Rational Points on the Intersection of Three Quadrics

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1 Introduction

Let $Q_1(\mathbf{x}), Q_2(\mathbf{x}), Q_3(\mathbf{x}) \in K[\mathbf{x}]$ be three quadratic forms in n variables $\mathbf{x} = (x_1, \ldots, x_n)$, defined over a number field K. This paper will be concerned with the Hasse principle and weak approximation for points over K on the intersection

$$\mathcal{R}:\ Q_1 = Q_2 = Q_3 = 0.$$

We begin by reviewing the situation for individual quadrics and pairs of quadrics. In the case of a single quadratic form the Hasse principle is always valid, while weak approximation holds for all nonsingular quadratic forms in $n \geq 3$ variables. When one has a pair of forms the Hasse principle may fail, even when the variety $Q_1 = Q_2 = 0$ is nonsingular, as is shown by the example

$$Q_1 = X_1X_2 - (X_3^2 - 5X_4^2), \quad Q_2 = (X_1 + X_2)(X_1 + 2X_2) - (X_3^2 - 5X_5^2)$$

(with $K = \mathbb{Q}$) due to Birch and Swinnerton-Dyer [4]. However it is known that for a nonsingular intersection defined by a pair of forms in 8 or more variables both the Hasse principle and weak approximation hold, see Heath-Brown [11, Theorem 1]. One cannot dispense with the smoothness condition here. The example

$$6X_1^2 - X_2^2 - X_3^2 = X_4^2 + \ldots + X_n^2 = 0$$

over the field $\mathbb{Q}$, has points in every completion, as soon as $n \geq 6$, but has no rational points.

The difficulty with the above example arises from the real completions, and it was shown by Colliot-Thélène, Sansuc and Swinnerton-Dyer [6, Theorem C] that any pair of quadratic forms defined over a totally complex number field K has a common zero over K (and hence satisfies the Hasse principle) as soon as the number of variables n is at least 9. This enables us to handle an intersection $\mathcal{R}$ of three quadrics over a totally complex number field K, using the method of Leep [13]. If $n \geq 21$, then the projective variety $Q_3 = 0$ automatically contains a linear space of dimension 8 defined over K, since it must split off 9 hyperbolic planes. However the pair of forms Q_1, Q_2 will have a common zero over K in this linear space, by the above mentioned result of Colliot-Thélène, Sansuc and Swinnerton-Dyer, and we deduce that any triple of quadratic forms in at least 21 variables, defined over a totally complex number field K, will have a common zero in K.

However we wish to handle general number fields, and so we will assume that the variety $\mathcal{R}$ is nonsingular, or more precisely, that we have a nonsingular complete intersection. Explicitly, we shall require that the matrix

$$\left(\begin{array}{c}\nabla Q_1(\mathbf{x})\\ \nabla Q_2(\mathbf{x})\\ \nabla Q_3(\mathbf{x})\end{array}\right)$$

has rank 3 for every point $[\mathbf{x}]$ in $\mathcal{R}(\overline{K})$. When this condition holds we will say that Q_1, Q_2, Q_3 is a "nonsingular system" of quadratic forms. This ensures that $\mathcal{R}$ is an absolutely irreducible variety of codimension 3 and degree 8. (See for example Hartshorne [8, §I.5 and Ex. II.8.4, p. 188, part c].) For nonsingular complete intersections over $\mathbb{Q}$ one can apply the well-known result of Birch [3], which was proved using the Hardy–Littlewood circle method. Indeed Birch's work was generalized to arbitrary number fields by Skinner [17], who explicitly considers the question of weak approximation. The outcome is that, for a nonsingular complete intersection, $\mathcal{R}$ will satisfy the Hasse principle and weak approximation provided that $n > d+24$, where d is the dimension of the "Birch singular locus". In the present context the Birch singular locus is the projective variety given by

$$\mathrm{rank}\left(\begin{array}{c}\nabla Q_1(\mathbf{x})\\ \nabla Q_2(\mathbf{x})\\ \nabla Q_3(\mathbf{x})\end{array}\right) < 3.$$

In fact, for a nonsingular system of 3 quadratic forms the Birch singular locus will have dimension at most 2, so that it suffices to have $n \geq 27$.

It transpires that in our situation it is automatic that one has weak approximation, providing that the Hasse principle holds. Specifically, if $\mathcal{R}$ is a nonsingular complete intersection with $n \geq 12$, and $\mathcal{R}(K)$ is non-empty, then one has weak approximation by the result of Skorobogatov [18, Theorem 3, p.214]. Indeed Skorobogatov's theorem is much more general than this, and handles many singular intersections of three quadrics.

Having described the relevant background we are ready to state our principal result.

Theorem *Let Q_1, Q_2, Q_3 be a nonsingular system of quadratic forms in n variables, defined over a number field K. Then the variety*

$$\mathcal{R} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3,$$

where $\mathcal{Q}_i$ is the quadric $Q_i = 0$, satisfies both the Hasse principle and weak approximation, as soon as $n \geq 19$.

Thus we get both an improvement on the range $n \geq 21$ mentioned above for the case of totally imaginary fields, and on the range $n \geq 27$ coming from the methods of Birch and Skinner. We should also observe at this point that $\mathcal{R}$ will have points over any completion K_v at a finite place v, as soon as $n \geq 17$. This is established by Heath-Brown [10, page 138] when $K = \mathbb{Q}$, and the proof for general number fields K is completely analogous. Thus as far as the Hasse principle is concerned our theorem only requires solvability in the real completions of K. In the same connection we mention that $\mathcal{R}$ has local points over K_v for finite places over primes $p \geq 37$ as soon as $n \geq 13$, see Heath-Brown [9, Corollary 1].

Our basic strategy for proving the theorem will be to try and find a linear space L of dimension 7, defined over K and lying in the quadric hypersurface $\mathcal{Q}_3$. If we can ensure that the variety $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$ is nonsingular, we can then apply Theorem 1 of Heath-Brown [11], as mentioned above. This approach will require us firstly to establish the smoothness condition, and secondly to ensure that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$ has points everywhere locally.

We conclude this introduction by recording our thanks to the anonymous referee, whose meticulous reading and helpful comments on the original version of the paper have led to a number of improvements.

2 Geometric Considerations

We begin by replacing the forms Q_1, Q_2, Q_3 by more convenient ones. We will write $\mathbf{Q} = (Q_1, Q_2, Q_3)$ for our triple of quadratic forms, and proceed to consider linear combinations $\mathbf{t}.\mathbf{Q} = t_1Q_1 + t_2Q_2 + t_3Q_3$, where we write $\mathbf{t}$ for the vector (t_1, t_2, t_3) for brevity. The determinant $d_1(\mathbf{t}) := \det(\mathbf{t}.\mathbf{Q})$ is a form in t_1, t_2, t_3 of degree n. We also define the determinant

$$\delta(X, Y; \mathbf{t}^{(1)}, \mathbf{t}^{(2)}) := \det(X\mathbf{t}^{(1)}.\mathbf{Q} + Y\mathbf{t}^{(2)}.\mathbf{Q})$$

and the discriminant

$$d_2(\mathbf{t}^{(1)}, \mathbf{t}^{(2)}) := \mathrm{Disc}(\delta(X, Y; \mathbf{t}^{(1)}, \mathbf{t}^{(2)})).$$

Thus $\delta(X, Y; \mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ is a form of degree n in X and Y, and $d_2(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ is bihomogeneous in the entries of $\mathbf{t}^{(1)}$ and $\mathbf{t}^{(2)}$.

It will be convenient to record some properties of nonsingular systems of two quadratic forms. The following lemma follows from Reid [14, Proposition 2.1], for example.

Lemma 2.1 *Let $R_1(x_1, \ldots, x_m)$ and $R_2(x_1, \ldots, x_m)$ be quadratic forms over an algebraically closed field k of characteristic zero, and suppose that they constitute a nonsingular system, so that $\nabla R_1(\mathbf{x})$ and $\nabla R_2(\mathbf{x})$ are linearly independent for any non-zero $\mathbf{x}$ satisfying $R_1(\mathbf{x}) = R_2(\mathbf{x}) = 0$. Then every non-trivial linear combination $aR_1 + bR_2$ has rank at least $m - 1$. Moreover $\det(XR_1 + YR_2)$ is not identically zero, and has distinct linear factors over k. Conversely, this last condition is equivalent to the statement that R_1 and R_2 form a nonsingular system.*

In particular we see that if $\mathbf{a}^{(1)}, \mathbf{a}^{(2)} \in K^3$ then the variety

$$\mathbf{a}^{(1)}.\mathbf{Q} = \mathbf{a}^{(2)}.\mathbf{Q} = 0$$

is a nonsingular complete intersection if and only if $d_2(\mathbf{a}^{(1)}, \mathbf{a}^{(2)}) \neq 0$.

We now use the following result of Aznar [1, §2].

Lemma 2.2 *Let $V \subset \mathbb{P}^{n-1}$ be a nonsingular complete intersection of codimension r, which is defined over a field k of characteristic zero. Then there is a system of generators $F_1, \ldots, F_r \in k[\mathbf{x}]$ of the ideal of V, with*

$$\deg F_1 \geq \cdots \geq \deg F_r,$$

such that the varieties

$$W_i: \quad F_1 = \cdots = F_i = 0, \quad (i \le r),$$

are nonsingular complete intersections.

When V is originally defined by r forms G_j of the same degree d, the proof of Aznar's result shows that one can take the F_i to be linear combinations of the G_i.

In our case Lemma 2.2 implies in particular that there are linear combinations $\mathbf{m}^{(1)}.\mathbf{Q}$ and $\mathbf{m}^{(2)}.\mathbf{Q}$, with $\mathbf{m}^{(1)}, \mathbf{m}^{(2)} \in \overline{K}^3$, such that both the hypersurface $\mathbf{m}^{(1)}.\mathbf{Q} = 0$ and the intersection $\mathbf{m}^{(1)}.\mathbf{Q} = \mathbf{m}^{(2)}.\mathbf{Q} = 0$ are nonsingular. In particular we will have $d_1(\mathbf{m}^{(1)}) \ne 0$, so that the form $d_1(\mathbf{t})$ does not vanish identically. Moreover it follows from Lemma 2.1 that the intersection $\mathbf{m}^{(1)}.\mathbf{Q} = \mathbf{m}^{(2)}.\mathbf{Q} = 0$ is nonsingular if and only if $d_2(\mathbf{m}^{(1)}, \mathbf{m}^{(2)})$ is non-zero, and so we deduce that the form $d_2(\mathbf{t}^{(1)}, \mathbf{t}^{(2)})$ does not vanish identically.

We therefore see that we can choose three vectors $\mathbf{m}^{(1)}, \mathbf{m}^{(2)}, \mathbf{m}^{(3)} \in K^3$ such that none of

$$d_1(\mathbf{m}^{(3)}), \;\; d_2(\mathbf{m}^{(1)}, \mathbf{m}^{(2)}), \;\; d_2(\mathbf{m}^{(1)}, \mathbf{m}^{(3)}), \;\; \text{or} \;\; \det\left(\mathbf{m}^{(1)}|\mathbf{m}^{(2)}|\mathbf{m}^{(3)}\right),$$

vanishes. Then the three forms $Q_i'(\mathbf{x}) = \mathbf{m}^{(i)}.\mathbf{Q}(\mathbf{x})$, for $i = 1, 2, 3$ are defined over K, and since $\det\left(\mathbf{m}^{(1)}|\mathbf{m}^{(2)}|\mathbf{m}^{(3)}\right) \ne 0$ they will generate the same linear system as do Q_1, Q_2, Q_3. Moreover, since $d_1(\mathbf{m}^{(3)}) \ne 0$ we see that the quadric $Q_3' = 0$ is nonsingular. Finally, if $d_2(\mathbf{m}^{(1)}, \mathbf{m}^{(2)})$ and $d_2(\mathbf{m}^{(1)}, \mathbf{m}^{(3)})$ are non-zero then the varieties $Q_1' = Q_2' = 0$ and $Q_1' = Q_3' = 0$ are nonsingular complete intersections. Thus without loss of generality, after replacing Q_1, Q_2, Q_3 by Q_1', Q_2', Q_3' we may assume that our original forms satisfy these conditions.

We will also require $\mathcal{Q}_3$ to contain suitable linear spaces defined over the real completions of K. Suppose as above that Q_1 and Q_3 form a nonsingular system, and that K_v is real. Then the argument of Heath-Brown [11, Lemma 12.1] shows that there is a real θ_v for which $(\cos\theta_v)Q_1 + (\sin\theta_v)Q_3 = 0$ is nonsingular and contains a linear space of dimension at least $(n-4)/2$ over K_v. (The argument of [11] does not explicitly produce a nonsingular quadratic form, but since the functions n_+ and n_- are everywhere locally minimal one can change θ_v slightly, if necessary.) By weak approximation in $\mathbb{P}^1(K)$ we deduce that there exists $c \in K$ for which $cQ_1 + Q_3 = 0$ is also nonsingular and contains linear spaces of dimension at least $(n-4)/2$ over each real completion K_v. We now replace Q_3 by $cQ_1 + Q_3$ so that $\mathcal{Q}_3$ is nonsingular and has linear spaces over each real completion, of dimension at least $(n-4)/2$. For finite places v it is automatic that $\mathcal{Q}_3$ contains linear spaces over K_v, of dimension at least $(n-5)/2$. We can therefore conclude that $\mathcal{Q}_3$ has a linear space over K, with dimension at least $(n-5)/2$. (We refer the reader to Serre [16, Chapter 4] for the relevant theory of quadratic forms, and in particular for the Hasse–Minkowski theorem.) The existence of a single such linear space is enough to ensure that there is one through every K-point of $\mathcal{Q}_3$. An alternative way to express the above facts is to say that Q_3 splits off at least $(n-5)/2$ hyperbolic planes over K.

From now on we will fix the forms Q_1, Q_2 and Q_3. We remind the reader that the varieties $\mathcal{R}$, $\mathcal{Q}_1 \cap \mathcal{Q}_2$ and $\mathcal{Q}_3$ are all nonsingular, and that we have arranged that $\mathcal{Q}_3$ contains a linear space, defined over K, of dimension at least $(n-5)/2$.

For the remainder of this section we work over an arbitrary algebraically closed field k of characteristic zero. After the above preliminary manoeuvres we are ready to prove our first key result.

It will be convenient to define

$$F_t = F_t(\mathcal{Q}_3) := \{L \in \mathbb{G}(t, n-1) : L \subset \mathcal{Q}_3\},$$

where $\mathbb{G}(t, n-1)$ is the Grassmannian of projective linear spaces of dimension t in $\mathbb{P}^{n-1}$.

Lemma 2.3 *Suppose that the forms Q_1, Q_2, Q_3 are such that the varieties $\mathcal{R} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3$, $\mathcal{Q}_1 \cap \mathcal{Q}_2$, and $\mathcal{Q}_3$ are all nonsingular. Then for every integer t in the range $3 \leq t \leq (n-5)/2$ there is an $L \in F_t$, such that $L \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ is a nonsingular complete intersection.*

For the proof we will need some information about t-planes lying in quadric hypersurfaces. We begin by introducing some notation. Let $Q(x_1, \ldots, x_n)$ be a quadratic form of rank r, over k, and let $\mathcal{Q}$ be the quadric $Q = 0$, with dimension $n-2$. Let $F(n, r, t)$ be the Fano variety of t-planes in $\mathcal{Q}$, and let $F(n, r, t; P)$ be the subvariety of such planes passing through a given point $P \in \mathcal{Q}$. The variety $F(n, r, t)$ will be non-empty when $t \leq n - r/2 - 1$. Write $d_0(n, t, r) = \dim F(n, t, r)$, which will be independent of the particular quadratic form Q. Similarly let $\dim F(n, t, r; P) = d_1(n, t, r)$ for P a smooth point of $\mathcal{Q}$. These dimensions are given by the following lemma.

Lemma 2.4 *We have the following statements.*

(i) $d_1(n, t, r) = d_0(n-2, t-1, r-2)$ *if* $t \geq 1$ *and* $r \geq 2$.

(ii) $d_0(n, t, r) = n - 2 - t + d_0(n-2, t-1, r-2)$ *if* $t \geq 1$ *and* $2 \leq r \leq 2n - 2t - 2$.

(iii) $d_0(n, t, r) = (t+1)(n - 2 - 3t/2)$ *if* $2t + 2 \leq r \leq 2n - 2t - 2$.

This result is easy and should be well-known. We include a proof for completeness.

We take $P = [\mathbf{e}_1]$ and extend to a basis $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of K^n. Then Q takes the shape $x_1 L(x_2, \ldots, x_n) + Q'(x_2, \ldots, x_n)$ with respect to this basis, with $L \neq 0$. A further change of basis simplifies this to $x_1 x_2 + Q''(x_3, \ldots, x_n)$. Here Q'' will have rank $r-2$. One then sees that t-planes in $\mathcal{Q}$ containing $\mathbf{e}_1$ correspond to $(t-1)$-planes in $Q'' = 0$, and the result (i) follows.

For part (ii) we consider the incidence correspondence

$$I = \{(P, L) : P \in L \in F(n, t, r)\}.$$

The projection π_2 onto the second factor takes I onto $F(n, t, r)$, and each fibre has dimension t, so that $d_0(n, t, r) = \dim(I) - t$. The condition $r \leq 2n - 2t - 2$ ensures that $d_0(n-2, t-1, r-2) \geq 0$, whence $d_1(n, t, r) \geq 0$ by part (i). Thus, for the projection π_1 onto the first factor, we see that $\pi_1(I)$ contains every smooth point of $\mathcal{Q}$. Hence $\pi_1(I) = \mathcal{Q}$, and $\pi_1^{-1}(P)$ will have dimension $d_1(n, t, r)$ for smooth points P. Thus $\dim(I) = \dim(\mathcal{Q}) + d_1(n, t, r)$, and the result follows from part (i).

Finally, part (iii) follows from part (ii) by induction on t, the result being clearly true for $t = 0$.

We can now move to the proof of Lemma 2.3 By Lemma 2.4 we have

$$\dim(F_t) = d_0(n,t,n) = (t+1)(n-2-3t/2)$$

if $2t+2 \le n$. We proceed to consider the variety $F^\dagger$ defined to be the set of t-planes $L \in F$ for which $L \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ is singular. Let

$$I := \{(L, [\mathbf{x}], [\mathbf{t}]) \in F_t \times \mathcal{R} \times \mathbb{P}^1 : [\mathbf{x}] \in L, \mathbf{y}^T(t_1 Q_1 + t_2 Q_2)\mathbf{x} = 0 \,\forall [\mathbf{y}] \in L\}.$$

If $L \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ has a singular singular point $\mathbf{x}$, then $\mathbf{x} \in L \cap \mathcal{R}$ and there is some $[\mathbf{t}] \in \mathbb{P}^1$ such that $\mathbf{y}^T(t_1 Q_1 + t_2 Q_2)\mathbf{x} = 0$ for every $[\mathbf{y}] \in L$. Thus if π_1 is the projection from I onto its first factor, one has $\pi_1(I) = F^\dagger$. It follows that $\dim(F^\dagger) \le \dim(I)$.

Now consider the projection $\pi_{2,3}$ onto the second and third factors. The fibre above the pair $([\mathbf{x}], [\mathbf{t}])$ will be

$$\{L \in F_t : [\mathbf{x}] \in L, \mathbf{y}^T(t_1 Q_1 + t_2 Q_2)\mathbf{x} = 0 \,\forall [\mathbf{y}] \in L\}. \tag{2.1}$$

We write $Q = t_1 Q_1 + t_2 Q_2$ and

$$H(\mathbf{x}, \mathbf{t}) = \{[\mathbf{y}] : \mathbf{y}^T Q \mathbf{x} = 0\}$$

for convenience. Thus the fibre (2.1) may be written as

$$\{L \in F_t : L \subseteq H(\mathbf{x}, \mathbf{t}), [\mathbf{x}] \in L\}.$$

Since $[\mathbf{x}] \in \mathcal{R}$, and $\mathcal{R}$ is nonsingular, we must have $Q\mathbf{x} \ne \mathbf{0}$, so that $H(\mathbf{x}, \mathbf{t})$ is a hyperplane. Hence $H(\mathbf{x}, \mathbf{t})$ intersects $\mathcal{Q}_3$ to produce a quadric hypersurface $\mathcal{Q}'$ in $\mathbb{P}^{n-2}$, with rank, r say, at least $n-2$. The point $[\mathbf{x}]$ must be a smooth point of $\mathcal{Q}'$, since otherwise $Q\mathbf{x}$ and $Q_3\mathbf{x}$ would be proportional, contradicting the fact that $[\mathbf{x}]$ is a smooth point of $\mathcal{R}$.

It follows that the dimension of the fibre (2.1) will be $d_1(n-1, t, r)$. According to Lemma 2.4 we have

$$d_1(n-1,t,r) = d_0(n-3, t-1, r-2) = t\big(n-5-3(t-1)/2\big)$$

if $2(t-1)+2 \le r-2 \le 2(n-3) - 2(t-1) - 2$. Since we are assuming that $t \le (n-5)/2$ the required condition on r certainly holds. We therefore deduce that

$$\begin{aligned}
\dim(F^\dagger) \le \dim(I) &= \dim(\mathcal{R} \times \mathbb{P}^1) + \dim \pi_{2,3}^{-1}([\mathbf{x}], [\mathbf{t}]) \\
&= \big((n-4)+1\big) + t\big(n-5-3(t-1)/2\big) \\
&= (t+1)\big(n-2-3t/2\big) - 1 \\
&= \dim(F_t) - 1
\end{aligned}$$

so that $F^\dagger$ must be a proper subvariety of F_t. This completes the proof of Lemma 2.3.

We turn now to our second key result.

Lemma 2.5 *Suppose that the forms Q_1, Q_2, Q_3 are such that the varieties $\mathcal{R} = \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap \mathcal{Q}_3$, $\mathcal{Q}_1 \cap \mathcal{Q}_2$, and $\mathcal{Q}_3$ are all nonsingular. Then for every non-negative integer $t \le (n-5)/2$ there is an $L \in F_t$ such that $(Q_3 L)^\perp \cap \mathcal{R}$ is nonsingular.*

Here we define

$$(Q_3L)^{\perp} := \{[\mathbf{y}] : \mathbf{y}^T Q_3 \mathbf{x} = 0, \forall [\mathbf{x}] \in L\}.$$

We prove this by induction on t. For the base case $t = 0$ the space L is a single point P, say. We will write $\mathcal{R}^*$ for the dual variety to $\mathcal{R}$, and Q_3^{-1} for the linear map whose matrix is the inverse of the matrix Q_3. The $L = \{P\}$ fulfils the conditions of the lemma provided that $P \in \mathcal{Q}_3$ and $P \notin Q_3^{-1}(\mathcal{R}^*)$. There will always be a suitable point P if $\mathcal{Q}_3 \not\subseteq Q_3^{-1}(\mathcal{R}^*)$. However $\mathcal{R}^*$ is a proper subvariety of $\mathbb{P}^{n-1}$. We claim that it cannot be a nonsingular quadric, whence we cannot have $\mathcal{Q}_3 = Q_3^{-1}(\mathcal{R}^*)$. To prove the claim we merely observe that if $\mathcal{R}^* = \mathcal{Q}$, say, then $\mathcal{R} = \mathcal{R}^{**} = \mathcal{Q}^*$. However $\mathcal{Q}^*$ is itself a nonsingular quadric, whereas $\mathcal{R}$ has degree 8. The lemma then follows in the case of dimension zero.

To establish the induction step we suppose we have a suitable space L of dimension t, and look for an appropriate L' of dimension $t+1$. Indeed we shall restrict our attention to linear spaces satisfying

$$L \subset L' \subseteq \mathcal{Q}_3 \cap (Q_3L)^{\perp}.$$

This requirement on L' allows us to restrict all our varieties to the subspace $(Q_3L)^{\perp}$. We write $\mathcal{Q}_i' = \mathcal{Q}_i \cap (Q_3L)^{\perp}$ for $i = 1, 2, 3$, and $\mathcal{R}' = \mathcal{R} \cap (Q_3L)^{\perp}$. Thus $\mathcal{R}'$ is nonsingular. More concretely, we choose $[\mathbf{e}_0], \ldots, [\mathbf{e}_t]$ spanning L, and extend these to a set of points $[\mathbf{e}_0], \ldots, [\mathbf{e}_{n-t-2}]$ spanning $(Q_3L)^{\perp}$, and then to a set $[\mathbf{e}_0], \ldots, [\mathbf{e}_{n-1}]$ spanning $\mathbb{P}^{n-1}$. We proceed to write

$$Q_i'(x_1, \ldots, x_{n-t-1}) = Q_i(\sum_{j=1}^{n-t-1} x_j \mathbf{e}_{j-1}) \quad (i = 1, 2, 3)$$

so that these quadratic forms correspond to the quadrics $\mathcal{Q}_i'$, seen as varieties in $\mathbb{P}^{n-t-2}$. We note in particular that Q_3' is singular, with rank $n - 2t - 2$. Indeed the quadric hypersurface $\mathcal{Q}_3' \subset \mathbb{P}^{n-t-2}$ will be a cone with vertex set $L \subset \mathbb{P}^{n-t-2}$.

We construct L' as $< L, [\mathbf{x}] >$ with $[\mathbf{x}] \in \mathcal{Q}_3'$ but $[\mathbf{x}] \notin L$. This will ensure that $L' \subseteq \mathcal{Q}_3'$ and that $\dim(L') = t + 1$. Moreover we will have $(Q_3L')^{\perp} = (Q_3L)^{\perp} \cap (Q_3\mathbf{x})^{\perp}$, so that $(Q_3L')^{\perp} \cap \mathcal{R}'$ will be nonsingular provided that $[Q_3\mathbf{x}]$ is not in $(\mathcal{R}')^*$. Since $\dim(L) = t < \dim(\mathcal{Q}_3') = n - t - 3$, the generic $[\mathbf{x}] \in \mathcal{Q}_3'$ will satisfy $[\mathbf{x}] \notin L$. Moreover it will also satisfy $[Q_3\mathbf{x}] \notin (\mathcal{R}')^*$ by a similar reasoning to that given in the dimension zero case above. Specifically, $(\mathcal{R}')^*$ is a proper subvariety of $\mathbb{P}^{n-t-2}$, so the only situation to rule out is that in which it is equal to $\mathcal{Q}_3'$. However if Q_3' has rank $n - 2t - 2$ then the dual $(\mathcal{Q}_3')^*$ will be a quadric of dimension $n - 2t - 4$, which cannot possibly be the variety $\mathcal{R}'$, since the latter will have degree 8.

It follows that we can take $L' =< L, [\mathbf{x}] >$ for a generic $[\mathbf{x}] \in \mathcal{Q}_3'$. This completes the induction step and so establishes the lemma.

Following on from Lemmas 2.3 and 2.5 we make the following definitions.

Definition *Let Q_1, Q_2, Q_3 be a nonsingular system of quadratic forms in n variables as above, and let $L \subseteq \mathbb{P}^{n-1}$ be a linear space contained in the quadric $\mathcal{Q}_3$. Then we say that L is "admissible" if and only if both $L \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ and $(Q_3L)^{\perp} \cap \mathcal{R}$ are nonsingular. If L has dimension t in the range $3 \leq t \leq 7$ we say that L is "chain-admissible" if there exist admissible linear spaces $L = L_t \subset L_{t+1} \subset \ldots \subset L_7$ with $\dim(L_i) = i$ for $t \leq i \leq 7$.*

We then have the following result.

Lemma 2.6 *For each $t \in [3,7]$ there is a Zariski-closed proper subset $Z_t \subset F_t$, such that $L \in F_t$ is chain-admissible if and only if $L \notin Z_t$.*

It is clear that the $L \in F_t$ for which $L \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ is singular form a closed subset A_t say, of F_t. Similarly those for which $(Q_3 L)^\perp \cap \mathcal{R}$ is singular form a closed subset B_t say. By Lemmas 2.3 and 2.5 these are proper closed subsets of F_t. Moreover, since F_t is absolutely irreducible, the union $A_t \cup B_t = C_t$, say, is also a closed proper subset of F_t. A linear subspace $L \in F_t$ is admissible if and only if $L \notin C_t$. We proceed to prove Lemma 2.6 by downwards induction, and it is clear that we can take $Z_7 = C_7$.

For $t < 7$ if $L \in F_t$ fails to be chain-admissible then either $L \in C_t$ (because L is not itself admissible) or L' fails to be chain-admissible for every $L' \in F_{t+1}$ containing L. This latter case holds precisely when $L' \in Z_{t+1}$ for every such L'. Write D_t for the set of $L \in F_t$ with the property that $L' \in Z_{t+1}$ for every $L' \in F_{t+1}$ containing L. We claim that D_t is a proper closed subset of F_t. Once this is established the induction step of the proof is completed by taking $Z_t = C_t \cup D_t$.

To handle D_t we consider

$$I := \{(L, L') \in F_t \times Z_{t+1} : \, L \subset L'\}.$$

For the projection onto the first factor we have

$$\dim(\pi_1^{-1}(L)) = \dim\{L' \in Z_{t+1} : \, L \subset L'\} \leq n - 2t - 4,$$

with equality exactly when $L' \in Z_{t+1}$ for every linear space $L' \in F_{t+1}$ containing L. Thus D_t is the set of L for which $\dim(\pi_1^{-1}(L))$ is maximal, whence D_t is Zariski-closed, by the upper semi-continuity property of the dimension of fibres (Harris [7, Theorem 11.12], for example). Moreover we have

$$\dim(D_t) \leq \dim(I) - (n - 2t - 4).$$

On the other hand, for the projection onto the second factor we have

$$\dim(\pi_2^{-1}(L')) = t + 1,$$

whence

$$\dim(I) = \dim(Z_{t+1}) + t + 1.$$

Since Z_{t+1} is a proper subset of F_{t+1} by the downward induction hypothesis, we deduce that

$$\begin{array}{rcl} \dim(D_t) & \leq & \dim(I) - (n - 2t - 4) \\ & = & \dim(Z_{t+1}) + t + 1 - (n - 2t - 4) \\ & < & \dim(F_{t+1}) - n + 3t + 5 \\ & = & (t+2)(n - 2 - 3(t+1)/2) - n + 3t + 5 \\ & = & (t+1)(n - 2 - 3t/2) \\ & = & \dim(F_t). \end{array}$$

Here we have used Lemma 2.4 to compute

$$\dim(F_{t+1}) = d_0(n, t+1, n) = (t+2)(n - 2 - 3(t+1)/2)$$

and

$$\dim(F_t) = d_0(n,t,n) = (t+1)(n-2-3t/2).$$

The above calculation shows that $\dim(D_t) < \dim(F_t)$, so that D_t is a proper subset of F_t, as required.

We conclude this section with two easy results in a similar vein.

Lemma 2.7 *There is a Zariski-closed proper subset $\mathcal{R}_0 \subset \mathcal{R}$ such that every $P \in \mathcal{R} - \mathcal{R}_0$ is contained in a chain-admissible linear space $L \in F_3 - Z_3$.*

Let $\mathcal{R}_0$ be the set of points $P \in \mathcal{R}$ such that $L \in Z_3$ for every $L \in F_3$ which contains P. We need to show that $\mathcal{R}_0$ is a proper closed subset of $\mathcal{R}$. It follows from Lemma 2.6 that there is at least one $L \in F_3 - Z_3$. Since $\dim(L) + \dim(\mathcal{R}) = 3 + (n-4) = n-1$ it follows that $L \cap \mathcal{R}$ is non-empty, containing P say. Then $P \in \mathcal{R} - \mathcal{R}_0$, so that $\mathcal{R}_0$ is a proper subset of $\mathcal{R}$.

To show that $\mathcal{R}_0$ is Zariski-closed we consider

$$I = \{(P,L) \in \mathcal{R} \times Z_3 : P \in L\}.$$

For the projection onto the first factor we have

$$\dim(\pi_1^{-1}(P)) \le d_1(n,3,n),$$

in the notation of Lemma 2.4, with equality exactly when $L \in Z_3$ for every $L \in F_3$ which contains P. Thus $\mathcal{R}_0$ is the set of P for which $\dim(\pi_1^{-1}(P))$ is maximal, and it follows that $\mathcal{R}_0$ is Zariski-closed, as claimed.

Lemma 2.8 *Given $L \in F_3 - Z_3$, there is a Zariski-closed proper subset $\mathcal{R}_1$ of $\mathcal{R} \cap (Q_3L)^\perp$, such that $< L, P > \in F_4 - Z_4$ for every $P \in \mathcal{R} \cap (Q_3L)^\perp - \mathcal{R}_1$. Similarly, given $L \in F_t - Z_t$, for some $t \in [3,6]$, there is a Zariski-closed proper subset E_t of $\mathcal{Q}_3 \cap (Q_3L)^\perp$, such that $< L, P > \in F_{t+1} - Z_{t+1}$ for every $P \in \mathcal{Q}_3 \cap (Q_3L)^\perp - E_t$.*

To prove the first part of the lemma we let

$$I := \{(P,L') \in (\mathcal{R} \cap (Q_3L)^\perp) \times Z_4 : P \in L', \, L \subset L'\},$$

and take $\mathcal{R}_1 = \pi_1(I)$, so that $\mathcal{R}_1 \subseteq \mathcal{R} \cap (Q_3L)^\perp$ is clearly Zariski-closed. Since $L \in F_3 - Z_3$ is chain-admissible there is at least one linear space $L_0 \in F_4 - Z_4$ containing L. Then $L_0 \cap \mathcal{R} = L_0 \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ has dimension 2, since L_0 is admissible, and similarly $L \cap \mathcal{R} = L \cap \mathcal{Q}_1 \cap \mathcal{Q}_2$ has dimension 1. We may therefore find a point $P \in L_0 \cap \mathcal{R} - L \cap \mathcal{R}$. We claim that $P \in \mathcal{R} \cap (Q_3L)^\perp - \mathcal{R}_1$, which shows that $\mathcal{R}_1$ is a proper subset of $\mathcal{R} \cap (Q_3L)^\perp$, and so proves the first part of the lemma. Since $L \subset L_0 \subset \mathcal{Q}_3$ it follows that $L_0 \subseteq (Q_3L)^\perp$, whence $P \in L_0 \cap \mathcal{R} \subseteq (Q_3L)^\perp \cap \mathcal{R}$. On the other hand if we had $P \in \mathcal{R}_1$ there would be a linear space L' such that $(P, L') \in I$. Then we would have $P \in L'$ and $L \subset L'$, and therefore $L' =< L, P >$, since $P \notin L$ by our choice of P. However the same reasoning shows that L_0 is also equal to $< L, P >$, so that $L' = L_0$. This gives us a contradiction, since $L' \in Z_4$ while $L_0 \in F_4 - Z_4$.

Turning to the second part of the lemma, we consider

$$I := \{(P,L') \in (\mathcal{Q}_3 \cap (Q_3L)^\perp) \times Z_{t+1} : P \in L', \, L \subset L'\},$$

and take $E_t = \pi_1(I)$, so that $E_t \subseteq \mathcal{Q}_3 \cap (Q_3L)^\perp$ is clearly Zariski-closed. Since $L \in F_t - Z_t$ is chain-admissible there is at least one linear space $L_0 \in F_{t+1} - Z_{t+1}$ containing L. We claim that $P \in \mathcal{Q}_3 \cap (Q_3L)^\perp - E_t$ for any point $P \in L_0 - L$, which will show that E_t is a proper subset of $\mathcal{Q}_3 \cap (Q_3L)^\perp$. Since $L \subset L_0 \subset \mathcal{Q}_3$ it follows that $L_0 \subseteq (Q_3L)^\perp$, whence $P \in \mathcal{Q}_3 \cap (Q_3L)^\perp$. On the other hand if we had $P \in E_t$ there would be a linear space L' such that $(P, L') \in I$. Then we would have $P \in L'$ and $L \subset L'$, and therefore $L' =< L, P >$, since $P \notin L$ by our choice of P. However the same reasoning shows that L_0 is also equal to $< L, P >$, so that $L' = L_0$. This gives us a contradiction, since $L' \in Z_{t+1}$ while $L_0 \in F_{t+1} - Z_{t+1}$.

3 Global 3-planes in $\mathcal{Q}_3$

In this section we shall make repeated use of three key principles. The first of these is the fact that we have weak approximation on quadrics. The second is that if V is an absolutely irreducible projective variety defined over the number field K, with a smooth point P_v in some completion K_v, then the K_v-points of V are Zariski-dense in any given neighbourhood of P_v. (This should be well-known, but the reader may see Browning, Dietmann and Heath-Brown [5, Lemma 3.4], for example. The proof is an application of the implicit function theorem.)

The third general principle is embodied in the following lemma.

Lemma 3.1 *Let V be a projective algebraic variety defined over a completion K_v of K by equations*

$$f_1(x_0, \ldots, x_m) = \ldots = f_r(x_0, \ldots, x_m) = 0,$$

and suppose that P is K_v-point on V at which the vectors $\nabla f_1, \ldots, \nabla f_r$ are linearly independent. Suppose we are given varieties $V^{(j)}$ defined over K_v by equations

$$f_1^{(j)}(x_0, \ldots, x_m) = \ldots = f_r^{(j)}(x_0, \ldots, x_m) = 0,$$

of bounded degree, in which $f_i^{(j)} \to f_i$ (under the metric induced from K_v) as $j \to \infty$. Then for sufficiently large j there are K_v-points $P^{(j)}$ on $V^{(j)}$ such that $P^{(j)} \to P$ as $j \to \infty$. Moreover $P^{(j)}$ will be a nonsingular point of $V^{(j)}$, in the sense above.

Again this should be well-known, but we present a proof for completeness. We suppose firstly that v is a finite place. We let $P = [\mathbf{t}]$ say, and we rescale $\mathbf{t}$ and the polynomials f_j so as to have v-adic integer coefficients. Since $f_i^{(j)} \to f_i$ it follows that $f_i^{(j)}$ also has integral coefficients if j is large enough. By hypothesis, the matrix formed from the rows $\nabla f_1(\mathbf{t}), \ldots, \nabla f_r(\mathbf{t})$ has rank r. We may therefore suppose without loss of generality that the determinant, Δ say, of the first r columns is non-zero. Let Δ_j be the corresponding determinant formed from $\nabla f_1^{(j)}(\mathbf{t}), \ldots, \nabla f_r^{(j)}(\mathbf{t})$. Thus $\Delta_j \to \Delta$, so that $|\Delta_j|_v = |\Delta|_v \neq 0$ if j is large enough.

We also set

$$\delta_j = \max\{|f_1^{(j)}(\mathbf{t})|_v, \ldots, |f_r^{(j)}(\mathbf{t})|_v\},$$

and note that δ_j tends to

$$\max\{|f_1(\mathbf{t})|_v, \ldots, |f_r(\mathbf{t})|_v\} = 0,$$

since $P \in V$. Thus if j is large enough we will have $\delta_j < |\Delta|_v^2$. This condition allows us to use Hensel's Lemma, which provides a point $P^{(j)} = [\mathbf{t}^{(j)}]$ on $V^{(j)}$, with

$$\max_i |t_i^{(j)} - t_i|_v \le \delta_j/|\Delta|_v. \tag{3.1}$$

It follows that $P^{(j)}$ tends to P as required. Since $\Delta_j \ne 0$ for large j the nonsingularity condition also holds.

When v is an infinite place we use a completely analogous argument, replacing Hensel's Lemma by Newton Approximation. We begin by normalizing so that the entries of $\mathbf{t}$, and the coefficients of the f_i, all have modulus at most 1. The condition $\delta_j < |\Delta|^2$ has to be replaced by $\delta_j < C|\Delta|^2$ with a constant C depending on m, r and the degrees of the polynomials involved. Similarly, the bound in (3.1) becomes $C'\delta_j/|\Delta|$ with a corresponding constant C'. With these changes the proof goes through as before.

For our theorem we will assume that we are given local points $[\mathbf{x}_v] \in \mathcal{R}(K_v)$ for every place v of K. We will also be given a finite set of places S and a (small) positive ε, and our challenge will be to find a point $[\mathbf{x}] \in \mathcal{R}(K)$ such that

$$|\mathbf{x} - \mathbf{x}_v|_v < \varepsilon \quad \text{for all } v \in S. \tag{3.2}$$

Without loss of generality we will include all infinite places in S, as well as all finite places above rational primes up to 37. In particular, S will be non-empty. From now on we will assume that the number n of variables in our quadratic forms satisfies $n \ge 19$.

The variety $\mathcal{R}$ is nonsingular, and for each $v \in S$ the K_v-points of $\mathcal{R}$ are therefore Zariski-dense in every neighbourhood of $[\mathbf{x}_v]$, by the second principle above. It follows that if $\mathcal{R}_0$ is as in Lemma 2.7 then there is a point $[\mathbf{x}'_v] \in \mathcal{R} - \mathcal{R}_0$, defined over K_v, in the neighbourhood $|\mathbf{x}_v - \mathbf{x}'_v|_v < \varepsilon/2$. Thus it suffices to find a K-point of $\mathcal{R}$ with

$$|\mathbf{x} - \mathbf{x}'_v|_v < \varepsilon/2 \quad \text{for all } v \in S,$$

where now there is a chain-admissible 3-plane $L_v \in F_3$ through $[\mathbf{x}'_v]$. We therefore change our notation, replacing $\mathbf{x}'_v$ by $\mathbf{x}_v$ and ε by $\varepsilon/2$ so that is still suffices to work with the condition (3.2). Note that L_v may be defined over $\overline{K_v}$ rather than K_v. None the less the existence of a single chain-admissible L_v shows that the generic 3-plane $L \subset \mathcal{Q}_3$ through $[\mathbf{x}_v]$ is also chain-admissible. We wish to deduce that there is a chain-admissible 3-plane L_v defined over K_v and passing through $[\mathbf{x}_v]$, by using the second of the principles described at the beginning of the present section. To do this we need to know firstly that there is at least one 3-plane $L_v \subset \mathcal{Q}_3$ defined over K_v and passing through $[\mathbf{x}_v]$, and secondly that the Fano variety of 3-planes $L \subset \mathcal{Q}_3$ passing through $[\mathbf{x}_v]$ is smooth. The first of these follows from the fact that Q_3 splits off at least $(n-5)/2$ hyperbolic planes over K_v. For the second, it suffices to note that the variety in question is a projective homogeneous space for an orthogonal group, and hence is smooth. Thus we may assume that L_v is defined over K_v.

Our plan now is to produce a sequence of 3-planes $L^{(m)}$ defined over K, which approximate L_v for each $v \in S$, in the following sense.

Lemma 3.2 *For each $v \in S$ let $L_v \in F_3 - Z_3$ be a chain-admissible 3-plane defined over K_v, and suppose that $[\mathbf{e}_{0,v}], \ldots, [\mathbf{e}_{3,v}]$ is a basis of L_v. Then there are sequences of K-points $[\mathbf{e}_0^{(m)}], \ldots, [\mathbf{e}_3^{(m)}]$ spanning chain-admissible 3-planes $L^{(m)} \subset \mathcal{Q}_3$, such that*

$$\lim_{m\to\infty} \mathbf{e}_i^{(m)} = \mathbf{e}_{i,v} \quad \textit{for } 0 \le i \le 3, \tag{3.3}$$

for every $v \in S$.

(We teach our students that a sequence can have at most one limit! There is of course some abuse of notation above, and strictly speaking we should have said that $\iota_v(\mathbf{e}_i^{(m)})$ tends to $\mathbf{e}_{i,v}$, where ι_v is the embedding of K^n into K_v^n.)

In effect Lemma 3.2 is saying that the Fano variety of 3-planes in $\mathcal{Q}_3$ satisfies weak approximation. Our argument would extend to the t-planes on a smooth quadric hypersurface in $\mathbb{P}^m$, for $t < (m-1)/2$. However weak approximation fails for $t = (m-1)/2$ in which case the Fano variety decomposes into two irreducible components, as was pointed out to the author by Dan Loughran. It seems likely that this weak approximation principle is known to others, and no claim to originality is being made. However since we require a special variant of the result we present a full proof.

We will produce the sequences $[\mathbf{e}_t^{(m)}]$ by induction on t, the case $t = 0$ merely being an instance of weak approximation on $\mathcal{Q}_3$. We therefore consider the induction step, and suppose we already have suitable sequences of vectors $\mathbf{e}_0^{(m)}, \ldots, \mathbf{e}_{t-1}^{(m)}$. Thus the conditions required for $\mathbf{e}_t^{(m)}$ are that

$$[\mathbf{e}_t^{(m)}] \in V^{(m)} := \{[\mathbf{x}] \in \mathcal{Q}_3 : \mathbf{x}^T Q_3 \mathbf{e}_i^{(m)} = 0 \text{ for } 0 \le i \le t-1\},$$

and that

$$\lim_{m\to\infty} \mathbf{e}_t^{(m)} = \mathbf{e}_{t,v} \tag{3.4}$$

for every $v \in S$. Notice that (3.3) (for $i \le t-1$) and (3.4) automatically ensure that the vectors $\mathbf{e}_0^{(m)}, \ldots, \mathbf{e}_t^{(m)}$ are linearly independent, if m is large enough. The variety $V^{(m)}$ is a quadric of rank at least $n - 2t \ge 5$, so that it must have smooth points over K_v for every finite place v. For every $v \in S$, and in particular for every infinite place, the varieties $V^{(m)}$ are approximations to

$$V_v := \{[\mathbf{x}] \in \mathcal{Q}_3 : \mathbf{x}^T Q_3 \mathbf{e}_{i,v} = 0 \text{ for } 0 \le i \le t-1\},$$

in the sense given by Lemma 3.1. Moreover $[\mathbf{e}_{t,v}]$ is a nonsingular point on V_v, in the sense of the lemma. It follows that for large enough m the variety $V^{(m)}$ has a smooth point $[\mathbf{f}_v^{(m)}]$ for every $v \in S$, such that

$$\lim_{m\to\infty} \mathbf{f}_v^{(m)} = \mathbf{e}_{t,v}.$$

In particular $V^{(m)}$ has points everywhere locally, and so has a K-point, by the Hasse principle.

We can now complete the induction step. Given $\eta > 0$ we choose $m_0(\eta)$ so that

$$|\mathbf{f}_v^{(m)} - \mathbf{e}_{t,v}|_v < \eta/2$$

for all $v \in S$, and all $m \geq m_0(\eta)$. Moreover we can use weak approximation on $V^{(m)}$ to find points $\mathbf{e}_t^{(m)}$ on $V^{(m)}(K)$ such that

$$|\mathbf{e}_t^{(m)} - \mathbf{f}_v^{(m)}|_v < \eta/2$$

for all $v \in S$, and all $m \geq m_0(\eta)$. Then for all $v \in S$ and all $m \geq m_0(\eta)$ we will have

$$|\mathbf{e}_t^{(m)} - \mathbf{e}_{t,v}|_v < \eta,$$

whence (3.4) holds, as required. Finally, since $L_v \notin Z_3$ we will have $L^{(m)} \notin Z_3$ for large enough m, by continuity, so that $L^{(m)}$ is also chain-admissible.

4 Completion of the Argument

Our strategy now is to consider the intersection of $\mathcal{Q}_1 \cap \mathcal{Q}_2$ with $L^{(m)}$. We begin with the following result.

Lemma 4.1 *Let $\varepsilon > 0$ be given. If m is large enough, for every $v \in S$ there will be a K_v-point $[\mathbf{y}_v^{(m)}] \in \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L^{(m)}$ such that*

$$|\mathbf{y}_v^{(m)} - \mathbf{x}_v|_v < \varepsilon/2.$$

This is a further application of Lemma 3.1. The varieties

$$V^{(m)} := \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L^{(m)}$$

are approximations to $V := \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L_v$, in the sense of the lemma, and $P = [\mathbf{x}_v]$ lies on V, since we have both $P \in \mathcal{R}$ and $P \in L_v$. Moreover L_v is chain-admissible, and hence in particular is admissible, whence V is nonsingular. The lemma therefore produces appropriate points $[\mathbf{y}_v^{(m)}]$.

We now fix a suitable m in Lemma 4.1, and write $L = L^{(m)}$ and $\mathbf{y}_v = \mathbf{y}_v^{(m)}$ accordingly. Thus L is a chain-admissible 3-plane, defined over K. Moreover for each $v \in S$ we have $[\mathbf{y}_v] \in \mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$. Finally, to prove our theorem it will suffice to find a point $[\mathbf{x}] \in \mathcal{R}(K)$ with

$$|\mathbf{y}_v - \mathbf{x}|_v < \varepsilon/2$$

for each $v \in S$.

For each $v \in S$ the variety $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$ has a local point, namely $[\mathbf{y}_v]$. Moreover it is nonsingular, since L is chain-admissible. Thus there are local points at all but finitely many places. Let T be the set of places for which there are no local points. Thus S and T are disjoint, so that T is a finite set of finite places v each of which lies over a prime $p \geq 37$. To handle this remaining set of places it suffices to intersect $\mathcal{Q}_1 \cap \mathcal{Q}_2$ with a suitable 4-plane, as our next result shows.

Lemma 4.2 *If T is nonempty there is a chain-admissible 4-plane L', defined over K, such that $L \subset L'$, for which $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L'$ has K_v-points for every $v \in T$.*

It follows of course that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L'$ has points over every completion of K. Naturally, if T were empty the same would already be true for $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$.

To prove Lemma 4.2 we choose K-points $[\mathbf{e}_0], \ldots, [\mathbf{e}_3]$ spanning L, and look for an additional K-point $[\mathbf{x}] = [\mathbf{e}_4]$ such that

$$L' := < [\mathbf{e}_0], \ldots, [\mathbf{e}_4] >$$

fulfils the necessary conditions. In order to have $L' \subset \mathcal{Q}_3$ we will require $[\mathbf{x}] \in \mathcal{Q}_3 \cap (Q_3 L)^{\perp}$ We would like the variety L' to be chain-admissible, and so we will require that $L' \notin Z_4$. However L itself is chain-admissible, so that there is at least one point P_0, (which might be defined over $\overline{K}$) for which

$$< L, P_0 > \in F_4 - Z_4.$$

Such a point P_0 will be smooth point of $\mathcal{Q}_3 \cap (Q_3 L)^{\perp}$. It follows that the set of points P for which $< L, P >$ is a chain-admissible 4-plane, is a nonempty Zariski-open subset (U, say) of $\mathcal{Q}_3 \cap (Q_3 L)^{\perp}$.

To arrange that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L'$ has a K_v-point it will be helpful if $[\mathbf{x}]$ is "near" to a K_v-point of $\mathcal{Q}_1 \cap \mathcal{Q}_2$. We would therefore like the variety

$$\left(\mathcal{Q}_3 \cap (Q_3 L)^{\perp}\right) \cap (\mathcal{Q}_1 \cap \mathcal{Q}_2) = (Q_3 L)^{\perp} \cap \mathcal{R}$$

to contain a K_v-point $[\mathbf{x}_v]$ for every $v \in T$. However $(Q_3 L)^{\perp}$ has dimension $n - 5 \geq 14$, so that $(Q_3 L)^{\perp} \cap \mathcal{R}$ is the zero locus of a system of three quadratic forms in at least 15 variables. There is therefore a K_v-point whenever v is a finite place above a prime $p \geq 37$, by Heath-Brown [9, Corollary 1]. (The reader should note that one only needs to know that some bound of the form $p \geq p_0$ suffices. This is a corollary of the famous Ax–Kochen Theorem [2]. However in the present situation one can now provide an explicit value of p_0. Indeed the work of Schuur [15] could also be used here.) Thus there are points $[\mathbf{x}_v] \in \mathcal{R} \cap (Q_3 L)^{\perp}$ for every $v \in T$. Indeed, since L is chain-admissible it is certainly admissible, so that $\mathcal{R} \cap (Q_3 L)^{\perp}$ must be nonsingular. If then follows that the K_v-points on $\mathcal{R} \cap (Q_3 L)^{\perp}$ will be Zariski-dense. In particular we can choose a K_v-point $[\mathbf{x}_v]$ in $\mathcal{R} \cap (Q_3 L)^{\perp} - \mathcal{R}_1$, with $\mathcal{R}_1$ as in Lemma 2.8. Thus, if we set $L_v =< L, [\mathbf{x}_v] >$, then L_v is admissible so that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L_v$ is nonsingular. In particular $[\mathbf{x}_v]$ will be a smooth point of $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L_v$.

Since $[\mathbf{x}_v] \in \mathcal{R} \cap (Q_3 L)^{\perp}$ it follows in particular that $[\mathbf{x}_v] \in \mathcal{Q}_3 \cap (Q_3 L)^{\perp}$. We now claim that $[\mathbf{x}_v]$ cannot be a singular point of $\mathcal{Q}_3 \cap (Q_3 L)^{\perp}$. For otherwise the vectors

$$Q_3 \mathbf{x}_v, Q_3 \mathbf{e}_0, \ldots, Q_3 \mathbf{e}_3$$

would be linearly dependent. Since Q_3 is nonsingular it would follow that

$$\mathbf{x}_v \in < \mathbf{e}_0, \ldots, \mathbf{e}_3 >,$$

so that $[\mathbf{x}_v] \in L$. However $[\mathbf{x}_v]$ was chosen to lie in $\mathcal{R}$, so that it would in particular be a K_v-point of $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$. We would therefore have a contradiction, since T was defined to be the set of places where $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$ had no local points.

Thus $[\mathbf{x}_v]$ is a smooth point of $\mathcal{Q}_3 \cap (Q_3 L)^{\perp}$. This variety certainly has at least one smooth K-point, since the quadratic form Q_3 was constructed to split off at least $(n - 5)/2 > 4$ hyperbolic planes over K. Thus we can use weak approximation on $\mathcal{Q}_3 \cap (Q_3 L)^{\perp}$ to produce a sequence of K-points $[\mathbf{x}^{(m)}]$, such that

$$\lim_{m\to\infty} \mathbf{x}^{(m)} = \mathbf{x}_v$$

for every $v \in T$. For large enough m a continuity argument shows that if $L^{(m)} =< L, [\mathbf{x}^{(m)}] >$ then $L^{(m)} \in F_4 - Z_4$. We also see that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L^{(m)}$ approximates $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L_v$ in the sense of Lemma 3.1. Moreover $[\mathbf{x}_v]$ is a smooth point of $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L_v$, whence Lemma 3.1 provides K_v-points $P^{(m)}$ on $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L^{(m)}$ as soon as m is large enough, for all $v \in T$. The lemma then follows on choosing $L' = L^{(m)}$ with a suitably large m.

As the final step in our argument we state the following result.

Lemma 4.3 *There is an admissible 7-plane L'' containing L', defined over K, and such that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L''$ has points over every completion of K.*

Before presenting the proof of the lemma, we show how it suffices for our theorem. Since $L'' \in F_7$ is admissible, the variety $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L''$ is contained in $\mathcal{R}$, and is a nonsingular intersection of two quadrics in $\mathbb{P}^7$. By construction it has points over K_v for every place v. Indeed the subvariety $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$ has a K_v-point $[\mathbf{y}_v]$ for every $v \in S$, and has K_v-points for all $v \notin T$, while $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L'$ has K_v-points for any remaining places $v \in T$. We may therefore apply the author's result [11, Theorem 1] described in the introduction, which shows that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L''$ satisfies the Hasse principle and weak approximation. This allows us to conclude that $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L''$ has K-points arbitrarily close to $[\mathbf{y}_v]$ for each $v \in S$. Our theorem therefore follows.

It remains to establish Lemma 4.3. We have already observed that either $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L$ has points everywhere locally, if T is empty, or $\mathcal{Q}_1 \cap \mathcal{Q}_2 \cap L'$ does. The linear spaces L and L' are chain-admissible, and are defined over K. It is therefore enough to show that if M is any chain-admissible linear space of dimension $t \in [3, 6]$, defined over K, then there is a chain admissible space $M' \supset M$ of dimension $t + 1$, also defined over K. Once this is proved we can use this repeatedly to go from L or L' to L''.

We recall that $\mathcal{Q}_3$ contains at least one 7-plane defined over K, whence M will be contained in such a 7-plane, M^*, say. We can choose a basis $\mathbf{e}_1, \ldots \mathbf{e}_n$ for K^n so that M is spanned by $[\mathbf{e}_1], \ldots, [\mathbf{e}_{t+1}]$ and M^* by $[\mathbf{e}_1], \ldots, [\mathbf{e}_8]$, and such that

$$Q_3(\sum_1^n X_i \mathbf{e}_i) = X_1 X_9 + \ldots + X_8 X_{16} + Q(X_{17}, \ldots, X_n)$$

for a suitable nonsingular form Q. Then $(Q_3 M)^{\perp}$ is spanned by $[\mathbf{e}_1], \ldots, [\mathbf{e}_8]$ and $[\mathbf{e}_{t+10}], \ldots, [\mathbf{e}_n]$, and one therefore sees that $[\mathbf{e}_8]$ will be a smooth K-point of $\mathcal{Q}_3 \cap (Q_3 M)^{\perp}$. Having shown that there is at least one smooth K-point on $\mathcal{Q}_3 \cap (Q_3 M)^{\perp}$ we deduce that the K-points are Zariski-dense, so that there is a K-point $P \in \mathcal{Q}_3 \cap (Q_3 M)^{\perp} - E_t$, in the notation of Lemma 2.8. We can then complete the proof of Lemma 4.3 by taking $M' =< M, P >$.

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