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Differential inclusions for the Schouten tensor and nonlinear eigenvalue problems in conformal geometry

Jonah A.J. Duncan [a], Luc Nguyen [b,*]

[a] *Johns Hopkins University, 404 Krieger Hall, Department of Mathematics, 3400 N. Charles Street, Baltimore, MD 21218, United States of America*

[b] *Mathematical Institute and St Edmund Hall, University of Oxford, Andrew Wiles Building, Radcliffe Observatory Quarter, Woodstock Road, OX2 6GG, UK*

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A B S T R A C T

Let g_0 be a smooth Riemannian metric on a closed manifold M^n of dimension $n \geq 3$. We study the existence of a smooth metric g conformal to g_0 whose Schouten tensor A_g satisfies the differential inclusion $\lambda(g^{-1}A_g) \in \Gamma$ on M^n, where $\Gamma \subset \mathbb{R}^n$ is a cone satisfying standard assumptions. Inclusions of this type are often assumed in the existence theory for fully nonlinear elliptic equations in conformal geometry. We assume the existence of a continuous metric g_1 conformal to g_0 satisfying $\lambda(g_1^{-1}A_{g_1}) \in \overline{\Gamma'}$ in the viscosity sense on M^n, together with a nondegenerate ellipticity condition, where $\Gamma' = \Gamma$ or Γ' is a cone slightly smaller than Γ. In fact, we prove not only the existence of metrics satisfying such differential inclusions, but also existence and uniqueness results for fully nonlinear eigenvalue problems for the Schouten tensor. We also give a number of geometric applications of our results. We show that the sign of a nonlinear eigenvalue for the σ_2 operator is a conformal invariant in three dimensions. We also give a generalisation of a theorem of Aubin & Ehrlick on pinching of

* Corresponding author.
E-mail addresses: jdunca33@jhu.edu (J.A.J. Duncan), luc.nguyen@maths.ox.ac.uk (L. Nguyen).

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the Ricci curvature, and an application in the study of Green's functions for fully nonlinear Yamabe problems.

Contents

1. Introduction

1.1. Main results

Let M^n be a closed manifold of dimension $n \geq 3$, g_0 a smooth Riemannian metric on M^n and $[g_0]$ the set of smooth metrics conformal to g_0. The Yamabe problem, solved through the combined works of Yamabe [54], Trudinger [49], Aubin [3] and Schoen [45], asserts that $[g_0]$ contains a metric g of constant scalar curvature R_g, where the constant is positive (resp. negative, resp. zero) if and only if $[g_0]$ contains a metric of positive (resp. negative, resp. zero) scalar curvature. It is well-known that the sign of any such constant coincides with the sign of the first eigenvalue of the conformal Laplacian of any metric in $[g_0]$, and the sign of the Yamabe invariant

$$Y(M^n, [g_0]) = \inf_{g \in [g_0]} \frac{\int_{M^n} R_g \, dv_g}{\mathrm{Vol}(M^n, g)^{\frac{n-2}{n}}}.$$

Since the work of Viaclovsky [51] and Chang, Gursky & Yang [9], there has been significant interest in fully nonlinear generalisations of the Yamabe problem involving the eigenvalues of the trace-modified Schouten tensor $A_{g_0}^t$, defined for $t \in (-\infty, 1]$ by

$$A_{g_0}^t = \frac{1}{n-2}\bigg(\mathrm{Ric}_{g_0} - \frac{tR_{g_0}}{2(n-1)}g_0\bigg). \tag{1.1}$$

Here, Ric_{g_0} denotes the Ricci curvature tensor of g_0. When $t = 1$, $A_{g_0}^1$ (henceforth denoted A_{g_0}) is the Schouten tensor, which arises in the Ricci decomposition of the Riemann curvature tensor. Of particular interest are elliptic equations of the form

$$\sigma_k^{1/k}(\lambda(g_u^{-1}A_{g_u}^t)) = \psi(x,u), \quad \lambda(g_u^{-1}A_{g_u}^t) \in \Gamma_k^+ \quad \text{on } M^n, \tag{1.2}$$

where $t \leq 1$ is fixed, $\psi > 0$ is a prescribed function and $g_u = e^{-2u}g_0 \in [g_0]$ is the unknown metric. In (1.2), $\lambda(g_u^{-1}A_{g_u}^t)$ denotes the eigenvalues of the $(1,1)$-tensor $g_u^{-1}A_{g_u}^t$, $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the k'th elementary symmetric polynomial, and

$$\Gamma_k^+ = \{\lambda \in \mathbb{R}^n : \sigma_j(\lambda) > 0 \text{ for all } 1 \leq j \leq k\}.$$

Note that $A_{g_u}^t$ and $A_{g_0}^t$ are related by the conformal transformation law

$$A_{g_u}^t = \nabla_{g_0}^2 u + \frac{1-t}{n-2}\Delta_{g_0} u\, g_0 - \frac{2-t}{2}|du|_{g_0}^2 g_0 + du \otimes du + A_{g_0}^t. \tag{1.3}$$

When $k = 1$ and ψ is a positive constant, (1.2) is the Yamabe equation in the case of positive Yamabe invariant. When $2 \leq k \leq n$ and $t \leq 1$, (1.2) is fully nonlinear and non-uniformly elliptic, and when ψ is constant it is often referred to as the (trace-modified, when $t < 1$) σ_k-Yamabe equation.[1] When $\psi(x,u) = \mu e^{2u}$ for a constant $\mu > 0$, we refer to (1.2) as the σ_k-Yamabe eigenvalue problem: in this case, both the determination of the function u and the constant μ are part of the existence problem.

In existing literature addressing (1.2), it has been customary to assume that there exists a conformal metric $g \in [g_0]$ satisfying

$$\lambda(g^{-1}A_g^t) \in \Gamma_k^+ \quad \text{on } M^n. \tag{1.4}$$

Note that when $k = 1$, (1.4) is precisely the assumption that g has positive scalar curvature. Under the assumption (1.4) with $t = 1$, the σ_k-Yamabe equation has been solved in various cases – see e.g. Chang, Gursky & Yang [8], Ge & Wang [17], Guan & Wang [21], Gursky & Viaclovsky [26], Li & Li [30], Li & Nguyen [33], and Sheng, Trudinger & Wang [47]. Under (1.4) the trace-modified σ_k-Yamabe equation and the σ_k-Yamabe eigenvalue problem have also been solved in various cases – see e.g. [26,30,33,50].

[1] Equation (1.2) is also elliptic when $t \geq n-1$, although this case is different in nature due to negativity of the scalar curvature, and is not considered in this paper.

For recent related works, see e.g. [6,11,15,24,35,36,39] and the references therein. We remark more generally that, since the work of Lions [42] and Lions, Trudinger & Urbas [43], fully nonlinear elliptic eigenvalue problems have attracted much interest – for a partial list of references, see [1,13,16,44,48] and the references therein.

It is an interesting and important problem to determine when $[g_0]$ contains a smooth metric satisfying (1.4) for $k \geq 2$; we will discuss previous integral-type results related to this problem slightly later in the introduction. In this paper, we are interested in using viscosity-type conditions to establish the existence of a smooth metric in $[g_0]$ satisfying

$$\lambda(g^{-1}A_g^t) \in \Gamma \quad \text{on } M^n, \tag{1.5}$$

where we have now replaced Γ_k^+ with a more general cone Γ satisfying the following properties:

$$\Gamma \subset \mathbb{R}^n \text{ is an open, convex, connected symmetric cone with vertex at the origin} \tag{1.6}$$

$$\Gamma_n^+ \subseteq \Gamma \subseteq \Gamma_1^+. \tag{1.7}$$

Namely, we assume the existence of a continuous metric $g_{\hat{u}} = e^{-2\hat{u}}g_0$ satisfying

$$\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \overline{\Gamma} \quad \text{in the viscosity sense on } M^n \tag{1.8}$$

(see Section 2 for the meaning of (1.8)). The notion of viscosity solutions to fully nonlinear Yamabe equations was first studied by Li in [32]. This notion is natural in light of recent existence results for non-classical viscosity solutions to related Yamabe-type equations in [20,34,40]. We also highlight that many of our results in this paper are new even in the case that $g_{\hat{u}}$ is smooth. Moreover, in light of [32] and subsequent work on viscosity solutions to fully nonlinear Yamabe equations, such an additional smoothness assumption would not substantially simplify our treatment.

Part of our motivation for the condition (1.8) stems from the expectation that, in the study of compactness issues for (1.2), an appropriately rescaled sequence of solutions to (1.2) (either for $t = 1$ or as $t \to 1$) converges to a possibly non-smooth metric $g_{\hat{u}} = e^{-2\hat{u}}g_0$ satisfying (1.8) for $\Gamma = \Gamma_k^+$. This has been observed in a number of situations, see for instance [8,16,26,30,33,35,50]. Since compactness issues play an important role in the existence theory for (1.2) and more general fully nonlinear elliptic equations, it is desirable to understand the gap between (1.8) and (1.5). The study of continuous metrics satisfying (1.8) is also closely related to the study of Green's functions for fully nonlinear Yamabe problems – see the recent work of Li & Nguyen [35], and Theorem 1.15 below for a related result.

In order to state our results, we first introduce some notation. For $\tau \in [0, 1]$, $\lambda \in \mathbb{R}^n$ and $e = (1, \ldots, 1) \in \mathbb{R}^n$, we define

$$\lambda^\tau = \tau\lambda + (1-\tau)\sigma_1(\lambda)e \quad \text{and} \quad \Gamma^\tau = \{\lambda \in \mathbb{R}^n : \lambda^\tau \in \Gamma\}.$$

We observe that since

$$A_g^t = \tau^{-1}\big[\tau A_g + (1-\tau)\sigma_1(\lambda(g^{-1}A_g))g\big] \quad \text{for } \tau = \tau(t) = \left(1 + \frac{1-t}{n-2}\right)^{-1},$$

it is equivalent to consider for $\tau \in (0,1]$ the existence of smooth conformal metrics $g \in [g_0]$ satisfying

$$\lambda(g^{-1}A_g) \in \Gamma^\tau \quad \text{on } M^n, \tag{1.9}$$

rather than considering (1.5) for $t \in (-\infty, 1]$. Clearly, $\Gamma^1 = \Gamma$ and $\Gamma^0 = \Gamma_1^+$. We note that the inclusion $\Gamma \subseteq \Gamma_1^+$ in (1.7) implies the monotonicity property $\Gamma^\tau \subseteq \Gamma^{\tau'}$ for $\tau' \leq \tau \leq 1$. Moreover, the properties (1.6) and (1.7) are inherited by Γ^τ for $\tau < 1$.

In addressing the existence of a conformal metric satisfying (1.9) we will consider a class of fully nonlinear elliptic equations, whose solutions satisfy (1.9). In fact, we will obtain existence and uniqueness results for corresponding fully nonlinear eigenvalue problems. To this end, suppose that $f : \mathbb{R}^n \to \mathbb{R}$ satisfies the following properties:

$$f \in C^\infty(\Gamma) \cap C^0(\overline{\Gamma}) \text{ is concave, 1-homogeneous and symmetric in the } \lambda_i, \tag{1.10}$$

$$f > 0 \text{ in } \Gamma, \quad f = 0 \text{ on } \partial\Gamma \quad \text{and} \quad f_{\lambda_i} > 0 \text{ in } \Gamma \text{ for } 1 \leq i \leq n. \tag{1.11}$$

(For the existence of such a defining function for Γ satisfying (1.6) and (1.7), see [35, Appendix A].) For

$$f^\tau(\lambda) := f(\lambda^\tau),$$

we then consider

$$f^\tau(\lambda(g_0^{-1}A_{g_{u_\tau}})) = \mu_\tau, \quad \lambda(g_0^{-1}A_{g_{u_\tau}}) \in \Gamma^\tau, \tag{1.12}$$

where we have denoted the constant in (1.12) by μ_τ rather than μ (we will see below in Theorem 1.1 that the constant $\mu_\tau > 0$ for which (1.12) admits a solution depends uniquely on τ). We note that when the pair (f, Γ) satisfies the properties (1.6), (1.7), (1.10) and (1.11), so too does the pair (f^τ, Γ^τ) for $\tau \in [0,1]$. Moreover, whenever $\tau < 1$, the pair $(f^\tau, \overline{\Gamma^\tau})$ is locally strictly elliptic (see e.g. [35, equation (A.1)]).

Theorem 1.1. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and let (f, Γ) satisfy (1.6), (1.7), (1.10) and (1.11). Suppose that there exists a metric $g_{\hat{u}} = e^{-2\hat{u}}g_0$, $\hat{u} \in C^0(M^n)$, satisfying (1.8). Then either*

1. *There exist $\mu_1 > 0$ and $u_1 \in C^\infty(M^n)$ satisfying (1.12) for $\tau = 1$, or*
2. *There exists a $C^{1,1}$ metric g conformal to g_0 with $\lambda(g^{-1}A_g) \in \partial\Gamma$ a.e. on M^n.*

Moreover, the former statement is equivalent to the existence of a smooth metric $g \in [g_0]$ *satisfying* (1.9) *for* $\tau = 1$*. Finally, the eigenfunction/eigenvalue pair is unique in the following sense: if* (u_1, μ_1)*,* $(\check{u}_1, \check{\mu}_1) \in C^\infty(M^n) \times (0, \infty)$ *both satisfy* (1.12)*, then* $\check{\mu}_1 = \mu_1$ *and* $\check{u}_1 = u_1 + c$ *for some constant* $c \in \mathbb{R}$*.*

Remark 1.2. We note that, by the strong comparison principle, if there exists a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma$ on M^n, then $Y(M^n, [g_0]) > 0$ and there is no C^0 metric g conformal to g_0 satisfying $\lambda(g^{-1}A_g) \in \mathbb{R}^n \backslash \Gamma$ in the viscosity sense on M^n. The converse also holds (without assuming (1.8)) – see Theorem 3.10.

Remark 1.3. Suppose $(f, \Gamma) = (\sigma_2^{1/2}, \Gamma_2^+)$ and that there exist $\mu_1 > 0$ and $u_1 \in C^\infty(M^n)$ satisfying (1.12) for $\tau = 1$. Then by uniqueness in Theorem 1.1, the constant μ_1 coincides with the constant $\lambda(g_0, \sigma_2)^{1/2}$ in [16, Theorem 1]. See Section 5 for the definition of $\lambda(g_0, \sigma_2)$.

Our second main result shows that, unless the metric $g_{\hat{u}}$ in Theorem 1.1 is smooth and Ricci flat, for each $\tau < 1$ there exists a smooth conformal metric satisfying (1.9):

Theorem 1.4. *In addition to the assumptions of Theorem 1.1, assume* $\Gamma \neq \Gamma_1^+$*. Then either:*

1. *There exist* $\mu_\tau > 0$ *and* $u_\tau \in C^\infty(M^n)$ *satisfying* (1.12) *for every* $\tau \in (0, 1)$*, or*
2. $\hat{u}$ *is a smooth solution to* $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$ *on* M^n*.*

Moreover, the eigenfunction/eigenvalue pair is unique in the following sense: if (u_τ, μ_τ)*,* $(\check{u}_\tau, \check{\mu}_\tau) \in C^\infty(M^n) \times (0, \infty)$ *both satisfy* (1.12)*, then* $\check{\mu}_\tau = \mu_\tau$ *and* $\check{u}_\tau = u_\tau + c$ *for some constant* $c \in \mathbb{R}$*.*

Remark 1.5. As a consequence of Theorem 1.4, it is easy to see that under the hypotheses of Theorem 1.4, Statement 1 above is equivalent to each of the following statements:

$1'$. There exist $\tau \in (0, 1)$ and $g \in [g_0]$ satisfying (1.9).
$1''$. For every $\tau \in (0, 1)$, there exists $g_\tau \in [g_0]$ satisfying (1.9).

To put Theorems 1.1 and 1.4 into context, we now briefly discuss some previous work on the existence of conformal metrics satisfying (1.4). In [9], Chang, Gursky and Yang studied the case $k = 2$, $n = 4$ and $t = 1$: they showed that if $Y(M^4, [g_0]) > 0$ and $\int_{M^4} \sigma_2(\lambda(g_0^{-1}A_{g_0}))\, dv_{g_0} > 0$, then there exists a conformal metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma_2^+$ on M^4. An alternative proof encompassing the case $t \leq 1$ was given by Gursky & Viaclovsky in [25]. Existence results under similar integral-type conditions were later established in three dimensions by Catino & Djadli [7] and Ge, Lin & Wang [16]. In [16], under the assumption that $R_{g_0} > 0$ and $\int_{M^3} \sigma_2(\lambda(g_0^{-1}A_{g_0}))\, dv_{g_0} > 0$, the authors showed the existence of a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma_2^+$ on M^3. If g_0 only satisfies $R_{g_0} > 0$ and $\int_{M^3} \sigma_2(\lambda(g_0^{-1}A_{g_0}))\,dv_{g_0} = 0$, and g_0 cannot be conformally deformed to a different background metric for which the previous case applies, then g_0 is an optimiser for the functional $Y_{2,1}([g_0])$ considered in [16] and therefore satisfies $\lambda(g_0^{-1}A_{g_0}) \in \partial\Gamma_2^+$ on M^3. It follows that $\lambda(g_0^{-1}A_{g_0}^t) \in \Gamma_2^+$ for all $t < 1$; compare with [7] for $t \leq t_0 \approx 0.7$. In dimensions $n \geq 5$, existence results are only known to hold under integral-type conditions which are assumed to hold for *all* metrics in $[g_0]$ – see [16,46]. (This is in contrast to condition (1.8) and the conditions discussed above in three and four dimensions, which are only assumed to hold for a *single* metric.) In particular, in [16] the existence of metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma_2^+$ in dimensions $n \geq 5$ is shown to hold if

$$Y(M^n, [g_0]) > 0 \quad \text{and} \quad \inf_{g \in [g_0],\, R_g > 0} \frac{\int_{M^n} \sigma_2(\lambda(g^{-1}A_g))\,dv_g}{\left(\int_{M^n} \sigma_1(\lambda(g^{-1}A_g))\,dv_g\right)^{\frac{n-4}{n-2}}} > 0.$$

As by-product of our method for establishing Theorems 1.1 and 1.4, we prove that if $Y(M^n, [g_0]) > 0$, then either there exists a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma_2^+$ on M^n, or there exist a $C^{1,1}$ metric g conformal to g_0 satisfying $\sigma_1(g^{-1}A_g) \geq 0$ and $\sigma_2(g^{-1}A_g) \leq 0$ a.e. on M^n – see Theorem 3.10 in Section 3.5 for the precise statement. As an application of Theorem 3.10, we recover the above mentioned result of Ge, Lin & Wang [16] in dimensions $n \geq 5$, and also extend this result to cover the corresponding trace-modified problems – see Theorem 5.2 in Section 5.2.

As a side remark, we also note that the integral conditions mentioned in the previous paragraph require some degree of differentiability of the conformal factor, whereas no differentiability is assumed in our condition (1.8).

The following is an immediate consequence of Theorem 1.4:

Corollary 1.6. *In addition to the hypotheses of Theorem 1.1 suppose $Y(M^n, [g_0]) > 0$. Then for each $\tau \in (0, 1)$, there exists a smooth metric $g_\tau \in [g_0]$ with $\lambda(g_\tau^{-1}A_{g_\tau}) \in \Gamma^\tau$ on M^n.*

As far as the authors are aware, even in the case that $\hat{u}$ is smooth, Corollary 1.6 was not previously known in dimensions $n \geq 5$ (the case $n = 3$ follows from [16], in light of the discussion above, and the case $n = 4$ follows from [25, Theorem 1.1]).

As remarked above, the strong comparison principle implies that if $g_{\hat{u}}$ satisfies $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$ in the viscosity sense on M^n, then there is no smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma$ on M^n. This fact, combined with Theorem 1.1, yields the following:

Corollary 1.7. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and let Γ satisfy (1.6) and (1.7). Suppose there exists a metric $g_{\hat{u}} = e^{-2\hat{u}}g_0$, $\hat{u} \in C^0(M^n)$, satisfying $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$ in the viscosity sense on M^n. Then there exists a $C^{1,1}$ metric $g = e^{-2u}g_0$ satisfying $\lambda(g^{-1}A_g) \in \partial\Gamma$ a.e. on M^n.*

Remark 1.8. It is an open problem as to whether all continuous viscosity solutions $g = e^{-2u}g_0$ to $\lambda(g^{-1}A_g) \in \partial\Gamma_k^+$ $(k \geq 2)$ belong to $C^{1,1}$, and if solutions u are unique up to addition of constants. The question of uniqueness is related to the known failure of the strong comparison principle for sub/supersolutions to the equation $\lambda(g^{-1}A_g) \in \partial\Gamma_k^+$ – see [41]. We note that, for $\tau \in (0,1)$, uniqueness and $C^{1,1}$ regularity of continuous viscosity solutions to $\lambda(g^{-1}A_g) \in \partial\Gamma^\tau$ follow from Theorem 1.10 and Proposition A.1.

We now highlight some elements of the proof of Theorem 1.4 (the ideas for Theorem 1.1 are similar). We first show in Section 3.1 that (1.8) and $\Gamma \neq \Gamma_1^+$ imply either $Y(M^n, [g_0]) > 0$ or $\hat{u}$ is smooth with $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$. In the latter case we are done. In the former case, rather than directly addressing (1.12), we first consider a class of equations for which we prove both existence and uniqueness under (1.8). More precisely, for $\tau \in (0,1)$ and $h = h(x,z) > 0$ satisfying conditions (C1)-(C3) in Section 3, we prove existence and uniqueness of solutions to

$$f^\tau\big(\lambda(g_0^{-1}A_{g_{u_\tau}})\big) = h(x, u_\tau), \quad \lambda(g_0^{-1}A_{g_{u_\tau}}) \in \Gamma^\tau \tag{1.13}$$

assuming (1.8) and $Y(M^n, [g_0]) > 0$ – see Theorem 3.2 in Section 3. For each $\tau \in (0,1)$, the solution to (1.12) is then obtained as a suitably rescaled limit of solutions to (1.13) in the case $h(x,z) = e^{\beta z}$ as $\beta \to 0^+$. Now, the uniqueness for (1.13) follows from (C1) (which is a properness condition on h) and the strong comparison principle. For existence we use a degree argument, which relies on obtaining *a priori* C^2 estimates on solutions u_τ which are uniform with respect to τ on any compact subset of $(0,1)$. These are obtained in Section 3.2, where the main task is to establish a lower C^0 bound (the upper C^0 bound follows a standard argument, and first/second derivative estimates follow from previous work of various authors). We assume for a contradiction that the lower C^0 bound fails along a sequence $\tau_i \to \tau < 1$, which we show implies the existence of a $C^{1,1}$ metric $\widetilde{g} = e^{-2\widetilde{u}}g_0$ satisfying $\lambda(\widetilde{g}^{-1}A_{\widetilde{g}}) \in \partial\Gamma^\tau$ a.e. on M^n. We now recall the assumption that $\hat{u} \in C^0(M^n)$ satisfies (1.8), i.e. $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \overline{\Gamma}$ in the viscosity sense on M^n. At this point, if one could show that $\widetilde{u} = \hat{u} + c$ for some constant c, a contradiction would follow from a geometric property of the cone Γ (see Lemma 3.4). For this purpose, we prove a strong comparison principle for $\tau < 1$ in Section 2 (see Theorem 2.3).

We note that the analogous strong comparison principle for $\tau = 1$ is false in general – see [41]. However, we will see that for cones Γ where such a comparison principle *can* be established, Theorem 1.1 can be improved. This is the case, for instance, when

$$(1, 0, \dots, 0) \in \Gamma, \tag{1.14}$$

due to the validity of our strong comparison principle (see Theorem 2.3′). As an important example, the cones Γ_k^+ do not satisfy (1.14) when $k \geq 2$, whereas the trace-modified cones $(\Gamma_k^+)^\tau$ do satisfy this condition for any $\tau \in [0,1)$. The relevance of condition (1.14) in the context of obtaining comparison principles for fully nonlinear elliptic and parabolic equations was previously recognised in [4]. The condition (1.14) is equivalent to $\overline{\Gamma_n^+}\backslash\{0\} \subset \Gamma$, and also equivalent to $\Gamma = (\widetilde{\Gamma})^\tau$ for some $\tau \in (0,1)$ and $\widetilde{\Gamma}$ satisfying (1.6) and (1.7); for a proof of these facts, we refer the reader to Appendix A. The latter characterisation allows for applications e.g. to fully nonlinear equations involving the Ricci tensor, as we will see later. Our improved result under (1.14) is as follows:

Theorem 1.9. *In addition to the hypotheses of Theorem 1.1, suppose also that (1.14) holds and $g_{\hat{u}}$ is not a $C^{1,1}$ solution to $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$ a.e. on M^n. Then there exist $\mu_1 > 0$ and $u_1 \in C^\infty(M^n)$ satisfying (1.12) for $\tau = 1$, and this is equivalent to the existence of a smooth metric $g \in [g_0]$ satisfying (1.9) for $\tau = 1$. Finally, the same uniqueness statement as in Theorem 1.1 holds.*

We will use the following uniqueness and regularity result in the proof of Theorem 1.9:

Theorem 1.10. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and suppose Γ satisfies (1.6), (1.7) and (1.14). Then continuous viscosity solutions $g_u = e^{-2u}g_0$ to the equation $\lambda(g_u^{-1}A_{g_u}) \in \partial\Gamma$ on M^n are unique up to addition of constants to u, and belong to $C^{1,1}(M^n)$.*

1.2. Applications

We now discuss some geometric applications of our main results. We start with two results that are consequences of Theorem 1.4 (more precisely, Corollary 1.6) and previous work of Ge, Lin & Wang [16], and concern the case $(f,\Gamma) = (\sigma_2^{1/2}, \Gamma_2^+)$. We refer the reader to Section 5 for the definition of the nonlinear eigenvalue $\lambda(\sigma_2, g_0)$ and the nonlinear Yamabe-type invariant $Y_{2,1}([g_0])$, which were previously studied in [16] (see also [17,18,23]).

Theorem 1.11. *Let (M^3, g_0) be a smooth, closed Riemannian 3-manifold. Then $\lambda(\sigma_2, g_0)$ is positive (resp. negative, resp. zero) if and only if $Y_{2,1}([g_0])$ is positive (resp. negative, resp. zero). In particular, the sign of $\lambda(\sigma_2, g_0)$ is a conformal invariant.*

The counterpart to Theorem 1.11 in dimensions $n \geq 4$ was previously obtained [16], where it was also shown for $n = 3$ that $\lambda(\sigma_2, g_0) > 0$ if and only if $Y_{2,1}([g_0]) > 0$. Our contribution is therefore in the case when $\lambda(\sigma_2, g_0)$ is zero or negative.

Remark 1.12. It was shown in [16] that when $Y_{2,1}([g_0]) = 0$, there exists a $C^{1,1}$ metric $g \in [g_0]$ satisfying $\sigma_2(\lambda(g^{-1}A_g)) \equiv 0$ and $R_g \geq 0$ a.e. on M^n. The converse for $n = 3$ also holds by an approximation argument similar to that in the proof of Theorem 5.2.

The relevance of Theorem 1.11 (and its higher dimensional counterpart in [16]) in relation to the σ_2-Yamabe problem is as follows. It is shown in [16, Theorem 2] that $Y_{2,1}([g_0]) > 0$ if and only if there exists $g \in [g_0]$ with $\lambda(g^{-1}A_g) \in \Gamma_2^+$ on M^n. Since there exists a solution to the σ_2-Yamabe problem under the assumption of a conformal metric satisfying $\lambda(g^{-1}A_g) \in \Gamma_2^+$, the existence of a solution σ_2-Yamabe problem in $[g_0]$ is therefore equivalent to positivity of $\lambda(\sigma_2, g_0)$. By Theorem 1.11, the work in [16] and Remark 1.12, the vanishing of $\lambda(\sigma_2, g_0)$ (equivalently, the vanishing of $Y_{2,1}([g_0])$) is equivalent to the existence of a $C^{1,1}$ solution to $\sigma_2(\lambda(g^{-1}A_g)) \equiv 0$, $R_g \geq 0$ a.e. on M^3. This is in direct analogy with the Yamabe problem, where the existence of a conformal metric with positive (resp. zero) constant scalar curvature is equivalent to the positivity (resp. the vanishing) of the first eigenvalue of the conformal Laplacian of any conformal metric.

Our next result both unifies and extends the work of Ge, Lin & Wang [16, Theorem 2] and Catino & Djadli [7, Theorem 1.3], as we will explain in more detail in Section 5:

Theorem 1.13. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ with $Y_{2,1}([g_0]) = 0$. Then there exists a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g^t) \in \Gamma_2^+$ if and only if $t < 1$.*

Theorem 1.13 also has geometric consequences regarding the existence of conformal metrics with pinched Ricci curvature – see Corollary 5.1.

Our next two results are applications of Theorem 1.9. The first of these applications generalises a result of Aubin & Ehrlick [2,12], who showed that if a manifold admits a smooth metric g_0 with $\mathrm{Ric}_{g_0} \geq 0$ on M^n and $\mathrm{Ric}_{g_0} > 0$ at some point on M^n, then it admits a conformal metric $g \in [g_0]$ with $\mathrm{Ric}_g > 0$ on M^n. We extend this result to a wider range of lower bounds, which are only required to be satisfied in the viscosity sense on M^n:

Theorem 1.14. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$. Suppose $\hat{g}$ is a continuous metric conformal to g_0 such that*

$$\mathrm{Ric}_{\hat{g}} \geq \alpha R_{\hat{g}} \hat{g} \quad \text{in the viscosity sense on } M^n \tag{1.15}$$

for some constant $\alpha < \frac{1}{2(n-1)}$, and that $\hat{g}$ is not a $C^{1,1}$ solution to $\lambda(\hat{g}^{-1}(\mathrm{Ric}_{\hat{g}} - \alpha R_{\hat{g}}\hat{g})) \in \partial\Gamma_n^+$ a.e. on M^n. Then there exists a smooth metric $g \in [g_0]$ such that $\mathrm{Ric}_g > \alpha R_g g$ on M^n.

Note that when $\alpha \geq 0$, all the relevant metrics in Theorem 1.14 have nonnegative Ricci curvature, and Theorem 1.14 can also be obtained by appealing to the work of [33] (rather than Theorem 1.9) and our strong comparison principle in Section 2 – see Section 5.4 for the details.

Our second application of Theorem 1.9 establishes a partial converse to [35, Theorem 1.2], where the existence of a Green's function for Γ satisfying (1.6) and (1.7) is obtained under the assumption that there exists a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma$ on M^n (together with a natural, necessary condition on Γ that is not relevant in the discussion below). For $m \geq 1$ distinct points $p_1, \ldots, p_m \in M^n$ and positive numbers

$c_1, \ldots, c_m$, we recall from [35] that a positive function $w \in C^0_{\mathrm{loc}}(M^n \backslash \{p_1, \ldots, p_m\})$ is a Green's function for Γ with poles $p_1, \ldots, p_m$ and strengths $c_1, \ldots, c_m$ if the metric $g = w^{\frac{4}{n-2}} g_0$ satisfies $\lambda(g^{-1}A_g) \in \partial\Gamma$ in the viscosity sense on $M^n \backslash \{p_1, \ldots, p_m\}$ and $\lim_{x \to p_i} d_g(x, x_i)^{n-2} w(x) = c_i$ for each $i = 1, \ldots, m$. We prove:

Theorem 1.15. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and suppose Γ satisfies (1.6), (1.7) and (1.14). Suppose for some finite set of points $\{p_1, \ldots, p_m\} \subset M^n$ there exists a Green's function for Γ with poles $p_1, \ldots, p_m$. Then there exists a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma$ on M^n.*

Remark 1.16. It would be interesting to determine whether Theorem 1.15 is true without condition (1.14), i.e. assuming $(1, 0, \ldots, 0) \in \partial\Gamma$.

The plan of the paper is as follows. In Section 2 we establish a strong comparison principle, which will be used as a tool in proving our main results. Section 3 is devoted to the proof of Theorems 1.1 and 1.4. We start by stating existence and uniqueness results for (1.13) – see Theorems 3.1 and 3.2. In Section 3.1 we show that (1.8) and $\Gamma \neq \Gamma_1^+$ imply either $g_{\hat{u}}$ is smooth with $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$, or $Y(M^n, [g_0]) > 0$. In Section 3.2 we obtain *a priori* estimates for (1.13), assuming $Y(M^n, [g_0]) > 0$. In Section 3.3 we prove Theorems 3.1 and 3.2. In Section 3.4, we use Theorems 3.1 and 3.2 to prove Theorems 1.1 and 1.4 by a suitable limiting argument. In Section 4 we present refinements of Theorems 1.1 and 3.1 under (1.14), and in particular we prove Theorem 1.9 and Theorem 1.10. In Section 5 we address some geometric consequences of main results – in particular, we prove Theorems 1.11–1.15.

2. A strong comparison principle

In this section we prove a strong comparison principle for subsolutions and supersolutions to the equation $\lambda(g^{-1}A_g) \in \partial\Gamma$ assuming (1.6), (1.7) and (1.14). As a consequence, this yields a strong comparison principle for the equation $\lambda(g^{-1}A_g) \in \partial\Gamma^\tau$ for $\tau \in (0,1)$, regardless of whether (1.14) is satisfied – see Proposition A.1 in Appendix A. Our strong comparison principle will be used to obtain *a priori* estimates in Section 3 and is also of independent interest. Our discussion will involve semicontinuous functions, although for the purpose of obtaining our main results in this paper, the reader may assume that all involved functions are continuous.

We begin by recalling the notion of viscosity subsolutions and viscosity supersolutions, which have appeared extensively in the literature. In what follows, for a domain $\Omega \subset M^n$, we denote by $\mathrm{USC}(\Omega)$ the set of upper semicontinuous functions $u : \Omega \to \mathbb{R} \cup \{-\infty\}$, and $\mathrm{LSC}(\Omega)$ the set of lower semicontinuous functions $u : \Omega \to \mathbb{R} \cup \{+\infty\}$.

For the convenience of the reader, we recall the following expression for A_{g_u}:

$$A_{g_u} = \nabla^2_{g_0} u - \frac{1}{2}|du|^2_{g_0} g_0 + du \otimes du + A_{g_0}. \tag{2.1}$$

Definition 2.1. Suppose Γ satisfies (1.6) and (1.7), and let $\Omega \subset M^n$ be a domain equipped with a smooth Riemannian metric g_0. For $u \in \mathrm{USC}(\Omega)$, we say that $g_u = e^{-2u}g_0$ satisfies

$$\lambda(g_u^{-1}A_{g_u}) \in \overline{\Gamma} \text{ in the viscosity sense on } \Omega \tag{2.2}$$

if for any $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ satisfying $u(x_0) = \varphi(x_0)$ and $u(x) \leq \varphi(x)$ near x_0, there holds

$$\lambda(g_\varphi^{-1}A_{g_\varphi})(x_0) \in \overline{\Gamma}.$$

For $u \in \mathrm{LSC}(\Omega)$, we say that $g_u = e^{-2u}g_0$ satisfies

$$\lambda(g_u^{-1}A_{g_u}) \in \mathbb{R}^n\backslash\Gamma \text{ in the viscosity sense on } \Omega \tag{2.3}$$

if for any $x_0 \in \Omega$ and $\varphi \in C^2(\Omega)$ satisfying $u(x_0) = \varphi(x_0)$ and $u(x) \geq \varphi(x)$ near x_0, there holds

$$\lambda(g_\varphi^{-1}A_{g_\varphi})(x_0) \in \mathbb{R}^n\backslash\Gamma.$$

We to refer to an upper semicontinuous (resp. lower semicontinuous) function u satisfying (2.2) (resp. (2.3)) as a *viscosity subsolution* (resp. *viscosity supersolution*) of the equation $\lambda(g_u^{-1}A_{g_u}) \in \partial\Gamma$ on Ω. We call $u \in C^0(\Omega)$ a *viscosity solution* of the equation $\lambda(g_u^{-1}A_{g_u}) \in \partial\Gamma$ if it is both a viscosity subsolution and a viscosity supersolution.

Remark 2.2. If $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$, then (2.2) (resp. (2.3)) is equivalent to $\lambda(g_u^{-1}A_{g_u}) \in \overline{\Gamma}$ (resp. $\lambda(g_u^{-1}A_{g_u}) \in \mathbb{R}^n\backslash\Gamma$) a.e. in Ω. Therefore, if $u \in C^{1,1}_{\mathrm{loc}}(\Omega)$, then u is a viscosity solution to $\lambda(g_u^{-1}A_{g_u}) \in \partial\Gamma$ if and only if $\lambda(g_u^{-1}A_{g_u}) \in \partial\Gamma$ a.e. in Ω. We refer to e.g. [38, Lemma 2.5] for a proof of these facts.

Our main result in this section is the following strong comparison principle:

Theorem 2.3. *Let $\Omega \subset M^n$ be a domain equipped with a smooth Riemannian metric g_0 and let Γ satisfy (1.6) and (1.7). Suppose that for some $\tau \in (0,1)$, $u_1 \in \mathrm{USC}(\Omega)$ satisfies*

$$\lambda(g_{u_1}^{-1}A_{g_{u_1}}) \in \overline{\Gamma^\tau} \quad \text{in the viscosity sense on } \Omega \tag{2.4}$$

and $u_2 \in C^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies

$$\lambda(g_{u_2}^{-1}A_{g_{u_2}}) \in \mathbb{R}^n\backslash\Gamma^\tau \quad \text{a.e. in } \Omega. \tag{2.5}$$

If $u_1 \leq u_2$ in Ω, then either $u_1 \equiv u_2$ in Ω or $u_1 < u_2$ in Ω.

Remark 2.4. As noted in the introduction, Theorem 2.3 does not hold for $\tau = 1$ [41].

We show in Appendix A that if Γ satisfies (1.6) and (1.7), then (1.14) holds if and only if there exists some $\widetilde{\Gamma} \subset \Gamma$ satisfying (1.6) and (1.7) and a number $\tau < 1$ such that $\Gamma = (\widetilde{\Gamma})^\tau$. Thus, we have the following equivalent formulation of Theorem 2.3:

Theorem 2.3′. *Let $\Omega \subset M^n$ be a domain equipped with a smooth Riemannian metric g_0 and let Γ satisfy (1.6), (1.7) and (1.14). Suppose that $u_1 \in \mathrm{USC}(\Omega)$ satisfies*

$$\lambda(g_{u_1}^{-1} A_{g_{u_1}}) \in \overline{\Gamma} \quad \text{in the viscosity sense on } \Omega \tag{2.6}$$

and $u_2 \in C^{1,1}_{\mathrm{loc}}(\Omega)$ satisfies

$$\lambda(g_{u_2}^{-1} A_{g_{u_2}}) \in \mathbb{R}^n \backslash \Gamma \quad \text{a.e. in } \Omega. \tag{2.7}$$

If $u_1 \leq u_2$ in Ω, then either $u_1 \equiv u_2$ in Ω or $u_1 < u_2$ in Ω.

Our proof of Theorem 2.3 is inspired by that of Li, Nguyen & Wang [38, Theorem 2.3], which in turn uses ideas from [4] and [37]. In [38], the authors obtain a strong comparison principle for fully nonlinear, locally strictly elliptic equations satisfying a non-degeneracy condition (see equation (2.3) therein). In our setting, the non-degeneracy condition of [38] is not satisfied.

2.1. A preliminary strong comparison principle

We begin with a preliminary comparison principle, analogous to [38, Proposition 2.6]. We note that whilst a counterpart to [38, Proposition 2.7] is also possible in our setting, it will not be needed in this paper.

Proposition 2.5. *Let $\Omega \subset \mathbb{R}^n$ be an open Euclidean ball, g_0 a smooth Riemannian metric on $\overline{\Omega}$, and suppose Γ satisfies (1.6) and (1.7). Suppose also that $u_1 \in \mathrm{USC}(\overline{\Omega})$ satisfies*

$$\lambda(g_{u_1}^{-1} A_{g_{u_1}}) \in \overline{\Gamma} \quad \text{in the viscosity sense on } \Omega \tag{2.8}$$

and that there exists a constant $\delta > 0$ such that $u_2 \in C^{1,1}_{\mathrm{loc}}(\Omega) \cap \mathrm{LSC}(\overline{\Omega})$ satisfies

$$\lambda(g_{u_2}^{-1} A_{g_{u_2}} + \delta I) \in \mathbb{R}^n \backslash \Gamma \quad \text{a.e. in } \Omega. \tag{2.9}$$

If $u_1 \leq u_2$ in Ω and $u_1 < u_2$ on $\partial\Omega$, then $u_1 < u_2$ in Ω.

We establish some notation before proving Proposition 2.5. Let $(x, z, p, B) \in \Omega \times \mathbb{R} \times \mathbb{R}^n \times \mathrm{Sym}_n(\mathbb{R})$. In what follows, for shorthand we write

$$\lambda(x, z, p, B) \in \overline{\Gamma} \quad \text{if} \quad \lambda\bigg[g_0^{-1}(x)\bigg(B - \frac{1}{2}|p|^2_{g_0(x)} g_0(x) + p \otimes p + A_{g_0(x)}(x)\bigg)\bigg] \in \overline{\Gamma}. \tag{2.10}$$

Note that when $z = u(x)$, $p = du(x)$ and $B = \nabla^2_{g_0(x)}u(x)$, the quantity inside the square parentheses in (2.10) is the $(1,1)$-Schouten tensor of the metric $e^{-2u}g_0$ at x, and $(z,p,B) = (u(x), du(x), \nabla^2_{g_0(x)}u(x))$ is the second order jet of u at x, henceforth denoted by $J^2_{g_0}[u](x)$.

The proof of Proposition 2.5 is an adaptation of the proof of [38, Proposition 2.6]:

Proof of Proposition 2.5. We suppose for a contradiction that $u_1 \leq u_2$ in Ω and $u_1 < u_2$ on $\partial\Omega$, but there is a point $\hat{x} \in \Omega$ such that $u_1(\hat{x}) = u_2(\hat{x})$. Let $\Omega' \Subset \Omega$ be a ball concentric to Ω such that $\hat{x} \in \Omega'$ and $u_1 < u_2$ on $\partial\Omega'$. For $\varepsilon > 0$ small and $x \in \Omega'$, we denote by $\hat{u}_\varepsilon(x)$ the sup-convolution of u_1 with respect to g_0:

$$\hat{u}_\varepsilon(x) := \sup_{y\in\Omega}\big(u_1(y) - \varepsilon^{-1}d_{g_0}(x,y)^2\big) \geq u_1(x). \tag{2.11}$$

It is well-known that for ε sufficiently small, the supremum in (2.11) is attained in Ω, namely for each $x \in \Omega'$ there exists a point $x^* = x^*(\varepsilon, x) \in \Omega$ such that

$$\hat{u}_\varepsilon(x) = u_1(x^*) - \varepsilon^{-1}d_{g_0}(x,x^*)^2. \tag{2.12}$$

Moreover, $\hat{u}_\varepsilon$ is punctually second order differentiable (see [5] for the definition) a.e. in Ω' with $\nabla^2_{g_0}\hat{u}_\varepsilon \geq -C(\Omega', g_0)\varepsilon^{-1}g_0$ a.e. in Ω', and $\hat{u}_\varepsilon$ converges monotonically to u_1 as $\varepsilon \to 0$. For a proof of these facts, see [5,37]. We will use the following lemma, whose proof is postponed until later:

Lemma 2.6. *With the notation above, there exists a sequence $(\varepsilon,\eta) \to (0,0)$, a corresponding sequence of points $\{y_{\varepsilon,\eta}\} \subset \Omega'$, and a constant C_1 independent of ε and η such that*

(1) $\hat{u}_\varepsilon$ and u_2 are punctually second order differentiable at $y_{\varepsilon,\eta}$,
(2) for $y^*_{\varepsilon,\eta} = x^*(\varepsilon, y_{\varepsilon,\eta})$,

$$\lambda\big(y^*_{\varepsilon,\eta}, \hat{u}_\varepsilon(y_{\varepsilon,\eta}) + \varepsilon^{-1}d_{g_0}(y_{\varepsilon,\eta}, y^*_{\varepsilon,\eta})^2, d\hat{u}_\varepsilon(y_{\varepsilon,\eta}), \nabla^2_{g_0}u_2(y_{\varepsilon,\eta}) + C_1\eta g_0\big) \in \overline{\Gamma},$$

(3) $\hat{u}_\varepsilon(y_{\varepsilon,\eta}) - u_2(y_{\varepsilon,\eta}) \to 0$, and
(4) $|d\hat{u}_\varepsilon(y_{\varepsilon,\eta}) - du_2(y_{\varepsilon,\eta})| \leq C_1\eta$.

Assuming the validity of Lemma 2.6 for now, we explain how to obtain the desired contradiction. First observe that since $u_2 \in C^{1,1}_{\mathrm{loc}}(\Omega)$, we have $|J^2_{g_0}[u_2](y_{\varepsilon,\eta})| \leq C\|u_2\|_{C^{1,1}(\overline{\Omega'},g_0)}$, where here and below C is a constant independent of ε and η, which may change from line to line. Therefore, after restricting to a subsequence of (ε,η) if necessary, there exists (y_0, J_0) for which

$$(y_{\varepsilon,\eta}, J^2_{g_0}[u_2](y_{\varepsilon,\eta})) \to (y_0, J_0). \tag{2.13}$$

To aid the exposition, let us assume for the moment that $u_1 \in C^0(\overline{\Omega})$. If $m : [0,\infty) \to [0,\infty)$ is a modulus of continuity for u_1 in Ω', i.e. $m(0) = 0$ and $|u_1(p) - u_1(q)| \leq m(d_{g_0}(p,q))$ for all $p, q \in \overline{\Omega'}$, then

$$\varepsilon^{-1} d_{g_0}(x,x^*)^2 \leq m\bigg(\Big(C(\Omega', g_0)\varepsilon \sup_{\overline{\Omega'}} |u_1|\Big)^{1/2}\bigg) \tag{2.14}$$

for all $x \in \Omega'$ (see [5,37]), from which it follows that

$$\lim_{\varepsilon \to 0} \varepsilon^{-1} d_{g_0}(x,x^*)^2 = 0 \quad \text{for all } x \in \Omega'. \tag{2.15}$$

Then by the third and fourth properties in Lemma 2.6, (2.15) with $x = y_{\varepsilon,\eta}$ and (2.13),

$$\big(y^*_{\varepsilon,\eta}, \hat{u}_\varepsilon(y_{\varepsilon,\eta}) + \varepsilon^{-1} d_{g_0}(y_{\varepsilon,\eta}, y^*_{\varepsilon,\eta})^2, d\hat{u}_\varepsilon(y_{\varepsilon,\eta}), \nabla^2_{g_0} u_2(y_{\varepsilon,\eta}) + C_1 \eta g_0\big) \to (y_0, J_0). \tag{2.16}$$

Appealing to (2.16) and the second property in Lemma 2.6, we have that $\lambda(y_0, J_0) \in \overline{\Gamma}$. Returning to (2.13), we therefore see that distance from $\lambda(y_{\varepsilon,\eta}, J^2_{g_0}[u_2](y_{\varepsilon,\eta}))$ to $\overline{\Gamma}$ tends to zero along a sequence $(\varepsilon, \eta) \to (0,0)$, contradicting (2.9). This completes the proof of Proposition 2.5 in the case $u_1 \in C^0(\overline{\Omega})$ (up to proving Lemma 2.6).

Let us now treat the general case where $u_1 \in \mathrm{USC}(\overline{\Omega})$. We amend the argument above to account for the lack of a modulus of continuity for u_1. In particular, (2.15) is in general not true for $u_1 \in \mathrm{USC}(\overline{\Omega})$ (for an example of where this limit is not zero, see Section 2 of [37]). Note that in our argument above, the continuity of u_1 is used only once, namely in using (2.15) to pass from (2.13) to (2.16) – the rest of the argument remains valid when $u_1 \in \mathrm{USC}(\overline{\Omega})$. In the semicontinuous case, to pass from (2.13) to (2.16) it instead suffices to show the weaker estimate

$$\liminf_{\varepsilon \to 0} \varepsilon^{-1} d_{g_0}(y_{\varepsilon,\eta}, y^*_{\varepsilon,\eta})^2 \leq \eta. \tag{2.17}$$

The inequality in (2.17) was previously obtained in the proof of [38, Proposition 2.6] – we give the argument here for the convenience of the reader. We will use a weaker version of (2.15) for $u_1 \in \mathrm{USC}(\overline{\Omega})$, namely that

$$\varepsilon^{-1} d_{g_0}(x,x^*)^2 \leq C \quad \text{for all } x \in \Omega' \tag{2.18}$$

(see [5,37]). So we may suppose for some fixed $\eta > 0$ and some sequence $\varepsilon_m \to 0$ that

$$\varepsilon_m^{-1} d_{g_0}(y_m, y^*_m)^2 \to d, \tag{2.19}$$

where $y_m := y_{\varepsilon_m,\eta}$. To prove (2.17), we need to show that $d \leq \eta$.

Let $\tau_m = \tau(\varepsilon_m, \eta)$ be defined as in (2.26). By (2.28), after restricting to a further subsequence if necessary, we have $\tau_m \to \tau_0 \in (-\infty, \eta]$, and also $y_m \to y_0 \in \overline{\Omega'}$. The latter limit, combined with (2.19), also implies $y_m^* \to y_0$, and hence by upper semicontinuity of u_1,

$$\limsup_{m\to\infty} u_1(y_m^*) \le u_1(y_0). \tag{2.20}$$

Therefore,

$$\begin{aligned} d &\overset{(2.19)}{=} \limsup_{m\to\infty} \varepsilon_m^{-1} d_{g_0}(y_m, y_m^*)^2 \overset{(2.11)}{=} \limsup_{m\to\infty} \big(u_1(y_m^*) - \hat{u}_{\varepsilon_m}(y_m)\big) \\ &\overset{(2.31)}{\le} \limsup_{m\to\infty} \big(u_1(y_m^*) - u_2(y_m) + \tau_m\big) \overset{(2.20)}{\le} u_1(y_0) - u_2(y_0) + \eta \le \eta, \end{aligned}$$

where to reach the last inequality we have used the fact that $u_1 \le u_2$ on Ω. As explained in the paragraph preceding (2.17), this completes the proof of Proposition 2.5 (up to proving Lemma 2.6). □

We now give the proof of Lemma 2.6:

Proof of Lemma 2.6. We first show that if $x \in \Omega'$ is any point where $\hat{u}_\varepsilon$ is punctually second order differentiable, then

$$\lambda(x^*, \hat{u}_\varepsilon(x) + \varepsilon^{-1} d_{g_0}(x, x^*)^2, d\hat{u}_\varepsilon(x), \nabla^2_{g_0} \hat{u}_\varepsilon(x)) \in \overline{\Gamma}. \tag{2.21}$$

Indeed, by definition of punctual second order differentiability at x, we have

$$\hat{u}_\varepsilon(\exp_x(z)) \le \hat{u}_\varepsilon(x) + d\hat{u}_\varepsilon(x)(z) + \frac{1}{2}\nabla^2_{g_0}\hat{u}_\varepsilon(x)(z,z) + o(|z|^2_{g_0}) \quad \text{as } z \to 0, \tag{2.22}$$

where $\exp_x : T_x\Omega \to \Omega$ is the exponential map at x with respect to g_0. On the other hand, by definition of $\hat{u}_\varepsilon$,

$$\hat{u}_\varepsilon(\exp_x(z)) \ge u_1(\exp_{x^*}(Pz)) - \varepsilon^{-1} d_{g_0}\big(\exp_x(z), \exp_{x^*}(Pz)\big)^2, \tag{2.23}$$

where $P : T_x\Omega \to T_{x^*}\Omega$ is the parallel transport map along the unique length-minimising geodesic from x to x^*. As shown in the proof of [35, Proposition 2.4], the first and second variation formulae for length imply

$$d_{g_0}\big(\exp_x(z), \exp_{x^*}(Pz)\big)^2 = d_{g_0}(x, x^*)^2 + o(|z|^2_{g_0}) \quad \text{as } z \to 0. \tag{2.24}$$

After substituting (2.24) into (2.23), and then (2.23) back into (2.22), we obtain

$$u_1(\exp_{x^*}(Pz)) \le \hat{u}_\varepsilon(x) + \varepsilon^{-1} d_{g_0}(x, x^*)^2 + d\hat{u}_\varepsilon(x)(z) + \frac{1}{2}\nabla^2_{g_0}\hat{u}_\varepsilon(x)(z,z) + o(|z|^2_{g_0}) \quad \text{as } z \to 0. \tag{2.25}$$

The inclusion (2.21) then follows from (2.25) and the fact that u_1 is a viscosity subsolution.

We now construct a sequence of points $y_{\varepsilon,\eta}$ with the properties stated in Lemma 2.6. For $\varepsilon, \eta > 0$, let $\tau = \tau(\varepsilon, \eta)$ be such that

$$\eta = \sup_{\Omega'}(\hat{u}_\varepsilon - u_2 + \tau). \tag{2.26}$$

Recalling that $\hat{x} \in \Omega'$ is such that $u_1(\hat{x}) = u_2(\hat{x})$, we have

$$\tau = u_1(\hat{x}) - u_2(\hat{x}) + \tau \leq \hat{u}_\varepsilon(\hat{x}) - u_2(\hat{x}) + \tau \leq \eta, \tag{2.27}$$

and since $u_1 \leq u_2$ in Ω,

$$\eta \geq \tau = \eta - \sup_{\Omega'}(\hat{u}_\varepsilon - u_2) \geq \eta - \sup_{\Omega'}(\hat{u}_\varepsilon - u_1). \tag{2.28}$$

Given $\varepsilon > 0$ sufficiently small so that $\hat{u}_\varepsilon - u_2 < 0$ on $\partial\Omega'$, since $\tau \leq \eta$ (by (2.27)), we may choose η sufficiently small so that

$$\xi := \hat{u}_\varepsilon - u_2 + \tau < 0 \quad \text{on } \partial\Omega'. \tag{2.29}$$

Now let $\xi^+ = \max(\xi, 0)$ and denote by Γ_{ξ^+} the concave envelope of ξ^+ in Ω' (with respect to the Euclidean structure). Note that ξ is semi-convex in Ω' and, by (2.29), $\xi \leq 0$ on $\partial\Omega'$, and hence by the Alexandrov-Bakelman-Pucci estimate in [5, Lemma 3.5], the concave envelope Γ_{ξ^+} is in $C^{1,1}(\Omega')$ and

$$\int\limits_{\{\xi = \Gamma_{\xi^+}\}} \det(-\partial^2\Gamma_{\xi^+}) \geq \frac{1}{C(\Omega')}(\sup_{\Omega'} \xi)^n > 0, \tag{2.30}$$

where $\partial^2\Gamma_{\xi^+}$ is the Hessian matrix of second-order partial derivatives of Γ_{ξ^+} with respect to the Euclidean coordinate system. Note that positivity in (2.30) follows from the fact that $0 < \eta = \sup_{\Omega'} \xi$, and it follows from (2.30) that the set $\{\xi = \Gamma_{\xi^+}\}$ has positive measure.

Now, since $\hat{u}_\varepsilon$ and u_2 are punctually second order differentiable a.e., for each sufficiently small $\varepsilon > 0$ and $\eta > 0$ as above, there exists some $y = y_{\varepsilon,\eta} \in \{\xi = \Gamma_{\xi^+}\}$ such that $\hat{u}_\varepsilon$ and u_2 are punctually second order differentiable at y. We will now see that

$$0 < \xi(y) = \hat{u}_\varepsilon(y) - u_2(y) + \tau \leq \eta, \tag{2.31}$$

$$|d\xi(y)| = |d\hat{u}_\varepsilon(y) - du_2(y)| \leq C\eta, \text{ and} \tag{2.32}$$

$$0 \geq \partial^2\xi(y) = \partial^2\hat{u}_\varepsilon(y) - \partial^2 u_2(y). \tag{2.33}$$

Indeed, to see (2.31), it is enough to observe that if $\xi(y) = 0$, then the concavity and nonnegativity of $\Gamma_{\xi+}$ would imply that $\Gamma_{\xi+} \equiv 0$, contradicting (2.30). (2.32) follows from (2.31) and concavity, and (2.33) holds since ξ is concave on the set $\{\xi = \Gamma_{\xi+}\}$.

Now, the third property in the lemma follows from (2.31) and the bound for τ in (2.28), the fourth property is precisely (2.32), and the first property is satisfied by construction. It remains to prove the second property. Let Γ_{ij}^k denote the Christoffel symbols of g_0 with respect to the Euclidean coordinate system. By (2.33),

$$\begin{aligned}[\nabla_{g_0}^2 u_2(y)]_{ij} &= [\partial^2 u_2(y)]_{ij} - \Gamma_{ij}^k(y)\partial_k u_2(y) \\ &\geq [\partial^2 \hat{u}_\varepsilon(y)]_{ij} - \Gamma_{ij}^k(y)\partial_k u_2(y) = [\nabla_{g_0}^2 \hat{u}_\varepsilon(y)]_{ij} + \Gamma_{ij}^k(y)\big(\partial_k \hat{u}_\varepsilon(y) - \partial_k u_2(y)\big)\end{aligned}$$

in the sense of matrices, and by using (2.32) to estimate $\partial_k \hat{u}_\varepsilon(y) - \partial_k u_2(y)$, we obtain

$$\nabla_{g_0}^2 u_2(y) \geq \nabla_{g_0}^2 \hat{u}_\varepsilon(y) - C_1 \eta g_0 \tag{2.34}$$

for some constant C_1 independent of ε, η.

Let $y^* = x^*(\varepsilon, y)$. By ellipticity, (2.21) and (2.34), we therefore have

$$\lambda\big(y^*, \hat{u}_\varepsilon(y) + \varepsilon^{-1} d_{g_0}(y, y^*)^2, d\hat{u}_\varepsilon(y), \nabla_{g_0}^2 u_2(y) + C_1 \eta g_0\big) \in \overline{\Gamma},$$

which is precisely the second property in the lemma. □

2.2. Proof of Theorem 2.3

With Proposition 2.5 established, we are now in a position to prove Theorem 2.3:

Proof of Theorem 2.3. We follow [38], which in turn uses ideas from [4,37]. We argue by contradiction. Assuming the conclusion is false, we can find a closed ball $\bar{B} \subset \Omega$ of some radius $R > 0$ for which there exists $\hat{x} \in \partial B$ with

$$u_1 < u_2 \text{ in } \bar{B}\backslash\{\hat{x}\} \quad \text{and} \quad u_1(\hat{x}) = u_2(\hat{x}).$$

Taking B smaller if necessary, we may assume it is contained inside a normal coordinate chart about its centre, and for the remainder of the proof we implicitly identify B with its image under this chart, assumed to be centred at the origin in $\mathbb{R}^n$.

We will deform u_2 into a strict supersolution $\tilde{u}$ (in the sense of (2.9)) in some open Euclidean ball A (contained in the image of the aforementioned chart) centred at $\hat{x}$ such that $u_1 < \tilde{u}$ on ∂A and $\inf_A(\tilde{u} - u_1) = 0$. These imply that $\tilde{u} - u_1$ attains its infimum (equal to zero) in A, which contradicts the conclusion of Lemma 2.5 that $u_1 < \tilde{u}$ in A.

We construct $\tilde{u}$ following ideas from [4], with a simplification due to the fact that the dependence of the Schouten tensor A_{g_u} on u is only through its first and second derivatives. Let $\alpha \gg 1$ be a constant, and define

$$E(x) = e^{-\alpha|x|^2} \quad \text{and} \quad h(x) = e^{-\alpha|x|^2} - e^{-\alpha R^2},$$

where $|x|^2 = x_1^2 + \cdots + x_n^2$. Also let $\mu > 0$ and $\nu \geq 0$ be constants, and define the following perturbation of u_2:

$$\tilde{u}(x) = \tilde{u}_{\mu,\nu}(x) := u_2(x) - \mu\big(h(x) - \nu\big). \tag{2.35}$$

We start by taking a ball A centred at $\hat{x}$ of sufficiently small radius R_A such that A is contained in the image of the aforementioned chart and $|x|$ is uniformly bounded away from 0 on A. For reasons that will become clear later, we also assume in what follows that the constants R_A, α and μ are chosen such that

$$R_A < \alpha^{-1} \quad \text{and} \quad \mu\alpha E(x) < 1 \text{ for } x \in A. \tag{2.36}$$

Note that the first of these conditions implies that $|h(x)| = O(E(x))$; here the implicit constant is universal, although throughout the proof we allow our implicit constants to depend on $\|u\|_{C^1(\overline{A})}$.

Recall that $\lambda(g_u^{-1}A_{g_u}) \in \Gamma^\tau$ if and only if $\lambda^\tau(g_u^{-1}A_{g_u}) \in \Gamma$, and by (2.1) we have

$$\lambda^\tau(g_{\tilde{u}}^{-1}A_{g_{\tilde{u}}}) = \lambda\bigg(g_{\tilde{u}}^{-1}\bigg(\tau\nabla_{g_0}^2\tilde{u} + (1-\tau)\Delta_{g_0}\tilde{u}\, g_0 - c_{n,\tau}|d\tilde{u}|_{g_0}^2\, g_0 + \tau d\tilde{u}\otimes d\tilde{u} + A_{g_0,\tau}\bigg)\bigg)$$

where $c_{n,\tau} = \frac{1}{2}(n-2-(n-3)\tau)$ and

$$A_{g_0,\tau} := \tau A_{g_0} + (1-\tau)\sigma_1(g_0^{-1}A_{g_0})g_0. \tag{2.37}$$

Likewise, we denote

$$\begin{aligned} A_{g_{\tilde{u}},\tau} &:= \tau A_{g_{\tilde{u}}} + (1-\tau)\sigma_1(g_{\tilde{u}}^{-1}A_{g_{\tilde{u}}})g_{\tilde{u}} \\ &= \tau\nabla_{g_0}^2\tilde{u} + (1-\tau)\Delta_{g_0}\tilde{u}\, g_0 - c_{n,\tau}|d\tilde{u}|_{g_0}^2\, g_0 + \tau d\tilde{u}\otimes d\tilde{u} + A_{g_0,\tau}. \end{aligned} \tag{2.38}$$

(Note that we write $\tau \in [0,1]$ as a subscript in (2.37) and (2.38), to avoid confusion with quantities in (1.1) and (1.3) that are defined for $t \in (-\infty, 1]$.)

For the remainder of the proof, computations are carried out at points $x \in A$ where u_2 is punctually second order differentiable; since $u_2 \in C^{1,1}_{\mathrm{loc}}(\Omega)$, the set of such points has full measure in A. We will use the following lemma, whose proof is postponed until later:

Lemma 2.7. *For $\tilde{u}$ defined in (2.35) and at points where u_2 is punctually second order differentiable,*

$$\begin{aligned} A_{g_{\tilde{u}},\tau}(x) &= A_{g_{u_2},\tau}(x) - 4\tau\mu\alpha^2E(x)x\otimes x - 4(1-\tau)\mu\alpha^2E(x)|x|^2g_0(x) \\ &\quad + O\big(\mu\alpha E(x)\big), \end{aligned} \tag{2.39}$$

where the implicit constant in the big-O term in (2.39) depends only on $\|u\|_{C^1(\overline{A})}$.

Assuming the validity of Lemma 2.7 for now, we explain how to complete the proof of Theorem 2.3. First observe that if α is taken sufficiently large, then the $O\big(\mu\alpha E(x)\big)$ terms in (2.39) are absorbed by the strictly negative term $-4(1-\tau)\mu\alpha^2 E(x)|x|^2 g_0(x)$. For R_A, α and μ satisfying the above constraints, after raising an index using $g_{\tilde{u}}^{-1}$, we therefore obtain for some positive function $p(x) = p_{R_A,\alpha,\mu}(x) \geq C^{-1} > 0$ the following inequality of $(1,1)$-tensors:

$$g_{\tilde{u}}^{-1} A_{g_{\tilde{u}},\tau}(x) \leq g_{\tilde{u}}^{-1} A_{g_{u_2},\tau}(x) - p(x) g_{\tilde{u}}^{-1} g_0(x) = g_{\tilde{u}}^{-1} A_{g_{u_2},\tau}(x) - p(x) e^{2\tilde{u}} I.$$

Therefore

$$g_{\tilde{u}}^{-1} A_{g_{\tilde{u}},\tau}(x) + p(x) e^{2\tilde{u}} I \leq g_{\tilde{u}}^{-1} A_{g_{u_2},\tau}(x),$$

and it follows from boundedness of p away from zero that for some constant $\delta > 0$,

$$g_{\tilde{u}}^{-1} A_{g_{\tilde{u}},\tau}(x) + \delta I < g_{\tilde{u}}^{-1} A_{g_{u_2},\tau}(x). \tag{2.40}$$

But by (2.5) and ellipticity, the inequality (2.40) implies

$$\lambda\big(g_{\tilde{u}}^{-1} A_{g_{\tilde{u}},\tau} + \delta I\big) \in \mathbb{R}^n \backslash \Gamma. \tag{2.41}$$

To obtain a contradiction to Proposition 2.5, we only need to choose ν so that

$$u_1 < \tilde{u}_{\mu,\nu} \text{ on } \partial A \quad \text{and} \quad \inf_A (\tilde{u}_{\mu,\nu} - u_1) = 0. \tag{2.42}$$

Denote $\nu_0 = \sup_A h \geq \sup_{A\cap B} h > 0$. Then for sufficiently small $\mu > 0$, it is easy to see that $u_1 < \tilde{u}_{\mu,\nu}$ on ∂A for all $0 \leq \nu \leq \nu_0$. Moreover $\inf_A(\tilde{u}_{\mu,0} - u_1) \leq 0 \leq \inf_A(\tilde{u}_{\mu,\nu_0} - u_1)$, so a value of $\nu \in [0, \nu_0]$ can be chosen so that $\inf_A(\tilde{u}_{\mu,\nu} - u_1) = 0$. This completes the proof of Theorem 2.3 (up to proving Lemma 2.7). □

We now give the proof of Lemma 2.7:

Proof of Lemma 2.7. Calculating $\partial_i \tilde{u}$ and $\partial_i\partial_j \tilde{u}$ explicitly, we see

$$\partial_i \tilde{u}(x) - \partial_i u_2(x) = 2\mu\alpha E(x) x_i \tag{2.43}$$

and

$$\partial_i\partial_j \tilde{u}(x) - \partial_i\partial_j u_2(x) = 2\mu\alpha E(x)\big(-2\alpha x_i x_j + \delta_{ij}\big). \tag{2.44}$$

It follows that

$$
\begin{aligned}
(\nabla^2_{g_0})_{ij}\tilde{u}(x) - (\nabla^2_{g_0})_{ij}u_2(x) &= \partial_i\partial_j\tilde{u}(x) - \partial_i\partial_j u_2(x) - \Gamma^k_{ij}(x)\big(\partial_k\tilde{u}(x) - \partial_k u_2(x)\big) \\
&\overset{(2.43)}{=} \partial_i\partial_j\tilde{u}(x) - \partial_i\partial_j u_2(x) - \Gamma^k_{ij}(x)\big(2\mu\alpha E(x)x_k\big) \\
&= \partial_i\partial_j\tilde{u}(x) - \partial_i\partial_j u_2(x) + O\big(\mu\alpha E(x)\big),
\end{aligned}
$$

where Γ^k_{ij} are the Christoffel symbols of the metric g_0, and we therefore obtain the identity

$$
\begin{aligned}
\tau\nabla^2_{g_0}\tilde{u}(x) + (1-\tau)\Delta_{g_0}\tilde{u}\,g_0(x) = \tau\nabla^2_{g_0}u_2(x) &+ (1-\tau)\Delta_{g_0}u_2(x)\,g_0(x) \\
&- 4\tau\mu\alpha^2E(x)x\otimes x - 4(1-\tau)\mu\alpha^2E(x)|x|^2g_0(x) \\
&+ O\big(\mu\alpha E(x)\big).
\end{aligned} \tag{2.45}
$$

Next we calculate $-c_{n,\tau}|d\tilde{u}|^2_{g_0}\,g_0 + \tau d\tilde{u}\otimes d\tilde{u}$. Expanding out $|d\tilde{u}|^2_{g_0}(x)$ using (2.43), and using (2.36) to assert $\mu^2\alpha^2E(x)^2 \leq \mu\alpha E(x)$, we see that

$$
|d\tilde{u}|^2_{g_0}(x) - |du_2|^2_{g_0}(x) = O\big(\mu\alpha E(x)\big) \tag{2.46}
$$

and likewise

$$
(d\tilde{u}\otimes d\tilde{u} - du_2\otimes du_2)(x) = O\big(\mu\alpha E(x)\big). \tag{2.47}
$$

Combining (2.45), (2.46) and (2.47) we obtain (2.39), as required. □

3. Proof of Theorems 1.1, 1.4 and related results

In this section we prove Theorems 1.1, 1.4 and some related results. As discussed in the introduction, a subtlety in the study of the nonlinear eigenvalue problem

$$
f^\tau(\lambda(g_0^{-1}A_{g_{u_\tau}})) = \mu_\tau, \quad \lambda(g_0^{-1}A_{g_{u_\tau}}) \in \Gamma^\tau \tag{3.1}
$$

is the non-existence of solutions for all but one value of μ_τ. In addition, when there exists a solution to (3.1), it is clear that the solution is not unique. As a means for establishing Theorems 1.1 and 1.4, we first consider a class of equations for which we prove both existence and uniqueness. More precisely, we consider

$$
f^\tau\big(\lambda(g_0^{-1}A_{g_{u_\tau}})\big) = h(x, u_\tau), \quad \lambda(g_0^{-1}A_{g_{u_\tau}}) \in \Gamma^\tau, \tag{3.2}
$$

where $h = h(x,z) : M^n \times \mathbb{R} \to (0,\infty)$ is any smooth positive function which is strictly proper and satisfies mild growth conditions as $z \to \pm\infty$:

(C1) $\frac{\partial h}{\partial z} > 0$ on $M^n \times \mathbb{R}$,
(C2) $\limsup\limits_{z\to-\infty}\sup\limits_{x\in M} h(x,z) = 0$ and $\limsup\limits_{z\to-\infty}\sup\limits_{x\in M}\big(|\nabla h(x,z)| + |\nabla^2 h(x,z)|\big) < \infty$,

(C3) $\liminf_{z\to+\infty} \inf_{x\in M} h(x,z) = +\infty$.

The properness condition (C1) has been widely used in the context of nonlinear elliptic equations, and is used in our argument to obtain uniqueness of solutions to (3.2). The growth conditions (C2) and (C3) are used in our argument to obtain existence of solutions to (3.2). An example of a function satisfying (C1)-(C3) is $h(x,z) = \tilde{h}(x)e^{\beta z}$ for $\beta > 0$, where $\tilde{h} > 0$ is any smooth positive function.

The following existence and uniqueness results for (3.2) serve as precursors to Theorems 1.1 and 1.4, respectively:

Theorem 3.1. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$, let $h = h(x,z)$ be a smooth positive function satisfying (C2) and (C3), and suppose (f,Γ) satisfies (1.6), (1.7), (1.10) and (1.11). Suppose there exists a metric $g_{\hat{u}} = e^{-2\hat{u}}g_0$, $\hat{u} \in C^0(M^n)$, satisfying (1.8). Then either:*

1. *There exists a smooth solution u_1 to (3.2) for $\tau = 1$, or*
2. *There exists a $C^{1,1}$ metric g conformal to g_0 with $\lambda(g^{-1}A_g) \in \partial\Gamma$ a.e. on M^n.*

The former statement is equivalent to the existence of a smooth metric $g \in [g_0]$ satisfying (1.9) for $\tau = 1$. Moreover, in the former case, for all $0 < \delta < 1$ and $\alpha \in (0,1)$, there exists a constant $C > 0$ depending only on n, g_0, δ, α and h such that solutions to (3.2) with $\tau \in [\delta, 1]$ satisfy

$$\|u_\tau\|_{C^{4,\alpha}(M^n,g_0)} \leq C. \tag{3.3}$$

Finally, if h also satisfies (C1), then solutions to (3.2) are unique for $\tau \in [0,1]$.

Theorem 3.2. *In addition to the hypotheses of Theorem 3.1, assume $\Gamma \neq \Gamma_1^+$. Then either:*

1. *There exists a smooth solution u_τ to (3.2) for every $\tau \in (0,1)$, or*
2. *$\hat{u}$ is a smooth solution to $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$ on M^n.*

Moreover, in the former case, for all $0 < \delta < T < 1$ and $\alpha \in (0,1)$, there exists a constant $C > 0$ depending only on $n, g_0, \delta, T, \alpha$ and h such that solutions to (3.2) with $\tau \in [\delta, T]$ satisfy

$$\|u_\tau\|_{C^{4,\alpha}(M^n,g_0)} \leq C. \tag{3.4}$$

Finally, if h also satisfies (C1), then solutions to (3.2) are unique for $\tau \in [0,1]$.

We begin in Section 3.1 by proving that, if $\Gamma \neq \Gamma_1^+$ and there exists a continuous metric conformal to g_0 satisfying (1.8), then either $Y(M^n,[g_0]) > 0$ or $g_{\hat{u}}$ is smooth with $\mathrm{Ric}_{g_{\hat{u}}} \equiv$

0 on M^n. In Section 3.2, we obtain the *a priori* estimates in Theorem 3.2. As alluded to in the introduction, this is the point at which we make use of the strong comparison principle obtained in Section 2. In Section 3.3 we complete the proof of Theorem 3.2 using a degree argument, after which Theorem 3.1 follows from a short additional argument. In Section 3.4, we prove Theorem 1.1 (resp. Theorem 1.4) by appealing to Theorem 3.1 (resp. Theorem 3.2) in the special case $h(x,z) = e^{\beta z}$ ($\beta > 0$) and applying a limiting argument as $\beta \to 0^+$. As a by-product of our methods for proving Theorems 3.1 and 3.2, we present in Section 3.5 a result of Kazdan-Warner type (which will be used later in Section 5).

3.1. A Yamabe-positive/Ricci-flat dichotomy

An important step in the proof of Theorems 3.1 and 3.2 (and hence Theorems 1.1 and 1.4) is to show that (1.8) and the assumption $\Gamma \neq \Gamma_1^+$ imply that either $Y(M^n,[g_0]) > 0$ or $g_{\hat{u}}$ is smooth with $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$:

Proposition 3.3. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and suppose Γ satisfies (1.6) and (1.7) with $\Gamma \neq \Gamma_1^+$. Suppose there exists a metric $g_{\hat{u}} = e^{-2\hat{u}}g_0$, $\hat{u} \in C^0(M^n)$, satisfying (1.8). Then either $Y(M^n,[g_0]) > 0$ or $g_{\hat{u}}$ is smooth with $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$ on M^n.*

We note that if $\hat{u} \in C^2(M^n)$, then the conclusion of Proposition 3.3 is clear in view of the following lemma with $t = 0$:

Lemma 3.4. *Suppose that Γ satisfies (1.6), (1.7) and $\Gamma \neq \Gamma_1^+$. Then for all $t \in [0,1)$, $\partial\Gamma^t \cap \overline{\Gamma} = \{(0,\ldots,0)\}$.*

Proof. Suppose for a contradiction that there exists some non-zero $\lambda \in \partial\Gamma^t \cap \overline{\Gamma}$, and let f be a defining function for Γ satisfying (1.10) and (1.11). Then $f(t\lambda + (1-t)\sigma_1(\lambda)e) = 0$ and $f(\lambda) \geq 0$. By concavity and homogeneity of f, it then follows that

$$0 = f(t\lambda + (1-t)\sigma_1(\lambda)e) \geq tf(\lambda) + (1-t)\sigma_1(\lambda)f(e) \geq (1-t)\sigma_1(\lambda)f(e). \tag{3.5}$$

Since $(1-t)f(e) > 0$, this implies $\sigma_1(\lambda) = 0$, i.e. $\lambda \in \partial\Gamma_1^+ \cap \overline{\Gamma}$.

Let $P \subset \mathbb{R}^n$ be the polygon with vertices consisting of all permutations of λ, that is, the convex hull of all permutations of λ. By symmetry, all vertices of P belong to $\partial\Gamma_1^+ \cap \overline{\Gamma}$, and by convexity of $\partial\Gamma_1^+ \cap \overline{\Gamma}$, it follows that $P \subseteq \partial\Gamma_1^+ \cap \overline{\Gamma}$. Moreover, the barycentre of P is at the origin, and in particular the origin belongs to the interior of P. Thus there is a neighbourhood of the origin relative to $\partial\Gamma_1^+$ which is contained in $\partial\Gamma_1^+ \cap \overline{\Gamma}$, and by homothety it follows that $\partial\Gamma_1^+ \cap \overline{\Gamma} = \partial\Gamma_1^+$. This implies that $\Gamma = \Gamma_1^+$, a contradiction. □

To handle the case that $\hat{u}$ is merely continuous, we first prove the following lemma:

Lemma 3.5. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and suppose Γ satisfies (1.6) and (1.7). If there exists a metric $g_{\hat{u}} = e^{-2\hat{u}}g_0$, $\hat{u} \in C^0(M^n)$, satisfying (1.8), then $Y(M^n, [g_0]) \geq 0$.*

Proof. We use some ideas from [35]. We suppose for a contradiction that the Yamabe invariant is negative, so we may assume $R_{g_0} < 0$ on M^n. We start by writing $g_{\hat{u}}$ in the form $g_{\hat{u}} = w^{\frac{4}{n-2}}g_0$, so that $w := e^{-(n-2)\hat{u}/2} \in C^0(M^n)$ is positive and

$$-\Delta_{g_0} w + c_n R_{g_0} w \geq 0 \quad \text{in the viscosity sense on } M^n, \tag{3.6}$$

where $c_n = \frac{n-2}{4(n-1)}$. We claim that (3.6) implies $R_{g_{\hat{u}}} \geq 0$ in the distributional sense, i.e.

$$\int_{M^n} -w\Delta_{g_0}\varphi \, dv_{g_0} \geq -\int_{M^n} c_n R_{g_0} w\varphi \, dv_{g_0} \quad \text{for all } 0 \leq \varphi \in C^\infty(M^n). \tag{3.7}$$

Taking $\varphi = 1$ in (3.7) and appealing to negativity of R_{g_0}, this yields the desired contradiction.

By a standard partition of unity argument, to obtain (3.7) it suffices to show that for any open set $\Omega \subset M^n$ contained in a single chart,

$$\int_{\Omega} -w\Delta_{g_0}\varphi \, dv_{g_0} \geq -\int_{\Omega} c_n R_{g_0} w\varphi \, dv_{g_0} \quad \text{for all } 0 \leq \varphi \in C_c^\infty(\Omega). \tag{3.8}$$

Let $\{w_j\} \subset C^\infty(\overline{\Omega})$ be a sequence of smooth functions converging uniformly to w in $\overline{\Omega}$ and such that $w_j < w$ in $\overline{\Omega}$. For each j, consider the functional $T_j : H^1(\Omega) \to \mathbb{R}$ defined by

$$T_j[\rho] = \int_{\Omega} \left(|\nabla_{g_0}\rho|_{g_0}^2 + c_n R_{g_0} w_j \rho\right) dv_{g_0}.$$

By the direct method, there exists a unique minimiser of T_j in $H^1(\Omega)$ subject to the constraints

$$\rho|_{\partial\Omega} = w_j|_{\partial\Omega} \quad \text{and} \quad \rho \geq w_j \text{ in } \Omega. \tag{3.9}$$

Moreover, this minimiser – henceforth denoted by ρ_j – satisfies in the weak sense

$$-\Delta_{g_0}\rho_j \geq -c_n R_{g_0} w_j \quad \text{in } \Omega \tag{3.10}$$

and

$$\begin{cases} -\Delta_{g_0}\rho_j = -c_n R_{g_0} w_j & \text{in } \{\rho_j > w_j\} \\ \rho_j = w_j < w & \text{on } \{\rho_j = w_j\}. \end{cases} \tag{3.11}$$

In particular, by elliptic regularity, ρ_j is smooth in $\{\rho_j > w_j\}$.

On the other hand, by (3.6), our negativity assumption on R_{g_0} and the fact that $w_j < w$, it follows that $-\Delta_{g_0} w + c_n R_{g_0} w_j \geq 0$ in the viscosity sense. Combining this with (3.11) yields

$$\begin{cases} \Delta_{g_0}(w - \rho_j) \leq 0 & \text{in the viscosity sense on } \{\rho_j > w_j\} \\ \rho_j < w & \text{on } \{\rho_j = w_j\}, \end{cases}$$

and by the comparison principle for viscosity solutions (see e.g. [5, Corollary 3.7]), it follows that $\rho_j \leq w$ on $\{\rho_j > w_j\}$. We also have $\rho_j = w_j < w$ on $\{\rho_j = w_j\}$, and since $\Omega = \{\rho_j \geq w_j\}$ by the constraint in (3.9), it follows that $\rho_j \leq w$ on Ω.

In summary, we have shown $w_j \leq \rho_j \leq w$ in Ω. By uniform convergence of w_j to w, it follows that ρ_j converges in $L^\infty(\Omega)$ to w. Testing (3.10) against a nonnegative test function $\varphi \in C_c^\infty(\Omega)$, integrating by parts and taking $j \to \infty$ then yields (3.8). $\square$

Proof of Proposition 3.3. By Lemma 3.5 we know $Y(M^n, [g_0]) \geq 0$. If $Y(M^n, [g_0]) > 0$ we are done, so suppose that $Y(M^n, [g_0]) = 0$. We may then assume that $R_{g_0} = 0$. Let $w \in C^0(M^n)$ be such that $g_{\hat{u}} = w^{\frac{4}{n-2}} g_0$, so that w is positive and by (3.6)

$$-\Delta_{g_0} w \geq 0 \quad \text{in the viscosity sense on } M^n. \tag{3.12}$$

By the strong maximum principle for viscosity supersolutions (see [5, Proposition 4.9]), w is locally constant on M^n and hence constant on M^n. It follows that $\hat{u}$ is smooth with $R_{g_{\hat{u}}} \equiv 0$, i.e. $\lambda(g_{\hat{u}}^{-1} A_{g_{\hat{u}}}) \in \partial\Gamma_1^+$. But by Lemma 3.4 with $t = 0$, $\lambda(g_{\hat{u}}^{-1} A_{g_{\hat{u}}}) \in \partial\Gamma_1^+ \cap \overline{\Gamma}$ implies $\lambda(g_{\hat{u}}^{-1} A_{g_{\hat{u}}}) = (0, \ldots, 0)$, and it follows that $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$. $\square$

3.2. Proof of the a priori estimates in Theorem 3.2

We now prove the *a priori* estimate claimed in Theorem 3.2 for solutions to (3.2), which we restate here for convenience:

Proposition 3.6. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ with $Y(M^n, [g_0]) > 0$. Let $h = h(x, z)$ be a smooth positive function satisfying (C2) and (C3), and suppose (f, Γ) satisfies (1.6), (1.7), (1.10), (1.11) and $\Gamma \neq \Gamma_1^+$. Suppose that there exists a metric $g_{\hat{u}} = e^{-2\hat{u}} g_0$, $\hat{u} \in C^0(M^n)$, satisfying (1.8). Then for all $0 < \delta < T < 1$ and $\alpha \in (0, 1)$, there exists a constant $C > 0$ depending only on $n, g_0, \delta, T, \alpha$ and h such that solutions u_τ to (3.2) with $\tau \in [\delta, T]$ satisfy*

$$\|u_\tau\|_{C^{4,\alpha}(M^n, g_0)} \leq C.$$

Remark 3.7. By Proposition 3.3, the assumption $Y(M^n, [g_0]) > 0$ in Proposition 3.6 is equivalent to $g_{\hat{u}}$ not being a smooth Ricci-flat metric.

Remark 3.8. If we replace the RHS of (3.2) with a τ-dependent function $h_\tau(x,z)$, then the conclusion of Proposition 3.6 still holds if (C2) is replaced by the conditions

$$\limsup_{z\to-\infty}\sup_{x\in M,\,\tau\in[0,1]} h_\tau(x,z)=0 \quad\text{and}\quad \limsup_{z\to-\infty}\sup_{x\in M,\,\tau\in[0,1]}(|\nabla h_\tau(x,z)|+|\nabla^2 h(x,z)|)<\infty,$$

and if (C3) is replaced by the condition $\liminf_{z\to+\infty}\inf_{x\in M,\,\tau\in[0,1]} h(x,z)=+\infty$.

We point out that the main task in Proposition 3.6 is the C^0 estimate. First and second derivative estimates depending on C^0 estimates were established in works such as [10,22,25,28,30,32,52,53], and once C^2 estimates are established, the equation (3.2) becomes uniformly elliptic, and higher order estimates follow from the theory of Evans and Krylov [14,29] (here we use the concavity assumption in (1.10)) and Schauder estimates.

Proof of Proposition 3.6. In light of the existing higher order estimates on solutions to (3.2) discussed above, it suffices to prove C^0 bounds on solutions to (3.2). We split the proof into two steps: in the first step, we obtain the uniform upper bound in the range $\tau\in[\delta,1]$, and subsequently we apply the estimates of Chen [10] to obtain uniform first and second derivative estimates in the range $\tau\in[\delta,1]$. In the second step we obtain the uniform lower bound in the range $\tau\in[\delta,T]$ for $T<1$.

Step 1: We begin by proving the uniform upper bound on solutions to (3.2), which we will see holds uniformly in $\tau\in[\delta,1]$.

We first observe that by concavity, symmetry and homogeneity of f, for all $\lambda\in\Gamma^\tau$

$$f^\tau(\lambda)=f(\lambda^\tau)\le f\left(\frac{\sigma_1(\lambda^\tau)}{n}e\right)+\nabla f\left(\frac{\sigma_1(\lambda^\tau)}{n}e\right)\cdot\left(\lambda^\tau-\frac{\sigma_1(\lambda^\tau)}{n}e\right)=\frac{f(e)}{n}\sigma_1(\lambda^\tau),$$

where we have used that $\nabla f(\frac{\sigma_1(\lambda^\tau)}{n}e)$ is parallel to e. Therefore, for any solution u_τ to (3.2) we have (using the notation in (2.37))

$$h(x,u_\tau(x))\le C\sigma_1(\lambda(g_0^{-1}A_{g_{u_\tau},\tau}))(x)\quad\text{for all } x\in M^n. \tag{3.13}$$

Now let $p\in M^n$ be a maximum point for u_τ. At p, the gradient terms in the expression for $A_{g_{u_\tau},\tau}$ (see (2.38)) vanish, and so (3.13) implies

$$\begin{aligned} h(p,u_\tau(p)) &\le C\sigma_1(\lambda(g_0^{-1}A_{g_0,\tau}))(p)+C(\tau+n(1-\tau))\Delta_{g_0}u_\tau(p)\\ &\le C\sigma_1(\lambda(g_0^{-1}A_{g_0,\tau}))(p), \end{aligned}\tag{3.14}$$

where to obtain the last inequality we have used the fact $\Delta_{g_0}u_\tau(p)\le 0$ (since p is a maximum point). The growth condition (C3) then implies an upper bound for $u_\tau(p)$.

With the uniform upper bound established, we may apply the first and second derivative estimates of Chen [10]. Indeed, taking $W=A_{g_{u_\tau}}$, $F(g_0^{-1}W)=f^\tau(g_0^{-1}W)$ and

$\Gamma = \Gamma^\tau$ in Case (a) of Theorem 1.1 therein, one obtains the first and second derivative estimate

$$|\nabla^2_{g_0} u_\tau|_{g_0} + |\nabla_{g_0} u_\tau|^2_{g_0} \leq C \quad \text{on } M^n \tag{3.15}$$

for all solutions u_τ to (3.2), $\tau \in [\delta, 1]$, where C depends on n, g_0, δ and an upper bound for

$$\sup_{x\in M,\, \tau\in[\delta,1]} \Big(h\big(x, u_\tau(x)\big) + \big|\nabla h\big(x, u_\tau(x)\big)\big| + \big|\nabla^2 h\big(x, u_\tau(x)\big)\big| \Big),$$

which by (C2) depends only on h and the uniform upper bound for u_τ obtained above.

Step 2: We now prove the uniform lower bound on solutions to (3.2) for $\tau \in [\delta, T]$, $T < 1$.

We assume for a contradiction that the uniform lower bound on solutions to (3.2) fails, so that for some sequence $\tau_i \to \tau \in [\delta, T]$ we have a corresponding sequence of solutions $\{u_i\}$ such that $\min_{M^n} u_i \to -\infty$. By the uniform first derivative estimate from Step 1, it follows that $\max_{M^n} u_i \to -\infty$ as well. We denote $g_i = e^{-2u_i} g_0$ and define the rescaled sequence

$$\widetilde{u}_i = u_i - \bar{u}_i, \quad \bar{u}_i := \frac{1}{\mathrm{Vol}(M^n, g_0)} \int\limits_{M^n} u_i \, dv_{g_0}.$$

Noting that $\widetilde{u}_i$ has zero average, the first and second derivative estimates in Step 1 imply that $\{\widetilde{u}_i\}$ is bounded in $C^2(M^n)$. Therefore, after restricting to a subsequence, for some $\widetilde{u} \in C^{1,1}(M^n)$ we have $\widetilde{u}_i \to \widetilde{u}$, where the convergence is in $C^{1,\alpha}(M^n)$ for all $\alpha < 1$. Denoting $\widetilde{g}_i = e^{-2\widetilde{u}_i} g_0 = e^{2\bar{u}_i} g_i$, we observe that by homogeneity of f, the functions $\widetilde{u}_i$ satisfy

$$f^{\tau_i}\big(\lambda(\widetilde{g}_i^{-1} A_{\widetilde{g}_i})\big) = h(x, u_i) e^{2(u_i - \bar{u}_i)} \quad \text{on } M^n. \tag{3.16}$$

Next observe that by our growth condition (C2), the RHS of (3.16) tends to zero uniformly as $i \to \infty$. It follows that the metric $\widetilde{g} = e^{-2\widetilde{u}} g_0$ satisfies

$$\lambda(\widetilde{g}^{-1} A_{\widetilde{g}}) \in \partial\Gamma^\tau \quad \text{in the viscosity sense on } M^n \tag{3.17}$$

(see Proposition 3.9 below). Since $\widetilde{u} \in C^{1,1}(M^n)$, it then follows (see e.g. [38, Lemma 2.5]) from (3.17) that

$$\lambda(\widetilde{g}^{-1} A_{\widetilde{g}}) \in \partial\Gamma^\tau \quad \text{a.e. on } M^n. \tag{3.18}$$

Now, by the assumption (1.8), there exists $\hat{u} \in C^0(M^n)$, $g_{\hat{u}} = e^{-2\hat{u}} g_0$, satisfying

$$\lambda(g_{\hat{u}}^{-1} A_{g_{\hat{u}}}) \in \overline{\Gamma} \quad \text{in the viscosity sense on } M^n. \tag{3.19}$$

It follows immediately from (3.19) that for τ as above,

$$\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \overline{\Gamma^{\tau}} \quad \text{in the viscosity sense on } M^n.$$

We wish to show that $\hat{u} - \widetilde{u}$ is constant. To this end, let $c \in \mathbb{R}$ be such that $\hat{u} \leq \widetilde{u} + c$ on M^n and $\hat{u}(x) = \widetilde{u}(x) + c$ for some $x \in M^n$. For this constant c, the set

$$\mathcal{C} = \{x \in M^n : \hat{u}(x) = \widetilde{u}(x) + c\}$$

is therefore non-empty. By continuity of $\hat{u}$ and $\widetilde{u}$, $\mathcal{C}$ is also closed. Moreover, since $\tau < 1$, for any $x \in \mathcal{C}$ we may apply the strong comparison principle of Theorem 2.3 (with $u_1 = \hat{u}$ and $u_2 = \widetilde{u} + c$) to a sufficiently small ball centred at x and conclude that $\mathcal{C}$ is open. Therefore $\mathcal{C} = M^n$, i.e. $\hat{u} = \widetilde{u} + c$ on M^n. In particular, $\hat{u} \in C^{1,1}(M^n)$, and so by [38, Lemma 2.5] and (3.19),

$$\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \overline{\Gamma} \quad \text{a.e. on } M^n. \tag{3.20}$$

Substituting $\widetilde{u} = \hat{u} - c$ into (3.18), we also see that $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma^{\tau}$ a.e. on M^n. By (3.20) and Lemma 3.4, this implies that $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$ a.e. on M^n, and taking the trace of this equation yields

$$0 = R_{g_{\hat{u}}}e^{-2\hat{u}} = R_{g_0} + 2(n-1)\Delta_{g_0}\hat{u} - (n-2)(n-1)|\nabla_{g_0}\hat{u}|_{g_0}^2 \quad \text{a.e. on } M^n.$$

Standard elliptic regularity (see e.g. [19, Theorem 9.19]) then implies $\hat{u} \in C^{\infty}(M^n)$, and thus $g_{\hat{u}}$ is a smooth metric with $R_{g_{\hat{u}}} \equiv 0$ on M^n. This contradicts positivity of $Y(M^n, [g_0])$, and the uniform lower bound is therefore established. □

Proposition 3.9. *Suppose that (f, Γ) satisfies (1.6), (1.7), (1.10) and (1.11). Suppose $\tau_i \to \tau \in (0, 1]$ and let $h_i \in C^0(M^n)$ be a sequence of positive functions converging uniformly to zero on M^n. Suppose that $g_{u_i} = e^{-2u_i}g_0$, $u_i \in C^0(M^n)$, is a sequence of solutions to*

$$f^{\tau_i}(\lambda(g_{u_i}^{-1}A_{g_{u_i}})) = h_i, \quad \lambda(g_{u_i}^{-1}A_{g_{u_i}}) \in \Gamma^{\tau_i} \quad \text{in the viscosity sense on } M^n, \tag{3.21}$$

and that $u_i \to u$ uniformly. Then $g = e^{-2u}g_0$ satisfies

$$\lambda(g^{-1}A_g) \in \partial\Gamma^{\tau} \quad \text{in the viscosity sense on } M^n.$$

Proof. When $h_i = 1$ for all i, the analogous result follows from the proof of [34, Theorem 1.3]. A simple modification of this argument yields Proposition 3.9; we omit the details here. □

3.3. Proof of Theorems 3.1 and 3.2

In this section we prove Theorems 3.1 and 3.2. We begin with the proof of Theorem 3.2:

Proof of Theorem 3.2. We split the proof into two steps. In Step 1, under the assumption that h satisfies (C1), we show that (3.2) has at most one solution. In Step 2, we use a degree argument to prove the existence statement.

Step 1: Suppose that for fixed $\tau \in [0,1]$, u and v are two solutions to (3.2). Suppose for a contradiction that $u > v$ somewhere in M^n. Then for some constant $c > 0$, $u \leq v + c$ on M^n with $u(x) = v(x) + c$ for some $x \in M^n$. Since $f^\tau(\lambda(g_0^{-1}A_{g_{v+c}})) = f^\tau(\lambda(g_0^{-1}A_{g_v})) = h(x,v) < h(x,v+c)$, $v+c$ is a supersolution to (3.2). Thus, the classical strong comparison principle implies $u = v + c$ on M^n, and therefore

$$\begin{aligned} h(x,v+c) = h(x,u) = f^\tau(\lambda(g_0^{-1}A_{g_u})) = f^\tau(\lambda(g_0^{-1}A_{g_{v+c}})) = f^\tau(\lambda(g_0^{-1}A_{g_v})) \\ = h(x,v). \end{aligned}$$

By (C1), this implies $c = 0$, which contradicts $c > 0$. Therefore $u \leq v$. The reverse inequality then follows by interchanging the roles of u and v.

Step 2: By Proposition 3.3, either $g_{\hat{u}}$ is smooth and $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$, or $Y(M^n,[g_0]) > 0$. In the former case we are done, so suppose we are in the latter case. Without loss of generality, we assume that $R_{g_0} > 0$.

In contrast to the work of [7,25,46], since our function is not assumed to satisfy the properness condition (C1), the continuity method is not applicable. We instead use a degree theory argument. Fix an arbitrary number $\alpha \in (0,1)$. Using the fact that $R_{g_0} > 0$, we first fix $\delta > 0$ for which $\lambda^\delta(g_0^{-1}A_{g_0}) \in \Gamma_n^+$ and set $h_0 = f^\delta(g_0^{-1}A_{g_0}) > 0$. Fix $T \in [\delta, 1)$ and consider for $\tau \in [\delta, T]$, $g_{u_\tau} = e^{-2u_\tau}g_0$, the equations

$$f^\tau\big(\lambda(g_0^{-1}A_{g_{u_\tau}})\big) = \frac{T-\tau}{T-\delta}h_0e^{2u_\tau} + \frac{\tau-\delta}{T-\delta}h, \quad \lambda(g_0^{-1}A_{g_{u_\tau}}) \in \Gamma^\tau. \tag{3.22}$$

By Proposition 3.6 and Remark 3.8, there exists a positive constant C such that every solution u_τ to (3.22) with $\tau \in [\delta, T]$ satisfies $\|u_\tau\|_{C^{4,\alpha}(M^n)} \leq C/2$. For this constant C and each $\tau \in [\delta, T]$, we then define

$$\mathcal{O}_\tau = \{u \in C^{4,\alpha}(M^n) : \lambda(g_0^{-1}A_{g_u}) \in \Gamma^\tau, \|u\|_{C^{4,\alpha}(M^n)} < C\}. \tag{3.23}$$

Now denote the RHS of the equation in (3.22) by h^τ and define

$$F_\tau[x,u,\nabla u,\nabla^2 u] := f^\tau(\lambda(g_0^{-1}A_{g_u}))(x) - h^\tau(x,u(x)),$$

so that solutions to (3.22) are precisely the zeros of F_τ. Then the degree $\deg(F_\tau, \mathcal{O}_\tau, 0)$ in the sense of [31] is well-defined an independent of $\tau \in [\delta, T]$.

We claim that $\deg(F_T, \mathcal{O}_T, 0) = 1$. By homotopy invariance, it suffices to show that $\deg(F_\delta, \mathcal{O}_\delta, 0) = 1$. To this end, first note that when $\tau = \delta$, $u_\delta \equiv 0$ is the unique solution to (3.22) with $\lambda(g_0^{-1}A_{g_{u_\delta}}) \in \Gamma^\delta$, where uniqueness follows from Step 1. Therefore, by Propositions 2.3 and 2.4 in [31], to prove $\deg(F_\delta, \mathcal{O}_\delta, 0) = 1$, it suffices to show that the linearisation of F_δ, as a mapping from $C^{2,\alpha}(M^n)$ to $C^\alpha(M^n)$, is invertible with no nonnegative eigenvalues. Indeed, for $u^s = u + s\varphi$ we compute using (2.38)

$$\mathcal{L}^\delta(\varphi) := \frac{d}{ds}\bigg|_{s=0} F_\delta[x, u^s, \nabla u^s, \nabla^2 u^s] = a_i^j (g_0^{-1}\nabla_{g_0}^2\varphi)_j^i + b^i(\nabla_{g_0}\varphi)_i + c\varphi, \quad (3.24)$$

where $a_i^j = \delta L(g_0^{-1}A_{g_u,\delta})_i^j + (1-\delta)\sigma_1(L(g_0^{-1}A_{g_u,\delta}))\delta_i^j$ is positive definite by the ellipticity assumption in (1.11) (here L denotes the linearisation of f), and $c = -\partial_u(h_0 e^{2u})$ is negative. It follows that $\mathcal{L}^\delta$ is invertible as a mapping $\mathcal{L}^\delta : C^{2,\alpha}(M^n) \to C^\alpha(M^n)$ with no nonnegative eigenvalues, as required.

We have shown that (3.22) admits a solution for $\tau = T$, and since $T < 1$ was arbitrary, it follows that there is a smooth solution u_τ to (3.2) for each $\tau < 1$. $\square$

We now give the proof of Theorem 3.1:

Proof of Theorem 3.1. The result is classical when $\Gamma = \Gamma_1^+$, so suppose $\Gamma \neq \Gamma_1^+$. By Theorem 3.2, either there exists a smooth solution $g_{u_\tau} = e^{-2u_\tau}g_0$ to (3.2) for each $\tau \in (0,1)$, or $g_{\hat{u}}$ is smooth with $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$. In the latter case, $g_{\hat{u}}$ clearly satisfies $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$, and we are done.

We now consider the former case. If there exists a $C^{1,1}$ metric g conformal to g_0 satisfying $\lambda(g^{-1}A_g) \in \partial\Gamma$ a.e. on M^n, then the classical strong comparison principle implies that there is no smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma$ on M^n, and we are done. So suppose that there is no $C^{1,1}$ metric g conformal to g_0 satisfying $\lambda(g^{-1}A_g) \in \partial\Gamma$ a.e. on M^n. As in Step 1 of the proof of Proposition 3.6, one obtains a uniform upper bound on solutions to (3.22) uniformly in $\tau \in [\delta, 1]$, and the first and second derivative estimates then follow uniformly in $\tau \in [\delta, 1]$ by [10]. Following the argument in Step 2 of the proof of Proposition 3.6, we then see that $\min_{M^n} u_\tau$ is bounded as $\tau \to 1$, otherwise one obtains a $C^{1,1}$ metric g conformal to g_0 satisfying $\lambda(g^{-1}A_g) \in \partial\Gamma$ a.e. on M^n, a contradiction. The degree argument, as carried out in Step 2 of the proof of Theorem 3.2, therefore yields a smooth solution to (3.2) with $\tau = 1$.

It remains to show that, under (1.8), the first statement in Theorem 3.1 is equivalent to the existence of a smooth metric $g \in [g_0]$ satisfying (1.9) for $\tau = 1$. The forward implication is clear. Conversely, the existence of a smooth metric $g \in [g_0]$ satisfying (1.9) for $\tau = 1$ and the classical strong comparison principle imply that there is no $C^{1,1}$ metric g conformal to g_0 with $\lambda(g^{-1}A_g) \in \partial\Gamma$ a.e. on M^n. Thus the first statement in Theorem 3.1 holds. $\square$

3.4. Proof of Theorems 1.1 and 1.4

In this section we prove Theorems 1.1 and 1.4. Our solutions will be constructed as suitably rescaled limits of solutions obtained in Theorems 3.1 and 3.2, respectively, therein taking $h(x,u) = e^{\beta u}$ and considering the limit $\beta \to 0^+$.

Proof of Theorem 1.4. By Theorem 3.2, either for each $\beta > 0$ and $\tau \in (0,1)$ there exists a smooth solution $g_{u_{\tau,\beta}} = e^{-2u_{\tau,\beta}}g_0$ to (3.2) with $h(x,z) = e^{\beta z}$, or $\hat{u}$ is smooth with $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$ on M^n. In the latter case we are done, so suppose we are in the former case. We fix $\tau \in (0,1)$ and henceforth write u_β as shorthand for $u_{\tau,\beta}$. Letting $v_\beta = u_\beta - \bar{u}_\beta$, where $\bar{u}_\beta = \mathrm{Vol}(M^n, g_0)^{-1}\int_{M^n} u_\beta \, dv_{g_0}$, we see that

$$f^\tau\big(\lambda(g_0^{-1}A_{v_\beta})\big) = f^\tau\big(\lambda(g_0^{-1}A_{u_\beta})\big) = e^{\beta u_\beta}. \tag{3.25}$$

Let $p \in M^n$ be a maximum point for u_β (equivalently, for v_β). As computed in Step 1 in the proof of Proposition 3.6, (3.25) implies $e^{\beta u_\beta(p)} \leq C\sigma_1(\lambda(g_0^{-1}A_{g_0,\tau}))(p)$ and therefore

$$e^{\beta u_\beta} \leq C \quad \text{on } M^n, \tag{3.26}$$

where here and for the rest of the proof C is a constant independent of β. As also discussed in the proof of Proposition 3.6, the *a priori* first and second derivative estimates of [10] on solutions v_β to (3.25) depend only on an upper bound for $e^{\beta u_\beta}$, and hence by (3.26) we have $|\nabla_{g_0}v_\beta|_{g_0}^2 + |\nabla_{g_0}^2 v_\beta|_{g_0} \leq C$. Since v_β has zero average, we therefore have the full C^2 estimate

$$\|v_\beta\|_{C^2(M^n,g_0)} \leq C. \tag{3.27}$$

Now, from (3.25) we see that

$$f^\tau\big(\lambda(g_{v_\beta}^{-1}A_{v_\beta})\big) = e^{2v_\beta}f^\tau\big(\lambda(g_0^{-1}A_{v_\beta})\big) = e^{\beta\bar{u}_\beta}e^{(\beta+2)v_\beta}. \tag{3.28}$$

Moreover, since $\beta > 0$, we can use Jensen's inequality to obtain from (3.26) the estimate

$$e^{\beta\bar{u}_\beta} \leq C.$$

If $\beta\bar{u}_\beta \to -\infty$ along some sequence $\beta \to 0$, then by (3.27), (3.28) and Proposition 3.9, we get $C^{1,\alpha}$ convergence along a further subsequence of v_β to some function $v_* \in C^{1,1}(M^n)$ satisfying

$$\lambda(g_{v_*}^{-1}A_{g_{v_*}}) \in \partial\Gamma^\tau \qquad \text{a.e. on } M^n,$$

which yields a contradiction exactly as in the proof of Proposition 3.6. Therefore, $\beta\bar{u}_\beta$ converges to some constant c along a sequence $\beta \to 0$. Again using (3.27) and (3.28), we get $C^{1,\alpha}$ convergence of v_β to some $v^* \in C^{1,1}(M^n)$, with v^* satisfying

$$f^\tau\big(\lambda(g_{v^*}^{-1}A_{g_{v^*}})\big) = e^c e^{2v^*}, \quad \lambda(g_{v^*}^{-1}A_{g_{v^*}}) \in \Gamma^\tau \quad \text{a.e. on } M^n. \tag{3.29}$$

By uniform ellipticity, v^* is smooth and (3.29) is satisfied everywhere on M^n. This completes the existence part of the proof with $\mu_\tau = e^c$.

We now prove the uniqueness part of Theorem 1.4. Assume for a contradiction that $\mu_\tau \neq \check{\mu}_\tau$ – without loss of generality, we may assume that $\mu_\tau < \check{\mu}_\tau$. After adding a constant to one of our solutions if necessary, we may also assume that $u_\tau \leq \check{u}_\tau$ and $u_\tau(x) = \check{u}_\tau(x)$ for some $x \in M^n$. Then $f^\tau\big(\lambda(g_{u_\tau}^{-1}A_{g_{u_\tau}})\big) = \mu_\tau e^{2u_\tau} < \check{\mu}_\tau e^{2\check{u}_\tau} = f^\tau\big(\lambda(g_{\check{u}_\tau}^{-1}A_{g_{\check{u}_\tau}})\big)$, and the classical strong comparison principle then implies $u_\tau = \check{u}_\tau$ on M^n, which contradicts $f^\tau\big(\lambda(g_{u_\tau}^{-1}A_{g_{u_\tau}})\big) < f^\tau\big(\lambda(g_{\check{u}_\tau}^{-1}A_{g_{\check{u}_\tau}})\big)$. Therefore $\mu_\tau = \check{\mu}_\tau$. The assertion $\check{u}_\tau = u_\tau + c$ for some constant $c \in \mathbb{R}$ then follows from a similar argument. □

Proof of Theorem 1.1. The argument is exactly the same as that in the proof of Theorem 1.4, except that we appeal to Theorem 3.1 instead of Theorem 3.2. □

3.5. A Kazdan-Warner type result

As a by-product of our method for establishing Theorems 3.1 and 3.2, we now prove a Kazdan-Warner type result which will be used in the proof of Theorem 5.2 in Section 5. It is a simple consequence of the classical strong comparison principle that if $g \in [g_0]$ satisfies $\lambda(g^{-1}A_g) \in \Gamma$ on *all* of M^n and Γ satisfies (1.6) and (1.7), then every metric $\widetilde{g} \in [g_0]$ must satisfy $\lambda(\widetilde{g}^{-1}A_{\widetilde{g}}) \in \Gamma$ *somewhere* on M^n. We prove the following:

Theorem 3.10. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ with $Y(M^n, [g_0]) > 0$, and suppose that Γ satisfies (1.6) and (1.7). Then either there exists a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma$ on M^n, or there exists $C^{1,1}$ metric g conformal to g_0 such that $R_g \geq 0$ and $\lambda(g^{-1}A_g) \in \mathbb{R}^n \backslash \Gamma$ a.e. on M^n.*

Proof. Suppose that there is no $C^{1,1}$ metric g conformal to g_0 such that $R_g \geq 0$ and $\lambda(g^{-1}A_g) \in \mathbb{R}^n\backslash\Gamma$ a.e. on M^n. Since $Y(M^n, [g_0]) > 0$, we may assume that $R_{g_0} > 0$ and, as in the proof of Theorem 3.2, we fix $\delta > 0$ for which $\lambda^\delta(g_0^{-1}A_{g_0}) \in \Gamma_n^+$ and set $h_0 = f^\delta(g_0^{-1}A_{g_0}) > 0$. Instead of (3.22), we then consider the path of equations

$$f^\tau(\lambda(g_0^{-1}A_{g_{u_\tau}})) = h_0 e^{2u_\tau}, \quad \lambda(g_0^{-1}A_{g_{u_\tau}}) \in \Gamma^\tau \tag{3.30}$$

for $\tau \in [\delta, 1]$, where $g_{u_\tau} = e^{-2u_\tau}g_0$.

For any $\alpha \in (0,1)$, let

$$\mathcal{U} = \{\tau \in [\delta, 1] : (3.30) \text{ has a solution } u_\tau \in C^{2,\alpha}(M^n)\}.$$

Since $u_\tau \equiv 0$ is a solution to (3.30) when $\tau = \delta$, $\mathcal{U}$ is non-empty.

We now show that $\mathcal{U}$ is closed. First note that (3.30) falls into the framework of Proposition 3.6. As explained in Step 1 in the proof of Proposition 3.6, solutions u_τ to (3.30) admit an upper bound uniformly in $\tau \in [\delta, 1]$, and consequently first and second derivative estimates uniformly in $\tau \in [\delta, 1]$ by [10]. Suppose for a contradiction the uniform lower bound on solutions to (3.30) fails for some sequence $\tau_i \to \tau_0 \leq 1$. Then, as explained in Step 2 in the proof of Proposition 3.6, one obtains a $C^{1,1}$ metric $g_{\widetilde{u}} = e^{-2\widetilde{u}} g_0$ satisfying $\lambda(g_{\widetilde{u}}^{-1} A_{g_{\widetilde{u}}}) \in \partial\Gamma^{\tau_0} \subseteq (\mathbb{R}^n \backslash \Gamma) \cap \overline{\Gamma_1^+}$ a.e. on M^n. This is a contradiction, since we assumed that no such metric exists. This establishes a uniform C^2 estimate on solutions to (3.30), and hence by [14,29], a uniform $C^{2,\beta}$ estimate for any $\beta \in (0,1)$. It follows that $\mathcal{U}$ is closed.

We next show that $\mathcal{U}$ is open. In fact, this follows immediately from the computation in (3.24), which implies that the linearised operator corresponding to (3.30) is invertible as a mapping from $C^{2,\alpha}(M^n)$ to $C^\alpha(M^n)$.

We have shown that $\mathcal{U}$ is non-empty, closed and open, and it follows that $\mathcal{U} = [\delta, 1]$. Thus (3.30) admits a solution for $\tau = 1$, and in particular $g_{u_1} \in [g_0]$ is a smooth metric satisfying $\lambda(g_{u_1}^{-1} A_{g_{u_1}}) \in \Gamma$ on M^n.

It remains to show that the two possibilities are mutually exclusive. But this follows from the classical strong comparison principle: if there exists a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1} A_g) \in \Gamma$ on M^n, then there is no LSC metric $\hat{g}$ conformal to g_0 satisfying $\lambda(\hat{g}^{-1} A_{\hat{g}}) \in \mathbb{R}^n \backslash \Gamma$ in the viscosity sense on M^n. $\square$

4. Some improvements when $(1, 0, \ldots, 0) \in \Gamma$

In this section we consider the following question: if the metric $g_{\hat{u}}$ in (1.8) is not a solution to the degenerate equation $\lambda(g_{\hat{u}}^{-1} A_{g_{\hat{u}}}) \in \partial\Gamma$ in the viscosity sense on M^n, then does there exist a smooth conformal metric $g \in [g_0]$ satisfying $\lambda(g^{-1} A_g) \in \Gamma$ on M^n? As stated in Theorem 1.9, we provide a positive answer to this question when condition (1.14) is satisfied, that is when $(1, 0, \ldots, 0) \in \Gamma$ (in this case, we can appeal to our strong comparison principle in Theorem 2.3′). We will also prove the uniqueness and regularity result stated in Theorem 1.10.

As a means for establishing Theorem 1.9, we will first prove the following refinement of Theorem 3.1:

Theorem 4.1. *In addition to the hypotheses of Theorem 3.1, suppose also that (1.14) holds and $g_{\hat{u}}$ is not a solution to $\lambda(g_{\hat{u}}^{-1} A_{g_{\hat{u}}}) \in \partial\Gamma$ on M^n in the viscosity sense. Then there exists a smooth solution u_1 to (3.2), and this is equivalent to the existence of a smooth metric $g \in [g_0]$ satisfying (1.9) for $\tau = 1$. Moreover, for all $0 < \delta < 1$ and $\alpha \in (0,1)$, there exists a constant $C > 0$ depending only on n, g_0, δ, α and h such that solutions to (3.2) with $\tau \in [\delta, 1]$ satisfy*

$$\|u_\tau\|_{C^{4,\alpha}(M^n, g_0)} \leq C. \tag{4.1}$$

If h also satisfies (C1), then solutions to (3.2) are unique.

As an immediate corollary of Theorem 4.1 and Proposition A.1 in Appendix A, we obtain the following (here we temporarily revert from using $\tau \in (0,1]$ to using $t \in (-\infty, 1]$, with $A^t_{g_u}$ given by the formula (1.3) when $g_u = e^{-2u}g_0$):

Corollary 4.2. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ and suppose $2 \leq k \leq n$. Suppose for some fixed $t \in (-\infty, 1)$ there exists a metric $g_{\hat{u}} = e^{-2\hat{u}}g_0$, $\hat{u} \in C^0(M^n)$, satisfying $\lambda(g_{\hat{u}}^{-1}A^t_{g_{\hat{u}}}) \in \overline{\Gamma_k^+}$ in the viscosity sense on M^n, but such that $g_{\hat{u}}$ is not a solution to $\lambda(g_{\hat{u}}^{-1}A^t_{g_{\hat{u}}}) \in \partial\Gamma_k^+$ in the viscosity sense on M^n. Then there exists a smooth metric $g_t \in [g_0]$ satisfying $\lambda(g_t^{-1}A^t_{g_t}) \in \Gamma_k^+$ on M^n.*

The first step in the proof of Theorem 4.1 is to extend the *a priori* estimates of Proposition 3.6 to $\tau = 1$ under our additional hypotheses. We state these estimates as a separate result:

Proposition 4.3. *In addition to the hypotheses of Proposition 3.6, suppose also that (1.14) holds and $g_{\hat{u}}$ is not a solution to $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$ on M^n in the viscosity sense. Then for all $0 < \delta < 1$ and $\alpha \in (0,1)$, there exists a constant $C > 0$ depending only on n, g_0, δ, α and h such that solutions u_τ to (3.2) with $\tau \in [\delta, 1]$ satisfy*

$$\|u_\tau\|_{C^{4,\alpha}(M^n, g_0)} \leq C_2. \tag{4.2}$$

Proof of Proposition 4.3. We first recall from Step 1 in the proof of Proposition 3.6 that solutions to (3.2) admit an upper C^0 bound and upper first and second derivative bounds uniformly in $\tau \in [\delta, 1]$. By Step 2 in the proof of Proposition 3.6, for any $T < 1$, solutions to (3.2) for $\tau \in [\delta, T]$ also admit a lower bound. Under only the hypotheses of Proposition 3.6, this lower bound may depend on T. To prove Proposition 4.3, we only need to prove that the lower bound on solutions to (3.2) holds uniformly as $\tau \to 1$ under our additional assumptions. As before, the $C^{4,\alpha}$ estimate then follows from the theory of Evans-Krylov [14,29] and classical Schauder estimates.

We suppose for a contradiction that the lower bound on solutions to (3.2) fails for a sequence $\tau_i \to 1$. Then, following the reasoning in Step 2 in the proof of Proposition 3.6, we obtain a $C^{1,1}$ metric $\widetilde{g} = e^{-2\widetilde{u}}g_0$ satisfying

$$\lambda(\widetilde{g}^{-1}A_{\widetilde{g}}) \in \partial\Gamma \quad \text{a.e. on } M^n. \tag{4.3}$$

Let $c \in \mathbb{R}$ be a constant such that $\hat{u} \leq \widetilde{u} + c$ on M^n and $\hat{u}(x) = \widetilde{u}(x) + c$ for some $x \in M^n$, where $\hat{u}$ is as in the statement of the proposition. As in Step 2 in the proof of Proposition 3.6, the strong comparison principle in Theorem 2.3′ (which applies since we now assume $(1, 0, \ldots, 0) \in \Gamma$) then yields $\hat{u} = \widetilde{u} + c$ on M^n.

On the other hand, since we assume $g_{\hat{u}}$ is not a solution to $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$ on M^n in the viscosity sense, there exists a point p and a function $\varphi \in C^2(M^n)$ that touches $\hat{u}$

from below at p such that $\lambda(g_\varphi^{-1}A_{g_\varphi})(p) \in \Gamma$. But $\varphi - c$ touches $\hat{u} - c = \widetilde{u}$ from below at p, and since $\widetilde{u}$ satisfies $\lambda(\widetilde{g}^{-1}A_{\widetilde{g}}) \in \partial\Gamma$ in the viscosity sense on M^n, it must be the case that $\lambda(g_{\varphi-c}^{-1}A_{g_{\varphi-c}})(p) \in \mathbb{R}^n \backslash \Gamma$. But this is equivalent to $\lambda(g_\varphi^{-1}A_{g_\varphi})(p) \in \mathbb{R}^n \backslash \Gamma$, a contradiction. This establishes the desired lower bound. □

We now complete the proofs of Theorem 4.1, Theorem 1.10 and Theorem 1.9:

Proof of Theorem 4.1. Since $g_{\hat{u}}$ is not a solution to $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$ on M^n in the viscosity sense, it cannot be the case that $g_{\hat{u}}$ is a smooth solution to $\mathrm{Ric}_{g_{\hat{u}}} \equiv 0$, and hence Proposition 3.3 implies $Y(M^n, [g_0]) > 0$. The hypotheses of Proposition 4.3 are therefore satisfied. The remainder of the proof of Theorem 4.1 is identical to that of Theorem 3.2, except one takes $T = 1$ in Step 2 and applies Proposition 4.3 instead of Proposition 3.6. □

Proof of Theorem 1.10. Suppose that there exists a continuous viscosity solution $g_u = e^{-2u}g_0$ to the equation $\lambda(g_u^{-1}A_{g_u}) \in \partial\Gamma$ on M^n. As a consequence of the classical strong comparison principle, it follows that there is no smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma$ on M^n. By Theorem 3.1, it follows that there exists a $C^{1,1}$ metric $g = e^{-2w}g_0$ satisfying $\lambda(g^{-1}A_g) \in \partial\Gamma$ a.e. on M^n.

Let $c \in \mathbb{R}$ and $x \in M^n$ be such that $u \leq w + c$ on M^n and $u(x) = w(x) + c$. Following the argument in Step 2 in the proof of Proposition 3.6, the strong comparison principle in Theorem 2.3′ then implies that $u = w + c$. In particular, this establishes the $C^{1,1}$ regularity of u. Now, if $g_v = e^{-2v}g_0$ is another continuous metric satisfying $\lambda(g_v^{-1}A_{g_v}) \in \partial\Gamma$ in the viscosity sense on M^n, then the same argument yields $v = w + d$ on M^n for some constant d, and therefore $u = v + c - d$ on M^n. This establishes the uniqueness statement. □

Proof of Theorem 1.9. By Theorem 1.10, the statement that $g_{\hat{u}}$ is not a $C^{1,1}$ solution to $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$ a.e. on M^n is equivalent to the statement that $g_{\hat{u}}$ is not a continuous solution to $\lambda(g_{\hat{u}}^{-1}A_{g_{\hat{u}}}) \in \partial\Gamma$ in the viscosity sense on M^n. The result then follows by arguing as in the proof of Theorem 1.4, taking u_β to be the solutions obtained in Theorem 4.1 with $h(x, u) = e^{\beta u}$. □

5. Geometric applications

In Sections 5.1 and 5.2, it will be more convenient to work with the parameter $t \in (-\infty, 1]$ rather than $\tau \in (0, 1]$ – see (1.3) for the formula for $A_{g_u}^t$ when $g_u = e^{-2u}g_0$.

5.1. The case $\Gamma = \Gamma_2^+$ and Ricci pinching (Theorem 1.13)

As discussed in the introduction, part of our motivation for establishing Theorems 1.1 and 1.4 stems from earlier work on the existence of conformal metrics satisfying

$\lambda(g^{-1}A_g^t) \in \Gamma_2^+$ on M^n. In [9], Chang, Gursky & Yang established the existence of a metric $g \in [g_0]$ with $\lambda(g^{-1}A_g) \in \Gamma_2^+$ on any closed 4-manifold satisfying

$$Y(M^4,[g_0]) > 0 \quad \text{and} \quad \int_{M^4} \sigma_2(\lambda(g_0^{-1}A_{g_0}))\, dv_{g_0} > 0. \tag{5.1}$$

Note that, in light of the Chern-Gauss-Bonnet formula in four dimensions, the integral in (5.1) is conformally invariant.

Ge, Lin & Wang [16] later established a similar result on closed 3-manifolds: if

$$R_{g_0} > 0 \quad \text{and} \quad \int_{M^3} \sigma_2(\lambda(g_0^{-1}A_{g_0}))\, dv_{g_0} > 0, \tag{5.2}$$

then there exists a conformal metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma_2^+$ on M^3. In fact, the authors obtain (see Case 1 of Theorem 2 therein) such a conformal metric in any dimension, assuming positivity of the following nonlinear Yamabe-type invariant:

$$Y_{2,1}([g_0]) := \begin{cases} \sup\limits_{g\in[g_0],\, R_g>0} \int_{M^3} \sigma_2(\lambda(g^{-1}A_g))\, dv_g \int_{M^3} \sigma_1(\lambda(g^{-1}A_g))\, dv_g & \text{if } n=3 \\ \int_{M^4} \sigma_2(\lambda(g^{-1}A_g))\, dv_g & \text{if } n=4 \\ \inf\limits_{g\in[g_0],\, R_g>0} \dfrac{\int_{M^n} \sigma_2(\lambda(g^{-1}A_g))\, dv_g}{\left(\int_{M^n} \sigma_1(\lambda(g^{-1}A_g))\, dv_g\right)^{\frac{n-4}{n-2}}} & \text{if } n\geq 5, \end{cases}$$

with the convention that $Y_{2,1}([g_0]) = -\infty$ if $Y(M^n,[g_0]) \leq 0$. When $Y_{2,1}([g_0]) = 0$, they also established (see Case 2 of Theorem 2 therein) the existence of a $C^{1,1}$ metric g conformal to g_0 satisfying $\lambda(g^{-1}A_g) \in \partial\Gamma_2^+$ a.e. on M^n. See also [7] in the case that $R_{g_0} > 0$ and $\int_{M^3} \sigma_2(g_0^{-1}A_{g_0})\, dv_{g_0} \geq 0$.

We now give the proof of Theorem 1.13:

Proof of Theorem 1.13. The assumption $Y_{2,1}([g_0]) = 0$ implies, in particular, that the Yamabe invariant $Y(M^n,[g_0])$ is positive. By [16, Theorem 2], the assumption $Y_{2,1}([g_0]) = 0$ also implies the existence of a $C^{1,1}$ metric $g = e^{-2u}g_0$ satisfying

$$\lambda(g^{-1}A_g) \in \partial\Gamma_2^+ \quad \text{a.e. on } M^n. \tag{5.3}$$

By Corollary 1.6, it follows from (5.3) that for each $t < 1$, there exists a smooth metric $g_t \in [g_0]$ satisfying $\lambda(g_t^{-1}A_{g_t}^t) \in \Gamma_2^+$ on M^n. Moreover, the non-existence of a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma_2^+$ on M^n follows from (5.3) and the classical strong comparison principle. □

The geometric significance of Theorem 1.13 is as follows. It is well-known in dimensions $n \geq 3$ that the Ricci tensor of a metric satisfying $\lambda(g^{-1}A_g^t) \in \Gamma_2^+$ on M^n satisfies the following pointwise pinching property:

$$\left(t - 2 + \frac{4}{n}\right) R_g g < 2\,\mathrm{Ric}_g < (2 - t) R_g g \tag{5.4}$$

(see e.g. [25, Proposition 5.2] for a proof of this fact). In this way, Theorem 1.13 asserts the existence of conformal metrics satisfying *pointwise* pinching of the Ricci tensor, given only an *integral* pinching condition on a background metric of positive scalar curvature. We state this pinching result as a further corollary:

Corollary 5.1. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 3$ with $Y_{2,1}([g_0]) \geq 0$. Then for all $t < 1$, there exists a smooth metric $g \in [g_0]$ satisfying $R_g > 0$ and (5.4).*

By the work of Hamilton in [27], in dimension $n = 3$ the pinching relation in (5.4) with $t \geq \frac{2}{3}$ implies that M^3 is diffeomorphic to a spherical space form.

5.2. Differential inclusions under positivity of Yamabe-type invariants

In this section we give an application of Theorem 3.10, in which we prove the existence of a smooth conformal metric satisfying $\lambda(g^{-1}A_g^t) \in \Gamma_2^+$ on M^n assuming positivity of a nonlinear Yamabe-type invariant. We restrict our attention to the case $n \geq 5$; for the cases $n = 3$ and $n = 4$ we refer to [7,9,16,25].

More precisely, for $t \leq 1$ and $n \geq 5$ we define

$$Y_2^t([g_0]) = \inf_{g \in [g_0],\, R_g > 0} \frac{\int_{M^n} \sigma_2(\lambda(g^{-1}A_g^t))\, dv_g}{\mathrm{Vol}(M^n, g)^{\frac{n-4}{n}}}, \tag{5.5}$$

with the convention that $Y_2^t([g_0]) = -\infty$ if $Y(M^n, [g_0]) \leq 0$. We prove:

Theorem 5.2. *Let (M^n, g_0) be a smooth, closed Riemannian manifold of dimension $n \geq 5$, and suppose $t \leq 1$. If $Y_2^t([g_0]) > 0$, then there exists a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g^t) \in \Gamma_2^+$ on M^n.*

Remark 5.3. In the case $t = 1$, Theorem 5.2 was previously established by Sheng in [46]. In this case, Theorem 5.2 implies the result of Ge, Lin & Wang [16] (discussed in the previous subsection) in dimensions $n \geq 5$, since $Y_{2,1}([g_0]) > 0$ implies $Y_2^1([g_0]) \geq CY_{2,1}([g_0])Y(M^n, [g_0])^{\frac{n-4}{n-2}} > 0$. In the case $t < 1$, a related statement was also proved in [46], although a condition different to $Y_2^t([g_0]) > 0$ was considered therein.

Proof of Theorem 5.2. Suppose $Y_2^t([g_0]) > 0$. We claim that the infimum in (5.5) is still positive if the infimum is taken over the larger set consisting of all $C^{1,1}$ conformal metrics

$g = u^{\frac{4}{n-2}} g_0$ satisfying $R_g \geq 0$ a.e. on M^n. Once this claim is proved, it follows that no $C^{1,1}$ metric g conformal to g_0 with $R_g \geq 0$ a.e. on M^n can satisfy $\sigma_2(\lambda(g^{-1}A_g^t)) \leq 0$ a.e. on M^n. Theorem 3.10 then implies the desired result.

To prove the claim, suppose $g = u^{\frac{4}{n-2}} g_0$ is a $C^{1,1}$ metric with $R_g \geq 0$ a.e. on M^n. Without loss of generality, we may assume $R_{g_0} > 0$. If L_{g_0} denotes the conformal Laplacian of g_0, then u satisfies the equation

$$L_{g_0} u = f := R_g u^{\frac{n+2}{n-2}} \quad \text{a.e. on } M^n.$$

Observe that $f \geq 0$ is not identically zero, otherwise we would have $u \equiv 0$ in view of the fact $R_{g_0} > 0$. Since $f \geq 0$ and f belongs to $L^\infty(M^n)$, we may convolve f with nonnegative smooth mollifiers and adjust the resulting mollifications by small positive constants to obtain a sequence f_i of smooth, positive functions on M^n such that $f_i \to f$ in $L^p(M^n, g_0)$ for all $p < \infty$. Let u_i denote the corresponding smooth solutions to

$$L_{g_0} u_i = f_i \quad \text{on } M^n.$$

Note that, since $R_{g_0} > 0$ and $f_i > 0$, the maximum principle implies $u_i > 0$. Moreover, since $f_i \to f$ in $L^p(M^n, g_0)$ for all $p < \infty$, standard elliptic theory implies that $u_i \to u$ in $W^{2,p}(M^n, g_0)$ for all $p < \infty$, and hence

$$0 < Y_2^t([g_0]) \leq \frac{\int_{M^n} \sigma_2(\lambda(g_{u_i}^{-1} A_{g_{u_i}}^t))\, dv_{g_{u_i}}}{\mathrm{Vol}(M^n, g_{u_i})^{\frac{n-4}{n}}} \longrightarrow \frac{\int_{M^n} \sigma_2(\lambda(g_u^{-1} A_{g_u}^t))\, dv_{g_u}}{\mathrm{Vol}(M^n, g_u)^{\frac{n-4}{n}}}$$

as $i \to \infty$. This proves the claim, and so completes the proof of Theorem 5.2. □

5.3. Conformal invariance of the sign of $\lambda(\sigma_2, g_0)$ (Theorem 1.11)

In this section we prove Theorem 1.11. In [16], the authors considered the following nonlinear eigenvalue for the σ_2 operator:

$$\lambda(\sigma_2, g_0) := \begin{cases} \sup\limits_{g = e^{-2u} g_0,\, R_g > 0} \dfrac{\int_{M^3} \sigma_2(\lambda(g^{-1}A_g))\, dv_g}{\int_{M^3} e^{4u}\, dv_g} & \text{if } n = 3 \\ \displaystyle\int_{M^4} \sigma_2(\lambda(g^{-1}A_g))\, dv_g & \text{if } n = 4 \\ \inf\limits_{g = e^{-2u} g_0,\, R_g > 0} \dfrac{\int_{M^n} \sigma_2(\lambda(g^{-1}A_g))\, dv_g}{\int_{M^n} e^{4u}\, dv_g} & \text{if } n > 5. \end{cases}$$

The interpretation of $\lambda(\sigma_2, g_0)$ as a nonlinear eigenvalue comes from [16, Theorem 1], where it is shown that if $\lambda(\sigma_2, g_0) > 0$, then $\lambda(\sigma_2, g_0)$ is achieved by a smooth metric $g_u = e^{-2u} g_0$ of positive scalar curvature satisfying $\sigma_2(\lambda(g_u^{-1} A_{g_u})) = \lambda e^{4u}$ on M^n. It is also shown that when $\lambda(\sigma_2, g_0) = 0$, $\lambda(\sigma_2, g_0)$ is achieved by a $C^{1,1}$ metric g_u of nonnegative scalar curvature satisfying $\sigma_2(\lambda(g_u^{-1}A_{g_u})) = 0$ a.e. on M^n.

In analogy with the case for the scalar curvature, where the sign of the Yamabe invariant of $[g_0]$ coincides with the sign of the first eigenvalue of the conformal Laplacian of any metric in $[g_0]$, one may expect a relationship between the signs of $Y_{2,1}([g_0])$ and $\lambda(\sigma_2, g_0)$. It is shown in [16, Lemma 3] that the sign of $Y_{2,1}([g_0])$ coincides with the sign of $\lambda(\sigma_2, g_0)$ when $n \geq 4$. For $n = 3$, it is also shown $\lambda(\sigma_2, g_0) > 0$ if and only if $Y_{2,1}([g_0]) > 0$, $\lambda(\sigma_2, g_0) \leq 0$ if and only if $Y_{2,1}([g_0]) \leq 0$, and $\lambda(\sigma_2, g_0) < 0$ implies $Y_{2,1}([g_0]) < 0$. Using our existence result in Theorem 3.2, we prove the remaining implication and hence establish Theorem 1.11:

Proof of Theorem 1.11. In light of the discussion above, we only need to show that $\lambda(\sigma_2, g_0) = 0$ implies $Y_{2,1}([g_0]) = 0$. For shorthand we denote

$$F[g] = \int\limits_{M^3} \sigma_2(\lambda(g^{-1}A_g))\, dv_g \int\limits_{M^3} \sigma_1(\lambda(g^{-1}A_g))\, dv_g$$

so that $Y_{2,1}([g_0]) = \sup_{g\in[g_0],\, R_g>0} F[g]$.

For $\tau \in (0,1)$, let $g_{u_\tau} = e^{-2u_\tau} g_0$ be the solutions obtained to (3.22) in the proof of Theorem 3.2 in the case that $f = \sigma_2^{1/2}$ and $Y(M^3, [g_0]) > 0$. A routine calculation yields

$$\sigma_2(\lambda(g_{u_\tau}^{-1}A_{g_{u_\tau}})) = (1-\tau)c(n,\tau)[\sigma_1(\lambda(g_{u_\tau}^{-1}A_{g_{u_\tau}}))]^2 - \tau^{-2}\sigma_2(\lambda^\tau(g_{u_\tau}^{-1}A_{g_{u_\tau}})),$$

where $c(n,\tau) = \frac{1}{2}\tau^{-2}(n-1)(2\tau + n(1-\tau))$ is bounded as $\tau \to 1$. Therefore

$$F[g_{u_\tau}] > -C(1-\tau) \int\limits_{M^3} [\sigma_1(\lambda(g_{u_\tau}^{-1}A_{g_{u_\tau}}))]^2\, dv_{g_{u_\tau}} \int\limits_{M^3} \sigma_1(\lambda(g_{u_\tau}^{-1}A_{g_{u_\tau}}))\, dv_{g_{u_\tau}}, \quad (5.6)$$

where here and for the remainder of the proof, C is a constant that remains bounded as $\tau \to 1$ (but may change from line to line).

Next, observe that we have the identities

$$\begin{aligned} R_{g_{u_\tau}}^2\, dv_{g_{u_\tau}} &= e^{u_\tau}\left(R_{g_0} + 4\Delta_{g_0}u_\tau - 2|\nabla_{g_0}u_\tau|_{g_0}^2\right)^2 dv_{g_0}, \\ R_{g_{u_\tau}}\, dv_{g_{u_\tau}} &= e^{-u_\tau}\left(R_{g_0} + 4\Delta_{g_0}u_\tau - 2|\nabla_{g_0}u_\tau|_{g_0}^2\right) dv_{g_0}. \end{aligned}$$

Moreover, by the first and second derivative estimates obtained on u_τ in Step 1 of the proof of Proposition 3.6 (which, we recall, hold uniformly for $\tau \leq 1$), we have $0 < R_{g_0} + 4\Delta_{g_0}u_\tau - 2|\nabla_{g_0}u_\tau|_{g_0}^2 \leq C$, and it then follows from the above that

$$F[g_{u_\tau}] \geq -C(1-\tau) \int\limits_{M^3} e^{u_\tau}\, dv_{g_0} \int\limits_{M^3} e^{-u_\tau}\, dv_{g_0}. \quad (5.7)$$

We claim that the product of integrals in (5.7) is bounded independently of $\tau \leq 1$. Once this is established, it will follow that $F[g_{u_\tau}] \geq -C(1-\tau)$. Since we already know by [16, Lemma 3] that $Y_{2,1}([g_0]) = \sup_{g\in[g_0],\, R_g>0} F[g] \leq 0$, and since $\tau < 1$ is arbitrary, the conclusion $Y_{2,1}([g_0]) = 0$ then follows.

To prove the claim, observe by the first derivative estimates obtained on u_τ in Step 1 of the proof of Proposition 3.6 (which are uniform for $\tau \leq 1$), there exists a constant C such that $|u_\tau(x) - u_\tau(y)| \leq C$ for all $x, y \in M^3$ and all $\tau < 1$, i.e.

$$-C - u_\tau(y) \leq -u_\tau(x) \leq C - u_\tau(y) \quad \text{for all } x, y \in M^3. \tag{5.8}$$

Taking exponentials in (5.8) (which preserves the inequalities) and then integrating against $dv_{g_0}(y)$ gives

$$0 < C^{-1} \int\limits_{M^3} e^{-u_\tau}\, dv_{g_0} \leq e^{-u_\tau(x)} \leq C \int\limits_{M^3} e^{-u_\tau}\, dv_{g_0} \quad \text{for all } x \in M^3,$$

or equivalently

$$0 < C^{-1} \leq e^{u_\tau(x)} \int\limits_{M^3} e^{-u_\tau}\, dv_{g_0} \leq C \quad \text{for all } x \in M^3.$$

Integrating against $dv_{g_0}(x)$ then gives

$$0 < C^{-1} \leq \int\limits_{M^3} e^{u_t}\, dv_{g_0} \int\limits_{M^3} e^{-u_t}\, dv_{g_0} \leq C,$$

as claimed. This completes the proof. □

5.4. A generalisation of a theorem of Aubin & Ehrlick (Theorem 1.14)

Proof of Theorem 1.14. The assumption (1.15) is equivalent to $\lambda(\hat{g}^{-1}A^t_{\hat{g}}) \in \overline{\Gamma_n^+}$ in the viscosity sense on M^n for $t = 2(n-1)\alpha < 1$. Since we assume that $\hat{g}$ is not a $C^{1,1}$ solution to $\lambda(\hat{g}^{-1}A^t_{\hat{g}}) \in \partial\Gamma_n^+$ a.e. on M^n, the desired conclusion then follows from Corollary 4.2. □

As a by-product of the proof of Theorem 1.14, the metric g obtained in Theorem 1.14 satisfies $\det(g_0^{-1}(\mathrm{Ric}_g - \alpha R_g g)) = c$ on M^n for some constant $c > 0$. When $0 \leq \alpha < \frac{1}{2(n-1)}$, as an alternative to the above proof, one can use [33] together with the strong comparison principle Theorem 2.3 to solve the trace-modified σ_n-Yamabe equation $\det(g^{-1}(\mathrm{Ric}_g - \alpha R_g g)) = 1$ on M^n as follows: the proof of [33, estimate (9)] shows that, as $\alpha \geq 0$, all solutions to the trace-modified σ_n-Yamabe equation are bounded from below. If they are not bounded from above, then a suitably rescaled limit will yield a $C^{1,1}$

solution to the equation $\lambda(g^{-1}(\mathrm{Ric}_g - \alpha R_g g)) \in \partial\Gamma_n^+$. By the strong comparison principle in Theorem 2.3, this contradicts the assumptions on $\hat{g}$ in Theorem 1.15. Existence then follows from a degree theory argument.

5.5. Differential inclusions vs. nonlinear Green's functions (Theorem 1.15)

Proof of Theorem 1.15. We assume for simplicity that the Green's function $w \in C^0_{\mathrm{loc}}(M^n\backslash\{p\})$ has a single pole at $p \in M^n$, of fixed but arbitrary strength. Observe that, if v is a positive function on M^n satisfying $\nabla_{g_0} v(p) = 0$ and $\nabla^2_{g_0} v(p) < 0$, then by (1.3) with $u = -\frac{2}{n-2}\log v$, the Schouten tensor of the metric $\alpha v^{\frac{4}{n-2}} g_0$ is positive definite near p for α large. In particular, after making a change of background metric if necessary, we may assume that g_0 satisfies $\lambda(g_0^{-1}A_{g_0}) \in \Gamma$ on some small geodesic ball $B_\varepsilon(p)$ centred at p.

Fix a large constant $M > 1 + \max_{M^n\backslash B_\varepsilon(p)} w$ and define $\widetilde{w} = \min\{w, M\}$. Clearly $\widetilde{w}$ is continuous, and we claim that the metric $\widetilde{g} = \widetilde{w}^{\frac{4}{n-2}} g_0$ satisfies $\lambda(\widetilde{g}^{-1}A_{\widetilde{g}}) \in \overline{\Gamma}$ in the viscosity sense on M^n. Once this claim is established, the existence of a smooth metric $g \in [g_0]$ satisfying $\lambda(g^{-1}A_g) \in \Gamma$ on M^n follows immediately from Theorem 4.1.

To this end, fix $x \in M^n$ and suppose that $\varphi \in C^2(M^n)$ is a positive function touching $\widetilde{w}$ from below at x. To prove the claim, we need to show that the metric $\bar{g} = \varphi^{\frac{4}{n-2}} g_0$ satisfies $\lambda(\bar{g}^{-1}A_{\bar{g}})(x) \in \overline{\Gamma}$. There are three cases to consider:

Case 1: $x \in \{w < M\}$. Since w is continuous away from p, the set $\{w < M\}$ is open and hence $\widetilde{w} \equiv w$ in a neighbourhood of x. Therefore, φ also touches w from below at x, and it follows from the fact that w is a viscosity subsolution that $\lambda(\bar{g}^{-1}A_{\bar{g}})(x) \in \overline{\Gamma}$.

Case 2: $x \in \{w > M\}$. Since $\widetilde{w}(x) = M$ and $\lambda(g_0^{-1}A_{g_0}) \in \Gamma$ in $B_\varepsilon(p)$, it follows in this case that $\lambda(\widetilde{g}^{-1}A_{\widetilde{g}})(x) \in \overline{\Gamma}$. Consequently, $\lambda(\bar{g}^{-1}A_{\bar{g}})(x) \in \overline{\Gamma}$.

Case 3: $x \in \{w = M\}$. We have $\varphi(x) = \widetilde{w}(x) = M = w(x)$ and $\varphi < \widetilde{w} \leq w$ near x (note $\widetilde{w} \leq w$ on all of M^n, by definition of $\widetilde{w}$). Therefore, φ touches w from below at x, and the conclusion follows as in Case 1. □

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Appendix A. Two characterisations of cones satisfying $(1, 0, \ldots, 0) \in \Gamma$

In this appendix we give two equivalent characterisations of cones Γ satisfying (1.6), (1.7) and $(1, 0, \ldots, 0) \in \Gamma$:

Proposition A.1. *Suppose* Γ *satisfies* (1.6) *and* (1.7)*. Then the following are equivalent:*

(1) $(1, 0, \ldots, 0) \in \Gamma$.
(2) $\overline{\Gamma_n^+} \backslash \{0\} \subset \Gamma$.
(3) There exists $\widetilde{\Gamma}$ *satisfying* (1.6) *and* (1.7) *and a number* $\tau < 1$ *for which* $\Gamma = (\widetilde{\Gamma})^\tau$.

Proof. The implication (2) $\implies$ (1) is immediate. Conversely, suppose $(1, 0, \ldots, 0) \in \Gamma$. By symmetry, the positive axes belong to Γ, and by convexity it then follows that $\partial\Gamma_n^+ \backslash \{0\} \subset \Gamma$. Since we assume $\Gamma_n^+ \subseteq \Gamma$, we therefore have $\overline{\Gamma_n^+} \backslash \{0\} \subset \Gamma$. Thus (1) $\implies$ (2).

Next we show (3) $\implies$ (1). Suppose that $\Gamma = (\widetilde{\Gamma})^\tau$ for some $\widetilde{\Gamma}$ satisfying (1.6) and (1.7) and some $\tau < 1$. Then

$$\tau(1, 0, \ldots, 0) + (1 - \tau)\sigma_1(1, 0, \ldots, 0)e = (1, 1 - \tau, \ldots, 1 - \tau) \in \Gamma_n^+, \tag{A.1}$$

and the assertion $(1, 0, \ldots, 0) \in \Gamma$ then follows from (A.1) and the assumption $\Gamma_n^+ \subseteq \Gamma$.

It remains to show (1) $\implies$ (3). Suppose Γ satisfies (1.6), (1.7) and $(1, 0, \ldots, 0) \in \Gamma$. We first claim that for $\tau' > 1$ sufficiently close to 1, $\Gamma^{\tau'}$ satisfies (1.6) and (1.7). In fact, (1.6) is immediate, and the inclusion $\Gamma^{\tau'} \subsetneq \Gamma_1^+$ in (1.7) follows from the fact $\Gamma^{\tau'} \subseteq \Gamma \subseteq \Gamma_1^+$. So it remains to show that $\Gamma_n^+ \subseteq \Gamma^{\tau'}$ for $\tau' > 1$ sufficiently close to 1. To this end, first observe that for $\tau' > 1$ sufficiently close to 1, $(1, 0, \ldots, 0) \in \Gamma^{\tau'}$. Indeed, $(1, 0, \ldots, 0) \in \Gamma^{\tau'}$ if and only if $(1, 1 - \tau', \ldots, 1 - \tau') \in \Gamma$, and this latter inclusion is clearly seen to hold for $\tau' > 1$ sufficiently close to 1, using openness of Γ and the fact that $(1, 0, \ldots, 0) \in \Gamma$. Now, since $\Gamma^{\tau'}$ is convex, symmetric and satisfies $(1, 0, \ldots, 0) \in \Gamma^{\tau'}$, the convex hull of all permutations of $(1, 0, \ldots, 0)$ is also contained in $\Gamma^{\tau'}$. By homothety, it then follows that $\Gamma_n^+ \subseteq \Gamma^{\tau'}$, as required.

By the previous paragraph, we may fix $\tau' > 1$ sufficiently close to 1 so that $\Gamma^{\tau'}$ satisfies (1.6) and (1.7). Then define $\widetilde{\Gamma} = \Gamma^{\tau'}$, and observe that $\Gamma = (\widetilde{\Gamma})^{\widetilde{\tau}}$ for

$$\widetilde{\tau} = \frac{n - (n-1)\tau'}{(n-1) - (n-2)\tau'} < 1. \tag{A.2}$$

Indeed, for any τ, $\lambda \in (\widetilde{\Gamma})^\tau$ if and only if $\tau\lambda + (1 - \tau)\sigma_1(\lambda)e \in \widetilde{\Gamma}$ which occurs if and only if

$$\tau'\big[\tau\lambda + (1 - \tau)\sigma_1(\lambda)e\big] + (1 - \tau')\sigma_1\big[\tau\lambda + (1 - \tau)\sigma_1(\lambda)e\big]e \in \Gamma, \tag{A.3}$$

since $\widetilde{\Gamma} = \Gamma^{\tau'}$. Collecting coefficients, we see that (A.3) is equivalent to

$$\tau'\tau\lambda + \big[\tau'(1 - \tau) + \tau(1 - \tau') + n(1 - \tau)(1 - \tau')\big]\sigma_1(\lambda)e \in \Gamma. \tag{A.4}$$

Taking $\tau = \widetilde{\tau}$ in (A.4), we see that quantity in the square parentheses vanishes, and hence $\lambda \in (\widetilde{\Gamma})^{\widetilde{\tau}}$ if and only if $\tau'\widetilde{\tau}\lambda \in \Gamma$, which occurs if and only if $\lambda \in \Gamma$ (since $a\Gamma = \Gamma$ for all $a > 0$). We have therefore shown (1) $\implies$ (3). $\square$

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