \to \mathrm{Val}_F \subset \Omega_F$. More precisely, we can even assume that the measure on M_F is concentrated on the union of Non-Archimedean norms and Archimedean norms extending $|\cdot|_\infty$ on $\mathbb{Q}$.

1.7 Heights

In this section we construct the correspondence between structures we have introduced. We start by defining globally valued fields via local terms.

Definition 1.7.1. A family of functions $R_t : (F^\times)^{n_t} \to \mathbb{R}$ indexed by $\mathbb{Q}$-tropical polynomials t (of arity n_t respectively), that are also denoted by

$$R_t(a) =: \int t(v(a))dv,$$

is called a family of *local terms* on F if it satisfies the following axioms:

1. R_t are compatible with permutations of variables and dummy variables.
2. (Linearity) $R_{t_1+t_2} = R_{t_1} + R_{t_2}, R_{qt} = qR_t$ for all $\mathbb{Q}$-tropical polynomials t_1, t_2, t and rational numbers q.
3. (Local-global positivity) If (ϕ, t) is a positive pair (see Definition 1.5.4), then

$$(\forall a)(\phi(a) \implies R_t(a) \geq 0).$$

We use the name *product formula field* to call a field with local terms that satisfy the following axiom:

(4) (Product formula) $\int v(x)dv = 0$ for all $x \in F^\times$.

Fields with local terms (resp. product formula fields) form a category where morphisms are embeddings of fields such that the predicates R_t of the bigger field restrict to the ones of the smaller one. We call local terms *trivial*, if all predicates R_t are zero.

Lemma 1.7.2. An admissible measure μ on F defines local terms on F via

$$R_t(a) := \int_{\Omega_F} t(v(a))d\mu(v).$$

Moreover, these only depend on the renormalisation class of μ.

Proof. Note that the map $v \mapsto t(v(a))$ can be bounded by a multiple of maximum of valuations of finitely many elements from $F^\times$, by Lemma 1.4.5. Hence, the integral will be finite by the first condition in the definition of admissibility.

We check the local-global positivity axiom. If (ϕ, t) is a positive pair and $F \models \phi(a)$, then $v \mapsto t(v(a))$ in non-negative on Ω_F, so the integral will be non-negative. We omit checking the other axioms of multiply valued field structures.

Assume that μ, μ' are in the same renormalisation class and keep the notation from Definition 1.6.9. Let f be a function supported on $\pi^{-1}(\overline{\Omega}_F(a))$ for some $a \in F^\times$ which is homogeneous with respect to the $\mathbb{R}_{>0}$-action, i.e., $f(rv) = rf(v)$ for $r \in \mathbb{R}_{>0}$. Then we have

$$\int_{\Omega_F} f(v)d\mu(v) = \int_{\overline{\Omega}_F(a)} f(s_a(\overline{v}))d\mu_a(\overline{v}) = \int_{\overline{\Omega}_F(a)} f(s_a(\overline{v}))\frac{d\mu_a}{d\mu'_a}d\mu'_a(\overline{v})$$

$$= \int_{\overline{\Omega}_F(a)} f(s_a(\overline{v}))\frac{s'_a}{s_a}d\mu'_a(\overline{v}) = \int_{\overline{\Omega}_F(a)} f(s'_a(\overline{v}))d\mu'_a(\overline{v}) = \int_{\Omega_F} f(v)d\mu'(v).$$

Now let f be a homogeneous (without loss of generality non-negative) function on Ω_F that is both μ, μ'-integrable. Take $a_1, \dots, a_n \in F^\times$ such that the integrals with respect to μ, μ' are approximated up to ε by the integrals over $\bigcup_{i=1}^n \Omega_F(a_i)$. The above calculation and the inclusion–exclusion principle shows that the integrals of f with respect to μ, μ' differ only at most 2ε, so they must be the same, as ε was arbitrary. If we take $f = \mathrm{ev}_\beta$ for some $\beta \in \mathrm{LDiv}_{\mathbb{Q}}(F)$, we are done. $\square$

Note that the first part of the above lemma works, even if μ satisfies only the first condition of being an admissible measure on F. Before we proceed further, we need to prove a few properties of heights (see Definition 1.1.1) which will allow us to pass from heights to positive functionals.

Lemma 1.7.3. Let h be a height on F. Then, h is invariant by duplication of coordinates, addition of zero coordinates, and multiplication of coordinates by roots of unity. More precisely, let $\xi \in F^n$, $x \in F$, and let $\lambda \in F^\times$ be a root of unity. Then,

$$h(\xi, x, x) = h(\xi, x, 0) = h(\xi, \lambda x) = h(\xi, x).$$

Proof. Let us start with duplication of coordinates. Using the fact that $h(1, 1) = 0$, we have

$$h(\xi, x, \xi, x) = h((\xi, x) \otimes (1, 1)) = h(\xi, x).$$

But, by monotonicity, $h(\xi, x) \leq h(\xi, x, x) \leq h(\xi, x, \xi, x)$, which shows that $h(\xi, x, x) = h(\xi, x)$. Now, we show that $h(1, 0) = h(1, \lambda) = 0$.

- By duplication of coordinates, $h(1, 0) = h(1, 0, 0, 0) = h((1, 0)^{\otimes 2}) = 2h(1, 0)$, thus $h(1, 0) = 0$.
- Let m be the order of λ in $F^\times$, and consider the tuple $\Lambda := (1, \lambda, \dots, \lambda^{m-1})$. Since we know $h(1, 1) = 0$, we may assume that $m \geq 2$. Then, up to permutation of coordinates, $\Lambda^{\otimes m} = (\Lambda, \dots, \Lambda)$, where the tuple Λ is repeated m times. Hence, by duplication of coordinates, $mh(\Lambda) = h(\Lambda)$, thus $h(\Lambda) = 0$. But, by monotonicity, $0 = h(1) \leq h(1, \lambda) \leq h(\Lambda)$, which shows that $h(1, \lambda) = 0$.

Now, by monotonicity of h, we have

$$h(\xi, x) \leq h(\xi, x, 0) \leq h((\xi, x) \otimes (1, 0)) = h(\xi, x) + h(1, 0),$$

and

$$h(\xi, x) \leq h(\xi, x, \lambda x) \leq h((\xi, x) \otimes (1, \lambda)) = h(\xi, x) + h(1, \lambda).$$

Therefore, $h(\xi, x, 0) = h(\xi, x, \lambda x) = h(\xi, x)$. $\square$

Lemma 1.7.4 (Generalized triangle inequality)**.** Let h be a height on F with Archimedean error e. Then, for all $n, m \in \mathbb{N}$, and $(x_1, \dots, x_n) \in (F^m)^n$, we have

$$h(x_1 + \ldots + x_n) \leq h(x_1, \dots, x_n) + \log_2(n)e.$$

Proof. First, we show that if the lemma is true for some $n \in \mathbb{N}$, it is also true for $2n$. Indeed, we then have

$$\begin{aligned} h(x_1 + \ldots + x_{2n}) &\leq h(x_1 + \ldots + x_n, x_{n+1} + \ldots + x_{2n}) + e \\ &= h((x_1, x_{n+1}) + (x_2, x_{n+2}) + \ldots + (x_n, x_{2n})) + e \\ &\leq h(x_1, \dots, x_{2n}) + (\log_2(n) + 1)e \\ h(x_1 + \ldots + x_{2n}) &\leq h(x_1, \dots, x_{2n}) + \log_2(2n)e. \end{aligned}$$

By induction, the lemma is true whenever n is a power of 2.
In the general case, monotonicity allows us to add any number of zero tuples, so

$$h(x_1 + \ldots + x_n) \leq h(x_1, \ldots, x_n) + \lceil \log_2(n) \rceil e.$$

Now, fix some $n, m \in \mathbb{N}$, and tuples $x_1 + \ldots + x_n \in F^m$. For all $p \in \mathbb{N}$, by distributivity of the Segre product relative to coordinate-by-coordinate addition, the p-th Segre power of the sum $x_1 + \ldots + x_n$ can be written

$$(x_1 + \ldots + x_n)^{\otimes p} = \sum_{1 \leq i_1, \ldots, i_p \leq n} x_{i_1} \otimes \ldots \otimes x_{i_p}.$$

On the other hand, the p-th Segre power is also distributive relative to concatenation, hence (up to permutation of coordinates), $(x_1, \ldots, x_n)^{\otimes p}$ is the concatenation of the tuples $x_{i_1} \otimes \ldots \otimes x_{i_p}$ for all $(i_1, \ldots, i_p) \in \{1, \ldots, n\}^p$.
Thus,

$$h((x_1 + \ldots + x_n)^{\otimes p}) \leq h((x_1, \ldots, x_n)^{\otimes p}) + \lceil \log_2(n^p) \rceil e,$$

i.e. $ph(x_1 + \ldots + x_n) \leq ph(x_1, \ldots, x_n) + \lceil p \log_2(n) \rceil e$. We conclude by dividing by p and taking the limit when p goes to infinity. $\square$

Proposition 1.7.5. Let $l : \mathrm{LDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$ be a positive functional. Define a height h_l on F by

$$\forall (x_1, \ldots, x_n) \in F^n \setminus \{0\},\ h_l(x_1, \ldots, x_n) = -l \left(\bigwedge_{\substack{1 \leq i \leq n \\ x_i \neq 0}} \widehat{\mathrm{div}}(x_i) \right).$$

Then, $l \mapsto h_l$ is a bijection from the set of positive functionals on F to the set of heights on F, which additionally restricts to a bijection from the set of global functionals to the set of global heights.

Proof. It is immediate that h_l satisfies the axioms in Definition 1.1.1, with Archimedean error $e = h_l(2, 1)$, since the tropical divisor $\widehat{\mathrm{div}}(x + y) - \widehat{\mathrm{div}}(x) \wedge \widehat{\mathrm{div}}(y) - \widehat{\mathrm{div}}(2) \wedge 0$ is positive.

Now, for a height h on F, we may recover a group morphism $l_h : \mathrm{ULat}_{\mathbb{Q}}(F) \to \mathbb{R}$ by setting

$$l_h \left(\bigwedge_{j=1}^{m} \widehat{\mathrm{div}}(b_j) - \bigwedge_{i=1}^{n} \widehat{\mathrm{div}}(a_i) \right) = h(a) - h(b),$$

for all $a \in (F^\times)^n, b \in (F^\times)^m$. This shows injectivity. To prove surjectivity, we need to show that l_h factors to a positive functional on F, i.e. that it is positive on the positive cone of $\mathrm{ULat}_{\mathbb{Q}}(F)$. For that, we use characterization (v) in Corollary 1.5.15. More precisely, let us show that for all $\varepsilon > 0$, if $\alpha \in \Gamma_c$, then $l_h(\alpha) \geq -\varepsilon e$, where e is the Archimedean error of h. Let $\alpha \in \Gamma_c$. By Definition 1.5.11, we may write

$$\alpha = \bigwedge_{j=1}^{m} \widehat{\mathrm{div}}(b_j) - \bigwedge_{i=1}^{n} \widehat{\mathrm{div}}(a_i),$$

where $a = (a_1, \ldots, a_n) \in (F^\times)^n$, and each b_j is a sum of exactly $\lfloor 2^\varepsilon \rfloor$ elements of the form a_i, $-a_i$ or 0, i.e. coordinates of the tuple $a' = (((-1,1) \otimes a), 0) \in \mathbb{R}^{2n+1}$. Note that by lemma 1.7.3, $h(a') = h(a)$.

Let $r := \lfloor 2^\varepsilon \rfloor$, and fix a bijection $\pi : \{1, \ldots, 2n+1\}^r \to \{1, \ldots, (2n+1)^r\}$. For all $j \in \{1, \ldots, r\}$ consider the tuple $d_j \in F^{(2n+1)^r}$ such that, for all $(i_1, \ldots, i_r) \in \{1, \ldots, (2n+1)\}^r$,

$$d_{j,\pi(i_1,\ldots,i_r)} = a'_{i_j}.$$

Since d_j only contains coordinates of a', we have $h(d_1, \ldots, d_r) \leq h(a') = h(a)$. On the other hand, every b_j is a sum of r coordinates of a', so by construction, b is contained in $d_1 + \ldots + d_r$. Thus, by monotonicity and Lemma 1.7.4,

$$\begin{aligned} h(b) &\leq h(d_1 + \ldots + d_r) \\ &\leq h(d_1, \ldots, d_r) + \log_2(r)e \\ h(b) &\leq h(a) + \varepsilon e. \end{aligned}$$

Thus, $l_h(\alpha) \geq -\varepsilon e$. By additivity of l_h and characterization (v) in Corollary 1.5.15, it follows that l_h is positive of the positive cone of $\mathrm{ULat}_{\mathbb{Q}}(F)$, which concludes the proof. Furthermore, it is immediate that h_l is global if and only if l is a global functional. ☐

Remark 1.7.6. The above proof also shows that for any height h, the Archimedean error e can always be taken to be $h(2, 1)$.

The main result of this section is the following theorem giving equivalence between the above introduced definitions.

Theorem 1.7.7. There is a bijective correspondence between the following structures on F:

1. Local terms,

2. Heights,

3. Positive functionals,
4. Families of local measures,
5. Renormalisation classes of admissible measures,
6. Equivalence classes of lattice valuations.

Moreover, this correspondence respects the 'global' ones.

Proof. We describe how to pass between structures with consecutive numbers in circle.

- To pass from (1) to (2), we define $h(a_1, \dots, a_n)$ to be $R_t(a_1, \dots, a_n)$ for $t = -\min(x_1, \dots, x_n)$. The axioms from Definition 1.1.1 are satisfied with Archimedean error $e = h(2, 1)$, since the lattice divisor $\widehat{\operatorname{div}}(x+y) - \widehat{\operatorname{div}}(x) \wedge \widehat{\operatorname{div}}(y) - \widehat{\operatorname{div}}(2) \wedge 0$ is positive.
- From (2) to (3) (even to get the equivalence between those two) we use Proposition 1.7.5.
- From (3) to (4), a positive functional on F yields a family of local measures $(\mu_a)_{a \in F^\times}$ as in Corollary 1.6.4.
- The passage from (4) to (5) is the content of Construction 1.6.11.
- Lemma 1.7.2 provides a correspondence from (5) to (1).
- From (5) to (6), let μ be an admissible measure on F. This makes $L^1(\Omega_F, \mu)$ an L^1-lattice, and the map $\underline{v} : x \mapsto (v \mapsto v(x))$ a lattice valuation (here Γ is the L^1-lattice generated by the image of $\underline{v}$).
- Finally, from (6) to (2), if $(\Gamma, \underline{v})$ is a lattice valuation on F, then we define a height on F by

  $$h(a_1, \dots, a_n) = -\int \underline{v}(a_1) \wedge \ldots \wedge \underline{v}(a_n).$$

  This clearly satisfies the axioms from Definition 1.1.1.

We omit the proof that by going in a cycle, we get back the same structure. □

Definition 1.7.8. We say that F is equipped with a *multiply valued field* structure (abbreviated MVF), if it is given with one of the equivalent structures from Theorem 1.7.7. We call an MVF F a *globally valued field* (abbreviated GVF), if the MVF structure is additionally global, or in other words it satisfies the product formula.

Corollary 1.7.9. If μ is a Radon measure on Ω_F satisfying the first axiom of being an admissible measure on F, then there is an admissible measure ν on F inducing the same MVF structure on F. Hence, an MVF structure on F can be seen as a Radon measure on Ω_F such that the evaluation functions are integrable, up to the equivalence relation given by integrating homogeneous functions on Ω_F as in Lemma 1.7.2.

Finally, we can compare globally valued fields to proper adelic curves, which we define below.

Definition 1.7.10. [24, Section 3.1] An *adelic structure on F* is a measure space $(\Omega, \mathcal{A}, \nu)$ together with a map $\phi : \omega \mapsto |\cdot|_\omega$ from Ω to M_F such that for all $a \in F^\times$ the function

$$\omega \mapsto \log |a|_\omega$$

is integrable (and measurable) on Ω. Moreover, we call the adelic structure *proper*, if for all $a \in F^\times$

$$\int_\Omega \log |a|_\omega d\nu(\omega) = 0.$$

Together, the data $(F, (\Omega, \mathcal{A}, \nu), \phi)$ is called an *adelic curve*.

Similarly as in Lemma 1.7.2, adelic structures on F induce multiply valued field structures by taking

$$R_t(a) := \int_\Omega t(-\log |a|_\omega) d\nu(\omega).$$

Moreover, by Remark 1.6.12, we get the following.

Corollary 1.7.11. If F is countable, then every multiply valued field structure on F is induced by an adelic structure. Similarly, in that case every GVF structure on F is induced by a proper adelic structure. Moreover, Ω can be assumed to be M_F.

1.8 Intersection theory

This section can be seen as a preview to some results appearing in Chapter 2 where I classify GVF structures on extensions of $\mathbb{Q}$ in terms of Arakelov geometry.

One can construct examples of globally valued fields using intersection theory or Arakelov intersection theory. If $\mathcal{X}$ is an arithmetic variety, let us denote by $\mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ the group of arithmetic $\mathbb{R}$-divisors of C^0-type, see Section 2.2 for precise definitions. Assume that F is a finitely generated extension of $\mathbb{Q}$. Let $\mathrm{ADiv}_{\mathbb{R}}(F)$ be the direct limit of $\mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$'s, over the system of arithmetic varieties $\mathcal{X}$ with their function field isomorphic to F. A GVF functional $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ is a linear map which vanishes on principal arithmetic divisors and is non-negative on effective arithmetic divisors. With this notation we have the following.

Theorem 1.8.1 (Corollary 2.3.18)**.** There is a natural bijection:

$$\big\{\text{GVF functionals } \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}\big\} \longleftrightarrow \big\{\text{GVF structures on } F\big\}$$

$$l \longmapsto \big(R_t^l(a) := l(\overline{\mathcal{D}}_t(a))\big)_t$$

where $\overline{\mathcal{D}}_t(a) = t(\widehat{\mathrm{div}}(a))$ if a is a tuple from $F^\times$ and t is a $\mathbb{Q}$-tropical polynomial.

The lattice structure on $\mathrm{ADiv}_{\mathbb{Q}}(F)$ is constructed in Section 2.3 and $\overline{\mathcal{D}}_t(a)$ is defined with respect to it. This result follows from Theorem 1.7.7 and Corollary 1.5.17, together with density of lattice divisors in the space of arithmetic divisors proved in Proposition 2.3.17. We postpone the proof to Chapter 2 where more details about arithmetic $\mathbb{R}$-divisors of C^0-type are given.

In particular, if $\mathcal{X}$ is an F-model of dimension d over $\mathrm{Spec}(\mathbb{Z})$ and $\overline{\mathcal{D}}$ is an arithmetically nef $\mathbb{R}$-divisor on $\mathcal{X}$, then the map

$$\overline{\mathcal{E}} \mapsto \overline{\mathcal{D}}^d \cdot \overline{\mathcal{E}}$$

defines a GVF structure on F, where the product taken is the arithmetic intersection product (see [45], [56] or Section 2.2.6), possibly calculated on a blowup of $\mathcal{X}$. In fact, any GVF structure can be approximated by such in the following sense.

Theorem 1.8.2 (Remark 2.2.72)**.** The set of GVF structures on F coming from arithmetically ample arithmetic divisors $\overline{\mathcal{D}}$ on F-models $\mathcal{X}$ (by the formula $\overline{\mathcal{E}} \mapsto \overline{\mathcal{D}}^d \cdot \overline{\mathcal{E}}$) is dense in the space of all GVF functionals $\mathrm{ADiv}_{\mathbb{Q}}(F) \to \mathbb{R}$, with respect to the pointwise convergence topology.

Let us remark that the example outlined in Equation (1.4) from the introduction comes from arithmetic intersection product by taking $\mathcal{X}$ to be $\mathbb{P}^1_{\mathbb{Z}}$ and $\overline{\mathcal{D}}$ to be $\widehat{\mathrm{div}}(z) \vee 0$ (equivalently $\mathcal{O}(1)$ with the Weil metric), where we write $z = \frac{x}{y}$ for the variable generating the function field $\mathbb{Q}(z)$ of $\mathbb{P}^1_{\mathbb{Z}}$.

Similarly, if K is a finitely generated field over a field k and we restrict our attention to GVF structures that are trivial when restricted to k, we get the following correspondence.

Theorem 1.8.3. [10] There is a natural bijection:

$$\big\{\text{GVF functionals } \mathrm{Div}_{\mathbb{Q}}(K/k) \to \mathbb{R}\big\} \longleftrightarrow \big\{\text{GVF structures on } K \text{ trivial on } k\big\},$$

$$l \longmapsto \big(R^l_t(a) := l(D_t(a))\big)_t$$

where $D_t(a) = t(\mathrm{div}(a))$ for tuples a from $F^\times$ and $\mathbb{Q}$-tropical polynomials t.

Here $\mathrm{Div}_{\mathbb{Q}}(K/k) = \varinjlim \mathrm{Div}_{\mathbb{Q}}(X)$ is the direct limit of ($\mathbb{Q}$-Cartier) divisors of normal projective varieties X over k together with isomorphisms $k(X) \simeq K$ over k. These are often called ($\mathbb{Q}$-Cartier) b-divisors, see [68]. The lattice structure on $\mathrm{Div}_{\mathbb{Q}}(K/k)$ is defined as follows: if $D, E \in \mathrm{Div}(X)$ are effective Cartier divisors, then we define $D \wedge E$ by the class of the exceptional fiber of the blowup $X' \to X$ of X at $D \cap E$. Equivalently, we define $D \wedge E$ as the class of $D \cap E$ on a blowup X' where $D \cap E$ is a Cartier divisor. For $\mathbb{Q}$-divisors, or non-effective divisors, we extend using axioms of a group lattice. The proof that it is well defined is the same as Lemma 2.3.5. A GVF functional $l : \mathrm{Div}_{\mathbb{Q}}(K/k) \to \mathbb{R}$ on the left hand side of the correspondence is a linear map which vanishes on principal divisors and is non-negative on the effective cone. Let us note, that there is a natural pairing $\mathrm{Val}_{K/k} \times \mathrm{Div}_{\mathbb{Q}}(K/k) \to \mathbb{R}$ satisfying the conclusion of Lemma 1.4.8, and such that for $D \in \mathrm{Div}_{\mathbb{Q}}(K/k)$ we have $D = 0$ if and only if, for all $v \in \mathrm{Val}_{K/k}$ we have $v(D) = 0$.

Proof of Theorem 1.8.3. By Theorem 1.7.7 the data of a GVF structure on K trivial on k is equivalent to the data of a GVF functional $l : \mathrm{LDiv}_{\mathbb{Q}}(K) \to \mathbb{R}$ that vanishes on $\mathrm{LDiv}_{\mathbb{Q}}(k)$. Let $I_k \subset \mathrm{LDiv}_{\mathbb{Q}}(K)$ be the ideal generated by $\mathrm{LDiv}_{\mathbb{Q}}(k)$ (in the sense of Definition 1.4.9). It is enough to prove that as group lattices

$$\mathrm{LDiv}_{\mathbb{Q}}(K)/I_k \simeq \mathrm{Div}_{\mathbb{Q}}(K/k).$$

We construct the isomorphism in the following way. First, there is a map $f : \mathrm{ULat}_{\mathbb{Q}}(K) \to \mathrm{Div}_{\mathbb{Q}}(K/k)$ coming from the universal property. Second, note that the ideal I from Definition 1.5.1 is contained in the kernel of f. Indeed if $\alpha \in I$, then $f(\alpha)$ pairs with any $v \in \mathrm{Val}_{K/k}$ to zero, so $f(\alpha) = 0$. Thus, f factors through $f : \mathrm{LDiv}_{\mathbb{Q}}(K) \to \mathrm{Div}_{\mathbb{Q}}(K/k)$, and from Corollary 1.5.9 we get that the kernel is exactly the ideal generated by $\mathrm{LDiv}_{\mathbb{Q}}(k)$, which finishes the proof. ☐

From the above proof we get the following.

Corollary 1.8.4. The quotient of $\mathrm{LDiv}_{\mathbb{Q}}(K)$ by the (group lattice) ideal generated by $\mathrm{LDiv}_{\mathbb{Q}}(k)$ is isomorphic to $\mathrm{Div}_{\mathbb{Q}}(K/k)$.

In fact, GVF functionals also vanish on numerically trivial divisors. We give a simple proof below.

Lemma 1.8.5. [10, Proposition 11.5] Let $l : \mathrm{Div}(K/k) \to \mathbb{R}$ be a GVF functional. Then for a numerically trivial divisor E, the equality $l(E) = 0$ holds.

Proof. Assume that E is a divisor on X with $k(X) \simeq K$ and fix an ample $\mathbb{Q}$-divisor A on X. Since E is numerically trivial, $\mathbb{Q}$-divisors $A \pm E$ are ample (by Kleiman's criterion). Hence, for some big enough n we get that $n(A \pm E)$ are rationally equivalent to effective $\mathbb{Q}$-divisors. Thus

$$n \cdot l(A \pm E) = l(n(A \pm E)) \geq 0.$$

Dividing by n, we get $|l(E)| \leq l(A)$. Since A was an arbitrary ample $\mathbb{Q}$-divisor, by replacing it by tA for arbitrary small $t > 0$, we get $l(E) = 0$. □

Hence, the left (and so also right) hand side of the correspondence in Theorem 1.8.3 can be equivalently described as

$$N_1^+(K/k) := \varprojlim N_1^+(X),$$

where $N_1^+(X)$ is the movable cone in $N_1(X)$ (i.e., the dual cone to the effective cone in $N^1(X)$). Let us point out that the cone $N_1^+(K/k)$ has appeared in [27] under the name $\mathrm{BPF}^{d-1}(\mathcal{X})$ where d is the transcendence degree of K over k.

Note that if D is an ample divisor on a d-dimensional X with $k(X) \simeq K$, then the formula

$$E \mapsto D^{d-1} \cdot E$$

defines a GVF structure on K and the set of such GVF structures on K (trivial on k) is dense.

Theorem 1.8.6. [10, Corollary 10.11] The set of GVF structures on K coming from ample divisors D on varieties X with $k(X) \simeq K$ (by the formula $E \mapsto D^{d-1} \cdot E$) is dense in the space of all GVF functionals $\mathrm{Div}_{\mathbb{Q}}(K/k) \to \mathbb{R}$, with respect to the pointwise convergence topology.

For an exposition of the proof of this theorem, see Section 2.1.2. In a different language, this was independently proven in [27, Theorem C].

Theorem 1.8.2 and Theorem 1.8.6 together with certain Bertini theorems, are crucial results implying that globally valued fields $\overline{\mathbb{Q}}[1], \overline{k(t)}[1]$ respectively (see Section 1.10 for the notation) are existentially closed (see Section 1.12).

In Chapter 3 (which is based on the joint paper with Pablo Destic and Nuno Hultberg [28]), the intersection theory introduced in [25] is studied from the continuous logic point of view.

1.9 Disintegration

Consider a field extension $K \subset L$ and assume that L is equipped with an MVF structure. One can describe restriction of this structure to K in a few ways, using Theorem 1.7.7. For example, if $(\mu'_a)_{a\in L}$ is a family of local measures on L, then for $a \in F$, one can look at restrictions $\Omega_{F'}(a) \to \Omega_F(a)$ and if μ_a denotes the pushforward of μ'_a, then $(\mu_a)_{a\in K}$ is the family of local measures on K corresponding to the restricted multiply valued field structure. Similarly, if μ' is a measure on Ω_L inducing the MVF structure on L, then for the partial map $r : \Omega_L \to \Omega_K$ restricting elements to K, the measure $\mu = r_*\mu'$ induces the restricted MVF structure on K. Also, one can look at the restriction of the corresponding positive functional, via the embedding from Corollary 1.5.6. Finally, the restriction of the MVF or height functions is just the restriction of their domains to the smaller field.

Let us introduce some notation regarding the partial restriction map $r : \Omega_L \to \Omega_K$. It is defined on

$$\Omega_{L:K} := \{v \in \Omega_L : v|_K \notin \{-\infty, 0, \infty\}^K\}.$$

We denote the complement of this open set by $\Omega_{L/K}$. Also, for $v \in \Omega_K$ we write $\Omega_{L:v}$ for the set of $w \in \Omega_L$ with $w|_K = v$. Hence

$$\Omega_{L:K} = \bigcup_{v\in\Omega_K} \Omega_{L:v}.$$

Let μ be an admissible measure on L. We can write μ as the sum of its restrictions to $\Omega_{L:K}, \Omega_{L/K}$. We denote these restrictions by $\mu_{L:K}, \mu_{L/K}$ respectively. Note that $\mu = \mu_{L:K} + \mu_{L/K}$.

Lemma 1.9.1. If $r_*\mu$ can be renormalised to an admissible measure ν on K, then μ can be renormalised to an admissible measure μ' on L such that $r_*\mu' = \nu$.

Proof. Note that local measures associated to ν and pushforwards of local measures associated to μ are the same. Hence, given sections $s_a : \overline{\Omega}_K(a) \to \Omega_K(a)$ for $a \in K^\times$

agreeing on overlaps, it is enough to construct sections $u_a : \overline{\Omega}_L(a) \to \Omega_L(a)$ (agreeing on overlaps) such that the diagram

$$\begin{tikzcd}
\overline{\Omega}_L(a) \arrow[r, "u_a"] \arrow[d, "r"'] & \Omega_L(a) \arrow[d, "r"] \\
\overline{\Omega}_K(a) \arrow[r, "s_a"] & \Omega_K(a)
\end{tikzcd}$$

commutes. This is achieved by copying Construction 1.6.11 for both K and L. Start by taking any sections u_a agreeing on overlaps (construct them as in Construction 1.6.11). Next, let $h_a : \overline{\Omega}_L(a) \to \mathbb{R}_{>0}$ be defined by

$$h_a(\overline{v}) = \frac{s_a(\overline{v}|_K)}{u_a(\overline{v})|_K}.$$

Since both s_a's and u_a's agree on overlaps, we get that h_a's also agree on overlaps, so by putting

$$u_a' := h_a \cdot u_a$$

we are done. $\square$

Thus, if ν is a global measure on K inducing a GVF structure, and there is a GVF structure on L extending the one on K, one can find a global measure μ on L such that $r_*\mu = \nu$.

Let us restrict our attention to the case where K, L are countable. Note that then Ω_K, Ω_L are locally compact, separable metric spaces (because a compact metric space is separable). Using the existence of conditional probability, we get the following description of GVF extensions.

Corollary 1.9.2. Let ν be a global measure on K. Then for every GVF structure on L extending the one on K, there is a measure $\mu_{L/K}$ on $\Omega_{L/K}$ and a family of measures μ_v on $\Omega_{L:K}$, for every $v \in \Omega_K$, such that:

1. $v \mapsto \mu_v$ is a Borel map and μ_v is a probability measure for ν-almost every element of Ω_K;
2. if $\mu_{L:K} := \nu \otimes \mu_v$, that is $\mu(A) = \int_{\Omega_{L:K}} \mu_v(A) d\nu(v)$, then $\mu := \mu_{L:K} + \mu_{L/K}$ is a globalising measure on L inducing the GVF structure there;
3. μ_v is concentrated on $r^{-1}(v)$ for ν-almost all $v \in \Omega_K$.

Proof. First we use Lemma 1.9.1 to get a global measure μ on K with $r_*\mu_{L:K} = \nu$. Next, we order $K^\times$ as $(a_i)_{i<\omega}$ and for every a_i we use the existence of conditional probability (see [63, Theorem A] for a precise statement) to get the measures μ_v for $v \in \pi^{-1}(\overline{\Omega}_K(a_i)\backslash\bigcup_{j<i}\overline{\Omega}_K(a_j))$ satisfying the assumptions locally (where $\pi : \Omega_K \to \overline{\Omega}_K$ is the projection map). By countability of K, the obtained family $(\mu_v)_{v\in\Omega_K}$ satisfies the assumptions. □

Definition 1.9.3. We call a GVF structure *discrete* on a field K, if it is induced by a measure $\mu = \sum_{i\in I}\delta_{v_i}$ being a sum of Dirac deltas at some family of pairwise non-equivalent valuations $\{v_i\}_{i\in I} \subset \mathrm{Val}_K$ such that for every $a \in K^\times$ the set $\{i \in I : v_i(a) \neq 0\}$ is finite and we have $\sum_i v_i(a) = 0$.

Corollary 1.9.2 gives a nice presentation of globally valued fields structures on finitely generated extensions of a countable discrete GVF.

Corollary 1.9.4. Let $K \subset L$ be a finitely generated extension of a discrete GVF structure on a countable K given by the family $\{v_i\}_{i\in I} \subset \mathrm{Val}_K$. Let X be a projective variety over K with function field isomorphic to L. Denote by X_i^{an} the Berkovich analytification of $X_i = X \otimes_K K_{v_i}$ for $i \in I$, and by X_0^{an} the Berkovich analytification of X treated as a variety over K with the trivial metric. Then, there exists a measure μ_0 on X_0^{an} and a family of probability measures μ_i on X_i^{an} such that

$$\mu = \mu_0 + \sum_{i\in I}\mu_i$$

is a global measure on L. Moreover, all these measures are concentrated on points in the analytifications mapping to the generic point $\eta : \mathrm{Spec}(L) \to X$ and the measures μ_i are uniquely determined by this property.

Conversely, any such family of measures with μ satisfying the product formula axiom, yields a GVF structure on L extending the one on K.

Remark 1.9.5. Note that the measure μ_0 above is not unique, because in X_0^{an} there are many points corresponding to equivalent pseudo-valuations on L. This is in contrast to X_i^{an} for $i \in I$, where any two different points represent non-equivalent pseudo-valuations on L, because all of them restrict to the same non-trivial valuation on K.

1.10 Finite extensions

In this section we study finite extensions of some globally valued fields.

Lemma 1.10.1. [10, Lemma 12.1] Let C be an irreducible curve over a field k with the function field $K = k(C)$ and fix $t \in K \setminus k$. Then, for every $r \geq 0$ there is a unique GVF structure on K which is trivial on k and satisfies $\mathrm{ht}(t) = r$.

Proof. This follows from the fact that $N_1^+(K/k) = N_1^+(C) = \mathbb{R}_{\geq 0}$. For an alternative proof, see [10]. $\square$

We write $\overline{k(t)}[r]$ to denote the unique GVF structure on $\overline{k(t)}$ with $\mathrm{ht}(t) = r$ and $\mathrm{ht}(a) = 0$ for $a \in k^\times$. Note that this notation depends on the choice of a transcendental element t.

The next results follows from Theorem 1.8.1, but we present an elementary proof avoiding the use Arakelov geometry.

Lemma 1.10.2. Let K be a number field. Then, for any $r \geq 0$, there is a unique GVF structure on K satisfying $\mathrm{ht}(2) = r \cdot \log 2$.

Proof. Let $S \subset \mathrm{Val}_K$ be a set of representatives of equivalence classes of valuations on K such that for all $a \in K^\times$ we have $\sum_{v \in S} v(a) = 0$. By Ostrowski's theorem any GVF structure on K is given by a choice of weights $w_v \geq 0$ such that for all $a \in K^\times$ we have $\sum_{v \in S} v(a) \cdot w_v = 0$.

Let W be the $\mathbb{R}$-vector space freely generated by S. A choice of weights as above corresponds to a linear functional on W that vanishes at the image of the map $f : K^\times \to W$, sending a to $\sum_{v \in S} v(a) \cdot v$. Let $N = \{\sum_{v \in S} a_v \cdot v : \sum_{v \in S} a_v = 0\}$ be the kernel of the 'summing of coordinates' functional on W. It is enough to show that the real vector space spanned by $f(K^\times)$ is N, as then, a linear functional $l : W \to \mathbb{R}$ that vanishes on $f(K^\times)$ is the same as a linear functional on $W/N \simeq \mathbb{R}$.

Consider $w = \sum_{v \in S} a_v \cdot v \in f(K^\times)$. We can write $w = w' + w''$ with $w' \in W$ having zero entries over Archimedean places, and $w'' \in W$ having zero entries over non-Archimedean places. The class of w' modulo N can be seen as an element of $\mathrm{Cl}(K) \otimes \mathbb{R} = 0$, as the class group of K is finite. On the other hand, by the Dirichlet unit theorem, the dimension of the real span of elements from $f(K^\times)$ with vanishing non-Archimedean entries is of dimension $r_1 + r_2 - 1$, where r_1, r_2 are the numbers of real and complex embeddings (up to conjugation) of K. Thus, the class of w'' modulo N is also trivial, which finishes the proof. $\square$

If $r = 1$, we call this unique GVF structure on a number field the *standard* GVF structure. Standard GVF structures are compatible with extensions of number fields by the uniqueness in Lemma 1.10.2. The union $\overline{\mathbb{Q}}$ of all number fields is thus also equipped with a unique GVF structure stratifying $\mathrm{ht}(2) = \log 2$, and we also call it *standard*. We write $\overline{\mathbb{Q}}[1]$ for the standard GVF structure on $\overline{\mathbb{Q}}$, and $\overline{\mathbb{Q}}[r]$ for the one satisfying $\mathrm{ht}(2) = r \cdot \log 2$.

For a general globally valued field, we only have uniqueness of a GVF structure on a finite extension additionally assuming Galois-invariance (see Example 1.10.6 for a counter-example without the Galois invariance assumption). To prove it we need a few lemmas.

Lemma 1.10.3. Let $K \subset F$ be a finite Galois extension with the Galois group G. Then the restriction map $\Omega_F^\circ \to \Omega_K^\circ$ is defined everywhere, and identifies Ω_K° with the topological quotient of Ω_F° by G.

Proof. First we prove the the restriction map is defined everywhere. Assume that $|\cdot|$ is a pseudo-absolute value on F that is $\{0, 1, \infty\}$-valued on K, but not on F. In particular, $|\cdot|$ must be non-Archimedean. Let $a \in F$ be such that $|a| > 1$ and let $b_i \in K$ be such that $\sum_{i \leq n} b_i a^i = 0$ and $b_n \neq 0$. Let i_0 be such that b_{i_0} is the biggest, in the sense that for $i \neq i_0$ we have $|b_{i_0}/b_i| \geq 1$. Dividing everything by b_{i_0} we get

$$|a|^{i_0} \leq \max_{i \neq i_0} |b_i/b_{i_0}| \cdot |a|^i \leq \max_{i \neq i_0} |a|^i.$$

It follows that $i_0 = n$, as $|a| > 1$. Thus without loss of generality $b_n = 1$ and $|b_i| \leq 1$. If there is i with $|b_i| = 1$, then $\max_{i<n} |b_i| \cdot |a|^i$ is uniquely achieved (say at i_1), so we get $|a|^n = |b_{i_1}| \cdot |a|^{i_1} = |a|^{i_1}$ which gives a contradiction. Hence for all i we have $|b_i| = 0$ and so $1 < |a|^n \leq \max_{i<n} |b_i| \cdot |a|^i = 0$, which gives a contradiction.

Now we prove that the restriction map is a topological quotient by G. By local compactness it is enough to prove that G acts on fibers transitively. In other words, we need to show that if $|\cdot|_1, |\cdot|_2$ are pseudo-absolute values on F that coincide on K, then there exists $g \in G$ such that $|g(\cdot)|_1 = |\cdot|_2$.

We split the problem in two cases, depending on $|\cdot|$ being non-Archimedean or Archimedean. First, assume the former. Then let $o_i \subset \mathcal{O}_i$ denote the inclusions $\{|x|_i \leq 1\} \subset \{|x|_i \neq \infty\}$ for $i = 1, 2$. Note that $\mathcal{O}_i$ is uniquely determined by o_i, as the ordered group $F^\times/o_i^\times$ has a unique convex non-zero Archimedean subgroup. By [31, Conjugation Theorem 3.2.15] there is a $g \in G$ such that $go_1 = o_2$. By uniqueness, we get that also $g\mathcal{O}_1 = \mathcal{O}_2$, and since both $|\cdot|_1, |\cdot|_2$ extend a non-trivial pseudo-absolute value $|\cdot|$ on K, we get that $|g^{-1}(\cdot)|_1 = |\cdot|_2$.

Now, we assume that $|\cdot|$ is Archimedean. Let $\mathcal{O}_1 = \{|x|_1 \neq \infty\}, \mathcal{O}_2 = \{|x|_2 \neq \infty\}$ and denote by $\mathcal{O}$ their intersections with K. By [31, Conjugation Theorem 3.2.15] there is a $g \in G$ such that $g\mathcal{O}_1 = \mathcal{O}_2$. Let us assume that $\mathcal{O}_1 = \mathcal{O}_2$, by replacing $|\cdot|_1$ with $|g^{-1}(\cdot)|_1$. We end up with the diagram

$$\begin{tikzcd}
F \arrow[r] & l \cup \{\infty\} \\
K \arrow[u, hook] \arrow[r] & k \cup \{\infty\} \arrow[u, hook]
\end{tikzcd}$$

where k, l are the residue fields of $\mathcal{O}, \mathcal{O}_1$ respectively. Use the notation $|\cdot|_1, |\cdot|_2$ for the two induced absolute values on l that extend the absolute value $|\cdot|$ on k. Denote by $G_1 < G$ the subgroup fixing $\mathcal{O}_1$ setwise. By [31, Lemma 5.2.6], using the fact that $k \subset l$ is separable as we are in characteristic zero, the natural map $G_1 \to \mathrm{Aut}(l/k)$ is surjective. Thus, it is enough to show that there exists $\sigma \in \mathrm{Aut}(l/k)$ such that $|\sigma(\cdot)|_1 = |\cdot|_2$. We can assume that absolute values $|\cdot|_1, |\cdot|_2$ come from two embeddings of l into $\mathbb{C}$ that restrict to the same embedding on k. By [31, Theorem 5.2.7.(2)] the extension $k \subset l$ is Galois, which proves the existence of the desired $\sigma \in \mathrm{Aut}(l/k)$ and finishes the proof. $\square$

Lemma 1.10.4. Let G be a finite group acting on a locally compact topological space X with a locally compact topological quotient $r : X \to Y = X/G$. Assume that ν is a Radon measure on Y. Then there is a unique G-invariant Radon measure μ on X such that $r_*\mu = \nu$.

Proof. Since ν is a Radon measure, without loss of generality we can assume that Y is compact. In that case ν is given by a functional $\nu : C(Y) \to \mathbb{R}$ and we prove that it has a unique G-invariant extension to $C(X)$. Note that the pullback $C(Y) \to C(X)$ maps $C(Y)$ bijectively onto $C(X)^G$. Also, there is a retraction $\mathrm{Tr} : C(X) \to C(X)^G \simeq C(Y)$ defined by $\mathrm{Tr}(f) := \frac{1}{|G|}\sum_{g\in G} g^* f$. Thus, a G-invariant functional $\mu : C(X) \to \mathbb{R}$ extending ν must be of the form $\mu(f) = \mu(\mathrm{Tr}(f)) = \nu(\mathrm{Tr}(f))$. This finishes the proof. $\square$

Note that the above proof works in the same way for a compact group G (using the Haar measure instead of the counting measure). Now we prove the existence and uniqueness of a Galois-invariant GVF extension. Existence can be concluded using [24, Section 3.4].

Proposition 1.10.5. Let K be a GVF and consider a finite Galois field extension $K \subset F$. Then there is a unique GVF structure on F extending the one on K that is invariant under the action of $G = \mathrm{Aut}(F/K)$.

Proof. By Lemma 1.10.3 the restriction $r : \Omega_F^\circ \to \Omega_K^\circ$ satisfies the assumptions of Lemma 1.10.4, so for any global measure ν on K there is a unique G-invariant measure μ on F such that $r_*\mu = \nu$. By Lemma 1.9.1 we get the uniqueness.

To prove existence it is enough to prove that the measure μ obtained as above satisfies the product formula. From the proof of Lemma 1.10.4 we get that for $a \in F^\times$ we have:

$$\int_{\Omega_F} \mathrm{ev}_a \, d\mu = \int_{\Omega_K} \mathrm{Tr}(\mathrm{ev}_a) d\nu,$$

where $\mathrm{ev}_a(v) = v(a)$. However, using $\mathrm{Tr}(\mathrm{ev}_a) = \mathrm{ev}_{\mathrm{N}(a)}$ (where $\mathrm{N}(a) \in K$ is the F/K-norm) we get the product formula, provided that we prove integrability of ev_a.

Let S_k be the symmetric functions for $k = 0, \ldots, n$ (with $n = [F : K]$) so that $\sum_{k \le n} S_k(a)a^k = 0$. Note that for any pseudo-absolute value $|\cdot|$ on F we have

$$|a|^n \le [n] \max_{k<n}(|S_k(a)| \cdot |a|^k) = [n]|S_{k_0}(a)| \cdot |a|^{k_0},$$

where by an expression in square brackets like $[n]$ we mean n if $|\cdot|$ is Archimedean and 1 otherwise. By dividing and taking $-\log$ we get

$$(n - k_0)v(a) \ge v(S_{k_0}(a)) - \log[n] \ge \min_{k<n} v(S_k(a)) - \log[n],$$

where $v = \log|\cdot|$. If we assume that $v(a) \le 0$, we get that $v(a) \ge (n - k_0)v(a) \ge \min_{k<n} v(S_k(a)) - \log[n]$. Hence, this gives a bound

$$-v(a)^- \le -\min_{k<n} v(S_k(a)) + \log[n].$$

By Remark 1.6.12 we can assume that the restriction of ν to Archimedean $v \in \Omega_K$ is concentrated on the set $\{v(2) = -\log 2\}$, so we can replace $\log[n]$ in the above bound by $-v(n)^-$. In this way we get integrability of the function $-\mathrm{ev}_a^-$ and by replacing a with a^{-1} the integrability of ev_a follows. □

Since we have uniqueness and existence for every finite Galois extension, there is also a unique Galois-invariant GVF extension to the algebraic closure. We call it the *symmetric extension.* Note that without assuming Galois-invariance, there can be multiple ways to extend a GVF structure to a finite field extension.

Example 1.10.6. Consider any GVF structure on $\mathbb{Q}(x, y)$ with $\mathrm{ht}(x) \ne \mathrm{ht}(y)$. If we look at the degree two extension $\mathbb{Q}(x+y, xy) \subset \mathbb{Q}(x, y)$, then the symmetric extension of the restriction of the starting GVF structure satisfies $\mathrm{ht}(x) = \mathrm{ht}(y)$, so yields a different GVF structure then the starting one.

1.11 Continuous logic

We outline unbounded continuous logic, at least for discretely metrized spaces. We closely follow [5], but in this simplified setting. For other introductions to continuous logic, see [8] or [38].

Definition 1.11.1. A *language* $\mathcal{L}$ consists of the following data. First, a tuple of function symbols f, constant symbols c and continuous predicate symbols P. Second, for each function or predicate symbol, its arity is specified, and it is equipped with a *gauge modulus* Δ_p, i.e. a right-continuous increasing function $\Delta_p : (0, \infty) \to (0, \infty)$. Finally, there are two distinguished predicates, a *gauge symbol* (or *height symbol*) ht of arity 1, and the binary symbol '=' for the metric which will be interpreted as the $\{0, 1\}$-valued discrete metric. For simplicity, we assume that ht is also a continuous predicate symbol.

Example 1.11.2. The language of globally valued fields consists of binary function symbols $+, \cdot$, constants $0, 1$, and continuous predicates $(h_n)_{n \in \mathbb{N}}$. We often omit n when discussing h_n. Later we will determine the gauge moduli for these symbols.

Definition 1.11.3. Let $\mathcal{L}$ be a language. An *$\mathcal{L}$-structure* $\mathfrak{M}$ is a set M together with interpretations of function, constant and predicate symbols from $\mathcal{L}$, and an interpretation of a gauge symbol $\mathrm{ht} : M \to \mathbb{R}_{\geq 0}$. Moreover, if f is a function symbol with gauge modulus Δ_f, then

$$(\forall r > 0)(\forall x \in M)(\mathrm{ht}(x) < r \implies \mathrm{ht}(f(x)) \leq \Delta_f(r)).$$

Similarly, for a predicate symbol P with gauge modulus Δ_P we demand that

$$(\forall r > 0)(\forall x \in M)(\mathrm{ht}(x) < r \implies |P(x)| \leq \Delta_P(r)).$$

Finally, for each constant symbol c we have a number Δ_c such that $\mathrm{ht}(c) \leq \Delta_c$. If the domain of f or P has more than one variable, we define $\mathrm{ht} : M^k \to \mathbb{R}_{\geq 0}$ by taking maximum of heights of coordinates and impose the same condition.

Example 1.11.4. Let F be equipped with a GVF structure, given by a global height function h. We equip it with an $\mathcal{L}$-structure, where $\mathcal{L}$ is the language of globally valued fields in the following way. For $x \in F$ we define

$$\mathrm{ht}(x) := h[1 : x].$$

Let $e > 0$ be such that $\mathrm{ht}(2) \leq e$. Then the following inequalities hold:

- $\mathrm{ht}(xy) \leq \mathrm{ht}(x) + \mathrm{ht}(y)$,
- $\mathrm{ht}(x+y) \leq \mathrm{ht}(x) + \mathrm{ht}(y) + e$.

To see this one can either prove it directly, or use Corollary 1.7.11 together with [30, Proposition 1.13]. These inequalities show that the function $\Delta(r) = 2r + e$ is valid gauge moduli for addition and multiplication symbols. For constants $0, 1$ we take $\Delta = 0$. For a predicate symbol h_n, $\Delta_n(r) = 1 + n \cdot r$ can serve as a gauge moduli for h_n interpreted as the height $h_n : F^n \to \{-1\} \cup \mathbb{R}_{\geq 0}$ with $h_n(0) = -1$.

Note that here we put $h(0) = -1$ instead of $h(0) = -\infty$, because the value $-\infty$ is not allowed in unbounded continuous logic.

Definition 1.11.5. Let $\mathcal{L}$ be a language. We define *$\mathcal{L}$-terms* inductively as follows:

- Any variable is an $\mathcal{L}$-term.
- Any constant c from $\mathcal{L}$ is an $\mathcal{L}$-term.
- If f is a function symbol from $\mathcal{L}$ of arity n, and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, then $f(t_1, \ldots, t_n)$ is an $\mathcal{L}$-term.

We then define *$\mathcal{L}$-formulas* inductively as follows:

- If P is a predicate symbol from $\mathcal{L}$ of arity n, and $t_1, \ldots, t_n$ are $\mathcal{L}$-terms, then $P(t_1, \ldots, t_n)$ is an $\mathcal{L}$-formula.
- As connectives we use continuous functions $\mathbb{R}^n \to \mathbb{R}$. So whenever $\varphi_1(x), \ldots, \varphi_n(x)$ are formulas in variables x and $u : \mathbb{R}^n \to \mathbb{R}$ is continuous, then $u(\varphi_1(x), \ldots, \varphi_n(x))$ is a formula.
- Let $f : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a compactly supported continuous function and $\varphi(x, y)$ be a formula in tuples of variables x, y. Then we can form formulas of variables y of the form:

$$\psi(y) = \inf_x f(\mathrm{ht}(x)) \cdot \varphi(x, y), \;\; \xi(y) = \sup_x f(\mathrm{ht}(x)) \cdot \varphi(x, y).$$

A formula is called *quantifier-free* if it does not use the quantifiers inf and sup. A variable x is said to occur freely in a formula ϕ if is has at least one occurrence which is not bound by a quantifier. A formula with no free variables is called a *sentence*. We denote by $F_n(\mathcal{L})$ (or F_n if $\mathcal{L}$ is clear from the context) the set of $\mathcal{L}$-formulas with n free variables $x_1, \ldots, x_n$, and by $F_n^{\mathrm{qf}}(\mathcal{L})$ (or F_n^{qf}) the set of quantifier-free formulas in

$F_n(\mathcal{L})$. For each $\mathcal{L}$-structure $\mathfrak{M}$ one can inductively define the *interpretation* of any formula $\varphi(x)$ evaluated on a tuple a from $\mathfrak{M}$ (denoted $\varphi(a)^{\mathfrak{M}}$ or just $\varphi(a)$). In the base of this induction, for an atomic formula $w_1 = w_2$ we define the value $(w_1 = w_2)^{\mathfrak{M}}(a)$ as 0 if $w_1(a) = w_2(a)$ in $\mathfrak{M}$ and 1 otherwise. Moreover, when interpreting quantifiers, we put

$$(\inf_x f(\mathrm{ht}(x)) \cdot \varphi(x, b))^{\mathfrak{M}} := \min(0, \inf_a f(\mathrm{ht}(a)) \cdot \varphi(a, b)^{\mathfrak{M}}),$$

and similarly with supremum. If $\mathfrak{M}$ has elements of unbounded height, this coincides with the natural interpretation. This modification is needed for Łoś's theorem to hold.

Moreover, for each formula $\varphi(x)$ we define inductively its *gauge moduli* Δ_φ so that

$$(\forall L - \text{structures } \mathfrak{M})(\forall a \in M)(\forall r > 0)(\mathrm{ht}(a) < r \implies |\varphi(a)| \leq \Delta_\varphi(r)).$$

We leave the details on how to exactly define Δ_φ to the reader.

Definition 1.11.6. A *theory* T in a language $\mathcal{L}$ is a set of conditions of the form $\varphi = 0$ for sentences φ. For an $\mathcal{L}$-structure $\mathfrak{M}$ we say that $\mathfrak{M}$ is a *model* of T (written $\mathfrak{M} \models T$), if in $\mathfrak{M}$ the conditions $\varphi^{\mathfrak{M}} = 0$ from T hold.

We also allow conditions of the form $\varphi \in C$ where C is a closed subset of $\mathbb{R}$, as this gives equivalent notion.

Example 1.11.7. Assume that $\mathcal{L}$ is a language and let f be a binary function symbol, P be a continuous predicate, and c be a constant. Let us describe how to write the statement

$$(\forall x, y)\big(f(x, y) = c \implies P(x) = 0\big)$$

as a theory T, i.e., how to find a theory T such that the models of T are exactly the $\mathcal{L}$-structures satisfying (in the naive sense) this statement.

Let $f_n : \mathbb{R}_{\geq 0} \to \mathbb{R}$ be a sequence of continuous non-increasing functions such that $f_n|_{[0,n]} = 1$ and $f_n|_{[n+1,\infty)} = 0$. For all natural n we consider the condition

$$\big[\sup_{x,y} f_n(\mathrm{ht}(x, y)) \cdot \varphi_n(x, y)\big] = 0,$$

where

$$\varphi_n(x, y) = \big[\min\big(1 - (f(x, y) = c), |P(x)|\big)\big].$$

The set T of all such conditions does the job, as if there was a model $\mathfrak{M}$ of T with $a, b \in M$ such that $f(a, b) = c$ in $\mathfrak{M}$ and $P(a) \neq 0$, then for n such that $\mathrm{ht}(a), \mathrm{ht}(b) < n$ the formula

$$\sup_{x,y} f_n(\mathrm{ht}(x, y)) \cdot \varphi_n(x, y)$$

would be evaluated to something bigger or equal $\min(1, |P(a)|)$ which is bigger than zero.

Definition 1.11.8. We denote by GVF_e the theory of globally valued fields in the language of globally valued fields with $\mathrm{ht}(2) \leq e$, see Example 1.11.4. It consists of the field axioms, the axioms from Definition 1.1.1, the axiom $\mathrm{ht}(x) = h[x : 1]$, axioms saying that $\mathrm{ht}(2) \leq e$ and $h_n(0) = -1$, and the product formula axiom (asserting that the height h is global). More precisely, one needs to translate these axioms to conditions of the form $\varphi = 0$, as in Example 1.11.7. We skip the details of this translation. By abuse of notation, if $\mathfrak{M} \models \mathrm{GVF}_e$ for some $e \geq 0$, we write $\mathfrak{M} \models \mathrm{GVF}$. Moreover, for every $\mathbb{Q}$-tropical polynomial t, we introduce a predicate R_t using height predicates h_n and Lemma 1.4.5.

Construction 1.11.9. Fix a language $\mathcal{L}$. Let I be a set, $\mathcal{U}$ be an ultrafilter on I and $\mathfrak{M}_i$ be $\mathcal{L}$-structures for $i \in I$. We define the ultraproduct $\mathfrak{M} := \prod \mathfrak{M}_i/\mathcal{U}$ as the quotient $M := N_0/\sim$ where

$$N_0 = \{(a_i)_{i\in I} \in \prod_{i\in I} M_i : \mathrm{ht}(a_i) \text{ is bounded}\}$$

and $(a_i)_{i\in I} \sim (b_i)_{i\in I}$ if the set of indices i where $a_i = b_i$ is in $\mathcal{U}$. We equip $\mathfrak{M}$ with an $\mathcal{L}$-structure interpreting constants as classes of constants in the product and functions as classes of values of functions in the product. For a continuous predicate symbol P, we interpret it by taking:

$$P([a])^{\mathfrak{M}} := \lim_{i\to\mathcal{U}} P(a(i))^{\mathfrak{M}_i},$$

where $[a_i]$ is the class of $(a(i))_{i\in I} \in N_0$ and $\lim_{i\to\mathcal{U}}$ is the ultralimit with respect to $\mathcal{U}$. These definitions make sense by Definition 1.11.3.

Theorem 1.11.10. (Łoś's theorem) Let φ be an n-ary $\mathcal{L}$-formula and $\mathfrak{M}_i$ be $\mathcal{L}$-structures for $i \in I$. Fix an ultrafilter $\mathcal{U}$ on I, let $\mathfrak{M} := \prod \mathfrak{M}_i/\mathcal{U}$, and $[a_1], \dots, [a_n] \in \mathfrak{M}$. Then,

$$\varphi([a_1], \dots, [a_n])^{\mathfrak{M}} = \lim_{i\to\mathcal{U}} \varphi(a_1(i), \dots, a_n(i))^{\mathfrak{M}_i}.$$

Proof. The proof is a standard induction over the complexity of the formula φ. We only describe how to deal with the quantifier case.

Consider a formula $\varphi(x) = \inf_y f(\mathrm{ht}(y)) \cdot \psi(x, y)$. Fix a tuple $[a] = ([a_1], \dots, [a_n])$ in M. First, assume that $\varphi([a]) < r$ and that in the minimum in the inductive interpretation is not attained at 0 (the other case is easy). This means that there is a tuple $[b]$ in M such that $f(\mathrm{ht}([b])) \cdot \psi([a],[b]) < r$. By continuity of f and the inductive hypothesis, we get that

$$\lim_{i \to \mathcal{U}} f(\mathrm{ht}\, b(i)) \cdot \psi(a(i), b(i)) < r.$$

Thus

$$\lim_{i \to \mathcal{U}} \inf_{y} f(\mathrm{ht}(y)) \cdot \psi(a(i), y) < r.$$

On the other hand, assume that $\lim_{i \to \mathcal{U}} \varphi(a(i)) < r$. Without loss of generality, assume that for all $i \in I$ there is a tuple $b(i)$ in M_i such that $f(\mathrm{ht}\, b(i)) \cdot \psi(a(i), b(i)) < r$. We have two cases now:

- either $\mathrm{ht}(b(i))$ is bounded,
- or it is not.

In the bounded case, there is a tuple $[b]$ in M such that $f(\mathrm{ht}([b])) \cdot \psi([a],[b]) < r$, which proves that $\varphi([a]) < r$. In the unbounded case, since f is compactly supported, we get that $0 < r$. By the definition of interpretation of quantifiers, we are done. $\square$

Example 1.11.11. Pick a sequence of real positive numbers $(r_i)_{i<\omega}$ converging to zero. Let F be the ultraproduct $\prod_i \overline{\mathbb{Q}}[r_i]/\mathcal{U}$ with respect to an ultrafilter $\mathcal{U}$ on ω. Then F is a characteristic zero, algebraically closed field, with the GVF structure trivial on $\overline{\mathbb{Q}}$. One can also consider a finite-to-zero characteristic variant, i.e., $K = \prod_p \overline{\mathbb{F}_p(t)}[1]/\mathcal{U}$, which is again, a characteristic zero, algebraically closed GVF, with the GVF structure trivial on $\overline{\mathbb{Q}}$.

Example 1.11.12. Jensen's formula (1.3):

$$\sum_{0<|a|<r} \mathrm{ord}_a(f) \log \frac{r}{|a|} + \frac{1}{2\pi} \int_0^{2\pi} -\log |f(re^{i\theta})| d\theta = -\log |f(0)|$$

can be used to define a GVF structure on a subfield of meromorphic functions $\mathcal{M}$. Indeed, fix an ultrafilter u on $\mathbb{R}_{>0}$ that avoids finite Lebesgue measure sets. Let $\eta(r)$ be a positive function that converges to 0 when $r \to \infty$. Put

$$\mathrm{ht}_{\eta,r}(f) := \sum_{0<|a|<r} \max(0, \mathrm{ord}_a(f)) \frac{\log(r/|a|)}{\eta(r)} + \frac{1}{2\pi} \int_0^{2\pi} \max(0, -\log |f(re^{i\theta})|) \frac{d\theta}{\eta(r)}$$

and $\mathrm{ht}_{\eta,u}(f) := \lim_{r \to u} \mathrm{ht}_{\eta,r}(f)$. Then we define

$$\mathcal{M}[\eta, u] = \{f \in \mathcal{M} : \mathrm{ht}_{\eta,u}(f) < \infty\}$$

and equip it with a GVF structure by defining heights $h : \mathbb{P}^n(\mathcal{M}[\eta, u]) \to \mathbb{R}$ similarly. By existential closedness of $\overline{\mathbb{C}(t)}[1]$ (see Section 1.12), $\mathcal{M}[\eta, u]$ satisfies the universal theory of $\overline{\mathbb{C}(t)}[1]$ (as $\mathcal{M}[\eta, u]$ embeds into an ultrapower of $\overline{\mathbb{C}(t)}[1]$). This formalizes a rudimentary part of Vojta's dictionary between number theory and value distribution theory, and sets a goal of formalizing more.

Definition 1.11.13. A set of conditions $\Sigma = \{\varphi_i \leq r_i : i \in I\}$ is *approximately finitely satisfiable* if for all finite $I_0 \subset I$ and $\varepsilon > 0$ the set of conditions $\{\varphi_i \leq r_i + \varepsilon : i \in I_0\}$ is satisfiable.

Corollary 1.11.14. (Compactness) Let Σ be a set of conditions in variables $(x_\alpha)_{\alpha<\kappa}$, and $(r_\alpha)_{\alpha<\kappa} \in \mathbb{R}^\kappa_{\geq 0}$. Assume that $\Sigma' = \Sigma \cup \{\mathrm{ht}(x_\alpha) \leq r_\alpha : \alpha < \kappa\}$ is approximately finitely satisfiable. Then, it is satisfiable. Furthermore, assume that for each finite subset σ of Σ' and $\varepsilon > 0$, σ is satisfiable up to ε in some model $\mathfrak{M}_{\sigma,\varepsilon} \models T$. Then, Σ' is satisfiable in some ultraproduct of the $\mathfrak{M}_{\sigma,\varepsilon}$.

Proof. Write $\Sigma' = \{\varphi_i \leq r_i : i \in I\}$, and let $P_\omega(I)$ be the set of finite subsets of I. For each $I_0 \in P_\omega(I)$ and $\varepsilon > 0$, let $\mathfrak{M}_{I_0,\varepsilon}$ be a model in which $\{\varphi \leq r_i + \varepsilon : i \in I_0\}$ is satisfiable, and let $\hat{I_0} \subset P_\omega(I)$ be the set of I_1 such that $I_0 \subset I_1$. Then, the set

$$\{\hat{I_0} \times (0, \varepsilon) : I_0 \in P_\omega(I), \varepsilon \in (0, 1)\}$$

is stable under finite intersections, so we can extend it to an ultrafilter $\mathcal{U}$ on $P_\omega(I) \times (0, 1)$.

For all $I_0 \in P_\omega(I)$ and $\varepsilon > 0$, let $\tilde{I_0} \in P_\omega(I)$ be the finite set of indices of the conditions $\mathrm{ht}(x_\alpha) \leq r_\alpha$, for α such that x_α occurs freely in φ_i for some $i \in I_0$. Now, let

$$\mathfrak{M}^* := \prod \mathfrak{M}_{I_0 \cup \tilde{I_0}, \varepsilon}/\mathcal{U}.$$

Define a sequence $([a_\alpha])_{\alpha<\kappa}$ in $\mathfrak{M}^*$ in the following way: for all $I_0 \in P_\omega(I)$ and $\varepsilon > 0$, set $([a_\alpha(I_0, \varepsilon)])_{\alpha<\kappa}$ to be a realisation in $\mathfrak{M}_{I_0 \cup \tilde{I_0}, \varepsilon}$ of $\{\varphi \leq r_i + \varepsilon : i \in I_0 \cup \tilde{I_0}\}$. Since we may choose any value of a_α if x_α does not occur freely in φ, we set $a_\alpha = 0$ in this case. Thus, for all $\alpha < \kappa$, and for all $(I_0, \varepsilon) \in P_\omega(I) \times (0, 1)$, we have $\mathrm{ht}(a_\alpha(i)) \leq r_\alpha + 1$, which ensures that $[a_\alpha]$ is indeed an element of $\mathfrak{M}^*$. Now, let $i \in I$, and $\varepsilon > 0$, the set $\widehat{\{i\}} \times (0, \varepsilon)$ is in $\mathcal{U}$, therefore by Łoś's theorem, $\varphi([a_\alpha] : \alpha < \kappa)^{\mathfrak{M}^*} \leq r_i + \varepsilon$. Since this is true for all $\varepsilon > 0$, $([a_\alpha]_{\alpha<\kappa}$ is indeed a realisation of Σ'. □

Definition 1.11.15. Let T be a theory in a language $\mathcal{L}$. We define the space of *n-types* $S_n(T)$ as the set of functions $f : F_n(\mathcal{L}) \to \mathbb{R}$ such that there is $\mathfrak{M} \models T$ and

$a \in M^n$ such that $f(\varphi) = \varphi(a)^{\mathfrak{M}}$ for all $\varphi \in F_n$. Thus, each function represents a consistent set of conditions $\varphi(c_1, \dots, c_n) = f(\varphi)$ in the language enriched by constants $c_1, \dots, c_n$. The space of n-types is equipped with the pointwise convergence topology. We write $S_{n,\mathrm{ht}\le r}(T)$ for the set of types with $\mathrm{ht}(c_i) \le r$ for $i = 1, \dots, n$. Note that

$$S_n(T) = \bigcup_{r\ge 0} S_{n,\mathrm{ht}\le r}(T)$$

and each $S_{n,\mathrm{ht}\le r}(T)$ is compact (by the compactness theorem). Moreover, $S_n(T)$ is locally compact, and equipped with the colimit topology with respect to the above union.

Similarly, we define the space of *quantifier-free types* $S_n^{\mathrm{qf}}(T)$ as the set of functions $f : F_n^{\mathrm{qf}}(\mathcal{L}) \to \mathbb{R}$ which are satisfiable in some model of T. Note that there is a restriction map $S_n(T) \to S_n^{\mathrm{qf}}(T)$ (where the latter is the set of quantifier-free n-types).

If $\mathfrak{M}$ is a model of T, we define $\mathcal{L}_M$ as the language obtained by adding constants $(c_a)_{a\in M}$ to $\mathcal{L}$. Then, $\mathfrak{M}$ is naturally an $\mathcal{L}_M$-structure by interpreting c_a as a. An $\mathcal{L}_M$-formula may also be called an $\mathcal{L}$-formula with parameters in M. Let T' be the theory in $\mathcal{L}_M$ consisting of T and the atomic diagram of $\mathfrak{M}$ (i.e., all quantifier-free sentences about tuples from M, written using constants $(c_a)_{a\in M}$). Note that models of T' correspond to inclusions $\mathfrak{M} \subset \mathfrak{N}$, where $\mathfrak{N}$ is a model of T. If T is clear from the context, we write $S_n(\varnothing)$ for $S_n(T)$ and if $\mathfrak{M} \models T$, we write $S_n(M) := S_n(T')$ with T' as above (we abuse notation slightly, as $S_n(M)$ depends on $\mathfrak{M}$, not only on the set M). Similarly, we define $S_n^{\mathrm{qf}}(M) := S_n^{\mathrm{qf}}(T')$. For $a \in M$, we define $\mathrm{tp}(a/M) \in S_n(M)$ and $\mathrm{qftp}(a/M) \in S_n^{\mathrm{qf}}(M)$ to be the respective functions $\varphi \mapsto \varphi(a)^{\mathfrak{M}}$.

Definition 1.11.16. Let $\mathfrak{M} \subset \mathfrak{N}$ be an extension of models of T. A function $f : N^n \to \mathbb{R}$ is called *M-definable* if there exists a continuous function $\tilde{f} : S_n(M) \to \mathbb{R}$ such that

$$\forall a \in N,\ f(a) = \tilde{f}(\mathrm{tp}(a/M)).$$

Similarly, f is called *quantifier-free M-definable* if there exists a continuous function $\tilde{f} : S_n^{\mathrm{qf}}(M) \to \mathbb{R}$ such that

$$\forall a \in N,\ f(a) = \tilde{f}(\mathrm{qftp}(a/M)).$$

Note that in both cases f can be defined by the same formula on any $\mathfrak{M} \subset \mathfrak{N}' \models T$.

Similarly, a subset $X \subset N^n$ is called *(quantifier-free) M-definable* if its indicator function is (quantifier-free) M-definable.

Note that we can treat $\varphi \in F_n(\mathcal{L}_M)$ as an M-definable function by putting $\varphi(p) := p(\varphi)$ for $p \in S_n(M)$. The induced map $F_n(\mathcal{L}_M) \to C(S_n(M))$ has a dense image in the compact-open topology. If two formulas $\varphi_1, \varphi_2 \in F_n(\mathcal{L})$ (without parameters) define the same function in all models of T, we say that they are *T-equivalent*.

We could define type spaces over models of the universal part of T, but in the GVF context that does not yield a more general notion, as GVF_e is a universal theory, at least if we add the inverse symbol to the language.

Remark 1.11.17. Let K be a GVF and let K_0 be the underlying field, treated as a structure in the language $\mathcal{L}_{\text{rings}} = (0, 1, +, \cdot)$. Then, the inclusion of the set of $\mathcal{L}_{\text{rings}}$-formulas to the set of formulas over the language of globally valued fields induces a restriction map $\pi : S_n^{\text{qf}}(K) \to S_n^{\text{qf}}(K_0)$, mapping $\text{qftp}(a/K)$ to the point $\text{qftp}(a/K_0) \in S_n^{\text{qf}}(K_0)$ corresponding to the irreducible subvariety $\text{loc}(a/K) = V \subset \mathbb{A}_K^n$ defined by polynomials that vanish on a. Since a generates $K(V)$ over K, $\text{qftp}(a/K)$ is uniquely determined by $\text{qftp}(a/K_0)$ and the heights of tuples from $K(V)$. Thus, there is a bijection between the fiber of π over V and the set of GVF structures on $K(V)$ extending the one on K.

Remark 1.11.18. It is worth mentioning that the only quantifier-free definable sets of bounded height in GVFs are the algebraically constructible sets. More precisely, if K is a GVF and $X \subset S_{n,\text{ht}\leq r}^{\text{qf}}(K)$ is a clopen subset, then it is induced by some constructible subset of K^n. Indeed, this is because the set of quantifier-free GVF types extending a given quantifier-free ACF type over K is convex, and hence connected. Moreover, since $-\min(h(x), 0)$ defines the characteristic function of $\{0\}$ in a GVF, the restriction map from quantifier-free GVF types to ACF types is continuous.

1.12 Existential closedness

In this section we present a few notions from discrete logic in the unbounded continuous setup. We follow [19, Chapter 3]. Let $\mathcal{L}$ be an (unbounded continuous logic) language and let T be an $\mathcal{L}$-theory. Consider an extension $\mathfrak{M} \subset \mathfrak{N}$ of models of T.

Definition 1.12.1. Let $\mathfrak{M} \subset \mathfrak{N}$ be an inclusion of $\mathcal{L}$-structures. We write $\mathfrak{M} \prec_\exists \mathfrak{N}$, if $\mathfrak{M}$ is *existentially closed in* $\mathfrak{N}$, which means that if $\varphi(x, y)$ is a quantifier-free formula in $\mathcal{L}$ and $b \in M^{|y|}$, then for all $\varepsilon > 0, r > 0$ the following holds:

$$(\exists a \in N^{|x|} : \text{ht}(a) \leq r)(\varphi(a, b)^{\mathfrak{N}} = 0) \implies (\exists a' \in M^{|x|} : \text{ht}(a') \leq r+\varepsilon)(|\varphi(a', b)^{\mathfrak{M}}| \leq \varepsilon).$$

Equivalently, this means that for any formula $\varphi(x, y)$ valued in $[0, 1]$ and any natural n, if

$$\mathfrak{N} \models \big((\sup_x f_n(\mathrm{ht}(x)) \cdot (1 - \varphi(x, b)\big) = 1,$$

then

$$\mathfrak{M} \models \big((\sup_x f_n(\mathrm{ht}(x)) \cdot (1 - \varphi(x, b)\big) = 1,$$

where f_n are the functions from Example 1.11.7.

The ε shows up in the definition, because with it we get the following lemma directly generalising the discrete logic case.

Lemma 1.12.2. Let $\mathfrak{M} \subset \mathfrak{N}$ be an inclusion of $\mathcal{L}$-structures. The following conditions are equivalent:

1. $\mathfrak{M} \prec_{\exists} \mathfrak{N}$.
2. In $S_n^{\mathrm{qf}}(M)$, the set of quantifier-free types realised in $\mathfrak{M}$ is dense in the set of quantifier-free types realised in $\mathfrak{N}$ for all $n \in \mathbb{N}$.
3. For some ultrapower $\mathfrak{M}^*$ of $\mathfrak{M}$, there is an embedding $\mathfrak{N} \hookrightarrow \mathfrak{M}^*$ over $\mathfrak{M}$.

Proof. (3) $\Rightarrow$ (1) follows directly from Łoś's theorem.

(1) $\Rightarrow$ (3) Let $(a_\alpha)_{\alpha<\kappa}$ be an enumeration of $\mathfrak{N}$, let $r_\alpha = \mathrm{ht}(x_\alpha)$ and let Σ be the set of conditions of the form $\varphi_i = 0$ in the variables $(x_\alpha)_{\alpha<\kappa}$, with parameters in M, which are satisfied by the a_α. Then, by (1), Σ is finitely approximately satisfiable in $\mathfrak{M}$, so by Corollary 1.11.14, Σ is satisfiable in an ultrapower of $\mathfrak{M}$.

(1) $\Rightarrow$ (2) Let $a \in N^n$, and let $U \subset S_n^{\mathrm{qf}}(M)$ be an open set containing $\mathrm{qftp}(a/M)$. Then, U contains a base open which can be expressed as a finite intersection of conditions of the form $\{|\varphi_i| < \delta\}$, say for $i \leq m$, where all φ_i are quantifier-free formulas with parameters in M. Then, Definition 1.12.1 applied to the formula $\varphi = \max(|\varphi_1 - \varphi_1(a)^{\mathfrak{N}}|, \ldots, |\varphi_m - \varphi_m(a)^{\mathfrak{N}}|\}$ with $r = \mathrm{ht}(a)$ and ε small enough, gives the existence of an $a' \in M^n$ such that for all $i \leq n$, $|\varphi(a')^{\mathfrak{M}}| < \delta$. Hence, $\mathrm{qftp}(a'/M) \in U$, which proves (2).

(2) $\Rightarrow$ (1) follows from the fact that the intersection of conditions $\mathrm{ht}(x) < r + \varepsilon$ and $|\varphi(x, b)| < \varepsilon$ defines a neighborhood of $\mathrm{qftp}(a/M)$ in $S_n(M)$. □

Similarly, we write $\mathfrak{M} \prec \mathfrak{N}$ and call the inclusion *elementary*, if for all tuples a in $\mathfrak{M}$ and all formulas $\varphi(x)$ we have $\varphi(a)^{\mathfrak{M}} = \varphi(a)^{\mathfrak{N}}$. If the embedding $\mathfrak{N} \hookrightarrow \mathfrak{M}^*$ in the above lemma is elementary, then $\mathfrak{M} \prec \mathfrak{N}$ by Łoś's lemma (and also vice versa).

Definition 1.12.3. An $\mathcal{L}$-structure $\mathfrak{M} \models T$ is *existentially closed* (with respect to T) if for all extensions $\mathfrak{M} \subset \mathfrak{N} \models T$ we have $\mathfrak{M} \prec_\exists \mathfrak{N}$. We call T *model complete*, if for all embeddings $\mathfrak{M} \subset \mathfrak{N}$ of models of T we have $\mathfrak{M} \prec \mathfrak{N}$.

Model complete theories also have the following characterisation.

Lemma 1.12.4. Let T be a theory. Then, T is model complete if and only if all models of T are existentially closed.

Proof. The forward direction is clear, since $\mathfrak{M} \prec \mathfrak{N}$ implies $\mathfrak{M} \prec_\exists \mathfrak{N}$.

Conversely, assume all models of T are existentially closed, let $\mathfrak{M} \subseteq \mathfrak{N}$ be an inclusion of models of T, and let φ be a sentence with parameters in M. We show that $\varphi^{\mathfrak{N}} = \varphi^{\mathfrak{M}}$ by induction on the complexity of φ.

If φ is quantifier-free, then by definition $\varphi^{\mathfrak{N}} = \varphi^{\mathfrak{M}}$.

If $\varphi = \inf_x f(\mathrm{ht}(x)) \cdot \psi(x)$, with ψ such that $\psi(a)^{\mathfrak{N}} = \psi(a)^{\mathfrak{M}}$ for all $a \in M^n$, then $\varphi^{\mathfrak{N}} = \min(0, \inf_{a \in N^n} f(\mathrm{ht}(a)) \cdot \psi(a)^{\mathfrak{N}}) \leq \varphi^{\mathfrak{M}}$. On the other hand, by Lemma 1.12.2, there is an ultraproduct $\mathfrak{M}^*$ of $\mathfrak{M}$ such that $\mathfrak{N}$ embeds into $\mathfrak{M}^*$ over $\mathfrak{M}$. The same argument as above shows that $\varphi^{\mathfrak{M}^*} \leq \varphi^N$. But, by Łoś's theorem, we have $\varphi^{\mathfrak{M}^*} = \varphi^{\mathfrak{M}}$, hence $\varphi^{\mathfrak{N}} = \varphi^{\mathfrak{M}}$.

If φ is of the form $\sup_x f(\mathrm{ht}(x)) \cdot \psi(x)$, the same argument applied to $-\psi(x)$ shows that $\varphi^{\mathfrak{N}} = \varphi^{\mathfrak{M}}$, which concludes the proof. ☐

Definition 1.12.5. We say that an $\mathcal{L}$-theory T^* is a *model companion* of T, if T^* is model complete and:

- all models of T admit an embedding into a model of T^*,
- all models of T^* admit an embedding into a model of T.

If a model companion of T exists, it is unique. Moreover, similarly as in the discrete logic, there is a following characterisation of existence of a model companion.

Lemma 1.12.6. Assume that T is *inductive*, i.e., that a sum of a chain of models of T is also a model of T. Let K be the class of existentially closed models of T. Then the following are equivalent:

1. K is an elementary class,
2. T has a model companion.

Moreover, if these conditions are satisfied, the model companion is the common theory of K.

Proof. This is a straightforward consequence of Lemma 1.12.4. □

We mention that if T is inductive, existentially closed models exist and every model of T can be embedded in such. Note that theories GVF_e are inductive, for $e \geq 0$. A very important question for the development of globally valued fields is the following conjecture.

Conjecture 1.12.7. For $e \geq 0$, the theory GVF_e has a model companion.

The affirmative answer to this question would open a path for the use of model theoretic methods in studying heights. In the first instance, it would follow that existentially closed globally valued fields are elementarily equivalent to one of the following (up to a scaling of the GVF structure): $\overline{\mathbb{Q}}$ with $\mathrm{ht}(2) = \log(2)$, $\overline{\mathbb{Q}(t)}$ with $\mathrm{ht}(t) = 1$, $\mathrm{ht}\,|_{\mathbb{Q}} = 0$, and $\overline{\mathbb{F}_p(t)}$ with $\mathrm{ht}(t) = 1$ for a prime p. This is because $\overline{\mathbb{Q}(t)}, \overline{\mathbb{F}_p(t)}$ and $\overline{\mathbb{Q}}[1]$ are existentially closed by [10, 71] respectively (for the proof of the latter, see Chapter 2). The proof methods in loc.cit. and the recent developments of adelic curves [52, 72] suggest a positive answer.

Let us note that existential closedness of the above GVFs and Lemma 1.12.2 imply the following result, which due to Corollary 1.7.11 may shed some light on the study of adelic curves.

Corollary 1.12.8. Every globally valued field K embeds in an ultrapower of:

- either $\overline{\mathbb{Q}}[r]$ if K is of characteristic zero and $r = \mathrm{ht}(2) > 0$,
- or $\overline{\mathbb{Q}(t)}[1]$ (with $\mathrm{ht}(2) = 0$) if K is of characteristic zero and $\mathrm{ht}(2) = 0$,
- or $\overline{\mathbb{F}_p(t)}[1]$ if K is of characteristic p.

The GVF in the middle dot could be replaced by any GVF of the form $\overline{k(t)}[1]$ with k of characteristic zero, for example by $\overline{\mathbb{C}(t)}[1]$. Below is another corollary that follows from existential closedness of $\overline{\mathbb{Q}}[1]$.

Corollary 1.12.9. Let X be a variety over $\mathbb{Q}$ and let h be a quantifier-free definable function on X (for example a Néron–Tate height on an abelian variety $X \subset A$ or the Weil height with respect to some embedding $X \subset \mathbb{P}^k$). Assume that there is an embedding $X \subset \mathbb{P}^k$ and numbers $C, D > 0$ such that $h_k \leq C \cdot h + D$ on $X(\overline{\mathbb{Q}})$, and that there are only finitely many elements in $X(\overline{\mathbb{Q}})$ of height smaller than h_0. Then these elements can be computed effectively (knowing C and D).

Proof. Let $\{x_1, \ldots, x_n\} = \{x \in X(\overline{\mathbb{Q}}) : h(x) < h_0\}$. Assume that $x \in X(L)$ for some GVF extension $\overline{\mathbb{Q}} \subset L$ satisfies $h(x) < h_0$. Then by existential closedness of $\overline{\mathbb{Q}}[1]$, there is $x' \in X(\overline{\mathbb{Q}}) \setminus \{x_1, \ldots, x_n\}$ such that $h(x') < h_0$, which gives a contradiction. Hence, $\{x_1, \ldots, x_n\}$ is the set of all solutions to the condition $h(x) < h_0$ on X in all GVF extensions of $\overline{\mathbb{Q}}[1]$.

Let L_d be a discrete first-order language with two sorts, one in the ring language and another one in an ordered field language. We also add constants on the ordered field sort R, indexed by the real numbers, and constants on the ring sort, indexed by the algebraic numbers. Let T_d be the first order theory given by the GVF axioms from Definition 1.1.1 (with $-\infty$ replaced by -1), but where the values of the heights land in the sort R, and where R is a real-closed field containing $\mathbb{R}$. Furthermore, we add the quantifier-free formulas about the GVF $\overline{\mathbb{Q}}$ to the theory T_d. Note that for any $M \models T_d$, the set $M_b := \{m \in M : \mathrm{ht}(m) \text{ is bounded}\}$ has a natural GVF structure, where one takes the height valued in R^M and post-composes it with the standard-part function. Let $E = C \cdot h_0 + D + 1$. Then, by the assumptions and the first paragraph of this proof,

$$T_d \models \{x \in X : h(x) < h_0 \wedge h_k(x) < E\} = \{x_1, \ldots, x_n\}.$$

Hence, by completeness (for discrete first order logic), there is an algorithm that will find the proof of this statement - a program listing all consequences of T_d will at some point arrive at this sentence for some $x_1, \ldots, x_n$. $\square$

Chapter 2

Existential closedness of algebraic numbers

This chapter is based on the author's paper [71]. Some changes are made to incorporate the results from Chapter 1, mostly in Section 2.3.2. The paper has been submitted to a journal.

2.1 Introduction

2.1.1 Main results

In this paper we study polynomial equations with height conditions. Recall that for a number $q \in \mathbb{Q}^{\times}$ its height is $\mathrm{ht}(q) := \log(\max(|a|, |b|))$, if $q = a/b$ for some coprime integers a, b. It measures the complexity of a number. This height extends to the absolute logarithmic Weil height on $\overline{\mathbb{Q}}$, and on tuples $\overline{a} \in \overline{\mathbb{Q}}^m$ we define it to be the maximum of heights of coordinates. Note that it is not invariant under (even linear) changes of coordinates. In general, if X is a variety over $\overline{\mathbb{Q}}$ and $f, g : X \to \mathbb{A}^m_{\overline{\mathbb{Q}}}$ are morphisms, we get an induced height for $x \in X(\overline{\mathbb{Q}})$ given by $\mathrm{ht}(f(x)) - \mathrm{ht}(g(x))$. If $h_1, \ldots, h_n$ are functions on $X(\overline{\mathbb{Q}})$ obtained in this way, one could ask the following question.

Question I. Given real numbers $r_1, \ldots, r_n$, does there exist $x \in X(\overline{\mathbb{Q}})$ with $h_1(x) = r_1, \ldots, h_n(x) = r_n$?

This question is important when one wants to compare various heights, for example when on an abelian variety one looks at the naive heights and the Néron–Tate height, see for example inequalities in [41, Part B].

Without conditions on heights, the (weak) Nullstellensatz provides the answer. More precisely, X has a $\overline{\mathbb{Q}}$-point if and only if X has an F-point, for some field extension $\overline{\mathbb{Q}} \subset F$, which is equivalent to non-emptiness of X.

Of course, even for a non-empty X, the answer cannot be positive for any tuple of real numbers, essentially for two reasons. First, the height function is not onto $\mathbb{R}$. Second, there may be some relations between heights $h_1, \ldots, h_n$, for example $h_1 + h_2 = h_3$ or $h_1 \geq 0$. To fix the first problem we ask instead whether there is a point approximating values $r_1, \ldots, r_n$ up to ε, for some $\varepsilon > 0$. To resolve the other problem, we need to make the definition of height robust enough, so that the necessary conditions on tuples $r_1, \ldots, r_n$ are simple to state.

The Weil height machine (see [41, Part B]) is an example of a robust formalism for heights on varieties over $\overline{\mathbb{Q}}$. For a projective variety X over $\overline{\mathbb{Q}}$ with a line bundle L, the Weil height machine associates a function $h_L : X(\overline{\mathbb{Q}}) \to \mathbb{R}$ which is well defined only up to a bounded term. If L is very ample, then h_L has a representative of the form described above, on the complement of the hypersurface given by the zeroes of a section of L. However, to make sense of our motivating question we need to have heights defined not up to bounded terms. Fortunately, one can equip L with an extra datum which will give a representative of h_L. For simplicity assume that X is over $\mathbb{Q}$ and choose a projective integral scheme $\mathcal{X}$ over $\mathbb{Z}$ with a line bundle $\mathcal{L}$, such that $\mathcal{X}_{\mathbb{Q}} = X, \mathcal{L}_{\mathbb{Q}} = L$. Such a scheme is called an *arithmetic variety.* Assume additionally that X is smooth over $\mathbb{Q}$ and that h is a conjugation invariant hermitian metric on the complex analytification of L. A pair $\overline{\mathcal{L}} = (\mathcal{L}, h)$ is called a *hermitian line bundle* on $\mathcal{X}$ and it defines a height function $h_{\overline{\mathcal{L}}} : X(\overline{\mathbb{Q}}) \to \mathbb{R}$. If x is a closed point in X, then $h_{\overline{\mathcal{L}}}(x)$ is the normalised arithmetic degree of the line bundle $\overline{\mathcal{L}}$ restricted to the curve defined by taking the closure of $\{x\}$ in $\mathcal{X}$. For the details of this approach, see [18]. We choose to work with slightly more general objects parametrising heights, namely *arithmetic* $\mathbb{R}$*-divisors of* C^0*-type* introduced by Moriwaki in [58]. These are pairs $\overline{\mathcal{D}} = (\mathcal{D}, g)$, where $\mathcal{D}$ is an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$ and g is a $\mathcal{D}$-Green's function of C^0-type. A trivial example of such object is a *principal* $\mathbb{R}$-divisor of C^0-type, i.e., a real combination of $\widehat{\operatorname{div}}(f) = (\operatorname{div}(f), -\log|f|^2)$ for f's in the function field of X. For a closed point $x \in X$, one defines $h_{\overline{\mathcal{D}}}(x)$ as the normalised (arithmetic) intersection of $\overline{\mathcal{D}}$ and the closure of $\{x\}$ in $\mathcal{X}$, see Subsection 2.2.4. Now we can ask a more refined version of the question above.

Question II. Given real numbers $r_1, \ldots, r_n$, $\varepsilon > 0$, and arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{D}}_1, \ldots, \overline{\mathcal{D}}_n$ on $\mathcal{X}$, does there exist $x \in X(\overline{\mathbb{Q}})$ with

$$|h_{\overline{\mathcal{D}}_i}(x) - r_i| < \varepsilon \text{ for all } i = 1, \ldots, n?$$

The arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ form a vector space over $\mathbb{R}$ denoted $\mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$. The main theorem of this paper gives the following answer.

Theorem A (Theorem 2.2.70)**.** Let $\overline{\mathcal{D}}_0, \ldots, \overline{\mathcal{D}}_n$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ with $\overline{\mathcal{D}}_0 = (\mathrm{div}(2), 0)$. Fix $\varepsilon > 0$, a closed strict subscheme $Z \subset X = \mathcal{X} \otimes \mathbb{Q}$, and a linear functional

$$l : \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R},$$

which is non-negative on effective arithmetic $\mathbb{R}$-divisors of C^0-type, zero on principal arithmetic $\mathbb{R}$-divisors of C^0-type, and sends $\overline{\mathcal{D}}_0$ to $\log(2)$. Then there exists a closed point $x \in X \setminus Z$ such that for all $i = 0, \ldots, n$

$$|h_{\overline{\mathcal{D}}_i}(x) - l(\overline{\mathcal{D}}_i)| < \varepsilon.$$

The theorem still holds if l is defined only on the real vector subspace of $\mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ generated by $\overline{\mathcal{D}}_0, \ldots, \overline{\mathcal{D}}_n$, provided that one of $\overline{\mathcal{D}}_1, \ldots, \overline{\mathcal{D}}_n$ is big in the sense of Definition 2.2.39, see Corollary 2.3.28. Note that the conditions on values of heights $h_{\overline{\mathcal{D}}_0}(x), \ldots, h_{\overline{\mathcal{D}}_n}(x)$ in Theorem A are optimal, if we expect to find a solution x general enough. Indeed, always $h_{\overline{\mathcal{D}}_0}(x) = \log(2)$; if $x \notin \mathrm{supp}(\mathcal{D}_i)$ and $\overline{\mathcal{D}}_i$ is effective, then $h_{\overline{\mathcal{D}}_i}(x) \geq 0$ for all $i = 1, \ldots, n$; and for any principal $\overline{\mathcal{E}}$ we have $h_{\overline{\mathcal{E}}}(x) = 0$, by the product formula.

One can consider another way of making sense out of Question I. Note that for a number $q \in \mathbb{Q}^{\times}$, the height $\mathrm{ht}(q)$ can be expressed in a way that does not involve a presentation as a quotient of two integers:

$$\mathrm{ht}(q) = -\min(-\log|q|, 0) + \sum_{p \text{ prime}} -\min(v_p(q), 0)\log p,$$

where v_p is the p-adic valuation on $\mathbb{Q}$ with $v_p(p) = 1$. More generally, let us consider the measure μ on the space of (non-Archimedean and Archimedean) valuations Val_K for a number field K, defined by the following formula

$$\mu := \frac{1}{[K : \mathbb{Q}]}\Bigg(\sum_{p \in \mathrm{Spec}(\mathcal{O}_K)} \delta_{\mathrm{ord}_p} \cdot \log \#\kappa(p) + \sum_{\sigma : K \to \mathbb{C}} \delta_{-\log|\sigma(\cdot)|}\Bigg).$$

The first sum is taken only over closed points, and the second sum is taken over all embeddings of fields (so that $-\log|\sigma(\cdot)|$ is an Archimedean valuation for an embedding σ). By $\kappa(p)$ we mean the residue field of the point p and δ denotes the Dirac delta. Then for $q \in K^\times$ we have

$$\mathrm{ht}(q) = \int_{\mathrm{Val}_K} -\min(v(q), 0) d\mu(v)$$

and

$$\int_{\mathrm{Val}_K} v(q) d\mu(v) = 0.$$

If $t(x_1, \dots, x_n)$ is an expression composed out of n free variables, addition, $\mathbb{Q}$-scalar multiplication and taking minima (i.e., it is a $\mathbb{Q}$-tropical polynomial, see Definition 1.4.1), for a tuple $a_1, \dots, a_n \in (K^\times)^n$ we can define the "t-height" of this tuple by

$$R_t(a_1, \dots, a_n) := \int_{\mathrm{Val}_K} t(v(a_1), \dots, v(a_n)) d\mu(v).$$

It does not depend on the number field K, so we get a function $R_t : (\overline{\mathbb{Q}}^\times)^n \to \mathbb{R}$. Note that (representatives of) Weil heights are included among these functions.

In general, a field F equipped with a collection of functions $R_t : (F^\times)^n \to \mathbb{R}$ for each $\mathbb{Q}$-tropical polynomial t is called a *globally valued field* (abbreviated GVF) if it satisfies the axioms from Definition 1.7.1. One denotes

$$R_t(\overline{a}) =: \int t(v(\overline{a})) dv$$

and the product formula axiom asserts that

$$\int v(a) dv = 0 \text{ for all } a \in F^\times.$$

These unbounded continuous logic structures were introduced by Ben Yaacov and Hrushovski in [10]. The notation with integrals is natural in the sense that every GVF structure on F comes from a unique (up to a renormalisation) measure μ on the space Ω_F of pseudo-valuations on F, such that for all $\mathbb{Q}$-tropical polynomials t and all tuples $\overline{a}$ in $F^\times$

$$\int t(v(\overline{a})) dv = \int_{\Omega_F} t(v(\overline{a})) d\mu(v).$$

For the details on this construction see Chapter 1, Theorem 1.7.7.

One can reformulate Question I in the language of globally valued fields in the following way.

Question III. Given real numbers $r_1, \ldots, r_n$, $\varepsilon > 0$, polynomials $f_1, \ldots, f_s, g \in \overline{\mathbb{Q}}[x_1, \ldots, x_m]$ and $\mathbb{Q}$-tropical polynomials (in m variables) $t_1, \ldots, t_n$, does there exist $\overline{a} = (a_1, \ldots, a_m) \in (\overline{\mathbb{Q}}^{\times})^m$ satisfying $f_1(\overline{a}) = \cdots = f_s(\overline{a}) = 0, g(\overline{a}) \neq 0$ with

$$|R_{t_i}(\overline{a}) - r_i| < \varepsilon \text{ for all } i = 1, \ldots, n?$$

The answer to this question cannot be positive for all tuples $r_1, \ldots, r_n$ as for example it may happen that $R_{t_1+t_2} = R_{t_1} + R_{t_2}$, or $R_{t_3} \geq 0$. Moreover, equations $f_1(\overline{a}) = 0, \ldots, f_s(\overline{a}) = 0$ can add some constrains on the heights. To see for which tuples $r_1, \ldots, r_n$ existence of solutions is possible/expected one can use a definition common in model theory. Namely, call equations

$$f_1(\overline{a}) = \cdots = f_s(\overline{a}) = 0, g(\overline{a}) \neq 0, R_{t_1}(\overline{a}) = r_1, \ldots, R_{t_n}(\overline{a}) = r_n$$

consistent, if there is a GVF extension $\overline{\mathbb{Q}} \subset F$ and a tuple $\overline{a}$ in F satisfying them. We prove that consistent equations can be approximately solved in $\overline{\mathbb{Q}}$, which in the language of model theory means (essentially by definition) that $\overline{\mathbb{Q}}$ is *existentially closed* as a GVF, see Definition 2.3.21. Note that this can bee seen as a (weak) Nullstellensatz with height conditions.

Theorem B (Theorem 2.3.24)**.** $\overline{\mathbb{Q}}$ is an existentially closed GVF.

This theorem is an arithmetic analogue of [10, Theorem 2.1] which states that the field $\overline{k(t)}$ equipped with a natural GVF structure coming from the product formula on curves over the base field k (which is an arbitrary field) is existentially closed. Observe that as an immediate corollary from Theorem B, one gets a global Fekete-Szegő type result, i.e., [10, Theorem 3.11] for number fields (in loc.cit. it is conditional on the existential closedness of the algebraic numbers).

Theorem B served as a motivation for this project and it is a consequence of Theorem A because of a translation theorem between the languages of globally valued fields and of functionals on arithmetic $\mathbb{R}$-divisors. To express the translation, fix a finitely generated extension $\mathbb{Q} \subset F$ and as in Definition 2.3.1, let $\mathrm{ADiv}_{\mathbb{R}}(F)$ be the injective limit of $\mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ for all (normal, generically smooth) arithmetic varieties $\mathcal{X}$ with function field isomorphic to F. We call $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ a *GVF functional*, if it satisfies the following conditions, already appearing in Theorem A:

- it sends principal arithmetic $\mathbb{R}$-divisors of C^0-type to zero,
- it is non-negative on effective arithmetic $\mathbb{R}$-divisors of C^0-type.

Additionally, we call such l *normalised*, if $l((\mathrm{div}(2), 0)) = \log 2$. For the details see Definition 2.3.10. In Definition 2.3.3 we introduce a lattice structure on the space of arithmetic $\mathbb{Q}$-divisors of C^0-type denoted $\mathrm{ADiv}_{\mathbb{Q}}(F) \subset \mathrm{ADiv}_{\mathbb{R}}(F)$. In particular for a $\mathbb{Q}$-tropical polynomial t (of arity m) the expression $t(\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m)$ makes sense for $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m \in \mathrm{ADiv}_{\mathbb{Q}}(F)$. This lattice structure has also been considered in [77, 67] in the language of (generalised or b-) Weil functions. We prove the following.

Theorem C (Corollary 2.3.18)**.** Let F be a finitely generated extension of $\mathbb{Q}$. There is a natural bijection

$$\big\{\text{GVF functionals } \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}\big\} \longleftrightarrow \big\{\text{GVF structures on } F\big\}$$

$$l \longmapsto \big(R^l_t(\overline{a}) := l(\overline{\mathcal{D}}_t(\overline{a}))\big)_{\mathbb{Q}\text{-tropical polynomials } t}$$

where $\overline{\mathcal{D}}_t(\overline{a}) = t(\widehat{\mathrm{div}}(a_1), \dots, \widehat{\mathrm{div}}(a_n))$ if $\overline{a} = (a_1, \dots, a_n)$.

This theorem formally extends to arbitrary extensions of $\mathbb{Q}$, see Remark 2.3.19. Over finite prime fields an analogous geometric description of globally valued fields follows from [10, Section 11].

At the end of the paper we present a new interpretation of the essential infimum function ζ. If $\overline{\mathcal{D}}$ is an arithmetic $\mathbb{R}$-divisor of C^0-type on a normal, generically smooth arithmetic variety $\mathcal{X}$, then the essential infimum of $\overline{\mathcal{D}}$ is defined to be

$$\zeta(\overline{\mathcal{D}}) := \sup_{\mathcal{Y} \subset \mathcal{X}} \inf_{x \in \mathcal{X}(\overline{\mathbb{Q}}) \setminus \mathcal{Y}} h_{\overline{\mathcal{D}}}(x),$$

where the supremum is taken over all Zariski-closed strict subschemes $\mathcal{Y} \subset \mathcal{X}$. However, since points approximate arbitrary GVF functionals by Theorem A, the following holds.

Theorem D. [64, Theorem 4.6] Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Then

$$\zeta(\overline{\mathcal{D}}) \leq \inf\{l(\overline{\mathcal{D}}) : l \text{ is a normalised GVF functional on } \mathrm{ADiv}_{\mathbb{R}}(F)\}.$$

Moreover, if $\mathcal{D}_{\mathbb{Q}}$ is big, this is an equality.

This result was proved by Qu and Yin in a different language (on their right-hand side some arithmetic intersection numbers appear instead of evaluations of GVF functionals), and it generalises [3, Corollary 4.2]. The author would like to thank François Ballaÿ for providing a reference. We decided to include the above theorem in this introduction to underline the new interpretation of the essential infimum function

ζ. To see our presentation of the proof of Theorem D using Theorem A, see Theorem 2.3.32. Note that Theorem A is stronger, in the sense that it allows to consider several different arithmetic $\mathbb{R}$-divisor of C^0-type at once.

Assume that $\mathcal{X}$ is a normal, generically smooth arithmetic variety of dimension $d+1$ and let $\overline{\mathcal{D}}$ be a pseudo-effective arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$, with $\mathcal{D}_{\mathbb{Q}}$ big. The essential infimum function is related to other arithmetic invariants via Zhang's inequality which states (in the form from [3, Theorem 7.2.(ii)]) that under the above assumptions

$$\zeta(\overline{\mathcal{D}}) \geq \frac{\widehat{\mathrm{vol}}(\overline{\mathcal{D}})}{(d+1)\,\mathrm{vol}(\mathcal{D}_{\mathbb{Q}})} =: \varphi(\overline{\mathcal{D}}).$$

We prove the following theorem which gives a characterisation of when the equality occurs in Zhang's inequality.

Theorem E (Theorem 2.3.36)**.** Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Consider the statements:

(1) $\varphi(\overline{\mathcal{D}}) = \zeta(\overline{\mathcal{D}})$;

(2) $D_{\overline{\mathcal{D}}}\varphi$ is a GVF functional;

(3) the infimum

$$\zeta(\overline{\mathcal{D}}) = \inf\{l(\overline{\mathcal{D}}) : l \text{ is a normalised GVF functional on } \mathrm{ADiv}_{\mathbb{R}}(F)\}$$

is achieved at a unique normalised GVF functional;

(4) ζ is Gateaux differentiable at $\overline{\mathcal{D}}$ in every direction.

Then (1) $\iff$ (2) $\implies$ (3) $\iff$ (4).

Standard methods show that Gateaux differentiability of ζ at $\overline{\mathcal{D}}$ gives equidistribution, see Lemma 2.3.38 and [18, Section 8]. From the logic viewpoint, equidistribution corresponds to uniqueness of a GVF functional (at least its non-generic part), satisfying some equations, in the case of Theorem E of the form $l(\overline{\mathcal{D}}) = \zeta(\overline{\mathcal{D}})$. It is an interesting question why the classical equidistribution theorems are formulated with respect to a single divisor only.

We choose to work with arithmetic $\mathbb{R}$-divisors of C^0-type in this paper, however, one could work in a more general context of adelic $\mathbb{R}$-divisors on varieties over number fields (see [59], [3]). We only work over $\mathbb{Q}$ as this is enough for existential closedness of $\overline{\mathbb{Q}}$, essentially by the claim in the proof of Theorem 2.3.24. Also, from the GVF point of view passing from arithmetic to adelic divisors does not make much difference, since GVF functionals on arithmetic $\mathbb{R}$-divisors of C^0-type extend uniquely to GVF functionals on adelic $\mathbb{R}$-divisors on the generic fiber of the corresponding arithmetic variety. This follows from [59, Theorem 4.1.3].

2.1.2 The technique used to prove Theorem A

In this subsection we describe the proof technique used to prove Theorem A. It is an arithmetic generalisation of the argument for existential closedness of $\overline{k(t)}$ from [10] for a fixed base field k. Assume that $\overline{k} = k$. For a projective variety X over k, let $N^1(X)$ be the Néron–Severi group of X tensored with $\mathbb{R}$. The crucial geometric ingredient in [10] is the following theorem.

Theorem. [10, Theorem 10.9, Corollary 10.11] Let X be a projective smooth variety with a dominant morphism to the projective line $\pi : X \to \mathbb{P}^1$ over k. Assume that $l : N^1(X) \to \mathbb{R}$ is a linear functional having the following properties:

- l is non-negative on classes of effective divisors;
- for $D = \pi^* p$, where $p \in \mathbb{P}^1$ is closed, we have $l(D) = \deg_{\mathbb{P}^1}(p) = 1$ (by algebraic closedness).

Fix $\varepsilon > 0$ and let $D_1, \dots, D_n$ be a basis of $N^1(X)$. Then there exists an irreducible curve $C \subset X$ such that for all $i = 1, \dots, n$ we have

$$\Big|\frac{\deg_C(D_i)}{\deg(C/\mathbb{P}^1)} - l(D_i)\Big| < \varepsilon.$$

Note that in this case, if η is the generic point of $\mathbb{P}^1$, then C is determined by its generic point $\eta' \in X_\eta$ and $\frac{\deg_C(D_i)}{\deg(C/\mathbb{P}^1)}$ can be thought of as $h_{D_i}(\eta')$.

To outline the proof of this theorem from [10], we recall the terminology concerning divisors and curves on a projective smooth variety X over k. For details, see [49]. Assume that X is of dimension $d+1$.

- By $\mathrm{Z}_1(X), \mathrm{Z}_d(X) = \mathrm{Z}^1(X)$ we denote the cycles of dimension 1 and d respectively, in X. This means that these are free abelian groups generated by integral subvarieties of the corresponding dimension.
- There is a bilinear intersection pairing $\cdot : \mathrm{Z}_1(X) \times \mathrm{Z}^1(X) \to \mathbb{Z}$. It induces the numerical equivalences $\equiv$ on $\mathrm{Z}_1(X), \mathrm{Z}^1(X)$ where

$$D \equiv D' \iff (\text{for all } C \in \mathrm{Z}_1(X))(C \cdot D = C \cdot D'),$$

$$C \equiv C' \iff (\text{for all } D \in \mathrm{Z}^1(X))(C \cdot D = C' \cdot D).$$

We use the following notation for the quotients tensored with reals: $N^1(X) := (\mathrm{Z}^1(X)/\equiv) \otimes \mathbb{R}, N_1(X) := (\mathrm{Z}_1(X)/\equiv) \otimes \mathbb{R}$. Note that $N_1(X), N^1(X)$ are finite dimensional, and the induced pairing $\cdot : N_1(X) \times N^1(X) \to \mathbb{R}$ is perfect of signature $(1, -1, -1, \ldots, -1)$, by the Hodge index theorem.

- Let $\mathrm{Eff}_1(X) \subset N_1(X), \mathrm{Eff}^1(X) \subset N^1(X)$ be the closed convex cones generated by effective curves and effective divisors in $N_1(X)$ and $N^1(X)$ respectively. Denote by $N_1^+(X), N_+^1(X)$ the dual cones. More precisely

  $$N_1^+(X) = \{C : (\text{for all } D \in \mathrm{Eff}^1(X))(C \cdot D \geq 0)\},$$

  $$N_+^1(X) = \{D : (\text{for all } C \in \mathrm{Eff}_1(X))(C \cdot D \geq 0)\}.$$

  We call elements of $N_1^+(X)$ movable curves classes, and elements of $N_+^1(X)$ numerically effective (nef) divisors.

- Any $D \in \mathrm{Z}^1(X)$ is a Cartier divisor (as X is smooth), so it induces a line bundle $\mathcal{O}(D)$ on X. Its volume is the quantity

  $$\mathrm{vol}(D) := \limsup_{m \to \infty} \frac{\dim_k H^0(X, \mathcal{O}(mD))}{m^{d+1}/(d+1)!}.$$

  It extends to a continuous function $\mathrm{vol} : N^1(X) \to \mathbb{R}$. The *big cone* is the set

  $$\mathrm{Big}(X) := \{D : \mathrm{vol}(D) > 0\}.$$

  It can be equivalently described as the (euclidean) interior of $\mathrm{Eff}^1(X)$. On the other hand, the interior of $N_+^1(X)$ is the convex cone generated by classes of ample divisors D, i.e., such that the linear system $|mD|$ induces an embedding into a projective space, for m big enough.

- The volume function $\mathrm{vol} : N^1(X) \to \mathbb{R}$ is differentiable on the big cone and the derivative at $D \in \mathrm{Big}(X)$ is given by $(d+1)\langle D^d \rangle$, where $\langle D^d \rangle : N^1(X) \to \mathbb{R}$ is the *positive intersection product*. It can be seen as an element $\langle D^d \rangle \in N_1^+(X)$. Moreover, $\langle D^d \rangle$ can be approximated by pushforwards of d'th self intersections of ample D' on blowups $f : X' \to X$ with $D' \leq f^*D$. If D itself is an ample Cartier divisor, then $\langle D^d \rangle$ can be given by the d'th self intersection of D. In this case one can use the Theorem of Bertini to get an irreducible curve. For the details of this point see [15], [10, Section 10].

Sketch of the proof of the Theorem. We divide the argument from [10] into the following steps.

1. Replace l by an approximation that is strictly positive on $\mathrm{Eff}^1(X) \setminus \{0\}$.
2. Consider the function
$$\gamma := \frac{\mathrm{vol}^{1/(d+1)}}{l} : N^1(X) \setminus \{0\} \to \mathbb{R}.$$
By the first point it is continuous, and for $t > 0$ we have $\gamma(D) = \gamma(tD)$. The restriction of γ to the unit sphere of $N^1(X)$ determines the whole function, and by compactness of the sphere it achieves maximum. If at B the maximum is achieved, then by calculating the derivative of (the logarithm of) γ at B we get
$$\frac{\langle B^d \rangle}{\mathrm{vol}(B)} = \frac{l}{l(B)}.$$
3. Pick a blowup $f : X' \to X$ and an ample B' on X' with $B' \leq f^*B$ such that the pushforward of the d'th self intersection of B' approximates $\langle B^d \rangle$ and $\mathrm{vol}(B')$ approximates $\mathrm{vol}(B)$. For simplicity assume that $X' = X, B' = B$. Let C be a d'th self intersection of B, which is irreducible by the Bertini theorem.
4. We get the equation
$$\frac{C}{\mathrm{vol}(B)} = \frac{l}{l(B)}.$$
By applying this equation on π^*p, where p is some closed point of $\mathbb{P}^1$, we get
$$\frac{\deg(C/\mathbb{P}^1)}{\mathrm{vol}(B)} = \frac{\deg_C(\pi^*p)}{\mathrm{vol}(B)} = \frac{l(\pi^*p)}{l(B)} = \frac{1}{l(B)},$$
where the first equality follows from the projection formula for π. Thus
$$\deg(C/\mathbb{P}^1) = \frac{\mathrm{vol}(B)}{l(B)}.$$
By putting everything together we get
$$\frac{C}{\deg(C/\mathbb{P}^1)} = l,$$
Note that in the first step we replaced l by an approximation, so at the end we get the conclusion if the approximation was good enough.

□

The proof of Theorem A uses the same steps, but we need to work with arithmetic $\mathbb{R}$-divisors of C^0-type, instead of Cartier divisors. Fortunately the necessary arithmetic variants of the geometric theorems used in the above sketch are available. However, in the arithmetic context a few additional steps are necessary. More precisely, in the arithmetic analogue of step (3) from the above sketch, a lower bound on the (geometric) volume of the generic part of the divisor is needed. This bound is obtained by using arithmetic Fujita approximation [22], [82] together with Zhang's inequality [84], see Lemma 2.2.59 and Proposition 2.2.68. The other crucial ingredients are differentiability of the arithmetic volume [23] (see also [45]) and arithmetic Bertini theorems [20] and [80]. Note that [20] was motivated by a question of Hrushovski at the SAGA in Orsay, regarding arithmetic type Bertini theorems.

2.1.3 Connections with other results

Existential closedness of $\overline{\mathbb{Q}}$ and $\overline{k(t)}$ sets a goal of axiomatizing existentially closed models of the theory GVF. Ben Yaacov has published research notes [7], [6] related to this aim. The model companion is conjectured to exist (see Conjecture 1.12.7), and it is possible that it is tame in some model theoretic sense. For example, by [11, Theorem 14.2], GVF structures are quantifier-free stable (i.e., quantifier-free formulas do not have the order property in the continuous logic sense).

On the other hand, a more geometrical approach to globally valued fields was developed by Chen and Moriwaki in the series of papers about *adelic curves* (see Definition 1.7.10) [24], [25], [26]. The authors in [25] introduced Arakelov intersection theory over an arbitrary countable adelic curve. Differentiability of the volume function in the context of adelic curves may lead to generalisations of the main result of this paper to a bigger class of GVF structures. The results in [53], [72] are very promising in this direction. Also, see Chapter 3 for a continuous logic approach to the intersection theory from [25] and an application in Arakelov geometry.

Existential closedness of $\overline{\mathbb{Q}}$ also has some immediate applications to understanding spectra of height functions. For example, it implies that if $\overline{\mathcal{D}}$ is a geometrically big arithmetic $\mathbb{R}$-divisor of C^0-type on an arithmetic variety $\mathcal{X}$, then the set

$$\{h_{\overline{\mathcal{D}}}(x) : x \in \mathcal{X}(\overline{\mathbb{Q}})\}$$

is dense in the interval $(\zeta(\overline{\mathcal{D}}), +\infty)$. More generally, for an n-tuple of arithmetic $\mathbb{R}$-divisors of C^0-type, the set of points in $\mathbb{R}^n$ that can be approximated by the values of heights of tuples from $\mathcal{X}$ is convex. This follows from the fact that the set of GVF structures on $\mathbb{Q}(\mathcal{X})$ extending the standard one on $\mathbb{Q}$ is convex (and closed under addition of GVF structures trivial on $\mathbb{Q}$). In particular, this automatically recovers the 'density' parts of the results from [46]. Furthermore, using the 'continuity of heights' from Chapter 3, one can also recover the density of values of the Faltings height above its essential infimum, i.e., the 'density' part of [16, Corollary 1.1].

2.1.4 Acknowledgements

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2.1.5 Outline

The article is organized as follows. In Section 2.2 we introduce all the necessary background on arithmetic $\mathbb{R}$-divisors of C^0-type. In Subsection 2.2.10 we give the proof of Theorem A. Next in Section 2.3 we introduce lattice divisors and use them to prove Theorem B using the translation from Theorem C. In Subsection 2.3.3 we prove Theorem D and Theorem E.

2.2 Intersection theory of arithmetic divisors

Notation. We stick with the following conventions.

1. If f is a function on a real vector space, then we write $D_x f(y)$ for the derivative of f at x in direction y, so that $D_x f$ is a linear map (if f is Gateaux differentiable at x).

2. If A is an abelian group, and $\mathbb{K}$ is one of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$ we write $A_{\mathbb{K}}$ for the tensor product $A \otimes_{\mathbb{Z}} \mathbb{K}$. If A is written multiplicatively, then for $f_1, \dots, f_n \in A$ and $\alpha_1, \dots, \alpha_n \in \mathbb{K}$ we write $f_1^{\alpha_1} \cdots f_n^{\alpha_n}$ for the product $f_1 \otimes \alpha_1 \cdot \ldots \cdot f_n \otimes \alpha_n$.

3. For a field F we call a function $v : F^{\times} \to \mathbb{R}$ a *non-Archimedean valuation* if it is a multiplicative homomorphism and satisfies the ultrametric inequality (with the convention $v(0) = \infty$).

4. By an *Archimedean valuation* on a field F we mean a function $v : F^\times \to \mathbb{R}$ of the form $v(-) = -\log|\sigma(-)|$, where $\sigma : F \to \mathbb{C}$ is an embedding of fields. Note that complex conjugate embeddings give the same Archimedean valuation and every Archimedean valuation comes from a pair of conjugate embeddings (or from a single real embedding).

5. By a *valuation* we mean either a non-Archimedean or an Archimedean valuation on a field and we denote by Val_F the space of valuations on a field F (with the topology coming from the embedding into $\mathbb{R}^{F\setminus\{0\}}$).

6. If $\mathcal{X}$ is a scheme over $\mathrm{Spec}(A)$ and we have a map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ we write $\mathcal{X}_B$ or $\mathcal{X} \otimes B$ for the fiber product. By a strict subscheme of $\mathcal{X}$ we mean a subscheme that is not equal to $\mathcal{X}$.

7. If $p \in \mathcal{X}$ is a point in a scheme, we write $\kappa(p)$ for the residue field. The function field of $\mathcal{X}$ is denoted by $\kappa(\mathcal{X})$.

8. For a scheme $\mathcal{X}$ we denote by $\mathrm{Z}_d(\mathcal{X})$ the group of d-dimensional cycles on $\mathcal{X}$, i.e., the free abelian group generated by dimension d integral subschemes of $\mathcal{X}$. Moreover, we denote by $\mathrm{Div}(\mathcal{X})$ the group of Cartier divisors on $\mathcal{X}$ and by $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$ the invertible subsheaf of the sheaf of rational functions on $\mathcal{X}$ corresponding to $\mathcal{D}$, as defined in [39, Chapter II, Section 6] or [24, Definition 2.4.2]. Note that then the rational function 1 defines a canonical rational section of $\mathcal{O}_{\mathcal{X}}(\mathcal{D})$.

9. If $\mathcal{L}$ is a line bundle on a scheme $\mathcal{X}$ and s is a rational section of $\mathcal{L}$, then as in [24, Remark 2.4.4], by $\mathrm{div}(s)$ we denote the Cartier divisor on $\mathcal{X}$ associated to the invertible subsheaf of the sheaf $\mathcal{K}$ of invertible rational functions on $\mathcal{X}$, defined by the image of the embedding $\mathcal{L} \to \mathcal{K}$ sending s to 1. We use the same notation for the cycle associated with this Cartier divisor.

10. An *arithmetic variety* in this paper is an integral (irreducible and reduced) scheme, finite type and projective over $\mathrm{Spec}(\mathbb{Z})$. It is *generically smooth* if the base change $\mathcal{X}_{\mathbb{Q}}$ is smooth over $\mathrm{Spec}(\mathbb{Q})$. We write $\deg(x) = [\kappa(x) : \mathbb{Q}]$ for a closed point of the generic fiber $x \in \mathcal{X}_{\mathbb{Q}} \subset \mathcal{X}$.

11. If $\mathcal{X}$ is an arithmetic variety and v is a non-Archimedean valuation on $\kappa(\mathcal{X})$, we write $\mathrm{supp}(v) \in \mathcal{X}$ for the image of the maximal ideal of the value ring

$\mathcal{O}_v \subset \kappa(\mathcal{X})$ via the unique map ϕ making the following diagram commute:

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(\mathcal{X})) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{Spec}(\mathcal{O}_v) \arrow[r] \arrow[ur, dashed, "\phi"] & \mathrm{Spec}(\mathbb{Z}).
\end{tikzcd}$$

Such map always exists by the valuative criterion of properness.

2.2.1 Hermitian line bundles on arithmetic varieties

For this subsection let $\mathcal{X}$ be a generically smooth arithmetic variety. Let $d+1$ be the dimension of $\mathcal{X}$ and let $X = \mathcal{X}_{\mathbb{Q}}$ be the generic fiber.

Warning 2.2.1. In this subsection $\mathcal{X}$ is **not** necessarily **normal**.

Remark 2.2.2. We use the following conventions regarding complex analytifications of arithmetic varieties.

1. By $\mathcal{X}(\mathbb{C})$ we mean the complex analytification of the base change $\mathcal{X}_{\mathbb{C}} = \mathcal{X} \otimes \mathbb{C}$. It is in a natural bijection with the set of maps $\mathrm{Spec}(\mathbb{C}) \to \mathcal{X}$. Complex conjugation induces a map $\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\mathbb{C})$ and by precomposition it induces *complex conjugation* $F_\infty : \mathcal{X}(\mathbb{C}) \to \mathcal{X}(\mathbb{C})$.

2. In the context of the first point, if $x \in X$ is a closed point and $\sigma : \kappa(x) \to \mathbb{C}$ is an embedding of fields, we denote by x^σ the point of $\mathcal{X}(\mathbb{C})$ corresponding to the composition
$$\mathrm{Spec}(\mathbb{C}) \to \mathrm{Spec}(\kappa(x)) \to X \to \mathcal{X}.$$

Definition 2.2.3. Let M be a complex manifold with a (complex) line bundle L. A *hermitian metric* h on L is a collection of hermitian metrics h_x on fibers L_x of L for all $x \in M$. It is *smooth* if for all open $U \subset M$ and all sections s of L over U, the function $|s|_h^2(x) := h_x(s(x), s(x))$ is a smooth function $U \to \mathbb{R}$. Since the data of a smooth hermitian metric is determined by functions $|\cdot|_h$ (and vice versa), we use the name "smooth hermitian metric" also for collections of norms of sections coming from some h as described above. The *Chern form* of $\overline{L} := (L, h)$ is a differential $(1,1)$-form $c_1(\overline{L})$ on M such that if s is an invertible (holomorphic) section of L over $U \subset M$, then $c_1(\overline{L})|_U = \frac{i}{2\pi}\partial\overline{\partial} \log |s|_h^2$.

Definition 2.2.4. Let $M = \mathcal{X}(\mathbb{C})$ and let $\overline{L} = (L, h)$ be a line bundle on M equipped with a hermitian metric h. We say that h is of *real type*, if for all $x \in M$ with $\overline{x} := F_\infty(x)$ and for all sections $s, s' \in L_x$ we have

$$h_x(s, s') = \overline{h_{\overline{x}}(F_\infty(s), F_\infty(s'))},$$

where the action on germs of sections is defined by precomposition with F_∞.

Definition 2.2.5. A *hermitian line bundle* $\overline{\mathcal{L}} = (\mathcal{L}, |\cdot|_{\overline{\mathcal{L}}})$ on $\mathcal{X}$ is a pair consisting of a line bundle $\mathcal{L}$ on $\mathcal{X}$ and a smooth hermitian metric of real type $|\cdot|_{\overline{\mathcal{L}}}$ on the complex analytification of the line bundle $\mathcal{L}_\mathbb{C}$ on $\mathcal{X}_\mathbb{C}$. A section $s \in H^0(\mathcal{X}, \mathcal{L})$ is *(strictly) small*, if $|s|_{\overline{\mathcal{L}}} \leq 1$ (resp. $|s|_{\overline{\mathcal{L}}} < 1$) on $\mathcal{X}(\mathbb{C})$. The (finite) set of small sections is denoted by $\widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}})$. The Chern form of the analytification of $\mathcal{L}_\mathbb{C}$ on $\mathcal{X}_\mathbb{C}$ is denoted by $c_1(\overline{\mathcal{L}})$.

Theorem 2.2.6. [18, Theorem (2.5)] Let $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d$ be hermitian line bundles on $\mathcal{X}$. There exists a unique family of linear maps

$$\widehat{\deg}(\overline{\mathcal{L}}_0 \cdots \overline{\mathcal{L}}_{n-1} | -) : \mathrm{Z}_n(\mathcal{X}) \to \mathbb{R},$$

for $n = 0, \ldots, d+1$ (for $n = 0$ we mean a map $\widehat{\deg}(-) : \mathrm{Z}_0(\mathcal{X}) \to \mathbb{R}$) satisfying the following properties:

1. For every integer $n \in \{0, \ldots, d\}$, every integral closed subscheme $\mathcal{Z}$ of $\mathcal{X}$ such that $\dim(\mathcal{Z}) = n+1$, every integer $m \neq 0$ and every regular meromorphic (i.e., defined on a dense open subset of $\mathcal{Z}$) section s of $\mathcal{L}_n^{\otimes m}|_{\mathcal{Z}}$, one has
$$m \, \widehat{\deg}(\overline{\mathcal{L}}_0 \cdots \overline{\mathcal{L}}_n | \mathcal{Z})$$
$$= \widehat{\deg}(\overline{\mathcal{L}}_0 \cdots \overline{\mathcal{L}}_{n-1} | \operatorname{div}(s)) + \int_{\mathcal{Z}(\mathbb{C})} - \log \|s\| \wedge c_1(\overline{\mathcal{L}}_0) \wedge \ldots \wedge c_1(\overline{\mathcal{L}}_{n-1}).$$
2. For a closed point z of $\mathcal{X}$ viewed as a 0-dimensional cycle, one has
$$\widehat{\deg}(z) = \log \#\kappa(z).$$

Moreover, these maps are multilinear and symmetric in the hermitian line bundles $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d$ and only depend on their isomorphism classes as hermitian line bundles.

Notation 2.2.7. If $n = d+1$ in the context of the above theorem, we skip $\mathcal{X}$ in the notation and write

$$\widehat{\deg}(\overline{\mathcal{L}}_0 \cdots \overline{\mathcal{L}}_d)$$

or just $\overline{\mathcal{L}}_0 \cdot \ldots \cdot \overline{\mathcal{L}}_d$.

Remark 2.2.8. For a morphism of generically smooth arithmetic varieties $f : \mathcal{Y} \to \mathcal{X}$, if $\overline{\mathcal{L}} = (\mathcal{L}, h)$ is a hermitian line bundle on $\mathcal{X}$, then one can pullback it to $\mathcal{Y}$ to get a hermitian line bundle $f^*\overline{\mathcal{L}} := (f^*\mathcal{L}, f^*h)$ on $\mathcal{Y}$.

Remark 2.2.9. Let $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_{l-1}$ be hermitian line bundles on $\mathcal{X}$. If $i : \mathcal{Z} \hookrightarrow \mathcal{X}$ is an l-dimensional irreducible reduced subscheme which is generically smooth, then the following intersection numbers are the same

$$\widehat{\deg}(\overline{\mathcal{L}}_0 \cdots \overline{\mathcal{L}}_{l-1} | \mathcal{Z}) = \widehat{\deg}(i^*\overline{\mathcal{L}}_0 \cdots i^*\overline{\mathcal{L}}_{l-1}).$$

Here on the right hand side we treat $\mathcal{Z}$ as a generically smooth arithmetic variety, and $i^*\overline{\mathcal{L}}_0, \dots, i^*\overline{\mathcal{L}}_{l-1}$ are hermitian line bundles on $\mathcal{Z}$. In this case we also write $\overline{\mathcal{L}}_0|_{\mathcal{Z}} \cdot \ldots \cdot \overline{\mathcal{L}}_{l-1}|_{\mathcal{Z}}$ for this intersection number.

Theorem 2.2.10. [18, Proposition (2.9)] Let $f : \mathcal{X}' \to \mathcal{X}$ be a generically finite morphism of arithmetic varieties, let $\mathcal{Z}$ be an integral closed subscheme of $\mathcal{X}'$ and let $l = \dim(\mathcal{Z})$. Assume that $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_{l-1}$ are hermitian line bundles on $\mathcal{X}$.

1. If $\dim(f(\mathcal{Z})) < l$, then $\widehat{\deg}(f^*\overline{\mathcal{L}}_0 \cdots f^*\overline{\mathcal{L}}_{l-1} | \mathcal{Z}) = 0$.
2. Otherwise, $\dim(f(\mathcal{Z})) = l$ and

$$\widehat{\deg}(f^*\overline{\mathcal{L}}_0 \cdots f^*\overline{\mathcal{L}}_{l-1} | \mathcal{Z}) = \widehat{\deg}(\overline{\mathcal{L}}_0 \cdots \overline{\mathcal{L}}_{l-1} | f_*(\mathcal{Z})).$$

where $f_*(\mathcal{Z}) = [\kappa(\mathcal{Z}) : \kappa(f(\mathcal{Z}))] f(\mathcal{Z})$ is an l-cycle on $\mathcal{X}$.

Definition 2.2.11. A hermitian line bundle $\overline{\mathcal{L}}$ on $\mathcal{X}$ is *ample* if the following three conditions are satisfied:

- $\mathcal{L}$ is relatively ample with respect to $\mathcal{X} \to \mathrm{Spec}(\mathbb{Z})$,
- $c_1(\overline{\mathcal{L}})$ is positive,
- $H^0(\mathcal{X}, n\mathcal{L})$ is generated (as a $\mathbb{Z}$-module) by strictly small sections of $n\overline{\mathcal{L}}$, for n big enough.

Definition 2.2.12. [80, Definition 1.1] Let $\overline{\mathcal{M}}, \overline{\mathcal{L}}$ be ample hermitian line bundles on $\mathcal{X}$. Denote by $\| \cdot \|$ the hermitian norm of $\overline{\mathcal{L}}$. A section $s \in \widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}})$ is $(\varepsilon, \overline{\mathcal{M}})$-*irreducible* if the following conditions are satisfied.

1. The d-cycle $\mathcal{Z} := \mathrm{div}(s)$ is irreducible (hence an arithmetic variety).

2. The following inequality holds

$$|\widehat{\deg}(\overline{\mathcal{M}}^d \cdot \overline{\mathcal{L}}) - \widehat{\deg}(\overline{\mathcal{M}}^d|\mathcal{Z})| < \varepsilon \cdot \widehat{\deg}(\overline{\mathcal{M}}^d|\mathcal{Z}).$$

By the inductive definition of the arithmetic intersection number (see Theorem 2.2.6), we can present the second condition in the following equivalent forms:

$$\int_{\mathcal{X}(\mathbb{C})} -\log\|s\|^2 \wedge c_1(\overline{\mathcal{M}})^d < \varepsilon \cdot \widehat{\deg}(\overline{\mathcal{M}}^d|\operatorname{div}(s))$$

or

$$\int_{\mathcal{X}(\mathbb{C})} -\log\|s\|^2 \wedge c_1(\overline{\mathcal{M}})^d < \frac{\varepsilon}{1+\varepsilon} \cdot \widehat{\deg}(\overline{\mathcal{M}}^d \cdot \overline{\mathcal{L}}).$$

Remark 2.2.13. The definition from [80, Definition 1.1] is more general and has an additional assumption that the current $\frac{i}{2\pi}\partial\overline{\partial}(-\log\|s\|^2) + \delta_{\mathcal{Z}(\mathbb{C})}$ is represented by a semi-positive form. However, in our case, by the Poincaré–Lelong formula we get

$$\frac{i}{2\pi}\partial\overline{\partial}(-\log\|s\|^2) + \delta_{\mathcal{Z}(\mathbb{C})} = [c_1(\overline{\mathcal{L}})]$$

as currents. Since $\overline{\mathcal{L}}$ is an ample hermitian line bundle, the form $c_1(\overline{\mathcal{L}})$ is positive, so the additional assumption is automatically satisfied in this case.

Theorem 2.2.14. [80, Remark 6.7.(ii)] Assume that $\dim(\mathcal{X}) \geq 2$. Let $\overline{\mathcal{L}}, \overline{\mathcal{M}}$ be ample hermitian line bundles on $\mathcal{X}$. Fix $\varepsilon > 0$. Then

$$\lim_{n\to\infty} \frac{\#\{s \in \widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) : s \text{ is } (\varepsilon, \overline{\mathcal{M}})\text{-irreducible}, \operatorname{div}(s)_{\mathbb{Q}} \text{ is smooth}\}}{\#\widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})} = 1.$$

Theorem 2.2.15. [20, Theorem 2.21]. Let $\overline{\mathcal{L}}$ be an ample hermitian line bundle on $\mathcal{X}$ and $\mathcal{Y} \subset \mathcal{X}$ be a closed strict subscheme. Then

$$\lim_{n\to\infty} \frac{\#\{s \in \widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n}) : \operatorname{div}(s) \text{ is not contained in } \mathcal{Y}\}}{\#\widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n})} = 1.$$

2.2.2 Arithmetic $\mathbb{R}$-divisors

Warning 2.2.16. For the rest of this section (unless stated otherwise) we impose **stronger** assumptions on $\mathcal{X}$. Namely here $\mathcal{X}$ is a generically smooth, **normal** arithmetic variety of dimension $d+1$.

Remark 2.2.17. We recall here definitions of objects related to $\mathcal{X}$ that we need.

1. Let $\mathcal{J}$ be a subsheaf of the sheaf of continuous real-valued functions on $\mathcal{X}(\mathbb{C})$ (with respect to the euclidean topology). A $\mathcal{J}$*- function* (or a $\mathcal{J}$-type function) on an euclidean open conjugation invariant subset $U \subset \mathcal{X}(\mathbb{C})$ is a continuous function $U \to \mathbb{R}$ lying in $\mathcal{J}(U)$ that is invariant under complex conjugation.

2. We use three subsheafs $\mathcal{J}$ of the sheaf of continuous (real-valued) functions on $\mathcal{X}(\mathbb{C})$, in the above context:

    - $\mathcal{J} = C^0$ - the full sheaf of continuous functions,
    - $\mathcal{J} = C^\infty$ - the subsheaf of smooth functions,
    - $\mathcal{J} = C^0 \cap \mathrm{PSH}$ - the subsheaf of plurisubharmonic continuous functions.

    For the definition of a plurisubharmonic function, see [59, Section 1.4].

3. Let $\mathbb{K}$ be one of $\mathbb{Z}, \mathbb{Q}, \mathbb{R}$. A $\mathbb{K}$-*Cartier divisor* on $\mathcal{X}$ is an element of the group $\mathrm{Div}(\mathcal{X})_\mathbb{K}$. A $\mathbb{K}$-*rational function* is an element of $\kappa(\mathcal{X})^\times_\mathbb{K}$. Let $\mathcal{D} = \sum_i \alpha_i \mathcal{D}_i \in \mathrm{Div}(\mathcal{X})_\mathbb{K}$ for $\alpha_i \in \mathbb{K}$ and Cartier divisors $\mathcal{D}_i$. If f_i is a local equation for $\mathcal{D}_i$ for all i, then we call $\prod_i f_i^{\alpha_i} \in \kappa(\mathcal{X})^\times_\mathbb{K}$ a local equation for $\mathcal{D}$. On the other hand, if $f \in \kappa(\mathcal{X})^\times_\mathbb{K}$, then we write (f) or $\mathrm{div}(f)$ for the corresponding $\mathbb{K}$-Cartier divisor. For $\mathcal{D} \in \mathrm{Div}(\mathcal{X})_\mathbb{K}$ we write $\mathcal{D}_\mathbb{Q} \in \mathrm{Div}(\mathcal{X}_\mathbb{Q})_\mathbb{K}$ for its restriction to the generic fiber.

4. For a $\mathbb{K}$-Cartier divisor $\mathcal{D}$ on $\mathcal{X}$, its *support* is the set

    $$\mathrm{supp}(\mathcal{D}) := \{p \in \mathcal{X} : f \notin (\mathcal{O}^\times_{\mathcal{X},p})_\mathbb{K} \text{ for a local equation } f \text{ of } \mathcal{D} \text{ at } p\}.$$

    If $\mathcal{D} = \sum_i \alpha_i \mathcal{D}_i$ is a decomposition as in the previous point with α_i's linearly independent over $\mathbb{Q}$, then $\mathrm{supp}(\mathcal{D}) = \bigcup_i \mathrm{supp}(\mathcal{D}_i)$ where the latter supp is the support of a Cartier divisor. For details see [59, Section 1.2].

5. A $\mathbb{Z}$-*Zariski open* set in $\mathcal{X}(\mathbb{C})$ is a set of the form $\mathcal{V}(\mathbb{C}) \subset \mathcal{X}(\mathbb{C})$ for some Zariski open $\mathcal{V} \subset \mathcal{X}$ (i.e., $\mathcal{V}$ open in the topology of the scheme $\mathcal{X}$). If $p \in \mathcal{X}(\mathbb{C})$, then a $\mathbb{Z}$-*Zariski neighbourhood* is an open neighbourhood of p being a $\mathbb{Z}$-Zariski open set.

Definition 2.2.18. Let $\mathcal{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$. A $\mathcal{D}$*-Green's function on* $\mathcal{X}$ *of* $\mathcal{J}$*-type* (if $\mathcal{J}$ is C^0, we skip it in the notation) is a function $g : \mathcal{U}(\mathbb{C}) \to \mathbb{R}$ for some Zariski open $\mathcal{U} \subset \mathcal{X}$ such that if $\mathcal{D} = \sum a_i \mathcal{D}_i$ for some real a_i's and Cartier divisors

$\mathcal{D}_i$'s, then for every $p \in \mathcal{X}(\mathbb{C})$ there is a $\mathbb{Z}$-Zariski neighbourhood $p \in \mathcal{V}(\mathbb{C}) \subset \mathcal{X}(\mathbb{C})$ with local equations f_i's for $\mathcal{D}_i$'s, such that the function

$$g(x) + \sum_i a_i \log |f_i(x)|^2$$

restricted to the intersection of $\mathcal{U}(\mathbb{C})$ and $\mathcal{V}(\mathbb{C})$, extends to a $\mathcal{J}$-function on $\mathcal{V}(\mathbb{C})$. We identify arithmetic $\mathbb{R}$-divisors of C^0-type $(\mathcal{D}, g) = (\mathcal{D}, g')$ if $g : \mathcal{U}(\mathbb{C}) \to \mathbb{R}, g' : \mathcal{U}'(\mathbb{C}) \to \mathbb{R}$ coincide on the intersection $(\mathcal{U} \cap \mathcal{U}')(\mathbb{C})$.

Remark 2.2.19. The identification in the above definition makes sense, because for a pair $(\mathcal{D}, g)$ as above, an extension $g' : \mathcal{U}'(\mathbb{C}) \to \mathbb{R}$ for $\mathcal{U} \subset \mathcal{U}'$ is unique if it exists. This follows from the fact that complex points of a Zariski open subset of $\mathcal{X}$ are dense in the euclidean topology on $\mathcal{X}(\mathbb{C})$. Thus, for every arithmetic $\mathbb{R}$-divisor of C^0-type, there exists the biggest Zariski open $\mathcal{U} \subset \mathcal{X}$ such that g can be defined on $\mathcal{U}(\mathbb{C})$ (just take the sum of all possible extensions and glue by uniqueness) and this open set is $\mathcal{X} \setminus \operatorname{supp}(\mathcal{D})$.

Definition 2.2.20. An *arithmetic* $\mathbb{R}$*-divisor of* C^0*-type [resp.* $\mathcal{J}$*-type] on* $\mathcal{X}$ is a pair $\overline{\mathcal{D}} = (\mathcal{D}, g)$, where $\mathcal{D}$ is an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$ and g is a $\mathcal{D}$-Green's function on $\mathcal{X}$ [of $\mathcal{J}$-type]. An arithmetic $\mathbb{R}$-divisor of C^0-type is called *principal*, if it can be written as an $\mathbb{R}$-combination of arithmetic $\mathbb{R}$-divisors (of C^∞-type) $\widehat{\operatorname{div}}(f) := (\operatorname{div}(f), -\log |f|^2)$ for some rational functions f on $\mathcal{X}$. Also, $(\mathcal{D}, g)$ is an *arithmetic* $\mathbb{Z}$*-divisor [resp.* $\mathbb{Q}$*-divisor] of* C^0*-type (or* $\mathcal{J}$*-type)* if it is an arithmetic $\mathbb{R}$-divisor of the corresponding type, with $\mathcal{D}$ being a Cartier [resp. $\mathbb{Q}$-Cartier] divisor on $\mathcal{X}$.

Remark 2.2.21. Let $\phi : \mathcal{Y} \to \mathcal{X}$ be a map of generically smooth, normal arithmetic varieties and let $\overline{\mathcal{D}} = (\mathcal{D}, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. If the pullback $\phi^*\mathcal{D}$ exists and g can be defined on an open subset $\mathcal{U}$ of $\mathcal{X}$ whose preimage is non-empty in $\mathcal{Y}$, one can equip $\phi^*\mathcal{D}$ with a Green's function being the pullback of g. In particular, if $\mathcal{Y} \to \mathcal{X}$ is a birational map, one can pullback arithmetic $\mathbb{R}$-divisors of C^0-type.

Notation 2.2.22. Arithmetic $\mathbb{R}$-divisors of C^0-type form an abelian group (with component-wise addition) and this group is denoted by $\operatorname{ADiv}_{\mathbb{R}}(\mathcal{X})$. The subgroup of principal arithmetic $\mathbb{R}$-divisors of C^0-type is denoted by $\operatorname{RDiv}_{\mathbb{R}}(\mathcal{X})$. We use the symbol $\equiv$ to indicate equality up to principal arithmetic $\mathbb{R}$-divisors of C^0-type. The quotient $\operatorname{ADiv}_{\mathbb{R}}(\mathcal{X}) / \operatorname{RDiv}_{\mathbb{R}}(\mathcal{X})$ is denoted by $\widehat{N}^1(\mathcal{X})$. Note that this notation is not standard, but we use it so that the proof of Theorem 2.2.70 looks similar to the geometric setting outlined in Subsection 2.1.2.

2.2.3 Positivity of $\mathbb{R}$-Cartier divisors

Definition 2.2.23. Let $\mathcal{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$ and let v be a non-Archimedean valuation on $\kappa(\mathcal{X})$. Let $p = \operatorname{supp}(v) \in \mathcal{X}$ and let $f \in \kappa(\mathcal{X})^\times_{\mathbb{R}}$ be a local equation for $\mathcal{D}$ at p. We then write $\beta(v, \mathcal{D})$ for the value $v(f)$ which is the $\mathbb{R}$-linear (multiplicative) extension of the valuation $v : \kappa(\mathcal{X})^\times \to \mathbb{R}$. Moreover, if $\overline{\mathcal{D}}$ is an $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$ we define $\beta(v, \overline{\mathcal{D}}) := \beta(v, \mathcal{D})$. On the other hand, if v is an Archimedean valuation on $\kappa(\mathcal{X})$ induced by $\sigma : \kappa(\mathcal{X}) \to \mathbb{C}$, then we define $\beta(v, \overline{\mathcal{D}}) := \frac{1}{2}g(p)$, where p is the point of $\mathcal{X}(\mathbb{C})$ coming from the composition

$$\operatorname{Spec}(\mathbb{C}) \xrightarrow{\operatorname{Spec}(\sigma)} \operatorname{Spec}(\kappa(\mathcal{X})) \longrightarrow \mathcal{X}.$$

Note g is defined on p since $\operatorname{Spec}(\kappa(\mathcal{X}))$ is the generic point of $\mathcal{X}$. Furthermore, since g is conjugation-invariant, the value $g(p)$ does not depend on the choice of σ inducing v.

Definition 2.2.24. Let $\mathcal{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$. The *Weil decomposition of* $\mathcal{D}$ is the codimension one cycle on $\mathcal{X}$ of the form

$$\sum_{\Gamma} \operatorname{ord}_{\Gamma}(\mathcal{D})\Gamma,$$

where the sum is taken over all codimension one integral subschemes of $\mathcal{X}$ and $\operatorname{ord}_\Gamma(\mathcal{D}) = \beta(\operatorname{ord}_\Gamma, \mathcal{D})$. This makes sense, as we assume that $\mathcal{X}$ is normal, which implies that the order of vanishing at an integral subscheme Γ of codimension one is a valuation on $\kappa(\mathcal{X})$.

Lemma 2.2.25. Let $\mathcal{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$. The following are equivalent.

(1) $\mathcal{D}$ is a positive $\mathbb{R}$-combination of effective Cartier divisors.

(2) For all $p \in \mathcal{X}$ if $f \in \kappa(\mathcal{X})^\times_{\mathbb{R}}$ is a local equation for $\mathcal{D}$ at p, then $f \in (\mathcal{O}_{\mathcal{X},p} \setminus \{0\})_{\mathbb{R}}$.

(3) For all non-Archimedean valuations v on $\kappa(\mathcal{X})$ we have $\beta(v, \mathcal{D}) \geq 0$.

(4) If $\mathcal{D} = \sum_\Gamma a_\Gamma \Gamma$ is the Weil decomposition of $\mathcal{D}$, then all a_Γ are non-negative.

Moreover, the same holds (with $\mathbb{Q}$-tensors instead of $\mathbb{R}$-tensors) for a $\mathbb{Q}$-Cartier divisor $\mathcal{D}$.

In the above by $(\mathcal{O}_{\mathcal{X},p} \setminus \{0\})_{\mathbb{R}}$ we mean the subsemigroup of $\kappa(\mathcal{X})^\times_{\mathbb{R}}$ generated by elements of the form g^r with $g \in \mathcal{O}_{\mathcal{X},p} \setminus \{0\}$ and $r > 0$.

Proof. Implications (1) $\implies$ (2) $\implies$ (3) $\implies$ (4) are skipped. The implication (4) $\implies$ (1) is [24, Proposition 2.4.16]. Implications in the $\mathbb{Q}$ case are skipped. $\square$

Definition 2.2.26. Let $\mathcal{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$. We say that $\mathcal{D}$ is *effective*, if it satisfies the equivalent conditions from Lemma 2.2.25. In this case we write $\mathcal{D} \geq 0$. Similarly, if $\mathcal{E}$ is another $\mathbb{R}$-Cartier divisor on $\mathcal{X}$, we write $\mathcal{D} \geq \mathcal{E}$, if $\mathcal{D} - \mathcal{E} \geq 0$. We also define the space of rational functions having poles at most given by $\mathcal{D}$ as

$$H^0(\mathcal{X}, \mathcal{D}) := \{f \in \kappa(\mathcal{X})^\times : \mathcal{D} + \operatorname{div}(f) \geq 0\} \cup \{0\}.$$

A similar definition applies to $\mathbb{R}$-Cartier divisors on $X = \mathcal{X}_\mathbb{Q}$.

Lemma 2.2.27. [58, Lemma 5.2.3] The following hold.

1. Let $\mathcal{Z}$ be a Weil divisor on $\mathcal{X}$. Then there is an effective Cartier divisor $\mathcal{A}$ on $\mathcal{X}$ such that $\mathcal{Z} \leq \mathcal{A}$.
2. Let $\mathcal{D}$ be a Cartier divisor on $\mathcal{X}$. Then there are effective Cartier divisors $\mathcal{A}$ and $\mathcal{B}$ on $\mathcal{X}$ such that $\mathcal{D} = \mathcal{A} - \mathcal{B}$.
3. Let $x_1, \dots, x_l$ be points of $\mathcal{X}$ and let $\mathcal{D}$ be a Cartier divisor on $\mathcal{X}$. Then there are effective Cartier divisors $\mathcal{A}$ and $\mathcal{B}$, and a non-zero rational function ϕ on $\mathcal{X}$ such that $\mathcal{D} + (\phi) = \mathcal{A} - \mathcal{B}$ and $x_1, \dots, x_l \notin \operatorname{supp}(\mathcal{A}) \cup \operatorname{supp}(\mathcal{B})$.

Lemma 2.2.28. Let $\overline{\mathcal{D}} = (\mathcal{D}, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Fix a point $x \in \mathcal{X}$. Then there exists an arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{D}}' = (\mathcal{D}', g')$ such that $\mathcal{D}' = \sum_{i=1}^m \alpha_i' \mathcal{D}_i'$ for some real α_i''s and effective Cartier $\mathcal{D}_i'$'s, and such that $\overline{\mathcal{D}} - \overline{\mathcal{D}}'$ is principal. Moreover, g' can be defined on $(\mathcal{X} \setminus \bigcup_{i=1}^m \operatorname{supp}(\mathcal{D}_i'))(\mathbb{C})$ and one can assume that $x \in \mathcal{X} \setminus \bigcup_{i=1}^m \operatorname{supp}(\mathcal{D}_i')$.

Proof. Let $\mathcal{D} = \sum_{i=1}^n \alpha_i \mathcal{D}_i$ for some real α_i's and Cartier (but not necessarily effective) divisors $\mathcal{D}_i$'s. By Lemma 2.2.27 there are effective Cartier divisors $\mathcal{A}_i, \mathcal{B}_i$'s and rational functions f_i's on $\mathcal{X}$, such that

$$\mathcal{D}_i = \mathcal{A}_i - \mathcal{B}_i + \operatorname{div}(f_i) \text{ for all } i = 1, \dots, n$$

and with $x \notin \operatorname{supp}(\mathcal{A}_i) \cup \operatorname{supp}(\mathcal{B}_i)$. Let $\mathcal{D}' = \sum_{i=1}^n \alpha_i(\mathcal{A}_i - \mathcal{B}_i)$. Let $\mathcal{U} = \mathcal{X} \setminus \operatorname{supp}(\mathcal{D})$ and let $\mathcal{V} = \mathcal{X} \setminus \bigcup_{i=1}^n \operatorname{supp}(\operatorname{div}(f_i))$. Let $g' = g + \sum_{i=1}^n \alpha_i \log |f_i|^2$ on $(\mathcal{U} \cap \mathcal{V})(\mathbb{C})$. Pick $P \in (\mathcal{U} \cap \mathcal{V})(\mathbb{C})$ and let $\mathcal{W}(\mathbb{C}) \subset (\mathcal{U} \cap \mathcal{V})(\mathbb{C})$ be a $\mathbb{Z}$-Zariski neighbourhood of P with local equations a_i, b_i for $\mathcal{A}_i, \mathcal{B}_i$ respectively (for all i) such that on $\mathcal{W}(\mathbb{C})$ a continuous extension of

$$g(x) + \sum_{i=1}^{n} \alpha_i(\log|a_i|^2(x) - \log|b_i|^2(x) + \log|f_i|^2(x))$$

can be defined. Then the function

$$g'(x) + \sum_{i=1}^{n} \alpha_i(\log|a_i|^2(x) - \log|b_i|^2(x))$$

can also be defined on $\mathcal{W}(\mathbb{C})$, which proves that $(\mathcal{D}', g')$ is an arithmetic $\mathbb{R}$-divisor of C^0-type. By Remark 2.2.19 g' can be defined on $\big(\mathcal{X} \setminus \bigcup_{i=1}^n(\mathrm{supp}(\mathcal{A}_i) \cup \mathrm{supp}(\mathcal{B}_i))\big)(\mathbb{C})$. Moreover, by construction $x \notin \bigcup_{i=1}^n(\mathrm{supp}(\mathcal{A}_i) \cup \mathrm{supp}(\mathcal{B}_i))$, which finishes the proof.

□

2.2.4 Heights with respect to arithmetic $\mathbb{R}$-divisors

Definition 2.2.29. Let $x \in X = \mathcal{X}_{\mathbb{Q}}$ be a closed point and let v be a valuation on $\kappa(x)$. Let $\overline{\mathcal{D}} = (\mathcal{D}, g)$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$ with $\mathcal{D}$ effective and $x \notin \mathrm{supp}(\mathcal{D})$. We define the *local degree* of $\overline{\mathcal{D}}$ with respect to v $\widehat{\deg}_v(\overline{\mathcal{D}}|\overline{\{x\}})$ in the following way:

- If v is Archimedean we put $\widehat{\deg}_v(\overline{\mathcal{D}}|\overline{\{x\}}) := \frac{1}{2}g(x^\sigma)$ where $\sigma : \kappa(x) \to \mathbb{C}$ is any embedding which induces v on $\kappa(x)$.
- If v is non-Archimedean we put $\widehat{\deg}_v(\overline{\mathcal{D}}|\overline{\{x\}}) = v(d|_x)$ where d is a local equation for $\mathcal{D}$ at the point $q \in \mathcal{X}$ defined as the image of the maximal ideal of the value ring $\mathcal{O}_v \subset \kappa(x)$ via the map ϕ in the following diagram

$$\begin{tikzcd}
\mathrm{Spec}(\kappa(x)) \arrow[r] \arrow[d] & \mathcal{X} \arrow[d] \\
\mathrm{Spec}(\mathcal{O}_v) \arrow[r] \arrow[ur, dashed, "\phi"] & \mathrm{Spec}(\mathbb{Z}).
\end{tikzcd}$$

  Existence and uniqueness of ϕ follows from the valuative criterion of properness. If d is an $\mathbb{R}$-rational function, we use the unique (multiplicative) $\mathbb{R}$-linear extension of v to calculate $v(d|_x)$. Note that q generalises to x, so d is a regular function on the neighbourhood of x and $d|_x \in \kappa(x)^\times_{\mathbb{R}}$.

Definition 2.2.30. Let $\overline{\mathcal{D}} = (\mathcal{D}, g)$ be an arithmetic $\mathbb{Z}$-divisor of C^0-type on $\mathcal{X}$ with $\mathcal{D}$ effective and let $x \in X = \mathcal{X}_{\mathbb{Q}}$ be a closed point of the generic fiber of $\mathcal{X}$. If

$x \notin \operatorname{supp}(\mathcal{D})$ and $\mathcal{D}$ is a Cartier divisor on $\mathcal{X}$ the *height* of x with respect to $\overline{\mathcal{D}}$ is defined to be

$$h_{\overline{\mathcal{D}}}(x) := \frac{1}{[\kappa(x) : \mathbb{Q}]}\Big(\log \#\big(\mathcal{O}_{\mathcal{C}}(\mathcal{D})/\mathcal{O}_{\mathcal{C}}\big) + \frac{1}{2}\sum_{\sigma:\kappa(x)\to\mathbb{C}} g(x^{\sigma})\Big),$$

where $\mathcal{C} = \overline{\{x\}} \subset \mathcal{X}$ is the 1-cycle corresponding to x and we treat $\mathcal{O}_{\mathcal{C}}(\mathcal{D})/\mathcal{O}_{\mathcal{C}}$ as a module over the ring of functions of $\mathcal{C} \cap \mathcal{D}$. For a general $\overline{\mathcal{D}}$, by Lemma 2.2.28 there is a decomposition $\overline{\mathcal{D}} = \sum_{i=1}^{m} \alpha_i \overline{\mathcal{D}}_i + \widehat{\operatorname{div}}(f)$ with α_i real, $\mathcal{D}_i$ effective Cartier, such that $x \notin \operatorname{supp}(\mathcal{D}_i)$ for all $i = 1, \dots, m$ and where f is an $\mathbb{R}$-rational function on $\mathcal{X}$. Then we put

$$h_{\overline{\mathcal{D}}}(x) := \sum_{i=1}^{m} \alpha_i h_{\overline{\mathcal{D}}_i}(x).$$

Note that this is well defined by the product formula on number fields. For details, see [59, Section 4.2]. Also, we define the *arithmetic degree* of $\overline{\mathcal{D}}$ with respect to x as the quantity $\widehat{\deg}(\overline{\mathcal{D}}|\overline{\{x\}})$ satisfying

$$h_{\overline{\mathcal{D}}}(x) = \frac{\widehat{\deg}(\overline{\mathcal{D}}|\overline{\{x\}})}{\deg(x)}.$$

Lemma 2.2.31. Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$ and let $x \in X = \mathcal{X}_{\mathbb{Q}}$ be a closed point of the generic fiber of $\mathcal{X}$ with $x \notin \operatorname{supp}(\mathcal{D})$. Let $K = \kappa(x)$ and denote by $\mathcal{O}_K$ its integer ring. Then

$$h_{\overline{\mathcal{D}}}(x) = \int_{\operatorname{Val}_K} \widehat{\deg}_v(\overline{\mathcal{D}}|\overline{\{x\}}) d\mu(v),$$

where μ is the measure

$$\mu = \frac{1}{[K : \mathbb{Q}]}\Bigg(\sum_{p \in \operatorname{Spec}(\mathcal{O}_K)} \delta_{\operatorname{ord}_p} \cdot \log \#\kappa(p) + \sum_{\sigma:K\to\mathbb{C}} \delta_{-\log|\sigma(-)|}\Bigg)$$

and Val_K is the space of valuations on K (see Example 2.3.20).

Proof. By Lemma 2.2.28 we can write $\overline{\mathcal{D}} = \sum_{i=1}^{m} \alpha_i \overline{\mathcal{D}}_i + \widehat{\operatorname{div}}(f)$ with α_i real, $\mathcal{D}_i$ effective Cartier, such that $x \notin \operatorname{supp}(\mathcal{D}_i)$ for all $i = 1, \dots, m$ and where f is an $\mathbb{R}$-rational function on $\mathcal{X}$. Note that

$$\operatorname{supp}(\operatorname{div}(f)) \subset \operatorname{supp}(\mathcal{D}) \cup \bigcup_{i=1}^{m} \operatorname{supp}(\mathcal{D}_i)$$

and x is not in the right hand side, so $x \notin \operatorname{supp}(\widehat{\operatorname{div}}(f))$. Since both the height and the integral are linear in $\overline{\mathcal{D}}$ and give zero on $\widehat{\operatorname{div}}(f)$ (see the product formula in Example 2.3.20), it is enough to check that the lemma holds for an arithmetic $\mathbb{Z}$-divisor of C^0-type $\overline{\mathcal{D}}$ with $\mathcal{D}$ effective. This is standard, but for convenience we provide a proof here.

By unraveling the definitions, we need to prove that

$$\log \#\big(\mathcal{O}_{\mathcal{C}}(\mathcal{D})/\mathcal{O}_{\mathcal{C}}\big) = \sum_{p \in \operatorname{Spec}(\mathcal{O}_K)} \widehat{\deg}_{\mathrm{ord}_p}(\overline{\mathcal{D}}|\overline{\{x\}}) \cdot \log \#\kappa(p).$$

Let $\mathcal{C}$ be the closure of x in $\mathcal{X}$ (with the reduced scheme structure). By the valuative criterion of properness (not directly, one has to use it for primes of $\mathcal{O}_K$), we get a diagonal map ϕ on the commutative diagram

$$\begin{tikzcd}
\operatorname{Spec}(K) \arrow[r, "x"] \arrow[d] & \mathcal{X} \arrow[d] \\
\operatorname{Spec}(\mathcal{O}_K) \arrow[r] \arrow[ur, dashed, "\phi"] & \operatorname{Spec}(\mathbb{Z}).
\end{tikzcd}$$

Since x is the generic point of $\mathcal{C}$, ϕ factors through $\mathcal{C}$. As $\mathcal{C} \to \operatorname{Spec}(\mathbb{Z})$ is projective and has finite fibers (as the composition with $\operatorname{Spec}(\mathcal{O}_K) \to \mathcal{C}$ has finite fibers), by [75, Theorem 19.1.7] we get that $\mathcal{C}$ is affine (and finite) over $\operatorname{Spec}(\mathbb{Z})$. Let $\mathcal{C} = \operatorname{Spec}(A)$. Note that then $\mathbb{Z} \subset A \subset \mathcal{O}_K \subset K$ and $\mathcal{O}_K$ is the integral closure of A. Let I, J be the ideals of $\mathcal{D}$ in $A, \mathcal{O}_K$ respectively (more precisely J is the ideal of the pullback of $\mathcal{D}$ to $\operatorname{Spec}(\mathcal{O}_K)$). Consider the diagram coming from the snake lemma

$$\begin{tikzcd}
 & 0 \arrow[r] \arrow[d] & 0 \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, rounded corners] & \\
0 \arrow[r] & I \arrow[r] \arrow[d, "f"'] & A \arrow[r] \arrow[d, "g"'] & A/I \arrow[r] \arrow[d, "h"'] & 0 \\
0 \arrow[r] & J \arrow[r] \arrow[d] & \mathcal{O}_K \arrow[r] \arrow[d] & \mathcal{O}_K/J \arrow[r] \arrow[d] & 0 \\
 & \operatorname{cok} f \arrow[r] & \operatorname{cok} g \arrow[r] & \operatorname{cok} h \arrow[r] & 0.
\end{tikzcd}$$

Note that $\log \#\big(\mathcal{O}_{\mathcal{C}}(\mathcal{D})/\mathcal{O}_{\mathcal{C}}\big) = \log \#(A/I)$. We now show that $\#(A/I) = \#(\mathcal{O}_K/J)$. If $q \in \operatorname{Spec}(A)$, then we can localise everything at q and as $\mathcal{D}$ is an effective Cartier divisor, it means that in A_q the ideal I_q is generated by a single element $a \in A_q$. Also J_q is generated by a in $(\mathcal{O}_K)_q$. Thus $(\operatorname{cok} f)_q \simeq (\operatorname{cok} g)_q$. We calculate

$$\log \#(\operatorname{cok} f) = \sum_{q \in \operatorname{Spec}(A)} \log \#(\operatorname{cok} f)_q = \sum_{q \in \operatorname{Spec}(A)} \log \#(\operatorname{cok} g)_q = \log \#(\operatorname{cok} g).$$

Using two exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \ker h \arrow[r] & \operatorname{cok} f \arrow[r] & \operatorname{cok} g \arrow[r] & \operatorname{cok} h \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r] & \ker h \arrow[r] & A/I \arrow[r] & \mathcal{O}_K/J \arrow[r] & \operatorname{cok} h \arrow[r] & 0 \end{tikzcd}$$

we get that

$$\log\#(\ker h) - \log\#(\operatorname{cok} f) + \log\#(\operatorname{cok} g) - \log\#(\operatorname{cok} h) = 0,$$

$$\log\#(\ker h) - \log\#(A/I) + \log\#(\mathcal{O}_K/J) - \log\#(\operatorname{cok} h) = 0.$$

Putting it all together we get that indeed $\log\#(A/I) = \log\#(\mathcal{O}_K/J)$. We calculate

$$\log\#(\mathcal{O}_K/J) = \sum_{p\in\operatorname{Spec}(\mathcal{O}_K)} \log\#(\mathcal{O}_K/J)_p = \sum_{p\in\operatorname{Spec}(\mathcal{O}_K)} \operatorname{ord}_p(J)\log\#\kappa(p),$$

where by $\operatorname{ord}_p(J)$ we mean the value of the valuation ord_p at a local equation for J at p. By the definition $\widehat{\deg}_{\operatorname{ord}_p}(\overline{\mathcal{D}}|\overline{\{x\}}) = \operatorname{ord}_p(J)$, which finishes the proof. □

Lemma 2.2.32. Let $x \in X = \mathcal{X}\otimes\mathbb{Q}$ be a closed point and let $\overline{\mathcal{D}} = (\operatorname{div}(2), 0)$. Then

$$h_{\overline{\mathcal{D}}}(x) = \log(2).$$

Proof. Note that $\overline{\mathcal{D}} = \widehat{\operatorname{div}}(2) + (0, 2\log(2))$. Hence we get

$$h_{\overline{\mathcal{D}}}(x) = h_{(0,2\log(2))}(x) = \frac{1}{2[\kappa(x):\mathbb{Q}]}\sum_{\sigma:\kappa(x)\to\mathbb{C}} 2\log(2) = \log(2).$$

□

Lemma 2.2.33. Assume that $\mathcal{X}'$ is a generically smooth, normal arithmetic variety. Let $f : \mathcal{X}' \to \mathcal{X}$ be a generically finite map and let $x' \in X' = \mathcal{X}'\otimes\mathbb{Q}$ be a closed point. Then for $x = f(x') \in X = \mathcal{X}\otimes\mathbb{Q}$ and for any arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{D}}$ on $\mathcal{X}$ we have

$$h_{f^*\overline{\mathcal{D}}}(x') = h_{\overline{\mathcal{D}}}(x).$$

Proof. As in Definition 2.2.30, we reduce to the case that $\overline{\mathcal{D}}$ is an arithmetic $\mathbb{Z}$-divisor of C^0-type with $\mathcal{D}$ effective and $x \notin \operatorname{supp}(\mathcal{D})$. Let $\mathcal{C} = \overline{\{x\}}$ and $\mathcal{C}' = \overline{\{x'\}}$. Then $\mathcal{C}' \to \mathcal{C}$ is a finite morphism of degree $\deg(x'/x) := \deg(x')/\deg(x)$. Both sides of the equality depend only on the restriction of $\overline{\mathcal{D}}$ to $\mathcal{C}$, so we can without loss of generality assume that $\mathcal{X} = \mathcal{C}, \mathcal{X}' = \mathcal{C}'$. In this case $\overline{\mathcal{D}}$ is a $\mathbb{Z}$-divisor of C^∞-type. Let $\overline{\mathcal{L}} = \mathcal{O}(\overline{\mathcal{D}})$ be the corresponding metrized line bundle. By Theorem 2.2.10 we get

$$\widehat{\deg}(f^*\overline{\mathcal{L}}|\mathcal{C}') = \widehat{\deg}(\overline{\mathcal{L}}|f_*\mathcal{C}') = \frac{\deg(x')}{\deg(x)}\cdot\widehat{\deg}(\overline{\mathcal{L}}|\mathcal{C}).$$

Thus

$$h_{f^*\overline{\mathcal{D}}}(x') = \frac{\widehat{\deg}(f^*\overline{\mathcal{L}}|\mathcal{C}')}{\deg(x')} = \frac{\widehat{\deg}(\overline{\mathcal{L}}|\mathcal{C})}{\deg(x)} = h_{\overline{\mathcal{D}}}(x).$$

□

2.2.5 Positivity of arithmetic $\mathbb{R}$-divisors

Definition 2.2.34. We use the following notions regarding an arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{D}} = (\mathcal{D}, g)$ on $\mathcal{X}$.

1. We write $\overline{\mathcal{D}} \geq 0$ if $\mathcal{D} \geq 0$ and $g \geq 0$. Note that this does not depend on the choice of g since if g' extends g and $g'(x) < 0$ for some $x \in \mathcal{X}(\mathbb{C})$, then the set $\{y \in \mathcal{X}(\mathbb{C}) : g'(y) < 0\}$ is open, so it has to intersect the domain of g which is a dense open subset of $\mathcal{X}(\mathbb{C})$. We then call $\overline{\mathcal{D}}$ *effective.* If $\overline{\mathcal{E}}$ is another arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$, then $\overline{\mathcal{D}} \geq \overline{\mathcal{E}}$ means that $\overline{\mathcal{D}} - \overline{\mathcal{E}} \geq 0$.

2. There is a natural norm $|\cdot|^{\overline{\mathcal{D}}}$ associating to each rational function $f \in H^0(\mathcal{X}, \mathcal{D})$ a function
$$|f|^{\overline{\mathcal{D}}} : \mathcal{X}(\mathbb{C}) \to \mathbb{R},$$
which is the unique continuous extension of $|f| \cdot \exp(-g/2)$ on $\mathcal{X}(\mathbb{C})$. For well definiteness, see [59, Proposition 1.4.2]. It induces a supremum norm
$$\|\cdot\|^{\overline{\mathcal{D}}}_{\sup} : H^0(\mathcal{X}, \mathcal{D}) \to \mathbb{R}.$$
A rational function $f \in H^0(\mathcal{X}, \mathcal{D})$ is called *small,* if $\|f\|^{\overline{\mathcal{D}}}_{\sup} \leq 1$. It is called *strictly small* if $\|f\|^{\overline{\mathcal{D}}}_{\sup} < 1$. The set of small sections is denoted by $\widehat{H}^0(\mathcal{X}, \overline{\mathcal{D}})$.

3. Assume that $\overline{\mathcal{D}}$ is an arithmetic $\mathbb{Z}$-divisor of C^∞-type. Let L be the (complex analytic) line bundle over $\mathcal{X}(\mathbb{C})$ induced by the base change of $\mathcal{O}_X(\mathcal{D}_{\mathbb{Q}})$ to $\mathbb{C}$ (recall that $X = \mathcal{X}_{\mathbb{Q}}$). Then for a Zariski open $\mathcal{U} \subset \mathcal{X}$ where $d \in \kappa(\mathcal{X})^\times$ is a local equation for $\mathcal{D}$ and a section s of L over $\mathcal{U}(\mathbb{C})$ one can define a norm of s by presenting $s = f \cdot d^{-1}$ for some holomorphic function $f : \mathcal{U}(\mathbb{C}) \to \mathbb{C}$ and by putting
$$|s|_{\overline{L}} := |f| \cdot \exp(-(g + \log|d|^2)/2),$$
where by $g + \log|d|^2$ we mean the unique continuous (even smooth) extension of this function to $\mathcal{U}(\mathbb{C})$. In fact, this metric comes from a unique hermitian product on L and the Chern form of the pair $(L, |\cdot|_{\overline{L}})$ is denoted by $c_1(\overline{\mathcal{D}})$ (see Definition 2.2.3). Note that $c_1(\overline{\mathcal{D}})|_{\mathcal{U}(\mathbb{C})} = \frac{i}{2\pi}\partial\overline{\partial}(g + \log|d|^2)$.

4. The *arithmetic volume* of $\overline{\mathcal{D}}$ is defined to be the following number
$$\widehat{\mathrm{vol}}(\overline{\mathcal{D}}) := \limsup_{m\to\infty} \frac{\log \# \widehat{H}^0(\mathcal{X}, m\overline{\mathcal{D}})}{m^{d+1}/(d+1)!}.$$
The arithmetic volume only depends on the class of $\overline{\mathcal{D}}$ in $\widehat{N}^1(\mathcal{X})$. Also, recall that *volume* of an $\mathbb{R}$-divisor $\mathcal{D}_{\mathbb{Q}}$ on the generic fiber $X = \mathcal{X}_{\mathbb{Q}}$ is
$$\mathrm{vol}(\mathcal{D}_{\mathbb{Q}}) := \limsup_{m\to\infty} \frac{\dim_{\mathbb{Q}} H^0(X, m\mathcal{D}_{\mathbb{Q}})}{m^d/d!}.$$

Remark 2.2.35. Note that for an arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{D}}$ on $\mathcal{X}$, it is effective if and only if for every valuation v on $\kappa(\mathcal{X})$ (non-Archimedean or Archimedean) we have $\beta(v, \overline{\mathcal{D}}) \geq 0$.

Notation 2.2.36. If $\overline{\mathcal{D}}$ is an arithmetic $\mathbb{Z}$-divisor of C^∞-type on $\mathcal{X}$, we denote by $\mathcal{O}(\overline{\mathcal{D}})$ the pair $(\mathcal{O}_{\mathcal{X}}(\mathcal{D}), |\cdot|)$, where $|\cdot|$ is the smooth hermitian metric of real type described in Definition 2.2.34. Note that in this case $\widehat{H}^0(\mathcal{X}, \overline{\mathcal{D}}) = \widehat{H}^0(\mathcal{X}, \mathcal{O}(\overline{\mathcal{D}}))$ (both definitions of small sections agree).

Remark 2.2.37. If $\overline{\mathcal{L}}$ is a hermitian line bundle on $\mathcal{X}$, then a rational section s of $\mathcal{L}$ defines an arithmetic $\mathbb{Z}$-divisor of C^∞-type on $\mathcal{X}$. Namely, take the pair $\widehat{\mathrm{div}}(s) = (\mathrm{div}(s), -\log|s|^2_{\overline{\mathcal{L}}})$. Then $\mathcal{O}(\widehat{\mathrm{div}}(s)) \simeq \overline{\mathcal{L}}$ as hermitian line bundles. Moreover, if $\overline{\mathcal{L}} = \mathcal{O}(\overline{\mathcal{D}})$ and s is the rational section corresponding to the rational function 1, then $\widehat{\mathrm{div}}(s) = \overline{\mathcal{D}} = (\mathcal{D}, g)$. To see that $-\log|s|^2_{\overline{\mathcal{L}}} = g$, take an open subset $\mathcal{U}$ where d is a local equation for $\mathcal{D}$ and where $d \in \mathcal{O}_{\mathcal{X}}(\mathcal{U})^\times$. Then $s = d \cdot d^{-1}$, so by the definition we have
$$|s|_{\overline{\mathcal{L}}} = |d| \cdot \exp(-(g + \log|d|^2)/2) = \exp(-g/2).$$

Remark 2.2.38. Let $\overline{\mathcal{L}} = \mathcal{O}(\overline{\mathcal{D}})$ for an arithmetic $\mathbb{Z}$-divisor of C^∞-type $\overline{\mathcal{D}}$ on $\mathcal{X}$. Then for a closed point in the generic fiber $x \in \mathcal{X}_{\mathbb{Q}} \subset \mathcal{X}$, Definition 2.2.30 and the arithmetic degree map from Theorem 2.2.6 agree, i.e.,
$$\widehat{\deg}(\overline{\mathcal{D}}|\overline{\{x\}}) = \widehat{\deg}(\overline{\mathcal{L}}|\overline{\{x\}}).$$

Definition 2.2.39. We gather here properties of arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ crucial for the proof of Theorem 2.2.70.

1. An arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{D}}$ on $\mathcal{X}$ is *big*, if $\widehat{\mathrm{vol}}(\overline{\mathcal{D}}) > 0$.

2. An arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{E}}$ on $\mathcal{X}$ is *pseudo-effective*, if for all big arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{B}}$ on $\mathcal{X}$, the sum $\overline{\mathcal{B}} + \overline{\mathcal{E}}$ is big.

3. We say that $\overline{\mathcal{D}} = (\mathcal{D}, g) \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ is *nef*, if the following conditions are satisfied:
    - $\mathcal{D}$ is relatively nef,
    - $\overline{\mathcal{D}}$ is an arithmetic $\mathbb{R}$-divisor of $(C^0 \cap \mathrm{PSH})$-type,
    - $h_{\overline{\mathcal{D}}}(x) \geq 0$ for all closed points $x \in X = \mathcal{X}_{\mathbb{Q}}$.
4. $\overline{\mathcal{D}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ is called *integrable*, if it is in a real vector space generated by nef arithmetic $\mathbb{R}$-divisors of C^0-type. The space of integrable $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ is denoted by $\mathrm{IDiv}_{\mathbb{R}}(\mathcal{X})$.
5. $\overline{\mathcal{D}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ is called *ample*, if there exists a decomposition $\overline{\mathcal{D}} = \sum_{i=1}^{m} \alpha_i \overline{\mathcal{A}}_i$ with $\alpha_i > 0$ for $i = 1, \ldots, m$ and where $\overline{\mathcal{A}}_i$ are arithmetic $\mathbb{Z}$-divisors of C^∞-type such that each $\mathcal{O}(\overline{\mathcal{A}}_i)$ is an ample hermitian line bundle.

Lemma 2.2.40. Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. If $\overline{\mathcal{D}}$ is big, then there is a natural number $n \neq 0$ and a rational function $f \in \kappa(\mathcal{X})^\times$ such that $\widehat{\mathrm{div}}(f) + n\overline{\mathcal{D}}$ is effective on $\mathcal{X}$. Moreover, $\mathcal{D}_{\mathbb{Q}}$ is big. Also, if $\overline{\mathcal{D}}$ is effective on $\mathcal{X}$, then it is pseudo-effective on $\mathcal{X}$.

Proof. We skip the proof of the first part. For the second part let $\overline{\mathcal{A}}$ be an ample $\mathbb{R}$-divisor of C^∞-type on $\mathcal{X}$ and using the continuity of arithmetic volume (see Theorem 2.2.54) pick a natural number n with $\widehat{\mathrm{vol}}(\overline{\mathcal{D}} - \frac{1}{n}\overline{\mathcal{A}}) > 0$. By the first part of the lemma, there is a $\mathbb{Q}$-rational function f on $\mathcal{X}$ with $\widehat{\mathrm{div}}(f) + \overline{\mathcal{D}} - \frac{1}{n}\overline{\mathcal{A}} \geq 0$. Thus $\mathrm{div}(f) + \mathcal{D}_{\mathbb{Q}} - \frac{1}{n}\mathcal{A}_{\mathbb{Q}} \geq 0$, so $\mathrm{vol}(\mathcal{D}_{\mathbb{Q}}) \geq \mathrm{vol}(\frac{1}{n}\mathcal{A}_{\mathbb{Q}}) > 0$. For the last part, let $\overline{\mathcal{B}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Since $\overline{\mathcal{D}}$ is effective, the rational function 1 is small as an element of $H^0(\mathcal{X}, \mathcal{D})$. Thus any small $s \in \widehat{H}^0(\mathcal{X}, m\overline{\mathcal{B}})$ is also a small element of $\widehat{H}^0(\mathcal{X}, m(\overline{\mathcal{B}} + \overline{\mathcal{D}}))$. Hence $\widehat{\mathrm{vol}}(\overline{\mathcal{B}} + \overline{\mathcal{D}}) \geq \widehat{\mathrm{vol}}(\overline{\mathcal{B}}) > 0$. □

Proposition 2.2.41. [58, Proposition 2.4.2] Let g be a $\mathcal{D}$-Green's function of $\mathcal{J}$-type (for $\mathcal{J}$ equal $C^0, \mathrm{PSH} \cap C^0$ or C^∞) on $\mathcal{X}$ and let

$$\mathcal{D} = b_1\mathcal{E}_1 + \cdots + b_r\mathcal{E}_r$$

be a decomposition such that $\mathcal{E}_1, \ldots, \mathcal{E}_r \in \mathrm{Div}(\mathcal{X})$ and $b_1, \ldots, b_r \in \mathbb{R}$. Note that $\mathcal{E}_i$ is not necessarily an irreducible Weil divisor. Then we have the following:

1. There are $g_1, \ldots, g_r$ such that g_i is an $\mathcal{E}_i$-Green's function of $\mathcal{J}$-type for each i and $g = b_1g_1 + \cdots + b_rg_r$.

2. If $\mathcal{E}_1, \dots, \mathcal{E}_r$ are effective, $b_1, \dots, b_r \geq 0, g \geq 0$, then there are $g_1, \dots, g_r$ such that g_i is a non-negative $\mathcal{E}_i$-Green's function of $\mathcal{J}$-type for each i and $g = b_1 g_1 + \cdots + b_r g_r$.

Lemma 2.2.42. Let $\mathcal{D}$ be an $\mathbb{R}$-Cartier divisor on $\mathcal{X}$. Then there exists a $\mathcal{D}$-Green's function g on $\mathcal{X}$.

Proof. It is enough to prove the lemma assuming that $\mathcal{D}$ is a Cartier divisor. In this case we can use an ample Cartier divisor $\mathcal{H}$ on $\mathcal{X}$ and take a natural number l such that $l\mathcal{H} + \mathcal{D}$ is ample. Since $\mathcal{D} = (l\mathcal{H} + \mathcal{D}) - l\mathcal{H}$, we can without loss of generality assume that $\mathcal{D}$ is ample. Then the corresponding line bundle $\mathcal{O}(\mathcal{D})$ has a hermitian metric, for example by [20, Corollary 2.11]. By taking the rational section of this bundle corresponding to the rational function 1, we are done (see Remark 2.2.37). □

Lemma 2.2.43. Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Fix $\varepsilon > 0$. Then there exist:

- ample arithmetic $\mathbb{R}$-divisors of C^∞-type $\overline{\mathcal{A}}, \overline{\mathcal{A}}'$;
- a C^0-function ϕ on $\mathcal{X}(\mathbb{C})$ with $\|\phi\|_{\sup} < \varepsilon$;

such that $\overline{\mathcal{D}} \equiv \overline{\mathcal{A}} - \overline{\mathcal{A}}' - (0, \phi)$ (equality modulo $\mathbb{R}$-rational equivalence).

Proof. By Proposition 2.2.41 we can write $\overline{\mathcal{D}} = \sum_i a_i \overline{\mathcal{D}}_i$ for some arithmetic $\mathbb{Z}$-divisors of C^0-type $\overline{\mathcal{D}}_i$ and $a_i \in \mathbb{R}$. Let ϕ_i be a C^0-function on $\mathcal{X}(\mathbb{C})$ such that $\overline{\mathcal{D}}_i + (0, \phi_i)$ are of C^∞-type and with $\|\phi_i\|_{\sup} < \delta$. By [20, Corollary 2.6] we can present $\mathcal{O}(\overline{\mathcal{D}}_i + (0, \phi_i))$ as a difference of two ample hermitian line bundles. By taking rational sections of those hermitian line bundles, we can write $\overline{\mathcal{D}}_i + (0, \phi_i)$ as a difference of ample arithmetic $\mathbb{Z}$-divisors of C^∞-type, at least up to a principal divisor. Let

$$\overline{\mathcal{D}}_i + (0, \phi_i) \equiv \overline{\mathcal{A}}_i - \overline{\mathcal{A}}_i'$$

be such a decomposition. Then we get

$$\overline{\mathcal{D}} \equiv \sum_i a_i (\overline{\mathcal{A}}_i - \overline{\mathcal{A}}_i') - (0, \sum_i a_i \phi_i).$$

Note that $\|\sum_i a_i \phi_i\|_{\sup} \leq \sum_i |a_i| \|\phi_i\|_{\sup} < (\sum_i |a_i|)\delta$. By taking δ small enough and rearranging the terms we get the result (note that a sum of ample arithmetic $\mathbb{R}$-divisors is ample). □

2.2.6 Intersection product and arithmetic volume

We use the intersection product of arithmetic divisors defined in [58] and extended by [45]. In our notation an arbitrary arithmetic $\mathbb{R}$-divisor of C^0-type appears on the right in contrary to [45]. More precisely, we have the following.

Theorem 2.2.44. There exists a symmetric multi-linear map

$$\widehat{\deg} : \mathrm{IDiv}_{\mathbb{R}}(\mathcal{X})^{\times(d+1)} \to \mathbb{R}$$

extending the intersection product of arithmetic $\mathbb{Z}$-divisors of C^∞-type from Theorem 2.2.6. Moreover, one can extend it even further, to a multi-linear function

$$\widehat{\deg} : \mathrm{IDiv}_{\mathbb{R}}(\mathcal{X})^{\times d} \times \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R}.$$

This extension is the unique one having the following property: If $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d$ are nef and $\overline{\mathcal{D}}_{d+1}$ is pseudo-effective, then $\widehat{\deg}(\overline{\mathcal{D}}_0 \cdots \overline{\mathcal{D}}_d \cdot \overline{\mathcal{D}}_{d+1}) \geq 0$. Also, it only depends on the classes of the divisors in $\widehat{N}^1(\mathcal{X})$.

Proof. This is [45, Lemma 2.5]. The fact that the product only depends on the classes of divisors in $\widehat{N}^1(\mathcal{X})$ follows for example from [58, Theorem 5.2.2 and Claim 6.4.2.2.(c)]. □

Notation 2.2.45. We usually omit $\widehat{\deg}$ in the notation, and just write $\overline{\mathcal{D}}_0 \cdot \ldots \cdot \overline{\mathcal{D}}_{d+1}$ for $\widehat{\deg}(\overline{\mathcal{D}}_0 \cdots \overline{\mathcal{D}}_d \cdot \overline{\mathcal{D}}_{d+1})$ where $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d$ are integrable arithmetic $\mathbb{R}$-divisors of C^0-type and $\overline{\mathcal{D}}_{d+1}$ is an arithmetic $\mathbb{R}$-divisor of C^0-type.

Lemma 2.2.46. [45, Lemma 2.3 and Lemma 2.5] Let $\pi : \mathcal{X}' \to \mathcal{X}$ be a birational morphism from a generically smooth normal arithmetic variety. Then

$$\overline{\mathcal{D}}_0 \cdot \ldots \cdot \overline{\mathcal{D}}_{d+1} = \pi^*\overline{\mathcal{D}}_0 \cdot \ldots \cdot \pi^*\overline{\mathcal{D}}_{d+1},$$

for all arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_{d+1}$ on $\mathcal{X}$ with $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_d$ integrable.

Theorem 2.2.47. [45, Theorem 6.4] Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$.

1. The following are equivalent.
   (a) $\overline{\mathcal{D}}$ is pseudo-effective.

   (b) For any normalized blowup $\phi : \mathcal{X}' \to \mathcal{X}$ and for any nef arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{H}}$ on $\mathcal{X}'$, we have
$$\widehat{\deg}(\overline{\mathcal{H}}^d \cdot \phi^*\overline{\mathcal{D}}) \geq 0.$$
   (c) For any blowup $\phi : \mathcal{X}' \to \mathcal{X}$ such that $\mathcal{X}'$ is normal, generically smooth, and for any ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathcal{H}}$ on $\mathcal{X}'$, we have
$$\widehat{\deg}(\overline{\mathcal{H}}^d \cdot \phi^*\overline{\mathcal{D}}) \geq 0.$$

2. Suppose that $\overline{\mathcal{D}}$ is pseudo-effective. The following are equivalent.
   (a) $\overline{\mathcal{D}}$ is principal.
   (b) There exist a blowup $\phi : \mathcal{X}' \to \mathcal{X}$ with normal, generically smooth $\mathcal{X}'$, and an ample arithmetic $\mathbb{R}$-divisor of C^∞-type $\overline{\mathcal{H}}$ on $\mathcal{X}'$, such that
$$\widehat{\deg}(\overline{\mathcal{H}}^d \cdot \phi^*\overline{\mathcal{D}}) = 0.$$

Lemma 2.2.48. Let $f : \mathcal{X}' \to \mathcal{X}$ be a blowup and $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Then $\widehat{\mathrm{vol}}(\overline{\mathcal{D}}) = \widehat{\mathrm{vol}}(f^*\overline{\mathcal{D}})$. Moreover, $\overline{\mathcal{D}}$ is pseudo-effective [effective] on $\mathcal{X}$ if and only if $f^*\overline{\mathcal{D}}$ is pseudo-effective [effective] on $\mathcal{X}'$.

Proof. The first part is standard, see for example [45, Section 2] or [59, Definition 4.3.2]. Now we prove the second part. Assume that $f^*\overline{\mathcal{D}}$ is pseudo-effective on $\mathcal{X}'$. Take a big arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{B}}$ on $\mathcal{X}$. Then, by the invariance of the arithmetic volume under birational maps, $f^*\overline{\mathcal{B}}$ is big, so we get
$$\widehat{\mathrm{vol}}(\overline{\mathcal{B}} + \overline{\mathcal{D}}) = \widehat{\mathrm{vol}}(f^*\overline{\mathcal{B}} + f^*\overline{\mathcal{D}}) > 0.$$
Thus $\overline{\mathcal{D}}$ is pseudo-effective on $\mathcal{X}$. The other implication follows from Theorem 2.2.47. To see that effectivity of $\overline{\mathcal{D}}, f^*\overline{\mathcal{D}}$ is equivalent, use Remark 2.2.35 and the definition of the pullback of an arithmetic $\mathbb{R}$-divisor of C^0-type. □

Lemma 2.2.49. Let $f : \mathcal{X}' \to \mathcal{X}$ be a blowup and $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Then $\overline{\mathcal{D}}$ is nef on $\mathcal{X}$ if and only if $f^*\overline{\mathcal{D}}$ is nef on $\mathcal{X}'$.

Proof. We check that the three conditions defining nefness are equivalent for $\overline{\mathcal{D}}$ and $f^*\overline{\mathcal{D}}$.

- Relative nefness: here equivalence follows from the projection formula and the fact that if $\mathcal{C}$ is a curve in $\mathcal{X}$ mapped to a closed point in $\mathrm{Spec}(\mathbb{Z})$, then there is a curve $\mathcal{C}'$ in $\mathcal{X}'$ whose push-forward gives a positive multiple of $\mathcal{C}$.

- ($C^0 \cap \mathrm{PSH}$)-type: here equivalence follows from [59, Proposition 1.4.1].
- Non-negative height on closed points: here equivalence follows from Lemma 2.2.33 and from the fact that f is surjective on closed points.

$\square$

Remark 2.2.50. Because of Lemma 2.2.48 and Lemma 2.2.49 we use the notions of bigness, pseudo-effectivity, effectivity and nefness without referring to an arithmetic variety where the divisor is defined.

2.2.7 Differentiability of the arithmetic volume

Results in this section follow easily from [45]. We provide some proofs for the reader's convenience.

Definition 2.2.51. We call a function $\alpha : \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R}$ *continuous*, if

$$\alpha(\overline{\mathcal{D}}) = \lim_{\varepsilon_1,\dots,\varepsilon_r,\|f\|_{\sup}\to 0} \alpha(\overline{\mathcal{D}} + \sum_{i=1}^{r} \varepsilon_i \overline{\mathcal{D}}_i + (0,f)),$$

for any arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{D}}, \overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_r$ on $\mathcal{X}$ and any f of C^0-type on $\mathcal{X}$.

Definition 2.2.52. Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Let $\hat{\Theta}(\overline{\mathcal{D}})$ be the set of pairs $(f : \mathcal{X}' \to \mathcal{X}, \overline{\mathcal{A}})$ consisting of a blowup $f : \mathcal{X}' \to \mathcal{X}$ together with a nef arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}'$ such that $f^*\overline{\mathcal{D}} - \overline{\mathcal{A}}$ is pseudo-effective. Let $\hat{\Theta}_{\mathrm{amp}}(\overline{\mathcal{D}})$ be the set of pairs $(f : \mathcal{X}' \to \mathcal{X}, \overline{\mathcal{A}})$ in $\hat{\Theta}(\overline{\mathcal{D}})$ where $\overline{\mathcal{A}}$ is an ample arithmetic $\mathbb{R}$-divisor of C^∞-type on $\mathcal{X}'$ and $f^*\overline{\mathcal{D}} - \overline{\mathcal{A}}$ is an effective arithmetic $\mathbb{Q}$-divisor of C^0-type on $\mathcal{X}'$. The *positive intersection d'th power of* $\overline{\mathcal{D}}$ is the function

$$\langle \overline{\mathcal{D}}^d \rangle : \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R}$$

defined in the following way:

$$\langle \overline{\mathcal{D}}^d \rangle \overline{\mathcal{E}} = \sup_{(f,\overline{\mathcal{A}})\in\hat{\Theta}_{\mathrm{amp}}(\overline{\mathcal{D}})} \overline{\mathcal{A}}^d \cdot f^*\overline{\mathcal{E}}, \text{ for nef and big } \overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X});$$

and for other $\overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ we extend by linearity and continuity in the sense of Definition 2.2.51.

Remark 2.2.53. The Definition 2.2.52 is equivalent to the one in [45]. This equivalence follows from [45, Proposition 3.9, Proposition 3.10.(3), proof of Theorem 5.3]. Moreover, if $\overline{\mathcal{D}}$ is nef in the above definition, then $\langle \overline{\mathcal{D}}^d \rangle \overline{\mathcal{E}} = \overline{\mathcal{D}}^d \cdot \overline{\mathcal{E}}$ by [45, Remark 3.8.(3)].

Theorem 2.2.54. [45, Theorem 5.3] The function $\widehat{\mathrm{vol}} : \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R}$ has the following properties:

1. It is continuous in the sense of Definition 2.2.51.
2. It is positively $(d+1)$-homogeneous, i.e.,

   $$\widehat{\mathrm{vol}}(t\overline{\mathcal{D}}) = t^{d+1}\widehat{\mathrm{vol}}(\overline{\mathcal{D}}),$$

   for any $t > 0$ and arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{D}}$ on $\mathcal{X}$.
3. It is differentiable on the big cone and at a big $\overline{\mathcal{D}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ the derivative is given by

   $$D_{\overline{\mathcal{D}}}\widehat{\mathrm{vol}}(\overline{\mathcal{E}}) = (d+1)\langle \overline{\mathcal{D}}^d \rangle \overline{\mathcal{E}}.$$

Lemma 2.2.55. Let $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ with $\overline{\mathcal{B}} - \overline{\mathcal{A}}$ pseudo-effective. Then $\widehat{\mathrm{vol}}(\overline{\mathcal{A}}) \leq \widehat{\mathrm{vol}}(\overline{\mathcal{B}})$.

Proof. Without loss of generality $\widehat{\mathrm{vol}}(\overline{\mathcal{A}}) > 0$. Fix $\varepsilon > 0$. We have

$$\widehat{\mathrm{vol}}(\overline{\mathcal{B}}) = \widehat{\mathrm{vol}}((1-\varepsilon)\overline{\mathcal{A}} + (\varepsilon\overline{\mathcal{A}} + \overline{\mathcal{B}} - \overline{\mathcal{A}})) \geq \widehat{\mathrm{vol}}((1-\varepsilon)\overline{\mathcal{A}}) = (1-\varepsilon)^{d+1}\widehat{\mathrm{vol}}(\overline{\mathcal{A}}),$$

where the inequality follows as in Lemma 2.2.40, as $\varepsilon\overline{\mathcal{A}} + \overline{\mathcal{B}} - \overline{\mathcal{A}}$ is big. By taking $\varepsilon \to 0$, we get the result. □

Definition 2.2.56. Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Following [45], we equip the set $\hat{\Theta}(\overline{\mathcal{D}})$ with a partial order defined in the following way:

$$(f_1 : \mathcal{X}_1 \to \mathcal{X}, \overline{\mathcal{A}}_1) \leq (f_2 : \mathcal{X}_2 \to \mathcal{X}, \overline{\mathcal{A}}_2)$$

if and only if there exists a blowup $f : \mathcal{X}' \to \mathcal{X}$ which factors as

$$\mathcal{X}' \xrightarrow{g_i} \mathcal{X}_i \xrightarrow{f_i} \mathcal{X}$$

so that on $\mathcal{X}'$ the arithmetic $\mathbb{R}$-divisor of C^0-type $g_2^*\overline{\mathcal{A}}_2 - g_1^*\overline{\mathcal{A}}_1$ is pseudo-effective.

Lemma 2.2.57. [45, Proposition 3.2] Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Then $\hat{\Theta}(\overline{\mathcal{D}})$ is filtered with respect to the order defined in Definition 2.2.56. Moreover, for any $(f_1 : \mathcal{X}_1 \to \mathcal{X}, \overline{\mathcal{A}}_1), (f_2 : \mathcal{X}_2 \to \mathcal{X}, \overline{\mathcal{A}}_2) \in \hat{\Theta}(\overline{\mathcal{D}})$ there is a blowup $f : \mathcal{X}' \to \mathcal{X}$ together with a nef arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{A}}$ such that f factors as in Definition 2.2.56 and

$$g_1^*\overline{\mathcal{A}}_2, g_2^*\overline{\mathcal{A}}_2 \leq \overline{\mathcal{A}} \leq f^*\overline{\mathcal{D}},$$

where the inequalities are written with respect to the pseudo-effective cone.

Lemma 2.2.58. [45, Proposition 3.1] Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Let $(f : \mathcal{X}' \to \mathcal{X}, \overline{\mathcal{A}}) \in \hat{\Theta}(\overline{\mathcal{D}})$ and fix a number $0 < \gamma < 1$. Then there is $\overline{\mathcal{B}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}')$ with $(f, \overline{\mathcal{B}}) \in \hat{\Theta}_{\mathrm{amp}}(\overline{\mathcal{D}})$ such that $f^*\overline{\mathcal{D}} - \overline{\mathcal{B}}$ is an effective arithmetic $\mathbb{Q}$-divisor of C^0-type and $\overline{\mathcal{B}} - \gamma\overline{\mathcal{A}}$ is pseudo-effective.

Lemma 2.2.59. Let $\overline{\mathcal{D}}_1, \ldots, \overline{\mathcal{D}}_n$ be big and nef arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$. Let $\overline{\mathcal{B}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Fix $\varepsilon > 0$. Then there is a pair $(f, \overline{\mathcal{A}}) \in \hat{\Theta}_{\mathrm{amp}}(\overline{\mathcal{B}})$ such that for all $i = 1, \ldots, n$

$$|\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i - \overline{\mathcal{A}}^d \cdot f^*\overline{\mathcal{D}}_i| < \varepsilon$$

and satisfying

$$\widehat{\mathrm{vol}}(\overline{\mathcal{B}}) \geq \widehat{\mathrm{vol}}(\overline{\mathcal{A}}) > \frac{1}{2}\widehat{\mathrm{vol}}(\overline{\mathcal{B}}).$$

Proof. First of all, we take $(f_0, \overline{\mathcal{A}}_0) \in \hat{\Theta}(\overline{\mathcal{B}})$ satisfying

$$\widehat{\mathrm{vol}}(\overline{\mathcal{B}}) \geq \widehat{\mathrm{vol}}(\overline{\mathcal{A}}_0) > \frac{3}{4}\widehat{\mathrm{vol}}(\overline{\mathcal{B}}).$$

Such exists by the arithmetic Fujita approximation theorem [22, Theorem 4.3], [82, Theorem C]; see also [45, Proposition 3.11.(1)]. Next, take pairs $(f_i, \overline{\mathcal{A}}_i) \in \hat{\Theta}_{\mathrm{amp}}(\overline{\mathcal{B}})$ such that

$$\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i - \varepsilon \leq \overline{\mathcal{A}}_i^d \cdot f_i^*\overline{\mathcal{D}}_i \leq \langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i$$

for all $i = 1, \ldots, n$. By Lemma 2.2.57 and induction, there is a pair $(f : \mathcal{X}' \to \mathcal{X}, \overline{\mathcal{A}}') \in \hat{\Theta}(\overline{\mathcal{D}})$ such that for all $i = 0, \ldots, n$ there exists a factorisation

$$\begin{tikzcd}\mathcal{X}' \arrow[r, "g_i"] & \mathcal{X}_i \arrow[r, "f_i"] & \mathcal{X}\end{tikzcd}$$

and

$$g_i^*\overline{\mathcal{A}}_i \leq \overline{\mathcal{A}}' \leq f^*\overline{\mathcal{D}}$$

with respect to the pseudo-effective cone. We can proceed by induction because of Lemma 2.2.48. Choose a real number $0 < \gamma < 1$ to be specified later. By Lemma 2.2.58 there is an ample $\overline{\mathcal{A}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}')$ such that $(f, \overline{\mathcal{A}}) \in \hat{\Theta}_{\mathrm{amp}}(\overline{\mathcal{B}})$ and $\overline{\mathcal{A}} - \gamma\overline{\mathcal{A}}'$ is pseudo-effective. Using Lemma 2.2.55 we get:

$$\widehat{\mathrm{vol}}(\overline{\mathcal{A}}) \geq \widehat{\mathrm{vol}}(\gamma\overline{\mathcal{A}}') = \gamma^d \widehat{\mathrm{vol}}(\overline{\mathcal{A}}') \geq \gamma^d \widehat{\mathrm{vol}}(\overline{\mathcal{A}}_0) > \gamma^d \cdot \frac{3}{4}\widehat{\mathrm{vol}}(\overline{\mathcal{B}}) > \frac{1}{2}\widehat{\mathrm{vol}}(\overline{\mathcal{B}}),$$

for γ such that $\gamma^d \cdot \frac{3}{4} > \frac{1}{2}$. The inequality $\widehat{\mathrm{vol}}(\overline{\mathcal{A}}) \leq \widehat{\mathrm{vol}}(\overline{\mathcal{B}})$ also follows from Lemma 2.2.55. Moreover for $i = 1, \dots, n$ we have

$$\begin{aligned} \langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i &\geq \overline{\mathcal{A}}^d \cdot f^*\overline{\mathcal{D}}_i \geq (\gamma\overline{\mathcal{A}}')^d \cdot f^*\overline{\mathcal{D}}_i \geq \gamma^d \cdot \overline{\mathcal{A}}_i^d \cdot f_i^*\overline{\mathcal{D}}_i \\ &\geq \gamma^d(\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i - \varepsilon) = \gamma^d \langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i - \gamma^d \varepsilon. \end{aligned}$$

Thus

$$|\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i - \overline{\mathcal{A}}^d \cdot f^*\overline{\mathcal{D}}_i| \leq (1 - \gamma^d)\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i + \gamma^d \varepsilon.$$

Let $M := \max(\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i : i = 1, \dots, n)$. Then for γ such that

$$(1 - \gamma^d)M + \gamma^d \varepsilon < 2\varepsilon$$

and using the fact that ε was arbitrary, we are done. □

Lemma 2.2.60. The following functions are continuous in the sense of Definition 2.2.51:

- $h_{(-)}(x)$ for a closed point $x \in \mathcal{X} \otimes \mathbb{Q}$.
- $\overline{\mathcal{D}}^d \cdot (-)$ for $\overline{\mathcal{D}}$ an ample arithmetic $\mathbb{R}$-divisor of C^∞-type on $\mathcal{X}$.
- $\langle \overline{\mathcal{B}}^d \rangle (-)$ for $\overline{\mathcal{B}}$ a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$.

Proof. All of the functions in question are linear, so it is enough to check that for α being one of them, we have

$$\lim_{\|f\|_{\sup} \to 0} \alpha((0, f)) = 0.$$

For the height function it follows directly from the definition. For the second function this is [45, Lemma 2.5]. For the last function this is by Definition 2.2.52. □

2.2.8 Essential infimum and Zhang's inequality

Definition 2.2.61. Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Use the notation $X = \mathcal{X} \otimes \mathbb{Q}$. The *essential infimum* of $\overline{\mathcal{D}}$ is the following number

$$\zeta(\overline{\mathcal{D}}) := \sup_{\mathcal{Y} \subset \mathcal{X}} \inf_{x \in \mathcal{X} \setminus \mathcal{Y}(\overline{\mathbb{Q}})} h_{\overline{\mathcal{D}}}(x),$$

where the supremum is taken taken over all closed strict subschemes $\mathcal{Y} \subset \mathcal{X}$. Note that ζ depends only on the rational class of an arithmetic $\mathbb{R}$-divisor of C^0-type and a priori has values in $\mathbb{R} \cup \{\pm\infty\}$.

Lemma 2.2.62. The essential infimum function ζ has the following properties:

1. If $\overline{\mathcal{D}}$ is an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$ and $f : \mathcal{X}' \to \mathcal{X}$ is a blowup, then $\zeta(\overline{\mathcal{D}}) = \zeta(f^*\overline{\mathcal{D}})$.
2. For any arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ on $\mathcal{X}$
$$\zeta(\overline{\mathcal{A}} + \overline{\mathcal{B}}) \geq \zeta(\overline{\mathcal{A}}) + \zeta(\overline{\mathcal{B}}).$$
3. It is positively 1-homogeneous.
4. It is non-negative on effective arithmetic $\mathbb{R}$-divisors of C^0-type.
5. For any arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ on $\mathcal{X}$ such that $\overline{\mathcal{B}}$ is big and $\overline{\mathcal{A}} \leq \overline{\mathcal{B}}$ with respect to the pseudo-effective cone, we have $\zeta(\overline{\mathcal{A}}) \leq \zeta(\overline{\mathcal{B}})$.
6. It has values in $\mathbb{R} \cup \{-\infty\}$.

Proof. The first property follows from the fact that blowups are isomorphisms on an open set and from Lemma 2.2.33. For the proof of properties two, three and four, see [3, Lemma 3.15]. For the fifth one fix arithmetic $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{A}}, \overline{\mathcal{B}}$ on $\mathcal{X}$ such that $\overline{\mathcal{B}}$ is big and $\overline{\mathcal{A}} \leq \overline{\mathcal{B}}$. Then we have

$$\begin{aligned}\zeta(\overline{\mathcal{B}}) = \lim_{\varepsilon \to 0^+} \zeta((1+\varepsilon)\overline{\mathcal{B}}) &= \lim_{\varepsilon \to 0^+} \zeta(\varepsilon\overline{\mathcal{B}} + (\overline{\mathcal{B}} - \overline{\mathcal{A}}) + \overline{\mathcal{A}}) \\ &\geq \liminf_{\varepsilon \to 0^+} \zeta(\varepsilon\overline{\mathcal{B}} + (\overline{\mathcal{B}} - \overline{\mathcal{A}})) + \zeta(\overline{\mathcal{A}}).\end{aligned}$$

By pseudo-effectivity of $\overline{\mathcal{B}} - \overline{\mathcal{A}}$ and bigness of $\overline{\mathcal{B}}$ the divisor $\varepsilon\overline{\mathcal{B}} + (\overline{\mathcal{B}} - \overline{\mathcal{A}})$ is big for $\varepsilon > 0$, hence the value of ζ on it is non-negative. Thus we get $\zeta(\overline{\mathcal{B}}) \geq \zeta(\overline{\mathcal{A}})$.

By Lemma 2.2.43 and the fact that values of ζ only depend on the rational equivalence classes of arithmetic $\mathbb{R}$-divisors of C^0-type, to get the last property it is enough to show that for an ample $\mathbb{Z}$-divisor of C^∞-type $\overline{\mathcal{A}}$ one has $\zeta(\overline{\mathcal{A}}) < \infty$. This follows from Zhang's inequalities, but for an elementary proof using Weil's height machine, see [14, Proposition 2.6]. $\square$

Lemma 2.2.63. [3, Lemma 3.16] Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$ and let $\overline{\mathcal{D}}_0 = (\mathrm{div}(2), 0)$. Then

1. $\zeta(\overline{\mathcal{D}} - \frac{r}{\log 2} \cdot \overline{\mathcal{D}}_0) = \zeta(\overline{\mathcal{D}}) - r$ for all real r;
2. if $\overline{\mathcal{D}}$ is pseudo-effective and $\mathcal{D}_{\mathbb{Q}}$ is big, then $\zeta(\overline{\mathcal{D}}) \geq 0$;
3. if $\mathcal{D}_{\mathbb{Q}}$ is big, the following inequality holds
$$\zeta(\overline{\mathcal{D}}) \geq \sup\{r : \overline{\mathcal{D}} - \frac{r}{\log 2} \cdot \overline{\mathcal{D}}_0 \text{ is pseudo-effective}\}.$$

Proof. The first point follows from additivity of height and Lemma 2.2.32. The second point is [3, Lemma 3.16]. The third point follows from the first two. □

Theorem 2.2.64. Let $\overline{\mathcal{D}}$ be a pseudo-effective arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$ with $\mathcal{D}_{\mathbb{Q}}$ big. Then the following inequality holds
$$\zeta(\overline{\mathcal{D}}) \geq \frac{\widehat{\mathrm{vol}}(\overline{\mathcal{D}})}{(d+1)\,\mathrm{vol}(\mathcal{D}_{\mathbb{Q}})}.$$

Proof. In this form the theorem follows from [3, Theorem 7.2.(ii)]. □

2.2.9 Convex geometry

Lemma 2.2.65. An open convex set in $\mathbb{R}^n$ is the interior of its closure.

Proof. This is classical, for a proof see for example [10, Lemma 5.6]. □

Lemma 2.2.66. Let V be a finite dimensional real vector space and let V^+ be a closed convex cone in V not containing any line. Let $l : V \to \mathbb{R}$ be a linear functional non-negative on V^+. Then for any vectors $v_0, \ldots, v_n \in V$ and for any $\varepsilon > 0$ there is a linear functional $l' : V \to \mathbb{R}$ strictly positive on $V^+ \setminus \{0\}$ such that for all $i = 0, \ldots, n$
$$|l(v_i) - l'(v_i)| \leq \varepsilon.$$

Moreover, if $v_0 \in V^+$, then one can additionally assume that $l'(v_0) \geq l(v_0)$.

Proof. Fix $v_0, \ldots, v_n \in V$ and $\varepsilon > 0$. It is enough to find a linear functional $m : V \to \mathbb{R}$ such that $V^+ \setminus \{0\} \subset \{v \in V : m(v) > 0\}$. Indeed, in this case we can just put $l' := l + \delta m$ for small enough positive δ. To define m one can equip V with a (non-degenerate, positively defined) scalar product and take m to be the product with a vector from the interior of the dual cone of V^+. For details, see for example [10, Lemma 5.7]. □

Lemma 2.2.67. Let V be a finite dimensional real vector space and let V^+ be a closed convex cone in V generating the whole space. Assume that $l : V \to \mathbb{R}$ is a linear functional strictly positive on $V^+ \setminus \{0\}$ and that $\alpha : V \to \mathbb{R}$ is a non-negative function which is:

1. non-zero,
2. continuous,
3. positively 1-homogeneous,
4. differentiable on $(V^+)^{\mathrm{o}}$,
5. vanishing outside $(V^+)^{\mathrm{o}}$.

Then there is a unique continuous continuation of the function $\frac{\alpha}{l}$ to $V \setminus \{0\}$. Moreover, there exists a point $p \in (V^+)^{\mathrm{o}}$ such that

$$\frac{D_p \alpha}{\alpha(p)} = \frac{l}{l(p)}.$$

Proof. Pick a euclidean norm on V and let S be the unit sphere of it. Since both l and α are positively 1-homogeneous, for the first part it is enough to show that $\frac{\alpha}{l}$ has a unique continuous extension to S. This follows from the fact that if $l(q) = 0$ for $q \in S$, then $q \in S \cap V \setminus V^+$ and $S \cap V \setminus V^+$ is a relatively open set in S, where α is zero. To prove the second part, let $p \in S$ be a point where the maximum of $\frac{\alpha}{l}$ is achieved (it exists by compactness of S). Then $\alpha(p), l(p) > 0$, because α isnon-zero. Point p is a local maximum of $\frac{\alpha}{l}$ and so also of $\log(\frac{\alpha}{l}) = \log(\alpha) - \log(l)$ (note that both functions are defined in a neighbourhood of p). Hence the derivative of $\log(\frac{\alpha}{l})$ at p vanishes and we get

$$\frac{D_p \alpha}{\alpha(p)} = D_p(\log(\alpha)) = D_p(\log(l)) = \frac{l}{l(p)}.$$

□

2.2.10 Approximating a global functional

For this subsection we fix a normal, generically smooth arithmetic variety $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z})$. Let $d + 1 \geq 2$ be the dimension of $\mathcal{X}$.

Proposition 2.2.68. Let $\overline{\mathcal{B}}, \overline{\mathcal{D}}_0, \ldots, \overline{\mathcal{D}}_n$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ with $\overline{\mathcal{B}}$ big. Fix $\varepsilon > 0$. Then there exists a blowup $f : \mathcal{X}' \to \mathcal{X}$ and an ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathcal{B}}'$ on $\mathcal{X}'$ such that:

- $\operatorname{vol}(\mathcal{B}'_{\mathbb{Q}}) > \frac{\widehat{\operatorname{vol}}(\overline{\mathcal{B}})}{2(d+1)\zeta(\overline{\mathcal{B}})} =: A$;
- $|\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i - \overline{\mathcal{B}}'^d \cdot f^*\overline{\mathcal{D}}_i| < A\varepsilon$ for all $i = 0, \dots, n$.

Proof. First, note that the expression

$$\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}} - \overline{\mathcal{B}}'^d \cdot f^*\overline{\mathcal{D}}$$

is linear and continuous in $\overline{\mathcal{D}}$, see Lemma 2.2.60. Moreover, it only depends on the rational equivalence class of $\overline{\mathcal{D}}$. Hence, by Lemma 2.2.43 we can without loss of generality assume that $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_n$ are ample arithmetic $\mathbb{R}$-divisors. In particular they are big and nef. Thus, by Lemma 2.2.59 for $A\varepsilon$ instead of ε, there is a pair $(f : \mathcal{X}' \to \mathcal{X}, \overline{\mathcal{A}}) \in \hat{\Theta}_{\text{amp}}(\overline{\mathcal{B}})$ such that for all $i = 0, \dots, n$

$$|\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i - \overline{\mathcal{A}}^d \cdot f^*\overline{\mathcal{D}}_i| < A\varepsilon$$

and

$$\widehat{\operatorname{vol}}(\overline{\mathcal{B}}) \geq \widehat{\operatorname{vol}}(\overline{\mathcal{A}}) > \frac{1}{2}\widehat{\operatorname{vol}}(\overline{\mathcal{B}}).$$

We use Theorem 2.2.64 to get the middle inequality of

$$\frac{\widehat{\operatorname{vol}}(\overline{\mathcal{B}})}{2(d+1)\operatorname{vol}(\mathcal{A}_{\mathbb{Q}})} < \frac{\widehat{\operatorname{vol}}(\overline{\mathcal{A}})}{(d+1)\operatorname{vol}(\mathcal{A}_{\mathbb{Q}})} \leq \zeta(\overline{\mathcal{A}}) \leq \zeta(f^*\overline{\mathcal{B}}) = \zeta(\overline{\mathcal{B}}).$$

The last inequality is by Lemma 2.2.62. We get

$$\operatorname{vol}(\mathcal{A}_{\mathbb{Q}}) > \frac{\widehat{\operatorname{vol}}(\overline{\mathcal{B}})}{2(d+1)\zeta(\overline{\mathcal{B}})} = A.$$

Note that $\overline{\mathcal{A}}$ is an ample arithmetic $\mathbb{R}$-divisor of C^∞-type on $\mathcal{X}'$, so it is of the form

$$\overline{\mathcal{A}} = \sum_j a_j \overline{\mathcal{A}}_j,$$

for some some $a_j \in \mathbb{R}$, and $\overline{\mathcal{A}}_j$ being ample arithmetic $\mathbb{Z}$-divisors of C^∞-type. Replacing a_j's by sufficiently good rational approximations, we can find an ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathcal{B}}'$ on $\mathcal{X}'$ satisfying the desired inequalities. This is because of multilinearity of the arithmetic intersection product and the continuity of the volume vol. □

The below proposition is proven in a similar way to [3, Theorem 4.1] and [64, Theorem 4.6].

Proposition 2.2.69. Let $\overline{\mathcal{B}}, \overline{\mathcal{D}}_0, \ldots, \overline{\mathcal{D}}_n$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ with $\overline{\mathcal{B}}$ being an ample arithmetic $\mathbb{Q}$-divisor of C^∞-type. Let $Z \subset X = \mathcal{X} \otimes \mathbb{Q}$ be a closed strict subscheme. Fix $\varepsilon > 0$. Then there exists a closed point $x \in X \setminus Z$ such that

$$\left| h_{\overline{\mathcal{D}}_i}(x) - \frac{\overline{\mathcal{B}}^d \cdot \overline{\mathcal{D}}_i}{\operatorname{vol}(\mathcal{B}_{\mathbb{Q}})} \right| < \varepsilon, \text{ for all } i = 0, \ldots, n.$$

Proof. First, note that $\frac{\overline{\mathcal{B}}^d \cdot \overline{\mathcal{D}}_i}{\operatorname{vol}(\mathcal{B}_{\mathbb{Q}})}$ does not change when we replace $\overline{\mathcal{B}}$ with a multiple of itself, so we can without loss of generality assume that $\overline{\mathcal{B}}$ is an ample $\mathbb{Z}$-arithmetic divisor of C^∞-type. Also, the expression

$$h_{\overline{\mathcal{D}}}(x) - \frac{\overline{\mathcal{B}}^d \cdot \overline{\mathcal{D}}}{\operatorname{vol}(\mathcal{B}_{\mathbb{Q}})}$$

is linear in $\overline{\mathcal{D}}$ and depends only on its rational equivalence class. Thus, by Lemma 2.2.60 and Lemma 2.2.43, we can without loss of generality assume that all $\overline{\mathcal{D}}_i$'s are ample arithmetic $\mathbb{Z}$-divisors of C^∞-type.

The above reduction allows us to work with ample hermitian line bundles

$$\mathcal{O}(\overline{\mathcal{B}}), \mathcal{O}(\overline{\mathcal{D}}_0), \ldots, \mathcal{O}(\overline{\mathcal{D}}_n) \text{ on } \mathcal{X}.$$

To ease the notation we do not write $\mathcal{O}(-)$ for the corresponding hermitian line bundles. Let C be a positive number, such that for all $i = 0, \ldots, n$ we have $C \cdot c_1(\overline{\mathcal{B}}) \geq c_1(\overline{\mathcal{D}}_i)$. Now we use Theorem 2.2.14 together with Theorem 2.2.15 for $\overline{\mathcal{L}} = \overline{\mathcal{B}}$ and $\mathcal{Y}$ being the closure of Z in $\mathcal{X}$, to find a small section $s_1 \in \widehat{H}^0(\mathcal{X}, \overline{\mathcal{L}}^{\otimes n_1})$ which is $(\delta, \overline{\mathcal{L}})$-irreducible (we specify δ later) and with $\operatorname{div}(s_1)$ generically smooth and not contained in $\mathcal{Y}$. Then for all $i = 0, \ldots, n$ we have

$$\overline{\mathcal{D}}_i \cdot \overline{\mathcal{B}}^d = \frac{1}{n_1} \overline{\mathcal{D}}_i \cdot \overline{\mathcal{B}}^{d-1} \cdot \widehat{\operatorname{div}}(s_1)$$

$$= \frac{1}{n_1}\Big(\overline{\mathcal{D}}_i|_{\operatorname{div}(s_1)} \cdot (\overline{\mathcal{B}}|_{\operatorname{div}(s_1)})^{d-1} + \int_{\mathcal{X}(\mathbb{C})} - \log \|s\| \wedge c_1(\overline{\mathcal{D}}_i) \wedge c_1(\overline{\mathcal{B}})^{d-1} \Big).$$

Note that in the above formula we treat $\operatorname{div}(s_1)$ as a cycle on $\mathcal{X}$ as in Theorem 2.2.6. By the choice of s_1 (and using one of the equivalent conditions from Definition 2.2.12) we get

$$0 \leq \int_{\mathcal{X}(\mathbb{C})} - \log \|s\| \wedge c_1(\overline{\mathcal{D}}_i) \wedge c_1(\overline{\mathcal{B}})^{d-1}$$

$$\leq C \cdot \int_{\mathcal{X}(\mathbb{C})} - \log \|s\| \wedge c_1(\overline{\mathcal{B}})^d \leq \frac{\delta}{1+\delta} \cdot n_1 C \cdot \overline{\mathcal{B}}^{d+1} < n_1 \varepsilon,$$

for δ small enough for the last inequality. Together we obtain

$$\big|\overline{\mathcal{D}}_i \cdot \overline{\mathcal{B}}^d - \frac{1}{n_1}\overline{\mathcal{D}}_i|_{\mathrm{div}(s_1)} \cdot (\overline{\mathcal{B}}|_{\mathrm{div}(s_1)})^{d-1}\big| < \varepsilon.$$

Now we use exactly the same strategy, to find a small section $s_2 \in \widehat{H}^0(\mathrm{div}(s_1), \overline{\mathcal{L}}|^{\otimes n_2}_{\mathrm{div}\,(s_1)})$ which satisfies

$$\big|\overline{\mathcal{D}}_i|_{\mathrm{div}(s_1)} \cdot (\overline{\mathcal{B}}|_{\mathrm{div}(s_1)})^{d-1} - \frac{1}{n_2}\overline{\mathcal{D}}_i|_{\mathrm{div}(s_1)\cap\mathrm{div}(s_2)} \cdot (\overline{\mathcal{B}}|_{\mathrm{div}(s_1)\cap\mathrm{div}(s_2)})^{d-2}\big| < \varepsilon.$$

We can do this since the cycle $\mathrm{div}(s_1)$ is irreducible, hence it is a (generically smooth, but not necessarily normal) arithmetic variety. Moreover, restrictions of ample hermitian line bundles are ample, for example by [20, Corollary 2.8]. In the above inequality $\mathrm{div}(s_1) \cap \mathrm{div}(s_2)$ is the cycle associated to zeroes of s_2 on the arithmetic variety $\mathrm{div}(s_1)$.

Repeating the above procedure d times, we find small sections $s_1, \ldots, s_d$ of

$$\overline{\mathcal{L}}^{\otimes n_1}, (\overline{\mathcal{L}}|_{\mathrm{div}(s_1)})^{\otimes n_2}, \ldots, (\overline{\mathcal{L}}|_{\mathrm{div}(s_1)\cap\ldots\cap\mathrm{div}(s_{d-1})})^{\otimes n_d}$$

respectively, such that the following hold:

1. $\mathrm{div}(s_1) \cap \cdots \cap \mathrm{div}(s_d)$ is an irreducible, generically smooth arithmetic variety of dimension 1, hence it is of the form $\overline{\{x\}}$ for some closed point $x \in X$. Moreover, we can assume that $x \notin Z$.

2. For all $i = 0, \ldots, n$ we have

$$\big|\overline{\mathcal{D}}_i \cdot \overline{\mathcal{B}}^d - \frac{1}{n_1 \cdots n_d}\widehat{\deg}(\overline{\mathcal{D}}_i|\overline{\{x\}})\big| < d\varepsilon. \tag{2.1}$$

By the inductive definition of the ordinary intersection product of line bundles on a variety X (see for example [18, Remark (2.6)]) and by ampleness of $\mathcal{B}_{\mathbb{Q}}$ we get

$$\begin{aligned}\mathrm{vol}(\mathcal{B}_{\mathbb{Q}}) = \mathcal{B}_{\mathbb{Q}}^d &= \frac{1}{n_1}\deg(\mathcal{B}_{\mathbb{Q}}^{d-1}|\,\mathrm{div}(s_1)_{\mathbb{Q}}) \\ &= \frac{1}{n_1 \cdot n_2}\deg(\mathcal{B}_{\mathbb{Q}}^{d-2}|\,\mathrm{div}(s_1)_{\mathbb{Q}} \cap \mathrm{div}(s_2)_{\mathbb{Q}}) \\ = \cdots &= \frac{1}{n_1 \cdots n_d}\deg(\mathrm{div}(s_1)_{\mathbb{Q}} \cap \cdots \cap \mathrm{div}(s_d)_{\mathbb{Q}}) \\ &= \frac{1}{n_1 \cdots n_d}\deg(x).\end{aligned}$$

Hence, using Equation (2.1) we end up with

$$\left|\frac{\overline{\mathcal{D}}_i \cdot \overline{\mathcal{B}}^d}{\mathrm{vol}(\mathcal{B}_{\mathbb{Q}})} - \frac{\widehat{\deg}(\overline{\mathcal{D}}_i | \overline{\{x\}})|}{\deg(x)}\right| < \frac{d}{\mathrm{vol}(\mathcal{B}_{\mathbb{Q}})}\varepsilon \text{ for all } i = 0, \dots, n.$$

By the definition of height we have

$$h_{\overline{\mathcal{D}}_i}(x) = \frac{\widehat{\deg}(\overline{\mathcal{D}}_i | \overline{\{x\}})}{\deg(x)},$$

so

$$\left|h_{\overline{\mathcal{D}}_i}(x) - \frac{\overline{\mathcal{D}}_i \cdot \overline{\mathcal{B}}^d}{\mathrm{vol}(\mathcal{B}_{\mathbb{Q}})}\right| < \frac{d}{\mathrm{vol}(\mathcal{B}_{\mathbb{Q}})}\varepsilon \text{ for all } i = 0, \dots, n.$$

This finishes the proof, as ε was arbitrary. $\square$

Theorem 2.2.70. Let $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_n$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ with $\overline{\mathcal{D}}_0 = (\mathrm{div}(2), 0)$. Fix $\varepsilon > 0$, a closed strict subscheme $Z \subset X = \mathcal{X} \otimes \mathbb{Q}$, and a linear functional

$$l : \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}) \to \mathbb{R},$$

which is non-negative on effective arithmetic $\mathbb{R}$-divisors of C^0-type, zero on principal arithmetic $\mathbb{R}$-divisors of C^0-type, and sends $\overline{\mathcal{D}}_0$ to $\log(2)$. Then there exists a closed point $x \in X \setminus Z$ such that for all $i = 0, \dots, n$

$$|h_{\overline{\mathcal{D}}_i}(x) - l(\overline{\mathcal{D}}_i)| < \varepsilon.$$

Proof. Put $M := \max(|l(\overline{\mathcal{D}}_0)|, \dots, |l(\overline{\mathcal{D}_n})|)$. Let V be the real span of classes of $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_n, \overline{\mathcal{M}}$ in $\widehat{N}^1(\mathcal{X})$, where $\overline{\mathcal{M}}$ is some big arithmetic $\mathbb{R}$-divisor of C^0-type in $\mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$. Let V^+ be the (euclidean) closure of the subset of classes of big arithmetic $\mathbb{R}$-divisors of C^0-type from V. It is a convex closed cone that does not contain any line in V by Theorem 2.2.47. Moreover, $\overline{\mathcal{D}}_0$ is effective, hence by Lemma 2.2.40 it lies in V^+. Thus, by Lemma 2.2.66 we can approximate l (by the assumptions it is well defined on V) arbitrarily close by linear functionals strictly positive on $V^+ \setminus \{0\}$ and non-decreasing the value on $\overline{\mathcal{D}}_0$. Since ε is arbitrary in the statement of the theorem, we can without loss of generality replace l with sufficiently good approximation, so we assume that l is strictly positive on $V^+ \setminus \{0\}$, $l(\overline{\mathcal{D}}_0) = \log(2) + \varepsilon'$ for some $0 \leq \varepsilon' < \varepsilon$, and that $\max(|l(\overline{\mathcal{D}}_0)|, \dots, |l(\overline{\mathcal{D}_n})|) < 2M$.

We use Lemma 2.2.67 with $\alpha = \widehat{\mathrm{vol}}^{1/(d+1)}$ and V^+, V, l as above. Note that the assumptions of the lemma are satisfied by Theorem 2.2.54 and by Lemma 2.2.65 which implies that $(V^+)^\circ$ is the set of classes of big arithmetic $\mathbb{R}$-divisors of C^0-type in V. Thus there exists $p \in (V^+)^\circ$ such that

$$\frac{D_p\alpha}{\alpha(p)} = \frac{l}{l(p)}.$$

If $\overline{\mathcal{B}}$ represents the class of p, then by Theorem 2.2.54 we get

$$\frac{\langle \overline{\mathcal{B}}^d \rangle}{\widehat{\mathrm{vol}}(\overline{\mathcal{B}})} = \frac{l}{l(\overline{\mathcal{B}})}.$$

By Proposition 2.2.68 there is a blowup $f : \mathcal{X}' \to \mathcal{X}$ and an ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathcal{B}}'$ on $\mathcal{X}'$ such that the following hold:

- $\mathrm{vol}(\mathcal{B}'_{\mathbb{Q}}) > \frac{\widehat{\mathrm{vol}}(\overline{\mathcal{B}})}{2(d+1)\zeta(\overline{\mathcal{B}})} =: A$;
- $|\langle \overline{\mathcal{B}}^d \rangle \overline{\mathcal{D}}_i - \overline{\mathcal{B}}'^d \cdot f^*\overline{\mathcal{D}}_i| < A\varepsilon$ for all $i = 1, \dots, n$.

Now, we can use Proposition 2.2.69 for the ample arithmetic $\mathbb{Q}$-divisor of C^∞-type $\overline{\mathcal{B}}'$ and arithmetic $\mathbb{R}$-divisors of C^0-type $f^*\overline{\mathcal{D}}_0, \dots, f^*\overline{\mathcal{D}}_n$ to find a closed point $x' \in X' \setminus f^{-1}(Z)$ (where $X' := \mathcal{X}' \otimes \mathbb{Q}$) which satisfies

$$\left| h_{f^*\overline{\mathcal{D}}_i}(x') - \frac{\overline{\mathcal{B}}'^d \cdot f^*\overline{\mathcal{D}}_i}{\mathrm{vol}(\mathcal{B}'_{\mathbb{Q}})} \right| < \varepsilon \text{ for all } i = 0, \dots, n.$$

By putting the above equations and inequalities together we get that for any natural i smaller or equal n we have

$$\left| h_{f^*\overline{\mathcal{D}}_i}(x') - \frac{\widehat{\mathrm{vol}}(\overline{\mathcal{B}})}{\mathrm{vol}(\mathcal{B}'_{\mathbb{Q}}) l(\overline{\mathcal{B}})} \cdot l(\overline{\mathcal{D}}_i) \right| < \left(1 + \frac{A}{\mathrm{vol}(\mathcal{B}'_{\mathbb{Q}})}\right)\varepsilon < 2\varepsilon. \qquad (2.2)$$

Let $C := \frac{\widehat{\mathrm{vol}}(\overline{\mathcal{B}})}{\mathrm{vol}(\mathcal{B}'_{\mathbb{Q}}) l(\overline{\mathcal{B}})}$. Applying the last inequality for $i = 0$ we get

$$|\log(2) - C(\log(2) + \varepsilon')| = |h_{f^*\overline{\mathcal{D}}_0}(x') - C \cdot l(\overline{\mathcal{D}}_0)| < 2\varepsilon,$$

because of Lemma 2.2.32. Thus by Lemma 2.2.71 we get that $|C - 1| < 3\varepsilon/\log(2)$. Recall that $\max(|l(\overline{\mathcal{D}}_0)|, \dots, |l(\overline{\mathcal{D}}_n)|) < 2M$. By using the bound on $|C - 1|$ in Equation (2.2) we get

$$|h_{f^*\overline{\mathcal{D}}_i}(x') - l(\overline{\mathcal{D}}_i)| < \left(2 + \frac{6M}{\log(2)}\right)\varepsilon.$$

Let $x = f(x') \in X$. By Lemma 2.2.33 we have $h_{f^*\overline{\mathcal{D}}_i}(x') = h_{\overline{\mathcal{D}}_i}(x)$. Since M is independent of ε (it only depends on the initial data), this finishes the proof of the theorem. $\square$

Lemma 2.2.71. Let $C, \varepsilon > 0$, $\varepsilon > \varepsilon' \geq 0$ and assume that $|1 - C(1+\varepsilon')| < 2\varepsilon$. Then $|C - 1| < 3\varepsilon$.

Proof. Skipped. $\square$

Remark 2.2.72. From the proof of Theorem 2.2.70 it follows that any functional l satisfying the assumptions, can be approximated (in the pointwise convergence topology) by functionals that are of the form $\overline{\mathcal{D}} \mapsto \overline{\mathcal{B}}^d \cdot f^*\overline{\mathcal{D}}$, where $\overline{\mathcal{B}}$ is an arithmetically ample $\mathbb{Q}$-divisor of C^∞-type on a blowup of $\mathcal{X}$, of degree one.

Note that from the proof of Theorem 2.2.70 it follows that l does not have to be defined on the whole space $\mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$. However, by the M.Riesz extension theorem it does not give a more general statement, see Subsection 2.3.3 and Corollary 2.3.28.

2.3 Applications in globally valued fields

2.3.1 Lattice divisors and globally valued field functionals

Fix a finitely generated field extension $\mathbb{Q} \subset F$ for this subsection.

Definition 2.3.1. We call $\mathcal{X}$ an *F-model*, if it is a generically smooth, normal arithmetic variety together with an isomorphism $\kappa(\mathcal{X}) \simeq F$. The system of F-models together with morphisms respecting isomorphisms of function fields with F is filtered. We use the following notations for direct limits of various groups of divisors with respect to this filtering

$$\mathrm{ADiv}_{\mathbb{K}}(F) := \varinjlim \mathrm{ADiv}_{\mathbb{K}}(\mathcal{X}) \text{ for } \mathbb{K} = \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{R};$$

$$\mathrm{RDiv}_{\mathbb{K}}(F) := \varinjlim \mathrm{RDiv}_{\mathbb{K}}(\mathcal{X}) \text{ for } \mathbb{K} = \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{R};$$

$$\widehat{N}^1(F) := \varinjlim \widehat{N}^1(\mathcal{X}).$$

We write $\mathrm{ADiv}^0_{\mathbb{K}}(\mathcal{X})$ for the group of arithmetic $\mathbb{K}$-divisors of C^0-type such that the underlying divisor is 0. This space can be naturally identified with the set of C^0-type functions on $\mathcal{X}(\mathbb{C})$. We define

$$\mathrm{ADiv}^0_{\mathbb{K}}(F) := \varinjlim \mathrm{ADiv}^0_{\mathbb{K}}(\mathcal{X}) \text{ for } \mathbb{K} = \mathbb{Z}, \mathbb{Q} \text{ or } \mathbb{R}.$$

Note that $\mathrm{ADiv}^0_{\mathbb{K}}(F)$ has a norm induced by the supremum norms on C^0-functions on (analytifications of) F-models. We use notation $\|\cdot\|_{\sup}$ for this norm.

Remark 2.3.2. Note that there is a natural exact sequence

$$0 \longrightarrow \mathrm{RDiv}_{\mathbb{R}}(F) \longrightarrow \mathrm{ADiv}_{\mathbb{R}}(F) \longrightarrow \widehat{N}^1(F) \longrightarrow 0.$$

Next we introduce a lattice structure on the space of arithmetic divisors on F-models. This structure has also been used in [77, 67] using the language of (generalised or b-) Weil functions.

Definition 2.3.3. Let $\overline{\mathcal{D}}, \overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{Q}}(F)$ and assume that $\mathcal{X}$ is an F-model such that $\overline{\mathcal{D}}, \overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{Q}}(\mathcal{X})$. We define the *lattice infimum* $\overline{\mathcal{D}} \wedge \overline{\mathcal{E}}$ of $\overline{\mathcal{D}} = (\mathcal{D}, g)$ and $\overline{\mathcal{E}} = (\mathcal{E}, h)$ in the following way.

1. Assume that $\mathcal{D}, \mathcal{E}$ are effective Cartier divisors on $\mathcal{X}$ and that $\mathcal{D} \cap \mathcal{E}$ is a Cartier divisor. Then we put $\overline{\mathcal{D}} \wedge \overline{\mathcal{E}} := (\mathcal{D} \cap \mathcal{E}, \min(g, h))$ as an element of $\mathrm{ADiv}_{\mathbb{Q}}(F)$.

2. Assume that $\mathcal{D}, \mathcal{E}$ are effective Cartier divisors on $\mathcal{X}$. Let $\mathcal{X}'$ be an F-model over $\mathcal{X}$ such that the intersection of the pullbacks of $\mathcal{D}, \mathcal{E}$ to $\mathcal{X}'$ is a Cartier divisor. Then use the above definition on $\mathcal{X}'$.

3. Assume that $\mathcal{D}, \mathcal{E}$ are Cartier divisors on $\mathcal{X}$. By Lemma 2.2.27 there exists an effective Cartier divisor $\mathcal{A}$ on $\mathcal{X}$ such that $\mathcal{D} + \mathcal{A}, \mathcal{E} + \mathcal{A}$ are effective. Equip $\mathcal{A}$ with an $\mathcal{A}$-Green's function a on $\mathcal{X}$. We can do this by Lemma 2.2.42. Define $\overline{\mathcal{D}} \wedge \overline{\mathcal{E}} := ((\overline{\mathcal{D}} + \overline{\mathcal{A}}) \wedge (\overline{\mathcal{E}} + \overline{\mathcal{A}})) - \overline{\mathcal{A}}$ for $\overline{\mathcal{A}} = (\mathcal{A}, a)$.

4. Assume that $\mathcal{D}, \mathcal{E}$ are $\mathbb{Q}$-Cartier divisors on $\mathcal{X}$. Let n be a natural number such that $n\mathcal{D}, n\mathcal{E}$ are Cartier divisors on $\mathcal{X}$. Put $\overline{\mathcal{D}} \wedge \overline{\mathcal{E}} := \frac{1}{n}(n\overline{\mathcal{D}} \wedge n\overline{\mathcal{E}})$.

We say that $\overline{\mathcal{D}} \wedge \overline{\mathcal{E}}$ *can be calculated on* $\mathcal{X}$, if we do not have to pass to a model $\mathcal{X}'$ above $\mathcal{X}$ to define it. More precisely, this means that there is natural number n and an effective Cartier divisor $\mathcal{A}$ on $\mathcal{X}$ such that $n\overline{\mathcal{D}}, n\overline{\mathcal{E}}$ are arithmetic $\mathbb{Z}$-divisors of C^0-type and the intersection $(n\mathcal{D} + \mathcal{A}) \cap (n\mathcal{E} + \mathcal{A})$ is a Cartier divisor on $\mathcal{X}$.

Moreover, if $\mathcal{D}, \mathcal{E}$ are $\mathbb{Q}$-Cartier divisors on $\mathcal{X}$, we define $\mathcal{D} \wedge \mathcal{E}$ using the same procedure.

Remark 2.3.4. Let $\mathcal{X}, \mathcal{X}'$ be F-models with a morphism $f : \mathcal{X}' \to \mathcal{X}$, and let $\overline{\mathcal{D}}, \overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{Q}}(\mathcal{X})$. If $\overline{\mathcal{D}} \wedge \overline{\mathcal{E}}$ can be calculated on $\mathcal{X}$, then $f^*\overline{\mathcal{D}} \wedge f^*\overline{\mathcal{E}}$ can be calculated on $\mathcal{X}'$. Indeed, it is enough to take a natural number n and a Cartier divisor $\mathcal{A}$ on $\mathcal{X}$ as in Definition 2.3.3, and consider the pullback $f^*\mathcal{A}$ on $\mathcal{X}'$.

Lemma 2.3.5. The lattice structure on $\mathrm{ADiv}_{\mathbb{Q}}(F)$ is well defined and does not depend on the choices in the definition.

Proof. Fix the notation from Definition 2.3.3. We check point by point, that the procedure does not depend on the choices made.

1. Assume that $\mathcal{D}, \mathcal{E}$ are effective Cartier divisors on $\mathcal{X}$ such that $\mathcal{D} \cap \mathcal{E}$ is an effective Cartier divisor. We need to show that $\min(g, h)$ is a $(\mathcal{D} \cap \mathcal{E})$-Green's function on $\mathcal{X}$. Let $\operatorname{Spec} A \subset \mathcal{X}$ be an open subset where $\mathcal{D}, \mathcal{E}, \mathcal{D} \cap \mathcal{E}$ are given respectively by d, e, f and assume that $x, y, d', e' \in A$ are such that $d = fd', e = fe', f = xd + ye$. Note that $f \cdot (1 - (xd' + ye')) = 0$, so $f = 0$ or $xd' + ye' = 1$, as A is an integral domain. If $f = 0$, then $d = e = 0$ and this cannot happen. We must show that the function
$$\min(g(p), h(p)) + \log |f|^2(p)$$
extends to a C^0-type function on $(\operatorname{Spec} A)(\mathbb{C})$. By the assumption we know that functions
$$g(p) + \log |d|^2(p) \text{ and } h(p) + \log |e|^2(p)$$
extend to a C^0-type functions on $(\operatorname{Spec} A)(\mathbb{C})$. Note that
$$g(p) + \log |d|^2(p) = g(p) + \log |f|^2(p) + \log |d'|^2(p),$$
$$h(p) + \log |e|^2(p) = h(p) + \log |f|^2(p) + \log |e'|^2(p).$$
Thus it is enough to show that for every $p \in (\operatorname{Spec} A)(\mathbb{C})$ we have $|d'|(p) \neq 0$ or $|e'|(p) \neq 0$. However, $xd' + ye' = 1$, so if $d'(p) = e'(p) = 0$, then $0 = x(p)d'(p) + y(p)e'(p) = 1$ which gives a contradiction.

2. Assume that $\mathcal{D}, \mathcal{E}$ are effective Cartier divisors on $\mathcal{X}$ and let $\mathcal{X}', \mathcal{X}''$ be F-models over $\mathcal{X}$ where intersections of pullbacks of $\mathcal{D}, \mathcal{E}$ are Cartier divisors. Without loss of generality $\mathcal{X}''$ is over $\mathcal{X}'$. Consider open sets $\operatorname{Spec} C \to \operatorname{Spec} B \to \operatorname{Spec} A$ inside $\mathcal{X}'' \to \mathcal{X}' \to \mathcal{X}$ respectively, and assume that $\mathcal{D}, \mathcal{E}$ have local equations $d, e \in A$ on $\operatorname{Spec}(A)$. Assume also, that on $\operatorname{Spec} B$ we have $(d, e) = (f)$ for some $f \in B$. Let $x, y, d', e' \in B$ be elements such that $d = fd', e = fe', f = xd + ye$. The same equations hold in C (for pullbacks), so (d, e) calculated in $\operatorname{Spec} B$ and pulled back to $\operatorname{Spec} C$ is the same as (d, e) calculated in $\operatorname{Spec} C$. Note that to define the Green's function for the intersection of pullbacks of $\mathcal{D}, \mathcal{E}$ on $\mathcal{X}''$ one can take the minimum of pullbacks of g, h to $\mathcal{X}''$, or the pullback of the minimum of pullbacks of g, h on $\mathcal{X}'$.

3. Assume that $\mathcal{D}, \mathcal{E}$ are Cartier divisors on $\mathcal{X}$ and let $\mathcal{A}, \mathcal{B}$ be effective Cartier divisors on $\mathcal{X}$ with $\mathcal{D}+\mathcal{A}, \mathcal{E}+\mathcal{A}$ effective. Replace $\mathcal{X}$ by an F-model $\mathcal{X}'$ over $\mathcal{X}$ where $(\mathcal{D}+\mathcal{A}) \cap (\mathcal{E}+\mathcal{A})$ is an effective Cartier divisor. To prove independence of $\overline{\mathcal{A}}$ in the definition of the lattice infimum (point 3.) it is enough to show that for any effective Cartier divisor $\mathcal{B}$ on $\mathcal{X}'$ we have
$$((\mathcal{D}+\mathcal{A}) \cap (\mathcal{E}+\mathcal{A})) - \mathcal{A} = ((\mathcal{D}+\mathcal{A}+\mathcal{B}) \cap (\mathcal{E}+\mathcal{A}+\mathcal{B})) - (\mathcal{A}+\mathcal{B}).$$
To ease the notation, replace $\mathcal{D}+\mathcal{A}, \mathcal{E}+\mathcal{A}$ by $\mathcal{D}, \mathcal{E}$ respectively. We want to prove that $\mathcal{D} \cap \mathcal{E} = ((\mathcal{D}+\mathcal{B}) \cap (\mathcal{E}+\mathcal{B})) - \mathcal{B}$ assuming that $\mathcal{D}, \mathcal{E}, \mathcal{D} \cap \mathcal{E}, \mathcal{B}$ are effective Cartier divisors on $\mathcal{X}$. Let $\operatorname{Spec} A \subset \mathcal{X}$ be an open subset where the effective Cartier divisors from the previous sentence are given respectively by $d, e, f, b \in A$. Also, assume that $\operatorname{Spec} A$ is chosen so that there are $x, y, d', e' \in A$ such that $d = fd', e = fe', f = xd + ye$. Note that then $bd = bfd', be = bfe', bf = xbd + ybe$, hence $(bf) = (be, bd)$, which gives the desired result. The fact that the $(\mathcal{D} \cap \mathcal{E})$-Green's function on $\mathcal{X}$ does not depend on $\overline{\mathcal{A}}$ follows from the fact that for real-valued functions g, h, j we have $\min(g+j, h+j) = \min(g, h) + j$.

4. Finally, assume that $\mathcal{D}, \mathcal{E}$ are $\mathbb{Q}$-Cartier and let $n \neq 0$ be a natural number such that $n\mathcal{D}, n\mathcal{E}$ are Cartier divisors on $\mathcal{X}$. Fix a natural number $m \neq 0$. We need to show that using the point 3. of Definition 2.3.3 we get $(n\mathcal{D}) \wedge (n\mathcal{E}) = \frac{1}{m}((mn\mathcal{D}) \wedge (mn\mathcal{E}))$. Replace $\mathcal{D}, \mathcal{E}$ by $n\mathcal{D}, n\mathcal{E}$ respectively, to assume that $n = 1$. By the previous points in this lemma, it is enough to prove that if $\mathcal{D}, \mathcal{E}$ are effective Cartier divisors on $\mathcal{X}$ with $\mathcal{D} \cap \mathcal{E}$ effective Cartier, then $(m\mathcal{D}) \cap (m\mathcal{E}) = m(\mathcal{D} \cap \mathcal{E})$. Let $\operatorname{Spec} A \subset \mathcal{X}$ be an open subset where $\mathcal{D}, \mathcal{E}, \mathcal{D} \cap \mathcal{E}$ are given respectively by d, e, f and assume that $x, y, d', e' \in A$ are such that $d = fd', e = fe', 1 = xd' + ye'$ (see point 1. in this lemma). Note that
$$1 = (xd' + ye')^{2m} = \sum_{i=0}^{2m} \binom{2m}{i} x^i y^{2m-i} (d')^i (e')^{2m-i} \in ((d')^m, (e')^m).$$
Thus, from the fact that $d^m = f^m (d')^m, e^m = f^m (e')^m$, we get that $(d^m, e^m) = (f^m)$, which finishes the proof. The fact that the $(\mathcal{D} \cap \mathcal{E})$-Green's function on $\mathcal{X}$ does not depend on m follows from the fact that $\min(mg, mh) = m \min(g, h)$.

□

Recall that in Definition 2.2.23 we have introduced a pairing $\beta(v, \overline{\mathcal{D}})$ between valuations v on F and arithmetic $\mathbb{R}$-divisors $\overline{\mathcal{D}} \in \operatorname{ADiv}_{\mathbb{R}}(\mathcal{X})$ on an F-model $\mathcal{X}$.

Lemma 2.3.6. Let v be a valuation on F. If $\overline{\mathcal{D}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$, then the value $\beta(v, \overline{\mathcal{D}})$ does not depend on the F-model on which $\overline{\mathcal{D}}$ is defined. Moreover, the function $\beta(v, -)$ has the following properties:

1. it is $\mathbb{R}$-linear,
2. for every $\overline{\mathcal{D}}, \overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{Q}}(F)$ we have $\beta(v, \overline{\mathcal{D}} \wedge \overline{\mathcal{E}}) = \min(\beta(v, \overline{\mathcal{D}}), \beta(v, \overline{\mathcal{E}}))$.

Proof. The fact that $\beta(v, \overline{\mathcal{D}})$ does not depend on the F-model follows from the definition of pullback, see Remark 2.2.21. Linearity is clear. To prove the second point, by Lemma 2.3.5 and $\mathbb{R}$-linearity we can reduce to the following situation. Assume that $\overline{\mathcal{D}}, \overline{\mathcal{E}}$ are arithmetic $\mathbb{Z}$-divisors on $\mathcal{X}$ with $\mathcal{D}, \mathcal{E}, \mathcal{D} \cap \mathcal{E}$ effective Cartier. If v is Archimedean, the statement follows from the definition, so assume that v is non-Archimedean. Let $p = \mathrm{supp}(v) \in \mathcal{X}$ and let $\operatorname{Spec} A \subset \mathcal{X}$ be an open neighbourhood of p where $\mathcal{D}, \mathcal{E}, \mathcal{D} \cap \mathcal{E}$ are given by $d, e, f \in A$ respectively. Assume additionally that $x, y, d', e' \in A$ are such that $d = fd', e = fe', 1 = xd' + ye'$ (see point 1. in Lemma 2.3.5). We need to show that $v(f) = \min(v(d), v(e))$. Note that

$$0 = v(1) = v(xd' + ye') \geq \min(v(x) + v(d'), v(y) + v(e')) \geq \min(v(d'), v(e'))$$

and

$$v(d) = v(f) + v(d'), v(e) = v(f) + v(e').$$

Note that if $v(d'), v(e') > 0$, then $0 \geq \min(v(d'), v(e')) > 0$, which gives a contradiction. Thus one of $v(d'), v(e')$ has to be zero, and we get the result. □

Lemma 2.3.7. Let $\mathcal{X}$ be an F-model with a closed point in the generic fiber $x \in X = \mathcal{X}_{\mathbb{Q}}$. Let $\overline{\mathcal{D}}, \overline{\mathcal{E}}, \overline{\mathcal{A}}$ be arithmetic $\mathbb{Z}$-divisors of C^0-type on $\mathcal{X}$ such that $\mathcal{A}, \mathcal{A} + \mathcal{D}, \mathcal{A} + \mathcal{E}$ are effective and $(\mathcal{A} + \mathcal{D}) \cap (\mathcal{A} + \mathcal{E})$ is principal, so that $\overline{\mathcal{D}} \wedge \overline{\mathcal{E}}$ can be calculated on $\mathcal{X}$. Assume that x does not belong to the support of any of $\mathcal{D}, \mathcal{E}, \mathcal{A}, \mathcal{A} + \mathcal{D}, \mathcal{A} + \mathcal{E}$ and let v be a valuation on $\kappa(x)$. Then

$$\widehat{\deg}_v(\overline{\mathcal{D}} \wedge \overline{\mathcal{E}} | \overline{\{x\}}) = \min(\widehat{\deg}_v(\overline{\mathcal{D}} | \overline{\{x\}}), \widehat{\deg}_v(\overline{\mathcal{E}} | \overline{\{x\}})).$$

Proof. We skip this proof. It is almost the same as the proof of Lemma 2.3.6. □

Definition 2.3.8. We denote by $\mathrm{LDiv}_{\mathbb{Q}}(F)$ the $\mathbb{Q}$-vector space generated by $\mathbb{Q}$-vector and lattice operations from principal arithmetic $\mathbb{Z}$-divisors $\widehat{\mathrm{div}}(f)$, for $f \in F$. We call these *lattice divisors* on F. For an F-model $\mathcal{X}$ we denote by $\mathrm{LDiv}_{\mathbb{Q}}(\mathcal{X})$ the inverse image of lattice divisors on F via the natural map $\mathrm{ADiv}_{\mathbb{Q}}(\mathcal{X}) \to \mathrm{ADiv}_{\mathbb{Q}}(F)$. We denote by $\mathrm{LDiv}^0_{\mathbb{Q}}(F), \mathrm{LDiv}^0_{\mathbb{Q}}(\mathcal{X})$ the subspaces where the corresponding divisor is 0.

Remark 2.3.9. In the above definition we use the same name and notation as in Definition 1.5.1. This is because both (divisible) group lattices are canonically isomorphic. Indeed, from the universal property, there is a surjective group lattice morphism from $\mathrm{ULat}_{\mathbb{Q}}(F)$ to the Arakelov lattice divisors $\mathrm{LDiv}_{\mathbb{Q}}(F)$ from Definition 2.3.8. Furthermore, by Remark 2.2.35 together with Corollary 1.5.17, the kernel is the same is in Definition 1.5.1.

Definition 2.3.10. Let l be a linear functional on a (real or rational) vector subspace of $\mathrm{ADiv}_{\mathbb{R}}(F)$ (resp. $\mathbb{Q}$ or $\mathbb{R}$-linear). We call l a *globally valued field functional* (abbreviated GVF functional), if it satisfies the following properties:

- it maps principal $\mathbb{R}$-divisors of C^0-type in its domain to 0,
- it is non-negative on effective arithmetic $\mathbb{R}$-divisors of C^0-type in its domain.

Moreover, we call l *normalized*, if its domain contains a pullback of a non-principal arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{D}}$ on $\mathrm{Spec}(\mathbb{Z})$ and l sends it to $\widehat{\deg}(\overline{\mathcal{D}})$. Here we mean the arithmetic degree from Theorem 2.2.44 or equivalently the arithmetic degree from Definition 2.2.30, for the arithmetic variety $\mathrm{Spec}(\mathbb{Z})$.

Note that this is as in Definition 1.6.3, however here the domain is more general.

Proposition 2.3.11. Let $\mathcal{X}$ be an F-model. Then $\mathrm{LDiv}^0_{\mathbb{Q}}(\mathcal{X})$ is a dense subset of $\mathrm{ADiv}^0_{\mathbb{R}}(\mathcal{X})$ with respect to the supremum norm.

Proof. Identify $\mathrm{ADiv}^0_{\mathbb{R}}(\mathcal{X})$ with continuous real-valued functions on $\mathcal{X}(\mathbb{C})/F_\infty$. By the Stone–Weierstrass theorem (max-closed), it is enough to check the following:

- a constant non-zero function is in $\mathrm{LDiv}^0_{\mathbb{Q}}(\mathcal{X})$;
- if $g \in \mathrm{LDiv}^0_{\mathbb{Q}}(\mathcal{X})$, then $ag \in \mathrm{LDiv}^0_{\mathbb{Q}}(\mathcal{X})$ for $a \in \mathbb{Q}$;
- if $g, h \in \mathrm{LDiv}^0_{\mathbb{Q}}(\mathcal{X})$, then $g + h, \max(g, h) \in \mathrm{LDiv}^0_{\mathbb{Q}}(\mathcal{X})$;
- the algebra $\mathrm{LDiv}^0_{\mathbb{Q}}(\mathcal{X})$ separates points of $\mathcal{X}(\mathbb{C})/F_\infty$.

For a constant non-zero function we can take $\widehat{\mathrm{div}}(2) \wedge \widehat{\mathrm{div}}(1)$. The second and third points are automatic. Now we prove the last point. Fix an embedding $\mathcal{X} \subset \mathbb{P}^n_{\mathbb{Z}}$ and assume that $\mathcal{X}$ is not contained in any hyperplane. Pick

$$x, y \in \mathcal{X}(\mathbb{C}) \subset \mathbb{P}^n_{\mathbb{Z}}(\mathbb{C}) = \{[z_0 : \cdots : z_n] | (\exists i = 0, \ldots, n)(z_i \neq 0)\}$$

which have different conjugation classes. Assume that $x_0 = y_0 = 1$ (possibly after a change of coordinates of $\mathbb{P}^n_{\mathbb{Z}}$). Let i, i' be indexes among $1, \ldots, n$ such that $x_i \neq y_i$ and $\overline{x_{i'}} \neq y_{i'}$. Note that we can without loss of generality assume that $i' = i$. Indeed, if $\overline{x_i} = y_i$ and $x_{i'} = y_{i'}$, then we can change the coordinates of $\mathbb{P}^n_{\mathbb{Z}}$ so that the new i'th coordinate is a sum of the i'th and i''th coordinate, and other coordinates stay the same. The new i'th coordinates of x and y become $x_i + x_{i'}, \overline{x_i} + x_{i'}$ respectively, and they have different conjugation classes as $x_i, x_{i'} \in \mathbb{C} \setminus \mathbb{R}$.

Let $X_0, \ldots, X_n$ be the natural projective coordinates on $\mathbb{P}^n_{\mathbb{Z}}$ and denote by f_j the restriction of $\frac{X_j}{X_0}$ to $\mathcal{X}$, for each $j = 0, \ldots, n$. This makes sense as $\mathcal{X}$ is not contained in any hyperplane. Let $\overline{\mathcal{E}}$ be the lattice divisor on F defined by the formula

$$\overline{\mathcal{E}} := \widehat{\operatorname{div}}(1) \wedge \widehat{\operatorname{div}}(f_1) \wedge \cdots \wedge \widehat{\operatorname{div}}(f_{i-1}) \wedge \widehat{\operatorname{div}}(f_{i+1}) \wedge \cdots \wedge \widehat{\operatorname{div}}(f_n).$$

Pick natural numbers a, b such that the following hold:

1. $\left|\frac{x_i+a}{x_i+b}\right| \neq \left|\frac{y_i+a}{y_i+b}\right|$;
2. for $c = a$ or $c = b$ we have

$$|x_i + c| > 1, |x_1|, \ldots, |x_n|;$$

$$|y_i + c| > 1, |y_1|, \ldots, |y_n|.$$

Such exists by Lemma 2.3.13 with N big enough. Consider the following lattice divisor on F

$$\overline{\mathcal{D}} := -(\overline{\mathcal{E}} \wedge \widehat{\operatorname{div}}(f_i + a)) + (\overline{\mathcal{E}} \wedge \widehat{\operatorname{div}}(f_i + b)).$$

Let $\mathcal{X}'$ be an F-model over $\mathcal{X}$ with $\overline{\mathcal{D}} \in \operatorname{LDiv}_{\mathbb{Q}}(\mathcal{X}')$. Let v be a non-Archimedean valuation on F.

Claim 2.3.12. The equality $\beta(v, \overline{\mathcal{D}}) = 0$ holds.

Proof of the claim. By Lemma 2.3.6 it is enough to show that

$$\min(\{v(f_j) : j \neq i\} \cup \{v(f_i + a)\}) = \min(\{v(f_j) : j \neq i\} \cup \{v(f_i + b)\}),$$

where j above varies form 0 to n with $f_0 = 1$. This follows from the ultrametric inequality for non-Archimedean valuations. □

In particular, the Weil decomposition of $\mathcal{D}$ is trivial, hence $\mathcal{D} = 0$ as $\mathcal{X}'$ is normal. Thus $\overline{\mathcal{D}} = (0, g)$ for some C^0-type function g on $\mathcal{X}'(\mathbb{C})$. Consider the function H defined by the formula

$$-\min(\{-\log|z_j| : j \neq i\} \cup \{-\log|z_i + az_0|\}) + \min(\{-\log|z_j| : j \neq i\} \cup \{-\log|z_i + bz_0|\}),$$

for $[z_0 : \cdots : z_n] \in \mathbb{P}^n_{\mathbb{Z}}(\mathbb{C})$. Note that it is a C^0-type function on $\mathbb{P}^n_{\mathbb{Z}}(\mathbb{C})$ (it does not depend on the representative of a point) and it restricts to a C^0-type function $h := H|_{\mathcal{X}(\mathbb{C})}$ on $\mathcal{X}(\mathbb{C})$. By Lemma 2.3.6 and Definition 2.2.23, for points $p \in \mathcal{X}'(\mathbb{C})$ factoring through the inclusion $\mathrm{Spec}(F) \to \mathcal{X}'$ (note that those are the same for $\mathcal{X}$ and $\mathcal{X}'$), we have

$$\begin{aligned} g(p) &= -\min(\{-\log|f_j|(p) : j \neq i\} \cup \{-\log|f_i + a|(p)\}) \\ &\quad + \min(\{-\log|f_j|(p) : j \neq i\} \cup \{-\log|f_i + b|(p)\}) \\ &= -\min(\{-\log|z_j|(p) : j \neq i\} \cup \{-\log|z_i + az_0|(p)\}) \\ &\quad + \min(\{-\log|z_j|(p) : j \neq i\} \cup \{-\log|z_i + bz_0|(p)\}) \\ &= H(p) = h(p), \end{aligned}$$

where the second inequality holds as $f_i = \frac{X_i}{X_0}|_{\mathcal{X}}$ and for p as above $z_0(p) \neq 0$. Since g, h are continuous, g must be a pullback of h to $\mathcal{X}'(\mathbb{C})$, as those agree on a dense subset. Thus $(0, h) \in \mathrm{LDiv}_{\mathbb{Q}}(\mathcal{X})$. Moreover, by the choice of a, b we get that

$$h(x) = \log\left|\frac{x_i + a}{x_i + b}\right| \neq \log\left|\frac{y_i + a}{y_i + b}\right| = h(y).$$

This finishes the proof. ☐

Lemma 2.3.13. Let $x \neq y \in \mathbb{C}$ be such that $\mathrm{Im}(x), \mathrm{Im}(y) \geq 0$. Let N be a natural number. Then there are natural numbers $a, b > N$ such that

$$\left|\frac{x + a}{x + b}\right| \neq \left|\frac{y + a}{y + b}\right|$$

Proof. Skipped. ☐

Lemma 2.3.14. Every GVF functional

$$\mathrm{ADiv}_{\mathbb{Q}}(F) \to \mathbb{R}$$

extends uniquely to a GVF functional

$$\mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}.$$

Proof. By Lemma 2.2.28 we get the uniqueness. For existence, use Proposition 2.2.41 to get surjectivity of the natural map $\mathrm{ADiv}_{\mathbb{Q}}(F) \otimes \mathbb{R} \to \mathrm{ADiv}_{\mathbb{R}}(F)$ and for the fact that the real span of the image of the effective cone of $\mathrm{ADiv}_{\mathbb{Q}}(F)$ spans the effective cone in $\mathrm{ADiv}_{\mathbb{R}}(F)$. ☐

Lemma 2.3.15. Let $\mathcal{D}$ be a $\mathbb{Q}$-Cartier divisor on an F-model $\mathcal{X}$. Then there exists an F-model $\mathcal{X}'$ over $\mathcal{X}$ and an a $\mathcal{D}$-Green's function g on $\mathcal{X}'$ such that $(\mathcal{D}, g) \in \mathrm{LDiv}_{\mathbb{Q}}(\mathcal{X}')$.

Proof. By presenting $\mathcal{D}$ as a rational number times a difference of two very ample Cartier divisors, we can reduce to the case where $\mathcal{D}$ is very ample. In particular it is globally generated. Pick a basis $f_1, \dots, f_n \in H^0(\mathcal{X}, \mathcal{D})$ of rational functions with poles at most given by $\mathcal{D}$.

Claim 2.3.16. $\mathcal{D} + \bigwedge_{i=1}^{n} \mathrm{div}(f_i) = 0$.

Proof of the claim. Recall that for a Cartier divisor $\mathcal{D}$ by rational functions with poles at most given by $\mathcal{D}$ we mean

$$H^0(\mathcal{X}, \mathcal{D}) = \{0\} \cup \{f \in \kappa(\mathcal{X})^\times \mid \mathcal{D} + \mathrm{div}(f) \geq 0\}.$$

Note that $\{\mathcal{D} + \mathrm{div}(f) \mid f \in H^0(\mathcal{X}, \mathcal{D})\}$ is the set of effective Cartier divisors linearly equivalent to $\mathcal{D}$. The fact that $\mathcal{O}(\mathcal{D})$ is globally generated implies that for every $p \in \mathcal{X}$ there is $\mathcal{E} \in \{\mathcal{D} + \mathrm{div}(f_i) \mid i = 1, \dots, n\}$ such that $p \notin \mathcal{E}$. Thus the ideal defining $\mathcal{E}$ is (1) at p. Since p was arbitrary, we get that the sum of ideals of effective Cartier divisors $\mathcal{E}_i = \mathcal{D} + \mathrm{div}(f_i)$ is (1). By the definition this means that (possibly passing to a blowup $\mathcal{X}'$ of $\mathcal{X}$)

$$0 = \bigwedge_{i=1,\dots,n} (\mathcal{D} + \mathrm{div}(f_i)) = \mathcal{D} + \bigwedge_{i=1,\dots,n} \mathrm{div}(f_i),$$

which finishes the proof of the claim. ☐

Thus, we can put $\overline{\mathcal{D}} := -\bigwedge_{i=1}^{n} \widehat{\mathrm{div}}(f_i)$ and this gives a desired $\mathcal{D}$-Green's function g on $\mathcal{X}'$. ☐

Proposition 2.3.17. Every GVF functional

$$\mathrm{LDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$$

extends uniquely to a GVF functional

$$\mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}.$$

Proof. Let $l : \mathrm{LDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$ be a GVF functional. By Lemma 2.3.14 it is enough to show that l has a unique extension to $\mathrm{ADiv}_{\mathbb{Q}}(F)$. Pick $\overline{\mathcal{D}} \in \mathrm{ADiv}_{\mathbb{Q}}(F)$ and let $\mathcal{X}$ be a generically smooth, normal, arithmetic variety with $\kappa(\mathcal{X}) \simeq F$ such that $\overline{\mathcal{D}} \in \mathrm{ADiv}_{\mathbb{Q}}(\mathcal{X})$. Let $\overline{\mathcal{D}} = (\mathcal{D}, g)$ on $\mathcal{X}$. By Lemma 2.3.15 (possibly replacing $\mathcal{X}$ by an F-model over it) there is a $\mathcal{D}$-Green's function g' on $\mathcal{X}$ with $(\mathcal{D}, g') \in \mathrm{LDiv}_{\mathbb{Q}}(\mathcal{X})$. The value of l on $(\mathcal{D}, g')$ is determined, so it is enough to define (and show uniqueness of) the value of the extension at $(0, g - g') \in \mathrm{ADiv}_{\mathbb{Q}}(\mathcal{X})$. Let h be the continuous extension of $g - g'$ to $\mathcal{X}(\mathbb{C})$. By Proposition 2.3.11 there is a pair of Green's functions for the 0 divisor g_1, g_2 on $\mathcal{X}$ such that:

- $g_1 \leq h \leq g_2$,
- $0 \leq g_2 - g_1 < \varepsilon$,
- $(0, g_1), (0, g_2) \in \mathrm{LDiv}_{\mathbb{Q}}(\mathcal{X})$.

To find these Green's functions, take g_1 approximating $h - \frac{1}{4}\varepsilon$ up to $\frac{1}{4}\varepsilon$ and similarly with g_2. Hence, any GVF extension of l assigns a value between $l((0, g_1))$ and $l((0, g_2))$ to $(0, h)$. Moreover,

$$|l((0, g_2)) - l((0, g_1))| = |l((0, g_2 - g_1))| < \varepsilon |l((0, 1))|.$$

Take a sequence $\varepsilon_n \to 0$ and for each ε_n pick $(g_1)_n, (g_2)_n$ as above, such that $(g_1)_n$ is increasing and $(g_2)_n$ is decreasing (one can do it since $\mathrm{LDiv}_{\mathbb{Q}}(F)$ is closed under maximum and minimum). For any GVF extension of l, we must have $l((0, h)) = \lim_n l((0, (g_i)_n)$ for $i = 1, 2$.

This proves that if an extension exists, it has to be given by the above construction. To prove existence one can either show that the above definition gives a GVF functional, or use the Riesz extension theorem (i.e., Theorem 2.3.26) to get one. □

Corollary 2.3.18. There is a natural bijection:

$$\big\{\text{GVF functionals } \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}\big\} \longleftrightarrow \big\{\text{GVF structures on } F\big\}$$

$$l \longmapsto \big(R^l_t(a) := l(\overline{\mathcal{D}}_t(a))\big)_t$$

where $\overline{\mathcal{D}}_t(a) = t(\widehat{\mathrm{div}}(a))$ if a is a tuple from $F^\times$ and t is a $\mathbb{Q}$-tropical polynomial.

Proof. As indicated in Section 1.8, this follows from Theorem 1.7.7 and Proposition 2.3.17 (see also Remark 2.3.9). □

Remark 2.3.19. One can generalise the above corollary to an arbitrary characteristic zero field F. Indeed, recall that we call a generically smooth, normal arithmetic variety $\mathcal{X}$ an *F-submodel*, if it is equipped with a fields embedding $\kappa(\mathcal{X}) \to F$ (as in Section 1.2). We equip F-submodels with a category structure where a morphism $\mathcal{X} \to \mathcal{Y}$ is a dominant map of arithmetic varieties that respects the field embeddings into F. Then we put

$$\mathrm{ADiv}_{\mathbb{R}}(F) = \varinjlim \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}),$$

where the limit is taken over the category of F-submodels with pullback maps on arithmetic $\mathbb{R}$-divisors of C^0-type. A linear functional $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ is called a GVF functional, if the natural restriction to $\mathrm{ADiv}_{\mathbb{R}}(\mathcal{X})$ is a GVF functional for all F-submodels $\mathcal{X}$. In this case $\mathrm{ADiv}_{\mathbb{R}}(F)$ also has a lattice structure. Moreover, there is a natural bijection

$$\big\{\text{GVF functionals } \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}\big\} \longleftrightarrow \big\{\text{GVF structures on } F\big\}$$

$$l \longmapsto \big(R^l_t(\overline{a}) := l(\overline{\mathcal{D}}_t(\overline{a}))\big)_{\mathbb{Q}\text{-tropical polynomials } t}$$

where $\overline{\mathcal{D}}_t(\overline{a}) = t(\widehat{\mathrm{div}}(a_1), \ldots, \widehat{\mathrm{div}}(a_n))$ if $\overline{a} = (a_1, \ldots, a_n)$.

2.3.2 Existential closedness of the algebraic numbers

In this section we prove the existential closedness of the algebraic numbers (with the standard GVF structure). Let us start with setting some notation.

Example 2.3.20 (Number fields)**.** Let K be a number field with the integer ring $\mathcal{O}_K$. Then, for $f \in K^*$ we have the following product formula

$$\sum_{p \in \mathrm{Spec}(\mathcal{O}_K)} \mathrm{ord}_p(f) \cdot \log \#\kappa(p) + \sum_{\sigma : K \to \mathbb{C}} -\log|\sigma(f)| = 0,$$

where in the first sum we only take closed points and the second sum is taken over embeddings of fields. This yields a GVF structure on K induced by the measure

$$\mu := \frac{1}{[K:\mathbb{Q}]}\Bigg(\sum_{p \in \mathrm{Spec}(\mathcal{O}_K)} \delta_{\mathrm{ord}_p} \cdot \log \#\kappa(p) + \sum_{\sigma : K \to \mathbb{C}} \delta_{-\log|\sigma(-)|}\Bigg).$$

This GVF structure is unique up to multiplication by a positive scalar by Lemma 1.10.2. We denote it by $K[1]$. It extends the GVF structure on $\mathbb{Q}$ given by the same formula. The composition of all finite extensions of $\mathbb{Q}$ yields the standard GVF structure on $\overline{\mathbb{Q}}$ denoted $\overline{\mathbb{Q}}[1]$. Note that this GVF structure is Galois-invariant, i.e., if $\overline{a}, \overline{a}'$ lie in the same Galois orbit, then $R_t(\overline{a}) = R_t(\overline{a}')$ for any $\mathbb{Q}$-tropical polynomial t of the corresponding arity.

In Definition 1.12.1 we defined existential closedness for a general (discrete, unbounded) continuous logic structure. In the context of globally valued fields (with the local terms presentation as in Definition 1.7.1), this specialises to the following.

Definition 2.3.21. Let F be a GVF. It is existentially closed, if for any GVF extension $F \subset K$ and for any finite tuple $\overline{a}$ in K the following condition holds: if $f_1, \dots, f_m \in F(\overline{x})$ and $g_1, \dots, g_n, h \in F[\overline{x}]$ satisfy

$$\begin{aligned} & R_{t_i}(f_i(\overline{a})) = r_i \text{ for } i = 1, \dots, m; \\ & g_1(\overline{a}) = \dots = g_n(\overline{a}) = 0; \\ & h(\overline{a}) \neq 0; \end{aligned} \tag{2.3}$$

for some $\mathbb{Q}$-tropical polynomials $t_1, \dots, t_m$, then for any $\varepsilon > 0$ there is a tuple $\overline{a}'$ in F such that

$$\begin{aligned} & |R_{t_i}(f_i(\overline{a}')) - r_i| < \varepsilon \text{ for } i = 1, \dots, m; \\ & g_1(\overline{a}') = \dots = g_n(\overline{a}') = 0; \\ & h(\overline{a}') \neq 0. \end{aligned} \tag{2.4}$$

Note that by Proposition 1.10.5 (just by the existence part) an existentially closed GVF must be algebraically closed.

Definition 2.3.22. Let F be a finitely generated extension of $\mathbb{Q}$ and fix an F-model $\mathcal{X}$. Let $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m$ be arithmetic $\mathbb{Q}$-divisors of C^0-type on $\mathcal{X}$, and t be a $\mathbb{Q}$-tropical polynomial. We say that *the inductive definition of* $t(\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m)$ *can be calculated on* $\mathcal{X}$, in the following cases:

- when $t(x_1, \dots, x_n) = x_i$, we always say so;
- when $t = \alpha \cdot t'$, we say so if the inductive definition of $t'(\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m)$ can be calculated on $\mathcal{X}$;
- when $t = t_1 + t_2$, we say so if the inductive definitions of

  $$t_1(\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m), t_2(\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m)$$

  can be calculated on $\mathcal{X}$;
- when $t = t_1 \wedge t_2$, we say so if the inductive definitions of

  $$\overline{\mathcal{D}} := t_1(\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m), \overline{\mathcal{E}} := t_2(\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m)$$

  can be calculated on $\mathcal{X}$, and $\overline{\mathcal{D}} \wedge \overline{\mathcal{E}}$ can be calculated on $\mathcal{X}$.

Lemma 2.3.23. Let $F=\mathbb{Q}(\overline{a})$ be a finitely generated extension of $\mathbb{Q}$ and let $\mathcal{X}$ be an F-model. Let $\overline{\mathcal{D}}=t(\widehat{\operatorname{div}}(f_1(\overline{a})),\ldots,\widehat{\operatorname{div}}(f_n(\overline{a})))$ for some $\mathbb{Q}$-tropical polynomial t and some rational functions $f_1,\ldots,f_n\in\mathbb{Q}(\overline{y})$ (where $\overline{y}$ are some independent variables). Assume that the inductive definition of $\overline{\mathcal{D}}$ can be calculated on $\mathcal{X}$. Let $x\in\mathcal{X}_{\mathbb{Q}}$ be a sufficiently generic closed point. Then

$$h_{\overline{\mathcal{D}}}(x)=R_t(f_1(\overline{a})|_x,\ldots,f_n(\overline{a})|_x),$$

where on the right hand side, $f_i(\overline{a})|_x$ is the evaluation of the rational function $f_i(\overline{a})$ at x for every $i=1,\ldots,n$, and R_t is calculated with respect to the GVF structure $\kappa(x)[1]$.

Proof. Let μ be the standard GVF measure on $\operatorname{Val}_{\kappa(x)}$ as described in Example 2.3.20 and let $\overline{f}=(f_1,\ldots,f_n)$. We calculate

$$R_t(\overline{f}(\overline{a})|_x)=\int t(v(\overline{f}(\overline{a})|_x))dv=\int_{\operatorname{Val}_{\kappa(x)}}t(v(\overline{f}(\overline{a})|_x))d\mu(v).$$

By Lemma 2.3.7 and induction, for a sufficiently generic x we get the equality with

$$\int_{\operatorname{Val}_{\kappa(x)}}\widehat{\deg}_v(\overline{\mathcal{D}}|\overline{\{x\}})d\mu(v)=h_{\overline{\mathcal{D}}}(x),$$

where the last equality holds by Lemma 2.2.31. □

Theorem 2.3.24. $\overline{\mathbb{Q}}[1]$ is an existentially closed GVF.

Proof. We start with the following reduction.

Claim 2.3.25. To prove existential closedness of $\overline{\mathbb{Q}}[1]$ it suffices to prove the following statement:

(*) For every finitely generated GVF extension $\mathbb{Q}[1]\subset K$ generated by a tuple $\overline{a}$, if $\overline{a}$ satisfies Equations (2.3) in Definition 2.3.21 (with $F=\mathbb{Q}$), then for any $\varepsilon>0$ there exists a tuple $\overline{a}'$ in $\overline{\mathbb{Q}}[1]$ satisfying Equations (2.4).

Proof of the claim. Let $\overline{\mathbb{Q}}[1]\subset K$ be a GVF extension and let $\overline{a}$ be a tuple in K. Assume that it satisfies Equations (2.3) from Definition 2.3.21 (with $F=\overline{\mathbb{Q}}$). Let b be an element in $\overline{\mathbb{Q}}$ such that f_i,g_j,h are defined using parameters from $\mathbb{Q}(b)$ for all i,j as in Definition 2.3.21. Fix $\varepsilon>0$. Use the condition (*) for the finitely generated extension $\mathbb{Q}[1]\subset\mathbb{Q}(b,\overline{a})$ to get a tuple $(b',\overline{a}')$ satisfying Equations (2.4) (but with b' replacing b in all functions f_i,g_j,h). Moreover, by adding some equations over $\mathbb{Q}$, we can assume that the minimal polynomial of b' is the same as that of b. Let σ be an automorphism of $\overline{\mathbb{Q}}$ taking b' to b. Since the GVF structure $\overline{\mathbb{Q}}[1]$ is Galois-invariant (see Example 2.3.20) the tuple $\sigma(\overline{a}')$ satisfies Equations (2.4), which finishes the proof of the claim. $\square$

We prove (*) for a finitely generated GVF extension $\mathbb{Q}[1] \subset F = \mathbb{Q}(\overline{a})$. By uniqueness of finite GVF extensions of $\mathbb{Q}[1]$, we can assume that F is not algebraic over $\mathbb{Q}$. By Corollary 2.3.18 a GVF structure on F corresponds to a GVF functional $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$. Continuous logic predicates in Equations (2.3) correspond to some finite family $\overline{\mathcal{D}}_1, \dots, \overline{\mathcal{D}}_m \in \mathrm{LDiv}_{\mathbb{Q}}(\mathcal{X})$ on some F-model $\mathcal{X}$ with $l(\overline{\mathcal{D}}_i) = R_{t_i}(\overline{f}_i(\overline{a}))$. By possibly passing to a different F-model, we can assume that the inductive definitions of these lattice arithmetic $\mathbb{Q}$-divisors of C^0-type can be calculated on $\mathcal{X}$. We can do this by Remark 2.3.4. Fix $\varepsilon > 0$. As F is of transcendence degree at least 1 over $\mathbb{Q}$, we can use Theorem 2.2.70 to find a closed point $x \in \mathcal{X}_{\mathbb{Q}}$ satisfying

$$|h_{\overline{\mathcal{D}}_i}(x) - l(\overline{\mathcal{D}}_i)| < \varepsilon \text{ for } i = 1, \dots, r$$

and being sufficiently generic so that one can apply Lemma 2.3.23. By putting $\overline{a}' = \overline{a}|_x$ we get

$$R_{t_i}(\overline{f}_i(\overline{a}')) = R_{t_i}(\overline{f}_i(\overline{a})|_x) = h_{\overline{\mathcal{D}}_i}(x) \text{ for all } i = 1, \dots, r.$$

This finishes the proof of (*) and hence proves that $\overline{\mathbb{Q}}[1]$ is an existentially closed GVF. $\square$

2.3.3 Essential infimum and GVF functionals

For this subsection, let F be a finitely generated extension of $\mathbb{Q}$ and let $\mathcal{X}$ be an F-model. Assume that the transcendence degree of F over $\mathbb{Q}$ is d (in particular $\mathcal{X}$ is of dimension $d+1$).

Theorem 2.3.26. (M.Riesz extension theorem) Let $\mathbb{K}$ be $\mathbb{R}$ or $\mathbb{Q}$ and work in the category of $\mathbb{K}$ vector spaces. Let E be a vector space, $W \subset E$ be a subspace, and let $K \subset E$ be a convex cone. Assume that a linear functional $l_0 : W \to \mathbb{R}$ non-negative on $W \cap K$ is given. If the condition

$$E \subset K + W$$

is satisfied, then there exists an extension $l : E \to \mathbb{R}$ of ϕ, non-negative on K. Moreover, l is unique, if for each $e \in E$ and $\varepsilon > 0$, there are $w, w' \in W$ with $w - e, e - w' \in K$ and $l_0(w - w') < \varepsilon$.

Proof. This is classical. For a proof, see [11, Lemma 5.8]. $\square$

Corollary 2.3.27. Let V be a real vector subspace of $\mathrm{ADiv}_{\mathbb{R}}(F)$ containing a big arithmetic $\mathbb{R}$-divisor $\overline{\mathcal{D}}$ of C^0-type. Assume that we are given a GVF functional

$$l_0 : V \to \mathbb{R}$$

Then there exists an extension $l_0 \subset l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ to a GVF functional l.

Proof. Let W be the image of V under the projection $\pi : \mathrm{ADiv}_{\mathbb{R}}(F) \to \widehat{N}^1(F)$. Note that l_0 factors through W as it is a GVF functional. Use Theorem 2.3.26 (the existence part) with $E = \widehat{N}^1(F)$ and K being the image of the cone of effective arithmetic $\mathbb{R}$-divisors of C^0-type under π. Take a class of an arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{E}}$ on an F-model $\mathcal{X}$. Note that by Theorem 2.2.54, for a natural number n we have (possibly by passing to a different F-model)

$$\widehat{\mathrm{vol}}(n\overline{\mathcal{D}} + \overline{\mathcal{E}}) = n^{d+1}\widehat{\mathrm{vol}}(\overline{\mathcal{D}} + \frac{1}{n}\overline{\mathcal{E}}),$$

and $\lim_n \widehat{\mathrm{vol}}(\overline{\mathcal{D}} + \frac{1}{n}\overline{\mathcal{E}}) = \widehat{\mathrm{vol}}(\overline{\mathcal{D}}) > 0$, so for n big enough $\overline{\mathcal{F}} := n\overline{\mathcal{D}} + \overline{\mathcal{E}}$ is big. In particular by Lemma 2.2.40 its π image is in K. Note that

$$\overline{\mathcal{E}} = \overline{\mathcal{F}} + (-n\overline{\mathcal{D}}),$$

and by the fact that $\pi(\overline{\mathcal{E}}) \in E$ was arbitrary, we get an extension function $\widehat{N}^1(F) \to \mathbb{R}$ non-negative on classes of effective arithmetic $\mathbb{R}$-divisors of C^0-type. By composing it with π we are done. $\square$

Corollary 2.3.28. Let $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_n$ be arithmetic $\mathbb{R}$-divisors of C^0-type on $\mathcal{X}$ with $\overline{\mathcal{D}}_0 = (\mathrm{div}(2), 0)$. Denote by V the real vector space generated by $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_n$ and assume that one of $\overline{\mathcal{D}}_0, \dots, \overline{\mathcal{D}}_n$ is big. Fix $\varepsilon > 0$, a closed strict subscheme $Z \subset X = \mathcal{X} \otimes \mathbb{Q}$, and a normalised GVF functional

$$l : V \to \mathbb{R},$$

Then there exists a closed point $x \in X \setminus Z$ such that for all $i = 0, \dots, n$

$$|h_{\overline{\mathcal{D}}_i}(x) - l(\overline{\mathcal{D}}_i)| < \varepsilon.$$

Construction 2.3.29. Let $\overline{\mathcal{B}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on an F-model $\mathcal{X}'$. We define a linear functional $l_{\overline{\mathcal{B}}} : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ by the following formula.

$$l_{\overline{\mathcal{B}}} : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$$

$$\overline{\mathcal{E}} \mapsto \langle (\phi^*\overline{\mathcal{B}})^d \rangle \cdot \overline{\mathcal{E}},$$

where $\overline{\mathcal{E}}$ is an arithmetic $\mathbb{R}$-divisor of C^0-type on $\phi : \mathcal{X}'' \to \mathcal{X}'$. By Lemma 2.2.46 $l_{\overline{\mathcal{B}}}$ is well defined. Moreover, by Theorem 2.2.54 we have

$$\langle (\phi^* \overline{\mathcal{B}})^d \rangle \cdot \overline{\mathcal{E}} = \frac{1}{d+1} D_{\overline{\mathcal{B}}} \widehat{\mathrm{vol}}(\overline{\mathcal{E}}) = \lim_{t \to 0^+} \frac{\widehat{\mathrm{vol}}(\phi^* \overline{\mathcal{B}} + t\overline{\mathcal{E}}) - \widehat{\mathrm{vol}}(\phi^* \overline{\mathcal{B}})}{(d+1)t}.$$

But for effective $\overline{\mathcal{E}}$ and $t > 0$ we have $\widehat{\mathrm{vol}}(\phi^* \overline{\mathcal{B}} + t\overline{\mathcal{E}}) - \widehat{\mathrm{vol}}(\phi^* \overline{\mathcal{B}}) \geq 0$, see the proof of Lemma 2.2.40. Thus $l_{\overline{\mathcal{B}}}(\overline{\mathcal{E}}) \geq 0$ and so $l_{\overline{\mathcal{B}}}$ is in fact a GVF functional. Note that if $\overline{\mathcal{B}}$ is nef, then $l_{\overline{\mathcal{B}}}(\overline{\mathcal{E}}) = (\phi^* \overline{\mathcal{B}})^d \cdot \overline{\mathcal{E}}$ by Remark 2.2.53.

Lemma 2.3.30. Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Then $\overline{\mathcal{D}}$ is pseudo-effective if and only if the following holds.

For all GVF functionals $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ we have $l(\overline{\mathcal{D}}) \geq 0$.

Moreover, $\overline{\mathcal{D}}$ is principal if and only if it satisfies the following.

For all GVF functionals $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ we have $l(\overline{\mathcal{D}}) = 0$.

Proof. We start with the first characterisation. Assume that $\overline{\mathcal{D}}$ is pseudo-effective and let $\overline{\mathcal{B}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type. Then for every $\delta > 0$ we have $\widehat{\mathrm{vol}}(\overline{\mathcal{D}} + \delta \overline{\mathcal{B}}) > 0$, so if l is a GVF functional on $\mathrm{ADiv}_{\mathbb{R}}(F)$ we have

$$l(\overline{\mathcal{D}}) = \lim_{\delta \to 0^+} l(\overline{\mathcal{D}} + \delta \overline{\mathcal{B}}) \geq 0,$$

where the last inequality follows from Lemma 2.2.40. On the other hand, if $\overline{\mathcal{D}}$ is not pseudo-effective, by Theorem 2.2.47 there is an F-model $\pi : \mathcal{X}' \to \mathcal{X}$ with a nef arithmetic $\mathbb{R}$-divisor $\overline{\mathcal{H}}$ such that $\overline{\mathcal{H}}^d \cdot \pi^* \overline{\mathcal{D}} < 0$. By Construction 2.3.29, $l = l_{\overline{\mathcal{H}}}$ is a GVF functional, and it satisfies $l(\overline{\mathcal{D}}) < 0$ which finishes the proof.

To prove the other characterisation, it is enough to prove that if all GVF functionals vanish on $\overline{\mathcal{D}}$, then $\overline{\mathcal{D}}$ is principal. Note that by the first part of the lemma we get that $\overline{\mathcal{D}}$ is pseudo-effective. Now the result follows from Theorem 2.2.47 and the proof of the first characterisation. $\square$

Remark 2.3.31. Fix the notation from Lemma 2.3.30 and let $\overline{\mathcal{D}}_0 = (\mathrm{div}(2), 0)$. Note that in both characterisations we could in fact check the corresponding condition only for normalised GVF functionals. Indeed, assume that the first condition holds for normalised GVF functionals on $\mathrm{ADiv}_{\mathbb{R}}(F)$. Take any GVF functional $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$. If $l(\overline{\mathcal{D}}_0) \neq 0$, we can rescale it (using a positive scalar as $\overline{\mathcal{D}}_0$ is effective) to be normalised, so $l(\overline{\mathcal{D}}) \geq 0$. If $l(\overline{\mathcal{D}}_0) = 0$, we can take any normalised GVF functional l' on $\mathrm{ADiv}_{\mathbb{R}}(F)$ and if we had $l(\overline{\mathcal{D}}) < 0$, then for n big enough $(l' + nl)(\overline{\mathcal{D}}) < 0$, which gives a contradiction.

By unravelling the proof of the first part of the below theorem, one can see that it is essentially the same as the proof of [64, Theorem 4.6] (but using Theorem 2.2.14 instead of [64, Theorem 3.9]). We decided to include this theorem here to underline the new interpretation of ζ using GVF functionals.

Theorem 2.3.32. [64, Theorem 4.6] Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Then

$$\zeta(\overline{\mathcal{D}}) \leq \inf\{l(\overline{\mathcal{D}}) : l \text{ is a normalised GVF functional on } \operatorname{ADiv}_{\mathbb{R}}(F)\}.$$

Moreover, if $\mathcal{D}_{\mathbb{Q}}$ is big, this is an equality.

Proof. For the first part, assume that there is a normalised GVF functional $l : \operatorname{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ such that $l(\overline{\mathcal{D}}) = r$. Fix a closed strict subscheme $\mathcal{Y} \subset \mathcal{X}$ and $\varepsilon > 0$. By Theorem 2.2.70 there is a closed point $x \in X = \mathcal{X}_{\mathbb{Q}}$ such that $x \notin \mathcal{Y}$ and $|h_{\overline{\mathcal{D}}}(x) - l(\overline{\mathcal{D}})| < \varepsilon$. We can use the theorem because l is normalised. Since $\mathcal{Y}$ was arbitrary, we get

$$\zeta(\overline{\mathcal{D}}) = \sup_{\mathcal{Y} \subset \mathcal{X}} \inf_{x \in \mathcal{X} \setminus \mathcal{Y}(\overline{\mathbb{Q}})} h_{\overline{\mathcal{D}}}(x) \leq r + \varepsilon.$$

As l and ε were arbitrary, we get the desired inequality.

For the other part, by Lemma 2.2.63 (using bigness of $\mathcal{D}_{\mathbb{Q}}$) we have

$$\zeta(\overline{\mathcal{D}}) \geq \sup\{r : \overline{\mathcal{D}} - \frac{r}{\log 2} \cdot \overline{\mathcal{D}}_0 \text{ is pseudo-effective}\},$$

for $\overline{\mathcal{D}}_0 = (\operatorname{div}(2), 0)$. By Lemma 2.3.30 we know that for any fixed $r \in \mathbb{R}$

$$\overline{\mathcal{D}} - \frac{r}{\log 2} \cdot \overline{\mathcal{D}}_0 \text{ is pseudo-effective}$$

if and only if for all GVF functionals $l : \operatorname{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ we have $l(\overline{\mathcal{D}} - \frac{r}{\log 2} \cdot \overline{\mathcal{D}}_0) \geq 0$. Note that for a normalised GVF functional $l : \operatorname{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ we have

$$l(\overline{\mathcal{D}} - \frac{r}{\log 2} \cdot \overline{\mathcal{D}}_0) = l(\overline{\mathcal{D}}) - r,$$

so this expression is ≥ 0 if and only if $l(\overline{\mathcal{D}}) \geq r$. By Remark 2.3.31 we get that

$$\zeta(\overline{\mathcal{D}}) \geq \sup\{r : (\forall l)(l(\overline{\mathcal{D}}) \geq r)\},$$

where we quantify over all normalised GVF functionals l on $\operatorname{ADiv}_{\mathbb{R}}(F)$. This gives the desired equality in the case where $\mathcal{D}_{\mathbb{Q}}$ is big. □

As a corollary one gets [64, Theorem 1.4] which is equivalent to [3, Corollary 4.2] without the semi-positivity assumption.

Corollary 2.3.33. [64, Theorem 1.4] Let $\overline{\mathcal{D}}$ be an arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$ with $\mathcal{D}_{\mathbb{Q}}$ big and let $\overline{\mathcal{D}}_0 = (\mathrm{div}(2), 0)$. Then

$$\zeta(\overline{\mathcal{D}}) = \sup\{r : \overline{\mathcal{D}} - \frac{r}{\log 2} \cdot \overline{\mathcal{D}}_0 \text{ is pseudo-effective}\}.$$

Proof. Note that by Lemma 2.2.63

$$\zeta(\overline{\mathcal{D}}) \geq r \iff \zeta(\overline{\mathcal{D}} - \frac{r}{\log 2}\overline{\mathcal{D}}_0) \geq 0,$$

which by Theorem 2.3.32 is equivalent to the inequality $\inf_l (l(\overline{\mathcal{D}}) - \frac{r}{\log 2}) \geq 0$ where the infimum is taken over all normalised GVF functionals on $\mathrm{ADiv}_{\mathbb{R}}(F)$. By Lemma 2.3.30 this is equivalent to $\overline{\mathcal{D}} - \frac{r}{\log 2}\overline{\mathcal{D}}_0$ being pseudo-effective, which finishes the proof. □

Construction 2.3.34. Let $\overline{\mathcal{D}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$ be big. Note that then $\mathcal{D}_{\mathbb{Q}}$ is big by Lemma 2.2.40. Let

$$\varphi(\overline{\mathcal{E}}) = \frac{\widehat{\mathrm{vol}}(\overline{\mathcal{E}})}{(d+1)\,\mathrm{vol}(\mathcal{E}_{\mathbb{Q}})}$$

for $\overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$ with big $\mathcal{E}_{\mathbb{Q}}$. Note that it is well defined since both $\widehat{\mathrm{vol}}$ and vol are invariant under birational pullbacks (see Lemma 2.2.48 and [49] for vol). By differentiability of $\widehat{\mathrm{vol}}$ from Theorem 2.2.54 and differentiability of vol proved in [15, Theorem A] we get a linear functional

$$D_{\overline{\mathcal{D}}}\varphi : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$$

which is the Gateaux derivative of φ calculated on an appropriate F-model. This linear functional is normalised and zero on principal arithmetic $\mathbb{R}$-divisors of C^0-type, but is not necessarily a GVF functional.

Lemma 2.3.35. Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Then there exists a normalised GVF functional $l : \mathrm{ADiv}_{\mathbb{R}}(F) \to \mathbb{R}$ such that $\zeta(\overline{\mathcal{D}}) = l(\overline{\mathcal{D}})$.

Proof. We give two proofs. First, we can find a sequence of normalised GVF functionals l_n on $\mathrm{ADiv}_{\mathbb{R}}(F)$ with $\lim_n l_n(\overline{\mathcal{D}}) = \zeta(\overline{\mathcal{D}})$ by Theorem 2.3.32. Pick a bound $C > 0$ such that $l_n(\overline{\mathcal{D}}) < C$ for all natural n. Let $\overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$. As in the proof of Corollary 2.3.27, take a natural number m such that $m\overline{\mathcal{D}} - \overline{\mathcal{E}}, m\overline{\mathcal{D}} + \overline{\mathcal{E}}$ are big. Then for all GVF functionals l on $\mathrm{ADiv}_{\mathbb{R}}(F)$ with $l(\overline{\mathcal{D}}) \leq C$ we have

$$-Cm \leq l(\overline{\mathcal{E}}) \leq Cm.$$

Denote by $m = m_{\overline{\mathcal{E}}}$ a number suitable for $\overline{\mathcal{E}}$. We get that the set of normalised GVF functionals l on $\mathrm{ADiv}_{\mathbb{R}}(F)$ with $l(\overline{\mathcal{D}}) \leq C$ embeds as a closed subset into

$$\prod_{\overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)} [-Cm_{\overline{\mathcal{E}}}, Cm_{\overline{\mathcal{E}}}].$$

This product is compact, so we can find a subnet of $(l_n)_{n\in\mathbb{N}}$ converging to a desired normalised GVF functional.

Another way to construct such a normalised GVF functional is to consider the vector space V generated by $\overline{\mathcal{D}}_0 = (\mathrm{div}(2), 0)$ and $\overline{\mathcal{D}}$ in $\mathrm{ADiv}_{\mathbb{R}}(F)$. Next, define l on V by putting $l(\overline{\mathcal{D}}_0) := \log(2), l(\overline{\mathcal{D}}) := \zeta(\overline{\mathcal{D}})$. By Corollary 2.3.33 we get that l is a GVF functional, and by Corollary 2.3.27 it has an extension to $\mathrm{ADiv}_{\mathbb{R}}(F)$, which finishes the second proof. ☐

Theorem 2.3.36. Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Consider the statements:

(1) $\varphi(\overline{\mathcal{D}}) = \zeta(\overline{\mathcal{D}})$;

(2) $D_{\overline{\mathcal{D}}}\varphi$ is a GVF functional;

(3) the infimum

$$\zeta(\overline{\mathcal{D}}) = \inf\{l(\overline{\mathcal{D}}) : l \text{ is a normalised GVF functional on } \mathrm{ADiv}_{\mathbb{R}}(F)\}$$

is achieved at the unique normalised GVF functional;

(4) ζ is Gateaux differentiable at $\overline{\mathcal{D}}$ in every direction.

Then (1) $\iff$ (2) $\implies$ (3) $\iff$ (4).

Proof. Let $\overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$. Note that for small real t the arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{D}} + t\overline{\mathcal{E}}$ is big. Let l_0 be a normalised GVF functional on $\mathrm{ADiv}_{\mathbb{R}}(F)$ with $l_0(\overline{\mathcal{D}}) = \zeta(\overline{\mathcal{D}})$. Such exists by Lemma 2.3.35. For real $t > 0$ small enough we then have

$$\frac{\zeta(\overline{\mathcal{D}} + t\overline{\mathcal{E}}) - \zeta(\overline{\mathcal{D}})}{t} \leq \frac{l_0(\overline{\mathcal{D}} + t\overline{\mathcal{E}}) - l_0(\overline{\mathcal{D}})}{t} = l_0(\overline{\mathcal{E}}). \tag{2.5}$$

To get the above inequality we use Theorem 2.3.32. Now we pass to proving equivalence of the conditions in the theorem.

(1) $\Rightarrow$ (2) : Assuming $\varphi(\overline{\mathcal{D}}) = \zeta(\overline{\mathcal{D}})$, by Theorem 2.2.64 we get

$$\frac{\varphi(\overline{\mathcal{D}} + t\overline{\mathcal{E}}) - \varphi(\overline{\mathcal{D}})}{t} \leq \frac{\zeta(\overline{\mathcal{D}} + t\overline{\mathcal{E}}) - \zeta(\overline{\mathcal{D}})}{t} \leq l_0(\overline{\mathcal{E}}).$$

Taking $t \to 0^+$ we get that $D_{\overline{\mathcal{D}}}\varphi \leq l_0$, but since they are linear (in particular odd) this implies that $D_{\overline{\mathcal{D}}}\varphi = l_0$ is a GVF functional.

$(2) \Rightarrow (1)$: From Theorem 2.3.32 we then get that $\zeta(\overline{\mathcal{D}}) \leq D_{\overline{\mathcal{D}}}\varphi(\overline{\mathcal{D}}) = \varphi(\overline{\mathcal{D}})$ which together with Theorem 2.2.64 gives the equality.

$(1) \Rightarrow (4)$: From the proof of $(1) \Rightarrow (2)$ we get that

$$l_0(\overline{\mathcal{E}}) = D_{\overline{\mathcal{D}}}\varphi(\overline{\mathcal{E}}) \leq \lim_{t \to 0^+} \frac{\zeta(\overline{\mathcal{D}} + t\overline{\mathcal{E}}) - \zeta(\overline{\mathcal{D}})}{t} \leq l_0(\overline{\mathcal{E}}).$$

For the negative t replace $\overline{\mathcal{E}}$ by $-\overline{\mathcal{E}}$.

$(4) \Rightarrow (3)$: By taking the limit $t \to 0$ in Equation (2.5) we get

$$D_{\overline{\mathcal{D}}}\zeta(\overline{\mathcal{E}}) \leq l_0(\overline{\mathcal{E}}).$$

Since both maps are linear, we get $D_{\overline{\mathcal{D}}}\zeta = l_0$, hence the uniqueness of l_0 follows.

$(3) \Rightarrow (4)$: Let V be the real vector subspace generated by $\overline{\mathcal{D}}_0 = (\mathrm{div}(2), 0), \overline{\mathcal{D}}, \overline{\mathcal{E}}$ in $\mathrm{ADiv}_{\mathbb{R}}(F)$. If the classes of $\overline{\mathcal{D}}_0, \overline{\mathcal{D}}, \overline{\mathcal{E}}$ are linearly dependent in $\widehat{N}^1(F)$, then differentiability of ζ at $\overline{\mathcal{D}}$ in direction $\overline{\mathcal{E}}$ follows from Lemma 2.2.62 and Lemma 2.2.63. Assume that those classes are linearly independent. Let V^+ be the (euclidean) closure of the big cone in V and pick a hyperplane P in V containing $\overline{\mathcal{D}}_0, \overline{\mathcal{D}}$, such that $Q := P \cap V^+$ is a compact convex set. This is possible as V^+ does not contain a line. By assumption and Theorem 2.3.26 we know that l_0 restricted to P is the unique affine function l on P having the following properties:

- $l(\overline{\mathcal{D}}_0) = \log(2)$,
- $l(\overline{\mathcal{D}}) = \zeta(\overline{\mathcal{D}})$,
- l is non-negative on Q.

Consider the function $f(t) = \zeta(\overline{\mathcal{D}} + t\overline{\mathcal{E}})$ on a neighbourhood of 0. It is concave by Lemma 2.2.62. We now show that if f is not differentiable at zero, then we can find a different affine function l having the listed properties. By moving the hypersurface P and rescaling $\overline{\mathcal{D}}_0, \overline{\mathcal{D}}, \overline{\mathcal{E}}$ we can without loss of generality assume that $\overline{\mathcal{D}}_0, \overline{\mathcal{D}}, \overline{\mathcal{E}} \in P$. In other words

$$P = \{a\overline{\mathcal{D}}_0 + b\overline{\mathcal{D}} + c\overline{\mathcal{E}} : a + b + c = 1\}.$$

Fix coordinates on P so that $\overline{\mathcal{D}}_0$ is the zero and the basis vectors are $e_1 = \overline{\mathcal{D}} - \overline{\mathcal{D}}_0, e_2 = \overline{\mathcal{E}} - \overline{\mathcal{D}}_0$. Consider the function

$$g(\varepsilon) := \sup\{r : \overline{\mathcal{D}} + \varepsilon e_2 + r e_1 \text{ is pseudo-effective}\}.$$

There is a positive t such that $g : (-t,t) \to \mathbb{R}$ is a concave function, as Q is a compact convex set. Note that

$$\overline{\mathcal{D}} + \varepsilon e_2 + re_1 = (1+r)\overline{\mathcal{D}} + \varepsilon\overline{\mathcal{E}} - (r+\varepsilon)\overline{\mathcal{D}}_0.$$

By Corollary 2.3.33 we know that $\overline{\mathcal{D}} + \varepsilon e_2 + re_1$ is pseudo-effective if and only if $\zeta(\overline{\mathcal{D}} + \varepsilon e_2 + re_1) \geq 0$, which by the above calculation and Lemma 2.2.63 is equivalent to

$$\zeta(\overline{\mathcal{D}} + \varepsilon e_2 + re_1) = \zeta((1+r)\overline{\mathcal{D}} + \varepsilon\overline{\mathcal{E}}) - (r+\varepsilon)\log(2)$$
$$= (1+r)\zeta(\overline{\mathcal{D}} + \frac{\varepsilon}{1+r}\overline{\mathcal{E}}) - (r+\varepsilon)\log(2) = (1+r)f(\frac{\varepsilon}{1+r}) - (r+\varepsilon)\log(2) \geq 0.$$

The supremum of r where this quantity is non-negative is achieved when we have the equality with zero, so we get the functional equation

$$(1+g(\varepsilon))f(\frac{\varepsilon}{1+g(\varepsilon)}) - (g(\varepsilon)+\varepsilon)\log(2) = 0.$$

Thus

$$f(\frac{\varepsilon}{1+g(\varepsilon)}) = \log(2)\frac{g(\varepsilon)+\varepsilon}{1+g(\varepsilon)}.$$

Note that $g(0)$ is strictly positive as $\widehat{\mathrm{vol}}(\overline{\mathcal{D}}) > 0$. If g was differentiable at 0, then f also would be, as

$$\frac{f(\frac{\varepsilon}{1+g(\varepsilon)}) - f(0)}{\frac{\varepsilon}{1+g(\varepsilon)}} = \log(2)\frac{\frac{g(\varepsilon)+\varepsilon}{1+g(\varepsilon)} - \frac{g(0)}{1+g(0)}}{\frac{\varepsilon}{1+g(\varepsilon)}}.$$

Thus g is not differentiable at 0. This means that there exist two different lines X_1, X_2 on P such that $X_i \cap Q = \{\overline{\mathcal{D}} + g(0)e_1\}$ for $i = 1,2$. Consider affine functions l_1, l_2 on P determined by the following conditions

- l_i vanishes on X_i,
- $l_i(\overline{\mathcal{D}}_0) = \log(2)$,
- $l_i(\overline{\mathcal{D}}) = \zeta(\overline{\mathcal{D}})$,

for $i = 1,2$. Note that such functions exists by Corollary 2.3.33. Moreover, they are non-negative on Q, since they are affine, their zero sets are supporting lines for Q, and they are positive on a point $\overline{\mathcal{D}}_0$ in Q. This gives a non-uniqueness of l_0, which finishes the proof. $\square$

Question 2.3.37. What is the set of all arithmetic big $\mathbb{R}$-divisors of C^0-type $\overline{\mathcal{D}}$ with $D_{\overline{\mathcal{D}}}\zeta$ Gateaux differentiable?

Before the next lemma, note that if $x \in \mathcal{X}_{\mathbb{Q}}$ is a sufficiently general closed point, then for $\overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$ one can make sense out of $h_{\overline{\mathcal{E}}}(x)$. Indeed, if $\overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(\mathcal{X}')$ for some F-model $\phi : \mathcal{X}' \to \mathcal{X}$, then if x is in the open subset where ϕ is an isomorphism, we can define $h_{\overline{\mathcal{E}}}(x) := h_{\overline{\mathcal{E}}}(\phi^{-1}(x))$.

The below lemma follows the lines of [18, Lemma (8.2)] or [23, Section 5] and proves that differentiability of ζ is enough for equidistribution (over all places).

Lemma 2.3.38. Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Assume that $(x_n)_{n\in\mathbb{N}}$ is a sequence of closed points in the generic fiber of $\mathcal{X}$ which converges to the generic point. If ζ is Gateaux differentiable at $\overline{\mathcal{D}}$ in directions $\overline{\mathcal{E}}, -\overline{\mathcal{E}} \in \mathrm{ADiv}_{\mathbb{R}}(F)$ and

$$\lim_n h_{\overline{\mathcal{D}}}(x_n) = \zeta(\overline{\mathcal{D}}),$$

then

$$\lim_n h_{\overline{\mathcal{E}}}(x_n) = D_{\overline{\mathcal{D}}}\zeta(\overline{\mathcal{E}}).$$

Proof. Let $t > 0$ be a real number. We calculate

$$\liminf_{n\to\infty} h_{\overline{\mathcal{E}}}(x_n) = \liminf_{n\to\infty} \frac{h_{\overline{\mathcal{D}}+t\overline{\mathcal{E}}}(x_n) - h_{\overline{\mathcal{D}}}(x_n)}{t}.$$

Note that $\liminf_{n\to\infty} h_{\overline{\mathcal{D}}+t\overline{\mathcal{E}}}(x_n) \geq \zeta(\overline{\mathcal{D}} + t\overline{\mathcal{E}})$ and $\liminf_{n\to\infty} h_{\overline{\mathcal{D}}}(x_n) = \zeta(\overline{\mathcal{D}})$. Thus we get

$$\liminf_{n\to\infty} \frac{h_{\overline{\mathcal{D}}+t\overline{\mathcal{E}}}(x_n) - h_{\overline{\mathcal{D}}}(x_n)}{t} \geq \frac{\zeta(\overline{\mathcal{D}} + t\overline{\mathcal{E}}) - \zeta(\overline{\mathcal{D}})}{t}.$$

Taking the limit $t \to 0^+$ we have

$$\liminf_{n\to\infty} h_{\overline{\mathcal{E}}}(x_n) \geq D_{\overline{\mathcal{D}}}\zeta(\overline{\mathcal{E}}).$$

Replacing $\overline{\mathcal{E}}$ with $-\overline{\mathcal{E}}$ we get

$$\limsup_{n\to\infty} h_{\overline{\mathcal{E}}}(x_n) \leq D_{\overline{\mathcal{D}}}\zeta(\overline{\mathcal{E}}).$$

Together those inequalities give $\lim_n h_{\overline{\mathcal{E}}}(x_n) = D_{\overline{\mathcal{D}}}\zeta(\overline{\mathcal{E}})$ which finishes the proof. ☐

Definition 2.3.39. Say that $\overline{\mathcal{D}}$ satisfies *logarithmic equidistribution*, if for any generic sequence $x_n \in \mathcal{X}(\overline{\mathbb{Q}})$ with $h_{\overline{\mathcal{D}}}(x_n) \to \zeta(\overline{\mathcal{D}})$, if $\overline{\mathcal{E}}$ is any arithmetic $\mathbb{R}$-divisor of C^0-type on an F-model, then $\lim_n h_{\overline{\mathcal{E}}}(x_n)$ exists and does not depend on $(x_n)_{n\in\mathbb{N}}$.

Corollary 2.3.40. Let $\overline{\mathcal{D}}$ be a big arithmetic $\mathbb{R}$-divisor of C^0-type on $\mathcal{X}$. Then $\overline{\mathcal{D}}$ satisfies logarithmic equidistribution, if and only if ζ is Gateaux differentiable at $\overline{\mathcal{D}}$.

Proof. Lemma 2.3.38 proves the implication from right to left.

On the other hand, if $\overline{\mathcal{D}}$ does not satisfy logarithmic distribution, then there are generic sequences $(x_n)_{n\in\mathbb{N}}, (x'_n)_{n\in\mathbb{N}}$ and an arithmetic $\mathbb{R}$-divisor of C^0-type $\overline{\mathcal{E}}$ on an F-model, such that

$$\lim_n h_{\overline{\mathcal{D}}}(x_n) = \lim_n h_{\overline{\mathcal{D}}}(x'_n) = \zeta(\overline{\mathcal{D}})$$

and

$$\lim_n h_{\overline{\mathcal{E}}}(x_n) \neq \lim_n h_{\overline{\mathcal{E}}}(x'_n).$$

In particular, there are two different GVF functionals on the linear span of $\overline{\mathcal{D}}, \overline{\mathcal{E}}$ in $\mathrm{ADiv}_{\mathbb{R}}(F)$ so Theorem 2.3.26 and Theorem 2.3.36 finish the proof. □

The bigness condition was recently relaxed in [4]. Moreover, the authors made progress towards Question 2.3.37 by finding new criteria (e.g. [4, Theorem 6.1]) for differentiability of the essential infimum ζ.

Chapter 3

Continuity of heights

This chapter is based on the paper [28] joint with Pablo Destic and Nuno Hultberg. Some changes are made, to incorporate the contents of the previous chapters (mostly in the introduction and Section 3.2.1). The author's contributions to this project consist of proving the quantifier-free definability of the arithmetic intersection pairing through the connection with adelic curves, and of the idea of using this definability to attack the Gualdi's conjecture. The paper has been submitted to a journal.

3.1 Introduction

In classical algebraic geometry Bezout's theorem states that generically the intersection of n hypersurfaces of degrees $d_1, \ldots, d_r$ in $\mathbb{P}^n$ is of degree $d_1 \ldots d_r$. More generally, degrees of fibers in a flat family do not vary.

Theorem 3.1.1 (Corollary III 9.10 [39])**.** Let $\mathcal{X} \to S$ be a flat projective family of varieties over a field K of relative dimension d. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$. Then, the degree of the fibres $\deg_{\mathcal{L}}(\mathcal{X}_s)$ is locally constant.

In particular, the map $s \mapsto \deg_{\mathcal{L}}(\mathcal{X}_s)$ is a continuous function on S with the Zariski topology. The same result is implied for the intersection of d different line bundles. In this work, we study an arithmetic variant of this 'continuity' of degrees. More precisely, we prove that arithmetic intersection numbers parametrized over a base form a quantifier-free definable formula in the theory GVF. Let us introduce some notation to explain this statement.

One can associate to a finite type scheme S over a GVF K, a locally compact Hausdorff space S_{GVF}, consisting of GVF quantifier-free types concentrated on S. We call it the GVF analytification of S. Note that for $K = \overline{\mathbb{Q}}$, a generic sequence

s_i in $S(\overline{\mathbb{Q}}) \subset S_{\mathrm{GVF}}$ converges to a point in S_{GVF} if and only if the value $h_{\overline{M}}(s_i)$ converges for every adelically metrized line bundle $\overline{M}$ on S. In Definition 3.3.21, for a flat family $\mathcal{X} \to S$ over a GVF K, we introduce the group of global line bundles on $\mathcal{X}$ over S. Every embedding $j : \mathcal{X} \to \mathbb{P}^n_S$ over S, induces a global line bundles on $\mathcal{X}$ over S. Following [84], we define semipositivity and integrability in this context. For example, the group of globally integrable line bundles on $\mathcal{X}$ over S is denoted by $\widehat{\mathrm{Pic}}^{\mathrm{int}}_{\mathbb{Q}}(\mathcal{X}/S)$. In Proposition 3.3.22 we define an intersection paring on tuples of globally integrable line bundles, and we prove the following.

Theorem 3.1.2 (Theorem 3.3.1)**.** Let $\mathcal{X} \to S$ be a flat projective morphism of finite type schemes over a GVF K of relative dimension d. Let $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d \in \widehat{\mathrm{Pic}}^{\mathrm{int}}_{\mathbb{Q}}(\mathcal{X}/S)$. Then, the map

$$\begin{aligned} S_{\mathrm{GVF}} &\to \mathbb{R} \\ s &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d | \mathcal{X}_s) \end{aligned}$$

is continuous.

Equivalently, this means that the intersection product $\widehat{\deg}(\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d | \mathcal{X}_s)$ (for $s \in S(F)$ with $K \subset F$ being a GVF extension) can be defined by a quantifier-free formula in the GVF language, with parameters from the base-field K. We believe that this fact may be important in axiomatizing the model companion of globally valued fields (if it exists). To see how the intersection pairing is defined in the case of globally integrable line bundles coming from embeddings into projective spaces, see Outline 0.1. In particular, the above theorem (more precisely a version where we do not fix a GVF structure on K, but only fix a uniform bound on $h(2 : 1)$) generalises Silverman's Height Limit Theorem [69, Theorem B] from limits of heights of points from a section of an abelian family, to limits of heights of cycles in such family. Another immediate application (that also uses the existential closedness of $\overline{\mathbb{Q}}$) is the density of the Faltings height above its essential infimum, see Section 2.1.3 for more details.

For the benefit of Arakelov geometers reading this thesis, let us phrase a version of the 'continuity of heights' over $\overline{\mathbb{Q}}$ that does not use globally valued fields in the statement.

Theorem 3.1.3. Let $\pi : \mathcal{X} \to S$ be a surjective morphism of projective varieties over $\overline{\mathbb{Q}}$, generically of relative dimension d. Let $(s_i)_i$ be a generic sequence of points in $S(\overline{\mathbb{Q}})$ such that for every adelic line bundle $\overline{M}$ on S the value $h_{\overline{M}}(s_i)$ converges. Let

$\overline{L}_0, \ldots, \overline{L}_d$ be integrable line bundles on $\mathcal{X}$. Then, the arithmetic intersection number on fibers

$$\widehat{\deg}(\overline{L}_0 \ldots \overline{L}_d | \mathcal{X}_{s_i})$$

converges. Its limit can be described as a certain arithmetic intersection number over a GVF.

Moreover, if $h_{\overline{M}}(s_i)$ converges to $\overline{L}^{\dim S}\overline{M}$ for some arithmetically nef line bundle $\overline{L}$ and every adelically metrized line bundle $\overline{M}$, then $\widehat{\deg}(\overline{L}_0 \ldots \overline{L}_d | \mathcal{X}_{s_i})$ converges to the intersection number $(\pi^*\overline{L})^{\dim S}\overline{L}_0 \ldots \overline{L}_d$.

The moreover part of the theorem corresponds to Proposition 3.3.26, and is essentially due to Chen and Moriwaki [25, Proposition 4.5.1]. Equidistribution theorems in Arakelov geometry provide us with various examples of sequences to which the above theorem can be applied. We note in particular Yuan's equidistribution in the form of [18, Lemma 8.2] gives such examples.

As an application, we look at heights of complete intersections of hypersurfaces twisted by generic tuples of roots of unity in a toric variety. The starting point of our considerations is [33], where a formula was given, for the height of a single hypersurface in a toric variety, with respect to a semipositively metrized toric divisor. However, Roberto Gualdi observed in his thesis [32] that it is not possible to extend a result of the same nature to an intersaction of more than one hypersurfaces. In fact, he gives examples of polynomials with the same associated arithmetic data, defining cycles of differing heights. As a remedy to this issue, he suggested to consider average heights of cycles with prescribed arithmetic data and formulated the following statement, which we prove in this article, as a conjecture.

Theorem 3.1.4 (Theorem 3.4.5)**.** Let $f_1, \ldots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K and let T be a proper toric variety with torus $\mathbb{T} = \mathbb{G}^n \subset T$. Denote by V_i the hypersurface defined by f_i and by ρ_i its Ronkin function. Let $(\zeta_{1,j}, \ldots, \zeta_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ for the Weil height and let $\overline{D}_0, \ldots, \overline{D}_{n-m}$ be semipositive toric adelic divisors on T with associated local roof functions $\theta_{0,v}, \ldots, \theta_{n-m,v}$. Then,

$$\lim_{j\to\infty} \widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_{n-m} \mid \zeta_{1,j}V_1 \cap \cdots \cap \zeta_{m,j}V_m)$$

$$= \sum_{v\in \mathrm{M}_K} n_v \,\mathrm{MI}_M(\theta_{0,v}, \ldots, \theta_{n-m,v}, \rho_1^{\vee}, \ldots, \rho_m^{\vee}).$$

The original conjecture can be found in [32, Conjecture 6.4.4]. Let us unravel the statement of this theorem. Assume for simplicity that $T = \mathbb{P}^n, n = m$ and $\overline{D}_0$ corresponds to the Weil projective height. For each place v on K, $\theta_{0,v}, \rho_1^\vee, \dots, \rho_m^\vee$ are concave functions on some compact convex sets (depending on the place v) contained in $\mathbb{R}^n$ identified with the character lattice M of $\mathbb{T} = \mathbb{G}^n$ tensored with $\mathbb{R}$ (in particular, it has a natural Haar measure dm of covolume one). The function MI_M is multilinear with respect to the sup-convolution of concave functions (an operation on concave functions that generalises the Minkowski sum) and satisfies

$$\mathrm{MI}(\theta, \dots, \theta) = (n+1)! \int_\Delta \theta(m) dm.$$

The Ronkin function of a Laurent polynomial f (with respect to a place v) is defined as a function $\rho_f : N_\mathbb{R} \to \mathbb{R}$ given by

$$\rho_f(u) = \int_{\mathrm{trop}^{-1}(u)} -\log |f(x)| d\sigma_u(x),$$

where N is the co-character lattice of $\mathbb{T}$ and $\mathrm{trop} : \mathbb{T}_v^{\mathrm{an}} \to N_\mathbb{R}$ is the tropicalization map (so if v is the real place of $\mathbb{Q}$ it corresponds to taking $-\log$'s of absolute values of coordinates on the tori $(\mathbb{C}^\times)^n$), and σ_u is a natural probabilistic measure on $\mathrm{trop}^{-1}(u)$ (The Haar measure in the archimedean case, and the Dirac delta at the Gauss point in the non-archimedean place). Legendre-Fenchel duality associates with ρ_f its dual $\rho_f^\vee$ which is supported on the Newton polygon of f inside $M_\mathbb{R}$. Thus, the equation from the conjecture describes the limit heights of twisted intersections as a sum of contributions from all places, of local convex theoretic data. Mixed integrals and further convex geometry tools are discussed in more detail in Section 3.2.4.

Roberto Gualdi and Martin Sombra have together proved partial results towards the conjecture. In [34] they prove the above result in the case $T = \mathbb{P}^2$, $m = 2$ and $f_1(x_1, x_2) = f_2(x_1, x_2) = x_1 + x_2 + 1$. Moreover, for this example, they compute that both sides of the identity in question are equal to the intriguing value $\frac{2\zeta(3)}{3\zeta(2)}$. They have also solved the $m = 2$ case of the conjecture in [35]. Their methods fundamentally differ from ours. In particular, their approach relies on local logarithmic equidistribution as in [29] while we modify the problem in such a way that we can apply Yuan's equidistribution theorem from [81].

The structure of this chapter is the following. In Section 3.2 we introduce globally valued fields, GVF analytifications, and give some examples. We also present necessary notions from Arakelov geometry of toric varieties. In Section 3.3 we use the theory of adelic curves to define the intersection product over arbitrary GVF and prove Theorem 3.3.1. We also prove Theorem 3.3.24 which relates global integrable line bundles (in our new sense) to integrable adelic line bundles over a number field (in the sense of Zhang). In Section 3.4 we use the previous results and perform calculations which allow us to conclude with Theorem 3.4.5. In Appendix 3.5 we prove an estimate on a variant of the Mahler measure of a polynomial, needed in the proof of Theorem 3.3.1. We believe this is known to experts, however we could not find a suitable reference. The appendix is self-contained and elementary.

3.1.1 Acknowledgements

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3.2 Preliminaries

This section will serve both to recall definitions and theorems, as well as to state and prove technical results related to the definitions just recalled.

3.2.1 Zhang divisors

Let F be a field. Recall that GVF structures on F correspond to *GVF functionals*, which are linear functionals

$$l : \mathrm{LDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$$

that are non-negative on the positive cone, and are zero on principal lattice divisors.

Assume that F is a finitely generated extension of $\mathbb{Q}$. In that case, $\mathrm{LDiv}_{\mathbb{Q}}(F)$ embeds into

$$\mathrm{ADiv}_{\mathbb{Q}}(F) = \varinjlim \mathrm{ADiv}_{\mathbb{Q}}(\mathcal{X}),$$

where $\mathrm{ADiv}_{\mathbb{Q}}(\mathcal{X})$ is the group of arithmetic $\mathbb{Q}$-divisors of C^0-type on $\mathcal{X}$, and the union is taken over the system of all arithmetic varieties $\mathcal{X}$ (i.e., normal, integral, flat and projective over $\mathrm{Spec}(\mathbb{Z})$) with an isomorphism $\kappa(\mathcal{X}) \simeq F$. Moreover, $\mathrm{ADiv}_{\mathbb{Q}}(F)$ has a (divisible) group lattice structure extending the group lattice structure of $\mathrm{LDiv}_{\mathbb{Q}}(F)$.

Let $X = (X, \pi_X)$ be a projective scheme over $\mathbb{Q}$ with a fixed isomorphism $\pi_X : \kappa(X) \to F$. To avoid confusion, we use the name *Zhang divisors/Zhang line bundles* for Cartier divisors on X equipped with adelic Green's functions/line bundles with adelic metrics, in the sense of [18, Remark (4.8)] (originally defined by Zhang [85]). Recall that a *v-adic Green's function g_v* for a divisor D on a variety X, is a continuous function $g_v : X_v^{\mathrm{an}} \setminus \mathrm{supp}(D) \to \mathbb{R}$ such that if $U \subset X$ is a Zariski open subset where D is the divisor of a rational function f, then $g_v + \log|f|_v$ extends to a continuous function on U_v^{an}. A Zhang divisor $\overline{D} = (D, (g_v)_{v\in \mathrm{M}_{\mathbb{Q}}})$ is a divisor D on X equipped with a family of continuous v-adic Green's functions g_v for $v \in \mathrm{M}_{\mathbb{Q}}$, such that there is an arithmetic variety $\mathcal{X}$ over $\mathrm{Spec}(\mathbb{Z})$ with $\mathcal{X} \otimes \mathbb{Q} = X$, and an arithmetic $\mathbb{Z}$-divisor $\overline{\mathcal{D}}$ of C^0-type with $\mathcal{D} \otimes \mathbb{Q} = D$, such that for all but finitely many places g_v coincides with the v-adic Green's function associated to $\overline{\mathcal{D}}$ (see [18, Section 4] for more details).

We denote the group of Zhang divisors on X by $\mathrm{ZDiv}(X)$ and by $\mathrm{ZDiv}_{\mathbb{Q}}(X)$ its tensor with $\mathbb{Q}$. Using the same methods as in Section 2.3.1, one can define a lattice structure on

$$\mathrm{ZDiv}_{\mathbb{Q}}(F) = \varinjlim \mathrm{ZDiv}_{\mathbb{Q}}(X),$$

where the direct limit is taken over the system of all X as above (with maps respecting the isomorphisms π_X). Then the group lattice $\mathrm{LDiv}_{\mathbb{Q}}(F)$ has a natural embedding into $\mathrm{ZDiv}_{\mathbb{Q}}(F)$. Moreover, every GVF functional extends uniquely to a positive functional on $\mathrm{ZDiv}_{\mathbb{Q}}(F)$ (similarly as in Proposition 2.3.17) so we get the following.

Corollary 3.2.1. On a finitely generated field F over $\mathbb{Q}$ there is a natural bijection

$$\big\{\text{GVF functionals } \mathrm{ZDiv}_{\mathbb{Q}}(F) \to \mathbb{R}\big\} \longleftrightarrow \big\{\text{GVF structures on } F\big\},$$

where by GVF functionals we mean the linear ones that are non-negative on the effective cone, and are zero on principal Zhang divisors.

For $\overline{D} \in \mathrm{ZDiv}_{\mathbb{Q}}(X)$ and $x \in X(\overline{\mathbb{Q}})$ we write $h_{\overline{D}}(x)$ for the height of x with respect to $\overline{D}$. We use the same notation for heights with respect to Zhang line bundles.

3.2.2 GVF analytification

Now, we describe a construction which recovers the space of quantifier-free types in the theory of globally valued fields, as in Remark 1.11.17.

Definition 3.2.2. Let K be a GVF and let X be a finite type scheme over K. We define the *GVF analytification* of X over K, denoted $X_{\mathrm{GVF},K}$ (or X_{GVF} if the base GVF is implied), to be the set of couples $x = (\pi(x), h_x)$ where $\pi(x)$ is a point of X, and h_x is a height on $\kappa(\pi(x))$ extending the height on K. If $x \in X_{\mathrm{GVF},K}$ and $U \subset X$ is an open containing $\pi(x)$, we will also denote by h_x the map

$$h_x : \begin{array}{ccc} \mathbb{A}(\mathcal{O}_X(U)) & \to & \mathbb{R} \cup \{-\infty\} \\ (f_1, \dots, f_n) & \mapsto & h_x(f_1(\pi(x)), \dots, f_n(\pi(x))) \end{array}.$$

We equip $X_{\mathrm{GVF},K}$ with the weakest topology such that

1. The map $\pi : X_{\mathrm{GVF},K} \to X$ is continuous onto X with the constructible topology.
2. For every Zariski open $U \subset X$, and every tuple $(f_1, \dots, f_n) \in \mathcal{O}_X(U)^n$, the map $x \mapsto h_x(f_1, \dots, f_n)$ is continuous.

Remark 3.2.3. If $X \subset \mathbb{A}^n_K$ is affine, it follows that $X_{\mathrm{GVF},K}$ can be naturally identified with the space S^{qf}_X of quantifier-free types on X. Since the space of quantifier-free types is locally compact, so is $X_{\mathrm{GVF},K}$.

Remark 3.2.4. Let $K \subset F$ be a GVF extension. Then, there is a canonical *analytification map* $X(F) \to X_{\mathrm{GVF},K}$, defined by taking $x \in X(F)$ to the point (x, h_x), where h_x is the restriction of the height on F to $\kappa(x)$. The image of x by this map will be denoted x^{an}.

3.2.3 Polarisations

Here we present how arithmetic intersection theory can induce GVF structures and how to interpret Yuan's equidistribution result as certain kind of uniqueness of a GVF structure.

Definition 3.2.5. Let F be a finitely generated characteristic zero field equipped with a GVF structure. We say that this GVF structure comes from a *polarisation* $(X, \overline{H}_1, \dots, \overline{H}_d)$, if X is a normal projective variety of dimension d over $\mathbb{Q}$ with function field F and $\overline{H}_i \in \mathrm{ZDiv}_{\mathbb{Q}}(X)$ are arithmetically nef Zhang divisors, such that the GVF functional

$$l : \mathrm{ZDiv}_{\mathbb{Q}}(F) \to \mathbb{R}$$

is given by

$$l(\overline{D}) = \overline{H}_1 \cdot \ldots \cdot \overline{H}_d \cdot \overline{D}.$$

The name "polarisation" comes from [56, Section 3.1], however here we also allow not necessarily model Zhang divisors. We also use the term polarisation if instead of $\overline{H}_i$'s we are given the corresponding Zhang line bundles $\mathcal{O}(\overline{H}_i)$. If the $\overline{H}_i$ occurs with multiplicity k_i we denote by $(X, \overline{H}_1^{k_1}, \ldots, \overline{H}_r^{k_r})$ for the corresponding polarisation.

Remark 3.2.6. A polarisation $(X, \overline{H}_1, \ldots, \overline{H}_d)$ induces a GVF structure structure on F that extends the standard GVF structure on $\mathbb{Q}$ (i.e., satisfies $\mathrm{ht}(2) = \log 2$) if and only if the geometric intersection number satisfies $H_1 \cdot \ldots \cdot H_d = 1$.

Yuan's equidistribution [81] gives examples of polarised GVF structures. Let us recall a version of it from [18, Lemma (8.2)].

Theorem 3.2.7. Let X be a projective variety of dimension d over $\mathbb{Q}$. Fix a semipositive Zhang divisor $\overline{D}$ with D ample and $\overline{D}^{d+1} = 0$. For any generic sequence $x_n \in X(\overline{\mathbb{Q}})$ with $h_{\overline{D}}(x_n) \to 0$, and any $\overline{M} \in \mathrm{ZDiv}_{\mathbb{Q}}(X)$ we have

$$\lim_n h_{\overline{M}}(x_n) = \frac{\overline{D}^d \cdot \overline{M}}{\deg(D)}.$$

For the next two corollaries, fix the context of the above theorem. Also, denote by F the function field of X.

Corollary 3.2.8. There is a unique GVF functional l on F extending the standard one on $\mathbb{Q}$ and satisfying $l(\overline{D}) = 0$.

Proof. Fix a GVF functional l on F with the above properties and a Zhang divisor $\overline{M} \in \mathrm{ZDiv}_{\mathbb{Q}}(F)$. Since the quantities from the assumptions of Theorem 3.2.7 are birational invariant, without loss of generality $\overline{M} \in \mathrm{ZDiv}_{\mathbb{Q}}(X)$. By existential closedness of $\overline{\mathbb{Q}}$ there is a generic sequence of elements $x_n \in X(\overline{\mathbb{Q}})$ such that $h_{\overline{D}}(x_n) \to l(\overline{D}) = 0$ and $h_{\overline{M}}(x_n) \to l(\overline{M})$. By Theorem 3.2.7 we get that

$$l(\overline{M}) = \frac{\overline{D}^d \cdot \overline{M}}{\deg(D)}.$$

On the other hand, l given by this formula on X and its blowups, is a GVF functional, which finishes the proof. □

Corollary 3.2.9. Fix a generic sequence $(x_n)_{n \in \mathbb{N}}$ satisfying the assumptions from Theorem 3.2.7. The sequence $x_n^{\mathrm{an}} \in X_{\mathrm{GVF}}$ converges to the point $x_\infty^{\mathrm{an}} \in X_{\mathrm{GVF}}$ defined by the generic point $x_\infty \in X(F)$ together with the GVF structure l on F. Moreover, l is induced by the polarisation $(X, \overline{D}, \ldots, \overline{D})$.

Yuan's equidistribution theorem can be applied to the case when $X = \mathbb{P}^d_K$ over a number field K and $\mathcal{O}(1)$ endowed with the Weil metrics (denoted by $\overline{\mathcal{O}(1)}$).

Corollary 3.2.10. There is a unique GVF structure on $\overline{\mathbb{Q}}(x_1, \ldots, x_d)$ extending the standard one on $\overline{\mathbb{Q}}$ and satisfying $\mathrm{ht}(x_1) = \cdots = \mathrm{ht}(x_d) = 0$.

Corollary 3.2.11. Fix a number field K. Let $\pi_i : \prod_{i=1}^e \mathbb{P}^d_K \to \mathbb{P}^d_K$ be the i-th projection and let $\overline{L} = \sum_i \pi_i^* \overline{\mathcal{O}(1)}$. Let $x_n \in \prod_{i=1}^e \mathbb{P}^d_K(\overline{\mathbb{Q}})$ be a generic sequence of small points, i.e., satisfying $h_{\overline{L}}(x_n) \to 0$. Let F be the function field of $\prod_{i=1}^e \mathbb{P}^d_K$ with the GVF structure coming from the polarisation $(\prod_{i=1}^e \mathbb{P}^d_K, \pi_1^* \overline{\mathcal{O}(1)}^d, \ldots, \pi_e^* \overline{\mathcal{O}(1)}^d)$. Then

$$\lim_n x_n^{\mathrm{an}} = x_\infty^{\mathrm{an}},$$

where $x_\infty^{\mathrm{an}} \in \mathbb{P}^d_{\mathrm{GVF}}$ is defined by the generic point $x_\infty \in \mathbb{P}^d_K(F)$. Moreover, naturally identifying F with $K(x_{ij} : i \leq d, j \leq e)$, the GVF structure on F is the unique one satisfying $\mathrm{ht}(x_{ij}) = 0$ and $\mathrm{ht}(2) = \log 2$.

3.2.4 Convex geometry and toric varieties

This section contains basic notions on Ronkin metrics on toric varieties and some calculations in convex geometry. For more details on Ronkin metrics we suggest [33]. For an in depth treatment of the Arakelov geometry of toric varieties, see [17].

Let us briefly recall basics of toric varieties here. Let $\mathbb{T} \cong \mathbb{G}_m^n$ be a split torus over a number field K. Denote by N its co-character lattice and by M its character lattice. We denote by $\langle m, u \rangle \in \mathbb{R}$ the natural pairing between $m \in M_\mathbb{R}$ and $u \in N_\mathbb{R}$ (so that $m \circ u(t) = t^{\langle m,u \rangle}$ for $m \in M, u \in N$). A *rational polyhedral cone* σ in $N_\mathbb{R}$ is a cone $\sigma \subset N_\mathbb{R}$ that is generated by a finite set of vectors $v_1, \ldots, v_n \in N$. A *complete fan* Σ in $N_\mathbb{R}$ is a finite collection $\Sigma = \{\sigma\}_{\sigma \in \Sigma}$ of rational polyhedral cones, closed under taking a face (i.e., an intersection with the zero set of a linear form that is non-negative on the ambient cone) of any cone, whose union is $N_\mathbb{R}$, and such that for every two $\sigma, \sigma' \in \Sigma$, their intersection $\tau = \sigma \cap \sigma'$ is a face of both σ and σ'. To a rational polyhedral cone σ one associates a finitely generated semigroup $\sigma^\vee \cap M$ (where $\sigma^\vee := \{m \in M_\mathbb{R} : \langle m, u \rangle \geq 0 \text{ for all } u \in \sigma\}$), and a finitely generated K-algebra $K[\sigma^\vee \cap M]$. For a complete fan Σ these glue to a proper variety X_Σ that contains each $X_\sigma = \mathrm{Spec}(K[\sigma^\vee \cap M])$ as a Zariski open set. In particular, we identify $\mathbb{T} \cong X_0 \subset X_\Sigma$ with the open subset corresponding to the trivial cone $\{0\}$ in Σ. Furthermore, X_Σ is equipped with a natural action of $\mathbb{T}$, and X_0 is the unique open (and dense) orbit of this action.

Definition 3.2.12. A *toric Cartier divisor* on a toric variety X is defined to be a Cartier divisor which is invariant under the action of the torus $\mu : \mathbb{T} \times X \to X$. This means that a Cartier divisor D is toric if $\mu^* D = \pi^* D$, where π denotes the projection map.

Definition 3.2.13. A *virtual support function* or *virtual polytope* with respect to a fan Σ on $N_{\mathbb{R}}$ is a function $N_{\mathbb{R}} \to \mathbb{R}$ that is linear and integral on each cone in Σ.

There is a natural bijection between toric Cartier divisors on X_Σ and virtual support function with respect to a complete fan Σ. If $f : N_{\mathbb{R}} \to \mathbb{R}$ is a virtual support function, that restricted to σ is given by $f(u) = \langle m_\sigma, u \rangle$, then one associates to it a toric Cartier divisor that restricted to X_σ is $\operatorname{div}(\chi^{-m_\sigma})$ (where for $m \in M$, χ^m is the associated affine function on $\mathbb{T}$, which we treat as a rational function on X_σ). For more details and for the proof of this correspondence, see [17, Section 3.3].

Let v be a place of K. A v-adic Green's function g_v for a toric Cartier divisor D is called *toric* if its restriction to $(X_0)^{\text{an}}$ factors through the *tropicalization map* $\operatorname{trop} : (X_0)^{\text{an}} \to N_{\mathbb{R}}$ taking $x \in (X_0)^{\text{an}}$ to $-\log |\chi^m(x)|_v$. For the details of tropicalization maps, see [17, Section 4.1].

Theorem 3.2.14. [17, Theorem 4.8.1.(1)] Let D be the toric divisor associated to the virtual support function Ψ. Then, the space of v-adic Green's functions for D is in bijection with continuous functions $\psi : N_{\mathbb{R}} \to \mathbb{R}$ such that $\psi - \Psi$ is bounded. Concave functions correspond to semipositive metrics under this bijection.

If $\overline{D}_v$ is a toric divisor with a semipositive v-adic Green's function, Legendre-Fenchel duality associates to ψ a concave function

$$\theta_{\overline{D}_v} : x \mapsto \inf_{u \in N_{\mathbb{R}}} (\langle x, u \rangle - \psi(u))$$

on the polytope $\Delta_\Psi \subset M_{\mathbb{R}}$ defined as

$$\Delta_\Psi := \{x \in M_{\mathbb{R}} : \langle x, u \rangle \geq \Psi(u) \text{ for all } u \in N_{\mathbb{R}}\}.$$

The function $\theta_{\overline{D}_v} : \Delta_\Psi \to \mathbb{R}$ is called the *roof function* of $\overline{D}_v$.

A collection of functions $(\psi_v)_{v \in \mathrm{M}_K}$ defines a Zhang metric on the toric divisor corresponding to Ψ if for all v the difference $|\psi_v - \Psi|$ is bounded and for almost all v we have $\psi_v = \Psi$. We call $\overline{D} \in \operatorname{ZDiv}(X_\Sigma)$ a *toric Zhang divisor*, if D is a toric Cartier divisor, and the metrics on $\overline{D}$ come from a collection of functions $(\psi_v)_{v \in \mathrm{M}_K}$ satisfying the above condition.

The vector space $M_{\mathbb{R}}$ carries a Haar measure normalized in such a way that M has covolume 1. We associate to a compact convex set Δ its volume $\mathrm{vol}(\Delta)$ with respect to the Haar measure. Recall that the Minkowski sum of subsets S_1, S_2 of a vector space V is defined by

$$S_1 + S_2 = \{s_1 + s_2 \mid s_1 \in S_1, s_2 \in S_2\}.$$

The volume is a homogeneous polynomial on the space of compact convex sets with Minkowski addition. Therefore, one can consider the following.

Definition 3.2.15. [17, Definition 2.7.14] The *mixed volume* is a multilinear form on compact convex sets defined by

$$\mathrm{MV}(\Delta_1, \dots, \Delta_n) = \sum_{j=1}^{n} (-1)^{n-j} \sum_{1 \le i_1 < \dots < i_j \le n} \mathrm{vol}(\Delta_{i_1} + \dots + \Delta_{i_j}).$$

It satisfies $\mathrm{MV}(\Delta, \dots, \Delta) = n! \, \mathrm{vol}(\Delta)$.

Given a concave function θ on a compact convex set Δ, we associate to it its integral $\int_\Delta \theta(m) dm$. Given concave functions θ_1 on Δ_1 and θ_2 on Δ_2, we define their sup-convolution by

$$\theta_1 \boxplus \theta_2(m) = \sup_{m_1 + m_2 = m} \theta(m_1) + \theta(m_2).$$

It defines a concave function on $\Delta_1 + \Delta_2$. The sup-convolution is defined this way so that for θ_1 and θ_2 positive concave functions the hypograph of $\theta_1 \boxplus \theta_2$ is the Minkowski sum of the hypographs of θ_1 and of θ_2. Similarly as in the case of mixed volumes, one can consider the following.

Definition 3.2.16. [17, Definition 2.7.16] The *mixed integral* is a multilinear form on concave functions $\theta_0, \dots, \theta_n$ on compact convex sets $\Delta_0, \dots, \Delta_n$ defined by

$$\mathrm{MI}(\theta_0, \dots, \theta_n) = \sum_{j=0}^{n} (-1)^{n-j} \sum_{0 \le i_0 < \dots < i_j \le n} \int_{\Delta_{i_1} + \dots + \Delta_{i_j}} \theta_{i_1} \boxplus \dots \boxplus \theta_{i_j}(m) dm.$$

It satisfies $\mathrm{MI}(\theta, \dots, \theta) = (n+1)! \int_\Delta \theta(m) dm$.

These notions are helpful to express arithmetic intersection numbers combinatorially.

Theorem 3.2.17. [17, Theorem 5.2.5] Let $\overline{D}_0, \ldots, \overline{D}_n$ be semipositive toric Zhang divisors on X_Σ. Then,

$$\widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_n | X_\Sigma) = \sum_{v \in \mathrm{M}_K} n_v \, \mathrm{MI}_M(\theta_{0,v}, \ldots, \theta_{n,v}),$$

where $\theta_{i,v}$ is the roof function of $\overline{D}_{i,v}$, for every $i = 0, \ldots, n$ and $v \in \mathrm{M}_K$.

For a non-zero Laurent polynomial $f \in K[M]$, we denote by $V(f)$ its vanishing locus on X_Σ. Let $\mathrm{NP}(f)$ be the Newton polytope of $f = \sum_m c_m \chi^m$, i.e., the convex hull of the positions m of its non-zero coefficients $c_m \neq 0$. Then, $\mathrm{NP}(f)$ defines a Cartier divisor on a suitable toric modification of X_Σ such that f gives rise to a regular section (see [17, Section 3.4] and [33, Lemma 5.1]). It vanishes precisely on $V(f)$ by [33, Theorem 4.3]. We now define the Ronkin metric on the divisor $D_{\mathrm{NP}(f)}$ corresponding to $\mathrm{NP}(f)$. For this we use that the fibers $\mathrm{trop}^{-1}(u)$ of the tropicalization map $\mathrm{trop} : \mathbb{T}_v^{\mathrm{an}} \to N_\mathbb{R}$ carry a natural choice of probability measure σ_u. In the non-archimedean case, it is concentrated on the Gauss point over u. In the archimedean case, it is induced by the Haar measure on $(S^1)^n$.

Definition 3.2.18. [33, Definition 2.7] Let f be a non-zero Laurent polynomial over K. The *Ronkin function* of f over a place v is the map $\rho_f : N_\mathbb{R} \to \mathbb{R}$ defined by

$$\rho_f : u \mapsto \int_{\mathrm{trop}^{-1}(u)} -\log |f(x)| d\sigma_u(x).$$

By [33, Proposition 2.10], ρ_f is a concave continuous function with bounded difference from $\Psi_{\mathrm{NP}(f)}$. By Theorem 3.2.14, it defines a semipositive Green's function for $D(f)$. The collection of v-adic Ronkin functions gives rise to a Zhang divisor R_f by [33, Lemma 5.11].

Theorem 3.2.19. (variation of [33, Theorem 5.12]) Let X_Σ be a proper toric variety. Let f be a Laurent polynomial with vanishing locus Z such that $\mathrm{NP}(f)$ defines a divisor on X_Σ. Let $\overline{D}_0, \ldots, \overline{D}_{n-1}$ be semipositive toric Zhang divisors on X_Σ. Then,

$$\widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_{n-1} | Z) = \widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_{n-1}, R_f | X_\Sigma).$$

Definition 3.2.20. Let $\gamma : V \to W$ be a homomorphism of finite dimensional real vector spaces and $f : V \to \mathbb{R} \cup \{-\infty\}$ be a closed concave function with compact support. We define the *direct image* of f along γ by

$$\gamma_* f(w) = \max_{v \in \gamma^{-1}(w)} f(v).$$

It is a closed concave function with compact domain in W.

Lemma 3.2.21. Let $\gamma : M \to M'$ be a map of lattices. Then, this induces a map on the rings of Laurent polynomials $K[\gamma] : K[M] \to K[M']$. Let $f \in K[M] \setminus \{0\}$. Then,

$$\rho_{K[\gamma](f)} = \rho_f \circ \gamma^\vee$$

and

$$\rho^\vee_{K[\gamma](f)} = \gamma_* \rho^\vee_f(m).$$

Proof. The first statement follows readily from the definitions. The second statement follows from the first using [17, Proposition 2.3.8(1)]. $\square$

We now prove generalizations of Lemma 1.11 and Proposition 1.12 in [33]. We apply this to write the formulas in the preceding section in the form used in [32, Conjecture 1].

Lemma 3.2.22. Let $\Delta_1, \dots, \Delta_k$ be polytopes contained in a k-dimensional rational subspace L and denote by π the projection away from this subspace to the quotient P. Let $Q_1, \dots, Q_{n-k}$ be polytopes in $M_\mathbb{R}$. Then,

$$\mathrm{MV}_M(\Delta_1, \dots, \Delta_k, Q_1, \dots, Q_{n-k}) = \mathrm{MV}_L(\Delta_1, \dots, \Delta_k) \cdot \mathrm{MV}_P(\pi(Q_1), \dots, \pi(Q_{n-k})).$$

Proof. By the definition of mixed volume one obtains

$$\begin{aligned}&\mathrm{MV}_M(\Delta_1, \dots, \Delta_k, Q_1, \dots, Q_{n-k})\\ &= \sum_{j=1}^{n-k} (-1)^{n-j} \sum_{1 \le i_1 < \dots < i_j \le n-k} \sum_{I \subset \{1,\dots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + Q_{i_1} + \dots + Q_{i_j}).\end{aligned}$$

Here Δ_I is taken to denote $\sum_{i \in I} \Delta_i$. It now suffices to show that

$$\sum_{I \subset \{1,\dots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + Q) = (-1)^k \,\mathrm{MV}_L(\Delta_1, \dots, \Delta_k) \operatorname{vol}(\pi(Q)).$$

For this take any $p \in P$ and denote by Q_p the preimage of p in Q. We may view $\sum_{I \subset \{1,\dots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + Q)$ as an integral over $\pi(Q)$, namely as

$$\int_{\pi(Q)} \sum_{I \subset \{1,\dots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + Q_p).$$

In order to conclude, we need to show that $\sum_{I \subset \{1,\dots,k\}} (-1)^{|I|} \operatorname{vol}(\Delta_I + R) = \mathrm{MV}_L(\Delta_1, \dots, \Delta_k)$ for any polytope R in the k-dimensional subspace spanned by the Δ_i. For this we decompose the expression by writing out the volumes as mixed volumes and order by the number of R occurring in the expansion.

$$\begin{aligned}&\sum_{I\subset\{1,\dots,k\}}(-1)^{|I|}\operatorname{vol}(\Delta_I+R)\\&=\sum_{s=0}^{k}\binom{k}{s}\sum_{I\subset\{1,\dots,k\}}(-1)^{|I|}\sum_{1\le j_1\le\cdots\le j_{k-s}\le k, j_i\in I}\binom{k-s}{J}\operatorname{MV}(\Delta_{j_1},\dots,\Delta_{j_{k-s}},R,\dots,R).\end{aligned}$$

Here $\binom{k-s}{J}$ is taken to denote the number of partitions of $k-s$ elements into partitions of form J, i.e., the multinomial coefficient for $k-s$ and $\#\{i|j_i=m\}$. We reorder the sum to sum over size $k-s$ multisets of $\{1,\dots,k\}$. Fix a multiset $1\le j_1\le\cdots\le j_{k-s}\le k$ of order $k-s$ in elements of $\{1,\dots k\}$. Then, its contribution to the sum is

$$\binom{k}{s}\binom{k-s}{J}\operatorname{MV}(\Delta_{j_1},\dots,\Delta_{j_{k-s}},R,\dots,R)\sum_{I\supseteq J}(-1)^{|I|},$$

where containment is understood on the level of underlying sets. By comparing to the expansion of $\prod_{I\setminus J}(1-1)$ we see that this vanishes for $J\neq I$ and is $(-1)^k$ for $J=I$. □

Lemma 3.2.23. Let the g_i be concave functions on polytopes Q_i and Δ_i as before. Then,

$$\operatorname{MI}_M(\iota_{\Delta_1},\dots,\iota_{\Delta_k},g_1,\dots,g_{n-k+1})=\operatorname{MV}_L(\Delta_1,\dots,\Delta_k)\cdot\operatorname{MI}_P(\pi_*g_1,\dots,\pi_*g_{n-k+1}).$$

Proof. The reduction to the previous Lemma is precisely as in [33, Proposition 1.12]. □

3.3 Intersection product

In this section we study the intersection product defined in [25] and prove that it varies continuously in flat families. We will treat the case of a trivial GVF in a separate section as the natural class of line-bundles differs from the non-trivial setting. Our main result is the following theorem.

Theorem 3.3.1. (Later appears as Theorem 3.3.23) Let $\mathcal{X}\to S$ be a flat projective morphism of finite type schemes over a GVF K of relative dimension d. Let

$\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d \in \widehat{\operatorname{Pic}}_{\mathbb{Q}}^{\operatorname{int}}(\mathcal{X}/S)$ be globally integrable line bundles on $\mathcal{X}$ over S. Then, the map

$$\begin{aligned} S_{\mathrm{GVF}} &\to \mathbb{R} \\ s &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d | \mathcal{X}_s) \end{aligned}$$

is continuous.

The above theorem can be applied in various settings of interest to number theorists because of the following.

Corollary 3.3.2. Suppose that S is projective over $\mathbb{Q}$ and $\mathcal{X} \to S$ a projective morphism. Then, any integrable Zhang line bundle on $\mathcal{X}$ is a globally integrable line bundle in $\widehat{\operatorname{Pic}}_{\mathbb{Q}}^{\operatorname{int}}(\mathcal{X}/S)$.

Proof. It is a weaker property to be an element of $\widehat{\operatorname{Pic}}_{\mathbb{Q}}^{\operatorname{int}}(\mathcal{X}/S)$ than of $\widehat{\operatorname{Pic}}_{\mathbb{Q}}^{\operatorname{int}}(\mathcal{X})$. Hence, the statement follows from Theorem 3.3.24. ☐

Remark 3.3.3. The property of being globally integrable is preserved under base change. In particular, one may apply Theorem 3.3.1 to integrable Zhang divisors on a projective $\mathcal{X}$ after restricting to the flat locus of $\mathcal{X} \to S$.

3.3.1 Lattice divisors

Let K be a countable GVF. We can assume that it is represented by a proper adelic curve $(K, (\Omega, \mathcal{A}, \nu), \phi)$ with $\Omega = M_K$, the trivial absolute value having zero mass, and the restriction of the measure to the archimedean places $\nu|_{\Omega_\infty}$ being supported at normalized valuations (i.e., satisfying $v(2) = -\log 2$). Moreover, whenever we consider a GVF extension $K \subset F$ in this subsection, we assume that F is also countable and the GVF structure on F is induced by an adelic curve structure with the same properties (see Section 1.9).

We recall ideas from the theory of adelic curves only briefly. We refer to [24, 25] for details. Let us start by recalling a definition.

Definition 3.3.4. Let X be a finite type K-scheme with a line bundle L. A *metric family* on L is a family $\varphi = (\varphi_\omega)_{\omega \in \Omega}$, where each φ_ω is a continuous metric on L_ω on X_ω^{an}. Here X_ω^{an} is the Berkovich analytification of $X_\omega = X \otimes_K K_\omega$, where K_ω is the completion of K with respect to the absolute value $\omega \in \Omega = M_K$. We call a pair $\overline{L} = (L, \varphi)$ a *metrized line bundle* on X over K.

This is as in [25, Definition 4.1.4], but we also allow non-projective X. Also, we naturally extend this definition to $\mathbb{Q}$-line bundles. Similarly, we extend the definition of a *Green's function family* of a ($\mathbb{Q}$-)Cartier divisor on X from [25, Definition 4.2.1], to a non-projective X, word for word. A ($\mathbb{Q}$-)Cartier divisor with a Green's function family is called a *metrized divisor*.

Chen and Moriwaki introduce *adelic line bundles* as a subset of metrized line bundles respecting the global nature of the adelic curve, see [25, Definition 4.1.9]. There are two conditions. Firstly, the variation of metrics along ω has to be measurable. This condition does not occur for number fields since their set of places is discrete. Finally, one needs a condition requiring that the family of metrics has finite distance to arising from a global model. This condition is referred to as being dominated. In order to obtain an intersection theory, one needs to demand the metrics at each place to be *integrable*, i.e. the difference of *semipositive* metrics. We follow the definition of semipositivity from [24, Section 2.3]. Note that this only allows for semipositive metrics on semiample line bundles.

Definition 3.3.5. Consider $\mathbb{P}^n = \mathbb{P}^n_K$ with coordinates $x_0, \dots, x_n$. We equip the anti-tautological line bundle $\mathcal{O}(1)$ with two families of metrics $\varphi = (|\cdot|_{\phi_\omega})_{\omega\in\Omega}, \psi = (|\cdot|_{\psi_\omega})_{\omega\in\Omega}$ defined in the following way. For a section $s \in H^0(\mathbb{P}^n, \mathcal{O}(1))$ identified with a linear form A we have

$$|s(z)|_{\varphi_\omega} := \frac{|A(z)|_\omega}{\max(|z_0|_\omega, \dots, |z_n|_\omega)},$$

for $z \in \mathbb{P}^{n,\mathrm{an}}_\omega$ and any $\omega \in \Omega$. The metric ψ is defined in the same way for non-archimedean ω, but for archimedean ω we set

$$|s(z)|_{\psi_\omega} := \frac{|A(z)|_\omega}{\sqrt{\sum_{i=0}^n |z_i|_\omega^2}},$$

for all $z \in \mathbb{P}^{n,\mathrm{an}}_\omega$. We call φ, ψ the Weil and the Fubini-Study metric respectively and use the notation $\overline{\mathcal{O}(1)} = (\mathcal{O}(1), \varphi), \overline{\mathcal{O}(1)}^{\mathrm{FS}} = (\mathcal{O}(1), \psi)$. We use the same notation for pullbacks of these adelic line bundles on $\mathbb{P}^n_S$ for any finite type K-scheme S. Both the Weil and the Fubini-Study metric define semipositive adelic line bundles.

For a place $\omega \in \Omega$ and Green's functions ϕ, ψ for a divisor D we denote by $d_\omega(\phi, \psi)$ the sup-norm of $\phi - \psi$ and call it the local distance of ϕ and ψ. For different metrics on a divisor over an adelic curve we denote by $d(\phi, \psi)$ the global distance of ϕ and ψ. It is defined as the upper integral $\int^+ d_\omega(\phi_\omega, \psi_\omega)\nu(d\omega)$ over the local distances (see [24, Definition A.4.1]).

From the point of view of globally valued fields the Weil metric is the fundamental object. We need to relate it to the Fubini-Study height to invoke calculations from that setting.

Lemma 3.3.6. Let $\alpha_n : \mathbb{P}^r \to \mathbb{P}^s$ be the n-th Veronese map. Consider two metrics on the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^r$, namely the Weil metric φ and the metric $\sqrt[n]{\alpha_n^*\psi}$ which is the n-th root of the pullback of the Fubini-Study metric from $\mathcal{O}(1)$ on $\mathbb{P}^s$. Then for non-archimedean places the two metrics are the same and for archimedean places $\sigma \in \Omega$ we have

$$d_\sigma(\varphi, \sqrt[n]{\alpha_n^*\psi}) \leq \frac{r \log n}{n},$$

for $n \geq r+1$.

Proof. We only calculate the archimedean case. Consider the section x_0 of $\mathcal{O}(1)$ on $\mathbb{P}^r$. By definition we have

$$|x_0(z)|_\varphi = \frac{|z_0|_\sigma}{\max(|z_0|_\sigma, \ldots, |z_n|_\sigma)}$$

and

$$|x_0(z)|_{\sqrt[n]{\alpha_n^*\psi}} = \sqrt[n]{\frac{|z_0|_\sigma^n}{\sqrt{\sum_{|I|=n} |z^I|_\sigma^2}}} = \frac{|z_0|_\sigma}{\sqrt[2n]{\sum_{|I|=n} |z^I|_\sigma^2}}.$$

Hence we calculate

$$\left| -\log \frac{|x_0(z)|_\varphi}{|x_0(z)|_{\sqrt[n]{\alpha_n^*\psi}}} \right| = \left| \log \frac{\max(|z_0|_\sigma, \ldots, |z_n|_\sigma)}{\sqrt[2n]{\sum_{|I|=n} |z^I|_\sigma^2}} \right|.$$

This is bounded by

$$|\log \sqrt[2n]{s+1}| = \left| \log \sqrt[2n]{\binom{r+n}{n}} \right| \leq \frac{1}{2n} |\log(r+1)n^r| \leq \frac{r \log n}{n},$$

where in the first inequality we use the fact that $\binom{r+n}{n} \leq (r+1)n^r$ and the second inequality holds for $n \geq r+1$. □

Lemma 3.3.7. Let $f : Q \to P$ be a morphism of projective schemes over K and let $\overline{L}$ be an adelic line bundle on P. Then $f^*\overline{L}$ is an adelic line bundle on Q.

If $\overline{L}$ is semipositive or integrable, so is $f^*\overline{L}$.

Proof. The fact that $f^*\overline{L}$ is an adelic line bundle is found in [26, Section 2.8.3 and 2.9.5]. The semipositivity assertion follows from [26, Lemma 6.1.2]. □

Fix a projective morphism $\pi : \mathcal{X} \to S$ of finite-type K-schemes, where S is not necessarily projective.

Definition 3.3.8. We say that a metrized family $\overline{\mathcal{L}}$ on $\mathcal{X}$ is *simple over* S, if there is a closed embedding $j : \mathcal{X} \to \mathbb{P}^n_S$ over S (for some n), such that $\overline{\mathcal{L}} = j^*\overline{\mathcal{O}(1)}$. The elements of the $\mathbb{Q}$-vector space of metrized $\mathbb{Q}$-line bundles on $\mathcal{X}$ generated by simple ones are called *lattice line bundles on* $\mathcal{X}$ *over* S and are denoted by $\mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)$. The space of metrized $\mathbb{Q}$-divisors coming from rational sections of such metrized $\mathbb{Q}$-line bundles is denoted by $\mathrm{LDiv}_{\mathbb{Q}}(\mathcal{X}/S)$ and its elements are called *lattice divisors on* $\mathcal{X}$ *over* S. If S is equal to $\mathrm{Spec}(K)$, we omit it in the notation. Moreover, if we want to emphasize the dependence on the GVF K, we use the notation $\mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)_K$.

Remark 3.3.9. It follows from Lemma 3.3.7 that simple metrized line bundles on $\mathcal{X}$ over S are semipositive on fibers. More precisely, if $K \subset F$ is a GVF extension, and $s \in S(F)$, then a simple metrized line bundle $\overline{\mathcal{L}}$ over S coming from an embedding $j : \mathcal{X} \to \mathbb{P}^n_S$ induces an adelic line bundle $\overline{\mathcal{L}}_s = j_s^*\overline{\mathcal{O}(1)}$ which is semipositive on $\mathcal{X}_s$ with respect to the GVF F.

We remark that for any GVF extension F/K, a finite type F-scheme T, and a morphism of K-schemes $T \to S$, there is a base-change map

$$\mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)_K \to \mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}_T/T)_F.$$

In particular for $s \in S(F)$, there is a specialisation map

$$\mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)_K \to \mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}_s)_F.$$

Let us describe these maps more precisely. Let $\overline{\mathcal{L}} = (\mathcal{L}, \varphi) \in \mathrm{LPic}_{\mathbb{Q}}(\mathcal{X}/S)_K$, for $\varphi = (\varphi_\omega)_{\omega \in M_K}$. This means that each φ_ω is a metric on $\mathcal{L}_\omega$ over $\mathcal{X}^{\mathrm{an}}_\omega$. To get a family of metrics $\psi = (\psi_v)_{v \in M_F}$ on the base-change $\mathcal{L}_T$ of $\mathcal{L}$ via the morphism $\mathcal{X}_T \to \mathcal{X}$ one proceeds as in [25, Example 4.1.8] (which works the same when $K' = F/K$ is not algebraic). Equivalently, one could take an embedding $j : \mathcal{X} \to \mathbb{P}^n_S$ that realises $\overline{\mathcal{L}} = j^*\overline{\mathcal{O}(1)}$ (or two embeddings such that it comes from the difference of pullbacks) and define $(\mathcal{L}_T, \psi)$ via the pullback of $\overline{\mathcal{O}(1)}$ through the map $j_T : \mathcal{X}_T \to \mathbb{P}^n_T$.

3.3.2 Heights of resultants

Chen and Moriwaki have constructed an intersection product of integrable adelic Cartier divisors in [25]. In this subsection we look closely at its definition which uses heights of certain resultants.

Theorem 3.3.10. [25, Theorem B] Let X be a projective scheme of pure dimension d over a GVF K. Then, there is a multilinear *adelic intersection product*

$$\mathrm{LPic}_{\mathbb{Q}}(X)^{d+1} \to \mathbb{R}.$$

Proof. Given finitely many elements of $\mathrm{LPic}_{\mathbb{Q}}(X)$ we can replace K by a countable subfield over which the corresponding embeddings to projective spaces (and X) are defined. Then we can represent the GVF K by an adelic curve. Since lattice line bundles are integrable, an arithmetic intersection number is defined by [25, Theorem B]. We show that the product on $\mathrm{LPic}_{\mathbb{Q}}(F)$ only depends on the induced GVF structure on K in Corollary 3.3.15. □

We write the intersection number of lattice line bundles $\overline{L}_0, \ldots, \overline{L}_d$ as $\overline{L}_0 \cdots \overline{L}_d$. We observe that the adelic intersection product is determined by its values on tuples of simple line bundles. Let us fix a tuple of such simple adelic Cartier divisors on a projective scheme X and analyse how to calculate their intersection.

Assume we are given closed embeddings $\xi_i : X \to \mathbb{P}^{r_i} = \mathbb{P}(V_i)$ for $i = 0, \ldots, d$, where V_i is a (r_i+1)-dimensional vector space over K with a distinguished basis. For a natural number n, denote by $\xi_i^{\otimes n} : X \to \mathbb{P}^{r_i(n)}$ the composition of ξ_i with the n-th Veronese map $\mathbb{P}^{r_i} = \mathbb{P}(V_i) \to \mathbb{P}(S^n V_i) = \mathbb{P}^{r_i(n)}$ and write $V_i(n) := S^n V_i$. Note that in this case we have

$$\dim V_i(n) = r_i(n) + 1 = \binom{r_i + n}{n} = O(n^{r_i}).$$

For each n we define the line bundle $L_i(n)$ to be the pullback $\xi_i^{\otimes n,*}\mathcal{O}(1)$. We pull back the Weil metric and the Fubini-Study metric to obtain adelic line bundles $\overline{L_i(n)}$ and $\overline{L_i(n)}^{\mathrm{FS}}$ respectively. For $n = 1$, we omit the n in the notation. We note that there is a canonical isomorphism $L_i(n) \cong L_i^{\otimes n}$.

Let $\delta_i(n)$ be the intersection number $L_0(n) \cdots L_{i-1}(n) \cdot L_{i+1}(n) \cdots L_d(n)$ and set $\delta_i = \delta_i(1)$. Since $L_i(n) \cong L_i^{\otimes n}$, we have $\delta_i(n) = n^d \cdot \delta_i$. Let

$$W(n) := S^{\delta_0(n)}(V_0(n)^\vee) \otimes_K \ldots \otimes_K S^{\delta_d(n)}(V_d(n)^\vee)$$

and note that using distinguished bases of V_i for $i = 0, \ldots, d$ we can naturally interpret elements of $W(n)$ as polynomials of multi-degree $(\delta_0(n), \ldots, \delta_d(n))$ on $V_0(n) \times \cdots \times V_d(n)$. There is a unique (up to scaling) element $R_n \in W(n)$ such that it vanishes on $(v_0, \ldots, v_d)$ if and only if the intersection $X \cap Z(v_0) \cap \cdots \cap Z(v_d)$ is non-empty (as a scheme), where $Z(v_i)$ is the pullback to X of the hyperplane in $\mathbb{P}(V_i(n))$ defined by the zero-set of the linear form v_i. We call it the resultant of X with respect to embeddings $\xi_i^{\otimes n}$. It determines a unique element $R_n \in \mathbb{P}(W(n))$ whose height calculates the adelic intersection product in the following way.

Remark 3.3.11. [25, Remark 4.2.13]

$$\overline{L_0(n)}^{\mathrm{FS}} \cdot \ldots \cdot \overline{L_d(n)}^{\mathrm{FS}} = \int_{\Omega \setminus \Omega_\infty} \log \|R_n\|_\omega \nu(d\omega)$$

$$+ \int_{\Omega_\infty} \nu(d\sigma) \int_{\mathbb{S}_0(n)_\sigma \times \cdots \times \mathbb{S}_d(n)_\sigma} \log |(R_n)_\sigma(z_0, \ldots z_d)| \eta_{\mathbb{S}_0(n)_\sigma}(dz_0) \otimes \cdots \otimes \eta_{\mathbb{S}_d(n)_\sigma}(dz_d)$$

$$+\nu(\Omega_\infty) \frac{1}{2} \sum_{i=0}^{d} \delta_i(n) \sum_{l=1}^{r_i(n)} \frac{1}{l},$$

where we use the notation from the cited remark, but with an additional variable n. This means that $\mathbb{S}_i(n)_\sigma$ is the unit sphere in $V_i(n)_\sigma$ with the sphere measure $\eta_{\mathbb{S}_i(n)_\sigma}$. Moreover, for a non-archimedean $\omega \in \Omega$, the norm $\| \cdot \|_\omega$ is the maximum of coefficients norm, with respect to the distinguished basis of $W(n)$, for example by [26, Proposition A.2.2]. Later we write $\eta(dz)$ for $\eta_{\mathbb{S}_0(n)_\sigma}(dz_0) \otimes \cdots \otimes \eta_{\mathbb{S}_d(n)_\sigma}(dz_d)$ and z for the tuple $z_0, \ldots, z_d$ (here n, d and σ are implicit).

Lemma 3.3.12. The adelic intersection product satisfies

$$\lim_n \frac{1}{n^{d+1}} \overline{L_0(n)}^{\mathrm{FS}} \cdot \ldots \cdot \overline{L_d(n)}^{\mathrm{FS}} = \overline{L_0(n)} \cdot \ldots \cdot \overline{L_d(n)}.$$

Proof. It suffices to show that the metrics on $\frac{1}{n}\overline{L_i(n)}^{\mathrm{FS}}$ converge with respect to the global distance to the metrics on $\overline{L_i}$. By Lemma 3.3.6 the global distance satisfies $d(\frac{1}{n}\overline{L_i(n)}^{\mathrm{FS}}, \overline{L_i}) \leq \nu(\Omega_\infty) \cdot \frac{r_i \log n}{n}$. □

We use the above formula and the lemma to calculate $\overline{D}_0 \cdot \ldots \cdot \overline{D}_d$ through resultants. More precisely we show the following.

Proposition 3.3.13. In the above context we have

$$\overline{L}_0 \cdot \ldots \cdot \overline{L}_d = \lim_n \frac{1}{n^{d+1}} \operatorname{ht}(R_n),$$

where we treat R_n's as tuples, using the distinguished basis of $W(n)$.

Proof. By Lemma 3.3.12 we only need to show that

$$|\overline{L_0(n)}^{\mathrm{FS}} \cdot \ldots \cdot \overline{L_d(n)}^{\mathrm{FS}} - \operatorname{ht}(R_n)| = o(n^{d+1}).$$

First we express $\operatorname{ht}(R_n)$ in a form of an integral

$$\operatorname{ht}(R_n) = \int_\Omega \log \|R_n\|_\omega \nu(d\omega)$$

$$= \int_{\Omega \setminus \Omega_\infty} \log \|R_n\|_\omega \nu(d\omega) + \int_{\Omega_\infty} \log \|R_n\|_\sigma \nu(d\sigma),$$

where $\| \cdot \|_\omega, \| \cdot \|_\sigma$ denote the maximum of coefficients norms (for a non-archimedean $\omega \in \Omega$ or archimedean $\sigma \in \Omega_\infty$), with respect to the distinguished basis of $W(n)$.

Claim 3.3.14. We have

$$\nu(\Omega_\infty) \frac{1}{2} \sum_{i=0}^{d} \delta_i(n) \sum_{l=1}^{r_i(n)} \frac{1}{l} = O(n^d \log n).$$

In particular, when divided by n^{d+1} converges to zero, for $n \to \infty$.

Proof. This follows from the fact that $\delta_i(n) = n^d \delta_i$ and

$$\sum_{l=1}^{r_i(n)} \frac{1}{l} = O(\log r_i(n)) = O(\log n^{r_i}) = O(\log n).$$

□

By Remark 3.3.11 we will be done if we show that

$$\Big| \int_{\mathbb{S}_0(n)_\sigma \times \cdots \times \mathbb{S}_d(n)_\sigma} \log |(R_n)_\sigma(z)| \eta(dz) - \log \|R_n\|_\sigma \Big| = o(n^{d+1}),$$

where the constant is independent of $\sigma \in \Omega_\infty$. But by Proposition 3.5.1 applied to the polynomial $(R_n)_\sigma$, we have

$$\Big| \int_{\mathbb{S}_0(n)_\sigma \times \cdots \times \mathbb{S}_d(n)_\sigma} \log |(R_n)_\sigma(z)| \eta(dz) - \log \|R_n\|_\sigma \Big| \leqslant \sum_{i=0}^{d} \delta_i(n) \left(\log(r_i(n)+1) + \sum_{k=1}^{r_i(n)-1} \frac{1}{k} \right),$$

where $\delta_i(n) = n^d \delta_i$ and $\log(r_i(n)+1) + \sum_{k=1}^{r_i(n)-1} \frac{1}{k} = O(\log(r_i(n))) = O(\log n)$ for all $i \leqslant d$, so

$$\sum_{i=0}^{d} \delta_i(n) \left(\log(r_i(n)+1) + \sum_{k=1}^{r_i(n)-1} \frac{1}{k} \right) = O(n^d \log n) = o(n^{d+1}),$$

where the given bound does not depend on σ. □

Corollary 3.3.15. The intersection product on $\mathrm{LPic}_{\mathbb{Q}}(X)$ only depends on the induced GVF structure on K.

Remark 3.3.16. By analysing precisely the proof of Proposition 3.3.13, one can see that in fact it shows existence of an absolute constant C such that

$$\left| \overline{L}_0 \cdot \ldots \cdot \overline{L}_d - \frac{1}{n^{d+1}} \mathrm{ht}(R_n) \right| \le C \cdot (1 + \nu(\Omega_\infty)) \cdot \max_i r_i \delta_i \cdot \frac{\log n}{n}.$$

Note that the number $\nu(\Omega_\infty)$ can be expressed as $\frac{\mathrm{ht}(2)}{\log 2}$ with respect to the induced GVF structure on K.

3.3.3 Definability of adelic intersection product

In this section we prove Theorem 3.3.1 for lattice line bundles. Let $\pi : \mathcal{X} \to S$ be a flat projective morphism with d-dimensional fibers. Suppose $S = \operatorname{Spec} A$ is an affine variety (so that A is an integral domain).

Let $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d \in \operatorname{LPic}(\mathcal{X}/S)$ be simple over S with embeddings $\alpha_i : \mathcal{X} \to \mathbb{P}_S^{k_i}$. For a field-valued point $s \in S(F)$ and any object Q over S, the notation $Q(s)$ denotes its base change to s. Denote by δ_i the intersection number

$$\deg(\mathcal{L}_0(s) \cdot \ldots \cdot \mathcal{L}_{i-1}(s) \cdot \mathcal{L}_{i+1}(s) \cdot \ldots \cdot \mathcal{L}_d(s) | \mathcal{X}_s)$$

for any $s \in S(F)$ in any field extension $K \subset F$. It is independent of the choice of s since the family $\pi : \mathcal{X} \to S$ is flat.

Lemma 3.3.17. There is a family of polynomials R_n with coefficients in A defined up to scalar in $A^\times$, such that for all $s \in S(F)$ we have

$$R_n(s) = R_{n,s},$$

where $R_{n,s}$ is the resultant R_n from Subsection 3.3.2, defined for the scheme $\mathcal{X}_s$ equipped with the family of embeddings $(\alpha_i)_s : \mathcal{X}_s \to \mathbb{P}^{k_i}_{\kappa(s)}$ for $i = 0, \dots, d$.

Proof. This is probably standard, but we could not find a reference so we sketch a proof here, based on the construction of resultants from [25, Section 1.6]. We use notation from loc.cit but with the base-field k replaced by A, X over k replaced by $\mathcal{X}$ over $S = \operatorname{Spec}(A)$, and L_i's replaced by $\mathcal{L}_i$'s.

First note that the whole Section 1.5 and Section 1.6 up to Proposition 1.6.2 of [25] go through word-for-word over A. It remains to prove the analogue of [25, Proposition 1.6.2] over A. This boils down to calculating the cycles

$$q_*(c_1(p^*\mathcal{L}_0) \cdots c_1(p^*\mathcal{L}_{i-1}) c_1(q^* q_i^*(\mathcal{O}_{E_i^\vee}(1))) c_1(p^*\mathcal{L}_{i+1}) \cdots c_1(p^*\mathcal{L}_d) \cap [\mathcal{X} \times_S \check{\mathbb{P}}])$$
$$= c_1(q_i^*(\mathcal{O}_{E_i^\vee}(1))) \cdot q_*(c_1(p^*\mathcal{L}_0) \cdots c_1(p^*\mathcal{L}_{i-1}) c_1(p^*\mathcal{L}_{i+1}) \cdots c_1(p^*\mathcal{L}_d) \cap [\mathcal{X} \times_S \check{\mathbb{P}}])$$

We look at the diagram

$$\begin{tikzcd} \mathcal{X} \times_S \check{\mathbb{P}} \arrow[r, "p"'] \arrow[d, "q"] & \mathcal{X} \arrow[d, "r"'] \\ \check{\mathbb{P}} \arrow[r, "s"] & S \end{tikzcd}$$

and use flat base-change to get the equality

$$q_*(c_1(p^*\mathcal{L}_0) \cdots c_1(p^*\mathcal{L}_{i-1}) c_1(p^*\mathcal{L}_{i+1}) \cdots c_1(p^*\mathcal{L}_d) \cap [\mathcal{X} \times_S \check{\mathbb{P}}])$$

$$= s^*r_*(c_1(\mathcal{L}_0)\cdots c_1(\mathcal{L}_{i-1})c_1(\mathcal{L}_{i+1})\cdots c_1(\mathcal{L}_d)\cap[\mathcal{X}]).$$

Let η be the generic point of S. By the flat base-change for the localisation map $\eta \to S$, we get

$$\begin{aligned} &r_*(c_1(\mathcal{L}_0)\cdots c_1(\mathcal{L}_{i-1})c_1(\mathcal{L}_{i+1})\cdots c_1(\mathcal{L}_d)\cap[\mathcal{X}]) \\ &\quad= \deg(\mathcal{L}_0(\eta)\cdots\mathcal{L}_{i-1}(\eta)\mathcal{L}_{i+1}(\eta)\cdots\mathcal{L}_d(\eta))[S] \end{aligned}$$

which is equal to $\delta_i \cdot [S]$. Hence the cycle in question is equal to

$$c_1(q_i^*(\mathcal{O}_{E_i^\vee}(1)))s^*(\delta_i\cdot[S]) = c_1(q_i^*(\mathcal{O}_{E_i^\vee}(1)))\cap\delta_i[\check{\mathbb{P}}] = c_1(q_i^*(\mathcal{O}_{E_i^\vee}(\delta_i)))\cap[\check{\mathbb{P}}],$$

which finishes the proof as in [25, Proposition 1.6.2].

□

Proposition 3.3.18. Let $\mathcal{X}\to S$ be a flat projective morphism of finite type schemes over a GVF K of relative dimension d. Let $\overline{\mathcal{L}}_0,\dots,\overline{\mathcal{L}}_d \in \mathrm{LPic}(\mathcal{X}/S)$ be lattice line bundles on X over S. Then, the map

$$\begin{aligned} S_{\mathrm{GVF}} &\to \mathbb{R} \\ s &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0(s),\dots,\overline{\mathcal{L}}_d(s)|\mathcal{X}_s) \end{aligned}$$

is continuous.

Proof. By Remark 3.2.3 (continuity with respect to the constructible topology) we may without loss of generality assume that S is an affine variety and the line bundles are simple over the base S.

Fix a net $(s_i)_i \in S_{\mathrm{GVF}}$ and $s\in S_{\mathrm{GVF}}$ such that $s_i\to s$. Using the notation from Lemma 3.3.17, put

$$I_{n,i} = \frac{1}{n^{d+1}}\,\mathrm{ht}(R_n(s_i)),\quad I_n = \frac{1}{n^{d+1}}\,\mathrm{ht}(R_n(s)),$$

$$I_i = \lim_n \frac{1}{n^{d+1}}\,\mathrm{ht}(R_n(s_i)),\quad I = \lim_n \frac{1}{n^{d+1}}\,\mathrm{ht}(R_n(s)).$$

We need to show that $\lim_i I_i = I$. Pick a positive number ε. First, note that there is a natural number n such that

$$|I - I_n| < \varepsilon$$

and

$$|I_i - I_{n,i}| < \varepsilon$$

for all i. Indeed, this is possible because by Remark 3.3.16 the above differences are bounded by $\frac{\log n}{n}$ times an absolute multiple of $(1+\nu(\Omega_\infty))\max_i k_i\delta_i$. Next, note that for i big enough we have

$$|I_{n,i} - I_n| < \varepsilon$$

because for a fixed n, $\mathrm{ht}(R_n(y))$ is a continuous function on S_{GVF}. Hence together we get

$$|I - I_i| < 3\varepsilon,$$

which finishes the proof as $\varepsilon > 0$ was arbitrary. □

3.3.4 Integrable divisors over globally valued fields

For applications, it is useful to consider not only lattice line bundles, but also to allow certain limit metrics. We will define global line bundles and globally semipositive line bundles over a GVF in the spirit of Zhang line bundles. Let $\mathcal{X} \to S$ be a projective morphism of finite type schemes over a GVF K.

Definition 3.3.19. A *lattice line bundle* $\overline{\mathcal{L}}$ on $\mathcal{X}$ over S is called *semipositive* if for every $s \in S_{\mathrm{GVF}}$ the family of metrics $\varphi|_{\mathcal{X}_s}$ consists of semipositive metrics over almost all places $\omega \in M_{\kappa(s)}$.

Definition 3.3.20. Let $\mathcal{L}$ be a line bundle on $\mathcal{X}$ over S. For every compact set $C \subset S_{\mathrm{GVF}}$, we define a pseudometric d_C on the space of metrics on $\mathcal{L}$. Let s be a section of $\mathcal{L}$ and let ϕ and ψ be two families of metrics on $\mathcal{L}$. Then, we define

$$d_C(\phi, \psi) = \sup_{z \in C} \int^{+}_{\Omega_{\kappa(z)}} \sup_{x \in \mathcal{X}^{\mathrm{an}}_{s,\omega}} |\log|s|_\phi - \log|s|_\psi|\nu(d\omega),$$

where $\int^{+}$ is the upper integral functional (i.e., the infimum of integrals of measurable functions bounding the ambient function from above, see [24, Definition A.4.1] for details).

Note that the set of (semipositive) adelic divisors is closed under this norm. The norm allows us to define notions of global and integrable divisors on $\mathcal{X}$ over S.

Definition 3.3.21. A *global line bundle* $\overline{\mathcal{L}}$ on $\mathcal{X}$ over S is defined to be a line bundle $\mathcal{L}$ with a metric family ϕ such that there is a sequence of lattice line bundles $\overline{\mathcal{L}}_k = (\mathcal{L}, \phi_k)$ such that

$$\lim_{k\to\infty} d_C(\phi_k, \phi) = 0$$

for every compact $C \subset S_{\mathrm{GVF}}$.

It is called *globally semipositive* if the metric families ϕ_k can be chosen semipositive. A global line bundle $\overline{\mathcal{L}}$ is called *globally integrable* if there are globally semipositive line bundles $\overline{\mathcal{L}}_+$ and $\overline{\mathcal{L}}_-$ and an isometry $\overline{\mathcal{L}} \cong \overline{\mathcal{L}}_+ \otimes \overline{\mathcal{L}}_-^\vee$.

We denote the isometry classes of global $\mathbb{Q}$-line bundles by $\widehat{\operatorname{Pic}}_{\mathbb{Q}}(\mathcal{X}/S)$. The subgroup of integrable line bundles is denoted by $\widehat{\operatorname{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(\mathcal{X}/S)$. If $S = \operatorname{Spec} K$, we often omit it in the notation. We furthermore denote the distance between two metric families by d.

Proposition 3.3.22. Let X be a projective scheme of pure dimension d over a countable GVF K. The intersection product on lattice divisors extends to a pairing

$$\widehat{\operatorname{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(X)^{d+1} \to \mathbb{R}.$$

Proof. By linearity, it suffices to construct the pairing for globally semipositive divisors. It suffices to show that on the set of semipositive lattice divisors the intersection product is continuous with respect to d.

Let $\overline{L} = (\mathcal{O}, \phi)$ be a lattice line bundle with trivial underlying line bundle and $d(\phi, 0) = C$ and let $\overline{L}_1, \ldots, \overline{L}_d \in \operatorname{LPic}_{\mathbb{Q}}{}^+(X)$ be semipositive lattice line bundles. Then,

$$|\overline{L} \cdot \overline{L}_1 \cdots \overline{L}_d| \leq C \deg(L_1 \cdots L_d).$$

This can be read off from the interpretation of the intersection number as the integral over local heights

$$\begin{aligned}
|\overline{L} \cdot \overline{L}_1 \cdots \overline{L}_d| &= \left| \int_\Omega \int_{X_\omega^{\mathrm{an}}} \log |1|_{\phi,\omega} \; c_1(\overline{L}_{1,\omega}) \cdots c_1(\overline{L}_{d,\omega}) \nu(d\omega) \right| \\
&\leq \int_\Omega^+ \sup_{x \in X_\omega^{\mathrm{an}}} |\log |1|_{\phi,\omega}(x)| \deg(\overline{L}_1 \cdots \overline{L}_d) \nu(d\omega) \\
&\leq C \deg(L_1 \cdots L_d).
\end{aligned}$$

□

We are finally ready to prove Theorem 3.3.1. We restate it for convenience.

Theorem 3.3.23. Let $\mathcal{X} \to S$ be a flat projective morphism of finite type schemes over a GVF K of relative dimension d. Let $\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d \in \widehat{\operatorname{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(\mathcal{X}/S)$ be globally integrable divisors on $\mathcal{X}$ over S. Then, the map

$$\begin{aligned}
S_{\mathrm{GVF}} &\to \mathbb{R} \\
s &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0, \ldots, \overline{\mathcal{L}}_d | \mathcal{X}_s)
\end{aligned}$$

is continuous.

Proof. We assume by linearity that $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d$ are all semipositive. Let $C \subset S_{\mathrm{GVF}}$ be a compact subset. For $k \in \mathbb{N}$, let $\overline{\mathcal{L}}_0^k, \dots, \overline{\mathcal{L}}_d^k$ be semipositive lattice line bundles such that $d_C(\overline{\mathcal{L}}_i^k, \overline{\mathcal{L}}_i)$ converges to zero. Then, the functions $\widehat{\deg}(\overline{\mathcal{L}}_0^k, \dots, \overline{\mathcal{L}}_d^k | \mathcal{X}_s)$ converge uniformly to $\widehat{\deg}(\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d | \mathcal{X}_s)$ on C. Since the former are continuous by Proposition 3.3.18, the latter is, too. We are done since S_{GVF} is locally compact by remark 3.2.3. □

For applications the following theorem is crucial. It follows from [84], but we have not found a suitable reference for the precise statement we need. Our reasoning uses some techniques found in the proof of the arithmetic Demailly theorem in [64] or from [20]. Note that we have the notion of semipositivity and integrability for both Zhang line bundles and global line bundles and that they are a priori different.

Theorem 3.3.24. Every integrable Zhang line bundle $\overline{L} = (L, \phi)$ on a projective variety over a number field is induced by an integrable global divisor.

Proof. We prove that arithmetically ample divisors are induced by integrable global divisors. Arithmetically ample divisors in turn are dense in semipositive Zhang divisors allowing us to finish the proof.

Definition 3.3.25. A hermitian line bundle $\overline{\mathcal{L}}$ over an arithmetic variety $\mathcal{X} \to \operatorname{Spec} \mathbb{Z}$ is called *arithmetically ample* if

1. $\mathcal{L}_{\mathbb{Q}}$ is ample,
2. the metrics on $\overline{\mathcal{L}}$ are semipositive at each place,
3. the height $\widehat{c}_1(\overline{\mathcal{L}}|_{\mathcal{Y}})^{\dim \mathcal{Y}} > 0$ for every irreducible horizontal subvariety $\mathcal{Y} \subseteq \mathcal{X}$.

A Zhang divisor is called *arithmetically ample* if it is induced by an arithmetically ample Hermitian line bundle.

We want to approximate an arithmetically ample arithmetic divisor $\overline{L}$ defined on the model $\mathcal{X}$ from below. For this we apply the maps $\iota_n : \mathcal{X} \to \mathbb{P}(H^0(nL))$. We endow the line bundle $\mathcal{O}(1)$ on $\mathbb{P}(H^0(nL))$ with the metric h_n induced by the supremum norm on $H^0(nL)$. Semipositivity implies precisely that the induced metrics on L converge uniformly to $\overline{L}$ by [84, Theorem 3.5]. This is always semipositive at all places and the underlying line bundle is ample. It is arithmetically ample since the subspace of integral sections in $H^0(nL)$ has a basis $s_1, \dots, s_N$ consisting of strictly small integral sections, at least for n large enough. We are reduced to proving the claim for $(\mathcal{O}(1), h_n)$ on $\mathbb{P}(H^0(nL))$. At finite places, the metric on $\mathcal{O}(1)$ agrees with the Weil metric for the basis $s_1 \dots, s_N$ (as in Claim 2.3.16). It remains to show the approximation at the infinite place. We can approximate the metric h_n by a smooth metric with everywhere positive curvature.

From now on we assume that L is $\mathcal{O}(1)$ on some projective space $\mathbb{P}^n$ with Weil metrics at finite places and a smooth metric with everywhere positive curvature at ∞. By Dini's theorem it suffices to prove pointwise approximation, i.e., for every point $x \in X(\mathbb{C})$ and $\epsilon > 0$ there exists an integer $N > 0$ and a small integral section $s \in H^0(\mathcal{O}(N))$ such that $-\frac{1}{N} \log |s(x)| < \epsilon$.

We prove first that for arbitrarily big N we can find $l \in H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(N))$ satisfying $-\log |l|_{\sup} \geq \epsilon$ and $-\frac{1}{N} \log |l(x)| < 2\epsilon$. Let $\overline{x}$ be the complex conjugate of x. We apply [84, Theorem 2.2] to $\mathcal{O}(1)$ and $Y = \{x, \overline{x}\}$ to obtain a holomorphic section s of $\mathcal{O}(N)$ with $-\log |s|_{\sup} = 0$ and $-\frac{1}{N} \log |s_N(x)|, -\frac{1}{N} \log |s_N(\overline{x})| < \epsilon/2$. The section $s_N \otimes \overline{s_N} \in H^0(\mathbb{P}^n_{\mathbb{C}}, \mathcal{O}(2N))$ is then a section $l \in H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(2N))$ satisfying $-\log |l|_{\sup} \geq 0$ and $-\frac{1}{2N} \log |l(x)| < \epsilon$. Rescaling proves the claim.

The vector space $H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(N))$ has a norm given by the supremum norm on $X(\mathbb{C})$. The global sections over $\mathbb{Z}$ form a lattice $\Lambda_N = H^0(\mathbb{P}^n_{\mathbb{Z}}, \mathcal{O}(N))$. By [84, Theorem 4.2], there exists $0 < r < 1$ such that for big enough N, there is a basis of Λ_N consisting of vectors of norm $< r^N$. For $r < r' < 1$, and big enough N it follows that for every $l \in H^0(\mathbb{P}^n_{\mathbb{R}}, \mathcal{O}(N))$ there exists $l' \in \Lambda_N$ with $|l - l'|_{\sup} < (r')^N$. We apply this to the section l constructed in the previous paragraph. Then, l' eventually satisfies $|l'|_{\sup} \leq 1$. Furthermore, for small enough ϵ in the construction of l we can ensure $-\log |r'| > 2\epsilon$. Then, for big enough N we have $-\frac{1}{N} \log |l'(x)| < 3\epsilon$ proving the theorem. □

Let us present a result due to Chen and Moriwaki that allows to calculate the adelic intersection product over a GVF structure that comes from a polarisation.

Let $(S, \overline{H}_1, \dots, \overline{H}_n)$ be a polarisation inducing a GVF structure on $F = \mathbb{Q}(S)$. Let X be a d-dimensional projective variety over F which is the generic fiber of a projective morphism $\pi : \mathcal{X} \to S$. Fix globally integrable line bundles $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d \in \widehat{\mathrm{Pic}}^{\mathrm{int}}_{\mathbb{Q}}(\mathcal{X}/S)_{\mathbb{Q}}$ and denote by $\overline{L}_0, \dots, \overline{L}_d$ their restriction to $\widehat{\mathrm{Pic}}^{\mathrm{int}}_{\mathbb{Q}}(X)_F$.

Proposition 3.3.26. [25, Proposition 4.5.1] The following equality holds:

$$\widehat{\deg}(\overline{L}_0, \dots, \overline{L}_d | X) = \overline{\mathcal{L}}_0 \cdot \ldots \cdot \overline{\mathcal{L}}_d \cdot \pi^* \overline{H}_1 \cdot \ldots \cdot \pi^* \overline{H}_n,$$

where the left hand side is the adelic intersection product over the globally valued field F and the right hand side is the arithmetic/Arakelov intersection product of integrable Zhang divisors on $\mathcal{X}$.

Proof. The case of the polarisation and the line bundles $\overline{L}_i$ being defined on a model over $\mathbb{Z}$ is [25, Proposition 4.5.1]. Our version follows from continuity of the intersection product on semipositive line bundles and continuity of the intersection number in families, cf. Theorem 3.3.1. $\square$

3.4 Average intersections

Let $f_1, \dots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K. Each f_i defines a hypersurface V_i inside a proper toric variety T with torus $\mathbb{T} = \mathbb{G}_m^n \subset T$. Let $(u_{1,j}, \dots, u_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ with respect to the Weil height on $\mathbb{T}^m \subset \mathbb{P}^{nm}$. For integrable Zhang divisors $\overline{D}_0, \dots, \overline{D}_{n-m}$ on T we want to compute

$$\lim_{j \to \infty} \widehat{\deg}(\overline{D}_0, \dots, \overline{D}_{n-m} | u_{1,j} V_1 \cap \dots \cap u_{m,j} V_m).$$

Denote the coordinates of the i-th factor of $\mathbb{G}_m^n$ by $w_{1,i}, \dots, w_{n,i}$. We let $V \subset T \times \mathbb{T}^m$ be the intersection of the vanishing loci of $f_i(z_1 w_{1,i}^{-1}, \dots, z_n w_{n,i}^{-1})$. We note that under $V \to \mathbb{T}^m$ the generic fibre has dimension $n-m$. The map $V \to \mathbb{T}^m$ is flat of relative dimension $n-m$ over a dense Zariski open $U \subseteq \mathbb{T}^m$. We define global line bundles $\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_{n-m} \in \widehat{\mathrm{Pic}}_{\mathbb{Q}}^{\mathrm{int}}(V/U)$ by pulling back $\mathcal{O}(\overline{D}_0), \dots, \mathcal{O}(\overline{D}_{n-m})$ to $T \times \mathbb{T}^m$ and restricting to V. This makes sense by Theorem 3.3.24. By Theorem 3.3.1, the map

$$\begin{aligned} U_{\mathrm{GVF}} &\to \mathbb{R} \\ u &\mapsto \widehat{\deg}(\overline{\mathcal{L}}_0, \dots, \overline{\mathcal{L}}_d | \mathcal{X}_u) \end{aligned}$$

is continuous. We note that on $(u_1, \dots, u_m) \in U(\overline{K})$ the above map is given by

$$(u_1, \dots, u_m) \mapsto \widehat{\deg}(\overline{D}_0, \dots, \overline{D}_{n-m} | u_1 V_1 \cap \dots \cap u_m V_m).$$

By Corollary 3.2.11, a generic small sequence $(u_{1,j}, \dots, u_{m,j})_j$ in $\mathbb{T}^m$ has the corresponding points on U_{GVF} converging to $K(w_{1,1}, \dots, w_{n,1}, \dots, w_{1,m}, \dots, w_{n,m})$ with the polarized GVF structure associated to $(\prod_{i=1}^m \mathbb{P}^n, \pi_1^* \overline{\mathcal{O}(1)}^n, \dots, \pi_m^* \overline{\mathcal{O}(1)}^n)$. Denote this limit point by $\eta^{\mathrm{can}} \in U_{\mathrm{GVF}}$.

Since the right hand side is the intersection over a polarized GVF it can be computed up to birational modification as the height of $V \subset T \times \prod_{i=1}^m \mathbb{P}^n$ with respect to $\pi_h^* \overline{D}_0 \dots \pi_h^* \overline{D}_{n-m} \cdot (\sum \pi_i^* \overline{\mathcal{O}(1)})^{nm}$ by Proposition 3.3.26. In other words, we get the following.

Lemma 3.4.1. Suppose that $\overline{D}_0, \ldots, \overline{D}_{n-m}$ are integrable Zhang divisors on T over K. Then,

$$\lim_{j\to\infty} \widehat{\deg}(\overline{D}_0, \ldots, \overline{D}_{n-m} | u_{1,j}V_1 \cap \cdots \cap u_{m,j}V_m)$$
$$= \widehat{\deg}(\pi_h^*\overline{D}_0 \ldots \pi_h^*\overline{D}_{n-m} \cdot \pi_1^*\overline{\mathcal{O}(1)}^n \cdots \pi_m^*\overline{\mathcal{O}(1)}^n | V).$$

On the right hand side, the $\overline{D}_i$ should be viewed as adelic divisors pulled back to $T \times \prod_{i=1}^m \mathbb{P}^n$.

We need to be careful when applying Fubini's theorem in the non-archimedean setting. This is because of the failure of $(X \times Y)^{\mathrm{an}} = X^{\mathrm{an}} \times Y^{\mathrm{an}}$. In our setting, there is a preferred very affine chart (i.e., an open subset isomorphic to a closed subscheme of a tori) given by the torus in the toric variety.

We sketch an argument that one can apply Fubini in this situation. This is based on [70, Proposition 3.4.21]. If $\alpha \in A^{(\dim X, \dim X)}(X^{\mathrm{an}})$ and $\beta \in A^{(\dim Y, \dim Y)}(Y^{\mathrm{an}})$ are smooth forms that are defined on very affine charts of integration $U \subseteq X$ and $V \subseteq Y$. Then, $U \times V \subseteq X \times Y$ is a very affine chart of integration for $\pi_X^*\alpha \wedge \pi_Y^*\beta$. Furthermore, $\mathrm{trop}(U \times V) = \mathrm{trop}(U) \times \mathrm{trop}(V)$ by Rosenlicht's theorem. Then, one needs to prove that one has the product measure on each polyhedron in the tropicalization. This is done by adapting [70, Lemma 3.4.16]. The general case follows by approximation. We refer to [37] for an introduction to the theory of forms in the non-archimedean setting.

Theorem 3.4.2. Let $\mathcal{R}_i$ denote the Ronkin divisor associated to $g_i = f_i(z_1 w_{1,i}^{-1}, \ldots, z_n w_{n,i}^{-1})$ on a suitable toric blowup X of $T \times \prod_{i=1}^m \mathbb{P}^n$. Let $\tilde{V}$ denote the common vanishing locus of the g_i. Then, we have an identity

$$\widehat{\deg}(\mathcal{R}_1 \ldots \mathcal{R}_m \pi_h^*\overline{D}_0 \ldots \pi_h^*\overline{D}_{n-m} \cdot \pi_1^*\overline{\mathcal{O}(1)}^n \cdots \pi_m^*\overline{\mathcal{O}(1)}^n | X)$$
$$= \widehat{\deg}(\pi_h^*\overline{D}_0 \ldots \pi_h^*\overline{D}_{n-m} \cdot \pi_1^*\overline{\mathcal{O}(1)}^n \cdots \pi_m^*\overline{\mathcal{O}(1)}^n | \tilde{V}).$$

Proof. In order not to overburden notation we omit the superscript denoting the analytification.

Let s_i denote the distinguished section of $\mathcal{O}(\mathcal{R}_i)$. The sections $g_i s_i$ of $\mathcal{O}(\mathcal{R}_i)$ have common vanishing locus $\tilde{V}$. By the iterative definition of the height, the equality of intersection numbers is equivalent to the vanishing of the integrals occurring in the height computations. These are of the form

$$\int_{\mathrm{div}(g_1) \cap \cdots \cap \mathrm{div}(g_{r-1})} \log|g_r s_r| c_1(\mathcal{R}_{r+1}) \ldots c_1(\mathcal{R}_m) \pi_h^* c_1(\overline{D}_0) \ldots \pi_h^* c_1(\overline{D}_{n-m}) \prod_{i=1}^m \pi_i^* c_1(\overline{\mathcal{O}(1)}).$$

Let us write ω for $c_1(\mathcal{R}_{r+1})\dots c_1(\mathcal{R}_m)\pi_h^* c_1(\overline{D}_0)\dots\pi_h^* c_1(\overline{D}_{n-m})\prod_{i\neq r}\pi_i^* c_1(\overline{\mathcal{O}(1)})$. To each of them we can apply Fubini to obtain

$$\begin{aligned}
&\int_{\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap X} \log|g_r s_r|\pi_r^* c_1(\overline{\mathcal{O}(1)})^n\omega \\
&= \int_{\left(\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap(\mathbb{T}\times\prod_{i=1}^{r-1}\mathbb{T})\right)\times\mathbb{T}\times\prod_{i=r+1}^{m}\mathbb{T}} \log|g_r s_r|\pi_r^* c_1(\overline{\mathcal{O}(1)})^n\omega \\
&= \int_{\left(\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap(\mathbb{T}\times\prod_{i=1}^{r-1}\mathbb{T})\right)\times\prod_{i=r+1}^{m}\mathbb{T}} \left(\int_{\mathbb{T}} \log|g_r s_r| c_1(\overline{\mathcal{O}(1)})^n\right)\omega \\
&= \int_{\left(\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap(\mathbb{T}\times\prod_{i=1}^{r-1}\mathbb{T})\right)\times\prod_{i=r+1}^{m}\mathbb{T}} \left(\int_{\mathbb{T}} \log|g_r s_r| d\sigma_0(x)\right)\omega.
\end{aligned}$$

The first equality follows since a Zariski closed subset with empty interior is a nullset with respect to a measure associated to a differential form.

We claim that the inner integral $\int_{\mathbb{T}}\log|g_r s_r|\sigma_0(x)$ vanishes at each fiber. Recall that $g_r(t,t_1,\dots,t_m) = f_r(tt_r^{-1})$. Let π_r denote the projection from $\mathbb{T}\times\prod_{i=1}^m\mathbb{T}$ to the r-th component $\mathbb{T}_r$ in the second factor.

We compute

$$\begin{aligned}
\log|s_r(t,t_1,\dots,t_m)| &= \int_{\mathrm{trop}^{-1}(\mathrm{trop}(t,t_1,\dots,t_m))} -\log|g_r(x)|d\sigma_{(t,t_1,\dots,t_m)}(x) \\
&= \int_{\mathrm{trop}^{-1}(\mathrm{trop}(t,t_r))\subset(\mathbb{T}\times\mathbb{T}_r)^{\mathrm{trop}}} -\log|f_r(xx_r^{-1})|d\sigma_{(t,t_r)}(x,x_r) \\
&= \int_{\mathrm{trop}^{-1}(\mathrm{trop}(tt_r^{-1}))\subset\mathbb{T}_r^{\mathrm{trop}}} -\log|f_r(x)|d\sigma_{tt_r^{-1}}(x) = \rho_r(tt_r^{-1}).
\end{aligned}$$

Here, ρ_r denotes the Ronkin function for f_r and $\mathbb{T}_r$ denotes the r-th factor of the torus.

Consider the fibre over an element

$$(t,t_1,\dots,t_{r-1},t_{r+1},\dots,t_m)\in\left(\mathrm{div}(g_1)\cap\cdots\cap\mathrm{div}(g_{r-1})\cap(\mathbb{T}\times\prod_{i=1}^{r-1}\mathbb{T})\right)\times\prod_{i=r+1}^{m}\mathbb{T}.$$

Over this fibre we evaluate the integral

$$\begin{aligned}
\int_{\mathbb{T}}\log|g_r(t,t_1,\dots,t_m)s_r(t,t_1,\dots,t_m)|\sigma_0(t_r) &= \int_{\mathbb{T}}\log|f_r(tt_r^{-1})| + \rho_r(tt_r^{-1})\sigma_0(t_r) \\
&= -\rho_r(t)+\rho_r(t) = 0.
\end{aligned}$$

□

Lemma 3.4.3. Let R_i denote the Ronkin line bundle associated to f_i and suppose it is already defined on T and assume $\overline{D}_0,\dots,\overline{D}_{n-m}$ are toric. Let X denote a suitable toric blow-up of $T \times \prod_{i=1}^{m} \mathbb{P}^n$ over which $\mathcal{R}_i$ are defined. We have an equality of intersection numbers

$$\widehat{\deg}(\mathcal{R}_1 \dots \mathcal{R}_m \pi_h^* \overline{D}_0 \dots \pi_h^* \overline{D}_{n-m} \cdot \pi_1^* \overline{\mathcal{O}(1)}^n \cdots \pi_m^* \overline{\mathcal{O}(1)}^n | X)$$

$$= \widehat{\deg}(R_1 \dots R_m \overline{D}_0 \dots \overline{D}_{n-m} | T)$$

Proof. By linearity, we assume that $\overline{D}_0, \dots, \overline{D}_{n-m}$ are all semipositive. Then, we interpret the left hand side in a combinatorial manner as in [17, Theorem 5.2.5]. Next, we apply Lemma 3.2.23. The occurring polytopes are m times n copies of the unit simplex, one set of copies for each factor in $\prod_{i=1}^{m} \mathbb{P}^n$. One sees immediately that the pushforward of the roof functions of the $\pi_h^* \overline{D}_i$'s yield the roof functions of the $\overline{D}_i$'s. Similarly, the pushforward of the roof function of $\mathcal{R}_i$ is the roof function of R_i by Lemma 3.2.21. This finishes the proof. □

Theorem 3.4.4. Let $f_1, \dots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K and let T be a proper toric variety with torus $\mathbb{T} = \mathbb{G}_m^n \subset T$. Suppose that $\mathrm{NP}(f_i)$ define divisors on T. Denote by V_i the hypersurface defined by f_i. Let $(\zeta_{1,j}, \dots, \zeta_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ with respect to the Weil height and $\overline{D}_0, \dots, \overline{D}_{n-m}$ be integrable toric Zhang divisors on T. Then,

$$\lim_{j \to \infty} \widehat{\deg}(\overline{D}_0, \dots, \overline{D}_{n-m} \mid \zeta_{1,j} V_1 \cap \dots \cap \zeta_{m,j} V_m) = \widehat{\deg}(R_1 \dots R_m \overline{D}_0 \dots \overline{D}_{n-m} | T).$$

Proof. This is a combination of Lemma 3.4.1, Theorem 3.4.2 and Lemma 3.4.3. □

Finally, we prove Conjecture 6.4.4. from [32].

Theorem 3.4.5. Let $f_1, \dots, f_m$ be Laurent polynomials in n variables with coefficients in a number field K and let T be a proper toric variety with torus $\mathbb{T} = \mathbb{G}^n \subset T$. Denote by V_i the hypersurface defined by f_i and by ρ_i its Ronkin function. Let $(\zeta_{1,j}, \dots, \zeta_{m,j})_j$ be a generic sequence of small points in $\mathbb{T}^m$ for the Weil height and let $\overline{D}_0, \dots, \overline{D}_{n-m}$ be semipositive toric Zhang divisors on T with associated local roof functions $\theta_{0,v}, \dots, \theta_{n-m,v}$. Then,

$$\lim_{j \to \infty} \widehat{\deg}(\overline{D}_0, \dots, \overline{D}_{n-m} \mid \zeta_{1,j} V_1 \cap \dots \cap \zeta_{m,j} V_m)$$

$$= \sum_{v \in \mathrm{M}_K} n_v \, \mathrm{MI}_M(\theta_{0,v}, \dots, \theta_{n-m,v}, \rho_1^\vee, \dots, \rho_m^\vee).$$

Proof. We apply the projection formula to restrict to the case, where the $\mathrm{NP}(f_i)$ define divisors on T. Then, the conjecture follows from Theorem 3.4.4 and Theorem 3.2.17. □

3.5 Appendix: Mahler measures of complex polynomials

In this appendix, we study some measures of complexity of complex polynomials. For a non-zero $P \in \mathbb{C}[X_1, \ldots, X_n]$, we define the logarithmic Mahler measure

$$m(P) = \int_{[0,1]^n} \log |P(e^{2i\pi t_1}, \ldots, e^{2i\pi t_n})| dt_1 \ldots dt_n,$$

and the logarithmic Fubini-Study Mahler measure

$$m_{\mathbb{S}_n}(P) = \int_{\mathbb{S}_n} \log |P(z_1, \ldots, z_n)| d\eta_n(z_1, \ldots, z_n),$$

where $\mathbb{S}_n$ is the unit sphere in $\mathbb{C}^n$ for the usual Euclidean norm, and η_n is the spherical measure on $\mathbb{S}_n$, normalized so that $\eta_n(\mathbb{S}_n) = 1$.

In [50], Pierre Lelong studied these two measures and gave a bound for the distance between them in terms of n and the degree of P.

In this appendix, we prove an analogue of Lelong's result for polynomials in $\overline{X}_1, \ldots, \overline{X}_n$, where each $\overline{X}_i$ is a tuple of abstract variables of length m_i. More precisely, we define the mixed Fubini-Study Mahler measure by

$$m_{\mathbb{S}_{m_1} \times \ldots \times \mathbb{S}_{m_n}}(P) = \int_{\mathbb{S}_{m_1} \times \ldots \mathbb{S}_{m_n}} \log |P(\overline{z}_1, \ldots, \overline{z}_n)| d\eta_{m_1}(\overline{z}_1) \wedge \ldots \wedge d\eta_{m_n}(\overline{z}_n).$$

This measure has already been studied by Rémond in [65], in which he bounds $m_{\mathbb{S}_{m_1} \times \ldots \times \mathbb{S}_{m_n}}(P)$ in terms of the absolute values of the coefficients of P. Our goal here is to prove a slightly different inequality, namely the following proposition.

Proposition 3.5.1. Let $P \in \mathbb{C}[\overline{X}_1, \ldots, \overline{X}_n]$ be a non-zero polynomial, where each $\overline{X}_i$ is a tuple of abstract variables of length m_i. For all $i \leq n$, let d_i be the degree of P in $\overline{X}_i$. Then,

$$\left| m_{\mathbb{S}_{m_1} \times \ldots \times \mathbb{S}_{m_n}} - \log \|P\| \right| \leqslant \sum_{i=1}^{n} d_i \left(\log(m_i + 1) + \frac{1}{2} \sum_{k=1}^{m_i - 1} \frac{1}{k} \right),$$

where $\|P\|$ is the maximum absolute value of the coefficients of P.

Let us start with the simpler case where each m_i is equal to 1. Let $P \in \mathbb{C}[X_1, \ldots, X_n]$ be a non-zero polynomial of degree d.

Lemma 3.5.2. Let

$$S(P) := \sup\{|P(z_1, \dots, z_n)| \,:\, (z_1, \dots, z_n) \in \mathbb{C}^n \text{ and } |z_i| \leqslant 1 \text{ for all } i\}.$$

Then,

$$\|P\| \leqslant S(P) \leqslant \binom{n+d}{n} \|P\|.$$

Proof. The rightmost inequality follows from the fact that a polynomial of degree d has at most $\binom{n+d}{n}$ non-zero coefficients.

For the other inequality, write $P = \sum_{k \in \mathbb{N}^n} a_n X^n$, and consider the integral

$$\begin{aligned}
I &= \int_{[0,1]^n} |P(e^{2i\pi t_1}, \dots, e^{2i\pi t_n})|^2 \, dt_1 \dots dt_n \\
&= \int_{[0,1]^n} P(e^{2i\pi t_1}, \dots, e^{2i\pi t_n}) \overline{P(e^{2i\pi t_1}, \dots, e^{2i\pi t_n})} \, dt_1 \dots dt_n \\
&= \int_{[0,1]^n} \left(\sum_{k,l \in \mathbb{N}^n} a_k \overline{a_l} \exp\left(2i\pi(k_1 - l_1)t_1 + \dots + 2i\pi(k_n - l_n)t_n\right) \right) dt_1 \dots dt_n \\
I &= \sum_{k,l \in \mathbb{N}^n} a_k \overline{a_l} \int_{[0,1]^n} \exp\left(2i\pi(k_1 - l_1)t_1 + \dots + 2i\pi(k_n - l_n)t_n\right) \, dt_1 \dots dt_n.
\end{aligned}$$

Now, for $k, l \in \mathbb{N}^n$ with $k \neq l$, there exists $1 \leqslant r \leqslant n$ with $k_r - l_r \neq 0$. So,

$$\begin{aligned}
&\int_{[0,1]^n} \exp\left(2i\pi(k_1 - l_1)t_1 + \dots + 2i\pi(k_n - l_n)t_n\right) \, dt_1 \dots dt_n \\
&\qquad = \int_{[0,1]^{n-1}} \exp\left(\sum_{s \neq r} 2i\pi(k_s - l_s)t_s \right) \left(\int_0^1 e^{2i\pi(k_r - l_r)t_r} dt_r \right) \prod_{s \neq r} dt_s = 0,
\end{aligned}$$

and for $k = l$,

$$\int_{[0,1]^n} \exp\left(2i\pi(k_1 - l_1)t_1 + \dots + 2i\pi(k_n - l_n)t_n\right) \, dt_1 \dots dt_n = 1.$$

So, finally

$$I = \sum_{k \in \mathbb{N}} |a_k|^2 \geqslant \|P\|^2.$$

But we also have

$$I = \int_{[0,1]^n} |P(e^{2i\pi t_1}, \dots, e^{2i\pi t_n})|^2 \, dt_1 \dots dt_n \leqslant S(P)^2.$$

Hence, $\|P\| \leqslant S(P)$. $\square$

Remark 3.5.3. In particular, Lemma 3.5.2 implies that

$$S(P) = \lim_{m\to+\infty} \|P^m\|^{1/m}.$$

Lemma 3.5.4. Assume that $n = 1$, i.e., $P = \sum_{k=0}^{d} a_k X^k \in \mathbb{C}[X]$. Then, for all $0 \leqslant k \leqslant d$:

$$|a_k| \leqslant \binom{d}{k} \exp(m(P))$$

Proof. Since $\mathbb{C}$ is algebraically closed, we can write $P = \lambda \prod_{i=1}^{\deg P}(X-\alpha_i)$, where $\lambda \in \mathbb{C}^\times$, $\alpha_1, \dots, \alpha_{\deg P} \in \mathbb{C}$. Now, by Jensen's formula, we have for each $1 \leqslant i \leqslant \deg P$,

$$\int_0^1 \log\left|e^{2i\pi t} - \alpha_i\right| dt = \max(0, \log|\alpha_i|).$$

So, by summing,

$$m(P) = \log|\lambda| + \sum_{i=1}^{\deg P} \max(0, \log|\alpha_i|).$$

Now, let $k \leqslant \deg P$. Then, the coefficient of X^k in P is equal to

$$a_k = (-1)^{\deg P - k} \lambda \sum_{I \subseteq \{1,\dots,\deg P\}} \prod_{i\in I} \alpha_i.$$

So, by the triangle inequality

$$|a_k| \leqslant |\lambda| \sum_{I \subseteq \{1,\dots,\deg P\}} \prod_{i\in I} |\alpha_i| \leqslant \binom{\deg P}{k} |\lambda| \prod_{i=1}^{\deg P} \max(1, \alpha_i) \leqslant \binom{d}{k} \exp(m(P).$$

$\square$

Lemma 3.5.5. Write $P = \sum_{m\in\mathbb{N}^n} a_m X_1^{m_1} \dots X_n^{m_n}$. Then, for every $m = (m_1, \dots, m_n) \in \mathbb{N}^n$, we have

$$|a_m| \leqslant \binom{d}{m_1, \dots, m_n} \exp(m(P)).$$

Proof. We prove this inequality by induction on n. The $n = 1$ case is the result of Lemma 3.5.4, so we may assume $n \geqslant 2$. The result is also immediate if a_m is zero, so assume it is not. Write $P = \sum_{k=0}^{d} P_k X_n^k$, where the P_k are in $\mathbb{C}[X_1, \dots, X_{n-1}]$. Then, we may write

$$m(P) = \int_{[0,1]^{n-1}} m(P(e^{2i\pi t_1}, \dots, e^{2i\pi t_{n-1}}, X)) dt_1 \dots dt_{n-1}.$$

By Lemma 3.5.4, we have for all $(t_1,\dots,t_{n-1}) \in [0,1]^{n-1}$, $|P_{m_n}(e^{2i\pi t_1},\dots,e^{2i\pi t_{n-1}})| \leqslant \binom{d}{m_n}\exp(m(P(e^{2i\pi t_1},\dots,e^{2i\pi t_{n-1}},X)))$. Since $P_{m_n}(e^{2i\pi t_1},\dots,e^{2i\pi t_{n-1}})$ is non-zero for almost all $(t_1,\dots,t_{n-1})$, we may write
$m(P(e^{2i\pi t_1},\dots,e^{2i\pi t_{n-1}},X)) \geqslant \log|P_{m_n}(e^{2i\pi t_1},\dots,e^{2i\pi t_{n-1}})| - \log\binom{d}{m_n}$ and integrate, yielding

$$m(P) \geqslant m(P_{m_n}) - \log\binom{d}{m_n},$$

i.e., $\exp(m(P_{m_n})) \leqslant \binom{d}{m_n}\exp(m(P))$ Since P_{m_n} has degree at most $d-m_n$, the induction hypothesis gives $|a_m| \leqslant \binom{d-m_n}{m_1,\dots,m_{n-1}}\exp(m(P_{m_n}))$. Since $\binom{d}{m_n}\binom{d-m_n}{m_1,\dots,m_{n-1}} = \binom{d}{m_1,\dots,m_n}$, this concludes. ☐

Corollary 3.5.6.

$$|m(P) - \log\|P\|| \leqslant d\log(n+1)$$

Proof. First, it is clear that $m(P) \leqslant \log(S(P)) \leqslant \log\|P\| + \log\binom{n+d}{n}$ by Lemma 3.5.2. A basic counting argument shows that $\binom{n+d}{n} \leqslant (n+1)^d$, hence $m(P) \leqslant \log\|P\| + d\log(n+1)$.

For the other direction, write $P = \sum\limits_{m\in\mathbb{N}^n} a_m X_1^{m_1}\dots X_n^{m_n}$. By Lemma 3.5.5, we have for all $m \in \mathbb{N}^n$,

$$|a_m| \leqslant \binom{d}{m_1,\dots,m_n}\exp(m(P))$$

A basic counting argument shows that $\binom{d}{m_1,\dots,m_n} \leqslant (n+1)^d$, hence by taking the maximum over all coefficients,

$$\|P\| \leqslant (n+1)^d\exp(m(P)),$$

i.e., $\log\|P\| \leqslant m(P) + d\log(n+1)$, which concludes the proof. ☐

Remark 3.5.7. If $P \in \mathbb{C}[X_1,\dots,X_n]$ is homogeneous, we may replace $n+1$ by n in the above inequality. Indeed, evaluating in $X_n = 1$ does not change $m(P)$ and $\|P\|$, so we may replace P with a polynomial in $n-1$ variables.

Lemma 3.5.8.

$$|m_{\mathbb{S}_n}(P) - \log\|P\|| \leqslant 2d\log(n+1).$$

Proof. It is clear from the definition that $m_{\mathbb{S}_n}(P(e^{2i\pi t_1}X_1,\dots,e^{2i\pi t_n}X_n)) = m_{\mathbb{S}_n}(P)$ for all $t_1,\dots,t_n \in [0,1]$. So, we may write

$$m_{\mathbb{S}_n}(P) = \int_{[0,1]^n} m_{\mathbb{S}_n}(P(e^{2i\pi t_1}X_1,\dots,e^{2i\pi t_n}X_n))dt_1\dots dt_n.$$

By Fubini, this is equal to

$$\int_{\mathbb{S}_n}\left(\int_{[0,1]^n}(P(z_1e^{2i\pi t_1},\ldots,z_ne^{2i\pi t_n}))dt_1\ldots dt_n\right)d\eta_n(\overline{z})=\int_{\mathbb{S}_n}m(P(z_1X_1,\ldots,z_nX_n))d\eta_n(\overline{z})$$

But, by Lemma 3.5.6, we have for all $\overline{z}\in\mathbb{S}_n$:

$$|m(P(z_1X_1,\ldots,z_nX_n))-\log\|P\||\leqslant d\log(n+1)$$

So, by integrating:

$$\left|m_{\mathbb{S}_n}(P)-\int_{\mathbb{S}_n}\log\|P(z_1X_1,\ldots,z_nX_n)\|d\eta_n(\overline{z})\right|\leqslant d\log(n+1).$$

Moreover, it is clear that for all $\overline{z}\in\mathbb{S}_n$, $\|P(z_1X_1,\ldots,z_nX_n)\|\leqslant\|P\|$, therefore $\int\limits_{\mathbb{S}_n}\log\|P(z_1X_1,\ldots,z_nX_n)\|d\eta_n(\overline{z})\leqslant\log\|P\|$.

On the other hand, if P is written as $\sum_{s\in\mathbb{N}^n}a_sX_1^{s_1}\ldots X_n^{s_n}$, we have for all s such that $a_s\neq 0$,

$$\begin{aligned}\int_{\mathbb{S}_n}\log\|P(z_1X_1,\ldots,z_nX_n)\|d\eta_n(\overline{z})&=\int_{\mathbb{S}_n}\left(\max_s\log|a_sz_1^{s_1}\ldots z_n^{s_n}|\right)d\eta_n(\overline{z})\\&\geqslant\max_s\int_{\mathbb{S}_n}\log|a_sz_1^{s_1}\ldots z_n^{s_n}|d\eta_n(\overline{z})\\&=\max_s\left(\log|a_s|+\sum_{i=1}^n s_i\int_{\mathbb{S}_n}\log|z_i|d\eta_n(\overline{z})\right)\\&=\max_s\left(\log|a_s|+|s|\int_{\mathbb{S}_n}\log|z_1|d\eta_n(\overline{z})\right)\end{aligned}$$

$$\int_{\mathbb{S}_n}\log\|P(z_1X_1,\ldots,z_nX_n)\|d\eta_n(\overline{z})\geqslant\log\|P\|+d\int_{\mathbb{S}_n}\log|z_1|d\eta_n(\overline{z}).$$

It remains to show that $\int\limits_{\mathbb{S}_n}\log|z_1|d\eta_n(\overline{z})\geq-\log(n+1)$. In fact, we even know from [50, Equation 2.28] that this integral evaluates to $-\frac{1}{2}\sum\limits_{i=1}^{n-1}\frac{1}{k}$. □

Now, we move on to the mixed case. Fix a non-zero polynomial $P\in\mathbb{C}[\overline{X}_1,\ldots,\overline{X}_n]$ and denote $d_i:=\deg_{\overline{X}_i}P$. Our goal is to adapt the result of Lemma 3.5.8 and find a bound for the distance between this measure and $\log\|P\|$ in terms of the d_i.

Lemma 3.5.9. Again, let

$$S(P):=\sup\{|P(\overline{z_1},\ldots,\overline{z}_n)|(\overline{z}_1,\ldots,\overline{z}_n)\in\mathbb{C}^{m_1+\ldots+m_n}\text{ and }|z_{i,j}|\leqslant 1\text{ for all }i,j\}.$$

Then,

$$\|P\|\leqslant S(P)\leqslant\left(\prod_{i=1}^n\binom{m_i+d_i}{m_i}\right)\|P\|$$

Proof. This follows from the fact that the number of non-zero coefficients of P is at most $\prod\limits_{i=1}^{n} \binom{m_i+d_i}{m_i}$, by the same argument as in the proof of Lemma 3.5.2. $\square$

Lemma 3.5.10. Write

$$P = \sum_{(k_1,\dots,k_n)\in\mathbb{N}^{m_1}\times\dots\times\mathbb{N}^{m_n}} a_{k_1,\dots,k_n}\overline{X}_1^{k_1}\dots\overline{X}_n^{k_n},$$

where $\overline{X}_i^{k_i} := \prod\limits_{j=1}^{m_i} X_{i,j}^{k_{i,j}}$. Then, for every $k = (k_1,\dots,k_n) \in \mathbb{N}^{m_1}\times\dots\times\mathbb{N}^{m_n}$, we have

$$|a_{k_1,\dots,k_n}| \leqslant \left(\prod_{i=1}^{n}\binom{d_i}{k_{i,1},\dots,k_{i,m_i}}\right)\exp(m(P)).$$

Proof. The proof is a straightforward induction on n, based on Lemma 3.5.5. $\square$

Corollary 3.5.11. The following inequality holds:

$$|m(P) - \log\|P\|| \leqslant \sum_{i=1}^{n} d_i\log(m_i+1).$$

Proof. First, it is clear that $m(P) \leqslant \log(S(P)) \leqslant \log\|P\| + \sum_{i=1}^{n}\log\binom{m_i+d_i}{m_i}$ by Lemma 3.5.2. As in the proof of Corollary 3.5.6, a counting argument shows that $\binom{m_i+d_i}{m_i} \leqslant (m_i+1)^{d_i}$, so $m(P) \leqslant \log\|P\| + \sum_{i=1}^{n} d_i\log(m_i+1)$.

Then, it follows from Lemma 3.5.10 and the fact that every multinomial coefficient $\binom{d_i}{k_1,\dots,k_{m_i}}$ is smaller or equal to $(m_i+1)^{d_i}$, that

$$\|P\| \leqslant \left(\prod_{i=1}^{n}(m_i+1)^{d_i}\right)\exp(m(P)),$$

i.e., $\log\|P\| \leqslant m(P) + \sum_{i=1}^{n} d_i\log(m_i+1)$, which concludes the proof. $\square$

Remark 3.5.12. If $P \in \mathbb{C}[\overline{X}_1,\dots,\overline{X}_n]$ is homogeneous in each of the tuples $\overline{X}_i$, we may replace m_i+1 by m_i in the above inequality. Indeed, evaluating in $X_{i,m_i} = 1$ does not change $m(P)$ nor $\|P\|$, so we may replace $\overline{X}_i$ by a (m_i-1)-tuple.

We are finally able to prove the main result of this appendix.

Proof of Proposition 3.5.1. We first prove the inequality

$$m_{\mathbb{S}_{m_1}\times\dots\times\mathbb{S}_{m_n}}(P) \leqslant \log\|P\| + \sum_{i=1}^{n} d_i\log(m_i+1)$$

exactly as in the proof of Corollary 3.5.11.

For the other inequality, we again reason by induction on n. If $n = 1$, the result follows directly from Lemma 3.5.8. So, assume $n \geqslant 2$. Write $P = \sum_{k\in\mathbb{N}^{m_n}} P_k\overline{X}_n^k$, where the P_k are in $\mathbb{C}[\overline{X}_1, \ldots, \overline{X}_{n-1}]$. Let $l \in \mathbb{N}^{m_n}$ be such that $\|\mathbb{P}_l\| = \|P\|$. Then, we have

$$m_{\mathbb{S}_{m_1}\times\ldots\times\mathbb{S}_{m_n}} = \int_{\mathbb{S}_1\times\ldots\times\mathbb{S}_{m_{n-1}}} m_{S_{m_n}}(P(\overline{z}_1, \ldots, \overline{z}_{n-1}, \overline{X}))d\eta_{m_1}(\overline{z}_1) \wedge \ldots \wedge d\eta_{m_{n-1}}(\overline{z_{n-1}})$$

By Lemma 3.5.8, we have for all $\overline{z}_1, \ldots, \overline{z}_{n-1}$,

$$\begin{aligned} m_{S_{m_n}}(P(\overline{z}_1, \ldots, \overline{z}_{n-1}, \overline{X})) &\geqslant \log\|P(\overline{z}_1, \ldots, \overline{z}_{n-1}, \overline{X})\| - d_n\left(\log(m_n+1) + \frac{1}{2}\sum_{k=1}^{m_n-1}\frac{1}{k}\right) \\ &\geqslant \log|P_l(\overline{z}_1, \ldots, \overline{z}_{n-1})| - d_n\left(\log(m_n+1) + \frac{1}{2}\sum_{k=1}^{m_n-1}\frac{1}{k}\right), \end{aligned}$$

so by integrating, we get

$$m_{\mathbb{S}_{m_1}\times\ldots\times\mathbb{S}_{m_n}} \geqslant m_{\mathbb{S}_{m_n}}(P_l) - d_n\left(\log(m_n+1) + \frac{1}{2}\sum_{k=1}^{m_n-1}\frac{1}{k}\right).$$

But, by induction hypothesis,

$$m_{\mathbb{S}_{m_n}}(P_l) \geqslant \log\|P_l\| - \sum_{i=1}^{n-1} d_i\left(\log(m_i+1) + \frac{1}{2}\sum_{k=1}^{m_i-1}\frac{1}{k}\right),$$

where $\|P_l\| = \|P\|$ by assumption, which concludes the proof. □

Bibliography

[1] Daniel Allcock and Jeffrey D. Vaaler. A Banach space determined by the Weil height. *Acta Arith.*, 136(3):279–298, 2009.

[2] Emil Artin and George Whaples. Axiomatic characterization of fields by the product formula for valuations. *Bull. Amer. Math. Soc.*, 51:469–492, 1945.

[3] François Ballaÿ. Successive minima and asymptotic slopes in Arakelov geometry. *Compos. Math.*, 157(6):1302–1339, 2021.

[4] François Ballaÿ and Martín Sombra. Approximation of adelic divisors and equidistribution of small points, 2024. https://arxiv.org/abs/2407.14978.

[5] Itaï Ben Yaacov. Continuous first order logic for unbounded metric structures. *J. Math. Log.*, 8(2):197–223, 2008.

[6] Itaï Ben Yaacov. Estimates on volumes of homogeneous polynomial spaces, 2018. https://arxiv.org/abs/1801.06994.

[7] Itaï Ben Yaacov. The Vandermonde determinant identity in higher dimension, 2024. https://arxiv.org/abs/1405.0993.

[8] Itaï Ben Yaacov, Alexander Berenstein, C. Ward Henson, and Alexander Usvyatsov. Model theory for metric structures. In *Model theory with applications to algebra and analysis. Vol. 2*, volume 350 of *London Math. Soc. Lecture Note Ser.*, pages 315–427. Cambridge Univ. Press, Cambridge, 2008.

[9] Itaï Ben Yaacov, Pablo Destic, Ehud Hrushovski, and Michał Szachniewicz. Globally valued fields: foundations, 2024. https://arxiv.org/abs/2409.04570.

[10] Itaï Ben Yaacov and Ehud Hrushovski. Globally valued function fields: existential closure, 2022. https://arxiv.org/abs/2212.07269.

[11] Itaï Ben Yaacov and Ehud Hrushovski. Globally valued fields: free amalgamation and quantifier-free stability. Unpublished notes, private correspondence, 2025.

[12] Vladimir G. Berkovich. *Spectral theory and analytic geometry over non-Archimedean fields*, volume 33 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1990.

[13] Jean-Benoît Bost and Huayi Chen. Concerning the semistability of tensor products in Arakelov geometry. *J. Math. Pures Appl. (9)*, 99(4):436–488, 2013.

[14] Sébastien Boucksom and Huayi Chen. Okounkov bodies of filtered linear series. *Compos. Math.*, 147(4):1205–1229, 2011.

[15] Sébastien Boucksom, Charles Favre, and Mattias Jonsson. Differentiability of volumes of divisors and a problem of Teissier. *J. Algebraic Geom.*, 18(2):279–308, 2009.

[16] José Ignacio Burgos Gil, Ricardo Menares, and Juan Rivera-Letelier. On the essential minimum of Faltings' height. *Math. Comp.*, 87(313):2425–2459, 2018.

[17] José Ignacio Burgos Gil, Patrice Philippon, and Martín Sombra. Arithmetic geometry of toric varieties. Metrics, measures and heights. *Astérisque*, (360):vi+222, 2014.

[18] Antoine Chambert-Loir. Chapter VII: Arakelov geometry, heights, equidistribution, and the Bogomolov conjecture. In *Arakelov geometry and Diophantine applications*, volume 2276 of *Lecture Notes in Math.*, pages 299–328. Springer, Cham, 2021.

[19] Chen Chung Chang and Jerome Keisler Keisler. *Model theory*, volume 73 of *Studies in Logic and the Foundations of Mathematics*. North-Holland Publishing Co., Amsterdam, third edition, 1990.

[20] François Charles. Arithmetic ampleness and an arithmetic Bertini theorem. *Ann. Sci. Éc. Norm. Supér. (4)*, 54(6):1541–1590, 2021.

[21] Zoé Chatzidakis and Ehud Hrushovski. Model theory of difference fields. *Trans. Amer. Math. Soc.*, 351(8):2997–3071, 1999.

[22] Huayi Chen. Arithmetic Fujita approximation. *Ann. Sci. Éc. Norm. Supér. (4)*, 43(4):555–578, 2010.

[23] Huayi Chen. Differentiability of the arithmetic volume function. *J. Lond. Math. Soc. (2)*, 84(2):365–384, 2011.

[24] Huayi Chen and Atsushi Moriwaki. *Arakelov geometry over adelic curves*, volume 2258 of *Lecture Notes in Mathematics*. Springer, Singapore, 2020.

[25] Huayi Chen and Atsushi Moriwaki. Arithmetic intersection theory over adelic curves, 2021. https://arxiv.org/abs/2103.15646.

[26] Huayi Chen and Atsushi Moriwaki. *Positivity in Arakelov geometry over adelic curves—Hilbert-Samuel formula and equidistribution theorem*, volume 355 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2024.

[27] Nguyen-Bac Dang and Charles Favre. Intersection theory of nef b-divisor classes. *Compos. Math.*, 158(7):1563–1594, 2022.

[28] Pablo Destic, Nuno Hultberg, and Michał Szachniewicz. Continuity of heights in families and complete intersections in toric varieties, 2024. https://arxiv.org/abs/2412.15988.

[29] Vesselin Dimitrov and Philipp Habegger. Galois orbits of torsion points near atoral sets. *Algebra Number Theory*, 18(11):1945–2001, 2024.

[30] Paolo Dolce and Francesco Zucconi. On the generalisation of Roth's theorem, 2023. https://arxiv.org/abs/2111.12409.

[31] Antonio J. Engler and Alexander Prestel. *Valued fields*. Springer Monographs in Mathematics. Springer-Verlag, Berlin, 2005.

[32] Roberto Gualdi. *Height of cycles in toric varieties*. Theses, Université de Bordeaux ; Universitat internacional de Catalunya, September 2018.

[33] Roberto Gualdi. Heights of hypersurfaces in toric varieties. *Algebra Number Theory*, 12(10):2403–2443, 2018.

[34] Roberto Gualdi and Martín Sombra. Limit heights and special values of the Riemann zeta function, 2023. https://arxiv.org/abs/2304.01966.

[35] Roberto Gualdi and Martín Sombra. Heights of complete intersections in toric varieties, 2024. https://arxiv.org/abs/2412.16308.

[36] Walter Gubler. Heights of subvarieties over M-fields. In *Arithmetic geometry (Cortona, 1994)*, volume XXXVII of *Sympos. Math.*, pages 190–227. Cambridge Univ. Press, Cambridge, 1997.

[37] Walter Gubler. Forms and current on the analytification of an algebraic variety (after Chambert-Loir and Ducros). In *Nonarchimedean and tropical geometry*, Simons Symp., pages 1–30. Springer, [Cham], 2016.

[38] Bradd Hart. An introduction to continuous model theory. In *Model theory of operator algebras*, volume 11 of *De Gruyter Ser. Log. Appl.*, pages 83–131. De Gruyter, Berlin, 2023.

[39] Robin Hartshorne. *Algebraic geometry.* Springer-Verlag, New York, 1977. Graduate Texts in Mathematics, No. 52.

[40] Edwin Hewitt and Karl Stromberg. *Real and abstract analysis*, volume No. 25 of *Graduate Texts in Mathematics.* Springer-Verlag, New York-Heidelberg, 1975. A modern treatment of the theory of functions of a real variable, Third printing.

[41] Marc Hindry and Joseph H. Silverman. *Diophantine geometry: An introduction*, volume 201 of *Graduate Texts in Mathematics.* Springer-Verlag, New York, 2000.

[42] Ehud Hrushovski. The Mordell-Lang conjecture for function fields. *J. Amer. Math. Soc.*, 9(3):667–690, 1996.

[43] Ehud Hrushovski. A logic for global fields. `https://people.maths.ox.ac.uk/hrushovski/Talks/SAGA16.pdf`, 2016. Séminaire Arithmétique et géométrie algébrique.

[44] Ehud Hrushovski. The elementary theory of the frobenius automorphisms, 2022. https://arxiv.org/abs/math/0406514.

[45] Hideaki Ikoma. On the concavity of the arithmetic volumes. *Int. Math. Res. Not. IMRN*, (16):7063–7109, 2015.

[46] Marcos Isai Morales Inostroza. Essential minimum in families of lines, 2022. https://arxiv.org/abs/2201.11174.

[47] Kiumars Kaveh and A. G. Khovanskii. Note on the Grothendieck group of subspaces of rational functions and Shokurov's Cartier b-divisors. *Canad. Math. Bull.*, 57(3):562–572, 2014.

[48] Ha Huy Khoai. Heights for p-adic meromorphic functions and value distribution theory. *Tạp chí Toán Học J. Math.*, 20(1):14–29, 1992.

[49] Robert Lazarsfeld. *Positivity in algebraic geometry. I*, volume 48. Springer-Verlag, Berlin, 2004. Classical setting: line bundles and linear series.

[50] Pierre Lelong. Mesure de Mahler et calcul de constantes universelles pour les polynômes de n variables. *Math. Ann.*, 299(4):673–695, 1994.

[51] Thibaud Lemanissier and Jérôme Poineau. *Espaces de Berkovich globaux—catégorie, topologie, cohomologie*, volume 353 of *Progress in Mathematics*. Birkhäuser/Springer, Cham, 2024.

[52] Chunhui Liu. Arithmetic Fujita approximation over adelic curves, 2024. https://arxiv.org/abs/2401.15271.

[53] Wenbin Luo. A relative bigness inequality and equidistribution theorem over function fields. *Int. J. Number Theory*, 21(2):487–514, 2025.

[54] Jacob Lurie. Seminar on the Fargues-Fontaine Curve, author's website.

[55] Peter Meyer-Nieberg. *Banach lattices.* Universitext. Springer-Verlag, Berlin, 1991.

[56] Atsushi Moriwaki. Arithmetic height functions over finitely generated fields. *Invent. Math.*, 140(1):101–142, 2000.

[57] Atsushi Moriwaki. The canonical arithmetic height of subvarieties of an abelian variety over a finitely generated field. *J. Reine Angew. Math.*, 530:33–54, 2001.

[58] Atsushi Moriwaki. Zariski decompositions on arithmetic surfaces. *Publ. Res. Inst. Math. Sci.*, 48(4):799–898, 2012.

[59] Atsushi Moriwaki. Adelic divisors on arithmetic varieties. *Mem. Amer. Math. Soc.*, 242(1144):v+122, 2016.

[60] Jérôme Poineau. La droite de Berkovich sur **Z**. *Astérisque*, (334):viii+xii+284, 2010.

[61] Jérôme Poineau. Dynamique analytique sur **Z**. I : Mesures d'équilibre sur une droite projective relative, 2024. https://arxiv.org/abs/2201.08480.

[62] Jérôme Poineau. Dynamique analytique sur **Z**. II : écart uniforme entre Lattès et conjecture de Bogomolov-Fu-Tschinkel, 2024. https://arxiv.org/abs/2207.01574.

[63] Renata Possobon and Christian S. Rodrigues. Geometric properties of disintegration of measures. *Ergodic Theory and Dynamical Systems*, page 1–30, 2024.

[64] Binggang Qu and Hang Yin. Arithmetic Demailly approximation theorem. *Adv. Math.*, 458:Paper No. 109961, 24, 2024.

[65] Gaël Rémond. Géométrie diophantienne multiprojective. In *Introduction to algebraic independence theory*, volume 1752 of *Lecture Notes in Math.*, pages 95–131. Springer, Berlin, 2001.

[66] Julia Robinson. The undecidability of algebraic rings and fields. *Proc. Amer. Math. Soc.*, 10:950–957, 1959.

[67] Min Ru and Paul Vojta. A birational Nevanlinna constant and its consequences. *Amer. J. Math.*, 142(3):957–991, 2020.

[68] V. V. Shokurov. Prelimiting flips. *Tr. Mat. Inst. Steklova*, 240:82–219, 2003.

[69] Joseph H. Silverman. Heights and the specialization map for families of abelian varieties. *J. Reine Angew. Math.*, 342:197–211, 1983.

[70] Martino Stoffel. Real-valued differential forms on non-archimedean abelian varieties with totally degenerate reduction, 2021.

[71] Michał Szachniewicz. Existential closedness of $\overline{\mathbb{Q}}$ as a globally valued field via Arakelov geometry, 2023. https://arxiv.org/abs/2306.06275.

[72] Antoine Sédillot. Differentiability of the χ-volume function over an adelic curve, 2023. https://arxiv.org/abs/2303.03377.

[73] Antoine Sédillot. *Study of projective varieties over adelic curves*. PhD thesis, Université Paris Cité, 2024.

[74] Michael Temkin. Relative Riemann-Zariski spaces. *Israel J. Math.*, 185:1–42, 2011.

[75] Ravi Vakil. The rising sea, foundations of algebraic geometry. `http://math.stanford.edu/~vakil/216blog/FOAGaug2922public.pdf`, 2022.

[76] Paul Vojta. *Diophantine approximations and value distribution theory*, volume 1239 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1987.

[77] Paul Vojta. Integral points on subvarieties of semiabelian varieties. I. *Invent. Math.*, 126(1):133–181, 1996.

[78] André Weil. Arithmetic on algebraic varieties. *Ann. of Math. (2)*, 53:412–444, 1951.

[79] André Weil. *Basic number theory*. Classics in Mathematics. Springer-Verlag, Berlin, 1995. Reprint of the second (1973) edition.

[80] Robert Wilms. On the irreducibility and distribution of arithmetic divisors, 2022. https://arxiv.org/abs/2211.03766.

[81] Xinyi Yuan. Big line bundles over arithmetic varieties. *Invent. Math.*, 173(3):603–649, 2008.

[82] Xinyi Yuan. On volumes of arithmetic line bundles. *Compos. Math.*, 145(6):1447–1464, 2009.

[83] Xinyi Yuan and Shou-Wu Zhang. Adelic line bundles on quasi-projective varieties, 2024. https://arxiv.org/abs/2105.13587.

[84] Shouwu Zhang. Positive line bundles on arithmetic varieties. *J. Amer. Math. Soc.*, 8(1):187–221, 1995.

[85] Shouwu Zhang. Small points and adelic metrics. *J. Algebraic Geom.*, 4(2):281–300, 1995.