LECTURE HALL GRAPHS AND THE ASKEY SCHEME

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Abstract. We establish, for every family of orthogonal polynomials in the q-Askey scheme and the Askey scheme, a combinatorial model for mixed moments and coefficients in terms of paths on the lecture hall graph. This generalizes the previous results of Corteel and Kim for the little q-Jacobi polynomials. We build these combinatorial models by bootstrapping, beginning with polynomials at the bottom and working towards Askey–Wilson polynomials which sit at the top of the q-Askey scheme. As an application of the theory, we provide the first combinatorial proof of the symmetries in the parameters of the Askey–Wilson polynomials.

1. Introduction

Orthogonal polynomials are classical objects that play a central role in a wide range of areas of mathematics. Since the pioneering work of Flajolet [12] and Viennot [22, 23] in the 1980s, researchers have found them to possess many interesting combinatorial properties. See [8, 23, 24] and references therein for more details.

Orthogonal polynomials can be defined as a sequence $\{p_n(x)\}_{n\ge0}$ of polynomials with $\deg p_n(x) = n$ such that there is a linear functional $\mathcal{L}$ satisfying $\mathcal{L}(p_n(x)p_m(x)) = K_n\delta_{n,m}$, where $K_n \ne 0$. Their *moments* $\{\sigma_n\}_{n\ge0}$ are defined by

$$\sigma_n = \frac{\mathcal{L}(x^n)}{\mathcal{L}(1)}.$$

More generally, the *mixed moments* $\{\sigma_{n,k}\}_{n,k\ge0}$ are defined by

$$\sigma_{n,k} = \frac{\mathcal{L}(x^n p_k(x))}{\mathcal{L}(p_k(x)^2)}.$$

By the orthogonality, we have

$$x^n = \sum_{k=0}^{n} \sigma_{n,k} p_k(x),$$

which can be taken as the definition of the mixed moments. We also define the *coefficients* (or *dual mixed moments*) $\{\nu_{n,k}\}_{n,k\ge0}$ by

$$p_n(x) = \sum_{k=0}^{n} \nu_{n,k} x^k.$$

Viennot [22, 23] found a combinatorial interpretation for $\sigma_{n,k}$ using Motzkin paths. To illustrate, suppose that $\{p_n(x)\}_{n\ge0}$ is a sequence of monic orthogonal polynomials satisfying the 3-term recurrence

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x).$$

A *Motzkin path* is a path on $\mathbb{Z}^2$ from (a, b) to (c, d) consisting of up steps $(1, 1)$, horizontal steps $(1, 0)$, and down steps $(1, -1)$ that never go below the line $y = 0$. The weight $w(p)$ of a Motzkin path p is the product of the weights of the steps in p, where the weight of an up step is always 1, the weight of a horizontal step at height k is b_k, and the weight of a down step starting at height k is λ_k. Viennot showed that $\sigma_{n,k}$ is the sum of $w(p)$ for all Motzkin paths p from $(0, 0)$ to (n, k). For example,

$$\sigma_{3,1} = b_0^2 + b_0b_1 + b_1^2 + \lambda_1 + \lambda_2$$

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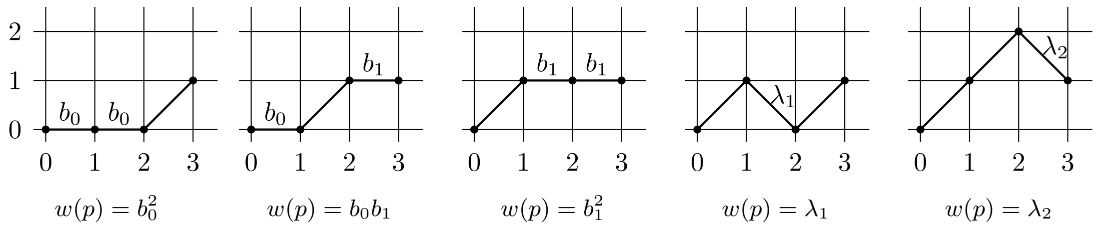

FIGURE 1. The Motzkin paths from $(0,0)$ to $(3,1)$ and their weights. The weights of horizontal and down steps are indicated.

is the generating function for all Motzkin paths from $(0,0)$ to $(3,1)$ as shown in Figure 1. Using this combinatorial interpretation for $\sigma_{n,k}$ one can show that the moments of Hermite, Charlier, and Laguerre polynomials are generating functions for perfect matchings, set partitions, and permutations, respectively. Viennot also found a combinatorial model for $\nu_{n,k}$ using lattice paths called Favard paths.

There is a natural way to extend univariate orthogonal polynomials to multivariate orthogonal polynomials using bialternant formulas. In the expansion of the multivariate orthogonal polynomials (resp. Schur polynomials) in terms of Schur polynomials (resp. multivariate orthogonal polynomials), the coefficients are given as determinants of $\nu_{n,k}$ (resp. $\sigma_{n,k}$). Therefore one may ask whether the Lindström–Gessel–Viennot (LGV) lemma [15] (see also [1, Chapter 32]) can be used to find an interpretation for the Schur coefficients using nonintersecting paths. However, neither Motzkin paths nor Favard paths give such a model, since their underlying graphs are not planar. One of the motivations of this paper is to find a planar graph that provides a lattice path interpretation for the mixed moments, so that it can be extended naturally to multivariate orthogonal polynomials.

Corteel and Kim [7] introduced the lecture hall graph in which paths are in natural bijection with lecture hall partitions and anti-lecture hall compositions. Lecture hall partitions with n nonnegative parts were originally defined in the enumeration of certain affine permutations coming from the affine Coxeter group $\tilde{C}_n$ [3]. They also have many connections with other combinatorial objects; see [20] for example.

Corteel and Kim [7] found a path model in the lecture hall graph for both mixed moments and coefficients of the little q-Jacobi polynomials. A nice feature of their combinatorial model is that it can be extended to give a tableau model for multivariate little q-Jacobi polynomials. Their discovery of the combinatorial model for little q-Jacobi polynomials, however, was somewhat by accident, because the coefficient $\nu_{n,k}$ of the little q-Jacobi polynomial happens to be equal to a formula, previously established in [9], for a generating function for truncated lecture hall partitions.

The goal of this paper is to develop a general interpretation of the mixed moments $\sigma_{n,k}$ and the coefficients $\nu_{n,k}$ of orthogonal polynomials as generating functions for paths in the lecture hall graph. We show that it is not a coincidence that paths on the lecture hall graph and little q-Jacobi polynomials are related. In fact, we prove that the relation is much stronger and that the orthogonal polynomials in the whole q-Askey scheme [18] can be studied this way.

We give a simple algorithm to construct a possible candidate giving a lecture hall path interpretation for both coefficients and mixed moments of orthogonal polynomials. This algorithm allows us to rediscover the lecture hall path interpretations for the little q-Jacobi polynomials without any prior knowledge of the generating function for truncated lecture hall partitions.

Our main result may be summarized in the following "meta" theorem. This is deliberately quite vague for now. The purpose of our manuscript is to develop the tools and results necessary to make it precise.

Theorem 1.1. *For every family of polynomials in the q-Askey scheme and in the Askey scheme, the mixed moments and coefficients have combinatorial models on the lecture hall graph.*

For example, Theorem 4.24 gives a combinatorial model for the mixed moments of the Askey–Wilson polynomials, and Proposition 2.8 together with this provides a model for their coefficients.

For certain polynomials in the q-Askey scheme, we can give different models: some will be just on the lecture hall graph of finite height; others will be on the graph of infinite height. For the polynomials in the q-Askey scheme, the models with finite height have the benefit that we can take the limit $q \to 1$ and those with infinite rows give a combinatorial model for the formal power series expansion of the mixed moments and the coefficients.

Here are some advantages of our lecture hall graph models. The first advantage of this method is that it gives a combinatorial model for the mixed moments and the coefficients with *the same graph model*, where southeast paths are used for mixed moments and northeast paths without consecutive east steps are used for coefficients. Note that Viennot's combinatorial models do not have this uniform property because Motzkin paths and Favard paths have different underlying graphs.

The second advantage is that it has a nice relationship with the q-Askey scheme. We start with the Stieltjes–Wigert polynomials and we establish simple lemmas that let us go up in the scheme. The only family that needs to be treated as a special case is that of the continuous q-Hermite polynomials.

The third advantage is that, since the lecture hall graph is planar, our combinatorial models can be naturally extended to study multivariate orthogonal polynomials using the Lindström–Gessel–Viennot lemma, which is not possible for Viennot's model. This was done in [7] for little q-Jacobi polynomials and we can now generalize this to the whole q-Askey scheme. In a forthcoming paper, we will give applications of our combinatorial models to multivariate versions of orthogonal polynomials, total positivity, and random matrix theory.

The last but not least advantage is that it gives a combinatorial model for the mixed moments of Askey–Wilson polynomials. Using this model we give the first combinatorial proof of the symmetry of the parameters a, b, c, d in the Askey–Wilson polynomials, which until now has been understood only analytically. We note that there is a combinatorial model, called staircase tableaux, for moments of Askey–Wilson polynomials with some reparametrization of a, b, c, d to $\alpha, \beta, \gamma, \delta$; see [11] and [10]. It is known that the new parameters $\alpha, \beta, \gamma, \delta$ are also symmetric, but finding a combinatorial proof is still open.

The paper is organized as follows. In Section 2 we define the lecture hall graph and we introduce mixed moments relative to different bases. In Section 3 we prove useful properties of the lecture hall graph, which will be used throughout the paper. In Section 4 we find weight systems for the mixed moments of Stieltjes–Wigert polynomials, q-Bessel polynomials, little q-Jacobi polynomials, big q-Jacobi polynomials, and Askey–Wilson polynomials, using a bootstrapping method. In Section 5 we give another bootstrapping method to find a lecture hall graph model for mixed moments of Askey–Wilson polynomials relative to continuous q-Hermite polynomials. In Section 6 we give another combinatorial model for mixed moments of Askey–Wilson polynomials. Using this model we give the first combinatorial proof of the symmetry of a, b, c, d in Askey–Wilson polynomials.

Finally, we remark that our method can also be applied to all orthogonal polynomials in the q-Askey scheme and in the Askey scheme. The remaining orthogonal polynomials will be covered in a forthcoming paper.

2. Preliminaries

In this section we give basic definitions and lemmas on the lecture hall graph. We then define mixed moments and coefficients of orthogonal polynomials relative to other bases. We will follow the standard notation of basic hypergeometric series; see for example [14].

2.1. Lecture hall graphs.

Lecture hall graphs. The lecture hall graph was first introduced by Corteel and Kim [7] in their study of little q-Jacobi polynomials. It has further been studied in [6] and [5].

For nonnegative integers t, i, j with $j \le i$, we denote

$$v_{i,j}^t = \left(i, t + \frac{j}{i+1}\right) \in \mathbb{Z} \times \mathbb{Q}.$$

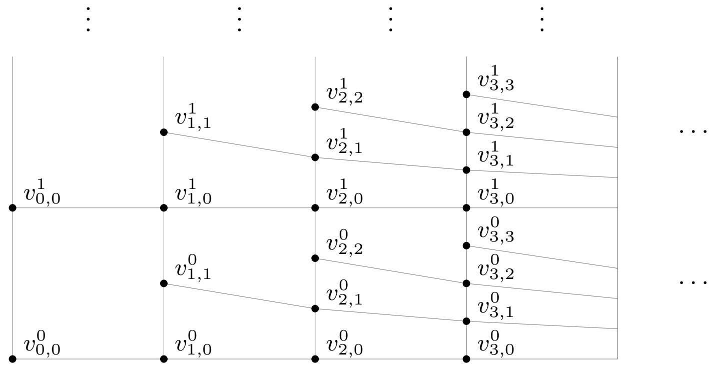

FIGURE 2. The lecture hall graph $\mathcal{G}$.

Definition 2.1. The *lecture hall graph* is the (undirected) graph $\mathcal{G} = (V, E)$, where

$$V = \{v^t_{i,j} : t, i, j \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq i\},$$
$$E = \{(v^t_{i,j}, v^t_{i,j+1}) : t, i, j \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq i\} \cup \{(v^t_{i,j}, v^t_{i+1,j}) : t, i, j \in \mathbb{Z}_{\geq 0}, 0 \leq j \leq i\}.$$

Here, $v^t_{i,i+1}$ is defined to be $v^{t+1}_{i,0}$; see Figure 2.

We consider three kinds of steps:

- an *east step* is a directed edge of the form $(v^t_{i,j}, v^t_{i+1,j})$ in $\mathcal{G}$,
- a *north step* is a directed edge of the form $(v^t_{i,j}, v^t_{i,j+1})$ or $(v^t_{i,i}, v^{t+1}_{i,0})$,
- a *south step* is a directed edge of the form $(v^t_{i,j}, v^t_{i,j-1})$ or $(v^t_{i,0}, v^{t-1}_{i,i})$.

For two vertices u and v in $\mathcal{G}$, a *path* from u to v is a sequence $(v_0, v_1, \dots, v_n)$ of vertices of $\mathcal{G}$ such that $v_0 = u$, $v_n = v$, and each (v_i, v_{i+1}) is an east, north, or south step. We also consider a path from $u \in \mathcal{G}$ to $v = (a, \infty)$, which is a sequence $(v_0, v_1, v_2, \dots)$ of vertices of $\mathcal{G}$ such that $v_0 = u$, $\lim_{n\to\infty} v_n = v$, and each (v_i, v_{i+1}) is an east, north, or south step. Similarly, a path from $u = (a, \infty)$ to $v \in \mathcal{G}$ is a sequence $(\dots, v_2, v_1, v_0)$ of vertices of $\mathcal{G}$ such that $v_0 = v$, $\lim_{n\to\infty} v_n = u$, and each (v_{i+1}, v_i) is an east, north, or south step.

Let $\mathrm{SE}(u \to v)$ denote the set of paths from u to v in $\mathcal{G}$ consisting of south and east steps. We also define $\mathrm{NE}^*(u \to v)$ to be the set of paths from u to v in $\mathcal{G}$ consisting of north and east steps with *no consecutive east steps*. Here, consecutive east steps means a pair (s_1, s_2) of steps of the form $s_1 = (v^t_{i,j}, v^t_{i+1,j})$ and $s_2 = (v^t_{i+1,j}, v^t_{i+2,j})$. For example, in Figure 3, the path $p_1 \in \mathrm{SE}((1,\infty) \to (5,0))$ has consecutive east steps $(v^2_{2,2}, v^2_{3,2})$ and $(v^2_{3,2}, v^2_{4,2})$ but the path $p_2 \in \mathrm{NE}^*((1,0) \to (5,\infty))$ does not have any consecutive east steps.

Paths in $\mathrm{SE}(u \to v)$ or $\mathrm{NE}^*(u \to v)$ are closely related to lecture hall partitions. To describe their connection we need some definitions.

A *lecture hall partition* is a nonnegative integer sequence $\lambda = (\lambda_1, \dots, \lambda_n)$ such that

$$\frac{\lambda_1}{n} \geq \frac{\lambda_2}{n-1} \geq \cdots \geq \frac{\lambda_n}{1}.$$

An *anti-lecture hall composition* is a nonnegative integer sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ such that

$$\frac{\alpha_1}{1} \geq \frac{\alpha_2}{2} \geq \cdots \geq \frac{\alpha_n}{n}.$$

As variations of the above objects, we consider the set $AL_{n,k}$ of *truncated anti-lecture hall compositions* and the set $RL^*_{n,k}$ of *truncated strict reverse lecture hall partitions* defined by

$$AL_{n,k} = \left\{ (\alpha_{k+1}, \dots, \alpha_n) : \frac{\alpha_{k+1}}{k+1} \geq \frac{\alpha_{k+2}}{k+2} \geq \cdots \geq \frac{\alpha_n}{n} \right\},$$
$$RL^*_{n,k} = \left\{ (\rho_{k+1}, \dots, \rho_n) : \frac{\rho_{k+1}}{k+1} < \frac{\rho_{k+2}}{k+2} < \cdots < \frac{\rho_n}{n} \right\}.$$

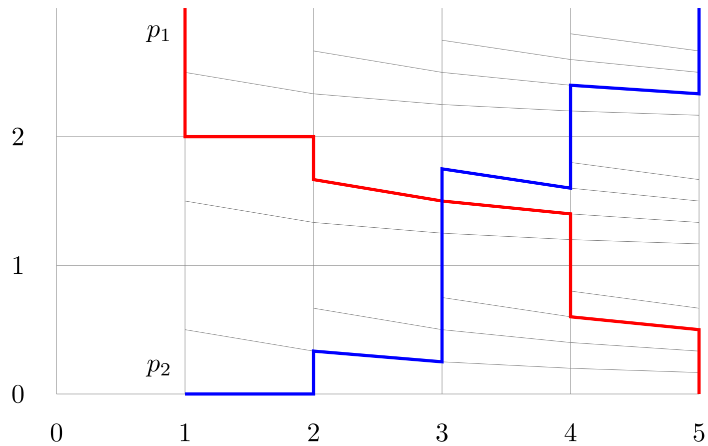

FIGURE 3. A path p_1 in $\mathrm{SE}((1,\infty) \to (5,0))$ and a path p_2 in $\mathrm{NE}^*((1,0) \to (5,\infty))$.

Now we define a map ϕ, which sends a path to a sequence of integers. Suppose that p is a path from (k,a) to (n,b) for some $k,n \in \mathbb{Z}_{\ge 0}$ and $a,b \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$. For $i = k+1,\dots,n$, suppose that the east step of p between $x = i-1$ and $x = i$ is the α_ith east step among all east steps between $x = i-1$ and $x = i$ from the bottom. In other words, if the east step of p between $x = i-1$ and $x = i$ is $(v^t_{i-1,j}, v^t_{i,j})$ in $\mathcal{G}$, then $\alpha_i = ti + j$. We define $\phi(p) = (\alpha_{k+1},\dots,\alpha_n)$. For example, if p_1 and p_2 are the paths in Figure 3, then $\phi(p_1) = (4,5,6,3)$ and $\phi(p_2) = (0,1,7,12)$. This map gives natural bijections from paths in a lecture hall graph to truncated anti-lecture hall compositions and truncated reverse lecture hall partitions.

Proposition 2.2. *The map ϕ induces the following two bijections:*

$$\phi : \mathrm{SE}((k,\infty) \to (n,0)) \to AL_{n,k},$$
$$\phi : \mathrm{NE}^*((k,0) \to (n,\infty)) \to RL^*_{n,k}.$$

Proof. Let $p \in \mathrm{SE}((k,\infty) \to (n,0))$ and $\phi(p) = (\alpha_{k+1},\dots,\alpha_n)$. By the construction of ϕ, for $i = k+1,\dots,n$, if the east step of p between $x = i-1$ and $x = i$ is $(v^t_{i-1,j}, v^t_{i,j})$, then $\alpha_i = ti+j$. Since $\alpha_i/i = t + j/i$ is the y-coordinate of the starting point $v^t_{i-1,j}$ of this east step, we have $\frac{\alpha_{k+1}}{k+1} \ge \frac{\alpha_{k+2}}{k+2} \ge \cdots \ge \frac{\alpha_n}{n}$. Thus $(\alpha_{k+1},\dots,\alpha_n) \in AL_{n,k}$. Conversely, for $(\alpha_{k+1},\dots,\alpha_n) \in AL_{n,k}$, we can reconstruct $p \in \mathrm{SE}((k,\infty) \to (n,0))$ using the relations $t = \lfloor \alpha_i/i \rfloor$ and $j = \alpha_i - i\lfloor \alpha_i/i \rfloor$. This shows the first bijection. The second bijection can be proved similarly. □

Our main goal is to find combinatorial models for mixed moments $\sigma_{n,k}$ and coefficients $\nu_{n,k}$ of orthogonal polynomials. We will see later in Proposition 2.8 that a combinatorial model for $\sigma_{n,k}$ using $\mathrm{SE}(u \to v)$ immediately gives a combinatorial model for $\nu_{n,k}$ using $\mathrm{NE}^*(u \to v)$, and vice versa. Hence, in this paper we will mostly consider the set $\mathrm{SE}(u \to v)$ rather than $\mathrm{NE}^*(u \to v)$. For brevity, we will write $p : u \to v$ to mean $p \in \mathrm{SE}(u \to v)$.

A *weight system* is a function w that assigns a weight $w(s)$ to each east step $s = (v^t_{i,j}, v^t_{i+1,j})$ in $\mathcal{G}$. We denote

$$w(t;i,j) := w(v^t_{i,j}, v^t_{i+1,j}).$$

In other words, $w(t;i,j)$ is the weight of the jth east step from the bottom, where the bottommost one is the 0th step, among the east steps in the region $\{(x,y) : i \le x \le i+1, t \le y < t+1\}$; see Figure 4. Given a weight system w, the weight $w(p)$ of a path p is defined to be the product of the weights of all east steps in p.

The following definition will be used throughout this paper.

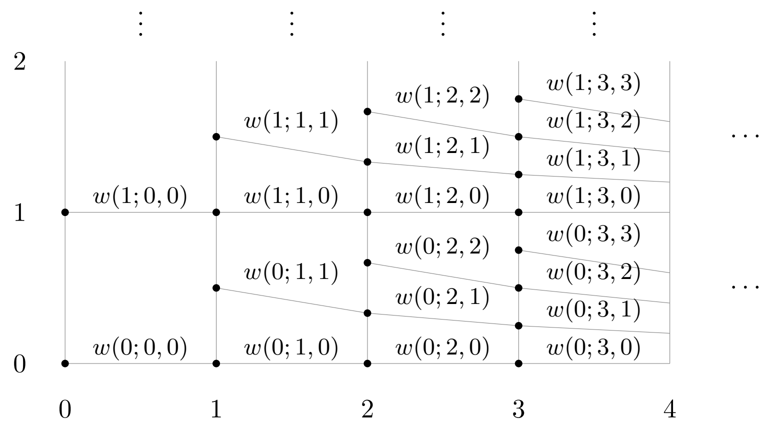

FIGURE 4. An illustration of $w(t; i, j)$.

Definition 2.3. For a weight system w and two nonnegative integers n and k, we define

$$h^w_{n,k} = \sum_{p\in \mathrm{SE}((k,\infty)\to(n,0))} w(p),$$

$$e^w_{n,k} = \sum_{p\in \mathrm{NE}^*((k,0)\to(n,\infty))} w(p).$$

By definition, we have $h^w_{n,k} = e^w_{n,k} = 0$ if $n < k$ and $h^w_{n,n} = e^w_{n,n} = 1$. Note that $h^w_{n,k}$ and $e^w_{n,k}$ generalize the *homogeneous symmetric function*

$$h_n(x_0, x_1, \dots) = \sum_{0\le i_0\le i_1\le\cdots\le i_{n-1}} x_{i_0}x_{i_1}\cdots x_{i_{n-1}}$$

and the *elementary symmetric function*

$$e_n(x_0, x_1, \dots) = \sum_{0\le i_0< i_1<\cdots< i_{n-1}} x_{i_0}x_{i_1}\cdots x_{i_{n-1}}$$

in the following sense: if w is the weight system defined by

$$w(t; i, j) = \begin{cases} x_t & \text{if } j = 0, \\ 0 & \text{otherwise,} \end{cases}$$

then we have

$$h_n(x_0, x_1, \dots) = h^w_{n,0}, \qquad e_n(x_0, x_1, \dots) = e^w_{n,0}.$$

The quantities $h^w_{n,k}$ and $e^w_{n,k}$ are dual to each other in the following sense.

Lemma 2.4. *We have the matrix identity*

$$(h^w_{n,k})^\infty_{n,k=0} = \left(((-1)^{n-k}e^w_{n,k})^\infty_{n,k=0}\right)^{-1}.$$

Equivalently, for $n \ge m$, we have

$$\sum_{r=m}^{n} h^w_{n,r}(-1)^{r-m}e^w_{r,m} = \delta_{n,m}.$$

Proof. The equivalence of the two identities is immediate from the fact that $(h^w_{n,k})^\infty_{n,k=0}$ is lower triangular. Thus it suffices to prove the second identity. Since this identity is clearly true for $n = m$, we assume that $n > m$. We follow the argument in the proof of [7, Proposition 3.5].

By Proposition 2.2, we can write

$$\sum_{r=m}^{n} h^w_{n,r}(-1)^{r-m}e^w_{r,m} = \sum_{(\rho,\alpha)\in X} W(\rho, \alpha), \tag{2.1}$$

where X is the set of pairs (ρ,α) of nonnegative integer sequences $\rho=(\rho_{m+1},\rho_{m+2},\dots,\rho_r)$ and $\alpha=(\alpha_{r+1},\alpha_{r+2},\dots,\alpha_n)$, for some r, such that

$$\frac{\rho_{m+1}}{m+1}<\frac{\rho_{m+2}}{m+2}<\cdots<\frac{\rho_r}{r},\qquad \frac{\alpha_{r+1}}{r+1}\ge\frac{\alpha_{r+2}}{r+2}\ge\cdots\ge\frac{\alpha_n}{n}, \tag{2.2}$$

and $W(\rho,\alpha)$ is defined by

$$W(\rho,\alpha)=(-1)^{r-m}\prod_{i=m+1}^{r}w(\lfloor\rho_i/i\rfloor;i,\rho_i-i\lfloor\rho_i/i\rfloor)\prod_{i=r+1}^{n}w(\lfloor\alpha_i/i\rfloor;i,\alpha_i-i\lfloor\alpha_i/i\rfloor).$$

We define a sign-reversing involution on X as follows. Suppose that $(\rho,\alpha)\in X$ is given as in (2.2). If $\alpha_{r+1}/(r+1)\le\rho_r/r$, then let $\alpha'=(\rho_r,\alpha_{r+1},\dots,\alpha_n)$ and $\rho'=(\rho_{m+1},\dots,\rho_{r-1})$, and otherwise let $\alpha'=(\alpha_{r+2},\dots,\alpha_n)$ and $\rho'=(\rho_{m+1},\dots,\rho_r,\alpha_{r+1})$. Then $(\rho',\alpha')\in X$ and $W(\rho',\alpha')=-W(\rho,\alpha)$. It is easy to check that the map $(\rho,\alpha)\mapsto(\rho',\alpha')$ is a sign-reversing involution on X with no fixed point if $n>m$. Thus (2.1) is equal to 0 and the proof is completed. ☐

The following simple lemma will be used frequently in this paper.

Lemma 2.5. *Suppose that w and w' are weight systems such that $w'(t;i,j)=C_i\cdot w(t;i,j)$ for all $t,i,j\in\mathbb{Z}_{\ge0}$ with $j\le i$. Then*

$$h^{w'}_{n,k}=C_kC_{k+1}\cdots C_{n-1}\cdot h^{w}_{n,k}.$$

Proof. This follows from the observation that

$$\begin{aligned}h^{w'}_{n,k}&=\sum_{p\in\mathrm{SE}((k,\infty)\to(n,0))}w'(p)\\&=\sum_{p\in\mathrm{SE}((k,\infty)\to(n,0))}C_kC_{k+1}\cdots C_{n-1}w(p)=C_kC_{k+1}\cdots C_{n-1}\cdot h^{w}_{n,k},\end{aligned}$$

because every $p\in\mathrm{SE}((k,\infty)\to(n,0))$ has exactly one east step between $x=i$ and $x=i+1$ for $i=k,k+1,\dots,n-1$. ☐

2.2. **Mixed moments relative to other bases.** Recall that the mixed moments $\{\sigma_{n,k}\}_{n,k\ge0}$ of orthogonal polynomials $\{p_n(x)\}_{n\ge0}$ with respect to a linear functional $\mathcal{L}$ are defined by

$$\sigma_{n,k}=\frac{\mathcal{L}(x^np_k(x))}{\mathcal{L}(p_k(x)^2)},$$

or equivalently,

$$x^n=\sum_{k=0}^{n}\sigma_{n,k}p_k(x).$$

By abuse of terminology, we extend the definition of the mixed moments $\sigma_{n,k}$ to polynomials that are not necessarily orthogonal polynomials.

Definition 2.6. Let $\{p_n(x)\}_{n\ge0}$ be a sequence of polynomials with $\deg p_n(x)=n$. The *mixed moments* $\{\sigma_{n,k}\}_{n,k\ge0}$ and the *coefficients* $\{\nu_{n,k}\}_{n,k\ge0}$ of $\{p_n(x)\}_{n\ge0}$ are defined by

$$x^n=\sum_{k=0}^{n}\sigma_{n,k}p_k(x),\qquad p_n(x)=\sum_{k=0}^{n}\nu_{n,k}x^k.$$

More generally, we also define mixed moments and coefficients by choosing a basis other than $\{x^n\}_{n\ge0}$.

Definition 2.7. Let $\{p_n(x)\}_{n\ge0}$ and $\{q_n(x)\}_{n\ge0}$ be sequences of polynomials with $\deg p_n(x)=\deg q_n(x)=n$. We define the *mixed moments* $\{\sigma_{n,k}\}_{n,k\ge0}$ and the *coefficients* $\{\nu_{n,k}\}_{n,k\ge0}$ of $\{p_n(x)\}_{n\ge0}$ *relative to* $\{q_n(x)\}_{n\ge0}$ by

$$q_n(x)=\sum_{k=0}^{n}\sigma_{n,k}p_k(x),\qquad p_n(x)=\sum_{k=0}^{n}\nu_{n,k}q_k(x).$$

Note that the original mixed moments $\{\sigma_{n,k}\}_{n,k\ge0}$ of $\{p_n(x)\}_{n\ge0}$ are the mixed moments of $\{p_n(x)\}_{n\ge0}$ relative to the standard basis polynomials $\{x^n\}_{n\ge0}$. Note also that, by definition, the mixed moments $\{\sigma_{n,k}\}_{n,k\ge0}$ and the coefficients $\{\nu_{n,k}\}_{n,k\ge0}$ of $\{p_n(x)\}_{n\ge0}$ relative to $\{q_n(x)\}_{n\ge0}$ are the entries of the change-of-basis matrices between the two bases $\{p_n(x)\}_{n\ge0}$ and $\{q_n(x)\}_{n\ge0}$ of the space of polynomials. Therefore the two matrices $(\sigma_{n,k})_{n,k=0}^\infty$ and $(\nu_{n,k})_{n,k=0}^\infty$ are inverses of each other. By this fact and Lemma 2.4, we obtain the following proposition.

Proposition 2.8. *Let $\sigma_{n,k}$ and $\nu_{n,k}$ be the mixed moments and coefficients of polynomials $\{p_n(x)\}_{n\ge0}$ (with respect to some basis $\{q_n(x)\}_{n\ge0}$). If w is a weight system such that*

$$\sigma_{n,k} = h^w_{n,k}, \qquad n \ge k \ge 0, \tag{2.3}$$

then

$$\nu_{n,k} = (-1)^{n-k} e^w_{n,k}, \qquad n \ge k \ge 0. \tag{2.4}$$

Conversely, if w is a weight system satisfying (2.4)*, then* (2.3) *also holds.*

Proposition 2.8 implies that if we have a lecture hall graph model for $\sigma_{n,k}$ then the same weight system also gives a lecture hall graph model for $(-1)^{n-k}\nu_{n,k}$ and vice versa. Therefore we will henceforth only focus on finding a lecture hall graph model for the mixed moments $\sigma_{n,k}$.

By the following lemma with $b_n(x) = x^n$, we can use the mixed moments $\{\tilde\sigma_{n,k}\}_{n,k\ge0}$ relative to $\{q_n(x)\}_{n\ge0}$ as an intermediate step to study the original mixed moments $\{\sigma_{n,k}\}_{n,k\ge0}$.

Lemma 2.9. *Let $\{p_n(x)\}_{n\ge0}$, $\{q_n(x)\}_{n\ge0}$, and $\{b_n(x)\}_{n\ge0}$ be sequences of polynomials with $\deg p_n(x) = \deg q_n(x) = \deg b_n(x) = n$ and let*

$$b_n(x) = \sum_{k=0}^n \sigma_{n,k} p_k(x),$$

$$b_n(x) = \sum_{k=0}^n \tau_{n,k} q_k(x),$$

$$q_n(x) = \sum_{k=0}^n \tilde\sigma_{n,k} p_k(x).$$

Then we have

$$\sigma_{n,k} = \sum_{r=k}^n \tau_{n,r}\tilde\sigma_{r,k}.$$

Proof. Observe that

$$\sum_{k=0}^n \sigma_{n,k} p_k(x) = b_n(x) = \sum_{r=0}^n \tau_{n,r} q_r(x) = \sum_{r=0}^n \tau_{n,r} \sum_{k=0}^r \tilde\sigma_{r,k} p_k(x).$$

Since $\{p_n(x)\}_{n\ge0}$ is a basis of the polynomial space, we obtain the lemma. □

We note that an equivalent statement of Lemma 2.9, for the case that $\{p_n(x)\}_{n\ge0}$ and $\{q_n(x)\}_{n\ge0}$ are orthogonal polynomials and $b_n(x) = x^n$, has been proved in [17, Proposition 2.2]. Lemma 2.9 allows us to find a lecture hall graph model for $\sigma_{n,k}$ using those for $\tau_{n,k}$ and $\tilde\sigma_{n,k}$. To give a precise statement we introduce some definitions.

Definition 2.10. The *height* of a weight system w is the smallest integer ℓ such that $w(t; i, j) = 0$ for all $t \ge \ell$. If there is no such integer, the height of w is defined to be ∞. For a weight system $w^{(1)}$ of height $\ell < \infty$ and any weight system $w^{(2)}$, we define $w^{(1)} \sqcup w^{(2)}$ to be the weight system obtained by adding $w^{(2)}$ on top of $w^{(1)}$, that is,

$$(w^{(1)} \sqcup w^{(2)})(t; i, j) = \begin{cases} w^{(1)}(t; i, j) & \text{if } t < \ell, \\ w^{(2)}(t - \ell; i, j) & \text{if } t \ge \ell. \end{cases}$$

Lemma 2.11. *Let* $\sigma_{n,k}, \tau_{n,k}$*, and* $\tilde{\sigma}_{n,k}$ *be given as in Lemma 2.9. Suppose that* $w^{(1)}$ *is a weight system of finite height with* $h_{n,k}^{w^{(1)}} = \tau_{n,k}$ *and that* $w^{(2)}$ *is a weight system with* $h_{n,k}^{w^{(2)}} = \tilde{\sigma}_{n,k}$*. Then* $h_{n,k}^{w^{(1)}\sqcup w^{(2)}} = \sigma_{n,k}$*.*

Proof. Let ℓ be the height of $w^{(1)}$. Since

$$\begin{aligned} h_{n,k}^{w^{(1)}\sqcup w^{(2)}} &= \sum_{p:(k,\infty)\to(n,0)} (w^{(1)} \sqcup w^{(2)})(p) \\ &= \sum_{r=k}^{n} \left(\sum_{p:(k,\infty)\to(r,\ell)} w^{(2)}(p) \right) \left(\sum_{p:(r,\ell)\to(n,0)} w^{(1)}(p) \right) = \sum_{r=k}^{n} \tau_{n,r}\tilde{\sigma}_{r,k}, \end{aligned}$$

we have $h_{n,k}^{w^{(1)}\sqcup w^{(2)}} = \sigma_{n,k}$. □

For a sequence $d = (d_0, d_1, d_2, \ldots)$, define

$$(x|d)^n = (x - d_0)(x - d_1)\cdots(x - d_{n-1}).$$

We will call the mixed moments $\tilde{\sigma}_{n,k}$ relative to $\{(x|d)^n\}_{n\geq 0}$ the *factorial mixed moments*. In most cases we will consider the original mixed moments $\sigma_{n,k}$, the factorial mixed moments $\tilde{\sigma}_{n,k}$ for some $(x|d)_n$, and the mixed moments relative to the continuous q-Hermite polynomials.

3. Properties of weight systems

In this section we prove useful properties of weight systems of height 1 and weight systems of infinite height. Firstly, we consider weight systems of height 1. In this case our lecture hall graph model is equivalent to a special case of the planar network of Fomin and Zelevinsky [13]. Using their result [13, Lemma 6], given a lower unitriangular matrix $(\sigma_{n,k})_{n,k=0}^{\infty}$, one can deduce that there is a unique weight system w of height 1 satisfying $h_{n,k}^{w} = \sigma_{n,k}$. We give an explicit formula for the weight system w using minors of the matrix $(\sigma_{n,k})_{n,k=0}^{\infty}$.

In the case of weight systems of infinite height, there can be many weight systems w satisfying $h_{n,k}^{w} = \sigma_{n,k}$. We provide a simple algorithm to discover a weight system of infinite height. Although our algorithm does not always produce a correct weight system, it does provide a valid one for all orthogonal polynomials in the q-Askey scheme except for the case of the continuous q-Hermite polynomials. In particular, this algorithm allows us to discover systematically the weight system for the mixed moments of little q-Jacobi polynomials in [7], which was first discovered "by accident" when they happened to come across the results in lecture hall partitions [9]. Furthermore, this algorithm can also be used to discover weight systems for big q-Jacobi polynomials and Askey–Wilson polynomials in this paper. We also provide various recurrences for weight systems which can be used to prove that a given weight system w indeed satisfies $h_{n,k}^{w} = \sigma_{n,k}$.

3.1. Weight systems of height 1.

In this subsection we will show that weight systems of height 1 are particularly useful because their weights are uniquely determined by $h_{n,k}^{w}$ and they can be used to build other weight systems. Observe that the lecture hall graph of height 1 can be identified with a staircase grid as shown in Figure 5. This implies that the total number of paths $p : (k, 1) \to (n, 0)$ is $\binom{n}{k}$.

In order to show the uniqueness for a weight system of height 1, we introduce some notation. Given a matrix $A = (a_{i,j})_{i,j=0}^{\infty}$, let $A_r(i, j)$ denote the minor of A with row indices $i, i+1, \ldots, i+r-1$ and column indices $j, j+1, \ldots, j+r-1$.

Proposition 3.1. *Let* $\{a_{n,k}\}_{n\geq k\geq 0}$ *be a triangular array of indeterminates and let* $A = (a_{n,k})_{n,k\geq 0}$*, where* $a_{n,k} = 0$ *if* $n < k$*. Then there is a unique weight system* w *of height* 1 *satisfying* $h_{n,k}^{w} = a_{n,k}$ *for all* $n \geq k \geq 0$*. Moreover, each* $w(0; i, j)$ *is the rational function in the indeterminates* $a_{n,k}$ *given by*

$$w(0; i, j) = \frac{A_{j+1}(i - j + 1, 0)A_j(i - j, 0)}{A_j(i - j + 1, 0)A_{j+1}(i - j, 0)}. \tag{3.1}$$

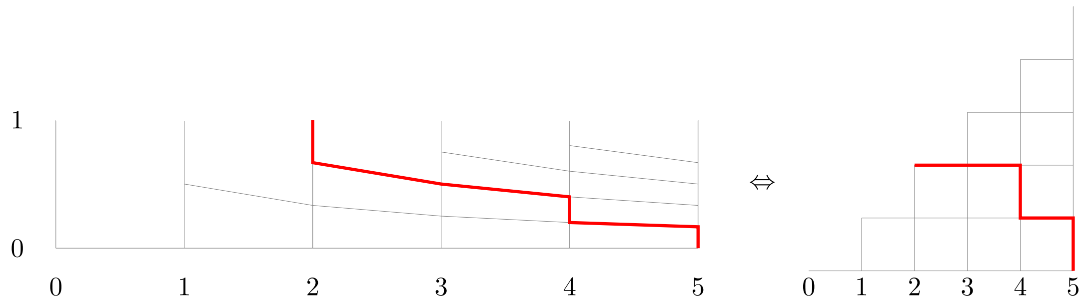

FIGURE 5. A path in the lecture hall graph of height 1 and its corresponding path in the staircase lattice.

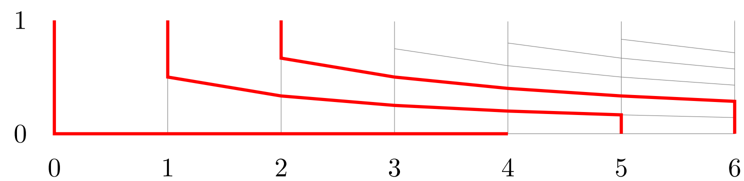

FIGURE 6. The unique family of nonintersecting paths from $(0,1),(1,1),(2,1)$ to $(4,0),(5,0),(6,0)$.

Proof. Suppose that w is a weight system of height 1 satisfying $h^w_{n,k} = a_{n,k}$ for all $n \ge k \ge 0$. Then, by the LGV lemma,

$$A_r(k,0) = \prod_{a=0}^{r-1}\prod_{b=0}^{k-1} w(0;a+b,a), \tag{3.2}$$

because there is only one nonintersecting family of paths from $(0,1),(1,1),\dots,(r-1,1)$ to $(k,0),(k+1,0),\dots,(k+r-1,0)$. More precisely, for $0 \le a \le r-1$, the unique path from $(a,1)$ to $(k+a,0)$ has the east steps of weights $w(0;a,a), w(0;a+1,a),\dots,w(0;a+k-1,a)$. See Figure 6 for an example. By substituting (3.2) into the right-hand side of (3.1), we get the left-hand side of (3.1). □

Remark 3.2. Our path model also has a close connection with totally positive matrices. Fomin and Zelevinsky [13] showed that a square matrix A is totally positive if and only if every "initial" minor is positive. Moreover, they showed that in this case there is a unique weight system on a certain planar graph that gives a combinatorial meaning to the (i,j)-entry of A. Our path model of height 1 for the mixed moment $\sigma_{n,k}$ is the unique weight system of Fomin and Zelevinsky for the lower unitriangular matrix $(\sigma_{n,k})_{n,k=0}^{\infty}$. As an application we obtain a total positivity of the matrix of the mixed moments (and also for the coefficients) of the big q-Jacobi polynomials with some reparametrization. More details will be given in our forthcoming paper.

Remark 3.3. Nakagawa et al. [19] used a lattice equivalent to a weight system of height 1 in their study of Macdonald's ninth variation of Schur functions.

The following proposition shows that given an array $\{a_{n,k}\}$, finding a height 1 weight system w with $h^w_{n,k} = a_{n,k}$ is equivalent to the same problem with $e^w_{n,k} = a_{n,k}$. Hence, by Proposition 3.1, there is also a unique weight system w of height 1 such that $e^w_{n,k} = a_{n,k}$.

Proposition 3.4. *Let w be a weight system of height* 1*. Define the weight system $\overline{w}$ of height* 1 *by $\overline{w}(0;i,j) = w(0;i,i-j)$ for $0 \le j \le i$. Then*

$$h^w_{n,k} = e^{\overline{w}}_{n,k}, \qquad e^w_{n,k} = h^{\overline{w}}_{n,k}.$$

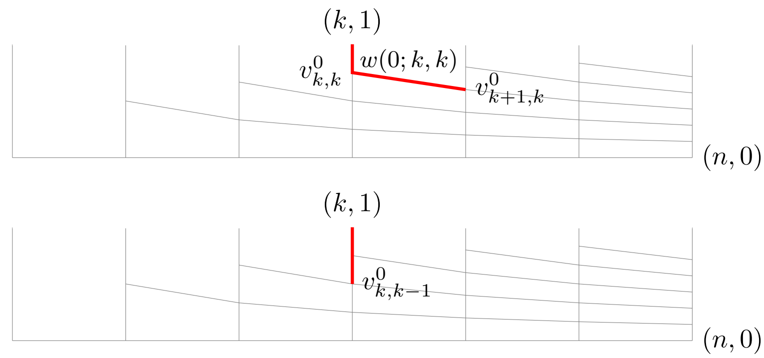

FIGURE 7. The beginning of a path $p : (k, 1) \to (n, 0)$ is drawn as in the first diagram if it has an east step of maximum height, which has weight $w(0; k, k)$. Otherwise, p is drawn as in the second diagram.

Proof. By Proposition 2.2, we have

$$h^w_{n,k} = \sum_{\lambda \in X} \prod_{i=k}^{n-1} w(0; i, \lambda_{i+1}),$$

where X is the set of integer sequences $\lambda = (\lambda_{k+1}, \ldots, \lambda_n)$ such that

$$1 > \frac{\lambda_{k+1}}{k+1} \geq \frac{\lambda_{k+2}}{k+2} \geq \cdots \geq \frac{\lambda_n}{n} \geq 0.$$

Similarly,

$$e^{\overline{w}}_{n,k} = \sum_{\mu \in Y} \prod_{i=k}^{n-1} \overline{w}(0; i, \mu_{i+1}),$$

where Y is the set of integer sequences $\mu = (\mu_{k+1}, \ldots, \mu_n)$ such that

$$0 \leq \frac{\mu_{k+1}}{k+1} < \frac{\mu_{k+2}}{k+2} < \cdots < \frac{\mu_n}{n} < 1.$$

For $\lambda \in X$, define $\psi(\lambda) = \mu$ by $\mu_i = i - 1 - \lambda_i$. It is straightforward to check that $\psi : X \to Y$ is a weight-preserving bijection, which implies the desired identity. □

The following two lemmas give recurrences for $h^w_{n,k}$, which can be used to prove an identity of the form $\sigma_{n,k} = h^w_{n,k}$.

Lemma 3.5. *Let w be a weight system of height 1. We have*

$$h^w_{n,k} = w(0; k, k) h^{w'}_{n-1,k} + h^{w'}_{n-1,k-1},$$

where $w'(0; i, j) = w(0; i + 1, j)$ for $i, j \in \mathbb{Z}_{\geq 0}$.

Proof. Since w is of height 1, we have

$$h^w_{n,k} = \sum_{p:(k,1)\to(n,0)} w(p).$$

Consider a path $p : (k, 1) \to (n, 0)$. If the first east step of p is the kth east step from the bottom, then p passes through $v^0_{k+1,k}$. Otherwise p passes through $v^0_{k,k-1}$ as shown in Figure 7. Thus,

$$h^w_{n,k} = w(0; k, k) \sum_{p:v^0_{k+1,k}\to(n,0)} w(p) + \sum_{p:v^0_{k,k-1}\to(n,0)} w(p). \tag{3.3}$$

Now consider a path $p : v^0_{k,k-1} \to (n, 0)$. Note that p is determined by its east steps. Let $p' : v^0_{k-1,k-1} \to (n - 1, 0)$ be the path obtained from p by replacing each east step $(v^0_{i,j}, v^0_{i+1,j})$ of p by an east step $(v^0_{i-1,j}, v^0_{i,j})$ as shown in Figure 8. By construction, we have $w(p) = w'(p')$.

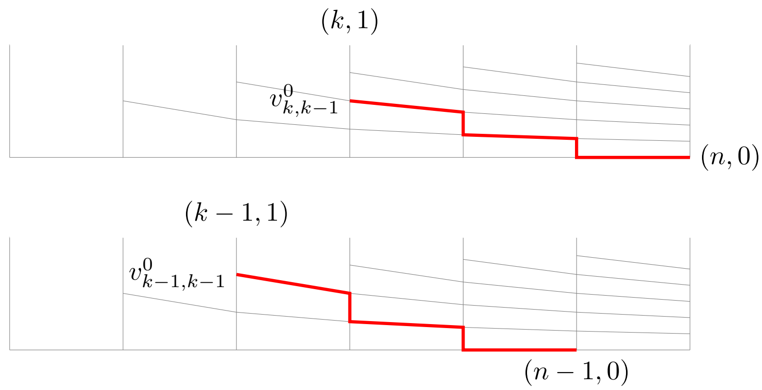

FIGURE 8. The correspondence between a path $p : v^0_{k,k-1} \to (n, 0)$ and a path $p : v^0_{k-1,k-1} \to (n-1, 0)$.

Since the map $p \mapsto p'$ is a bijection from the set of paths $p : v^0_{k,k-1} \to (n, 0)$ to the set of paths $p' : v^0_{k-1,k-1} \to (n-1, 0)$, we have

$$\sum_{p:v^0_{k,k-1}\to(n,0)} w(p) = \sum_{p':v^0_{k-1,k-1}\to(n-1,0)} w'(p') = h^{w'}_{n-1,k-1}. \tag{3.4}$$

The lemma then follows from (3.3) and (3.4). □

Lemma 3.6. *Let w be a weight system of height 1. We have*

$$h^w_{n,k} = w(0; n-1, 0)h^w_{n-1,k} + h^{w^+}_{n-1,k-1},$$

where $w^+(0; i, j) = w(0; i+1, j+1)$ for $i, j \in \mathbb{Z}_{\geq 0}$.

Proof. This can be proved similarly as in the proof of Lemma 3.5 except that we consider the last east step of $p : (k, 1) \to (n, 0)$ instead of the first east step. □

Using Lemma 3.6 we can prove the following proposition, which states that a weight system for mixed moments can be obtained from a weight system for factorial mixed moments by adding a weight system of height 1 at the bottom.

Proposition 3.7. *Fix a polynomial sequence $\{p_n(x)\}_{n\geq 0}$ with $\deg(p_n(x)) = n$. Let $\sigma_{n,k}$ and $\tilde{\sigma}_{n,k}$ be the mixed moments and the factorial mixed moments of $\{p_n(x)\}_{n\geq 0}$, i.e.,*

$$x^n = \sum_{k=0}^{n} \sigma_{n,k} p_k(x), \qquad (x|d)^n = \sum_{k=0}^{n} \tilde{\sigma}_{n,k} p_k(x).$$

Suppose that $\tilde{\sigma}_{n,k} = h^{\widetilde{w}}_{n,k}$ for a weight system $\widetilde{w}$. Then $\sigma_{n,k} = h^w_{n,k}$, where w is the weight system defined by

$$w(t; i, j) = \begin{cases} d_j & \text{if } t = 0, \\ \widetilde{w}(t-1; i, j) & \text{if } t \geq 1. \end{cases}$$

Proof. By Lemma 2.11, it suffices to show that

$$x^n = \sum_{k=0}^{n} h^{w_1}_{n,k} (x|d)^k, \tag{3.5}$$

where w_1 is the weight system of height 1 defined by $w_1(0; i, j) = d_j$. We will prove this by induction on n.

If $n = 0$, then $k = 0$ and both sides of (3.5) are equal to 1. Let $n \geq 1$ and suppose that (3.5) holds for $n - 1$. By Lemma 3.6,

$$h^{w_1}_{n,k} = d_0 h^{w_1}_{n-1,k} + h^{w_1^+}_{n-1,k-1}.$$

Then, by the induction hypothesis,

$$\begin{aligned}\sum_{k=0}^{n} h^{w_1}_{n,k}(x|d)^k &= d_0 \sum_{k=0}^{n-1} h^{w_1}_{n-1,k}(x|d)^k + (x-d_0)\sum_{k=1}^{n} h^{w_1^+}_{n-1,k-1}(x|d^+)^{k-1}\\ &= d_0 x^{n-1} + (x-d_0)x^{n-1} = x^n,\end{aligned}$$

where $d^+ = (d_1, d_2, \ldots)$. Thus (3.5) also holds for n. □

3.2. **Weight systems of infinite height.** In this subsection we consider weight systems of infinite height. To see the difference between weight systems of infinite height and those of height 1, suppose that given a triangular array $\{a_{n,k}\}_{n\ge k\ge 0}$ we want to find a weight system w such that $h^w_{n,k} = a_{n,k}$. If w is a weight system of height 1, then it is uniquely determined and we can use the formula in Proposition 3.1 to get the weight system. If, on the other hand, w is of infinite height, the weight w is not uniquely determined in general. How can we find such a weight system then? We propose the following algorithm to construct a possible candidate w.

Algorithm 1 (Finding a weight system of infinite height)**.** Fix a monomial ordering for the monomials of the parameters in $\{a_{n,k}\}_{n\ge k\ge 0}$. Define a weight system w such that the weight of the jth east step from the bottom between $x = n-1$ and $x = n$ to be the jth term of $a_{n,n-1}$ according to the monomial ordering.

It may seem too optimistic to hope that such a simple algorithm provides a correct weight system. However, this algorithm gives a valid weight system of infinite height for all orthogonal polynomials in the q-Askey scheme except the continuous q-Hermite polynomials. Once we have a candidate weight system w explicitly constructed via this algorithm, we can prove the identity $h^w_{n,k} = a_{n,k}$ using recursions such as the one in the following lemma.

Lemma 3.8. *For any weight system w and a positive integer ℓ, we have*

$$h^w_{n,k} = \sum_{r=k}^{n} h^{w_\ell}_{n,r} h^{w_\ell^+}_{r,k},$$

where w_ℓ is the weight system of height ℓ defined by

$$w_\ell(t;i,j) = \begin{cases} w(t;i,j) & \text{if } t<\ell,\\ 0 & \text{if } t\ge \ell,\end{cases}$$

and w_ℓ^+ is the weight system defined by $w_\ell^+(t;i,j) = w(t+\ell;i,j)$.

Proof. Consider a path $p : (k,\infty) \to (n,0)$. Let r be the largest integer such that p passes through (r,ℓ). Then we can decompose p as $p = p_1p_2$, where $p_1 : (k,\infty) \to (r,\ell)$ and $p_2 : (r,\ell) \to (n,0)$, and p_2 starts with a south step. Thus,

$$\begin{aligned} h^w_{n,k} &= \sum_{p:(k,\infty)\to(n,0)} w(p)\\ &= \sum_{r=k}^{n} \sum_{p_1:(k,\infty)\to(r,\ell)} w(p_1) \sum_{\substack{p_2:(r,\ell)\to(n,0)\\ p_2 \text{ starts with a south step}}} w(p_2)\\ &= \sum_{r=k}^{n} \sum_{p_1:(k,\infty)\to(r,0)} w_\ell^+(p_1) \sum_{p_2:(r,\ell)\to(n,0)} w_\ell(p_2)\\ &= \sum_{r=k}^{n} h^{w_\ell}_{n,r} h^{w_\ell^+}_{r,k}. \end{aligned}$$

□

If there is a weight system w_1 of height 1 and a weight system w of infinite height with the "shifting property", which is given in (3.6) below, we can construct a weight system of any height.

Proposition 3.9. *Suppose that* $\{\sigma_{n,k}\}_{n,k\ge 0}$ *satisfies*

$$\sigma_{n,k} = h^{w}_{n,k} = h^{w_1}_{n,k}$$

for a weight system w_1 *of height* 1 *and a weight system* w *of infinite height such that*

$$w(t;i,j) = C_i^t w(0;i,j), \tag{3.6}$$

for all $t,i,j\in\mathbb{Z}_{\ge 0}$ *with* $j\le i$. *Then, for any integer* $\ell\ge 1$, *we have*

$$\sigma_{n,k} = h^{w^{(\ell)}}_{n,k},$$

where $w^{(\ell)}$ *is the weight system of height* ℓ *given by*

$$w^{(\ell)}(t;i,j) = \begin{cases} w(t;i,j) & \text{if } 0\le t<\ell-1,\\ C_i^{\ell-1}w_1(0;i,j) & \text{if } t=\ell-1.\end{cases}$$

Proof. Using the notation in Lemma 3.8 with ℓ replaced by $\ell-1$, we have

$$h^{w}_{n,k} = \sum_{r=k}^{n} h^{w_{\ell-1}}_{n,r} h^{w^+_{\ell-1}}_{r,k}.$$

By (3.6), we have $w^+_{\ell-1}(t;i,j) = w(t+\ell-1;i,j) = C_i^{\ell-1}w(t;i,j)$. Thus, by Lemma 2.5 and the assumption, we obtain

$$h^{w^+_{\ell-1}}_{r,k} = C_k^{\ell-1}C_{k+1}^{\ell-1}\cdots C_{r-1}^{\ell-1}h^{w}_{r,k} = C_k^{\ell-1}C_{k+1}^{\ell-1}\cdots C_{r-1}^{\ell-1}h^{w_1}_{r,k} = h^{w'}_{r,k},$$

where w' is the weight system of height 1 defined by $w'(0;i,j) = C_i^{\ell-1}w_1(0;i,j)$. Hence,

$$h^{w}_{n,k} = \sum_{r=k}^{n} h^{w_{\ell-1}}_{n,r} h^{w'}_{r,k}.$$

On the other hand, applying Lemma 3.8 to the weight system $w^{(\ell)}$, we also have

$$h^{w^{(\ell)}}_{n,k} = \sum_{r=k}^{n} h^{w_{\ell-1}}_{n,r} h^{w'}_{r,k}.$$

Combining the results, we obtain $\sigma_{n,k} = h^{w}_{n,k} = h^{w^{(\ell)}}_{n,k}$. ☐

4. A bootstrapping method from Stieltjes–Wigert to Askey–Wilson

In this section, we construct lecture hall graph models for the mixed moments of Stieltjes–Wigert polynomials, q-Bessel polynomials, little q-Jacobi polynomials, big q-Jacobi polynomials, and Askey–Wilson polynomials. We first find a lecture hall graph model of height 1 for the mixed moments of Stieltjes–Wigert polynomials, and models of height 1 and infinite height for q-Bessel polynomials. We then prove "expanding lemmas" (Lemma 4.10 and Lemma 4.11), which turn a weight system of height 1 to a weight system of height 2. Using the expanding lemmas, we obtain lecture hall graph models of height 1 and infinite height for little q-Jacobi polynomials, big q-Jacobi polynomials, and Askey–Wilson polynomials.

4.1. **Stieltjes–Wigert.**

The monic *Stieltjes–Wigert polynomials* are defined by

$$S_n(x;q) = (-1)^n q^{-n^2}{}_1\phi_1\left(\begin{matrix} q^{-n}\\ 0\end{matrix};q,-q^{n+1}x\right).$$

The mixed moments $\sigma_{n,k}$ and the coefficients $\nu_{n,k}$ of Stieltjes–Wigert polynomials are given by

$$\sigma_{n,k} = q^{k^2-n^2+\binom{n-k}{2}}\begin{bmatrix} n\\ k\end{bmatrix}_q,$$

$$\nu_{n,k} = (-1)^{n-k}q^{k^2-n^2}\begin{bmatrix} n\\ k\end{bmatrix}_q.$$

Hence the mixed moments of Stieltjes–Wigert polynomials are q-binomial coefficients with some factors. The q-binomial coefficients satisfy the following recurrences:

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q = \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q . \tag{4.1}$$

We begin with two simple weight systems giving q-binomial coefficients with some possible factors.

Lemma 4.1. *Let w be the weight system of height* 1 *defined by $w(0;i,j) = q^j$. Then*

$$h^w_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Proof. Note that the weight $w(0;i,j) = q^j$ is independent of i and $w(0;k,k) = q^k$. Therefore, by applying Lemma 3.5, we obtain the recurrence

$$h^w_{n,k} = q^k h^w_{n-1,k} + h^w_{n-1,k-1}$$

for $1 \le k \le n-1$, which is the same recurrence as the first one in (4.1). Since $h^w_{n,k}$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q$ have the same initial conditions, namely, $h^w_{n,0} = h^w_{n,n} = 1$, we conclude that $h^w_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$. □

Lemma 4.2. *Let w be the weight system of height* 1 *defined by $w(0;i,j) = q^{i-j}$. Then*

$$h^w_{n,k} = q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q .$$

Proof. Let w' be the weight system of height 1 defined by $w'(0;i,j) = q^{-j}$. By Lemma 4.1, $h^{w'}_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{-k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}_q$. Since $w(0;i,j) = q^i w'(0;i,j)$, by Lemma 2.5,

$$h^w_{n,k} = q^{k+(k+1)+\cdots+(n-1)} h^{w'}_{n,k} = q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q . \qquad \square$$

We now give a weight system for the mixed moments of Stieltjes–Wigert polynomials.

Proposition 4.3. *Let w be the weight system of height* 1 *defined by $w(0;i,j) = q^{-i-j-1}$. Then*

$$h^w_{n,k} = \sigma_{n,k}.$$

Proof. Let w' be the weight system of height 1 defined by $w'(0;i,j) = q^{i-j}$. By Lemma 4.2, $h^{w'}_{n,k} = q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q$. Since $w(0;i,j) = q^{-2i-1} w'(0;i,j)$, by Lemma 2.5,

$$h^w_{n,k} = q^{-(2k+1)-(2k+3)-\cdots-(2n-1)} h^{w'}_{n,k} = q^{k^2-n^2+\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q = \sigma_{n,k}. \qquad \square$$

Since $\sigma_{n,k}$ is a polynomial in q, we do not consider its weight system of infinite height.

4.2. **q-Bessel polynomials.** The (monic) *q-Bessel polynomials* (also known as *alternative q-Charlier polynomials*) are defined by

$$p_n(x;a;q) = \frac{1}{(-1)^n q^{-\binom{n}{2}} (-aq^n;q)_n} {}_2\phi_1 \left(\begin{matrix} q^{-n}, -aq^n \\ 0 \end{matrix} ; q, qx \right). \tag{4.2}$$

Let $\sigma^b_{n,k}(a;q)$ and $\nu^b_{n,k}(a;q)$ denote the mixed moments and the dual mixed moments of the q-Bessel polynomials:

$$x^n = \sum_{k=0}^n \sigma^b_{n,k}(a;q) p_k(x;a;q), \qquad p_n(x;a;q) = \sum_{k=0}^n \nu^b_{n,k}(a;q) x^k.$$

These quantities have simple product formulas.

Proposition 4.4. *We have*

$$\sigma^b_{n,k}(a;q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{(-aq^{2k+1};q)_{n-k}} = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-aq;q)_{2k}}{(-aq;q)_{n+k}}, \tag{4.3}$$

$$\nu^b_{n,k}(a;q) = (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{(-aq^{n+k};q)_{n-k}} = (-1)^{n-k} q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-a;q)_{n+k}}{(-a;q)_{2n}}. \tag{4.4}$$

Proof. By (4.2), we have

$$\nu^b_{n,k}(a;q) = \frac{1}{(-1)^n q^{-\binom{n}{2}} (-aq^n;q)_n} \cdot \frac{(q^{-n};q)_k (-aq^n;q)_k}{(q;q)_k} q^k,$$

which is equivalent to (4.4). To prove (4.3) let $a_{n,k}$ and $b_{n,k}$ be the right hand side of (4.3) and (4.4), respectively. Then it suffices to show that

$$\sum_{j=0}^m b_{m,j} a_{j,n} = \delta_{m,n}.$$

Since the above sum is 0 if $m < n$, we may assume $m \ge n$. Then

$$\begin{aligned}\sum_{j=0}^m b_{m,j} a_{j,n} &= \frac{(-aq;q)_{2n}}{(-a;q)_{2m}} \sum_{j=n}^m (-1)^{m-j} q^{\binom{m-j}{2}} \begin{bmatrix} m \\ j \end{bmatrix}_q \begin{bmatrix} j \\ n \end{bmatrix}_q \frac{(-a;q)_{m+j}}{(-aq;q)_{n+j}} \\ &= \frac{(-a;q)_{m+n}}{(-a;q)_{2m}} \begin{bmatrix} m \\ n \end{bmatrix}_q (-1)^{m-n} q^{\binom{m-n}{2}} {}_2\phi_1 \left(\begin{matrix} q^{-m+n}, -aq^{m+n} \\ -aq^{2n+1} \end{matrix} ; q, q \right) \\ &= \frac{(-a;q)_{m+n}}{(-a;q)_{2m}} \begin{bmatrix} m \\ n \end{bmatrix}_q (-1)^{m-n} q^{\binom{m-n}{2}} \frac{(q^{-m+n+1};q)_{m-n}}{(-aq^{2n+1};q)_{m-n}} = \delta_{m,n},\end{aligned}$$

where the following identity [14, (1.5.2)] is used:

$${}_2\phi_1 \left(\begin{matrix} q^{-n}, b \\ c \end{matrix} ; q, cq^n/b \right) = \frac{(c/b;q)_n}{(c;q)_n}.$$

□

We will find several weight systems for $\sigma^b_{n,k}(a;q)$. We first give a weight system of height 1.

Proposition 4.5. *Let w be the weight system of height 1 defined by*

$$w(0;i,j) = \frac{q^j(1+aq^i)}{(1+aq^{i+j})(1+aq^{i+j+1})}.$$

Then the mixed moments of the q-Bessel polynomials are given by

$$\sigma^b_{n,k}(a;q) = h^w_{n,k}.$$

Proof. Let us write $h^w_{n,k} = h^w_{n,k}(a)$. By Lemma 3.5,

$$h^w_{n,k}(a) = w(0;k,k) h^{w'}_{n-1,k}(a) + h^{w'}_{n-1,k-1}(a).$$

Since $w'(0;i,j) = w(0;i+1,j)$, which is equal to $w(0;i,j)$ with a replaced by aq, we have $h^{w'}_{n,k}(a) = h^w_{n,k}(aq)$. Thus

$$h^w_{n,k}(a) = \frac{q^k(1+aq^k)}{(1+aq^{2k+1})(1+aq^{2k})} h^w_{n-1,k}(aq) + h^w_{n-1,k-1}(aq).$$

By induction, it suffices to show that $\sigma^b_{n,k}(a;q)$ also satisfies the same recurrence, where the initial conditions are $h^w_{n,k}(a) = \sigma^b_{n,k}(a;q) = 1$ if $n = k$, and $h^w_{n,k}(a) = \sigma^b_{n,k}(a;q) = 0$ if $n < k$. By (4.3), the recurrence we need to establish is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-aq;q)_{2k}}{(-aq;q)_{n+k}} = \frac{q^k(1+aq^k)}{(1+aq^{2k+1})(1+aq^{2k})} \begin{bmatrix} n-1 \\ k \end{bmatrix}_q \frac{(-aq^2;q)_{2k}}{(-aq^2;q)_{n+k-1}} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q \frac{(-aq^2;q)_{2k-2}}{(-aq^2;q)_{n+k-2}},$$

or equivalently,

$$(1+aq^{2k})\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k(1+aq^k)\begin{bmatrix} n-1 \\ k \end{bmatrix}_q + (1+aq^{n+k})\begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$

This follows from the two recurrences in (4.1). □

To find a weight system of infinite height for $\sigma^b_{n,k}(a;q)$, we need some lemmas. The following lemma allows us to find a weight system of infinite height for $\{\sigma_{n,k}\}_{n,k\ge 0}$ if a certain recurrence is satisfied.

Lemma 4.6 (Construction of a weight system of infinite height)**.** *Suppose that* $\{\sigma_{n,k}\}_{n,k\ge 0}$ *satisfy the following recurrence:*

$$\sigma_{n,k} = \sum_{r=k}^{n} h^{w_1}_{n,r}\sigma_{r,k}\prod_{i=k}^{r-1} C_i.$$

for some weight system w_1 *of height* 1*. Then*

$$\sigma_{n,k} = h^w_{n,k},$$

where w *is the weight system of infinite height defined by*

$$w(t;i,j) = C_i^t w_1(0;i,j).$$

Proof. Since $\sigma_{n,n} = h^w_{n,n} = 1$, it suffices to show that $h^w_{n,k}$ satisfy the same recurrence, namely,

$$h^w_{n,k} = \sum_{r=k}^{n} h^{w_1}_{n,r} h^w_{r,k}\prod_{i=k}^{r-1} C_i.$$

This can be proved similarly as in the proof of Proposition 3.9, hence we omit the details. □

The mixed moments $\sigma^b_{n,k}(a;q)$ satisfy a recurrence of the form in Lemma 4.6.

Lemma 4.7. *We have*

$$\sigma^b_{n,k}(a;q) = \sum_{r=k}^{n}\begin{bmatrix} n \\ r \end{bmatrix}_q (-a)^{r-k}q^{r^2-k^2}\sigma^b_{r,k}(a;q).$$

Proof. By (4.3), the identity we need to show is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(-aq;q)_{2k}}{(-aq;q)_{n+k}} = \sum_{r=k}^{n}\begin{bmatrix} n \\ r \end{bmatrix}_q (-a)^{r-k}q^{r^2-k^2}\begin{bmatrix} r \\ k \end{bmatrix}_q \frac{(-aq;q)_{2k}}{(-aq;q)_{r+k}}.$$

The above identity can be rewritten as

$$\frac{1}{(-aq^{2k+1};q)_{n-k}} = \sum_{i=0}^{n-k}\begin{bmatrix} n-k \\ i \end{bmatrix}_q (-a)^i q^{i^2+2ik}\frac{1}{(-aq^{2k+1};q)_i}.$$

This is equivalent to the following form of the q-binomial theorem; see [2, p. 77, Exercise 101]:

$$\frac{1}{(aq;q)_n} = \sum_{k=0}^{n} a^k q^{k^2}\begin{bmatrix} n \\ k \end{bmatrix}_q \frac{1}{(aq;q)_k}.$$ □

We are now ready to give a weight system of infinite height for $\sigma^b_{n,k}(a;q)$. Note that this weight system can be constructed using Algorithm 1 with the monomial ordering defined by $q < a$. See Figure 9 for the weight system in the following proposition.

Proposition 4.8. *Let* w *be the weight system defined by*

$$w(t;i,j) = (-aq^{2i+1})^t q^j.$$

The mixed moments of the q*-Bessel polynomials satisfy*

$$\sigma^b_{n,k}(a;q) = h^w_{n,k}.$$

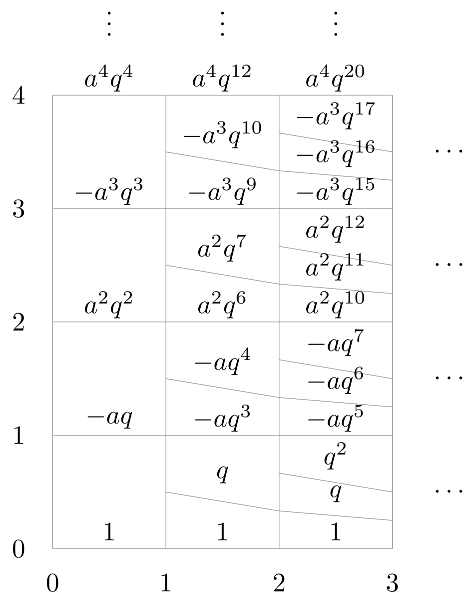

FIGURE 9. The weight system for the mixed moments of q-Bessel polynomials.

Proof. Let w_1 be the weight system of height 1 given by $w_1(0;i,j) = q^j$ so that $h^{w_1}_{n,k} = \begin{bmatrix} n \\ k \end{bmatrix}_q$ by Lemma 4.1. We can rewrite the recurrence in Lemma 4.7 as

$$\sigma^b_{n,k}(a;q) = \sum_{r=k}^{n} h^{w_1}_{n,r}\sigma^b_{r,k}(a;q)\prod_{i=k}^{r-1}(-aq^{2i+1}).$$

Then by Lemma 4.6 we obtain the desired result. □

Since we have both a weight system w_1 of height 1 and a weight system w of infinite height for $\sigma^b_{n,k}$, we can construct a weight system of any height using Proposition 3.9.

Corollary 4.9. *For a positive integer ℓ, let w_ℓ be the weight system of height ℓ defined by*

$$w_\ell(t;i,j) = \begin{cases} (-aq^{2i+1})^t q^j & \text{if } t < \ell - 1, \\ \frac{(-aq^{2i+1})^t q^j (1+aq^i)}{(1+aq^{i+j})(1+aq^{i+j+1})} & \text{if } t = \ell - 1. \end{cases}$$

Then

$$\sigma^b_{n,k}(a;q) = h^{w_\ell}_{n,k}.$$

Proof. The weight system w in Proposition 4.8 satisfies

$$w(\ell-1;i,j) = (-aq^{2i+1})^{\ell-1} w(0;i,j).$$

Thus the result follows from Proposition 4.5, Proposition 4.8, and Proposition 3.9. □

4.3. **Expanding weight systems.** In this subsection we prove some lemmas which allow us to expand one row of a weight system to two rows. Here, a row means the part of a weight system between $y = \ell - 1$ and $y = \ell$ for some $\ell \geq 1$.

See Figure 10 for the weight systems in the following lemma.

Lemma 4.10 (Expanding lemma 1)**.** *Let w_1 and w_2 be the weight systems defined by*

$$w_1(t;i,j) = \begin{cases} (1+bq^i)q^j & \text{if } t = 0, \\ 0 & \text{if } t \geq 1, \end{cases} \qquad w_2(t;i,j) = \begin{cases} q^j & \text{if } t = 0, \\ bq^{i+j} & \text{if } t = 1, \\ 0 & \text{if } t \geq 2. \end{cases}$$

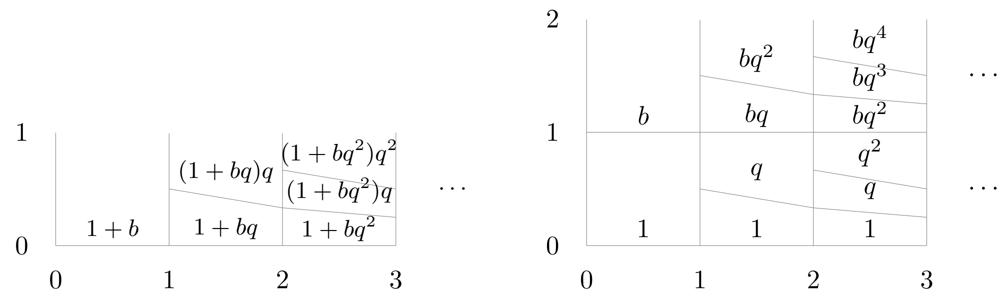

FIGURE 10. Two weight systems giving the same generating functions.

Then

$$\sum_{p:(k,1)\to(n,0)} w_1(p) = \sum_{p:(k,2)\to(n,0)} w_2(p).$$

Proof. Observe that w_1 is obtained from the weight system in Lemma 4.1 by multiplying $(1+bq^i)$ for each step between $x = i-1$ and $x = i$ for $i \geq 1$. Thus, by Lemma 2.5 and Lemma 4.1, we have

$$\sum_{p:(k,1)\to(n,0)} w_1(p) = \begin{bmatrix} n \\ k \end{bmatrix}_q (1+bq^k)(1+bq^{k+1})\cdots(1+bq^{n-1}). \tag{4.5}$$

By Lemma 3.8 with $\ell = 1$, we have

$$\sum_{p:(k,2)\to(n,0)} w_2(p) = \sum_{r=k}^{n} \sum_{p:(k,1)\to(r,0)} w'(p) \sum_{p:(r,1)\to(n,0)} w''(p),$$

where w' and w'' are the weight systems of height 1 defined by $w'(0;i,j) = bq^{i+j}$ and $w''(0;i,j) = q^j$. By Lemma 2.5 and Lemma 4.1 again, we have

$$\sum_{p:(k,1)\to(r,0)} w'(p) = b^{r-k} q^{k+(k+1)+\cdots+(r-1)} \begin{bmatrix} r \\ k \end{bmatrix}_q, \qquad \sum_{p:(r,1)\to(n,0)} w''(p) = \begin{bmatrix} n \\ r \end{bmatrix}_q.$$

Thus,

$$\begin{aligned}
\sum_{p:(k,2)\to(n,0)} w_2(p) &= \sum_{r=k}^{n} b^{r-k} q^{k+(k+1)+\cdots+(r-1)} \begin{bmatrix} r \\ k \end{bmatrix}_q \begin{bmatrix} n \\ r \end{bmatrix}_q \\
&= \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{r=k}^{n} b^{r-k} q^{\binom{r-k}{2}+k(r-k)} \begin{bmatrix} n-k \\ r-k \end{bmatrix}_q \\
&= \begin{bmatrix} n \\ k \end{bmatrix}_q \sum_{r=0}^{n-k} b^{r} q^{\binom{r}{2}+kr} \begin{bmatrix} n-k \\ r \end{bmatrix}_q \\
&= \begin{bmatrix} n \\ k \end{bmatrix}_q (1+bq^k)(1+bq^{k+1})\cdots(1+bq^{n-1}),
\end{aligned}$$

where the q-binomial theorem is used in the last equality. By (4.5) and the above formula, we obtain the lemma. □

For an integer k, let $\chi_o(k) = 1$ if k is odd and $\chi_o(k) = 0$ otherwise. Similarly, let $\chi_e(k) = 1$ if k is even and $\chi_e(k) = 0$ otherwise.

Lemma 4.11 (Expanding lemma 2)**.** *Let w_1 and w_2 be the weight systems defined by*

$$\begin{aligned}
w_1(t;i,j) &= a_{i,t}(1+bq^i)q^j, \\
w_2(t;i,j) &= a_{i,\lfloor t/2\rfloor}(bq^i)^{\chi_o(t)} q^j,
\end{aligned}$$

where $a_{i,j}$ is an arbitrary quantity that depends on i and j. Then

$$\sum_{p:(k,\infty)\to(n,0)} w_1(p) = \sum_{p:(k,\infty)\to(n,0)} w_2(p).$$

Proof. Consider a path $p : (k,\infty) \to (n,0)$. Let $t_1 > t_2 > \cdots > t_m$ be the integers such that p has at least one east step in the region $\{(x,y) : t_s \le y < t_s + 1\}$ for each s. Then the restriction of p to this region is a path $p_s : (k_{s-1}, t_s + 1) \to (k_s, t_s)$ starting with a south step, for some integers $k = k_0 < k_1 < \cdots < k_m = n$. Observe that $w_1(p) = w_1(p_1)\cdots w_1(p_m)$. Let $p'_s : (k_{s-1}, 1) \to (k_s, 0)$ be the path obtained by translating p_s by t_s units downwards. By the definition of w_1, we have

$$w_1(p_s) = a_{k_{s-1},t_s} a_{k_{s-1}+1,t_s} \cdots a_{k_s-1,t_s} w'_1(p'_s),$$

where w'_1 is the weight system of height 1 defined by $w'_1(0;i,j) = (1 + bq^i)q^j$. Therefore,

$$\sum_{p:(k,\infty)\to(n,0)} w_1(p) = \sum_{m\ge0}\sum_{(\mathbf{t},\mathbf{k})} \left(\prod_{s=1}^{m}\prod_{i=k_{s-1}}^{k_s-1} a_{i,t_s}\right) h^{w'_1}_{k_m,k_{m-1}} h^{w'_1}_{k_{m-1},k_{m-2}} \cdots h^{w'_1}_{k_1,k_0}, \tag{4.6}$$

where the last sum is over all pairs $(\mathbf{t},\mathbf{k})$ of tuples $\mathbf{t} = (t_1 > t_2 > \cdots > t_m)$ and $\mathbf{k} = (k = k_0 < k_1 < \cdots < k_m = n)$.

For the weight system w_2, we consider a different decomposition of a path $p : (k,\infty) \to (n,0)$. Let $t_1 > t_2 > \cdots > t_m$ be the integers such that p has at least one east step in the region $\{(x,y) : 2t_s \le y < 2t_s + 2\}$ for each s. Then the restriction of p to this region is a path $p_s : (k_{s-1}, 2t_s + 2) \to (k_s, 2t_s)$ starting with a south step, for some integers $k = k_0 < k_1 < \cdots < k_m = n$, and we have $w_2(p) = w_2(p_1)\cdots w_2(p_m)$. Let $p'_s : (k_{s-1}, 2) \to (k_s, 0)$ be the path obtained by translating p_s by $2t_s$ units downwards. By the definition of w_2, we have

$$w_2(p_s) = a_{k_{s-1},t_s} a_{k_{s-1}+1,t_s} \cdots a_{k_s-1,t_s} w'_2(p'_s),$$

where w'_2 is the weight system of height 2 defined by

$$w'_2(t;i,j) = \begin{cases} q^j & \text{if } t = 0,\\ bq^{i+j} & \text{if } t = 1,\\ 0 & \text{if } t \ge 2.\end{cases}$$

Therefore, we have

$$\sum_{p:(k,\infty)\to(n,0)} w_2(p) = \sum_{m\ge0}\sum_{(\mathbf{t},\mathbf{k})} \left(\prod_{s=1}^{m}\prod_{i=k_{s-1}}^{k_s-1} a_{i,t_s}\right) h^{w'_2}_{k_m,k_{m-1}} h^{w'_2}_{k_{m-1},k_{m-2}} \cdots h^{w'_2}_{k_1,k_0}, \tag{4.7}$$

where the last sum is the same as in (4.6).

By Lemma 4.10, we have $h^{w'_1}_{n,k} = h^{w'_2}_{n,k}$ for all n and k. Therefore, the right-hand sides of (4.6) and (4.7) are equal, which completes the proof. □

4.4. **Little q-Jacobi.** Let $p_n(x;a,b;q)$ be the monic *little q-Jacobi polynomial*:

$$p_n(x;a,b;q) = \frac{(aq;q)_n}{(-1)^n q^{-\binom{n}{2}}(abq^{n+1};q)_n}\, {}_2\phi_1\left(\begin{matrix} q^{-n}, abq^{n+1} \\ aq \end{matrix}; q, qx\right). \tag{4.8}$$

We denote by $\sigma^L_{n,k}(a,b;q)$ and $\nu^L_{n,k}(a,b;q)$ the mixed moments and the dual mixed moments of the little q-Jacobi polynomials, respectively:

$$x^n = \sum_{k=0}^{n} \sigma^L_{n,k}(a,b;q) p_k(x;a,b;q), \qquad p_n(x;a,b;q) = \sum_{k=0}^{n} \nu^L_{n,k}(a,b;q) x^k.$$

The following lemma will be used to find formulas for $\sigma^L_{n,k}(a,b;q)$ and $\nu^L_{n,k}(a,b;q)$.

Lemma 4.12. *Let $\{z_i\}_{i\ge0}$ be a sequence of nonzero quantities. If the inverse of a matrix $(a_{i,j})_{i,j=0}^{\infty}$ is $(b_{i,j})_{i,j=0}^{\infty}$, then the inverse of the matrix $(a_{i,j}z_i/z_j)_{i,j=0}^{\infty}$ is $(b_{i,j}z_i/z_j)_{i,j=0}^{\infty}$.*

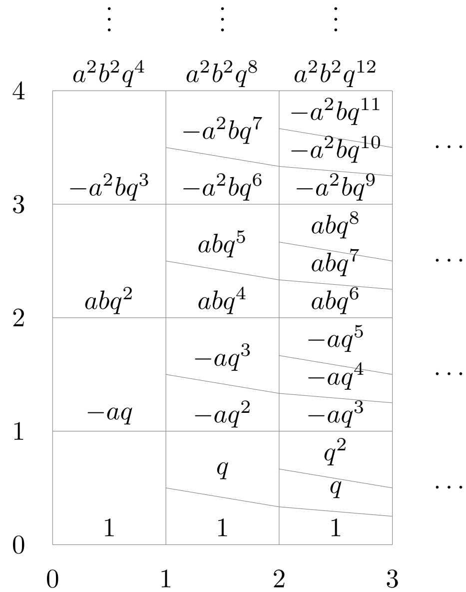

FIGURE 11. The weight system for the mixed moments of little q-Jacobi polynomials.

Proof. Let D be the diagonal matrix with diagonal entries z_i. Then

$$\left((a_{i,j}z_i/z_j)_{i,j=0}^{\infty}\right)^{-1} = \left(D(a_{i,j})_{i,j=0}^{\infty}D^{-1}\right)^{-1} = D(b_{i,j})_{i,j=0}^{\infty}D^{-1} = (b_{i,j}z_i/z_j)_{i,j=0}^{\infty}. \qquad \square$$

The following result has been proved in [7, Lemma 2.5] using the q-Saalschütz summation formula. We provide another proof using Lemma 4.12 because the same technique will be used later for big q-Jacobi polynomials and Askey–Wilson polynomials.

Proposition 4.13. *We have*

$$\sigma^L_{n,k}(a,b;q) = \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq^{k+1};q)_{n-k}}{(abq^{2k+2};q)_{n-k}}, \tag{4.9}$$

$$\nu^L_{n,k}(a,b;q) = (-1)^{n-k}q^{\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq^{k+1};q)_{n-k}}{(abq^{n+k+1};q)_{n-k}}. \tag{4.10}$$

Equivalently,

$$\sigma^L_{n,k}(a,b;q) = (aq^{k+1};q)_{n-k}\sigma^b_{n,k}(-abq;q), \tag{4.11}$$

$$\nu^L_{n,k}(a,b;q) = (aq^{k+1};q)_{n-k}\nu^b_{n,k}(-abq;q), \tag{4.12}$$

where $\sigma^b_{n,k}(a;q)$ and $\nu^b_{n,k}(a;q)$ are the mixed moments and the coefficients of the q-Bessel polynomials in (4.3) *and* (4.4).

Proof. By (4.8) and (4.4), we have (4.10) and (4.12). The equivalence of (4.9) and (4.11) follows from (4.3). Hence, we only need to prove (4.11). Since $(\sigma^b_{n,k}(-abq;q))_{n,k}$ and $(\nu^b_{n,k}(-abq;q))_{n,k}$ are inverses of each other, we obtain (4.11) by Lemma 4.12 and (4.12). $\square$

Now we give a weight system of height 1 for $\sigma^L_{n,k}(a,b;q)$.

Proposition 4.14. *Let w_1 be the weight system of height* 1 *defined by*

$$w_1(0;i,j) = \frac{q^j(1-abq^{i+1})(1-aq^{i+1})}{(1-abq^{i+j+1})(1-abq^{i+j+2})}.$$

Then the mixed moments of the little q-Jacobi polynomials satisfy

$$\sigma^L_{n,k}(a,b;q) = h^{w_1}_{n,k}.$$

Proof. By Proposition 4.5, we have

$$\sigma^b_{n,k}(-abq;q) = h^{w_1'}_{n,k},$$

where

$$w_1'(0;i,j) = \frac{q^j(1-abq^{i+1})}{(1-abq^{i+j+1})(1-abq^{i+j+2})}.$$

Observe that $w_1(0;i,j) = (1-aq^{i+1})w_1'(0;i,j)$. Since every path from (k,∞) to $(n,0)$ has exactly one east step between $x=i$ and $x=i+1$ for each $i=k,k+1,\dots,n-1$, we obtain

$$h^{w_1}_{n,k} = (aq^{k+1};q)_{n-k} h^{w_1'}_{n,k} = (aq^{k+1};q)_{n-k}\sigma^b_{n,k}(-abq;q) = \sigma^L_{n,k}(a,b;q),$$

where (4.11) is used for the last equality. □

Using the expanding lemma (Lemma 4.10) we obtain a weight system of infinite height for $\sigma^L_{n,k}(a,b;q)$. We note that this is also discoverable by Algorithm 1 using the monomial ordering $q<a<b$. See Figure 11 for the weight system in the following proposition.

Proposition 4.15. [6] *Let w be the weight system defined by*

$$\begin{aligned} w(t;i,j) &= (-a)^{\lceil t/2\rceil}(-b)^{\lfloor t/2\rfloor}q^{(i+1)t+j} \\ &= \left(abq^{2i+2}\right)^m q^j \times \begin{cases} 1 & \text{if } t=2m, \\ -aq^{i+1} & \text{if } t=2m+1. \end{cases} \end{aligned}$$

Then the mixed moments of the little q-Jacobi polynomials satisfy

$$\sigma^L_{n,k}(a,b;q) = h^w_{n,k}.$$

Proof. By the same argument as in the proof of Proposition 4.14, we obtain from Proposition 4.8 and (4.11) that

$$\sigma^L_{n,k}(a,b;q) = h^{w'}_{n,k},$$

where

$$w'(t;i,j) = (1-aq^{i+1})(abq^{2i+2})^t q^j.$$

By Lemma 4.11, we have $h^{w'}_{n,k} = h^w_{n,k}$, where

$$w(t;i,j) = (abq^{2i+2})^{\lfloor t/2\rfloor}(-aq^{i+1})^{\chi_o(t)}q^j = (-a)^{\lceil t/2\rceil}(-b)^{\lfloor t/2\rfloor}q^{(i+1)t+j},$$

as desired. □

Remark 4.16. Using Corollary 4.9, it is possible to obtain a weight system of any height for $\sigma^L_{n,k}(a,b;q)$. The same method also works for big q-Jacobi polynomials and Askey–Wilson polynomials. We omit the details.

4.5. **Big q-Jacobi.** Let $p_n(x;a,b,c;q)$ be the monic big q-Jacobi polynomial:

$$p_n(x;a,b,c;q) = \frac{(aq,cq;q)_n}{(abq^{n+1};q)_n}\,{}_3\phi_2\left(\begin{matrix} q^{-n}, abq^{n+1}, x \\ aq, cq \end{matrix}; q, q\right). \tag{4.13}$$

We denote by $\sigma^B_{n,k}(a,b,c;q)$ and $\nu^B_{n,k}(a,b,c;q)$ the mixed moments and the dual mixed moments of the big q-Jacobi polynomials, respectively:

$$x^n = \sum_{k=0}^n \sigma^B_{n,k}(a,b,c;q)p_k(x;a,b,c;q), \qquad p_n(x;a,b,c;q) = \sum_{k=0}^n \nu^B_{n,k}(a,b,c;q)x^k.$$

We will first consider the factorial mixed moments $\tilde\sigma^B_{n,k}(a,b,c;q)$ and factorial dual mixed moments $\tilde\nu^B_{n,k}(a,b,c;q)$ defined by

$$(x|\mathbf{q})^n = \sum_{k=0}^n \tilde\sigma^B_{n,k}(a,b,c;q)p_k(x;a,b,c;q), \qquad p_n(x;a,b,c;q) = \sum_{k=0}^n \tilde\nu^B_{n,k}(a,b,c;q)(x|\mathbf{q})^k,$$

where $\mathbf{q} = (1,q^{-1},q^{-2},\dots)$ so that

$$(x|\mathbf{q})^k = (x-1)(x-q^{-1})\cdots(x-q^{-k+1}).$$

Proposition 4.17. *We have*

$$\tilde{\sigma}^B_{n,k}(a,b,c;q) = (-1)^{n-k} q^{\binom{k}{2}-\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq^{k+1};q)_{n-k}(cq^{k+1};q)_{n-k}}{(abq^{2k+2};q)_{n-k}}, \quad (4.14)$$

$$\tilde{\nu}^B_{n,k}(a,b,c;q) = q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(aq^{k+1},cq^{k+1};q)_{n-k}}{(abq^{n+k+1};q)_{n-k}}. \quad (4.15)$$

Equivalently,

$$\tilde{\sigma}^B_{n,k}(a,b,c;q) = (-1)^{n-k} q^{\binom{k}{2}-\binom{n}{2}} (cq^{k+1};q)_{n-k} \sigma^L_{n,k}(a,b;q), \quad (4.16)$$

$$\tilde{\nu}^B_{n,k}(a,b,c;q) = (-1)^{n-k} q^{\binom{k}{2}-\binom{n}{2}} (cq^{k+1};q)_{n-k} \nu^L_{n,k}(a,b;q), \quad (4.17)$$

where $\sigma^L_{n,k}(a,b;q)$ and $\nu^L_{n,k}(a,b;q)$ are the mixed and dual mixed moments of the little q-Jacobi polynomials given in (4.9) *and* (4.10)*.*

Proof. This can be proved similarly as in the proof of Proposition 4.13. By (4.13) and (4.12), we obtain (4.15) and (4.17). The equivalence of (4.14) and (4.16) follows from (4.11). Hence, we only need to prove (4.16). Since $(\sigma^L_{n,k}(a,b;q))_{n,k}$ and $(\nu^L_{n,k}(a,b;q))_{n,k}$ are inverses of each other, we obtain (4.16) by Lemma 4.12 and (4.17). □

We give a weight system of height 1 and a weight system of infinite height for $\tilde{\sigma}^B_{n,k}(a,b,c;q)$ in the next two propositions.

Proposition 4.18. *Let $\widetilde{w}_1$ be the weight system of height* 1 *defined by*

$$\widetilde{w}_1(0;i,j) = \frac{-q^{j-i}(1-abq^{i+1})(1-aq^{i+1})(1-cq^{i+1})}{(1-abq^{i+j+1})(1-abq^{i+j+2})}.$$

Then the factorial mixed moments of the big q-Jacobi polynomials satisfy

$$\tilde{\sigma}^B_{n,k}(a,b,c;q) = h^{\widetilde{w}_1}_{n,k}.$$

Proof. This can be proved similarly as in the proof of Proposition 4.14 using that proposition and (4.16). □

See Figure 12 for the weight system in the following proposition.

Proposition 4.19. *Let*

$$\begin{aligned}\widetilde{w}(t;i,j) &= -(-a)^{\lfloor (t+2)/4\rfloor}(-b)^{\lfloor t/4\rfloor}(-c)^{\chi_o(t)} q^{(i+1)\lceil t/2\rceil - i+j} \\ &= (abq^{2i+2})^m q^j \times \begin{cases} -q^{-i} & \text{if } t=4m, \\ cq & \text{if } t=4m+1, \\ aq & \text{if } t=4m+2, \\ -acq^{i+2} & \text{if } t=4m+3. \end{cases}\end{aligned}$$

Then the factorial mixed moments of the big q-Jacobi polynomials satisfy

$$\tilde{\sigma}^B_{n,k}(a,b,c;q) = h^{\widetilde{w}}_{n,k}.$$

Proof. By Proposition 4.15 and (4.16) we have

$$\tilde{\sigma}^B_{n,k}(a,b,c;q) = h^{w'}_{n,k},$$

where

$$w'(t;i,j) = -q^{-i}(1-cq^{i+1}) \cdot (-a)^{\lceil t/2\rceil}(-b)^{\lfloor t/2\rfloor} q^{(i+1)t+j}.$$

By Lemma 4.11, we can replace the weight system w' by $\widetilde{w}$, where

$$\begin{aligned}\widetilde{w}(t;i,j) &= -(-cq^{i+1})^{\chi_o(t)}(-a)^{\lceil\lfloor t/2\rfloor/2\rceil}(-b)^{\lfloor\lfloor t/2\rfloor/2\rfloor} q^{(i+1)\lfloor t/2\rfloor - i+j} \\ &= -(-a)^{\lfloor (t+2)/4\rfloor}(-b)^{\lfloor t/4\rfloor}(-c)^{\chi_o(t)} q^{(i+1)\lceil t/2\rceil - i+j},\end{aligned}$$

as desired. □

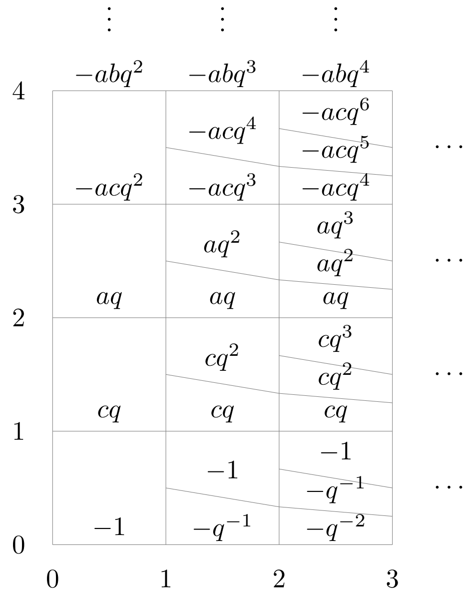

FIGURE 12. The lecture hall graph model for the factorial mixed moments of big q-Jacobi polynomials.

By Proposition 3.7, we can construct a weight system for $\sigma^B_{n,k}(a,b,c;q)$ from the weight system $\widetilde{w}$ in Proposition 4.19 by adding one row coming from the factorial basis $\{(x|\mathbf{q})^n\}_{n\ge 0}$ at the bottom. Miraculously, the new row and the bottom row of $\widetilde{w}$ cancel each other and the resulting weight system is $\widetilde{w}$ with bottom row removed as described in the next proposition. We note that this weight system is also discoverable by Algorithm 1 using the monomial ordering $q<c<a<b$. See Figure 13 for the weight system in the following proposition.

Proposition 4.20. *Let*

$$\begin{aligned} w(t;i,j) &= -(-a)^{\lfloor (t+3)/4\rfloor}(-b)^{\lfloor (t+1)/4\rfloor}(-c)^{\chi_e(t)}q^{(i+1)\lfloor t/2\rfloor+j+1} \\ &= (abq^{2i+2})^m q^j \times \begin{cases} cq & \text{if } t=4m, \\ aq & \text{if } t=4m+1, \\ -acq^{i+2} & \text{if } t=4m+2, \\ -abq^{i+2} & \text{if } t=4m+3. \end{cases} \end{aligned}$$

Then the mixed moments of the big q-Jacobi polynomials satisfy

$$\sigma^B_{n,k}(a,b,c;q) = h^w_{n,k}.$$

Proof. Let w' be the weight system defined by

$$w'(t;i,j) = \begin{cases} \widetilde{w}(t-1;i,j) & \text{if } t\ge 1, \\ q^{-j} & \text{if } t=0, \end{cases}$$

where $\widetilde{w}$ is the weight system in Proposition 4.19. In other words, w' is the weight system obtained from $\widetilde{w}$ by adding the weight system w_1 of height 1 defined by $w_1(0;i,j)=q^{-j}$ at the bottom as shown in Figure 14. By Proposition 3.7 and Proposition 4.19, we have $\sigma^B_{n,k}(a,b,c;q) = h^{w'}_{n,k}$. Thus it suffices to show that $h^{w'}_{n,k} = h^w_{n,k}$.

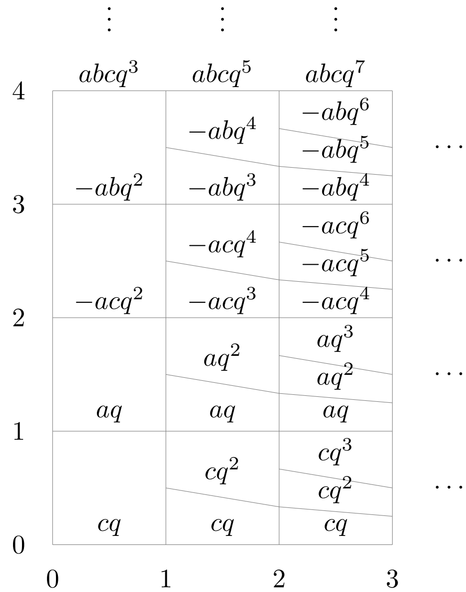

FIGURE 13. The weight system for the mixed moments of big q-Jacobi polynomials.

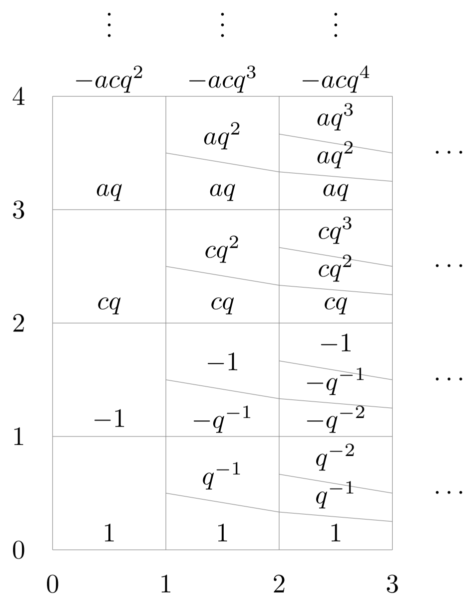

FIGURE 14. The weight system w' in the proof of Proposition 4.20.

Since $\lceil (t+1)/2 \rceil = \lfloor t/2 \rfloor + 1$, comparing the definitions of $w(t;i,j)$ and $\widetilde{w}(t;i,j)$, we have $w(t;i,j) = \widetilde{w}(t+1;i,j) = w'(t+2;i,j)$ for all $t \geq 0$. Thus, by Lemma 3.8 with $\ell = 2$, we have

$$h_{n,k}^{w'} = \sum_{r=k}^{n} h_{n,r}^{w'_2} h_{r,k}^{w},$$

where w_2' is the weight system of height 2 given by

$$w_2'(t;i,j) = \begin{cases} w'(t;i,j) & \text{if } t < 2, \\ 0 & \text{if } t \geq 2. \end{cases}$$

Therefore, to prove $h^{w'}_{n,k} = h^{w}_{n,k}$, it suffices to show that $h^{w_2'}_{n,r} = \delta_{n,r}$.

Let $\overline{w_1}$ be the weight system of height 1 defined by $\overline{w_1}(0;i,j) = w_1(0;i,i-j)$. Observe that $w_2'(0;i,j) = q^{-j} = w_1(0;i,j)$ and $w_2'(1;i,j) = -q^{j-i} = -w_1(0;i,i-j) = -\overline{w_1}(0;i,j)$. By Lemma 3.8 with $\ell = 1$ and Lemma 2.5, we have

$$h^{w_2'}_{n,r} = \sum_{m=r}^{n} h^{w_1}_{n,m} (-1)^{m-r} h^{\overline{w_1}}_{m,r}.$$

Then, by Proposition 3.4 and Lemma 2.4, we obtain

$$h^{w_2'}_{n,r} = \sum_{m=r}^{n} h^{w_1}_{n,m} (-1)^{m-r} e^{w_1}_{m,r} = \delta_{n,r},$$

which completes the proof. □

4.6. **Askey–Wilson polynomials.** The monic *Askey–Wilson polynomials* $p_n(x;a,b,c,d|q)$ are defined by

$$p_n(x;a,b,c,d|q) = \frac{(ab,ac,ad;q)_n}{2^n a^n (abcdq^{n-1};q)_n} {}_4\phi_3 \left(\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix} ; q, q \right), \tag{4.18}$$

where $x = \cos\theta = (e^{i\theta} + e^{-i\theta})/2$. Let $f = (f_0, f_1, \dots)$ be the sequence given by $f_j = (aq^j + a^{-1}q^{-j})/2$. Then it is easy to check that

$$(ae^{i\theta};q)_k (ae^{-i\theta};q)_k = (-2a)^k q^{\binom{k}{2}} (x|f)^k.$$

We denote by $\sigma^{AW}_{n,k}(a,b,c,d;q)$ and $\nu^{AW}_{n,k}(a,b,c,d;q)$ the mixed moments and the dual mixed moments of the Askey–Wilson polynomials, respectively:

$$x^n = \sum_{k=0}^{n} \sigma^{AW}_{n,k}(a,b,c,d;q) p_k(x;a,b,c,d|q), \qquad p_n(x;a,b,c,d|q) = \sum_{k=0}^{n} \nu^{AW}_{n,k}(a,b,c,d;q) x^k.$$

As in the case of big q-Jacobi polynomials, we will first consider the factorial mixed moments $\tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q)$ and factorial dual mixed moments $\tilde{\nu}^{AW}_{n,k}(a,b,c,d;q)$ defined by

$$(x|f)^n = \sum_{k=0}^{n} \tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q) p_k(x;a,b,c,d|q), \qquad p_n(x;a,b,c,d|q) = \sum_{k=0}^{n} \tilde{\nu}^{AW}_{n,k}(a,b,c,d;q) (x|f)^k.$$

Proposition 4.21. *We have*

$$\tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q) = (-2a)^{k-n} q^{\binom{k}{2}-\binom{n}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(abq^k, acq^k, adq^k;q)_{n-k}}{(abcdq^{2k};q)_{n-k}}, \tag{4.19}$$

$$\tilde{\nu}^{AW}_{n,k}(a,b,c,d;q) = (2a)^{k-n} q^{k(k-n)} \begin{bmatrix} n \\ k \end{bmatrix}_q \frac{(abq^k, acq^k, adq^k;q)_{n-k}}{(abcdq^{n+k-1};q)_{n-k}}. \tag{4.20}$$

Equivalently,

$$\tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q) = (2a)^{k-n} (adq^k;q)_{n-k} \tilde{\sigma}^{B}_{n,k}(ab/q, cd/q, ac/q;q), \tag{4.21}$$

$$\tilde{\nu}^{AW}_{n,k}(a,b,c,d;q) = (2a)^{k-n} (adq^k;q)_{n-k} \tilde{\nu}^{B}_{n,k}(ab/q, cd/q, ac/q;q), \tag{4.22}$$

where $\tilde{\sigma}^{B}_{n,k}(ab/q, cd/q, ac/q)$ and $\tilde{\nu}^{B}_{n,k}(ab/q, cd/q, ac/q)$ are the factorial mixed and dual mixed moments of the big q-Jacobi polynomials in (4.14) *and* (4.15).

Proof. This can be proved in a similar way to Proposition 4.17. □

As in the previous subsections, we give a weight system of height 1 for $\tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q)$.

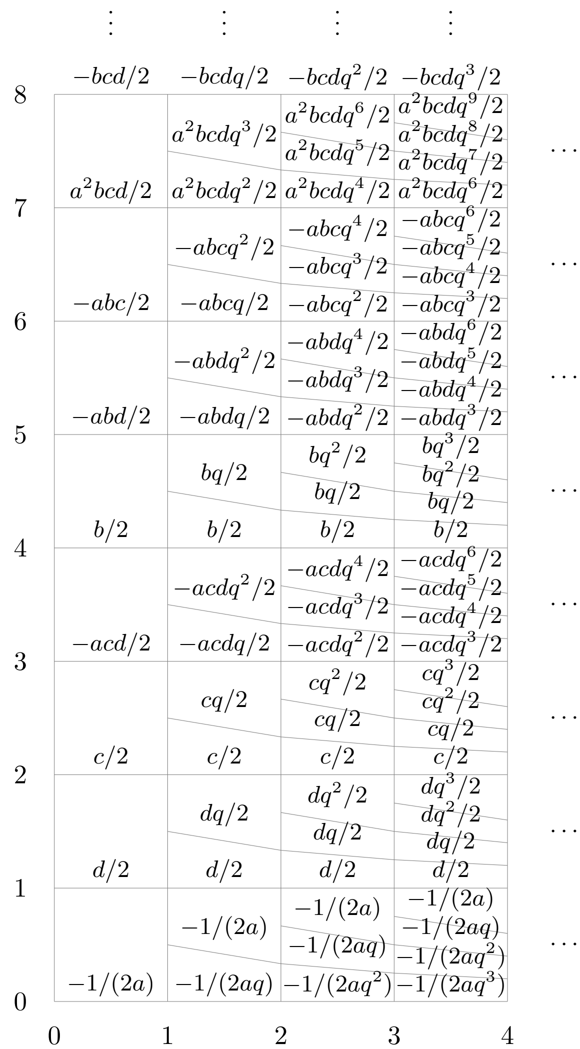

FIGURE 15. The lecture hall graph for the factorial moments of Askey–Wilson polynomials.

Proposition 4.22. *Let*

$$\widetilde{w}_1(0;i,j) = \frac{q^{j-i}(1-abq^i)(1-acq^i)(1-adq^i)(1-abcdq^{i-1})}{-2a(1-abcdq^{i+j-1})(1-abcdq^{i+j})}.$$

Then the factorial mixed moments of the Askey–Wilson polynomials satisfy

$$\tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q) = h^{\widetilde{w}_1}_{n,k}.$$

Proof. This can be proved in a similar way to Proposition 4.18 using that proposition and (4.21). □

Now we give a weight system of infinite height for $\tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q)$. This is also discoverable by Algorithm 1 using the monomial ordering $q < d < c < a < b$. See Figure 15 for the weight system in the following proposition.

Proposition 4.23. *Let*

$$\begin{aligned}\widetilde{w}(t;i,j) &= -(2a)^{-1}q^{(i+1)\lfloor (t+2)/4\rfloor -i+j}\\ &\qquad \times (-ab/q)^{\lfloor (t+4)/8\rfloor}(-cd/q)^{\lfloor t/8\rfloor}(-ac/q)^{\chi_o(\lfloor t/2\rfloor)}(-adq^i)^{\chi_o(t)}\\ &= \frac{1}{2}\left(abcdq^{2i}\right)^m q^j \times \begin{cases} -a^{-1}q^{-i} & \text{if } t=8m,\\ d & \text{if } t=8m+1,\\ c & \text{if } t=8m+2,\\ -acdq^i & \text{if } t=8m+3,\\ b & \text{if } t=8m+4,\\ -abdq^i & \text{if } t=8m+5,\\ -abcq^i & \text{if } t=8m+6,\\ a^2bcdq^{2i} & \text{if } t=8m+7.\end{cases}\end{aligned}$$

Then the factorial mixed moments of the Askey–Wilson polynomials satisfy

$$\tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q) = h^{\widetilde{w}}_{n,k}.$$

Proof. By (4.21) and Proposition 4.19, we have

$$\tilde{\sigma}^{AW}_{n,k}(a,b,c,d;q) = h^{\widetilde{w}'}_{n,k},$$

where w' is the weight system given by

$$\widetilde{w}'(t;i,j) = (2a)^{-1}(1-adq^i)\cdot(-1)(-ab/q)^{\lfloor (t+2)/4\rfloor}(-cd/q)^{\lfloor t/4\rfloor}(-ac/q)^{\chi_o(t)}q^{(i+1)\lceil t/2\rceil -i+j}.$$

By Lemma 4.11, $h^{\widetilde{w}'}_{n,k} = h^{\widetilde{w}}_{n,k}$, where $\widetilde{w}$ is the weight system given by

$$\begin{aligned}&\widetilde{w}(t;i,j)\\ &= -(2a)^{-1}(-adq^i)^{\chi_o(t)}(-ab/q)^{\lfloor (\lfloor t/2\rfloor+2)/4\rfloor}(-cd/q)^{\lfloor \lfloor t/2\rfloor/4\rfloor}(-ac/q)^{\chi_o(\lfloor t/2\rfloor)}q^{(i+1)\lceil \lfloor t/2\rfloor/2\rceil -i+j}\\ &= -(2a)^{-1}(-adq^i)^{\chi_o(t)}(-ab/q)^{\lfloor (t+4)/8\rfloor}(-cd/q)^{\lfloor t/8\rfloor}(-ac/q)^{\chi_o(\lfloor t/2\rfloor)}q^{(i+1)\lfloor (t+2)/4\rfloor -i+j},\end{aligned}$$

as desired. □

Finally, by Proposition 3.7 and Proposition 4.23, we obtain a weight system for the original mixed moments of Askey–Wilson polynomials. See Figure 16 for the weight system in the following theorem.

Theorem 4.24. *Let*

$$w(t;i,j) = \begin{cases} (aq^j + a^{-1}q^{-j})/2 & \text{if } t=0,\\ \widetilde{w}(t-1;i,j) & \text{if } t\ge 1,\end{cases}$$

where $\widetilde{w}$ is the weight system given in Proposition 4.23. Then the mixed moments of the Askey–Wilson polynomials satisfy

$$\sigma^{AW}_{n,k}(a,b,c,d;q) = h^{w}_{n,k}.$$

5. Another bootstrapping method from continuous q-Hermite polynomials

In this section, we provide another bootstrapping method to find a combinatorial model for mixed moments of Askey–Wilson polynomials relative to the continuous q-Hermite polynomials, which are one of the most well-studied families of orthogonal polynomials in the q-Askey scheme. At the end of this section, we give a lecture hall graph model for the mixed moments of the continuous q-Hermite polynomials. As applications of the results in this section, we prove Corollary 6.2, which establishes the total positivity of the matrix of mixed moments of Askey–Wilson polynomials relative to continuous q-Hermite polynomials, and we also give the first combinatorial proof of the symmetry of a,b,c,d in the Askey–Wilson polynomials in Section 6.

Our results build on those of Kim and Stanton [17], who used a bootstrapping method to compute the moments of Askey–Wilson polynomials from continuous q-Hermite polynomials.

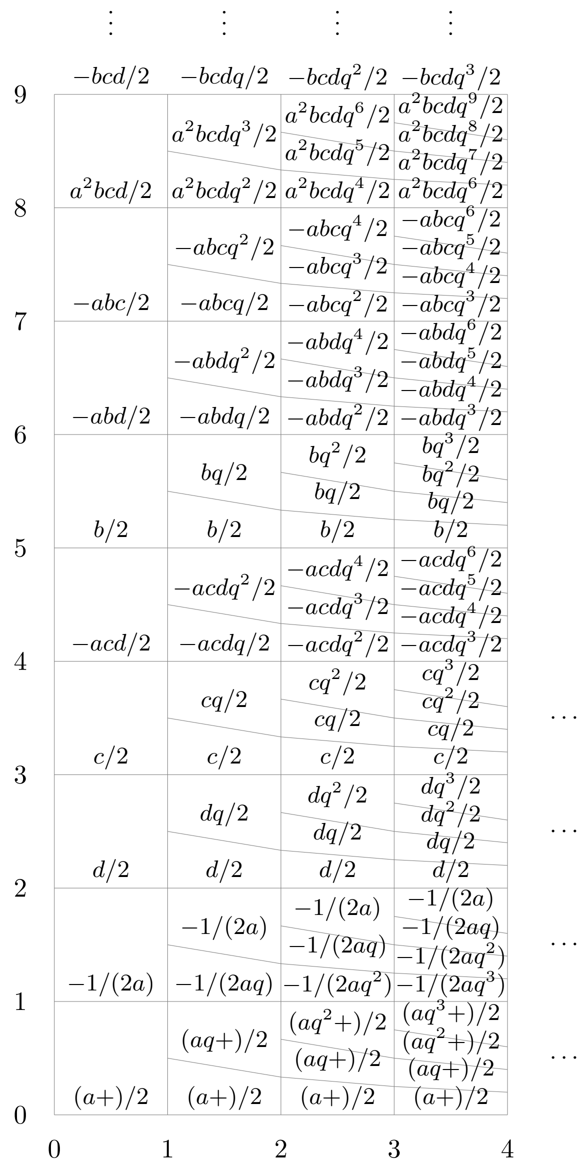

FIGURE 16. The lecture hall graph for the moments of Askey–Wilson polynomials. Here, $(x+)$ means $x + x^{-1}$.

5.1. **Definitions.** The (monic) *continuous q-Hermite polynomials* $H_n(x|q)$, *continuous big q-Hermite polynomials* $H_n(x;a|q)$, *Al–Salam–Chihara polynomials* $Q_n(x;a,b|q)$, and *continuous dual* *q-Hahn polynomials* $p_n^{dH}(x;a,b,c|q)$ are defined by

$$\begin{aligned}
H_n(x|q) &= \frac{e^{in\theta}}{2^n}\,{}_2\phi_0\left(\begin{matrix} q^{-n},0 \\ - \end{matrix};q,q^ne^{-2i\theta}\right),\\
H_n(x;a|q) &= \frac{1}{(2a)^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},ae^{i\theta},ae^{-i\theta} \\ 0,0 \end{matrix};q,q\right) = \frac{e^{in\theta}}{2^n}\,{}_2\phi_0\left(\begin{matrix} q^{-n},ae^{i\theta} \\ - \end{matrix};q,q^ne^{-2i\theta}\right),\\
Q_n(x;a,b|q) &= \frac{(ab;q)_n}{(2a)^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},ae^{i\theta},ae^{-i\theta} \\ ab,0 \end{matrix};q,q\right),\\
p_n^{dH}(x;a,b,c|q) &= \frac{(ab,ac;q)_n}{(2a)^n}\,{}_3\phi_2\left(\begin{matrix} q^{-n},ae^{i\theta},ae^{-i\theta} \\ ab,ac \end{matrix};q,q\right).
\end{aligned}$$

Here, $x=\cos\theta$. Note that these polynomials are special cases of the Askey–Wilson polynomials $p_n(x;a,b,c,d|q)$ defined in (4.18), namely,

$$\begin{aligned}
H_n(x|q) &= p_n(x;0,0,0,0|q),\\
H_n(x;a|q) &= p_n(x;a,0,0,0|q),\\
Q_n(x;a,b|q) &= p_n(x;a,b,0,0|q),\\
p_n^{dH}(x;a,b,c|q) &= p_n(x;a,b,c,0|q).
\end{aligned}$$

Let $\sigma_{n,k}^{H}(q)$ be the mixed moments of the continuous q-Hermite polynomials $\{H_n(x|q)\}_{n\ge0}$:

$$x^n = \sum_{k=0}^{n} \sigma_{n,k}^{H}(q) H_k(x|q).$$

The Touchard–Riordan-like formula due to Josuat-Vergès [16, Proposition 5.1] and Cigler and Zeng [4, Proposition 15] (see also [17, (12)]) can be stated as

$$\sigma_{n,k}^{H}(q) = \frac{1}{2^{n-k}} \sum_{r=k}^{n} \left(\binom{n}{\frac{n-r}{2}} - \binom{n}{\frac{n-r}{2}-1} \right) (-1)^{(r-k)/2} q^{\binom{(r-k)/2+1}{2}} \begin{bmatrix} \frac{r+k}{2} \\ \frac{r-k}{2} \end{bmatrix}_q,$$

where we define $\binom{i}{j} = \begin{bmatrix} i \\ j \end{bmatrix}_q = 0$ unless both i and j are integers.

We denote by $\sigma_{n,k}^{bH,H}(a;q)$, $\sigma_{n,k}^{Q,H}(a,b;q)$, $\sigma_{n,k}^{dH,H}(a,b,c;q)$, and $\sigma_{n,k}^{AW,H}(a,b,c,d;q)$ the mixed moments of continuous big q-Hermite polynomials, Al–Salam–Chihara polynomials, continuous dual q-Hahn polynomials, and Askey–Wilson polynomials relative to the continuous q-Hermite polynomials:

$$H_n(x|q) = \sum_{k=0}^{n} \sigma_{n,k}^{bH,H}(a;q) H_k(x;a|q), \tag{5.1}$$

$$H_n(x|q) = \sum_{k=0}^{n} \sigma_{n,k}^{Q,H}(a,b;q) Q_k(x;a,b|q),$$

$$H_n(x|q) = \sum_{k=0}^{n} \sigma_{n,k}^{dH,H}(a,b,c;q) p_k^{dH}(x;a,b,c|q),$$

$$H_n(x|q) = \sum_{k=0}^{n} \sigma_{n,k}^{AW,H}(a,b,c,d;q) p_k(x;a,b,c,d|q).$$

5.2. **Another bootstrapping method.** By re-normalizing the results in [17, (8)-(11)], we have

$$H_n(x|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{a}{2}\right)^{n-k} H_k(x;a|q), \tag{5.2}$$

$$H_n(x;a|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{b}{2}\right)^{n-k} Q_k(x;a,b|q),$$

$$Q_n(x;a,b|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{c}{2}\right)^{n-k} (abq^k;q)_{n-k} p_k^{dH}(x;a,b,c|q),$$

$$p_n^{dH}(x;a,b,c|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{d}{2}\right)^{n-k} \frac{(abq^k, acq^k, bcq^k;q)_{n-k}}{(abcdq^{2k};q)_{n-k}} p_k(x;a,b,c,d|q).$$

We will first find weight systems for the coefficients in the sums of the above equations. Then, using Lemma 2.11, we can build a weight system for $\sigma_{n,k}^{AW,H}(a,b,c,d;q)$ by successively stacking these weight systems.

See Figures 17, 18, and 19 for the weight systems $w^{(1)}$, $w^{(2)}$, and $w^{(3)}$, respectively in the following lemma.

Lemma 5.1. *Let $w^{(1)}$, $w^{(2)}$, $w^{(3)}$, and $w^{(4)}$ be the weight systems defined as follows:*

$$w^{(1)}(t;i,j) = \begin{cases} aq^j/2 & \text{if } t = 0, \\ 0 & \text{if } t \geq 1, \end{cases} \tag{5.3}$$

$$w^{(2)}(t;i,j) = \begin{cases} bq^j/2 & \text{if } t = 0, \\ 0 & \text{if } t \geq 1, \end{cases}$$

$$w^{(3)}(t;i,j) = \begin{cases} cq^j/2 & \text{if } t = 0, \\ -abcq^{i+j}/2 & \text{if } t = 1, \\ 0 & \text{if } t \geq 2, \end{cases}$$

$$w^{(4)}(t;i,j) = \frac{d}{2}(abcdq^{2i})^{\lfloor t/8 \rfloor}(-bcq^i)^{\chi_o(\lfloor t/4 \rfloor)}(-acq^i)^{\chi_o(\lfloor t/2 \rfloor)}(-abq^i)^{\chi_o(t)}q^j$$

$$= \frac{1}{2}\left(abcdq^{2i}\right)^m q^j \times \begin{cases} d & \text{if } t = 8m, \\ -abdq^i & \text{if } t = 8m+1, \\ -acdq^i & \text{if } t = 8m+2, \\ a^2bcdq^{2i} & \text{if } t = 8m+3, \\ -bcdq^i & \text{if } t = 8m+4, \\ ab^2cdq^{2i} & \text{if } t = 8m+5, \\ abc^2dq^{2i} & \text{if } t = 8m+6, \\ -a^2b^2c^2dq^{3i} & \text{if } t = 8m+7. \end{cases}$$

Then

$$h_{n,k}^{w^{(1)}} = \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{a}{2}\right)^{n-k}, \tag{5.4}$$

$$h_{n,k}^{w^{(2)}} = \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{b}{2}\right)^{n-k}, \tag{5.5}$$

$$h_{n,k}^{w^{(3)}} = \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{c}{2}\right)^{n-k} (abq^k;q)_{n-k}, \tag{5.6}$$

$$h_{n,k}^{w^{(4)}} = \begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{d}{2}\right)^{n-k} \frac{(abq^k, acq^k, bcq^k;q)_{n-k}}{(abcdq^{2k};q)_{n-k}}. \tag{5.7}$$

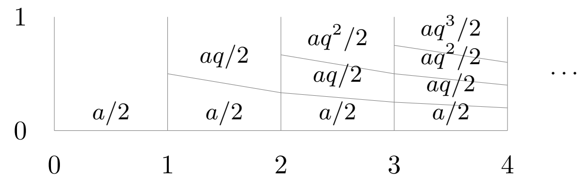

FIGURE 17. The weight system $w^{(1)}$.

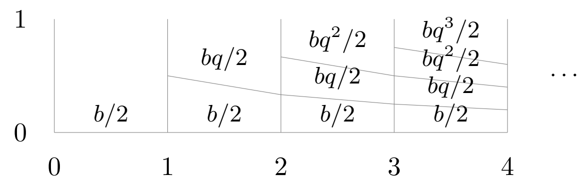

FIGURE 18. The weight system $w^{(2)}$.

Proof. By Lemma 4.1 and Lemma 2.5, we have (5.4) and (5.5). By Lemma 4.1 and Lemma 4.10, we have (5.6). To prove (5.7), observe that its right-hand side is

$$\begin{bmatrix} n \\ k \end{bmatrix}_q \left(\frac{d}{2}\right)^{n-k} \frac{(abq^k, acq^k, bcq^k; q)_{n-k}}{(abcdq^{2k}; q)_{n-k}} = \left(\frac{d}{2}\right)^{n-k} (abq^k, acq^k, bcq^k; q)_{n-k} \cdot \sigma^b_{n,k}(-abcd/q; q),$$

where $\sigma^b_{n,k}(a; q)$ is the mixed moment of q-Bessel polynomials in (4.3). Thus, by Proposition 4.8, if we define $w_1(t; i, j) = (abcdq^{2i})^t q^j$, then

$$h^{w_1}_{n,k} = \sigma^b_{n,k}(-abcd/q; q).$$

By Lemma 2.5, if we define $w_2(t; i, j) = (abcdq^{2i})^t q^j (1 - abq^i)(1 - acq^i)(1 - bcq^i)d/2$, then

$$h^{w_2}_{n,k} = \left(\frac{d}{2}\right)^{n-k} (abq^k, acq^k, bcq^k; q)_{n-k} \cdot \sigma^b_{n,k}(-abcd/q; q).$$

Now by applying Lemma 4.11 to $w_2(t; i, j)$ three times with the factors $(1 - bcq^i)$, $(1 - acq^i)$, and $(1 - abq^i)$ in this order, we obtain that $h^{w_2}_{n,k} = h^{w^{(4)}}_{n,k}$, which completes the proof of (5.7). □

By Lemma 2.11 and Lemma 5.1, we obtain the following lemma.

Lemma 5.2. *Let $w^{(1)}, w^{(2)}, w^{(3)}$, and $w^{(4)}$ be the weight systems given in Lemma 5.1. Then*

$$\begin{aligned}
\sigma^{bH,H}_{n,k}(a; q) &= h^{w^{(1)}}_{n,k}, \\
\sigma^{Q,H}_{n,k}(a, b; q) &= h^{w^{(1)} \sqcup w^{(2)}}_{n,k}, \\
\sigma^{dH,H}_{n,k}(a, b, c; q) &= h^{w^{(1)} \sqcup w^{(2)} \sqcup w^{(3)}}_{n,k}, \\
\sigma^{AW,H}_{n,k}(a, b, c, d; q) &= h^{w^{(1)} \sqcup w^{(2)} \sqcup w^{(3)} \sqcup w^{(4)}}_{n,k}.
\end{aligned}$$

Rewriting $h^{w^{(1)} \sqcup w^{(2)} \sqcup w^{(3)} \sqcup w^{(4)}}_{n,k}$ in Lemma 5.2 we obtain the following combinatorial model for $\sigma^{AW,H}_{n,k}(a, b, c, d; q)$.

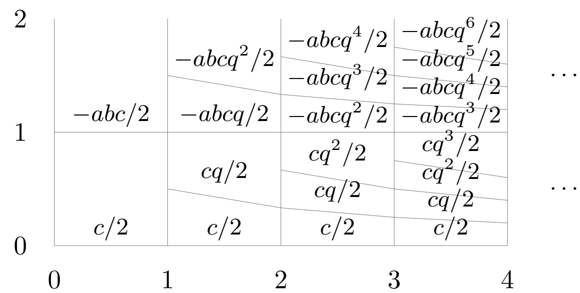

FIGURE 19. The weight system $w^{(3)}$.

Theorem 5.3. *Let $w^{AW,H}$ be the weight system defined by*

$$w^{AW,H}(t;i,j) = (-a^2q^i)^{\delta_{t,0}}\frac{d}{2}(abcdq^{2i})^{\lfloor (t-4)/8\rfloor}(-bcq^i)^{\chi_e(\lfloor t/4\rfloor)}(-acq^i)^{\chi_o(\lfloor t/2\rfloor)}(-abq^i)^{\chi_o(t)}q^j$$

$$= \frac{1}{2}\left(abcdq^{2i}\right)^m q^j \times \begin{cases} a & \text{if } t = 8m = 0, \\ -a^{-1}q^{-i} & \text{if } t = 8m \neq 0, \\ b & \text{if } t = 8m+1, \\ c & \text{if } t = 8m+2, \\ -abcq^i & \text{if } t = 8m+3, \\ d & \text{if } t = 8m+4, \\ -abdq^i & \text{if } t = 8m+5, \\ -acdq^i & \text{if } t = 8m+6, \\ a^2bcdq^{2i} & \text{if } t = 8m+7. \end{cases}$$

Then we have

$$h_{n,k}^{w^{AW,H}} = \sigma_{n,k}^{AW,H}(a,b,c,d;q).$$

5.3. **A weight system for continuous q-Hermite polynomials.** In order to obtain a weight system for the original mixed moment $\sigma_{n,k}^{AW}(a,b,c,d;q)$ of Askey–Wilson polynomials from Theorem 5.3, we can simply add a weight system for the mixed moment $\sigma_{n,k}^{H}(q)$ of continuous q-Hermite polynomials below the weight system $w^{AW,H}$. In this subsection we find a weight system for $\sigma_{n,k}^{H}(q)$. To do this we first consider continuous big q-Hermite polynomials $H_n(x;a|q)$.

Let $\sigma_{n,k}^{bH}(a;q)$ be the mixed moment of continuous big q-Hermite polynomials:

$$x^n = \sum_{k=0}^{n}\sigma_{n,k}^{bH}(a;q)H_k(x;a|q).$$

Since $H_n(x;a|q) = p_n(x;a,0,0,0|q)$, setting $b = c = d = 0$ in Theorem 4.24 gives the following proposition; see Figure 20 for the weight system.

Proposition 5.4. *Let w^{bH} be the weight system of height 2 defined by*

$$w^{bH}(t;i,j) = \begin{cases} (aq^j + a^{-1}q^{-j})/2 & \text{if } t = 0, \\ -1/(2aq^{i-j}) & \text{if } t = 1. \end{cases}$$

Then

$$\sigma_{n,k}^{bH}(a;q) = h_{n,k}^{w^{bH}}.$$

To find a weight system for $\sigma_{n,k}^{H}(q)$, we compare two polynomials $H_n(x|q)$ and $H_n(x;a|q)$. See Figure 21 for the weight system in the following lemma.

Lemma 5.5. *Let w' be the weight system of height 1 defined by*

$$w'(0;i,j) = -\frac{aq^{i-j}}{2}.$$

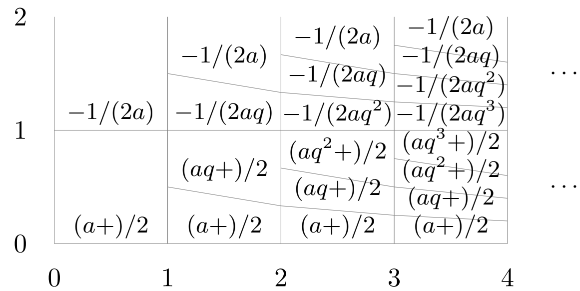

FIGURE 20. The lecture hall graph for the mixed moments of continuous big q-Hermite polynomials. Here, $(x+)$ means $x + x^{-1}$.

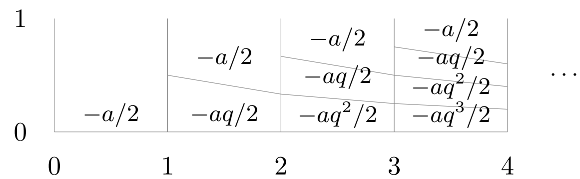

FIGURE 21. The weight system w' in Lemma 5.5.

Then

$$H_n(x;a|q) = \sum_{k=0}^{n} h_{n,k}^{w'} H_k(x|q).$$

Proof. Let $w^{(1)}$ be the weight system given in (5.3). By (5.1) and Lemma 5.2, we have

$$H_n(x|q) = \sum_{k=0}^{n} h_{n,k}^{w^{(1)}} H_k(x;a|q).$$

By Proposition 2.8, the above equation implies

$$H_n(x;a|q) = \sum_{k=0}^{n} (-1)^{n-k} e_{n,k}^{w^{(1)}} H_k(x|q). \tag{5.8}$$

Since $w^{(1)}$ is a weight system of height 1, by Proposition 3.4, we have $e_{n,k}^{w^{(1)}} = h_{n,k}^{\overline{w^{(1)}}}$, where $\overline{w^{(1)}}$ is the weight system of height 1 defined by $\overline{w^{(1)}}(0;i,j) = w^{(1)}(0;i,i-j) = aq^{i-j}/2$. Since $w'(0;i,j) = -\overline{w^{(1)}}(0;i,j)$, (5.8) is equivalent to the identity in this proposition. □

By Lemma 2.11, a weight system for $\sigma_{n,k}^{H}(q)$ can be obtained by adding the weight system w' in Lemma 5.5 at the top of a weight system of $\sigma_{n,k}^{bH}(a;q)$. Therefore, by Proposition 5.4, $w^{bH} \sqcup w'$ is a weight system for $\sigma_{n,k}^{H}(q)$. However, since $\sigma_{n,k}^{H}(q)$ is independent of a, we can replace a by any number. In particular, if we set $a = 1$ in the weight system $w^{bH} \sqcup w'$, we obtain the following proposition; see Figure 22 for the weight system.

Proposition 5.6. *Let w^H be the weight system of height* 3 *defined by*

$$w^H(t;i,j) = \begin{cases} (q^j + q^{-j})/2 & \text{if } t = 0, \\ -q^{j-i}/2 & \text{if } t = 1, \\ -q^{i-j}/2 & \text{if } t = 2. \end{cases}$$

Then

$$\sigma_{n,k}^{H}(q) = h_{n,k}^{w^H}.$$

Using Theorem 5.3 and Proposition 5.6, we obtain another weight system for the original mixed moments $\sigma_{n,k}^{AW}(a,b,c,d;q)$ of Askey–Wilson polynomials.

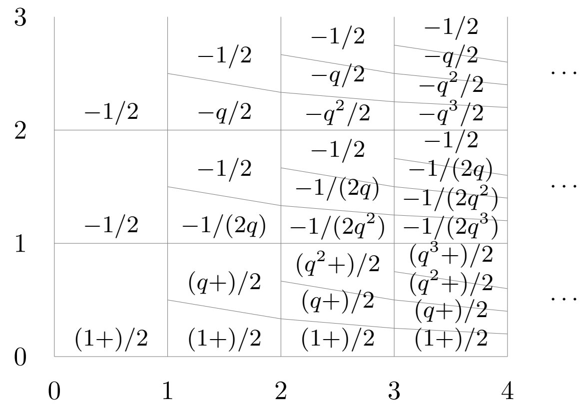

FIGURE 22. The weight system w^H for the mixed moments of continuous q-Hermite polynomials. Here, $(x+)$ means $x + x^{-1}$.

Theorem 5.7. *Let $w^{AW} = w^H \sqcup w^{AW,H}$, where $w^{AW,H}$ is the weight system in Theorem 5.3 and w^H is the weight system in Proposition 5.6. Then we have*

$$h_{n,k}^{w^{AW}} = \sigma_{n,k}^{AW}(a, b, c, d; q).$$

6. Combinatorial properties of Askey–Wilson polynomials

In this section, we study in more detail the weight system $w^{AW,H}$ in Theorem 5.3 for the mixed moment $\sigma_{n,k}^{AW,H}$ of Askey–Wilson polynomials relative to continuous q-Hermite polynomials. We then find a more efficient combinatorial model for $\sigma_{n,k}^{AW,H}$. Using our new combinatorial model we give the first combinatorial proof of the well-known fact that the mixed moments (and hence the coefficients as well) of Askey–Wilson polynomials are symmetric in a, b, c, d. As applications we also find some interesting properties of Askey–Wilson polynomials.

6.1. **Another combinatorial model for mixed moments.** Let $\tilde{\sigma}_{n,k}^{AW,H}(a, b, c, d; q)$ be the following rescaled mixed moments of Askey–Wilson polynomials relative to continuous q-Hermite polynomials:

$$\tilde{\sigma}_{n,k}^{AW,H}(a, b, c, d; q) := \frac{2^{n-k}}{\mathbf{i}^{n-k}} \sigma_{n,k}^{AW,H}(\mathbf{i}a, \mathbf{i}b, \mathbf{i}c, \mathbf{i}d; q),$$

where $\mathbf{i}$ is the imaginary number $\sqrt{-1}$. Using the substitution $(a, b, c, d) \mapsto (\mathbf{i}a, \mathbf{i}b, \mathbf{i}c, \mathbf{i}d)$, we can restate Theorem 5.3 as follows; see Figure 23 for the weight system.

Theorem 6.1. *Let $\widetilde{w}^{AW,H}$ be the weight system defined by*

$$\widetilde{w}^{AW,H}(t; i, j) = \left(abcdq^{2i}\right)^m q^j \times \begin{cases} a & \text{if } t = 8m = 0, \\ a^{-1}q^{-i} & \text{if } t = 8m \neq 0, \\ b & \text{if } t = 8m+1, \\ c & \text{if } t = 8m+2, \\ abcq^i & \text{if } t = 8m+3, \\ d & \text{if } t = 8m+4, \\ abdq^i & \text{if } t = 8m+5, \\ acdq^i & \text{if } t = 8m+6, \\ a^2bcdq^{2i} & \text{if } t = 8m+7. \end{cases} \tag{6.1}$$

Then we have

$$h_{n,k}^{\widetilde{w}^{AW,H}} = \tilde{\sigma}_{n,k}^{AW,H}(a, b, c, d; q).$$

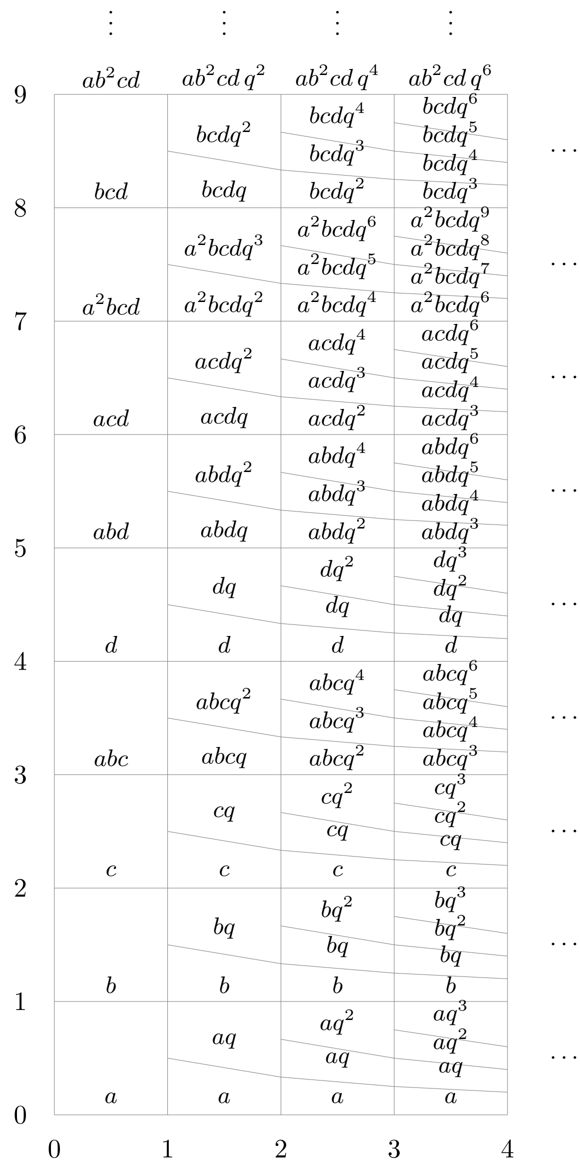

FIGURE 23. The lecture hall graph for the mixed moments of (rescaled) Askey–Wilson polynomials relative to continuous q-Hermite moments. Except for the first row, the rows $2, 3, \ldots, 9$ repeat modulo 8 with an extra factor of $abcdq^{2i}$ for every 8 rows.

Observe that $\widetilde{w}^{AW,H}(t; i, j)$ in (6.1) is a monomial in a, b, c, d, and q. Thus Theorem 6.1 implies that $\tilde{\sigma}_{n,k}^{AW,H}(a, b, c, d; q)$ is a formal power series in a, b, c, d, q with nonnegative integer coefficients. More generally, we have the following corollary.

Corollary 6.2. *The matrix*

$$\left(\tilde{\sigma}_{n,k}^{AW,H}(a, b, c, d; q)\right)_{n,k=0}^{\infty}$$

is totally positive. More precisely, every minor of this matrix is a formal power series in a, b, c, d, and q with nonnegative integer coefficients.

We will find another combinatorial model for $\tilde{\sigma}^{AW,H}_{n,k}(a,b,c,d;q)$ derived from Theorem 6.1. To do this, we need some definitions. First, observe that every weight $\widetilde{w}^{AW,H}(t;i,j)$ in Theorem 6.1 is of the form $a^{i_1}b^{i_2}c^{i_3}d^{i_4}q^j$ such that $i_1+i_2+i_3+i_4$ is odd and $|i_r-i_s|\le 1$ for all $1\le r,s\le 4$.

Definition 6.3. Let $\mathcal{T}$ denote the set of quadruples (t_1,t_2,t_3,t_4) of nonnegative integers satisfying the following conditions:

(1) $t_1+t_2+t_3+t_4$ is odd and
(2) $|t_i-t_j|\le 1$ for all $1\le i,j\le 4$.

We define a total order $\le$ on $\mathcal{T}$ by

$$(t_1,t_2,t_3,t_4)\le(s_1,s_2,s_3,s_4)\quad\text{if and only if}\quad (t_4,t_3,t_2,-t_1)\le_{\text{lex}}(s_4,s_3,s_2,-s_1),$$

where $\le_{\text{lex}}$ is the lexicographic order. In other words, $(t_1,t_2,t_3,t_4)\le(s_1,s_2,s_3,s_4)$ if and only if $t_4<s_4$ or ($t_4=s_4$ and $t_3<s_3$) or ($t_4=s_4, t_3=s_3$ and $t_2<s_2$) or ($t_4=s_4,t_3=s_3,t_2=s_2$ and $-t_1\le -s_1$).

For example, $(3,3,4,3)\le(3,3,3,4)$ and $(4,3,3,3)\le(2,3,3,3)$. The motivation of the above definition is the following lemma, which says that $(\mathcal{T},\le)$ is essentially the totally ordered set on the tuples (t_1,t_2,t_3,t_4) of the powers of a,b,c,d appearing in the weights $\widetilde{w}^{AW,H}(t;i,j)$ ordered by their heights t.

Lemma 6.4. *For a nonnegative integer t, define $\kappa(t)$ to be the tuple (t_1,t_2,t_3,t_4) of integers satisfying*

$$a^{t_1}b^{t_2}c^{t_3}d^{t_4}=(abcd)^m\times\begin{cases} a & \text{if } t=8m=0,\\ a^{-1} & \text{if } t=8m\ne 0,\\ b & \text{if } t=8m+1,\\ c & \text{if } t=8m+2,\\ abc & \text{if } t=8m+3,\\ d & \text{if } t=8m+4,\\ abd & \text{if } t=8m+5,\\ acd & \text{if } t=8m+6,\\ a^2bcd & \text{if } t=8m+7.\end{cases}$$

Then $\kappa:\mathbb{Z}_{\ge 0}\to\mathcal{T}$ is a bijection such that, for $t,s\in\mathbb{Z}_{\ge0}$, we have $t\le s$ if and only if $\kappa(t)\le\kappa(s)$. Moreover, if $\kappa(t)=(t_1,t_2,t_3,t_4)$, then for all $i,j\in\mathbb{Z}_{\ge0}$ with $j\le i$, we have

$$\widetilde{w}^{AW,H}(t;i,j)=a^{t_1}b^{t_2}c^{t_3}d^{t_4}q^{i(t_1+t_2+t_3+t_4-1)/2}\cdot q^j.$$

Proof. Both statements are immediate from the definition of the totally ordered set $\mathcal{T}$ and Theorem 6.1. □

From now on, we will use the parameters a_1,a_2,a_3,a_4 in place of a,b,c,d, respectively. For any quadruple $T=(t_1,t_2,t_3,t_4)$ of nonnegative integers, we define

$$\mathbf{a}^T:=a_1^{t_1}a_2^{t_2}a_3^{t_3}a_4^{t_4},$$
$$|T|:=t_1+t_2+t_3+t_4.$$

For integers $n\ge k\ge 0$, let

$$\mathcal{T}_{n,k}=\{(T_k,T_{k+1},\dots,T_{n-1})\in\mathcal{T}^{n-k}:T_k\ge T_{k+1}\ge\cdots\ge T_{n-1}\}.$$

Note that the set $\mathcal{T}_{n,k}$ depends only on the difference $n-k$. However, we keep both n and k in the notation $\mathcal{T}_{n,k}$ to emphasize that we index the elements of a tuple $(T_k,T_{k+1},\dots,T_{n-1})\in\mathcal{T}_{n,k}$ from k to $n-1$.

For $\mathbf{T}=(T_k,\dots,T_{n-1})\in\mathcal{T}_{n,k}$, the *sum* $s(\mathbf{T})$ and the *multiplicity* $m(\mathbf{T})$ of $\mathbf{T}$ are defined by

$$s(\mathbf{T}):=T_k+\cdots+T_{n-1},$$
$$m(\mathbf{T}):=(m_1,\dots,m_r),$$

where $T_k+\cdots+T_{n-1}$ means the component-wise addition and $m_1,\dots,m_r$ are the positive integers such that $m_1+\cdots+m_r=n-k$ and

$$T_k=\cdots=T_{k+m_1-1}>T_{k+m_1}=\cdots=T_{k+m_1+m_2-1}>\cdots>T_{k+m_1+\cdots+m_{r-1}}=\cdots=T_{n-1}. \quad (6.2)$$

We will use the following notation:

$$\|\mathbf{T}\|_{n,k}:=\sum_{i=k}^{n-1}\frac{i(|T_i|-1)}{2},$$

$$\begin{bmatrix} n \\ k, m(\mathbf{T}) \end{bmatrix}_q := \begin{bmatrix} n \\ k, m_1,\dots,m_r \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![m_1]_q!\cdots[m_r]_q!},$$

where $m(\mathbf{T})=(m_1,\dots,m_r)$.

We are now ready to state another combinatorial formula for $\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q)$.

Theorem 6.5. *We have*

$$\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q)=\sum_{\mathbf{T}\in\mathcal{T}_{n,k}}\mathbf{a}^{s(\mathbf{T})}q^{\|\mathbf{T}\|_{n,k}}\begin{bmatrix} n \\ k, m(\mathbf{T}) \end{bmatrix}_q.$$

Proof. Let $w=\widetilde{w}^{AW,H}$ be the weight system in Theorem 6.1 so that

$$\tilde{\sigma}^{AW,H}_{n,k}(a,b,c,d;q)=\sum_{p:(k,\infty)\to(n,0)}w(p).$$

Consider a path $p:(k,\infty)\to(n,0)$. For $k\le i\le n-1$, let $w(t^{(i)};i,j^{(i)})$ be the weight of the east step of p between $x=i$ and $x=i+1$. Then p is determined by the numbers $t^{(i)}$'s and $j^{(i)}$'s. Let $T_i=\kappa(t^{(i)})=(t_1^{(i)},t_2^{(i)},t_3^{(i)},t_4^{(i)})$ be the tuple defined in Lemma 6.4, which satisfies

$$\widetilde{w}^{AW,H}(t^{(i)};i,j^{(i)})=a^{t_1^{(i)}}b^{t_2^{(i)}}c^{t_3^{(i)}}d^{t_4^{(i)}}q^{i(t_1^{(i)}+t_2^{(i)}+t_3^{(i)}+t_4^{(i)}-1)/2}\cdot q^{j^{(i)}}.$$

Since $t^{(k)}\ge\cdots\ge t^{(n-1)}$, we have $\mathbf{T}=(T_k,\dots,T_{n-1})\in\mathcal{T}_{n,k}$. Let $m(\mathbf{T})=(m_1,\dots,m_r)$. Then (6.2) is equivalent to

$$t^{(c_0)}=\cdots=t^{(c_1-1)}>t^{(c_1)}=\cdots=t^{(c_2-1)}>\cdots>t^{(c_{r-1})}=\cdots=t^{(c_r-1)},$$

where $c_\ell=k+m_1+\cdots+m_\ell$. Since the y-coordinates of the starting points of the east steps must be weakly decreasing, this implies that

$$c_\ell\ge j^{(c_\ell)}\ge j^{(c_\ell+1)}\ge\cdots\ge j^{(c_{\ell+1}-1)}\ge 0. \quad (6.3)$$

Note that

$$\sum q^{j^{(c_\ell)}+\cdots+j^{(c_{\ell+1}-1)}}=\begin{bmatrix} c_{\ell+1} \\ m_{\ell+1} \end{bmatrix}_q,$$

where the sum is over all tuples $(j^{(c_\ell)},\dots,j^{(c_{\ell+1}-1)})$ satisfying (6.3). Hence, the sum of $w(p)$ for all paths $p:(k,\infty)\to(n,0)$ corresponding to a fixed tuple $\mathbf{T}\in\mathcal{T}_{n,k}$ with $m(\mathbf{T})=(m_1,\dots,m_r)$ is equal to

$$\mathbf{a}^{s(\mathbf{T})}\begin{bmatrix} c_1 \\ m_1 \end{bmatrix}_q\cdots\begin{bmatrix} c_r \\ m_r \end{bmatrix}_q q^{\|\mathbf{T}\|_{n,k}}. \quad (6.4)$$

Since $\left[{c_1 \atop m_1}\right]_q\cdots\left[{c_r \atop m_r}\right]_q=\left[{n \atop k,m_1,\dots,m_r}\right]_q=\left[{n \atop k,m(\mathbf{T})}\right]_q$, summing (6.4) over all $\mathbf{T}\in\mathcal{T}_{n,k}$ gives the theorem. ∎

Using Theorem 6.5, we obtain the following "shifting" property of $\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q)$. Here, $[\mathbf{a}^{\mathbf{s}}]f(a_1,a_2,a_3,a_4)$ means the coefficient of $\mathbf{a}^{\mathbf{s}}$ in $f(a_1,a_2,a_3,a_4)$.

Corollary 6.6. *Let* $\mathbf{s}=(s_1,s_2,s_3,s_4)$. *Then*

$$[\mathbf{a}^{\mathbf{s}}]\tilde{\sigma}^{AW,H}_{n+j,k+j}(a_1,a_2,a_3,a_4;q)=q^{j(|\mathbf{s}|-(n-k))/2}\frac{(q^{n+1};q)_j}{(q^{k+1};q)_j}[\mathbf{a}^{\mathbf{s}}]\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q).$$

Proof. By Theorem 6.5,

$$[\mathbf{a}^{\mathbf{s}}]\tilde{\sigma}^{AW,H}_{n+j,k+j}(a_1,a_2,a_3,a_4;q) = \sum_{\substack{\mathbf{T}\in\mathcal{T}_{n+j,k+j}\\ s(\mathbf{T})=\mathbf{s}}} q^{\|\mathbf{T}\|_{n+j,k+j}} \begin{bmatrix} n+j \\ k+j, m(\mathbf{T})\end{bmatrix}_q .$$

Since the two sets $\mathcal{T}_{n+j,k+j}$ and $\mathcal{T}_{n,k}$ are identical, we obtain the desired formula from

$$q^{\|\mathbf{T}\|_{n+j,k+j}} \begin{bmatrix} n+j \\ k+j, m(\mathbf{T})\end{bmatrix}_q = q^{j(|\mathbf{s}|-(n-k))/2}\frac{(q^{n+1};q)_j}{(q^{k+1};q)_j}\cdot q^{\|\mathbf{T}\|_{n,k}}\begin{bmatrix} n \\ k, m(\mathbf{T})\end{bmatrix}_q ,$$

which is straightforward to verify by reindexing $\|\mathbf{T}\|_{n+j,k+j} = \sum_{i=k+j}^{n+j-1}\frac{i(|T_i|-1)}{2}$ via $i\mapsto i'+j$. □

6.2. **The symmetry of a,b,c,d in mixed moments.** As another application of Theorem 6.5, we give a combinatorial proof of the symmetry of the parameters of a_1,a_2,a_3,a_4 in $\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q)$. To this end we introduce some definitions.

For a function $f(a_1,a_2,a_3,a_4)$ in the variables a_1,a_2,a_3,a_4 and a permutation $\tau\in\mathfrak{S}_4$, let

$$\tau\cdot f(a_1,a_2,a_3,a_4) = f(a_{\tau(1)},a_{\tau(2)},a_{\tau(3)},a_{\tau(4)}).$$

Our goal is to show that $\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q)$ is symmetric in a_1,a_2,a_3,a_4, that is, for any permutation $\tau\in\mathfrak{S}_4$,

$$\tau\cdot\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q) = \tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q). \tag{6.5}$$

Since the simple transpositions generate $\mathfrak{S}_4$, it suffices to prove (6.5) for the cases $\tau=(1,2)$, $\tau=(2,3)$, and $\tau=(3,4)$.

An *interval* of $\mathcal{T}$ is a subset I of $\mathcal{T}$ such that if $T_1,T_2\in I$ and $T_1\le T\le T_2$, then $T\in I$. For an interval I of $\mathcal{T}$, we define

$$\mathcal{T}_{n,k}(I) := \mathcal{T}_{n,k}\cap I^{n-k} = \{(T_k,T_{k+1},\dots,T_{n-1})\in I^{n-k} : T_k\ge T_{k+1}\ge\cdots\ge T_{n-1}\}$$

and

$$\sigma_{n,k}(I) = \sum_{\mathbf{T}\in\mathcal{T}_{n,k}(I)} \mathbf{a}^{s(\mathbf{T})} q^{\|\mathbf{T}\|_{n,k}}\begin{bmatrix} n \\ k, m(\mathbf{T})\end{bmatrix}_q . \tag{6.6}$$

For two intervals I_1 and I_2 of $\mathcal{T}$ we write $I_1<I_2$ if $T_1<T_2$ for all $T_1\in I_1$ and $T_2\in I_2$. An *increasing interval partition* of $\mathcal{T}$ is an infinite sequence $(I_0,I_1,\dots)$ of nonempty intervals $I_0<I_1<\cdots$ in $\mathcal{T}$ such that $I_0\cup I_1\cup\cdots=\mathcal{T}$. Note that in this case the intervals I_j's are pairwise disjoint and indeed form a partition of $\mathcal{T}$.

Lemma 6.7. *Let $(I_0,I_1,\dots)$ be an increasing interval partition of $\mathcal{T}$. Then*

$$\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q) = \sum_{n=n_0\ge n_1\ge\cdots\ge k}\prod_{j\ge0}\sigma_{n_j,n_{j+1}}(I_j),$$

where the sum is over all infinite sequences $(n_0,n_1,\dots)$ of integers such that $n=n_0\ge n_1\ge\cdots\ge k$ and $n_j=k$ for all sufficiently large j.

Proof. Let $X(n_0,n_1,\dots)$ be the set of tuples $\mathbf{T}=(T_k,\dots,T_{n-1})\in\mathcal{T}_{n,k}$ such that the number of elements of $\mathbf{T}$ in I_j is n_j-n_{j+1} for all $j\ge0$. Then by Theorem 6.5,

$$\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q) = \sum_{n=n_0\ge n_1\ge\cdots\ge k}\sum_{\mathbf{T}\in X(n_0,n_1,\dots)} \mathbf{a}^{s(\mathbf{T})}q^{\|\mathbf{T}\|_{n,k}}\begin{bmatrix} n \\ k, m(\mathbf{T})\end{bmatrix}_q . \tag{6.7}$$

Consider $\mathbf{T}=(T_k,\dots,T_{n-1})\in X(n_0,n_1,\dots)$. For $j\ge0$, let

$$\mathbf{T}_j = (T_{n_{j+1}},T_{n_{j+1}+1},\dots,T_{n_j-1}).$$

Then

$$\mathbf{a}^{s(\mathbf{T})}q^{\|\mathbf{T}\|_{n,k}} = \prod_{j\ge0}\mathbf{a}^{s(\mathbf{T}_j)}q^{\|\mathbf{T}_j\|_{n_j,n_{j+1}}}$$

and

$$\prod_{j\ge 0}\begin{bmatrix} n_j \\ n_{j+1}, m(\mathbf{T}_j)\end{bmatrix}_q = \prod_{j\ge 0}\frac{[n_j]_q!}{[n_{j+1}]_q!}\prod_{m\in m(\mathbf{T}_j)}\frac{1}{[m]_q!} = \frac{[n]_q!}{[k]_q!}\prod_{m\in m(\mathbf{T})}\frac{1}{[m]_q!} = \begin{bmatrix} n \\ k, m(\mathbf{T})\end{bmatrix}_q.$$

Therefore

$$\mathbf{a}^{s(\mathbf{T})}q^{\|\mathbf{T}\|_{n,k}}\begin{bmatrix} n \\ k, m(\mathbf{T})\end{bmatrix}_q = \prod_{j\ge 0}\mathbf{a}^{s(\mathbf{T}_j)}q^{\|\mathbf{T}_j\|_{n_j,n_{j+1}}}\begin{bmatrix} n_j \\ n_{j+1}, m(\mathbf{T}_j)\end{bmatrix}_q.$$

This shows that

$$\sum_{\mathbf{T}\in X(n_0,n_1,\dots)}\mathbf{a}^{s(\mathbf{T})}q^{\|\mathbf{T}\|_{n,k}}\begin{bmatrix} n \\ k, m(\mathbf{T})\end{bmatrix}_q = \prod_{j\ge 0}\sigma_{n_j,n_{j+1}}(I_j). \tag{6.8}$$

By (6.7) and (6.8) we obtain the lemma. □

By Lemma 6.7, in order to prove that $\tilde{\sigma}^{AW,H}_{n,k}(a_1,a_2,a_3,a_4;q)$ is invariant under a permutation $\tau\in\mathfrak{S}_4$, it suffices to show that there is an increasing interval partition $(I_0,I_1,\dots)$ of $\mathcal{T}$ such that $\tau\cdot\sigma_{r,s}(I_j)=\sigma_{r,s}(I_j)$ for any integers $r\ge s\ge 0$ and $j\ge 0$. We will find such an increasing interval partition of $\mathcal{T}$ for the three cases that $\tau=(2,3)$, $\tau=(3,4)$, and $\tau=(1,2)$ in this order.

First, we consider the simplest case $\tau=(2,3)$.

Lemma 6.8. *Let* $\mathbf{I}=(I_0,I_0^+,I_0^{++},I_1^{--},I_1^-,I_1,I_1^+,I_1^{++},I_2^{--},I_2^-,I_2,I_2^+,I_2^{++},\dots)$ *be the increasing interval partition of* $\mathcal{T}$ *given by*

$$\begin{aligned} I_t &:= \{(t_1,t_2,t_3,t_4)\in\mathcal{T} : t_4=t_3=t_2=t\},\\ I_t^+ &:= \{(t_1,t_2,t_3,t_4)\in\mathcal{T} : t_4=t, \{t_2,t_3\}=\{t,t+1\}\},\\ I_t^{++} &:= \{(t_1,t_2,t_3,t_4)\in\mathcal{T} : t_4=t,\ t_2=t_3=t+1\},\\ I_t^- &:= \{(t_1,t_2,t_3,t_4)\in\mathcal{T} : t_4=t, \{t_2,t_3\}=\{t,t-1\}\},\\ I_t^{--} &:= \{(t_1,t_2,t_3,t_4)\in\mathcal{T} : t_4=t,\ t_2=t_3=t-1\}. \end{aligned}$$

Let $\tau=(2,3)$*. Then for all integers* $n\ge k\ge 0$ *and* $I\in\mathbf{I}$*, we have*

$$\tau\cdot\sigma_{n,k}(I)=\sigma_{n,k}(I).$$

Proof. By the definition (6.6) of $\sigma_{n,k}(I)$, it suffices to find a bijection $\theta:\mathcal{T}_{n,k}(I)\to\mathcal{T}_{n,k}(I)$ such that if $\theta(\mathbf{T})=\mathbf{S}$ then $\tau\cdot\mathbf{a}^{s(\mathbf{T})}=\mathbf{a}^{s(\mathbf{S})}$, $\|\mathbf{T}\|_{n,k}=\|\mathbf{S}\|_{n,k}$, and $m(\mathbf{T})=m(\mathbf{S})$. To do this consider a tuple $\mathbf{T}=(T_k,\dots,T_{n-1})\in\mathcal{T}_{n,k}(I)$. For $k\le i\le n-1$, let $T_i=(t_1^{(i)},t_2^{(i)},t_3^{(i)},t_4^{(i)})$ and define $T_i'=(t_1^{(i)},t_3^{(i)},t_2^{(i)},t_4^{(i)})$. Note that $|T_i|=|T_i'|$ for all i. We define $\theta(\mathbf{T})$ to be the weakly decreasing rearrangement $\mathbf{S}$ of $(T_k',\dots,T_{n-1}')$ in the total order $\le$ on $\mathcal{T}$. By the construction, $\theta:\mathcal{T}_{n,k}(I)\to\mathcal{T}_{n,k}(I)$ is a bijection such that if $\theta(\mathbf{T})=\mathbf{S}$ then $\tau\cdot\mathbf{a}^{s(\mathbf{T})}=\mathbf{a}^{s(\mathbf{S})}$ and $m(\mathbf{T})=m(\mathbf{S})$. Thus it remains to show that $\|\mathbf{T}\|_{n,k}=\|\mathbf{S}\|_{n,k}$.

If I is one of I_t, I_t^{++}, or I_t^{--}, then every $T\in I$ satisfies $t_2=t_3$, so $T'=T$ and hence $\mathbf{T}=\mathbf{S}$; in particular $\|\mathbf{T}\|_{n,k}=\|\mathbf{S}\|_{n,k}$. If $I=I_t^+$ (resp. $I=I_t^-$), then every element $T\in I$ is either $(t,t,t+1,t)$ or $(t,t+1,t,t)$ (resp. $(t,t,t-1,t)$ or $(t,t-1,t,t)$). Hence we always have $|T|=|S|$ and therefore

$$\|\mathbf{S}\|_{n,k}=\sum_{i=k}^{n-1}\frac{i(|S_i|-1)}{2}=\sum_{i=k}^{n-1}\frac{i(|T_i|-1)}{2}=\|\mathbf{T}\|_{n,k},$$

which completes the proof. □

Before finding an increasing interval partition of $\mathcal{T}$ for $\tau=(3,4)$, we need some definitions. Let $\mathfrak{S}(0^r,1^s)$ be the set of words $\pi=\pi_1\cdots\pi_{r+s}$ consisting of r 0's and s 1's. For such a word $\pi=\pi_1\cdots\pi_{r+s}$, an *inversion* is a pair (i,j) of integers $1\le i<j\le r+s$ such that $\pi_i>\pi_j$. Let $\operatorname{inv}(\pi)$ denote the number of inversions of π. It is well known [21, 1.7.1 Proposition] that

$$\sum_{\pi\in\mathfrak{S}(0^r,1^s)}q^{\operatorname{inv}(\pi)}=\begin{bmatrix} r+s \\ r\end{bmatrix}_q. \tag{6.9}$$

Now we are ready to consider the case $\tau=(3,4)$.

Lemma 6.9. *Let* $\mathbf{I} = (I_0, I_0^+, I_1, I_1^+, \dots)$ *be the increasing interval partition of* $\mathcal{T}$ *given by*

$$I_t := \{(t_1, t_2, t_3, t_4) \in \mathcal{T} : t_3 = t_4 = t\},$$
$$I_t^+ := \{(t_1, t_2, t_3, t_4) \in \mathcal{T} : \{t_3, t_4\} = \{t, t+1\}\}.$$

Let $\tau = (3, 4)$*. Then for all integers* $n \ge k \ge 0$ *and* $I \in \mathbf{I}$*, we have*

$$\tau \cdot \sigma_{n,k}(I) = \sigma_{n,k}(I).$$

Proof. The case $I = I_t$ can be proved similarly as in the proof of Lemma 6.8. For the case $I = I_t^+$, we investigate the summand in

$$\sigma_{n,k}(I_t^+) = \sum_{\mathbf{T} \in \mathcal{T}_{n,k}(I_t^+)} \mathbf{a}^{s(\mathbf{T})} q^{\|\mathbf{T}\|_{n,k}} \begin{bmatrix} n \\ k, m(\mathbf{T}) \end{bmatrix}_q.$$

Note that I_t^+ has exactly four elements T_1, T_2, T_3, T_4, where

$$T_1 = (t+1, t+1, t, t+1) > T_2 = (t, t, t, t+1) > T_3 = (t+1, t+1, t+1, t) > T_4 = (t, t, t+1, t).$$

Therefore, every element $\mathbf{T} \in \mathcal{T}_{n,k}(I_t^+)$ is of the form $\mathbf{T} = (T_1^{m_1}, T_2^{m_2}, T_3^{m_3}, T_4^{m_4})$, where $T_i^{m_i}$ means the sequence $T_i, \dots, T_i$ of m_i occurrences of T_i. Then

$$\mathbf{a}^{s(\mathbf{T})} = (a_1 a_2 a_3 a_4)^{(n-k)t} (a_1 a_2 a_4)^{m_1} a_4^{m_2} (a_1 a_2 a_3)^{m_3} a_3^{m_4},$$
$$\|\mathbf{T}\|_{n,k} = \sum_{i=k}^{n-1} i(2t) + \sum_{i=k}^{k+m_1-1} i + \sum_{i=k+m_1+m_2}^{k+m_1+m_2+m_3-1} i,$$
$$\begin{bmatrix} n \\ k, m(\mathbf{T}) \end{bmatrix}_q = \begin{bmatrix} n \\ k, m_1, m_2, m_3, m_4 \end{bmatrix}_q.$$

Letting $M_1 = m_1 + m_2$, $M_2 = m_3 + m_4$, and $N = m_1 + m_3$, we can rewrite the above equations as follows:

$$\mathbf{a}^{s(\mathbf{T})} = (a_1 a_2 a_3 a_4)^{(n-k)t} a_1^N a_2^N a_3^{M_2} a_4^{M_1},$$
$$\|\mathbf{T}\|_{n,k} = 2t\left(\binom{n}{2} - \binom{k}{2}\right) + kN + \binom{N}{2} + m_2 m_3,$$
$$\begin{bmatrix} n \\ k, m(\mathbf{T}) \end{bmatrix}_q = \begin{bmatrix} n \\ k, M_1, M_2 \end{bmatrix}_q \begin{bmatrix} M_1 \\ m_1 \end{bmatrix}_q \begin{bmatrix} M_2 \\ m_3 \end{bmatrix}_q.$$

Therefore,

$$\sigma_{n,k}(I_t^+) = \sum_{\substack{M_1 + M_2 = n-k \\ 0 \le N \le n-k}} f(M_1, M_2, N) g(M_1, M_2, N), \tag{6.10}$$

where

$$f(M_1, M_2, N) = (a_1 a_2 a_3 a_4)^{(n-k)t} a_1^N a_2^N a_3^{M_2} a_4^{M_1} q^{2t\left(\binom{n}{2} - \binom{k}{2}\right) + kN + \binom{N}{2}} \begin{bmatrix} n \\ k, M_1, M_2 \end{bmatrix}_q,$$
$$g(M_1, M_2, N) = \sum_{(m_1, m_2, m_3, m_4) \in X(M_1, M_2, N)} q^{m_2 m_3} \begin{bmatrix} M_1 \\ m_1 \end{bmatrix}_q \begin{bmatrix} M_2 \\ m_3 \end{bmatrix}_q, \tag{6.11}$$

and $X(M_1, M_2, N)$ is the set of tuples (m_1, m_2, m_3, m_4) of nonnegative integers such that $m_1 + m_2 = M_1$, $m_3 + m_4 = M_2$, and $m_1 + m_3 = N$.

Since $\tau \cdot f(M_1, M_2, N) = f(M_2, M_1, N)$ and $\tau \cdot g(M_1, M_2, N) = g(M_1, M_2, N)$, applying τ to (6.10) yields

$$\tau \cdot \sigma_{n,k}(I_t^+) = \sum_{\substack{M_1 + M_2 = n-k \\ 0 \le N \le n-k}} f(M_2, M_1, N) g(M_1, M_2, N). \tag{6.12}$$

By (6.10) and (6.12), in order to show that $\tau \cdot \sigma_{n,k}(I_t^+) = \sigma_{n,k}(I_t^+)$, it suffices to verify that $g(M_1, M_2, N) = g(M_2, M_1, N)$. More generally, we will prove that $g(M_1, M_2, N)$ is independent of M_1 and M_2 by showing the following claim:

$$g(M_1, M_2, N) = \sum_{\pi \in \mathfrak{S}(0^N, 1^{n-k-N})} q^{\mathrm{inv}(\pi)} = \begin{bmatrix} n-k \\ N \end{bmatrix}_q. \tag{6.13}$$

To see this, consider $\pi = \pi_1 \cdots \pi_{n-k} \in \mathfrak{S}(0^N, 1^{n-k-N})$. Let m_1 (resp. m_2) be the number of 0's (resp. 1) in $\pi' = \pi_1 \cdots \pi_{M_1}$ and let m_3 (resp. m_4) be the number of 0's (resp. 1) in $\pi'' = \pi_{M_1+1} \cdots \pi_{n-k}$. Then $(m_1, m_2, m_3, m_4) \in X(M_1, M_2, N)$ and $\mathrm{inv}(\pi) = m_2 m_3 + \mathrm{inv}(\pi') + \mathrm{inv}(\pi'')$. Hence, the right-hand side of (6.13) is equal to

$$\sum_{(m_1,m_2,m_3,m_4) \in X(M_1,M_2,N)} q^{m_2 m_3} \sum_{\pi' \in \mathfrak{S}(0^{m_1}, 1^{m_2})} q^{\mathrm{inv}(\pi')} \sum_{\pi'' \in \mathfrak{S}(0^{m_3}, 1^{m_4})} q^{\mathrm{inv}(\pi'')}.$$

By (6.9), this is equal to the right-hand side of (6.11). Therefore the claim (6.13) holds and the proof is completed. ☐

Finally we consider the case $\tau = (1, 2)$, which is similar to the case $\tau = (3, 4)$.

Lemma 6.10. *Let $\mathbf{I} = (I_0, I_0^+, I_1, I_1^+, \dots)$ be the increasing interval partition of $\mathcal{T}$ given by*

$$I_t := \{(t_1, t_2, t_3, t_4) \in \mathcal{T} : t_3 = t_4 = t\},$$
$$I_t^+ := \{(t_1, t_2, t_3, t_4) \in \mathcal{T} : \{t_3, t_4\} = \{t, t+1\}\}.$$

Let $\tau = (1, 2)$. Then for all integers $n \geq k \geq 0$ and $I \in \mathbf{I}$, we have

$$\tau \cdot \sigma_{n,k}(I) = \sigma_{n,k}(I).$$

Proof. Since every tuple $(t_1, t_2, t_3, t_4) \in I_t^+$ satisfies $t_1 = t_2$, the case $I = I_t^+$ can be proved similarly as in the proof of Lemma 6.8. To prove the case $I = I_t$ we use a similar approach as in the proof of Lemma 6.9. For $t \geq 1$, I_t has exactly four elements T_1, T_2, T_3, T_4, where

$$T_1 = (t, t+1, t, t) > T_2 = (t-1, t, t, t) > T_3 = (t+1, t, t, t) > T_4 = (t, t-1, t, t).$$

For $t = 0$, the tuples T_2 and T_4 have a negative coordinate and hence do not lie in $\mathcal{T}$, so $I_0 = \{T_1, T_3\}$; the analysis below remains valid in this case with the convention $m_2 = m_4 = 0$. Therefore, every element $\mathbf{T} \in \mathcal{T}_{n,k}(I_t)$ is of the form $\mathbf{T} = (T_1^{m_1}, T_2^{m_2}, T_3^{m_3}, T_4^{m_4})$ and

$$\mathbf{a}^{s(\mathbf{T})} = (a_1 a_2 a_3 a_4)^{(n-k)t} a_1^{m_3 - m_2} a_2^{m_1 - m_4},$$
$$\begin{bmatrix} n \\ k, m(\mathbf{T}) \end{bmatrix}_q = \begin{bmatrix} n \\ k, m_1, m_2, m_3, m_4 \end{bmatrix}_q,$$
$$\|\mathbf{T}\|_{n,k} = \sum_{i=k}^{n-1} i(2t-1) + \sum_{i=k}^{k+m_1-1} i + \sum_{i=k+m_1+m_2}^{k+m_1+m_2+m_3-1} i.$$

Letting $M_1 = m_1 + m_2$, $M_2 = m_3 + m_4$, and $N = m_1 + m_3$, we can rewrite the above equations as follows:

$$\mathbf{a}^{s(\mathbf{T})} = (a_1 a_2 a_3 a_4)^{(n-k)t} a_1^{N - M_1} a_2^{N - M_2},$$
$$\begin{bmatrix} n \\ k, m(\mathbf{T}) \end{bmatrix}_q = \begin{bmatrix} n \\ k, M_1, M_2 \end{bmatrix}_q \begin{bmatrix} M_1 \\ m_1 \end{bmatrix}_q \begin{bmatrix} M_2 \\ m_3 \end{bmatrix}_q,$$
$$\|\mathbf{T}\|_{n,k} = (2t-1)\left(\binom{n}{2} - \binom{k}{2}\right) + kN + \binom{N}{2} + m_2 m_3.$$

Then by the same argument in the proof of Lemma 6.9, we obtain $\tau \cdot \sigma_{n,k}(I_t) = \sigma_{n,k}(I_t)$. ☐

By Lemmas 6.8, 6.9, and 6.10, we establish the symmetry of a, b, c, d in the mixed moments of Askey–Wilson polynomials. Let $\nu_{n,k}^{AW,H}(a, b, c, d; q)$ be the coefficient in the expansion

$$p_n(x; a, b, c, d|q) = \sum_{k=0}^{n} \nu_{n,k}^{AW,H}(a, b, c, d; q)\, H_k(x|q).$$

Theorem 6.11. *The rescaled mixed moment* $\tilde{\sigma}_{n,k}^{AW,H}(a_1, a_2, a_3, a_4; q)$ *is symmetric in* a_1, a_2, a_3, a_4. *Equivalently, the mixed moment* $\sigma_{n,k}^{AW,H}(a, b, c, d; q)$ *(and also the coefficient* $\nu_{n,k}^{AW,H}(a, b, c, d; q)$*) relative to continuous* q*-Hermite polynomials is symmetric in* a, b, c, d.

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