Liouville Type Theorem for Stationary Navier-Stokes Equations

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Abstract

It is shown that any smooth solution to the stationary Navier-Stokes system in R^3 with the velocity field, belonging globally to L_6 and $BM0^{-1}$, must be zero.

1 Introduction

Let us consider smooth solutions to the stationary Navier-Stokes system

$$u \cdot \nabla u - \Delta u = \nabla p, \qquad \mathrm{div} u = 0 \tag{1.1}$$

in $\mathbb{R}^3$ with the additional condition at infinity: $u(x) \to 0$ as $|x| \to \infty$. Then the question is weather or not u must be identically zero. The point addressed in this short note is under which additional assumptions the answer to the above question is positive.

In the monograph [3], it has been shown that the condition

$$u \in L_{\frac{9}{2}}(\mathbb{R}^3) \tag{1.2}$$

implies $u = 0$.

A plausible conjecture is that a sufficient condition for the positive answer could be as follows

$$\int\limits_{\mathbb{R}^3} |\nabla u|^2 dx < \infty. \tag{1.3}$$

At the moment of writing this note, a proof of whether or not (1.3) is a sufficient condition for u to be identically zero is not known yet. We are going to show that however under an additional condition it is true. To describe that condition, we need the following definition. We say that a divergence free vector valued field u belongs to the space $BMO^{-1}(\mathbb{R}^3)$ if there exists skew symmetric tensor d from $BMO(\mathbb{R}^3)$ such that

$$u = \operatorname{div} d = (d_{ij,j}).$$

It is known, see for instance [7], that if $d \in BMO(\mathbb{R}^3)$, then

$$\Gamma(s) := \sup_{x_0\in\mathbb{R}^3, 0<r} \Big(\frac{1}{|B(r)|} \int\limits_{B(x_0,r)} |d - [d]_{x_0,r}|^s dx\Big)^{\frac{1}{s}} < \infty$$

for each $1 \leq s < \infty$. Here, we denote by $B(x_0, r)$ the ball of radius r centred at point x_0 and $[d]_{x_0,r}$ is the mean value of d over the ball $B(x_0, r)$.

Our aim is to prove the following result.

Theorem 1.1. *The following statements are true:*
(A) any smooth divergence free vector-valued field $u \in BMO^{-1}(\mathbb{R}^3)$*, satisfying condition (1.3) and the system (1.1), is identically equal to zero;*
(B) any smooth divergence free vector-valued field $u \in BMO^{-1}(\mathbb{R}^3)$*, satisfying the condition*

$$u \in L_6(\mathbb{R}^3) \tag{1.4}$$

and the system (1.1), is identically equal to zero.

By the known inequality

$$\|v\|_{6,\Omega} \leq c\|\nabla v\|_{2,\Omega},$$

being valid for any $v \in C_0^\infty(\mathbb{R}^3)$, the statement (A) follows from the statement (B).

Before proving Theorem 1.1, let us show that the assumptions in (B) do not follow immediately from the condition (1.2) of [3]. Indeed, we can let

$$w = \sin\Big((|x|^2 + 1)^{\frac{1}{4}-\varepsilon}\Big)(1, 1, 1)$$

and

$$v = \operatorname{rot} w.$$

Direct calculations show that $v \in BMO^{-1}(\mathbb{R}^3) \cap L_6(\mathbb{R}^3)$ but $v \notin L_{\frac{9}{2}}(\mathbb{R}^3)$.

Finally, we would like to mention that there are many interesting papers devoted to the above or related questions, see, for example, [5], [6], [2], and [1].

2 Proof of Main Result

2.1 Caccioppoli Type Inequality

This is the main technical part of the proof. We take an arbitrary ball $B(x_0, R) \in \mathbb{R}^3$ and a non-negative cut off function $\varphi \in C_0^\infty(B(x_0, R))$ with the following properties $\varphi(x) = 1$ in $B(x_0, \varrho)$, $\varphi(x) = 0$ out of $B(x_0, r)$, and $|\nabla \varphi(x)| \leq c/(r - \varrho)$ for any $R/2 \leq \varrho < r \leq R$.

We let

$$\overline{u} = u - u_0, \qquad \overline{d} = d - [d]_{x_0,R},$$

where u_0 is an arbitrary constant.

We also know that, for any $2 < s < \infty$, there exists a constant $c_0 = c_0(s) > 0$ and a function $w \in W^1_s(B(x_0, r))$, vanishing on $\partial B(x_0, r)$, such that $\operatorname{div} w = \nabla \varphi \cdot \overline{u}$ and

$$\int\limits_{B(x_0,r)} |\nabla w|^s dx \leq c_0 \int\limits_{B(x_0,r)} |\nabla \varphi \cdot \overline{u}|^s dx \leq \frac{c_0}{(r-\varrho)^s} \int\limits_{B(x_0,R)} |\overline{u}|^s dx. \qquad (2.1)$$

Now, we can test the Navier-Stokes equations (1.1) with the function $\varphi \overline{u} - w$, integrate by parts in $B(x_0, r)$, and find the following identity

$$\int\limits_{B(x_0,r)} \varphi |\nabla u|^2 dx = -\int\limits_{B(x_0,r)} \nabla u : (\nabla \varphi \otimes \overline{u}) dx + \int\limits_{B(x_0,r)} \nabla w : \nabla u dx +$$

$$-\int\limits_{B(x_0,r)} (u \cdot \nabla u) \cdot \varphi \overline{u} dx + \int\limits_{B(x_0,r)} (u \cdot \nabla u) \cdot w dx = I_1 + I_2 + I_3 + I_4.$$

I_1 and I_2 can be estimated easily. As a result, we find

$$|I_1| + |I_2| \leq \Big(\int\limits_{B(x_0,r)} |\nabla u|^2 dx \Big)^{\frac{1}{2}} \frac{R^{3\frac{s-2}{2s}}}{r-\varrho} \Big(\int\limits_{B(x_0,R)} |\overline{u}|^s dx \Big)^{\frac{1}{s}}.$$

To estimate I_3 and I_4, we are going to use the skew symmetry of the matrix d. We have

$$|I_3| = \Big| \int\limits_{B(x_0,r)} \overline{d}_{jm,m}\overline{u}_{i,j}\overline{u}_i\varphi dx\Big| = \Big| \int\limits_{B(x_0,r)} \overline{d}_{jm}\overline{u}_{i,j}\overline{u}_i\varphi_{,m} dx\Big| \leq$$

$$\leq \frac{1}{r-\varrho}\Big(\int\limits_{B(x_0,r)} |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,r)} |\overline{d}|^2|\overline{u}|^2 dx\Big)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{r-\varrho}\Big(\int\limits_{B(x_0,r)} |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,r)} |\overline{u}|^s dx\Big)^{\frac{1}{s}}\Big(\int\limits_{B(x_0,r)} |\overline{d}|^{\frac{2s}{s-2}} dx\Big)^{\frac{s-2}{2s}} \leq$$

$$\leq c\frac{R^{3\frac{s-2}{2s}}}{r-\varrho}\Big(\int\limits_{B(x_0,r)} |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,R)} |\overline{u}|^s dx\Big)^{\frac{1}{s}}\Gamma(2s/(s-2)).$$

It remains to evaluated I_4:

$$|I_4| = \Big| \int\limits_{B(x_0,r)} \overline{d}_{jm,m}\overline{u}_{i,j}w_i dx\Big| = \Big| \int\limits_{B(x_0,r)} \overline{d}_{jm}\overline{u}_{i,j}w_{i,m} dx\Big| \leq$$

$$\leq \frac{1}{r-\varrho}\Big(\int\limits_{B(x_0,r)} |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,r)} |\overline{d}|^2|\nabla w|^2 dx\Big)^{\frac{1}{2}} \leq$$

$$\leq \frac{1}{r-\varrho}\Big(\int\limits_{B(x_0,r)} |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,r)} |\nabla w|^s dx\Big)^{\frac{1}{s}}\Big(\int\limits_{B(x_0,r)} |\overline{d}|^{\frac{2s}{s-2}} dx\Big)^{\frac{s-2}{2s}} \leq$$

$$\leq c(s)\frac{R^{3\frac{s-2}{2s}}}{r-\varrho}\Big(\int\limits_{B(x_0,r)} |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,R)} |\overline{u}|^s dx\Big)^{\frac{1}{s}}\Gamma(2s/(s-2)).$$

So, we have the following inequality

$$\int\limits_{B(x_0,\varrho)} |\nabla u|^2 dx \leq c(s)\frac{R^{3\frac{s-2}{2s}}}{r-\varrho}\Big(\int\limits_{B(x_0,r)} |\nabla u|^2 dx\Big)^{\frac{1}{2}}\Big(\int\limits_{B(x_0,R)} |\overline{u}|^s dx\Big)^{\frac{1}{s}}.$$

We can apply Young's inequality and conclude

$$\int\limits_{B(x_0,\varrho)} |\nabla u|^2 dx \leq \frac{1}{4}\int\limits_{B(x_0,r)} |\nabla u|^2 dx + c(s)\frac{1}{(r-\varrho)^2}R^{3\frac{s-2}{s}}\Big(\int\limits_{B(x_0,R)} |\overline{u}|^s dx\Big)^{\frac{2}{s}}.$$

The later inequality is valid for any $R/2 \leq \varrho < r \leq R$. It is known, see for instance [4] (this is just a matter of suitable iterations), that such an inequality implies the following Caccioppoli type inequality

$$\int\limits_{B(x_0,R/2)} |\nabla u|^2 dx \leq c(s) R^{3\frac{s-2}{s}-2} \Big(\int\limits_{B(x_0,R)} |\overline{u}|^s dx \Big)^{\frac{2}{s}} \tag{2.2}$$

which holds for any $B(x_0, R)$ in $\mathbb{R}^3$.

2.2 (A) implies (B)

Indeed, if we let $s = 6$ and $u_0 = 0$, then (2.2) takes the form

$$\int\limits_{B(x_0,R/2)} |\nabla u|^2 dx \leq c(s) \Big(\int\limits_{B(x_0,R)} |u|^6 dx \Big)^{\frac{1}{3}} \leq c(s) \Big(\int\limits_{\mathbb{R}^3} |u|^6 dx \Big)^{\frac{1}{3}}.$$

Passing $R \to \infty$, we conclude that (1.3) holds.

2.3 Proof of (A)

Now, we let $s = 3$ and $u_0 = [u]_{x_0,R}$ and use the Gagliardo-Nirenberg type inequality

$$\Big(\int\limits_{B(x_0,R)} |\overline{u}|^3 dx \Big)^{\frac{1}{3}} \leq c \Big(\int\limits_{B(x_0,R)} |\nabla u|^{\frac{3}{2}} dx \Big)^{\frac{2}{3}}$$

with a universal positive constant c. Now, (2.2) can reduced to the following reverse Hölder inequality

$$\frac{1}{|B(R/2)|} \int\limits_{B(x_0,R/2)} |\nabla u|^2 dx \leq c \Big(\frac{1}{|B(R)|} \int\limits_{B(x_0,R)} |\nabla u|^{\frac{3}{2}} dx \Big)^{\frac{4}{3}}$$

with a constant c that is independent of x_0 and R. We let $h := |\nabla u|^{\frac{3}{2}} \in L_{\frac{4}{3}}(\mathbb{R}^3)$ and let $M(h)$ is the maximal function of h, i.e.,

$$M_h(x_0) = \sup_{R>0} \int\limits_{B(x_0,R)} h(x) dx.$$

Then from the above inequality, it follows that

$$M_{h^{\frac{4}{3}}}(x_0) \leq c M_h^{\frac{4}{3}}(x_0).$$

It is known, see [7], that the right hand side of the latter inequality is integrable in $\mathbb{R}^3$ and the corresponding integral is bounded from above by the quantity

$$\int\limits_{B(x_0,R)} h^{\frac{4}{3}} dx = \int\limits_{B(x_0,R)} |\nabla u|^2 dx$$

times a universal constant. So, this means that $M_{h^{\frac{4}{3}}} \in L_1(\mathbb{R}^3)$. Since $h^{\frac{4}{3}} \in L_1(\mathbb{R}^3)$, it is possible only if h is identically equal to zero, see [7]. So, ∇u is identically equal to zero.

References

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